Osculating varieties of Veronesean and their higher secant varieties.

A.Bernardi, M.V.Catalisano, A.Gimigliano, M.Idà

Abstract. We consider the k-osculating varieties $O_{k,n.d}$ to the (Veronese) $d-$uple embeddings of $\mathbb{P}^n$. We study the dimension of their higher secant varieties via inverse systems (apolarity). By associating certain 0-dimensional schemes $Y \subset \mathbb{P}^n$ to $O^s_{k,n,d}$ and by studying their Hilbert function we are able, in several cases, to determine whether those secant varieties are defective or not.

0. Introduction.

Let us consider the following case of a quite classical problem: given a generic form f of degree d in $R := k[x_0, ..., x_n]$, what is the minimum s for which it is possible to write $f = L_1^{d-k}F_1 + ... + L_s^{d-k}F_s$, where $L_i \in R_1$ and $F_i \in R_k$? When $k = 0$ this is known as the "Waring problem for forms" (the original Waring problem is for integers), and it has been solved via results in [**AH**], e.g. see [**IK**] or [**Ge**].

In this generality, the problem is part of what was classically called "finding canonical forms for an $(n+1)$-ary d-ic" (e.g. see [**W**]). The following examples illustrate cases where the answer to the problem is not the expected one.

Example 1. One would expect (by a dimension count) that a generic $f \in (K[x_0, \dots, x_4])_3$ could be written as $f = L_1F_1 + L_2F_2$ with $L_i \in R_1$ and $F_i \in R_2$, but actually we need tree addenda: $f = L_1F_1 + L_2F_2 + L_3F_3$.

Example 2. We can't write a generic $f \in (K[x_0, \dots, x_5])_3$ as $f = L_1F_1 + L_2F_2 + L_3F_3$, but only as $f = L_1F_1 + \cdots + L_4F_4$.

Example 3. One would expect that a generic $f \in (K[x_0, \dots, x_6])_4$ could be written as $f = L_1F_1 + L_2F_2 + L_3F_3$, with $L_i \in R_1$ and $F_i \in R_3$, but not only is it impossible to write f as a sum of three addenda, but is it not even possible to write it as a sum of four. In fact f can only be written as $f = L_1F_1 + \cdots + L_5F_5$.

All the examples above comes from our Proposition 3.4.

Our approach to the problem is via the study of the dimension of higher secant varieties $O^s_{k,n.d}$ to $O_{k,n.d}$, the k^{th}-osculating variety to the (Veronese) $d-$uple embeddings of $\mathbb{P}^n$, since giving an answer to this geometrical problem implies getting the solution to the problem on forms.

We would like to notice that those secant varieties can reach a very high defectivity (e.g. see the example after Prop. 3.4), a phenomenon that does not happen for smooth varieties.

We use inverse system (apolarity) to reduce this problem to the study of the postulation of certain 0-dimensional schemes $Y \subset \mathbb{P}^n$, namely we reduce the evaluation of $\dim O^s_{k,n,d}$ to the evaluation of

$\dim|\mathcal{O}_{\mathbb{P}^n}(d)\otimes\mathcal{I}_Y|$ where $Y=Z_1+...+Z_s$ is a 0-dimensional subscheme of $\mathbb{P}^n$ such that, for each $i=1,...,s$, $(k+1)P_i\subset Z_i\subset(k+2)P_i$ and $l(Z_i)=\binom{k+n}{n}+n$.

We conjecture that the "bad behavior" of Y is always related to the scheme given by the fat points $(k+1)P_i$ or $Z_i\subset(k+2)P_i$ not being regular (see Conjecture 3.13). By using this idea, we are able to describe the behavior of the s^{th}-secant variety of $O_{k,n.d}$ for many values of (k,n,d).

In the case of $\mathbb{P}^2$, using known results on fat points, we are able to classify all the defective $O^s_{k,2.d}$ for small values of s ($s\leq 6$ and $s=9$, see Coroll. 3.16).

1. Preliminaries.

1.1. Notation.

i) In the following we set $R:=k[x_0,...,x_n]$, where $k=\bar{k}$ and char$k=0$, hence R_d will denote the forms of degree d on $\mathbb{P}^n$.

ii) If $X\subseteq\mathbb{P}^N$ is an irreducible projective variety, an m-fat point on X is the $(m-1)^{th}$ infinitesimal neighborhood of a smooth point P in X, and it will be denoted by mP (i.e. the scheme mP is defined by the ideal sheaf $\mathcal{I}^m_{P,X}\subset\mathcal{O}_X$).
Let $\dim X=n$; then, mP is a 0-dimensional scheme of length $\binom{m-1+n}{n}$.
If Z is the union of the $(m-1)^{th}$-infinitesimal neighborhoods in X of s generic points of X, we shall say for short that Z is union of s generic m-fat points on X.

iii) If $X\subseteq\mathbb{P}^N$ is a variety and P is a smooth point on it, the projectivized tangent space to X at P is denoted by $T_{X,P}$.

iv) We denote by $<U,V>$ both the linear span in a vector space or in a projective space of two linear subspaces U,V.

v) If X is a 0-dimensional scheme, we denote by $l(X)$ its length, while its support is denoted by supp$X$.

1.2. Definition. Let $X\subseteq\mathbb{P}^N$ be a closed irreducible projective variety; the $(s-1)^{th}$ *higher secant variety* of X is the closure of the union of all linear spaces spanned by s points of X, and it will be denoted by X^s. Let $\dim X=n$; the *expected dimension* for X^s is

$$\text{expdim}X^s:=\min\{N,sn+s-1\}$$

where the number $sn+s-1$ corresponds to ∞^{sn} choices of s points on X, plus ∞^{s-1} choices of a point on the $\mathbb{P}^{s-1}$ spanned by the s points. When this number is too big, we expect that $X^s=\mathbb{P}^N$. Since it is not always the case that X^s has the expected dimension, when $\dim X^s<\min\{N,sn+s-1\}$, X^s is said to be *defective.*

A classical result about secant varieties is Terracini's Lemma (see [**Te**], or, e.g. [**A**]), which we give here in the following form:

1.3. Terracini's Lemma: *Let X be an irreducible variety in $\mathbb{P}^N$, and let $P_1,...,P_s$ be s generic points on X. Then, the projectivised tangent space to X^s at a generic point $Q\in<P_1,...,P_s>$ is the linear span in $\mathbb{P}^N$ of the tangent spaces T_{X,P_i} to X at P_i, $i=1,...,s$, hence*

$$\dim X^s=\dim<T_{X,P_1},...,T_{X,P_s}>.$$

1.4. Corollary. *Let $(X, \mathcal{L})$ be an integral, polarized scheme. If $\mathcal{L}$ embeds X as a closed scheme in $\mathbb{P}^N$, then*

$$\dim X^s = N - \dim h^0(\mathcal{I}_{Z,X} \otimes \mathcal{L})$$

where Z is union of s generic 2-fat points in X.

Proof. By Terracini's Lemma, $\dim X^s = \dim < T_{X,P_1}, ..., T_{X,P_s} >$, with $P_1, ..., P_s$ generic points on X. Since X is embedded in $\mathbb{P}^N = \mathbb{P}(H^0(X, \mathcal{L})^*)$, we can view the elements of $H^0(X, \mathcal{L})$ as hyperplanes in $\mathbb{P}^N$; the hyperplanes which contain a space T_{X,P_i} correspond to elements in $H^0(\mathcal{I}_{2P_i,X} \otimes \mathcal{L})$, since they intersect X in a subscheme containing the first infinitesimal neighborhood of P_i. Hence the hyperplanes of $\mathbb{P}^N$ containing the subspace $< T_{X,P_1}, ..., T_{X,P_s} >$ are the sections of $H^0(\mathcal{I}_{Z,X} \otimes \mathcal{L})$, where Z is the scheme union of the first infinitesimal neighborhoods in X of the points P_i's. □

1.5. Definition. Let $X \subset \mathbb{P}^N$ be a variety, and let $P \in X$ be a smooth point; we define *the k^{th} osculating space to X at P* as the linear space generated by $(k+1)P$, and we denote it by $O_{k,X,P}$; hence $O_{0,X,P} = \{P\}$, and $O_{1,X,P} = T_{X,P}$, the projectivised tangent space to X at P.

Let $X_0 \subset X$ be the dense set of the smooth points where $O_{k,X,P}$ has maximal dimension. The *k^{th} osculating variety to X* is defined as:

$$O_{k,X} = \overline{\bigcup_{P \in X_0} O_{k,X,P}}.$$

2. Osculating varieties to Veronesean, and their higher secant varieties.

2.1. Notation.

i) We will consider here Veronese varieties, i.e. embeddings of $\mathbb{P}^n$ defined by the linear system of all forms of a given degree d: $\nu_d : \mathbb{P}^n \to \mathbb{P}^N$, where $N = \binom{n+d}{n} - 1$. The d-ple Veronese embedding of $\mathbb{P}^n$, i.e. $Im\nu_d$, will be denoted by $X_{n,d}$.

ii) In the following we set $O_{k,n,d} := O_{k,X_{n,d}}$, so that the $(s-1)^{th}$ higher secant variety to the k^{th} osculating variety to the Veronese variety $X_{n,d}$ will be denoted by $O^s_{k,n,d}$.

2.2. Remark. From now on $\mathbb{P}^N = \mathbb{P}(R_d)$; a form M will denote, depending on the situation, a vector in R_d or a point in $\mathbb{P}^N$.

We can view $X_{n,d}$ as given by the map $(\mathbb{P}^n)^* \to \mathbb{P}^N$, where $L \to L^d$, $L \in R_1$. Hence

$$X_{n,d} = \{L^d, \quad L \in R_1\}.$$

Let us assume (and from now on this assumption will be implicit) that $d \geq k$; at the point $P = L^d$ we have (see [**Se**], [**CGG**] sec.1, [**BF**] sec.2):

$$O_{k,X_{n,d},P} = \{L^{d-k}F, \quad F \in R_k\}. \qquad (*)$$

Notice that $O_{k,X_{n,d},P}$ has maximal dimension $\dim R_k - 1 = \binom{k+n}{n} - 1$ for all $P \in X_{n,d}$. This can also be seen in the following way: the fat point $(k+1)P$ on $X_{n,d}$ gives independent conditions to the hyperplanes of $\mathbb{P}^N$, since it gives independent conditions to the forms of degree d in $\mathbb{P}^n$.

Hence, $O_{k,n,d} = \bigcup_{P\in X_{n,d}} O_{k,X_{n,d},P}$.

As we have already noticed, for $k = 0$ $(*)$ gives $O_{k,X_{n,d},P} = \{P\} = \{L^d\}$, and for $k = 1$ it becomes $O_{k,X_{n,d},P} = T_{X_{n,d},P} = \{L^{d-1}F, \quad F \in R_1\}$.

In general, we have:

$$O_{k,n,d} = \{L^{d-k}F, \quad L \in R_1, \quad F \in R_k\}.$$

Hence,

$$O^s_{k,n,d} = \{L_1^{d-k}F_1 + ... + L_s^{d-k}F_s, \quad L_i \in R_1, \quad F_i \in R_k, \quad i = 1, ..., s\}.$$

In the following we also need to know the tangent space $T_{O_{k,n,d},Q}$ of $O_{k,n,d}$ at the generic point $Q = L^{d-k}F$ (with $L \in R_1, \quad F \in R_k$); one has that the affine cone over $T_{O_{k,n,d},Q}$ is $W = W(L,F) =< L^{d-k}R_k, L^{d-k-1}FR_1 >$ (see [**CGG**] sec.1, [**BF**] sec.2)).

2.3. Lemma. The dimension of $O_{k,n,d}$ is always the expected one, that is,

$$\dim O_{k,n,d} = \min\{N, \ n + \binom{k+n}{n} - 1\}$$

Proof. By 2.2, $\dim O_{k,n,d} = \dim W(L,F) - 1$, for a generic choice of L, F, so that we can assume that L does not divide F. When $\mathbb{P}(W) \neq \mathbb{P}^N$, we have $\dim W = \dim L^{d-k}R_k + \dim L^{d-k-1}FR_1 - \dim L^{d-k}R_k \cap L^{d-k-1}FR_1 = \binom{k+n}{n} + (n+1) - 1 = \binom{k+n}{n} + n$, since there is only the obvious relation between LR_k and FR_1, namely $LF - FL = 0$.

2.4. Consider the classic Waring problem for forms, i.e. "if we want to write a generic form of degree d as a sum of powers of linear forms, how many of them are necessary?" The problem is completely solved. In fact, $X^s_{n,d} = \{L_1^d + ... + L_s^d, \quad L_i \in R_1\}$ (see previous remark), hence the Waring problem is equivalent to the problem of computing $\dim X^s_{n,d}$. By Corollary 1.4 we have that $\dim X^s_{n,d} = N - \dim H^0(\mathcal{I}_{Z,\mathbb{P}^n} \otimes \mathcal{O}(d)) = H(Z,d) - 1$, where Z is a scheme of s generic 2-fat points in $\mathbb{P}^n$, and $H(Z,d)$ is the Hilbert function of Z in degree d. Since $H(Z,d)$ is completely known (see [**AH**]), we are done.

More generally, one could ask which is the least s such that a form of degree d can be written as $L_1^{d-k}F_1 + ... + L_s^{d-k}F_s$, with $L_i \in R_1$ and $F_i \in R_k$ for $i = 1, ..., s$; since by Remark 2.2 the variety $O^s_{k,n,d}$ parameterizes exactly the forms in R_d which can be written in this way, this is equivalent to answering, for each k, n, d, to the following question:

Find the least s, for each k, n, d, for which $O^s_{k,n,d} = \mathbb{P}^N$.

We are interested in a more complete description of the stratification of the forms of degree d parameterized by those varieties, namely in answering the following question:

Describe all s for which $O^s_{k,n,d}$ is defective, i.e. for which $\dim O^s_{k,n,d} < \mathrm{expdim} O^s_{k,n,d}$.

Notice that, since $d \geq k$, one has $\dim O_{k,n,d} = N$ if and only if $\binom{d+n}{n} \leq n + \binom{k+n}{n}$, hence for all such k, n, d and for any s we have $\dim O^s_{k,n,d} = \text{expdim} O^s_{k,n,d} = N$.
So we have to study this problem when $\binom{d+n}{n} > n + \binom{k+n}{n}$, $s \geq 2$; it is easy to check that whenever $n \geq 2$ this condition is equivalent to $d \geq k+1$; on the other hand the case $n = 1$ (osculating varieties of rational normal curves) can be easily described (all the $O^s_{k,1,d}$'s have the expected dimension, see next section), thus the question becomes:

Question Q(k,n,d): *For all k, n, d such that $d \geq k+1$, $n \geq 2$, describe all s for which*

$$\dim O^s_{k,n,d} < \min\{\, N,\ s(n + \binom{k+n}{n} - 1) + s - 1\} = \min\{\, \binom{d+n}{n} - 1,\ s\binom{k+n}{n} + sn - 1\}.$$

2.5. Remark. Terracini's Lemma 1.4 says that $\dim O^s_{k,n,d} = N - h^0(\mathcal{I}_X \otimes \mathcal{O}_{\mathbb{P}^N}(1))$, where X is a generic union of 2-fat points on $O_{k,n,d}$; we are not able to handle directly the study of $h^0(\mathcal{I}_X \otimes \mathcal{O}_{\mathbb{P}^N}(1))$, nevertheless, Terracini's Lemma 1.3 says that the tangent space of $O^s_{k,n,d}$ at a generic point of $< P_1, ..., P_s >$, $\quad P_i \in O_{k,n,d}$, is the span of the tangent spaces of $O_{k,n,d}$ at P_i; if $T_{O_{k,n,d},P_i} = \mathbb{P}(W_i)$, then

$$\dim O^s_{k,n,d} = \dim < T_{O_{k,n,d},P_1}, ..., T_{O_{k,n,d},P_s} >= \dim < W_1, ..., W_s > -1$$

We want to prove, via Macaulay's theory of "inverse systems", (see [**I**], [**IK**], [**Ge**], [**CGG**], [**BF**]) that, for a single W_i, $\dim W_i = N + 1 - h^0(\mathbb{P}^n, \mathcal{I}_Z(d))$ where $Z = Z(k,n)$ is a certain 0-dimensional scheme that we will analyze further, and dim $< W_1, ..., W_s >= N + 1 - h^0(\mathbb{P}^n, \mathcal{I}_Y(d))$ where $Y = Y(k,n,s)$ is a generic union in $\mathbb{P}^n$ of s 0-dimensional schemes isomorphic to Z. Hence,

$$\dim O^s_{k,n,d} = \dim\ < W_1, ..., W_s > -1 = N - h^0(\mathbb{P}^n, \mathcal{I}_Y(d)).$$

So, one strategy in order to answer to the question $Q(k,n,d)$ for a given (k,n,d) is the following:

1^{st} step: try to compute directly dim $< W_1, ..., W_s >$; if this is not possible, then

2^{nd} step: use the theory of inverse systems (classically *apolarity*):

Compute $W^\perp \subset R_d$, with respect to the perfect pairing $\phi : R_d \times R_d \to k$, where:
- W is a vector subspace of R_d,
- $\phi(f, g) := \Sigma_{I \in A_{n,d}} f_I g_I$, where $A_{n,d} := \{(i_0, ..., i_n) \in \mathbb{N}^{n+1}, \Sigma_j i_j = d\}$, with any fixed ordering; this gives a monomial basis $\{x_0^{i_0} \cdot ... \cdot x_n^{i_n}\}$ for the vector space R_d; if $f \in R_d$, $f = \Sigma_{i_0,...,i_n \in A_{n,d}} f_{i_0,...,i_n} x_0^{i_0} \cdot ... \cdot x_n^{i_n}$, we write for short $f = \Sigma f_I \mathbf{x}^I$, with $I = (i_0, ..., i_n)$.

Then, consider $I_d := W^\perp \subset R_d$. It generates an ideal $(I_d) \subset R$; in this way we define the scheme $Z(k,n,d) \subset \mathbb{P}^n$ by setting: $I_{Z(k,n,d)} := (I_d)^{sat}$. We will show that these schemes do not depend on d.

3^{rd} step, compute the postulation for a generic union of s schemes $Z(k,n,d)$ in $\mathbb{P}^n$.

Recall that $[< W_1, ..., W_s >]^\perp = W_1^\perp \cap ... \cap W_s^\perp$.

2.6. Lemma. *For all k, n and $d \geq k+2$, we have:*

$$(k+1)O \subset Z(k,n,d) \subset (k+2)O,$$

where $Z(k,n,d)$ was defined in 2.5, and $O = \text{supp } Z(k,n,d) \in \mathbb{P}^n$.

Proof. Let $W = < L^{d-k}R_k,\, L^{d-k-1}FR_1 > \subset R_d$ be the affine cone over $T_{O_{k,n,d},Q}$ at a generic point $Q = L^{d-k}F$, with $L \in R_1$, $F \in R_k$. Without loss of generality we can choose $L = x_0$, so that $W = x_0^{d-k-1}(x_0R_k + FR_1)$, hence $x_0^{d-k}R_k \subset W \subset x_0^{d-k-1}R_{k+1}$. So, for any (k,n,d),

$$(x_0^{d-k-1}R_{k+1})^{\perp} \subset W^{\perp} \subset (x_0^{d-k}R_k)^{\perp}. \qquad (**)$$

Now, denoting by $\mathfrak{p}$ the ideal $(x_1, ..., x_n)$, we have:

$$(x_0^{d-t}R_t)^{\perp} =< \{x_0^{i_0} \cdot ... \cdot x_n^{i_n} \mid \Sigma_j i_j = d, i_0 \leq d-t-1\} >=$$
$$< (\mathfrak{p}^d)_d,\, x_0(\mathfrak{p}^{d-1})_{d-1}, ..., x_0^{d-t-1}(\mathfrak{p}^{t+1})_{t+1} >= (\mathfrak{p}^{t+1})_d.$$

Now let us view everything in $(**)$ as the degree d part of a homogeneous ideal; we get:

$$(\mathfrak{p}^{k+2})_d \subset (I_{Z(k,n,d)})_d \subset (\mathfrak{p}^{k+1})_d.$$

Let $(x_1, ..., x_n)$ be local coordinates in $\mathbb{P}^n$ around the point $O = (1, 0, ..., 0)$; the above inclusions give, in terms of 0-dimensional schemes in $\mathbb{P}^n$:

$$(k+1)O \subset Z(k,n,d) \subset (k+2)O.$$

2.7. Lemma. For any k, n, d with $d \geq k+2$, the length of $Z = Z(k,n,d)$ is:

$$l(Z) = dimW = \binom{k+n}{n} + n.$$

Proof. One $(k+2)$-fat point always imposes independent conditions to the forms of degree $d \geq k+1$. Since $Z \subset (k+2)O$, then $h^1(\mathcal{I}_Z(d)) = 0$ immediately follows.

Now we have seen that our problem can be translated into a problem of studying certain schemes $Z(k,n,d) \subset \mathbb{P}^n$; we want to check that actually these schemes are the same for all $d \geq k+2$, say $Z(k,n,d) = Z(k,n)$.

2.8. Lemma. *For any k, n and $d \geq k+2$, we have $Z(k,n,d) = Z(k,n,k+2)$. Henceforth we will denote $Z(k,n) = Z(k,n,d)$, for all $d \geq k+2$.*

Proof. By the previous lemmata we already know that $Z(k,n,d)$ and $Z(k,n,k+2)$ have the same support and the same length, hence it is enough to show that $Z(k,n,d) \subset Z(k,n,k+2)$ (as schemes) in order to conclude. This will be done if we check that $I(Z(k,n,k+2))_d \subset I(Z(k,n,d))_d$; in fact, since both ideals are generated in degrees $\leq d$, this will imply that $I(Z(k,n,k+2))_j \subset I(Z(k,n,d))_j$, $\forall j \geq d$, hence the inclusion will hold also between the two saturations, implying $Z(k,n,d) \subset Z(k,n,k+2)$.

Let $f \in I(Z(k,n,k+2))_d$, then $f = h_1g_1 + ... + h_rg_r$, where $h_j \in R_{d-k-2}$ and $g_j \in I(Z(k,n,k+2))_{k+2}$; since $I(Z(k,n,d))_d$ is the perpendicular to $W =< L^{d-k}R_k, L^{d-k-1}FR_1 >$, it is enough to check that $h_jg_j \in W^{\perp}$, $j = 1, ..., r$. Without loss of generality we can assume $L = x_0$; hence, since $g_j \in < L^2R_k, LFR_1 >^{\perp}$,

$g_j = x_0 g' + g''$, with $g', g'' \in k[x_1, ..., x_n]$ and $g' \in (FR_1)^\perp$. It will be enough to prove $x_0^{i_0}...x_n^{i_n} g_j = x_0^{i_0+1}...x_n^{i_n} g' + x_0^{i_0}...x_n^{i_n} g'' \in W^\perp$, $\forall i_0, ..., i_n$ such that $i_0 + ... + i_n = d-k-2$. It is clear that $x_0^{i_0}...x_n^{i_n} g'' \in W^\perp$, since $i_0 \le d-k-2$; on the other hand, $x_0^{i_0+1}...x_n^{i_n} g' \in (x_0^{d-k} R_k)^\perp$ again by looking at the degree of x_0, while $x_0^{i_0+1}...x_n^{i_n} g' \in (x_0^{d-k-1} FR_1)^\perp$ since $g' \in (FR_1)^\perp$.

2.9. Remark. From the lemmata above it follows that in order to study the dimension of $O^s_{k,n,d}$, $\forall d \ge k+2$, we only need to study the postulation of unions of schemes $Z(k,n)$. For $d = k+1$, we will work directly on W, see Proposition 3.4.

What we got is a sort of "generalized Terracini" for osculating varieties to Veronesean, since the formula $\dim O^s_{k,n,d} = N - h^0(\mathcal{I}_Y(d))$ reduces to the one in Corollary 1.4 for $k = 0$. Instead of studying 2-fat points on $O_{k,n,d}$ (see Remark 2.5), we can study the schemes $Y \subset \mathbb{P}^n$.

2.10. Notation. Let $Y \subset \mathbb{P}^n$ be a 0-dimensional scheme; we say that Y is *regular* in degree d, $d \ge 0$, if the restriction map $\rho : H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \to H^0(\mathcal{O}_Y(d))$ has maximal rank, i.e. if $h^0(\mathcal{I}_Y(d)).h^1(\mathcal{I}_Y(d)) = 0$. We set $exp\ h^0(\mathcal{I}_Y(d)) := max\ \{0, \binom{d+n}{n} - l(Y)\}$; hence to say that Y is regular in degree d amounts to saying that $h^0(\mathcal{I}_Y(d)) = exp\ h^0(\mathcal{I}_Y(d))$.

Since we always have $h^0(\mathcal{I}_Y(d)) \ge exp\ h^0(\mathcal{I}_Y(d))$, we write

$$h^0(\mathcal{I}_Y(d)) = exp\ h^0(\mathcal{I}_Y(d)) + \delta,$$

where $\delta = \delta(Y, d)$; hence whenever $\binom{d+n}{n} - l(Y) \ge 0$, we have $\delta = h^1(\mathcal{I}_Y(d))$, while if $\binom{d+n}{n} - l(Y) \le 0$, $\delta = \binom{d+n}{n} - l(Y) + h^1(\mathcal{I}_Y(d))$; in any case, by setting $exp\ h^1(\mathcal{I}_Y(d)) := max\ \{0, l(Y) - \binom{d+n}{n}\}$, we get: $h^1(\mathcal{I}_Y(d)) = exp\ h^1(\mathcal{I}_Y(d)) + \delta$.

2.11. Remark. *For any k, n, d such that $d \ge k+1$, let $Y = Y(k,n,s) \subset \mathbb{P}^n$ be the 0-dimensional scheme defined in 2.5 for $Z = Z(k,n)$, and $\delta = \delta(Y, d)$. Then*

$$\dim O^s_{k,n,d} = \text{expdim} O^s_{k,n,d} - \delta.$$

In particular, $\dim O^s_{k,n,d} = \text{expdim} O^s_{k,n,d}$ *if and only if:*

$h^0(\mathcal{I}_Y(d)) = 0,$ *when* $\binom{d+n}{n} \le s\binom{k+n}{n} + sn;$

$h^0(\mathcal{I}_Y(d)) = N + 1 - l(Y) = \binom{d+n}{n} - s\binom{k+n}{n} - sn$ *(i.e.* $h^1(\mathcal{I}_Y(d)) = 0$*),* *when* $\binom{d+n}{n} \ge s\binom{k+n}{n} + sn.$

3. A few results and a conjecture.

First let us consider the cases where the question $\mathbf{Q(k,n,d)}$ has already been answered.

$\mathbf{Q(k,1,d)}$. In this case every $O^s_{k,1,d}$, with $d \ge k+2$, has the expected dimension; in fact here $Z(k,1) = (k+2)O$, and the scheme $Y = \{s\ (k+2)\text{-fat points}\} \subset \mathbb{P}^1$ is regular in any degree d. Notice that for $d = k+1$ we trivially have $O_{k,1,k+1} = \mathbb{P}^N$.

$\mathbf{Q(1,n,d)}$. Here the variety $O_{1,n,d}$ is the tangential variety to the Veronese $X_{n,d}$. It is shown in [**CGG**] that $Z(1,n)$ is a "$(2,3)-$scheme" (i.e. the intersection in $\mathbb{P}^n$ of a 3-fat point with a double line); this is easy to see, e.g. by choosing coordinates so that $L = x_0,\ F = x_1$.
The postulation of generic unions of such schemes in $\mathbb{P}^n$, and hence the defectivity of $O^s_{1,n,d}$, has been studied. Moreover, a conjecture regarding all defective cases is stated there:

Conjecture ([**CGG**]). $O^s_{1,n,d}$ is not defective, except in the following cases:
1) for $d = 2$ and $n \geq 2s,\ s \geq 2$;
2) for $d = 3$ and $n = s = 2, 3, 4$.

In [**CGG**] the conjecture is proved for $s \leq 5$ (any d, n), for $s \geq \frac{1}{3}\binom{n+2}{2} + 1$ (any d, n); for $d = 2$ (any s, n), for $d \geq 3$ and $n \geq s + 1$, for $d \geq 4$ and $s = n$. In [**B**], the conjecture is proved for $n = 2, 3$ (any s, d).

$\mathbf{Q(2,2,d)}$. In [**BF**] it is proved that, for any $(s, d) \neq (2, 4)$, $O^s_{2,2,d}$ has the expected dimension.

Now we are going to prove some other cases.

The following (quite immediate) lemma describes what can be deduced about the postulation of the scheme Y from information on fat points:

3.1 Lemma. *Let $P_1, ..., P_s$ be generic points in $\mathbb{P}^n$, and set $X := (k+1)P_1 \cup ... \cup (k+1)P_s$, $T := (k+2)P_1 \cup ... \cup (k+2)P_s$. Now let Z_i be a 0-dimensional scheme supported on P_i , $(k+1)P_i \subset Z_i \subset (k+2)P_i$, with $l(Z_i) = l((k+1)P_i) + n$ for each $i = 1, ..., s$, , and set $Y := Z_1 \cup ... \cup Z_s$. Then:*

Y is regular in degree d if one of the following a) or b) holds:
a) $h^1(\mathcal{I}_T(d)) = 0$ (hence $\binom{d+n}{n} \geq s\binom{k+n+1}{n}$) ;
b) $h^0(\mathcal{I}_X(d)) = 0$ (hence $\binom{d+n}{n} \leq s\binom{k+n}{n}$).

Y is not regular in degree d, with defectivity δ, if one of the following c) or d) holds:
c) $h^1(\mathcal{I}_X(d)) > exp\ h^1(\mathcal{I}_Y(d)) = max\{0, l(Y) - \binom{d+n}{n}\}$; in this case $\delta \geq h^1(\mathcal{I}_X(d))) - exp\ h^1(\mathcal{I}_Y(d)$.
d) $h^0(\mathcal{I}_T(d)) > exp\ h^0(\mathcal{I}_Y(d)) = max\{0, \binom{d+n}{n} - l(Y)\}$; in this case $\delta \geq h^0(\mathcal{I}_T(d)) - exp\ h^0(\mathcal{I}_Y(d))$.

Proof. The statement follows by considering the cohomology of the exact sequences:

$$0 \to \mathcal{I}_T(d) \to \mathcal{I}_Y(d) \to \mathcal{I}_{Y,T}(d) \to 0$$

and

$$0 \to \mathcal{I}_Y(d) \to \mathcal{I}_X(d) \to \mathcal{I}_{X,Y}(d) \to 0$$

where we have: $h^1(\mathcal{I}_{Y,T}(d)) = h^1(\mathcal{I}_{X,Y}(d)) = 0$ since those two sheaves are supported on a 0-dimensional scheme.

3.2. Lemma. *Let $s \geq n+2$ and $d < k+1+2\frac{k+1}{n}$. Then $O^s_{k,n,d}$ is not defective and $O^s_{k,n,d} = \mathbb{P}^N$.*

Proof. Let $Y \subset \mathbb{P}^n$ be as in 2.5; we have to prove that $h^0(\mathcal{I}_Y(d)) = 0$ in our hypotheses.

Let $P_1, ..., P_s$ be the support of Y; we can always choose a rational normal curve $C \subset \mathbb{P}^n$ containing $n+2$ of the P_i's . For any hypersurface F given by a section of $\mathcal{I}_Y(d)$, since $nd < (k+1)(n+2)$, by Bezout we get $C \subset F$. But we can always find a rational normal curve containing $n+3$ points in $\mathbb{P}^n$, so this would imply that any $P \in \mathbb{P}^n$ is on F, i.e. $\mathcal{I}_Y(d) = 0$.

3.3. Lemma. *Assume $s = n+1$; if $d \leq k+1+\frac{k+2}{n}$, then $O^s_{k,n,d} = \mathbb{P}^N$.*

Proof. Since $d \geq k+1$, we can set $d = k+j$ with $j > 0$; let $W_i =< L_i^j R_k,\, L_i^{j-1} F_i R_1 >$ with $F_i \in R_k$ for $i = 1, ..., s$; since $s = n+1$, without loss of generality we can assume that $L_1 = x_0, ..., L_{n+1} = x_n$.
Hence $W_1 + ... + W_s$ contains $U := x_0^j R_k + ... + x_n^j R_k$; now in U the missing monomials are $x_0^{i_0} \cdot ... \cdot x_n^{i_n}$ with $i_l \leq j-1$ for each $l = 0, ..., n$, and $d = \deg(x_0^{i_0} \cdot ... \cdot x_n^{i_n}) \leq (n+1)(j-1)$. Hence if $d \geq (n+1)(j-1)$, i.e. $d < k+1+\frac{k+1}{n}$, we get $U = R_d$.
If $d = (n+1)(j-1)$ the only missing monomial in U is $x_0^{j-1} \cdot ... \cdot x_n^{j-1}$, hence it is enough to choose one of the F_i's in a proper way to get $W_1 + ... + W_s = R_d$.
If $d = (n+1)(j-1)-1$, i.e. $d = k+1+\frac{k+2}{n}$, the $n+1$ missing monomials in U are $x_0^{j-1} \cdot ... \cdot x_i^{j-2} ... \cdot x_n^{j-1}$ with $i = 0, ..., n$ and again it is possible to choose the F_i's so that $W_1 + ... + W_s = R_d$.

Case $\mathbf{Q(k, n, k+1)}$. The description for $k = 1$ given in [**CGG**], together with following proposition, describe this case completely.

3.4. Proposition. *If $s \geq 2$, $k \geq 2$ and $d = k+1$, consider the secant variety $O^s_{k,n,d} \subset \mathbb{P}^N$; then:*

A) if $s \leq n-1$ and its expected dimension is $s\binom{k+n}{n} + sn - 1$, then $O^s_{k,n,k+1}$ is defective with defect
$$\delta = s^2 - s + s\binom{k+n}{n} + \binom{n-s+d}{d} - N;$$
B) if $s \leq n-1$ and the expected dimension is $N = \binom{d+n}{n} - 1$ then
i) $O^s_{d-1,n,d}$ is defective with defect $\delta = \binom{n-s+d}{d} - s(n-s+1)$ if $s < \frac{1}{d}\binom{n-s+d}{d-1}$;
ii) $O^s_{d-1,n,d} = \mathbb{P}^N$ if $s \geq \frac{1}{d}\binom{n-s+d}{d-1}$;
C) if $s \geq n$ then $O^s_{d-1,n,d} = \mathbb{P}^N$.

Proof.

A) We have that $W = W_1 + \ldots + W_s =< x_0 R_k, \ldots, x_{s-1} R_k; F_1 R_1, \ldots, F_s R_1 >$ in R_d. We can suppose that the F_i's, $i = 1, \ldots, s$ are generic in $K[x_s, \ldots, x_n]_d := S_d$, and we have that $\frac{R_d}{W} \cong \frac{S_d}{(F_1, \ldots, F_s)_d}$. Then, since $(F_1, ..., F_s)_d =< F_1 S_1, \ldots, F_s S_1 >$ and the F_i's are generic, $\dim(F_1, ..., F_s)_d = \min\left\{\binom{n-s+d}{d}, s(n-s+1)\right\}$.

¿From this, and from our hypothesis about the expected dimension, we immediately get that $\dim W = N - \binom{n-s+d}{d} + s(n-s+1)$, and hence that the defectivity is $\delta = s^2 - s + s\binom{k+n}{n} + \binom{n-s+d}{d} - N$.

B) If $s\binom{n+d-1}{n} + ns \geq \binom{n+d}{n}$ we expect that $O^s_{d-1,n,d} = \mathbb{P}^N$. Proceeding as in the previous case, in order to compute $\dim W$ we can actually just consider the vector space $< F_1 S_1, \ldots, F_s S_1 >$; whose dimension is $min\left\{\binom{n-s+d}{d}, s(n-s+1)\right\}$; so we get that

i) if $s(n-s+1) < \binom{n-s+d}{d}$, then $O^s_{d-1,n,d}$ is defective. This happens if and only if $s < \frac{1}{d}\binom{n-s+d}{d-1}$, in this case the defect is $\delta = \binom{n-s+d}{d} - s(n-s+1)$.

ii) if $s(n-s+1) \geq \binom{n-s+d}{d}$, then $O^s_{d-1,n,d} = \mathbb{P}^N$ (for example this is always true for $d \geq n$);

C) It suffices to prove that $O^s_{d-1,n,d} = \mathbb{P}^N$ for $s = n$.

If $s = n$ and $d = k+1$, the subspace $W_1 + \cdots + W_s$ can be written as $< x_0 R_k, F_1 R_1, \ldots, x_{n-1} R_k, F_n R_1 >$, which turns out to be equal to $< x_0 R_k, \ldots, x_{n-1} R_k, x_n^{k+1} >= R_{k+1}$ so $O^n_{d-1,n,d} = \mathbb{P}^N$.

Example: The osculating 4^{th}-variety of $X_{6,5} \subset \mathbb{P}^{461}$

Let us consider the secant varieties of the 4^{th}-osculating variety $O_{4,6,5}$. We begin with $O^2_{4,6,5}$; we are in case A of Prop. 3.4, and we expect that $\dim O^2_{4,6,5} = 431$, but we get that the defectivity is $\delta = 86$ so that $\dim O^2_{4,6,5} = 345$.

When $s = 3, 4$ we are in case B of Prop. 3.4, and $\delta = 44$ for $O^3_{4,6,5}$, while $\delta = 9$ for $O^4_{4,6,5}$. Eventually, $O^5_{4,6,5} = \mathbb{P}^{461}$

So, even if we expect that $O^3_{4,6,5}$ should fill up $\mathbb{P}^N$, even the 4-secant variety doesn't.

In terms of forms we get that neither we can write a generic $f \in (K[x_0, \dots, x_6])_5$ as $f = L_1F_1 + L_2F_2 + L_3F_3$ with $L_i \in R_1$ and $F_i \in R_4$ (as we expect), nor as $f = L_1F_1 + \cdots + L_4F_4$, but we need five addenda.

Case $\mathbf{Q(k, 2, k+2)}$:

3.5. Corollary. *Assume $d = k+2$ and $n = 2$. Then, $O^s_{k,2,k+2}$ is not defective for $s \geq 3$ and $k \geq 1$, and $O^s_{k,2,k+2}$ is defective for $s = 2$ and $k \geq 1$.*

Proof. By 3.2 and 3.3, $O^s_{k,2,k+2}$ is not defective for $s \geq 3$ and $d \geq 3$, i.e. $k \geq 2$; the case $k = 1$ is already known by [**B**].
For $s = 2$ and $k \geq 1$, let $Y = Y(k, 2) \subset \mathbb{P}^2$ be the 0-dimensional scheme defined in 2.5; it is easy to check that *exp* $h^0(\mathcal{I}_Y(d)) =$ *exp* $h^0(\mathcal{I}_T(d)) = 0$, T denoting the generic union of two $(k+2)$-fat points in $\mathbb{P}^2$. Since T is not regular in degree $d = k+2$ for any $k \geq 1$, we conclude by lemma 3.1 d) that $O^s_{k,n,k+2}$ is defective with defectivity $\geq h^0(\mathcal{I}_T(d)) = 1$ (the only section is given by the $(k+2)$-ple line through the two points).

Case $\mathbf{Q(k, 3, k+2)}$:

3.6. Corollary. *Assume $d = k+2$ and $n = 3$. Then, $O^s_{k,3,k+2} = \mathbb{P}^N$ for $s \geq n+1 = 4$ and $k \geq 1$, while $O^s_{k,3,k+2}$ is defective for $s \leq 3$.*

Proof. The case $s \leq 3$ will be treated in Prop.3.10.
If $s = 4$ and $k = 1$, $O^4_{1,3,3} = \mathbb{P}^N$ by [**CGG**], (4.6). If $s = 4$ and $k = 2$, we have $O^4_{2,3,4} = \mathbb{P}^N$ by lemma 3.3.
If $s \geq 5$ and $k \geq 1$, or $s = 4$ and $k \geq 3$, the thesis follows by lemmata 3.2 and 3.3, respectively.

Case $\mathbf{Q(k, 4, k+2)}$:

3.7. Corollary. *Assume $d = k+2$ and $n = 4$. Then, $O^s_{k,4,k+2} = \mathbb{P}^N$ for $s \geq 5$ and $k \geq 1$, while $O^s_{k,4,k+2}$ is defective for $s \leq 4$.*

Proof. The case $s \leq 4$ will be given by Prop.3.10.
If $s \geq 5$ and $k = 1$, $O^s_{1,4,3} = \mathbb{P}^N$ by [**CGG**], (4.6) and (4.5). If $s = 5$ and $k = 2, 3$, we have $O^5_{k,4,k+2} = \mathbb{P}^N$ by Lemma 3.3.
If $s \geq n+2 = 6$ and $k \geq 2$, or $s = 5$ and $k \geq 4$, thesis follows by Lemmata 3.2 and 3.3, respectively.

Case $\mathbf{Q(k, 2, k+3)}$:

3.8. Corollary. *Assume $d = k+3$ and $n = 2$. Then:*

i) for $s=2$ and $k=1,2$: $\dim O^2_{k,2,k+3} = s\binom{k+2}{2}+2s-1$ (the expected one);
ii) for $s=2$ and $k\geq 3$: $O^2_{k,2,k+3}$ is defective;
iii) for $s\geq 3$ and $k\geq 1$: $O^s_{k,2,k+3}=\mathbb{P}^N$.

Proof.
If $s\geq n+2=4$ and $k\geq 2$, or $s=3$ and $k\geq 4$, the thesis follows by Lemmata 3.2 and 3.3, respectively.
If $s\geq 3$ and $k=1$, $O^s_{1,2,k+3}=\mathbb{P}^N$ by [**CGG**], (4.5).
If $s=3$ and $k=2,3$, we have $O^2_{k,2,k+3}=\mathbb{P}^N$ by lemma 3.3.
If $s=2$ and $k=1$, or $s=2$ and $k=2$, $O^2_{k,2,k+3}\neq\mathbb{P}^N$ is not defective by [**CGG**], (4.6) and [**BF**], Theorem 1, respectively.
If $s=2$ and $k\geq 3$, then, in the notations of lemma 3.1, we have :
for $k=3,4$ $exp\ h^1(\mathcal{I}_X(d)) = exp\ h^1(\mathcal{I}_Y(d))=0$, and the union X of 2 $(k+1)$-fat points is not regular in degree $d=k+3$;
for $k\geq 5$ $exp\ h^0(\mathcal{I}_Y(d)) = exp\ h^0(\mathcal{I}_T(d))=0$, and the union T of 2 $(k+2)$-fat points is not regular in degree $d=k+3$;
so we conclude by 3.1, c) and d).

For $s\leq n+1$, we have several partial results:

3.9. Proposition. *If $s\leq n+1$, $d\geq 2k+1$ and $k\geq 2$ then $O^s_{k,n,d}$ is regular.*

Proof. We have to study the dimension of the vector space $W_1+\cdots+W_s = < L_1^{d-k}R_k, L_1^{d-k-1}F_1R_1,\ldots, L_s^{d-k}R_k, L_s^{d-k-1}F_sR_1 >$, where $L_1,\ldots,L_s$ are generic in R_1 and $F_1,\ldots,F_s$ are generic in R_k. Since $s\leq n+1$, without loss of generality we may suppose $L_i=x_{i-1}$ for $i=1,\ldots,s$. Since $d\geq 2k+1$, for $\beta=d-k\geq 3$, the vector space $W_1+\cdots+W_s$ can be written as $< x_0^\beta R_k, x_0^{\beta-1}F_1R_1,\ldots,x_{s-1}^\beta R_k, x_{s-1}^{\beta-1}F_sR_1 >$. If we show that for a particular choice of $F_1,\ldots,F_s\in R_k$ the dimension of $W_1+\cdots+W_s = expdim(O^s_{k,n,d})+1$ we can conclude by semi-continuity that $O^s_{k,n,d}$ has the expected dimension. Let us consider the case $F_i=x_ix_{i+1}\widetilde{F}_i$ for $i=1,\ldots,s-2$, $F_{s-1}=x_{s-1}x_0\widetilde{F}_{s-1}$ and $F_s=x_0x_1\widetilde{F}_s$, where the $\widetilde{F}_j$'s are generic forms in R_{k-2}, $j=1,\ldots,n+1$. Let $< x_i^\beta R_k >=: A_i$ and $< x_i^{\beta-1}F_{i+1}R_1 >=: A'_i$, $i=0,\ldots,s-1$; then we get $A'_i = < x_i^{\beta-1}x_{i+1}x_{i+2}\widetilde{F}_{i+1}R_1 >$, $i=0,\ldots,s-3$; $A'_{s-2} = < x_{s-2}^{\beta-1}x_{s-1}x_0\widetilde{F}_{s-1}R_1 >$ and $A'_{s-1} = < x_{s-1}^{\beta-1}x_0x_1\widetilde{F}_sR_1 >$. Now $W_1+\cdots+W_s = \sum_{j=0}^{s-1}A_j+\sum_{j=0}^{s-1}A'_j$. We can easily notice that $A'_i\cap(\sum_{j=0}^{s-1}A_j+\sum_{j=0,j\neq i}^{s-1}A'_j) = A_i\cap(\sum_{j=0,j\neq i}^{s-1}A_j+\sum_{j=0}^{s-1}A'_j) = A_i\cap A'_i = < x_i^\beta R_k > \cap < x_i^{\beta-1}x_{i+1}x_{i+2}\widetilde{F}_{i+1}R_1 > = < x_i^\beta x_{i+1}x_{i+2}\widetilde{F}_{i+1} >$ for any fixed $i=0,\ldots,s-3$ (analogously if $i=s-2,s-1$). So we have found exactly s relations and we can conclude that $\dim(W_1+\cdots+W_s) = \dim(\sum_{j=0}^{s-1}A_j)+\dim(\sum_{j=0}^{s-1})A'_j) - s = s\binom{k+n}{n}+s(n+1)-s$, which is exactly the expected dimension.

3.10. Proposition. *If $s\leq n$ and $k+2\leq d\leq 2k$ then $O^s_{k,n,d}$ is defective with defect δ such that:*
A) $\delta\geq\binom{n-s+d}{d}$ if the expected dimension is $\binom{d+n}{n}-1$;
B) $\delta\geq\binom{s}{2}\binom{2k-d+n}{n}$ if the expected dimension is $s\binom{k+n}{n}+sn-1$.

Proof. Let $\beta:=d-k\geq 2$; we can rewrite the vector space $W_1+\cdots+W_s$ as follows: $< x_0^\beta R_k, x_0^{\beta-1}F_1R_1, \ldots, x_{s-1}^\beta R_k, x_{s-1}^{\beta-1}F_sR_1 >$.

A) We can observe that $k[x_s,...,x_n]_d \cap (W_1+\cdots+W_s)=\{0\}$, so if we expect that $O^s_{k,n,d}=\mathbb{P}^N$ we get a defect $\delta \geq \binom{n-s+d}{d}$.

B) Suppose now that $s\left[\binom{k+n}{n}+n\right] < \binom{d+n}{n}$. If $O^s_{k,n,d}$ were to have the expected dimension we would not be able to find more relations among the W_i's other than $x_i^\beta F_{i+1} \in < x_i^\beta R_k > \cap < x_i^{\beta-1}F_{i+1}R_1 >$, for $i=0,\ldots,s-1$ (as it happens in Proposition 3.9). But it's easy to see that $x_i^\beta x_j^\beta F \in < x_i^\beta R_k > \cap < x_j^\beta R_k >$ with $i \neq j$ and $F \in R_{k-\beta}$. We have exactly $\binom{s}{2}$ such terms for any choice of $F \in R_{k-\beta}$. We can also suppose that the $F_i \in R_k$ that appear in $W_1+\cdots+W_s$ are different from $x_j^\beta F$ for any $F \in R_{k-\beta}$ and $j=0,\ldots,s-1$, because $F_1,\ldots,F_s$ are generic forms of R_k. Then we can be sure that the form $x_i^\beta x_j^\beta F$ belonging to $< x_i^\beta R_k > \cap < x_j^\beta R_k >$ isn't one of the $x_i^\beta F_{i+1}$ that belongs to $< x_i^\beta R_k > \cap < x_i^{\beta-1}F_{i+1}R_1 >$. Now $dim(R_{k-\beta})=\binom{k-\beta+n}{n}$ so we can find $\binom{s}{2}\binom{k-\beta+n}{n}$ independent forms that give defectivity. Hence in case $s\left[\binom{k+n}{n}+n\right] < \binom{d+n}{n}$ we have $dim(O^s_{k,n,d}) \leq expdim - \binom{s}{2}\binom{k-\beta+n}{n} = expdim - \binom{s}{2}\binom{2k-d+n}{n}$.

3.11. Proposition. *If $s=n+1$, $k+2 \leq d \leq 2k$ and $expdim(O^{n+1}_{k,n,d})=(n+1)\left(\binom{k+n}{n}+n\right)-1$ then $O^{n+1}_{k,n,d}$ is defective with defect $\delta \geq \binom{n+1}{2}\binom{2k-d+n}{n}$.*

Proof. The proof of this fact is the same as case B) of the previous proposition.

3.12. Proposition. *If $s=n+1$, $n \geq \frac{k+2}{d-k-2}$, $k+2<d\leq 2k$ and $expdim(O^{n+1}_{k,n,d})=N$ then $O^{n+1}_{k,n,d}$ is defective with defect $\delta \geq \binom{(n+1)(d-k-1)-(d+1)}{n}$.*

Proof. If $k+2<d\leq 2k$, then $2<\beta:=d-k\leq k$ and we have to study the dimension of $W_1+\cdots+W_{n+1} = < x_0^\beta R_k, x_0^{\beta-1}F_1R_1,\ldots,x_n^\beta R_k, x_n^{\beta-1}F_{n+1}R_1 >$. It is easy to see that a monomial of the form $f=x_0^{\beta_0}\cdots x_n^{\beta_n}$ with $\sum_{i=0}^n \beta_i = d$ and $0\leq\beta_i\leq\beta-2$ for all $i\in\{0,\ldots,n\}$ is a form of degree d which does not belong to $W_1+\cdots+W_{n+1}$. In fact f can be written as $x_0^{d-(\gamma_0+k+2)}\cdots x_n^{d-(\gamma_n+k+2)}$ with $\sum_{i=0}^n\gamma_i = nd-(n+1)(k+2)$ and $\gamma_i\geq 0$ for all $i=0,\ldots,n$ and these forms are exactly $\binom{n+(n+1)(d-k-2)-d}{n}=\binom{(n+1)(d-k-1)-(d+1)}{n}$. In order for these forms to exist, one needs that $(n+1)(d-k-2)-d\geq 0$, i.e. that $n\geq\frac{k+2}{d-k-2}$. This is sufficient to show that if we expect that $O^{n+1}_{k,n,d}=\mathbb{P}^N$, and if $n\geq\frac{k+2}{d-k-2}$ and $k+2<d\leq 2k$, then $O^{n+1}_{k,n,d}$ is defective.

Let's notice that what we just saw is not sufficient to say that the defect δ is exactly equal to $\binom{(n+1)(d-k-1)-(d+1)}{n}$, because in $R_d \setminus < W_1+\cdots W_{n+1} >$ we can find also monomials like $x_0^{\beta_0}\cdots x_n^{\beta_n}$ with $\sum_{i=0}^n\beta_i=d$, at least one $\beta_i=\beta-1$ and each of the others $\beta_j\leq\beta-2$. Hence $\delta\geq\binom{(n+1)(d-k-1)-(d+1)}{n}$.

All the results on defectivity lead us to formulate the following:

3.13 Conjecture. *$O^s_{k,n,d}$ is defective only if Y is as in case c) or d) of Lemma 3.1.*

The conjecture amounts to say that the defectivity of Y can only occur if defectivity of the fat points schemes X or T imposes it.

3.14. Remark. In many examples the defectivity of Y is exactly the one imposed by X or by T (i.e. the inequalities on δ in Lemma 3.1 are equalities), but this is not always the case: for example if we consider the variety $O^2_{4,5,6}$ (see the example after Prop. 3.4), here we get that the corresponding scheme Y has defectivity 86 in degree 5. Here we have that X is given by two 5-fat points in $\mathbb{P}^6$, and it is easy to check that $h^0(\mathcal{I}_X(5))=126$ (all 5-tics through X can be viewed as cones over a 5-tic of a $\mathbb{P}^4$), so that its defectivity is 84. Hence, even if Y is "forced" to be defective by X, its defectivity is bigger, i.e. Y should impose to 5-tics 12 conditions more than X, but it imposes only ten conditions more.

It is easy to find similar behavior if $d = k + 1$, for instance for $n = 8$, $s = 3$, $d = k + 1 = 2$ or $n = 10$, $s = 3$, $d = k + 1 = 2$.

In the case of $\mathbb{P}^2$, we are able to prove our conjecture for small values of s:

3.15. Theorem. *Let X, Y be as above, $n = 2$ and $s = 3, 4, 5, 6$ or 9; then:*

$$H(Y, d) = \min\{H(X, d) + 2s, \binom{d+2}{2}\}.$$

The proof mainly uses la méthode d'Horace (e.g. see [**Hi**]) on the scheme Y. For a detailed proof, see [**Be**] and [**BC**].

Notice that this result implies that Y can be defective only when X is.

In general, it is quite a hard problem to determine, and even to give a conjecture upon, the postulation for an union of s m-fat points in $\mathbb{P}^n$.

For what concerns $\mathbb{P}^2$, there is a conjecture for the postulation of a generic union of fat points (e.g. see [**Ha**]). For a generic union $A \subset \mathbb{P}^2$ of s m-fat points with $s \geq 10$, the conjecture says that A is regular in any degree d. This has been proved for $m \leq 20$ in [**CCMO**]. For $s \leq 9$ all the defective cases are known (e.g. see [**Ha**] or [**CCMO**] for a complete list).

This allows us to list all the defective cases for some values of s (for related results see also [**BF2**]):

3.16 Corollary. *Let $n = 2$, $s \leq 6$ or $s = 9$. Then $O^s_{k,2,d}$ is defective if and only if:*

$s = 2$, $\quad k = 1$ *and* $d = 3$, *or* $\quad k \geq 2$ *and* $k + 2 \leq d \leq 2k$.

$s = 3$, $\quad \frac{3k+5}{2} \leq d \leq 2k$.

$s = 5$, $\quad 2k + 4 \leq d \leq \frac{5k+3}{2}$.

$s = 6$, $\quad k \equiv 2$ *(mod 5) and* $\frac{12(k+1)}{5} \leq d \leq \frac{5k+3}{2}$, *or* $k \not\equiv 2$ *(mod 5) and* $\frac{12(k+1)}{5} + 1 \leq d \leq \frac{5k+3}{2}$.

The case $s = 2$ is given by Propositions 3.4, 3.8, 3.9 and 3.10, while the other cases follow from Theorem 3.15 and the classification in [**CCMO**]. Notice that there are no defective cases for $s = 4$ or $s = 9$. In case $s = 2$ defectivity is forced exactly by defectivity of X or T.

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A.Bernardi, Dip. Matematica, Univ. di Milano, Italy, email: bernardi@mat.unimi.it

M.V.Catalisano, DIPEM, Univ. di Genova, Italy, e-mail: catalisano@dimet.unige.it

A.Gimigliano, Dip. di Matematica and C.I.R.A.M., Univ. di Bologna, Italy, e-mail: gimiglia@dm.unibo.it

M.Idà, Dip. di Matematica, Univ. di Bologna, Italy, e-mail: ida@dm.unibo.it