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IDEALS GENERATED BY 2-MINORS: BINOMIAL EDGE IDEALS AND POLYOMINO IDEALS

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To my loved three F...

Family, Fede, and Fuji.

Contents

INTRODUCTION

From the pioneering work by Richard Stanley [65] in 1975, combinatorial commutative algebra has been an active area of research that, as the name suggests, lies at the intersection of commutative algebra with combinatorics. New methods have evolved out of an influx of ideas from such diverse areas as polyhedral geometry, theoretical physics, representation theory, homological algebra, symplectic geometry, graph theory, integer programming, symbolic computation, and statistics. Also a broad range of books and lectures has been devoted to the resulting combinatorial techniques for dealing with polynomial rings, semigroup rings, and determinantal rings, see, e.g., [5], [6], [19], [20], [29], [31], [36], [54], [66], [67], and [69].

Since the early 1990s, a classical object in commutative algebra has been the study of binomial ideals. Frequently, methods used in combinatorics can be exploited to address problems arising from the study of binomial ideals. One of the first articles where binomial ideals appeared is [28], where the relation ideals of semigroup rings were identified as binomial ideals. But, a first systematic treatment of binomial ideals and toric rings is given in the Sturmfels' book [67], with applications to convex polytopes and integer programming. Eisenbud and Sturmfels, in [20], develop a general theory of binomial ideals and their primary decomposition, that will be the starting point for a lot of further works. In that work, lattice ideals were first systematically studied. They are binomial ideals with generators given by all the elements of an integer lattice. Note that toric ideals are lattice ideals for which the lattice is the kernel of an integer matrix. The study of lattice ideals is a rich subject on its own, see [54], [67] for the general theory and [53] for recent developments. In [40], Hoşten and Shapiro introduce lattice basis ideals which are binomial ideals with generators given by the elements of a basis of a saturated integer lattice. Another intensively-studied class of binomial ideals is that of the ideals generated by a subset of 2-minors of an $m \times n$-matrix X_{mn} of indeterminates. They are a generalization of the determinantal ideals with $k = 2$, which are ideals generated by all the k-minors of X_{mn} [6]. Hoşten and Sullivan in [40] consider the ideals of adjacent minors and Diaconis, Eisenbud, and Sturmfels in [15] compute the primary decomposition of these ideals. This dissertation is devoted to the study of two classes of ideals of 2-minors: binomial edge ideals and polyomino ideals.

Binomial edge ideals arise from finite graphs and this class of binomial ideals is one of the most studied. Their appeal results from the fact that the homological properties of these ideals reflect nicely the combinatorics of the underlying graphs. They were introduced in 2010 by Herzog, Hibi, Hreinsdóttir, Kahle and Rauh in [30], and independently by Ohtani in [55]. In the last decade, several works devoted to the study of the algebraic and homological properties of these ideals have been produced. Given a simple graph G, with vertex set $V(G) = [n]$ and edge set $E(G)$, the binomial edge ideal J_G of G is the ideal in $S = \mathbb{K}[x_1, \dots, x_n, y_1, \dots, y_n]$ generated by all $f_{ij} = x_i y_j - x_j y_i$, with $i < j$ and $\{i, j\} \in E(G)$. Looking at the definition, it can be seen immediately that the ideal of 2-minors of a $2 \times n$-matrix may be interpreted as the binomial edge ideal of a complete graph on $[n]$, where a complete graph K_n on $[n]$ is such that $E(K_n) = \{\{i, j\} \mid 1 \leq i < j \leq n\}$. Whereas, the ideal of adjacent minors may be interpreted as the binomial edge ideal of a path graph, where a path graph P_n on $[n]$ is such that $E(P_n) = \{\{i, i+1\} \mid 1 \leq i \leq n-1\}$. Moreover, binomial edge ideals arise naturally in the study of conditional independence ideals, and they generalize a class which has been studied by Fink [24].

A basic result on binomial edge ideals is that the initial ideal of J_G is squarefree, which implies that J_G is a radical ideal. In particular, in [30], the authors completely describe the Gröbner basis of J_G with respect to the monomial order induced by $x_1 > x_2 > \cdots x_n > y_1 > y_2 > \cdots > y_n$. Moreover, they provide the minimal prime ideals of J_G and the Krull dimension of S/J_G by means of particular subsets of the vertices of G, the cutsets of G. A subset T of $V(G)$ is said to be a *cutset* of G if the number of connected components, $c(T)$, induced by removing T from $V(G)$ is greater than the number of connected components induced by removing $T \setminus \{i\}$ from $V(G)$, for all $i \in T$. Denote by $\mathcal{C}(G)$ all the cutsets of G, then

$$\dim S/J_G = \max_{T \in \mathcal{C}(G)} \{n - c(T) + |T|\}.$$

In general, all the cutsets of a graph, and consequently the induced connected components, must be found in order to apply this formula. In [59], an algorithm for this aim is described, but, unfortunately, it is exponential in time and space. For some classes of graphs, this computation could result easier, but not trivial. In Subsection 3.1.2, we will provide a linear in time and space algorithm (see Theorem 3.1.7) that computes the Krull dimension of S/J_G, when G is a block graph. A connected induced subgraph of G that has no cutset of cardinality 1 and is maximal with respect to this property is a *block*. A connected induced subgraph of G that is a complete graph is called a *clique*. A graph is called *block graph*, or *clique tree*, if all its blocks are cliques. The idea of our algorithm is to find a minimal prime ideal of minimum height since it induces the Krull dimension of S/J_G.

Another fundamental invariant that has been studied intensively is the Castelnuovo-Mumford regularity of binomial edge ideals. Matsuda and Murai, in [52], first investigated the regularity, proving, for any connected graph G on $[n]$, that $\ell \leq \operatorname{reg} S/J_G \leq n-1$, where ℓ is the length of the longest induced path of G. In the same paper, they conjecture that $\operatorname{reg} S/J_G$ is exactly $n-1$ if and only if G is a path of length n. In [45], Kiani and Saeedi Madani give a positive answer to Matsuda-Murai conjecture. In [62], it is conjectured that $\operatorname{reg} S/J_G \leq c(G)$, where $c(G)$ is the number of maximal cliques of G. In [23], Ene and Zarojanu prove this for some classes of graphs, including block graphs. Furthermore, Kiani and Saeedi Madani characterized all graphs whose binomial edge ideal have regularity 2 and regularity 3, see [61] and [63]. In a very recent work [22], a new upper bound for any connected graph G has been proved by Ene, Rinaldo, and Terai, that is $\operatorname{reg} S/J_G \leq n - \dim \Delta(G)$, where $\Delta(G)$ is the clique complex of G, that is the simplicial complex of all cliques of G. It is still an open problem to determine an explicit formula for the regularity of binomial edge ideals in terms of the combinatorics of the graph.

Other important invariants, strictly related to the regularity, which are provided by the graded finite free resolution are the extremal Betti numbers of J_G. Let M be a finitely graded S-module. A Betti number $\beta_{i,i+j}(M) \neq 0$ is called *extremal* if $\beta_{k,k+\ell} = 0$ for all pairs $(k, \ell) \neq (i, j)$, with $k \geq i, \ell \geq j$. A nice property of the extremal Betti numbers is that M has a unique extremal Betti number if and only if $\beta_{p,p+r}(M) \neq 0$, where $p = \operatorname{proj\,dim} M$ and $r = \operatorname{reg} M$. Over the last few years, extremal Betti numbers have been studied by different researchers, also motivated by Ene, Hibi, and Herzog's conjecture ([21], [35]) on the equality of the extremal Betti numbers of J_G and $\text{in}_<(J_G)$. Some works in this direction are [3], [14], and [17], but the question has been completely and positively solved by Conca and Varbaro in [10]. For some classes of graphs, the extremal Betti numbers of J_G are explicitly provided, for instance by Dokuyucu, in [17], and by Hoang, in [38]. In [35], Herzog and Rinaldo compute one of the distinguished extremal Betti numbers of the binomial edge ideal of a block graph and classify all block graphs admitting precisely one extremal Betti number, by listing the forbidden induced subgraphs (which are 4 in total). A natural lower bound for the regularity of any block graph arises from these results. At the same time, Jayanthan et al in [42] and in [43] obtain a related result for trees, a subclass of block graphs.

Inspired by the results in [35], we define a new class of graphs, the *flower graphs* (see Definition 3.1.11), for which we compute the superextremal Betti numbers and the regularity. As a consequence, we then obtain new lower bounds for the regularity of any block graph (see Theorem 3.1.15 and Corollary 3.1.17). All these facts have been exploited by Kumar, in a very recent work [46], to classify all generalized block graphs that admit a unique extremal Betti number, where generalized block graphs are the generalization of block graphs and were introduced in [45]. Moreover, he proves that the Castelnuovo-Mumford regularity of binomial edge ideal of a generalized block graph is bounded below by $m(G) + 1$, where $m(G)$ is the number of minimal cutset.

Finally, in Subsection 3.1.4, we state one of the main results of this part. Indeed, we provide an efficient method to compute the Castelnuovo-Mumford regularity of any binomial edge ideal of block graphs (see Theorem 3.1.20) by means of a unique block graph traversal.

In Section 3.2, we study the extremal Betti numbers for binomial edge ideals of some classes of Cohen-Macaualy graphs: cone, bipartite and fan graphs. In general, it is hard to identify Cohen–Macaulay binomial edge ideals. A full classification of such ideals seems to be impossible. Cone graphs were introduced and investigated by Rauf and Rinaldo in [58]. They construct Cohen-Macaulay graphs by means of the formation of cones: connecting all the vertices of two disjoint Cohen-Macaulay graphs to a new vertex, the resulting graph is Cohen-Macaulay. For these graphs, we give the regularity and also the Cohen-Macaulay type (see Lemma 3.2.4 and Proposition 3.2.5). Bipartite and fan graphs are studied by Bolognini, Macchia and Strazzanti in [4]. They classify the bipartite graphs whose binomial edge ideal is Cohen-Macaulay. In particular, they present a family of bipartite graphs F_m whose binomial edge ideal is Cohen-Macaulay, and they prove that, if G is connected and bipartite, then J_G is Cohen-Macaulay if and only if G can be obtained recursively by gluing a finite number of graphs of the form F_m via two operations. In the same article, they describe a new family of Cohen-Macaulay binomial edge ideals associated with non-bipartite graphs, the fan graphs. For both these families, in [41], Jayanthan and Kumar compute a precise expression for the regularity. In Subsection 3.2.3, we provide the unique extremal Betti number of the binomial edge ideal of Cohen-Macaulay bipartite and fan graphs. In addition, we exploit the unique extremal Betti number of S/J_{F_m} to describe completely its Hilbert-Poincaré series.

The other class of ideals generated by some 2-minors, that we will consider in this work, is that of polyomino ideals. Polyominoes are two-dimensional objects obtained by joining edge by edge squares of the same size. They are studied from the point of view of combinatorics, e.g. in tiling problems of the plane, as well as from the point of view of commutative algebra, associating binomial ideals to polyominoes. The latter were first introduced by Qureshi in [56]. An *inner interval* of a polyomino $\mathcal{P}$ is an interval $[a, b] \subset \mathbb{N}^2$ such that all the cells of $[a, b]$ belong to $\mathcal{P}$, as well. Given an inner interval $[a, b]$ of $\mathcal{P}$, an *inner 2-minor* of $\mathcal{P}$ is the binomial $x_a x_b - x_c x_d \in S = \mathbb{K}[x_v | v \text{ is a corner of } \mathcal{P}]$, where c and d are the other two corners of $[a, b]$. The *polyomino ideal* of $\mathcal{P}$ is the binomial ideal of S generated by all the inner 2-minors of $\mathcal{P}$.

Two pending and of interest questions regarding polyomino ideals are to classify those that are prime and to prove if they are radical ideals. In this work, we focus on the first question. We said that a polyomino is prime if its polyomino ideal is prime. In [33], [34] and [57], the authors prove that simple polyominoes are prime. Roughly speaking, a *simple polyomino* is a polyomino without holes. Whereas, polyominoes having one or more holes are called *multiply connected polyominoes*, using the terminology adopted in [25], an introductory book on polyominoes.

In general, giving a complete characterization of the primality of multiply connected polyomino ideals does not seem to be so easy. A family of prime polyominoes obtained by removing a convex polyomino by a given rectangle has been showed in [37] and [64].

In Section 4.1, we give a necessary condition for the primality of the polyomino ideal. This condition is related to a sequence of inner intervals contained in the polyomino, called a *zig-zag walk* (see Definition 4.1.2). In particular, we prove that if the coordinate ring $\mathbb{K}[\mathcal{P}] = S/I_{\mathcal{P}}$ is a domain then $\mathcal{P}$ should have no zig-zag walks (see Proposition 4.1.5). We conjecture that this is also a sufficient condition for $\mathbb{K}[\mathcal{P}]$ to be a domain. We verify this conjecture computationally for all the multiply connected polyominoes with at most 14 cells. In support of our conjecture, we define a new infinite family of polyominoes, called *grid polyominoes*, that are obtained by removing inner intervals from a given rectangle in a way that avoids the existence of zig-zag walks. By using a Gröbner basis technique and lattice ideals, we prove that grid polyominoes are prime (see Theorem 4.2.9).

Moreover, we present a toric ideal associated with a polyomino, generalizing Shikama's definition in [64]. This toric ideal contains the polyomino ideal (see Proposition 4.1.1). Moreover, if the polyomino contains a zig-zag walk, the binomial associated with the zig-zag walk belongs to the toric ideal and the above inclusion is strict.

This thesis is organized as follows:

- Chapter 1 and 2 are both devoted to introducing all the definitions, notions, and well-known results that will be used in the other chapters. In particular, we briefly recall classical objects in commutative algebra, such as Gröbner basis, Krull dimension, depth, and the minimal graded free resolution of a module, together with some homological invariants related to it. We introduce the binomial ideals, including ideals generated by some 2-minors, and we focus on the presentation of binomial edge ideals and polyomino ideals. Moreover, we summarize basic terminologies arising from graph theory.

- Chapter 3 and 4 contain all and only the original results of this dissertation. In

Chapter 3, we will study Krull dimension, regularity and Betti numbers of binomial edge ideals of some classes of graphs. Whereas, in Chapter 4, we will investigate the primality of polyomino ideals.

Chapter 1
PRELIMINARIES

1.1 BASIC COMMUTATIVE ALGEBRA

In this section, we recall basis concepts from commutative algebra which are relevant for the subjects treated in the later chapters. We begin with a review on Krull dimension, regular sequences and depth of a module. We then describe the relationship, known as Auslander–Buchsbaum formula, between the depth of a graded S-module M and its projective dimension, where S is a polynomial ring, and study in more detail the finite minimal graded free S-resolution of M. The regularity and the graded Betti numbers of M will be defined via this resolution.

Throughout this dissertation all rings are considered to be commutative and with unity. For further detail on the results presented in this section, we refer the reader to classical books in commutative algebra, see [1], [5], [12], [18], [20], [29], and [31].

1.1.1 Dimension, regular sequences and depth

Let R be a ring and $I \subset R$ be an ideal of R. We denote by $\operatorname{Spec}(R)$ the *spectrum* of R, that is the set of all prime ideals of R, and by $\operatorname{Min}(R)$ the set of all the minimal prime ideals of R. Let $\mathfrak{p} \in \operatorname{Spec}(R)$. The *height* of $\mathfrak{p}$, denoted by height $\mathfrak{p}$, is the supremum of lengths of chains of prime ideals descending from $\mathfrak{p}$. Whereas, the height of any ideal $I \subseteq R$ is defined as follows

$$\text{height } I = \min\{\text{height } \mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec}(R) \text{ and } I \subset \mathfrak{p}\}.$$

The *Krull dimension* of R, denoted by $\dim R$, is defined as

$$\dim R = \sup\{\text{height } \mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec}(R)\}.$$

In general, it holds $\dim R/I + \text{height } I \leq \dim R$. The *Krull dimension* of an R-module M is defined as the maximal length of the chains of ideals $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $M_{\mathfrak{p}} \neq 0$, or, equivalently,

$$\dim M = \dim(R/\operatorname{Ann}_R(M)),$$

where $\mathrm{Ann}_R(M) = \{r \in R \mid \text{ for all } m \in M,\ rm = 0\}$ is the *annihilator* of M. Another equivalent definition of the Krull dimension of a graded module M can be given by means of its Hilbert series, as we will show afterwards.

Examples 1.1.1 *(a) Let $S = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial over a field $\mathbb{K}$ with n indeterminates. The following*

$$(x_1, \ldots, x_n) \supset (x_1, \ldots, x_{n-1}) \supset \cdots \supset (x_1) \supset (0)$$

is a decreasing sequence of prime ideals, then $\dim S \geq n$. Actually, no prime ideals chain of greater length exists, and the Krull dimension of S is exactly n (see Theorem [18, Corollary 10.13]).

(b) Let $S = \mathbb{K}[x_1, x_2, x_3]$, and consider the prime ideals $\mathfrak{p}_0 = (x_1)$ and $\mathfrak{p}_1 = (x_2, x_3)$ of S. Let $I = \mathfrak{p}_0 \cap \mathfrak{p}_1 \subset S$. Since height $\mathfrak{p}_0 = 1$ *and* height $\mathfrak{p}_1 = 2$, *it follows* height $I = 1$.

A *graded ring* is a ring R together with a family $(R_i)_{i \in \mathbb{Z}}$ of $\mathbb{K}$-vector spaces, such that $R = \bigoplus_{i=0}^{\infty} R_i$, $R_0 = \mathbb{K}$, and $R_i R_j \subseteq R_{i+j}$ for all $i, j \in \mathbb{Z}$. A graded ring R is *standard graded* if $R = \mathbb{K}[R_1]$. The simplest example of standard graded ring is given by the polynomial ring over $\mathbb{K}$ in the indeterminate $x_1, \ldots, x_n$ graded by 1.

A *graded R-module* is an R-module M together with a family $(M_i)_{i \in \mathbb{Z}}$ of $\mathbb{K}$-vector spaces and such that $M = \bigoplus_{i \in \mathbb{Z}} M_i$, with $R_i M_j \subset M_{i+j}$ for all $i, j \in \mathbb{Z}$. Thus each M_i is a $\mathbb{K}$-module. An element x of M is *homogeneous (of degree i)* if $x \in M_i$ for some integer i, and we write $\deg x = i$. Any element $y \in M$ has a unique presentation $y = \sum_i y_i$, where $y_i \in M_i$ for all $i \in \mathbb{Z}$, and all but a finite number of the y_i are 0. The nonzero components y_i are called the *homogeneous components* of y.

Given a graded R-module M and an integer a, the graded R-module $M(a)$ shifted by a is the R-module M equipped with the new grading $M(a)_j = M_{a+j}$ for all $j \in \mathbb{Z}$.

The following objects are relevant examples of graded rings. An ideal $I \subset R$ is called a *graded ideal*, if $I = \bigoplus_{j \in \mathbb{Z}} I_j$, where $I_j = I \cap R_j$ for all $j \in \mathbb{Z}$. An ideal $I \subset R$ is graded if and only if I is generated by homogeneous elements of R if and only if for $f \in I$, all the homogeneous components of f also belong to I.

Example 1.1.2 *The polynomial ring* $S = \mathbb{K}[x_1, \ldots, x_n]$ *can be graded by assigning to* x_i *the degree* $a_i \in \mathbb{N}_{>0}$*. Let* $n = 3$*, and* $a_1 = 1, a_2 = 3, a_3 = 4$*. Then the polynomial* $f = x_1x_2 - x_3$ *is homogeneous of degree 4. If* S_i *denotes the set of all monomials of* S *of degree* i*, then* S *inherits a structure of* $\mathbb{Z}$*-graded ring, and* $I = (f)$ *is a graded ideal of* S*.*

From now on, our general assumption is that S denotes the standard graded polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ in n indeterminates over a field $\mathbb{K}$, $\mathfrak{m} = (x_1, \ldots, x_n)$ the graded maximal ideal of S, and M a finitely generated graded S-module.

Definition 1.1.3 *A sequence* $\underline{f} = f_1, \ldots, f_m \in \mathfrak{m}$ *of homogeneous elements of* S *is called an* M*-regular sequence or simply* M*-sequence, if* f_i *is a nonzerodivisor on* $M/(f_1, \ldots, f_{i-1})M$ *for all* $i = 1, \ldots, n$*. The maximal possible length of an* M*-sequence is called the depth of* M*, denoted* depth M*.*

Examples 1.1.4 *(a) The typical example of a regular sequence of* S *is the sequence of the indeterminates* $x_1, \ldots, x_n$*.*

(b) Let $n = 4$*, and* $M = \mathbb{K}[x_1, \ldots, x_4]/(x_2(x_4 - x_1))$*. The sequence* $x_1, x_2x_3 - x_4^2$ *is an* M*-regular sequence.*

Under our general assumption, any permutation of a regular sequence is regular. In general, it holds depth $M \leq \dim M$. A relevant class of modules are those for which it holds the equality.

Definition 1.1.5 *An* S*-module* M *is called Cohen-Macaulay if*

$$\operatorname{depth} M = \dim M.$$

An important property of a Cohen–Macaulay module is that it has no embedded prime ideal. Moreover, all minimal prime ideals have the same height. Rings with this property are called *unmixed*. On the other hand, an unmixed module need not to be Cohen–Macaulay, as the following example shows.

Example 1.1.6 *Let* $S = \mathbb{K}[x_1, x_2, x_3, x_4]$ *and* $I = (x_1, x_2) \cap (x_3, x_4) \subset S$*.* S/I *is unmixed but is not Cohen-Macaulay, since* depth $S/I = 1$ *but* $\dim S/I = 2$*.*

The following result summarizes two fundamental properties of Cohen-Macaulay rings.

Theorem 1.1.7. *The following holds:*

1. *If R be a Cohen-Macaulay ring, then* $\operatorname{height} I + \dim R/I = \dim R$, *for any graded ideal I.*
2. *Let $\underline{f}$ be a regular R-sequence. R is Cohen-Macaulay if and only if $R/(\underline{f})$ is Cohen-Macaulay.*

1.1.2 Free resolution, Castelnuovo-Mumford regularity and Hilbert function

A *complex* $\mathbb{F}$ of S-modules is a sequence of modules F_i and maps $\varphi_i : F_i \to F_{i-1}$ such that, for all i, it holds $\varphi_{i-1} \circ \varphi_i = 0$. The *$i$-th homology* of $\mathbb{F}$, denoted $H_i(\mathbb{F})$, is the module

$$H_i(\mathbb{F}) = \ker \varphi_i / \mathrm{Im}\ \varphi_{i+1}.$$

A *free S-module* is a module F which is a direct sum of modules of the form $S(d)$, for some $d \in \mathbb{Z}$. A *free resolution* of an S-module M is a complex

$$\mathbb{F} : \cdots \to F_i \xrightarrow{\varphi_i} F_{i-1} \to \cdots \to F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \to 0$$

of free S-modules F_i such that $H_i(\mathbb{F}) = 0$ for all $i \neq 0$, and $H_0(\mathbb{F}) \cong M$. If there exists an integer $\ell > 0$ such that $F_{\ell+1} = 0$ and $F_i \neq 0$ for all $0 < i < \ell$, then we say that $\mathbb{F}$ is a *finite free resolution of length ℓ*.

A *finitely generated graded free S-module* is a module F which admits a finite basis of homogeneous elements. If the basis elements are of degree $a_1, \ldots, a_r$, then $F \cong \bigoplus_{j=1}^{r} S(-a_j)$.

Theorem 1.1.8 (Hilbert syzygy theorem)**.** *Any finitely generated graded S-module admits a finite graded free resolution of length $< n$, by finitely generated free modules.*

A *graded free S-resolution* of M is a free resolution $\mathbb{F}$ of M, where M is a graded S-module, the F_i are graded free modules, and the maps are homogeneous maps of degree 0, that is $\varphi_i((F_i)_n) \subset (F_{i-1})_n$. Such a resolution cannot be unique, since it depends on the choice of the basis of M, and, consequently, of the modules F_i. A graded free S-resolution of M is called *minimal* if Im $\varphi_i \subset \mathfrak{m}F_{i-1}$. The minimal graded free S-resolutions $\mathbb{F}$ of M is, due to Hilbert syzygy theorem, of length $< n$, and is unique, up to isomorphism. The latter implies that the number of generators of each degree required for the free modules F_i depends only on M, and $\mathbb{F}$ can be rewritten in the following form

$$\mathbb{F} : \cdots \to \bigoplus_j S(-j)^{\beta_{i,j}} \to \cdots \to \bigoplus_j S(-j)^{\beta_{1,j}} \to \bigoplus_j S(-j)^{\beta_{0,j}} \to M \to 0.$$

The numbers $\beta_{i,j}$ are called the *graded Betti numbers* of M and they are denoted $\beta_{i,j}(M)$.

Construction 1.1.9 *Let M be a graded S-module. Choose a homogeneous minimal system of generators $g_1, \ldots, g_t$ of M with $\deg g_i = d_i$. Define $F_0 = \bigoplus_{i=1}^t S(-d_i)$ with homogeneous basis $f_1, \ldots, f_t$ and $\deg f_i = d_i$. The map $f_i \mapsto g_i$ induces a surjective homogeneous map φ_0 from F_0 to M. The kernel K_0 of φ_o is again a finitely graded S-module. Choose a homogeneous minimal system of generators $g'_1, \ldots, g'_t$ of K_0 with $\deg g'_i = d'_i$. Set $F_1 = \bigoplus_{i=1}^{t'} S(-d'_i)$ with homogeneous basis $f'_1, \ldots, f'_t$ and $\deg f'_i = d'_i$. Define $\varphi_1 : F_1 \to F_0$ by $\varphi_1(f'_i) = g'_i$. By repeating this procedure, one gets a graded free resolution $\mathbb{F}$ of M. Since, for all $i \geq 0$, one has $\varphi_{i+1}(F_{i+1}) \subseteq \mathfrak{m}F_i$, $\mathbb{F}$ is a minimal graded free resolution of M.*

Example 1.1.10 *Let $S = \mathbb{K}[x_1, \ldots, x_4]$ and $I = (g_1, g_2, g_3)$, where $g_1 = x_1x_3 - x_2^2$, $g_2 = x_4x_3 - x_1x_2$, and $g_3 = x_4x_2 - x_1^2$. We now construct the minimal free resolution of S/I. Let $F_0 = S$. The kernel of the map $\varphi_0 : S \to S/I$ is I, which is minimally generated by $g_1, \ldots, g_3$. So $F_1 = S(-2)^3$. The generator of I can be viewed as the 2×2 minors of the 2×3-matrix:*

$$M = \begin{pmatrix} x_4 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}.$$

One can show that the only relations among the 3 generators are

$$x_4g_1 - x_1g_2 + x_2g_3 = 0, \qquad \text{and} \qquad x_1g_1 - x_2g_2 + x_3g_3 = 0.$$

So $F_2 = S(-3)^2$. Since the map $\varphi_2 : F_2 \to F_1$ is injective, there are no further relations. The minimal graded free resolution of S/I is thus

$$0 \to S(-3)^2 \to S(-2)^3 \to S \to S/I \to 0.$$

Given a complex $\mathbb{F}$ of finitely generated free modules and a S-module N, then $\mathbb{F} \otimes_S N$, $N \otimes_S \mathbb{F}$, $\mathrm{Hom}_S(\mathbb{F}, N)$, and $\mathrm{Hom}_S(N, \mathbb{F})$ are still complexes with complex maps induced by $\varphi \otimes_S N$, $N \otimes_S \varphi$, $\mathrm{Hom}_S(\varphi, N)$, and $\mathrm{Hom}_S(N, \varphi)$, respectively. If $\mathbb{F}$ is a minimal graded free resolution of M, it holds

$$\mathrm{Tor}_i^S(N, M) \cong H_i(N \otimes \mathbb{F}),$$

where Tor stands for the derived functor Tor. This is a useful tool to compute the graded Betti number of M, due to the following result:

Proposition 1.1.11. *Let $\mathbb{F}$ be the minimal graded free resolution of M, then any minimal set of homogeneous generators of F_i contains exactly* $\dim_{\mathbb{K}} \mathrm{Tor}_i^S(\mathbb{K}, M)_j$ *generators of degree j. Consequently,*

$$\beta_{i,j}(M) = \dim_{\mathbb{K}} \mathrm{Tor}_i^S(\mathbb{K}, M)_j.$$

Due to the Hilbert Syzygy Theorem and the previous proposition, it follows that $\beta_{i,j}(M) = 0$ for all i and j, with $i > n$. This means that there are only finitely many pairs (i, j) for which $\beta_{i,j}(M) \neq 0$.

The *projective dimension* of M, denoted $\mathrm{proj\,dim}\ M$, is

$$p = \mathrm{proj\,dim}\ M = \max\{i \mid \beta_{i,j}(M) \neq 0\},$$

and the *Castelnuovo-Mumford regularity* of M, denoted $\mathrm{reg}\ M$, is

$$r = \mathrm{reg}\ M = \max\{j - i \mid \beta_{i,j}(M) \neq 0 \text{ for some } i\}.$$

Moreover, if M is a Cohen-Macaulay module then the *Cohen-Macaulay type* of M, denoted CM-type(M), is

$$\text{CM-type}(M) = \sum_j \beta_{p,p+j}(M)$$

In the following, we briefly say regularity of M, instead of Castelnuovo-Mumford regularity of M. The (graded) Betti numbers of M are usually displayed by means of a table called *Betti diagram*, as Table 1.1 shows.

	0	$\cdots$	i	$\cdots$	p
0	$\beta_{0,0}$	$\cdots$	$\beta_{i,i}$	$\cdots$	$\beta_{p,p}$
$\vdots$	$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\vdots$
j	$\beta_{0,j}$	$\cdots$	$\beta_{i,i+j}$	$\cdots$	$\beta_{p,p+j}$
$\vdots$	$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\vdots$
r	$\beta_{0,r}$	$\cdots$	$\beta_{i,i+r}$	$\cdots$	$\beta_{p,p+r}$

Table 1.1: Betti diagram.

Example 1.1.12 *Let S/I be as in Example 1.1.10. Then the Betti numbers are $\beta_{0,0} = 1$, $\beta_{1,2} = 3$, $\beta_{2,3} = 2$, and $\beta_{i,j} = 0$ for all other (i, j).*

	0	1	2
0	1	-	-
1	-	3	2

Therefore, $\operatorname{proj\,dim} S/I = 2$, *and* $\operatorname{reg} S/I = 1$.

A Betti number $\beta_{i,i+j}(M) \neq 0$ is called *extremal* if $\beta_{k,k+\ell}(M) = 0$ for all pairs $(k, \ell) \neq (i, j)$, with $k \geq i, \ell \geq j$. In Table 1.2, the nonzero Betti numbers in the corners are the extremal Betti numbers. M has a unique extremal Betti number, $\widehat{\beta}(M)$, if and only if $\beta_{p,p+r}(M) \neq 0$.

By definition of projective dimension and regularity of M, there exist unique numbers i and j such that $\beta_{i,i+r}(M)$ and $\beta_{p,p+j}(M)$ are extremal Betti numbers. We call them the *distinguished extremal Betti numbers* of M. Let k be the maximal integer j such that $\beta_{i,j} \neq 0$ for some i. It is clear that $\beta_{i,k}(M)$ is an extremal Betti number for all i with $\beta_{i,k} \neq 0$, and that there is at least one such i. These Betti numbers are distinguished by the fact that they are positioned on the diagonal $\{(i, k-i) | i = 0, \ldots, k\}$ in the Betti diagram, and that all Betti numbers on the right lower side of the diagonal are zero. The Betti numbers $\beta_{i,k}$, for $i = 0, \ldots, k$, are called *superextremal*, regardless of whether they are zero or not. In Table 1.3, $\beta_{i,m} \neq 0$ is one of the superextremal Betti numbers, the others are on the same diagonal and are displayed using the symbol $*$. We refer the reader to [29, Chapter 11] for further details.

	0	1	$\cdots$	$\hat{\text{i}}$	$\cdots$	i	$\cdots$	$\cdots$	p
0	$\beta_{0,0}$	$\beta_{1,1}$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\beta_{p,p}$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\vdots$
$\hat{\text{j}}$	$\beta_{0,\hat{j}}$	$\beta_{1,1+\hat{j}}$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\beta_{p,p+\hat{j}} \neq 0$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	0
j	$\beta_{0,j}$	$\beta_{1,1+j}$	$\cdots$	$\cdots$	$\cdots$	$\beta_{i,i+j} \neq 0$	0	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\cdots$	$\cdots$	0	$\cdots$	$\cdots$	$\vdots$
r	$\beta_{0,r}$	$\beta_{1,1+r}$	$\cdots$	$\beta_{\hat{i},\hat{i}+r} \neq 0$	0	0	$\cdots$	$\cdots$	0

Table 1.2: Extremal Betti numbers.

	0	$\cdots$	$\cdots$	i	$\cdots$	$\cdots$	p
0	$\beta_{0,0}$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$*$	0
$\vdots$	$\vdots$	$\cdots$	$\cdots$	$\cdots$	$*$	0	$\vdots$
m-i	$\beta_{0,m-i}$	$\cdots$	$\cdots$	$\beta_{i,m} \neq 0$	0	$\cdots$	$\vdots$
$\vdots$	$\vdots$	$\cdots$	$*$	0	$\cdots$	$\cdots$	$\vdots$
r	$\beta_{0,r}$	$*$	0	$\cdots$	$\cdots$	$\cdots$	0

Table 1.3: Superextremal Betti numbers.

Theorem 1.1.13 (Auslander-Buchsbaum)**.** *Let M be a finitely generated graded S-module. Then*

$$\operatorname{proj\,dim}\, M + \operatorname{depth}\, M = n.$$

If M is Cohen–Macaulay of dimension d, then the Auslander–Buchsbaum theorem implies that $\operatorname{proj\,dim}\, M = n - d$.

Theorem 1.1.14. *Let M be a finitely generated graded Cohen–Macaulay S-module. Then M admits only one extremal Betti number which is $\widehat{\beta}(M) = \beta_{p,p+r}(M)$.*

Sometimes we need to compare the regularity of modules in a short exact sequence.

Proposition 1.1.15. *If $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is a short exact sequence of graded S-modules, then*

1. $\operatorname{reg} A \leq \max\{\operatorname{reg} B, \operatorname{reg} C + 1\}$;
2. $\operatorname{reg} B \leq \max\{\operatorname{reg} A, \operatorname{reg} C\}$;
3. $\operatorname{reg} C \leq \max\{\operatorname{reg} A - 1, \operatorname{reg} B\}$;

An immediate consequence of the above proposition is that

$$\operatorname{reg} A \oplus B \leq \max\{\operatorname{reg} A, \operatorname{reg} B\}.$$

We finish this subsection with a review on the Hilbert function, a classical tool in Commutative Algebra, that captures many useful numerical invariants. It plays an important role in Algebraic Geometry as well, and it is becoming increasingly relevant also in Computational Algebra.

The numerical function $H_M : \mathbb{Z} \to \mathbb{N}$ with $H_M(i) = \dim_{\mathbb{K}} M_i$ is called the *Hilbert function* of M. The formal Laurent series

$$\mathrm{HS}_M(t) = \sum_{i\in\mathbb{Z}} H_M(i)t^i$$

is called the *Hilbert-Poincaré series* of M. Due to the Hilbert-Serre's theorem, the Hilbert-Poincaré series is a rational function, that is $\mathrm{HS}_M(t) = p(t)/(1-t)^n$, where $p(t) \in \mathbb{Q}[t]$. After cancellation, we obtain a presentation

$$\mathrm{HS}_M(t) = \frac{h(t)}{(1-t)^d}, \qquad \text{where } h(t) \in \mathbb{Q}[t] \text{ and } h(1) \neq 0.$$

The number d is the Krull dimension of M. Let $h(t) = \sum_{i=0}^{c} h_i t^i$. The coefficient vector $(h_0, h_1, \ldots, h_c)$ is called the h-vector of M. The Hilbert function, and thus the Hilbert-Poincaré series, is *additive* relatively to exact sequences: if $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is a short exact sequence of graded S-modules, then

$$H_B(t) = H_A(t) + H_C(t), \text{ and } \mathrm{HS}_B(t) = \mathrm{HS}_A(t) + \mathrm{HS}_C(t).$$

By using the additivity of the Hilbert-Poincaré series, the polynomial $p(t)$ is related to the graded Betti numbers of M in the following way

$$p(t) = \sum_{i,j} (-1)^i \beta_{i,j}(M) t^j.$$

The degree of HS_M as rational function, denoted $a(M)$, that is $a(M) = \deg p(t) - n = \deg h(t) - d$, is called a-invariant of M. It holds

$$a(M) \leq \text{reg } M - \text{depth } M. \tag{1.1}$$

The equality holds if M is Cohen-Macaulay. In this case, $\dim M = \text{depth } M$, and then $\deg h(t) = \text{reg } M$.

Example 1.1.16 *Let $S = \mathbb{K}[x_1, x_2, x_3]$ be the standard graded polynomial ring and $I = (x_1^2, x_1x_2, x_1x_3, x_3^3) \subset S$. The Hilbert function of S/I is given by*

$$H_{S/I}(t) = \begin{cases} 1, & \textit{if } t = 0; \\ 3, & \textit{if } t \geq 1. \end{cases}$$

In fact, for $n = 2$, the monomials of degree 2 in S/I are x_2^2, x_2x_3, x_3^2, and for any degree $n \geq 3$, the monomials of degree n in S/I are $x_2^n, x_2^{n-1}x_3, x_2^{n-2}x_3^2$. It follows that the Hilbert-Poincaré series is

$$\mathrm{HS}_{S/I}(t) = \sum_{i \geq 0} 3t^i = \frac{1+2t}{1-t}.$$

In this case, $a(S/I) = 0$. A free minimal graded resolution of S/I, computed using Macaulay2 [27], looks like

$$0 \to S(-4) \to S(-3)^3 \oplus S(-4) \to S(-2)^3 \oplus S(-3) \to S \to S/I \to 0,$$

and then the Betti table is

	0	1	2	3
0	1	-	-	-
1	-	3	3	1
2	-	1	1	-

Looking at the Betti table, we get reg $S/I = 2$ *and* proj dim $S/I = 3$. *Since* $\dim S/I = 1$, *and by Auslander-Buchsbaum formula,* depth $S/I = 3 -$ proj dim $S/I = 0$, *S/I is not Cohen-Macaulay and, in particular, the inequality (1.1) is strict.*

1.1.3 Gröbner Bases

The theory of Gröbner basis for polynomial rings was developed by Bruno Buchberger in 1965. Thenceforth, Gröbner bases, together with initial ideals, provided new methods in many different branches of mathematics. They have been used not only for computational purposes but also to deduce theoretical results in commutative algebra and combinatorics. In this subsection, we recall some definitions and relevant results about Gröbner basis.

Let $S = \mathbb{K}[x_1, \dots, x_n]$. A *monomial order* on S is a total order $<$ on the set of all monomial of S, denoted by Mon S, such that

- $1 < m$, for all $m \in \text{Mon } S$;
- if $m_1, m_2 \in \text{Mon } S$ with $m_1 < m_2$, then $m_1 m_3 < m_2 m_3$ for all $m_3 \in \text{Mon } S$.

If $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, then $\mathbf{x}^{\boldsymbol{\alpha}} \in \text{Mon } S$ stands for the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

Examples 1.1.17 *Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n), \boldsymbol{\beta} = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$, with $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$, and $\mathbf{x}^{\boldsymbol{\alpha}}, \mathbf{x}^{\boldsymbol{\beta}} \in$ Mon S.*

(i) The lexicographic order on S induced by $x_1 > x_2 > \cdots > x_n$, denoted $<_{\text{lex}}$, is defined as $\mathbf{x}^{\boldsymbol{\alpha}} <_{\text{lex}} \mathbf{x}^{\boldsymbol{\beta}}$ if the leftmost nonzero component of the vector $\boldsymbol{\alpha} - \boldsymbol{\beta}$ is negative.

(ii) The graded lexicographic order on S induced by $x_1 > x_2 > \cdots > x_n$, denoted $<_{\text{grlex}}$, is defined as $\mathbf{x}^{\boldsymbol{\alpha}} <_{\text{grlex}} \mathbf{x}^{\boldsymbol{\beta}}$ if either (i) $\sum_{i=1}^n \alpha_i < \sum_{i=1}^n \beta_i$ or (ii) $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i$ and the leftmost nonzero component of the vector $\boldsymbol{\alpha} - \boldsymbol{\beta}$ is negative.

(iii) The graded reverse lexicographic order on S induced by $x_1 > x_2 > \cdots > x_n$, denoted $<_{\text{grevlex}}$, is defined as $\mathbf{x}^{\boldsymbol{\alpha}} <_{\text{grevlex}} \mathbf{x}^{\boldsymbol{\beta}}$ if either (i) $\sum_{i=1}^n \alpha_i < \sum_{i=1}^n \beta_i$ or (ii) $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i$ and the rightmost nonzero component of the vector $\boldsymbol{\alpha} - \boldsymbol{\beta}$ is positive.

Let $<$ be a fixed monomial order. Let $f = \sum_{i=1}^k a_i \mathbf{x}^{\boldsymbol{\alpha}_i}$ be a polynomial in S. The nonzero monomials $a_i \mathbf{x}^{\boldsymbol{\alpha}_i}$ in f are called *terms* of f. The *initial monomial* of f, denoted

$\text{in}_<(f)$, is the greatest term of f with respect to the monomial order $<$. The *leading coefficient*, $\text{lc}(f)$, of f is the coefficient of $\text{in}_<(f)$.

Let $I \subseteq S$ be an ideal of S. The *initial ideal* of I, denoted $\text{in}_<(I)$, with respect to $<$ is

$$\text{in}_<(I) = (\text{in}_<(f) \mid f \in I)\,.$$

In general, if $I = (f_1, \ldots, f_k)$ is an ideal of S, the initial ideal of I is not the ideal generated by $\{\text{in}_<(f_1), \ldots, \text{in}_<(f_k)\}$, as the following example shows.

Example 1.1.18 *Let $S = \mathbb{K}[x_1, x_2, x_3]$, and let $<$ be the lexicographic order with $x_1 > x_2 > x_3$. Let $I = (f_1, f_2)$, with $f_1 = x_1^3 - x_2$, $f_2 = x_1^2x_2 - x_3$. The polynomial $f = x_1x_3 - x_2^2 \in I$, but $\text{in}_<(f) = x_1x_3 \notin (\text{in}_<(f_1), \text{in}_<(f_2)) = (x_1^3, x_1^2x_2)$.*

Let I be an ideal of S. A *Gröbner basis* of I with respect to $<$ is a finite set of nonzero polynomials $\mathcal{G} = \{g_1, \ldots, g_s\} \subset I$ such that

$$\text{in}_<(I) = (\text{in}_<(g_i) \mid g_i \in \mathcal{G})\,.$$

A Gröbner basis of I always exists because the monomial ideal $\text{in}_<(I)$ is finitely generated.

Theorem 1.1.19. *Let I be an ideal of S and $\mathcal{G} = \{g_1, \ldots, g_s\}$ be a Gröbner basis of I with respect to a monomial order $<$. Then $I = (g_1, \ldots, g_s)$, that is $\mathcal{G}$ is a system of generators of I.*

As Example 1.1.18 shows, the converse of Theorem 1.1.19 is not true: in general, a system of generators of I is not a Gröbner basis of I. From now on, let $<$ be a fixed monomial order on S and we omit to say "with respect to $<$", if there is no danger of confusion. The *division algorithm* plays a fundamental role in theory of Gröbner basis.

Theorem 1.1.20 (The division algorithm)**.** *Let $g_1 \ldots, g_s, f \in S$ be nonzero polynomials. There exist $f_1, \ldots, f_s, f' \in S$ such that*

$$f = f_1g_1 + f_2g_2 + \cdots + f_sg_s + f' \tag{1.2}$$

and the following conditions are satisfied:

1. *If $f' \neq 0$, then none of the initial monomials* $\mathrm{in}_<(g_i)$, *with* $1 \leq i \leq s$, *divides any term of* f'.
2. *If* $f_i \neq 0$, *then, for all* $1 \leq i \leq s$, $\mathrm{in}_<(f) \geq \mathrm{in}_<(f_i g_i)$

The right-hand side of the Equation (1.2) is said to be a *standard expression* of f with respect to $g_1, \dots, g_s$, and f' a *remainder* of f with respect to $g_1, \dots, g_s$. One also says that f *reduces* to f' with respect to $g_1, \dots, g_s$. In general, the standard expression, and consequently the remainder, is not unique. However, if $\mathcal{G} = \{g_1, \dots, g_s\}$ is a Gröbner basis of an ideal I, then any nonzero polynomial $f \in S$ has a unique remainder with respect to $\mathcal{G}$. Moreover, f reduces to 0 with respect to $\mathcal{G}$ if and only if $f \in I$.

The highlights of the theory of Gröbner bases must be Buchberger's criterion and Buchberger's algorithm. Starting from a system of generators of an ideal, the algorithm supplies the effective procedure to compute a Gröbner basis of the ideal. The discovery of the algorithm is one of the most relevant achievements of Buchberger.

Let $f, g \in S$. The polynomial

$$S(f,g) = \frac{\mathrm{lcm}(\mathrm{in}_<(f), \mathrm{in}_<(g))}{\mathrm{lc}(f)\mathrm{in}_<(f)} f - \frac{\mathrm{lcm}(\mathrm{in}_<(f), \mathrm{in}_<(g))}{\mathrm{lc}(g)\mathrm{in}_<(g)} g$$

is called the *S-polynomial* of f and g.

Theorem 1.1.21 (Buchberger's criterion)**.** *Let I be a nonzero ideal of S and $\mathcal{G} = \{g_1, \dots, g_s\}$ a system of generators of I. Then $\mathcal{G}$ is a Gröbner basis if and only if for all $i \neq j$, $S(g_i, g_j)$ reduced to 0 with respect to $g_1, \dots, g_s$.*

The Buchberger's algorithm is an immediate consequence of the Theorem 1.1.21, and it works as follows:

Let $\mathcal{G}$ be a system of generators of a nonzero ideal I of S.

- Step 1: For each pair of distinct polynomials in $\mathcal{G}$ compute the S-polynomial and a remainder of it.
- Step 2: If all S-polynomials reduce to 0, then the algorithm ends and $\mathcal{G}$ is a Gröbner basis of I. Otherwise, join one of the nonzero remainders to the system of generators, call this new set $\mathcal{G}$ and go back to Step 1.

The Buchberger's algorithm terminates after a finite number of steps, since any monomial ideal is finitely generated. The following result can be used to shorten the calculation of Gröbner basis significanlty.

Lemma 1.1.22. *Let f and g be nonzero polynomials and suppose that* $\mathrm{in}_<(f)$ *and* $\mathrm{in}_<(g)$ *are relatively prime, that is* $\mathrm{lcm}(\mathrm{in}_<(f), \mathrm{in}_<(g)) = \mathrm{in}_<(f)\mathrm{in}_<(g)$*. Then $S(f,g)$ reduces to 0 with respect to f, g.*

Gröbner basis cannot be unique. In fact, if $\mathcal{G} = \{g_1, \ldots, g_s\}$ is a Gröbner basis of I, then any finite subset of nonzero polynomials of I which contains $\mathcal{G}$ is again a Gröbner basis of I. By imposing more constraints on the possible generators it is possible to define minimal and then reduced Gröbner bases.

A *minimal Gröbner basis* of I is a Gröbner basis $\mathcal{G} = \{g_1, \ldots, g_s\}$ of I such that

(i) $\mathrm{lc}(g_i) = 1$, for all $1 \leq i \leq s$,

(ii) $\{\mathrm{in}_<(g_1), \ldots, \mathrm{in}_<(g_s)\}$ is a minimal set of generators of $\mathrm{in}_<(I)$.

A minimal Gröbner basis exists, but may be not unique. For example, if $\{g_1, g_2\}$ is a minimal Gröbner basis of I with $\mathrm{in}_<(g_1) < \mathrm{in}_<(g_2)$, then $\{g_1, g_2 + g_1\}$ is again a minimal Gröbner basis of I.

A *reduced Gröbner basis* of I is a Gröbner basis $\mathcal{G} = \{g_1, \ldots, g_s\}$ of I such that

(i) $\mathrm{lc}(g_i) = 1$, for all $1 \leq i \leq s$,

(ii) if $i \neq j$, then none of the terms of g_j is divided by $\mathrm{in}_<(g_i)$.

A reduced Gröbner basis is a minimal Gröbner basis. However, the converse is false. A reduced Gröbner basis exists and is uniquely determined: the importance of reduced Gröbner bases lies in their uniqueness.

It is worth to underline that the Gröbner basis depends on the chosen monomial order, as the following simple example shows

Example 1.1.23 *Let $S = \mathbb{K}[x_1, x_2, x_3, x_4]$ and $I = (f_1, f_2)$, where $f_1 = x_1x_4 - x_2$ and $f_2 = x_3 - x_4^2$.*

Fix $<_{\mathrm{lex}}$ induced by $x_1 > \cdots > x_4$. With respect to $<_{\mathrm{lex}}$, the initial monomials of f_1 and f_2, that are x_1x_4 and x_3 respectively, are relatively prime. By Lemma 1.1.22, $\{f_1, f_2\}$ is a (reduced) Gröbner basis of I.

Now, fix $<_{\text{grevlex}}$ induced by $x_1 > \cdots > x_4$. With respect to $<_{\text{grevlex}}$, $\text{in}_<(f_1) = x_1x_4$ and $\text{in}_<(f_2) = x_4^2$. Performing the first step of the Buchberger's algorithm, one gets

$$S(f_1, f_2) = -x_4f_1 - x_1f_2 = x_2x_4 - x_1x_3 = f_3.$$

Since $\text{in}_<(f_3) = x_2x_4$ is relatively prime with $\text{in}_<(f_1)$ and $\text{in}_<(f_2)$, by Lemma 1.1.22, $\{f_1, f_2, f_3\}$ is a (reduced) Gröbner basis of I.

We conclude this subsection furnishing a way to compute the reduced Gröbner basis of an ideal quotient, in a specific case. We recall the definition of *ideal quotients*. If I is an ideal of S and f is a polynomial in S, then the following two subsets of S are again ideals:

$$(I : f) = \{g \in S \mid fg \in I\},$$
$$(I : f^\infty) = \{g \in S \mid f^r g \in I \text{ for some } r \in \mathbb{N}\}.$$

The second one is also called the *saturation* of I with respect to f.

Lemma 1.1.24. *[67, Lemma 12.1] Fix the graded reverse lexicographic monomial order induced by $x_1 > \cdots > x_n$, and let $\mathcal{G} = \{g_1, \ldots, g_s\}$ be the reduced Gröbner basis of a homogeneous ideal $I \subset S$. Then the set*

$$\{g_i \in \mathcal{G} \mid x_n \textit{ does not divide } g_i\} \cup \{g_i/x_n \mid g_i \in \mathcal{G} \textit{ and } x_n \textit{ divides } g_i\}$$

is a Gröbner basis of $(I : x_n)$. A Gröbner basis of $(J : x_n^\infty)$ is obtained by dividing each element $g_i \in \mathcal{G}$ by the highest power of x_n that divides g_i.

1.2 GRAPH THEORY

In this section, we summarize basic terminologies on finite graphs.

A *undirected graph* G is a pair (V, E), where V is a set whose elements are called *vertices*, and E is a set of unordered pairs of vertices, whose elements are called *edges*. The vertex set and the edge set of G are often denoted by $V(G)$ and $E(G)$, respectively.

If the edges are considered as ordered pairs of vertices, than G is called a *directed* graph. A *finite graph* is a graph G with a finite vertex set. In this dissertation, we will consider only finite undirected graphs, and we will briefly call them graphs.

Let G and H be two graphs. H is called a *subgraph* of G if $V(H) \subset V(G)$ and $E(H) \subset E(G)$. A subgraph H of G is called an *induced subgraph* if H contains all the edges $\{u, v\} \in E(G)$, with $u, v \in V(H)$. In this case, H is said to be the subgraph induced by $V(H)$.

Two graphs G_1 and G_2 are *isomorphic* if there exists a bijective map ϕ from $V(G_1)$ to $V(G_2)$ such that $\{u, v\} \in E(G_1)$ if and only if $\{\phi(u), \phi(v)\} \in E(G_2)$.

Given two graphs G_1, G_2, their *intersection* is the graph $G_1 \cap G_2$ such $V(G_1 \cap G_2) = V(G_1) \cap V(G_2)$ and $E(G_1 \cap G_2) = E(G_1) \cap E(G_2)$. Whereas, their *union* is the graph $G_1 \cup G_2$ such $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.

If $e = \{u, v\} \in E(G)$, with $u, v \in V(G)$, we say that u and v are *adjacent* and the edge e is *incident* with, or joins, u and v. The *degree* of $v \in V(G)$, denoted $\deg(v)$, is the number of edges incident with v. A vertex with degree zero is called an *isolated* vertex, and a vertex with degree 1 is called a *leaf* of G.

Example 1.2.1 *In Figure 1.1, the vertices v_1 and v_3 have degree 1, whereas $\deg(v_2) = 2$. The vertex v_4 is an isolated vertex.*

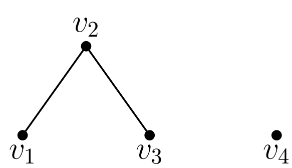


Figure 1.1

A *walk* of length n in G is a sequence of vertices $v_0, \dots, v_n \in V(G)$, such that for each $1 \leq i \leq n$, $\{v_{i-1}, v_i\} \in E(G)$. A *path* of length n, denoted P_n, is a walk of length n whose vertices are all distinct.

Example 1.2.2 *In Figure 1.2, some examples of paths are showed.*

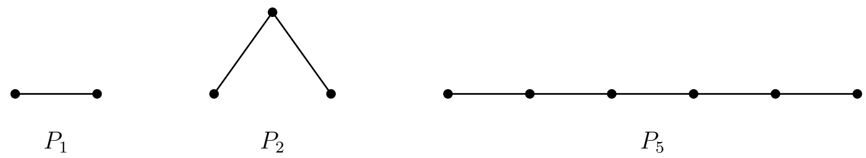


Figure 1.2: Paths.

We say that G is *connected* if for every pair of vertices u and v of G, there is a path in G from u to V. Every graph G can be written as $G = \cup_{i=1}^{r} G_i$, where $G_1, \ldots, G_r$ are the maximal connected induced subgraphs of G, also called the *connected component* of G.

A *cycle* of length n, denoted C_n is a walk $v_0, \ldots, v_n$, with $n \geq 3$, $v_n = v_0$, and all the vertices $v_0, \ldots, v_{n-1}$ are distinct. A cycle is *even* (respectively, *odd*) if its length is even (respectively, odd), that is, if it has an even (respectively, odd) number of vertices.

Example 1.2.3 *In Figure 1.3, some examples of cycles are showed.*

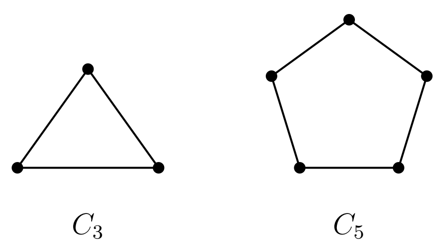


Figure 1.3: Cycles.

A *chord* of a cycle C_n in the graph G is an edge of G joining two non-adjacent vertices of C_n. A graph is called *chordal* if every cycle of G of length greater than 3 has a chord in G. Any induced subgraph of a chordal graph is chordal, as well.

Example 1.2.4 *In Figure 1.4, it is displayed a chordal graph.*

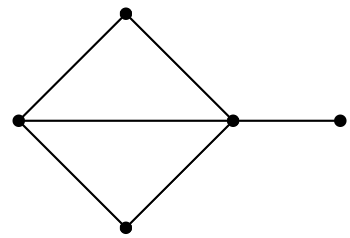


Figure 1.4: A chordal graph.

The *complete* graph, denoted K_n, is the graph such that every pair of its n vertices is adjacent.

Example 1.2.5 *In Figure 1.5, some examples of complete graphs are showed.*

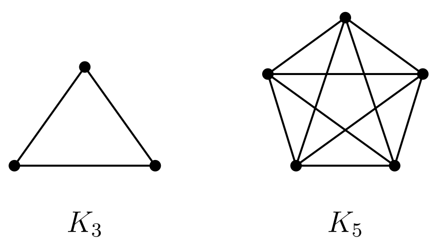


Figure 1.5: Complete graphs.

A *forest* is a graph without cycles. A *tree* is a connected forest. A particular tree is a graph consisting of three different edges that share a common vertex, and it is called *claw*, see Figure 1.6.

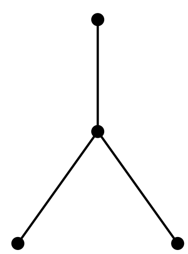


Figure 1.6: Claw.

A graph G is *bipartite* if its vertex set $V(G)$ can be partitioned into two disjoint subsets A and B such that every edge of G has one vertex in A and one vertex in B. A graph G is bipartite if and only if all the cycles of G are even.

Example 1.2.6 *In Figure 1.7, it is showed an example of a bipartite graph.*

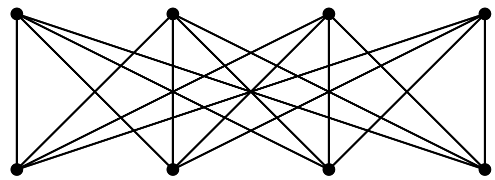


Figure 1.7: A bipartite graph.

Let G be a graph and $v \notin G$. The *cone* of v on G, namely $\text{cone}(v, G)$, is the graph with vertices $V(G) \cup \{v\}$ and edges $E(G) \cup \{\{v, w\} \mid w \in V(G)\}$.

Example 1.2.7 *In Figure 1.8, an example of a cone graph is showed.*

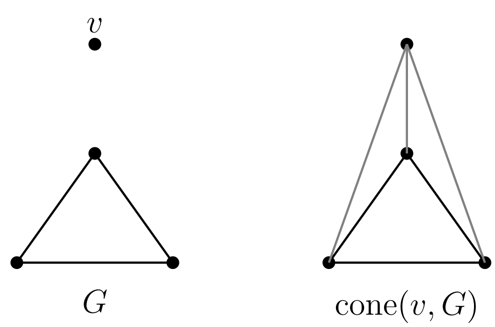


Figure 1.8: A cone graph.

Let G be a graph. A subset C of $V(G)$ is called a *clique* of G if for all $u, v \in C$, with $u \neq v$, one has $\{u, v\} \in E(G)$. A *maximal clique* is a clique that cannot be extended by including one more adjacent vertex. The *clique degree* of v, denoted by $\text{cdeg}(v)$, is the number of maximal cliques to which v belongs. A vertex v is called a *free vertex* of G if $\text{cdeg}(v) = 1$, and is called an *inner vertex* of G if $\text{cdeg}(v) > 1$.

A *finite simplicial complex* Δ is a collection of subsets of a finite set of vertices V such that:

1. $\{v\} \in \Delta$, for all $v \in V$;

2. $H \in \Delta$ and $G \subset H$ implies $G \in \Delta$.

The *clique complex* $\Delta(G)$ of G is the simplicial complex of all its cliques. A clique C of G is called *face* of $\Delta(G)$ and its *dimension* is given by $|C|-1$. The maximal faces of $\Delta(G)$ with respect to inclusion are called *facets* of $\Delta(G)$, and they are the maximal cliques of G. The set of all the facets is denoted by $\mathcal{F}(\Delta(G))$. The dimension of $\Delta(G)$ is the maximum of the dimensions of all facets.

A set $T \subset V(G)$ is called *cutset* of G if $c(T \setminus \{v\}) < c(T)$ for each $v \in T$, where $c(T)$ denotes the number of connected components induced by removing T from G. We denote by $\mathcal{C}(G)$ the set of all cutsets of G. When $T \in \mathcal{C}(G)$ consists of one vertex v, v is called a *cutpoint*.

A connected subgraph of G that has no cutpoint and is maximal with respect to this property is a *block*. G is called *block graph* (or *clique tree*) if all its blocks are complete graphs. One can see that a graph G is a block graph if and only if it is a chordal graph in which every two maximal cliques have at most one vertex in common. Let G be a block graph, an *endblock* of G is a block having at most one cutpoint.

Example 1.2.8 *In Figure 1.9, it is showed an example of block graph. It consists of 11 blocks. The vertex v is an inner vertex, with* $\mathrm{cdeg}(v) = 3$*. Whereas, the vertex w is a free vertex. The block K_4 is an endblock of the graph.*

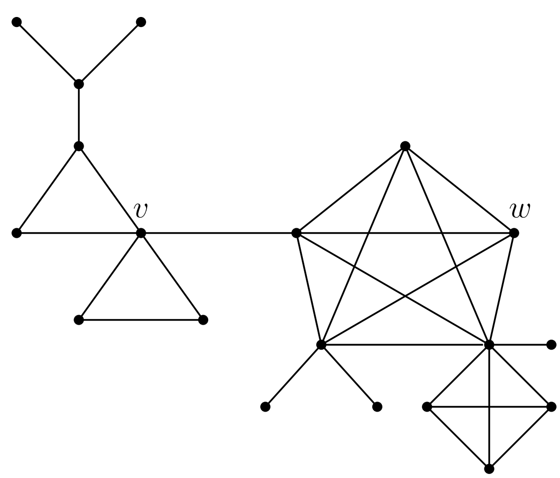


Figure 1.9: A block graph

A connected chordal graph is said to be a *generalized block graph* if for every $F_i, F_j, F_k \in \mathcal{F}(\Delta(G))$, if $F_i \cap F_j \cap F_k \neq \emptyset$, then $F_i \cap F_j = F_i \cap F_k = F_j \cap F_k$. One could see that all block graphs are generalized block graphs.

Definition 1.2.9 *A graph G is decomposable if exists a decomposition*

$$G = G_1 \cup G_2 \tag{1.3}$$

with $V(G_1) \cap V(G_2) = \{v\}$ such that v is a free vertex of G_1 and G_2. If G is not decomposable, we call it indecomposable. By a recursive decomposition (1.3) applied to each G_1 and G_2, after a finite number of steps we obtain

$$G = G_1 \cup \cdots \cup G_r \tag{1.4}$$

where $G_1, \ldots, G_r$ are indecomposable and for $1 \leq i < j \leq r$ either $V(G_i) \cap V(G_j) = \emptyset$ or $V(G_i) \cap V(G_j) = \{v_{ij}\}$, where v_{ij} is a free vertex of G_i and G_j. The decomposition (1.4) is unique up to ordering and we say that G is decomposable into indecomposable graphs $G_1, \ldots, G_r$.

Chapter 2
BINOMIAL IDEALS

In this chapter, we introduce first the large class of binomial ideals. They are ideals generated by binomials and their structure can be interpreted directly from their generators. Since the early 1990s binomial ideals have been widely studied. In [20], a comprehensive analysis of their algebraic properties is given. The study of binomial ideals is motivated by the frequency with which they occur in interesting context, for instance, any toric variety is defined by binomials. Moreover, they find application in an area of research, called computational algebraic statistics, introduced first in [16]. Ideals generated by a subset of 2-minors of a $m \times n$-matrix of indeterminates and lattice ideals are other examples of important classes of binomial ideals. Secondly, we present the two families of binomial ideals that will be object of our study: binomial edge ideals and polyomino ideals. Both of them may be viewed as ideals generated by a subset of 2-minors of a $(2 \times n)$-matrix of indeterminates. The former are ideals attached to a finite graph. The latter are ideals attached to a polyomino. For both of them, definitions, preliminary and well-known results on their algebraic properties arising from their combinatorial structure are given.

2.1 BINOMIAL, TORIC AND LATTICE IDEALS

In this section, we introduce binomial, toric and lattice ideals, and discuss some of their properties.

Let $S = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring over a field $\mathbb{K}$. A *binomial* belonging to S is a polynomial $u - v$, where u and v are monomials in S. A *binomial ideal* is an ideal of S generated by binomials. Any binomial ideal is generated by a finite number of binomials. One of the nice properties is the following

Proposition 2.1.1. *[20, Proposition 1.1] Let I be a binomial ideal of S. The reduced Gröbner basis of I with respect to an arbitrary monomial order consists of binomials.*

There is no analogue of Proposition 2.1.1 for ideals generated by polynomials with more than two terms. One immediate application of Proposition 2.1.1 is a test for binomiality.

Corollary 2.1.2. *[20, Corollary 1.2] Let $<$ be a monomial order on S. An ideal $I \subset S$ is binomial if and only if the reduced Gröbner basis for I consists of binomials.*

If $I \subset S$ is a binomial ideal generated by the binomials $f_1, \ldots, f_r$, any binomial $\mathbf{x}^\alpha - \mathbf{x}^\beta$ belonging to I can be written as linear combinations of the binomial generators with coefficients which are monomials with unitary scalars:

$$\mathbf{x}^\alpha - \mathbf{x}^\beta = \sum_{k=1}^{s} (-1)^{e_k} \mathbf{x}^{\gamma_\mathbf{k}} f_{i_k},$$

where $e_k \in \{0,1\}$, $\gamma_k \in \mathbb{N}^n$, $1 \leq i_k \leq r$, and $\gamma_\mathbf{p} f_{i_p} \neq \gamma_\mathbf{q} f_{i_q}$, for all $p \neq q$. For a detailed proof, see [31, Lemma 3.8].

Let $T = \mathbb{K}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$ be the Laurent polynomial ring over $\mathbb{K}$ in the variables $t_1, \ldots, t_d$, and $A \in \mathbb{Z}^{d \times n}$ be a matrix with column vectors $\mathbf{a}_j$. We define a $\mathbb{K}$-algebra homomorphism

$$\pi : S \longrightarrow T \quad \text{with} \quad x_j \mapsto \mathbf{t}^{\mathbf{a}_j}.$$

the image of π is the $\mathbb{K}$-subalgebra $\mathbb{K}[\mathbf{t}^{\mathbf{a}_1}, \ldots, \mathbf{t}^{\mathbf{a}_n}]$ of T, denoted $\mathbb{K}[A]$ and called *toric ring* of A. The kernel of π, denoted I_A, is called *toric ideal* of A.

Given $\mathbf{b} = (b_1, \ldots, b_n)^{\mathrm{t}} \in \mathbb{Z}^n$, let $f_\mathbf{b} \in S$ denote the polynomial defined by

$$f_\mathbf{b} = \mathbf{x}^{\mathbf{b}^+} - \mathbf{x}^{\mathbf{b}^-},$$

where $\mathbf{b}^+$ denotes the vector obtained from $\mathbf{b}$ by replacing all negative components of $\mathbf{b}$ by zero, and $\mathbf{b}^- = -(\mathbf{b} - \mathbf{b}^+)$.

Theorem 2.1.3. *[31, Theorem 3.2] Any toric ideal is a binomial ideal. More precisely, let $A \in \mathbb{Z}^{d \times n}$. Then I_A is generated by the binomials $f_\mathbf{b}$ with $\mathbf{b} \in \mathbb{Z}^n$ and $A\mathbf{b} = 0$.*

Example 2.1.4 *Let*

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix} \in \mathbb{Z}^{2 \times 3}.$$

The toric ring is $\mathbb{K}[A] = \mathbb{K}[t_1, t_1^2 t_2, t_1 t_2^3]$. Since $\ker(A) = \langle (5, -3, 1) \rangle$, *we get that the toric ideal is $I_A = (x_1^5 x_3 - x_2^3) \subset \mathbb{K}[x_1, x_2, x_3]$.*

It is clear that any toric ideal is a prime ideal. Actually, for the binomial ideals, it is true also the converse.

Theorem 2.1.5. *[31, Theorem 3.4] Let $I \subset$ be a prime binomial ideal. Then I is a toric ideal.*

Now, we give another interpretation of toric ideals by means of lattice ideals. A *lattice* $\mathcal{L}$ is a subgroup of $\mathbb{Z}^n$. In particular, $\mathcal{L}$ is a free abelian group of rank $m \le n$. The *lattice ideal* of $\mathcal{L}$, denoted $I_{\mathcal{L}}$, is the following binomial ideal in S given by

$$I_{\mathcal{L}} = (f_{\mathbf{b}} \mid \mathbf{b} \in \mathcal{L}) .$$

Any toric ideal is a lattice ideal. Indeed, if I_A is a toric ideal, then, by Theorem 2.1.3, I_A is generated by the binomials $f_{\mathbf{b}}$ with $A\mathbf{b} = 0$, that is $I_A = I_{\mathcal{L}}$, with $\mathcal{L} = \{\mathbf{b} \mid A\mathbf{b} = 0\}$. On the other hand, not all lattice ideal are toric ideals. One simple such example is the ideal $I_{\mathcal{L}}$ for $\mathcal{L} \subset \mathbb{Z}^2$ with basis $(2, -2)^{\mathrm{t}}$. Here, $I_{\mathcal{L}} = (x_1^2 - x_2^2) \in \mathbb{K}[x_1, x_2]$. If $I_{\mathcal{L}}$ would be a toric ideal it would be a prime ideal. But $x_1^2 - x_2^2 = (x_1 - x_2)(x_1 + x_2)$, and so $I_{\mathcal{L}}$ is not a prime ideal.

We have the following general result:

Theorem 2.1.6. *[31, Theorem 3.17] Let $\mathcal{L} \subset \mathbb{Z}^n$ be a lattice. The following conditions are equivalent:*

1. *the abelian group $\mathbb{Z}^n/\mathcal{L}$ is torsionfree;*
2. *$I_{\mathcal{L}}$ is a prime ideal.*

The equivalent conditions hold if and only if $I_{\mathcal{L}}$ is a toric ideal.

A lattice $\mathcal{L} \subset \mathbb{Z}^n$ is called *saturated* if for $\mathbf{c} \in \mathbb{Z}^n$ and $a \in \mathbb{Z}$, $a\mathbf{c} \in \mathcal{L}$ implies $\mathbf{c} \in \mathcal{L}$. This is equivalent to saying there exists a $d \times n$ integral matrix A with rank $A = d = n - \operatorname{rank} \mathcal{L}$ and $\mathcal{L} = \ker A \cap \mathbb{Z}^n$. The lattice ideal $I_{\mathcal{L}}$ is prime if and only if $\mathcal{L}$ is a saturated lattice.

Let $\mathcal{B} = \{\mathfrak{b}_1, \ldots, \mathfrak{b}_m\}$ be a basis of a lattice $\mathcal{L} \subset \mathbb{Z}^n$. The ideal

$$I_{\mathcal{B}} = (f_{\mathfrak{b}_i} \mid \mathfrak{b}_i \in \mathcal{B})$$

is called the *lattice basis ideal* of $\mathcal{L}$. In general, $I_{\mathcal{B}}$ is strictly contained in $I_{\mathcal{L}}$.

Example 2.1.7 *Let $A = (1, 2, 1) \in \mathbb{Z}^{1\times 3}$. The toric ideal I_A is the lattice ideal $I_{\mathcal{L}}$ of the lattice $L \in \mathbb{Z}^3$ with basis $\mathcal{B} = \{(1, -1, 1), (0, 1, -2)\}$. Then $I_{\mathcal{B}} = (x_1x_3 - x_2, x_2 - x_3^2) \subset \mathbb{K}[x_1, x_2, x_3]$, while $I_{\mathcal{L}}$ contains the binomial $x_1 - x_3$ which does not belong to $I_{\mathcal{B}}$.*

Given a lattice $\mathcal{L}$, the lattice ideal of $\mathcal{L}$ can be computed from the lattice basis ideal of $\mathcal{L}$ by taking the saturation with respect to the product of all variables

Theorem 2.1.8. *[31, Corollary 3.22] Let $\mathcal{L} \subset \mathbb{Z}^n$ be a lattice. Then,*

$$I_{\mathcal{L}} = \left(I_{\mathcal{B}} : \left(\prod_{i=1}^{n} x_i \right)^{\infty} \right).$$

Return to the Example 2.1.7. Applying Theorem 2.1.8, one gets $I_{\mathcal{L}} = (x_1 - x_3, x_2 - x_3^2)$.

As last part of this section, we give a brief overview on some classes of ideals generated by a subset of minors of a $m \times n$-matrix of indeterminates. Let

$$X_{mn} = \begin{pmatrix} x_{11} & \cdots & \cdots & x_{1n} \\ \vdots & \cdots & \cdots & \vdots \\ x_{m1} & \cdots & \cdots & x_{mn} \end{pmatrix}$$

be an $m \times n$-matrix of indeterminates x_{ij} which generate the polynomial ring $\mathbb{K}[x_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n]$. The ideals generated by all k-minors of X_{mn} are called *determinantal ideals* and they have been studied from many different points of view, for a comprehensive exposition see [6] and [5, Chapter 7]. These ideals are Cohen–Macaulay prime ideals, and their Krull dimension is $(k-1)(m+n-k+1)$ [39]. Similar determinantal ideals generated by even more general sets of minors have been also investigated. There are many variations such as ladder determinantal ideals [9], and mixed ladder determinantal ideals [26] where the ideals of (mixed) minors in a ladder-shape region in X_{mn} are studied. In both cases, these ideals are prime and Cohen–Macaulay.

An *adjacent k-minor* of X_{mn} is the determinant of a submatrix of X_{mn} with row indices $r_1, \ldots, r_k$ and column indices $c_1, \ldots, c_k$ where these indices are consecutive integers, that is $r_{i+1} = r_i + 1$ and $c_{j+1} = c_j + 1$, for all $2 \leq i, j \leq k$. Let $I_{mn}(k)$ be the ideal generated by all adjacent k-minors of X_{mn}.

As opposed to the ideal of all k-minors, the ideal $I_{mn}(k)$ is far from being a prime ideal. This ideal first appeared in [15] for the case $k = 2$ where primary decompositions of $I_{2n}(2)$ and $I_{44}(2)$ were given. The motivation for studying $I_{mn}(2)$ comes from its applications in algebraic statistics. $I_{mn}(2)$ is a binomial ideal and is a very special instance of a lattice basis ideal, and minimal primes of lattice basis ideals have been characterized in [40].

The ideal generated by all adjacent 2-minors of X_{mn} is a lattice basis ideal, and the corresponding lattice ideal is just the ideal of all 2-minors of X_{mn}. For ideals generated by any set of adjacent 2-minors of X_{mn}, holds the following

Proposition 2.1.9. *[31, Proposition 8.1] Let $\mathscr{C}$ be any set of adjacent 2-minors of X_{mn}, and $I_{\mathscr{C}}$ be the ideal generated by all the elements in $\mathscr{C}$. Then*

(a) $I_{\mathscr{C}}$ is a lattice basis ideal;

(b) $I_{\mathscr{C}}$ is a prime ideal if and only if all x_{ij} are nonzerodivisors modulo $I_{\mathscr{C}}$.

2.2 BINOMIAL EDGE IDEALS

Binomial edge ideals were introduced in 2010 by Herzog, Hibi, Hreinsdóttir, Kahle and Rauh in [30], and independently by Ohtani in [55]. They are a natural generalization of the ideals of 2-minors of a $2 \times n$-matrix: their generators are those 2-minors whose column indices correspond to the edges of a graph. Related to binomial edge ideals are the ideals of adjacent minors considered by Hoşten and Sullivant [40]. When the graph is a path, binomial edge ideals may be interpreted as an ideal of adjacent minors. This particular class of binomial edge ideals has also been considered by Diaconis, Eisenbud and Sturmfels in [15], where they compute the primary decomposition of this ideal.

In these last years, many algebraic and homological properties of binomial edge ideals have been widely investigated, such as their minimal prime ideals, Krull dimension, Castelnuovo-Mumford regularity and the projective dimension, see for instance [30], [21], [61], [45], and [58].

In this section, we summarize the main results on binomial edge ideals, needed for further sections.

2.2.1 The reduced Gröbner basis and the minimal prime ideals of a binomial edge ideal

Let $S = \mathbb{K}[\{x_i, y_j\}_{1\leq i,j\leq n}]$ be the polynomial ring in $2n$ variables with coefficients in a field $\mathbb{K}$. Let G be a graph with vertex set $[n] = \{1, \ldots, n\}$ and edge set $E(G)$. Define

$$f_{ij} = x_i y_j - x_j y_i \in S.$$

The *binomial edge ideal* of G, denoted J_G, is the ideal generated by all the binomials f_{ij}, for $i < j$ and $\{i, j\} \in E(G)$.

Example 2.2.1 *Let $G = K_3$ be the complete graph with $V(G) = [3]$. Then,*

$$J_G = (x_1y_2 - x_2y_1, x_2y_3 - x_3y_2, x_1y_3 - x_3y_1) \in \mathbb{K}[x_1, x_2, x_3, y_1, y_2, y_3].$$

One can easily observe that S/J_G is not a domain, in general. For example, if $G = P_2$, with $E(G) = \{\{1, 2\}, \{2, 3\}\}$, then $y_2(x_1y_3 - x_3y_1) \in J_G = (x_1y_2 - x_2y_1, x_2y_3 - x_3y_2)$, while neither y_2 nor $x_1y_3 - x_3y_1$ belongs to J_G. S/J_G is a domain if and only if G is a complete graph on $[n]$, that is J_G is the determinantal ideal generated by all 2-minors of X_{2n}.

Firstly, we give a necessary and sufficient condition for J_G to having a quadratic Gröbner basis.

Theorem 2.2.2. *[30, Theorem 1.1] Let G be a graph on $[n]$, and let $<$ be the lexicographic order on $S = \mathbb{K}[\{x_i, y_j\}_{1\leq i,j\leq n}]$ induced by $x_1 > x_2 > \cdots > x_n > y_1 > y_2 > \cdots > y_n$. The following conditions are equivalent:*

1. *the generators f_{ij} of J_G form a quadratic Gröbner basis;*
2. *for all edges $\{i, j\}$ and $\{k, \ell\}$, with $i < j$ and $k < \ell$, one has $\{j, \ell\} \in E(G)$ if $i = k$, and $\{i, k\} \in E(G)$ if $j = \ell$.*

Condition 2. of Theorem 2.2.2 depends on the labeling of the vertices.

Example 2.2.3 *Let G_1, G_2 be two paths of length 2, with $E(G_1) = \{\{1, 2\}, \{2, 3\}\}$ and $E(G_2) = \{\{1, 2\}, \{1, 3\}\}$. The graphs G_1 and G_2 are isomorphic, but G_1 satisfies condition 2, while G_2 does not, indeed $\{1, 2\}, \{1, 3\} \in E(G_2)$, but $\{2, 3\} \notin E(G_2)$.*

A graph G on $[n]$ is *closed with respect to the given labeling of the vertices*, if G satisfies condition 2. of Theorem 2.2.2. A graph G with $|V(G)| = n$, is *closed*, if its vertices can be labeled by the integer $1, 2, \ldots, n$ such that G is closed with respect to this labeling. In Theorem 2.2.2 the role of the lexicographic order on S is fundamental. In [13], the authors prove that the existence of a quadratic Gröbner basis is not related to the lexicographic order on S. Indeed, the closed graphs are the only graphs for which the binomial edge ideal has a quadratic Gröbner basis with respect to some monomial order on S.

Theorem 2.2.4. *[13, Theorem 3.4] Let G be a graph. The following are equivalent:*

1. *G is closed on $[n]$;*
2. *J_G has a quadratic Gröbner basis with respect to some monomial order on S.*

To compute explicitly the reduced Gröbner basis of J_G we need to introduce the following concept. Let G be a graph on $[n]$, and $i, j \in V(G)$, with $i < j$. A path $i = i_0, i_1, \ldots, i_r = j$ from i to j is called *admissible* if

(i) $i_k \neq i_\ell$, for $k \neq \ell$;

(ii) for each $k = 1, \ldots, r-1$ one has either $i_k < i$ or $i_k > j$;

(iii) for any proper subset $\{j_1, \ldots, j_s\}$ of $\{i_1, \ldots, i_{r-1}\}$, the sequence $i, j_1, \ldots, j_s, j$ is not a path.

Given an admissible path

$$\pi : i = i_0, i_1, \ldots, i_r = j$$

from i to j, we associate the monomial

$$u_\pi = \left(\prod_{i_k > j} x_{i_k}\right)\left(\prod_{i_\ell < j} y_{i_\ell}\right).$$

Example 2.2.5 *Let G be the graph in Figure 2.1. An admissible path from 1 and 3 is $\pi : 1, 4, 5, 3$ and $u_\pi = x_4 x_5$, an admissible path from 1 and 4 is $\pi' : 1, 5, 4$ and $u_{\pi'} = x_5$, an admissible path from 2 and 5 is $\pi'' : 2, 1, 5$ and $u_{\pi''} = y_1$, and an admissible path from 3 to 5 is $\pi''' : 3, 2, 1, 5$ and $u_{\pi'''} = y_1 y_2$. There is no other admissible path of G, except for the edges of G.*

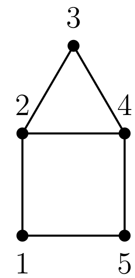


Figure 2.1

Theorem 2.2.6. *[30, Theorem 2.1] Let G be a graph on $[n]$. Let $<$ be the lexicographic order on S induced by $x_1 > x_2 > \cdots > x_n > y_1 > y_2 > \cdots > y_n$. The set of binomials*

$$\mathcal{G} = \bigcup_{i<j} \{u_\pi f_{ij} \mid \pi \text{ is an admissible path from } i \text{ to } j\}$$

is a reduced Gröbner basis of J_G.

Return to Example 2.2.5. The reduced Gröbner basis of J_G is

$$\mathcal{G} = \{f_{1,2}, f_{1,5}, f_{2,3}, f_{2,4}, f_{3,4}, f_{4,5}, u_\pi f_{1,3}, u_{\pi'} f_{1,4}, u_{\pi''} f_{2,5}, u_{\pi'''} f_{3,5}\}.$$

As an immediate consequence of Theorem 2.2.6 and since $\text{in}_<(J_G)$ is a square-free monomial ideal, one gets

Corollary 2.2.7. *[30, Corollary 2.2] Let G be a graph on $[n]$. Then, J_G is a radical ideal.*

It follows that J_G is the intersection of its minimal prime ideals. We want to determine such prime ideals. We denote by $\mathcal{C}(G)$ the set of all cutsets of G. Let $T \in \mathcal{C}(G)$ and define

$$P_T(G) = \left(\bigcup_{i \in T} \{x_i, y_i\}, J_{\tilde{G}_1}, \ldots, J_{\tilde{G}_{c(T)}} \right) \subseteq S$$

where $G_1, \ldots, G_{c(T)}$ are the connected components induced by removing T from G, and $\tilde{G}_i$, for $i = 1, \ldots, c(T)$, denotes the complete graph on $V(G_i)$. Obviously, $P_T(G)$ is a prime ideal. In fact, each $J_{\tilde{G}_j}$ is the ideal of 2-minors of a generic $2 \times n_j$-matrix, with $n_j = |V(G_j)|$. Since all the prime ideals $J_{\tilde{G}_j}$, as well as the prime ideals $(\bigcup_{i \in T}\{x_i, y_i\})$ are prime ideals in pairwise different sets of variables, $P_T(G)$ is a prime ideal, too.

Lemma 2.2.8. *[30, Lemma 3.1] Let G be a connected graph on $[n]$ and $T \in \mathcal{C}(G)$. Then, $P_T(G)$ is a prime ideal of height $n - c(T) + |T|$.*

Theorem 2.2.9. *[30, Theorem 3.2, Corollary 3.9] Let G be a connected graph on $[n]$ and $T \in \mathcal{C}(G)$. Then, $P_T(G)$ is a minimal prime ideal of J_G, and*

$$J_G = \bigcap_{T \in \mathcal{C}(G)} P_T(G). \tag{2.1}$$

It follows that J_G is prime if and only if each connected component of G is a complete graph. Moreover, J_G is unmixed if and only if, for all $T \in \mathcal{C}(G)$, one has $c(T) = |T| + 1$. Lemma 2.2.8 and Theorem 2.2.9 yield a way to compute the Krull dimension of J_G.

Corollary 2.2.10. *Let G be a graph on $[n]$. Then,*

$$\dim S/J_G = \max_{T \in \mathcal{C}(G)} \{n + c(T) - |T|\}. \tag{2.2}$$

In particular, $\dim S/J_G \geq n + c$, where c is the number of connected components of G.

Let G be a graph on $[n]$ with c connected components. Since any Cohen-Macaulay ideal is unmixed, if J_G is Cohen-Macaulay, then $\dim(S/J_G) = n + c$.

Example 2.2.11 *Let G be the graph as in Figure 2.2.*

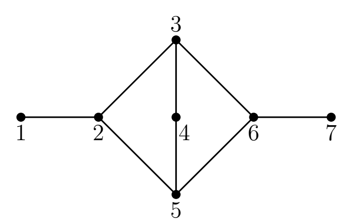


Figure 2.2

The set of all the cutsets of G is $\mathcal{C}(G) = \{\emptyset, \{2\}, \{6\}, \{2,6\}, \{3,5\}, \{2,4,6\}\}$. For any $T \in \mathcal{C}(G)$, one gets $n + c(T) - |T| = 8$, that is $\dim S/J_G = 8$ and, in particular, J_G is unmixed. But, one can verify that J_G is not Cohen-Macaulay.

Proposition 2.2.12. *[60, Corollary 1.1], [4, Remark 3.1] Let G be a graph and $u \in V(G)$ be an inner vertex of G. Then,*

$$J_G = J_{G'} \cap Q_u,$$

where G' is the graph obtained from G by connecting all the vertices adjacent to u, and $Q_v = \bigcap_{T\in\mathcal{C}(G), u\in T} P_T(G)$. Moreover, if u is a cutpoint of G, then

$$J_G = J_{G'} \cap ((x_u, y_u) + J_{G''}), \tag{2.3}$$

where G'' is the graph obtained from G by removing u.

The decomposition (2.3) will be extensively exploited to prove almost all the results in Chapter 3. For this reason, we introduce the following

Set-up 2.2.13 *Let G be a graph on $[n]$ and $u \in V(G)$ a cutpoint of G. We denote by*

G' the graph obtained from G by connecting all the vertices adjacent to u,

G'' the graph obtained from G by removing u,

H the graph obtained from G' by removing u.

Remark 2.2.14 *Using the notation introduced in Set-up 2.2.13, since $J_{G'} + ((x_u, y_u) + J_{G''}) = ((x_u, y_u) + J_H)$, the decomposition (2.3) of J_G leads to the short exact sequence*

$$0 \longrightarrow S/J_G \longrightarrow S/J_{G'} \oplus S/((x_u, y_u) + J_{G''}) \longrightarrow S/((x_u, y_u) + J_H) \longrightarrow 0. \tag{2.4}$$

From (2.4), we get the following long exact sequence of Tor modules

$$\begin{aligned} \cdots \to T_{i+1,i+1+(j-1)}(S/((x_u, y_u) + J_H)) \to T_{i,i+j}(S/J_G) \to \\ T_{i,i+j}(S/J_{G'}) \oplus T_{i,i+j}(S/((x_u, y_u) + J_{G''})) \to T_{i,i+j}(S/((x_u, y_u) + J_H)) \to \cdots \end{aligned} \tag{2.5}$$

where $T^S_{i,i+j}(M)$ stands for $\operatorname{Tor}^S_{i,i+j}(M, \mathbb{K})$ for any S-module M, and S is omitted if it is clear from the context.

2.2.2 On the regularity of binomial edge ideals

Before introducing the results on the regularity of the binomial edge ideals, we state the following result on the depth of the binomial edge ideals of chordal graphs.

Theorem 2.2.15. *[21, Theorem 1.1] Let G be a chordal graph on $[n]$ with the property that any two distinct maximal cliques intersect in at most one vertex. Then* depth $S/J_G = n+c$, *where c is the number of connected components of G. Moreover, the following conditions are equivalent:*

1. *J_G is unmixed;*
2. *J_G is Cohen-Macaulay;*
3. *each vertex of G is the intersection of at most two maximal cliques.*

Note that the class of graphs in Theorem 2.2.15 are exactly the block graphs.

Our next goal is to present lower and an upper bounds for the regularity of binomial edge ideals appeared in [52]. Even if several results have been found for particular classes of binomial edge ideals, as, for instance, closed graphs, here we state only general results. Let $G_1, \dots, G_r$ be the connected components of G. If $S_i = \mathbb{K}[\{x_j, y_j\}_{j \in V(G_i)}]$, for $i = 1, \dots, r$, then $S/J_G \cong \bigotimes_{i=1}^{r} S_i/J_{G_i}$. This equality shows that it is enough to consider connected graphs. Whereas, the following result suggests considering indecomposable graphs.

Lemma 2.2.16. *[42, Theorem 3.1] Let G be decomposable into $G_1, \dots, G_r$. Then,*

$$\text{reg } S/J_G = \sum_{i=1}^{r} \text{reg } S/J_{G_i}.$$

By comparing the Betti numbers of J_G and J_H, when H is any induced subgraph of G, we obtain a lower bound for the regularity of G.

Lemma 2.2.17. *[52, Corollary 2.2] Let G be a graph on $[n]$ and let H be an induced subgraph of G. Then, $\beta_{i,j}(S/J_G) \geq \beta_{i,j}(S/J_H)$, for all i, j.*

An immediate consequence is the following:

Corollary 2.2.18. *Let G be a graph on $[n]$ and let H be an induced subgraph of G. Then,* reg $(S/J_G) \geq$ reg (S/J_H).

In [52], a lower and upper bound for the regularity of a generic binomial edge ideal is given. Only for some classes of graphs, a precise formula for the regularity is known, for instance, for closed graphs and Cohen-Macaulay binomial edge ideals of bipartite graphs. In [61] and [63], Saeedi Madani and Kiani characterize all graphs whose binomial edge ideals, as well as their initial ideals, have regularity 2 and 3.

Theorem 2.2.19. *[52, Theorem 1.1] Let G be a connected graph on $[n]$. Then,*

$$\ell \leq \text{reg } S/J_G \leq n-1,$$

where ℓ is the length of the longest induced path of G.

While drawing up this thesis, a new upper bound has been proved by Ene, Rinaldo, and Terai in [22].

Theorem 2.2.20. *[22, Theorem 2.1] Let G be a connected graph on $[n]$. Then,*

$$\text{reg } S/J_G \leq n - \dim \Delta(G),$$

where $\Delta(G)$ is the clique complex of G.

If G is a closed graph, the lower bound in Theorem 2.2.19 is reached, and in particular it holds $\text{reg } S/J_G = \text{reg } S/\text{in}_<(J_G) = \ell$.

Example 2.2.21 *Both inequalities in Theorem 2.2.19 could be strict, indeed consider the graph G in Figure 2.3. G has 6 vertices, the length of the longest induced path of G is 3 but, using Macaulay2 [27], one gets* $\text{reg } S/J_G = 4$. *It follows* $3 < \text{reg } S/J_G < 5$.

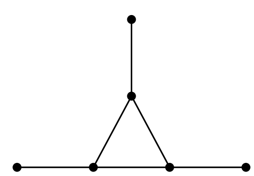


Figure 2.3

Due to Theorem 2.2.19, if the binomial edge ideal of a connected graph has regularity equal to 2, then it is a complete graph. Actually, it is true also the converse.

Proposition 2.2.22. *Let G be a connected graph on $[n]$. Then,* $\text{reg } S/J_G = 1$ *if and only if G is a complete graph.*

The upper bound for regularity in Theorem 2.2.19 is reached only for path graph. It was conjectured by Matsuda and Murai in [52], and settled in affirmative by Kiani and Saeedi Madani in [45].

Theorem 2.2.23. *[45, Theorem 3.4] Let G be a graph on $[n]$. Then,* $\operatorname{reg} S/J_G = n-1$ *if and only if G is a path graph.*

In [23], the authors proved the conjecture posed in [62] for closed graphs and block graphs. For these graphs, the regularity of S/J_G is bounded below by the length of the longest induced path of G and above by $c(G)$, where $c(G)$ is the number of maximal cliques of G.

2.2.3 Cohen-Macaulayness

The classification of Cohen–Macaulay binomial edge ideals in terms of the underlying graphs is still widely open and it seems rather hopeless to give a full classification. However, many authors have studied classes of Cohen-Macaulay binomial edge ideals in terms of the associated graph, see e.g. [2], [4], [21], [45], [58], [59], and [60].

In this subsection, we collect some of results concerning the unmixedness and Cohen-Macaulayness of classes of binomial edge ideals, as closed and bipartite graphs.

By [21, Theorem 3.1], it holds that if G is a connected graph on $[n]$ which is closed with respect to the given labeling, then J_G is Cohen-Macaulay if and only if $\operatorname{in}_<(J_G)$ is Cohen-Macaulay. Moreover, closed graphs with Cohen-Macaulay binomial edge ideal have the nice property that $\beta_{ij}(J_G) = \beta_{ij}(\operatorname{in}_<(J_G))$ for all i, j (see [21, Proposition 3.2]).

In [58], the authors investigate binomial edge ideals of graphs obtained by gluing of subgraphs and the formation of cones.

Theorem 2.2.24. *[58, Theorem 2.7] Let G be decomposable into G_1 and G_2. Then*

$$\operatorname{depth} S/J_G = \operatorname{depth} S_1/J_{G_1} + \operatorname{depth} S_2/J_{G_2} - 2,$$

where $S_i = \mathbb{K}[\{x_j, y_j\}_{j\in V(H_i)}]$, for $i = 1, 2$. Moreover, J_G is Cohen-Macaulay if and only if J_{G_1} and J_{G_2} are Cohen-Macaulay.

Lemma 2.2.25. *[58, Lemma 3.4] Let $H = \bigsqcup_{i=1}^{r} H_i$ be a graph with H_i connected components with $r \geq 1$, and let $G = \operatorname{cone}(v, H)$. If J_G is unmixed has at most two connected components.*

Due to Lemma 2.2.25, if G is a cone, namely $G = \text{cone}(v, H)$, it is necessary for G being Cohen-Macaulay that H has at most two connected components.

Lemma 2.2.26. *[58, Corollary 3.6, Corollary 3.7] Let $H = H_1 \sqcup H_2$ on $[n]$ such that H_1 and H_2 are connected graphs and let $G = \text{cone}(v, H)$. Then,*

$$\dim S/J_G = \max\{\dim S_1/J_{H_1} + \dim S_2/J_{H_2}, n+1\},$$

where $S_i = \mathbb{K}[\{x_j, y_j\}_{j \in V(H_i)}]$, for $i = 1, 2$. Moreover, J_G is unmixed if and only if J_{H_1} and J_{H_2} are unmixed.

The next result shows how to construct Cohen-Macaulay graphs by means of the formation of cones.

Theorem 2.2.27. *[58, Theorem 3.8] Let $H = H_1 \sqcup H_2$ such that H_1 and H_2 are connected graphs and let $G = \text{cone}(v, H)$. If J_{H_1} and J_{H_2} are Cohen-Macaulay, then J_G is Cohen-Macaulay.*

It is still an open question whether the converse of Theorem 2.2.27 is true.

In [4], a complete classification of Cohen-Macaulay binomial edge ideals of bipartite graphs is given, and the next subsection is devoted to present this class of binomial edge ideals.

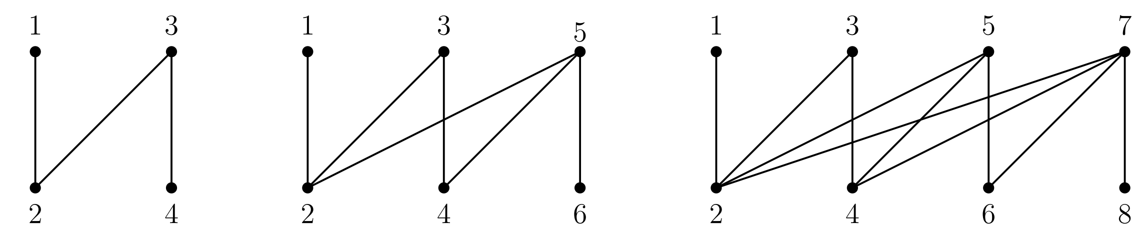
2.2.4 Binomial edge ideal of bipartite and fan graphs

In [4], Bolognini, Macchia, and Strazzanti study unmixed and Cohen-Macaulay binomial edge ideal of bipartite graphs. A first distinguishing fact about bipartite graphs with binomial edge ideal unmixed is that they have exactly two leaves. This, in particular, means that the graph has at least two cutpoints. They exhibit an explicit and recursive construction in graph-theoretical terms of all Cohen-Macaulay binomial edge ideals of bipartite graphs. Moreover, an other family of Cohen-Macaulay graphs appear in that work: the fan graphs. Afterwards, in [41], the regularity of both Cohen-Macaulay binomial edge ideals of bipartite and fan graphs is investigated. This section is devoted to recall definitions and to collect all these results, while in Section 3.2.3, we provide the unique extremal

Betti number of these classes of Cohen-Macaulay binomial edge ideals.

For every $m \geq 1$, let F_m denote the graph on the vertex set $[2m]$ and with edge set $E(F_m) = \{\{2i, 2j-1\} \mid i = 1, \dots, m,\ j = i, \dots, m\}$. Any F_m is a bipartite graph.

Example 2.2.28 *In Figure 2.4, some examples of bipartite graphs F_m, for $m \in \{2, 3, 4\}$, are displayed.*



(a) The graph F_2 (b) The graph F_3 (c) The graph F_4

Figure 2.4: Bipartite graphs F_m.

In [4], they prove that if G is connected and bipartite, then J_G is Cohen-Macaulay if and only if G can be obtained recursively by gluing a finite number of graphs of the form F_m via two operations.
Operation $*$: For $i = 1, 2$, let G_i be a graph with at least one leaf f_i. We denote by $G = (G_1, f_1) * (G_2, f_2)$ the graph G obtained by identifying f_1 and f_2.

Example 2.2.29 *In Figure 2.5, it is displayed the graph obtained by gluing F_3 and F_4 via the operation $*$.*

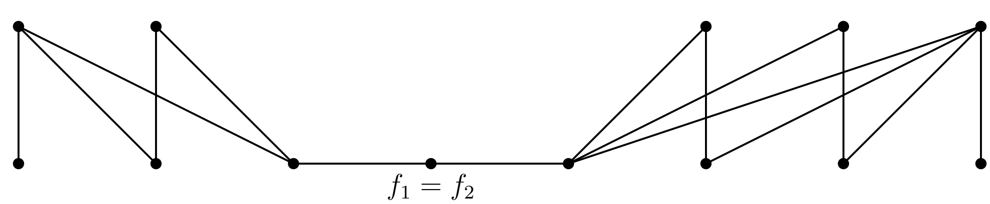


Figure 2.5: The graph $F_3 * F_4$.

Operation $\circ$: For $i = 1, 2$, let G_i be a graph with at least one leaf f_i, v_i its neighbour and assume $\deg_{G_i}(v_i) \geq 3$. We denote by $G = (G_1, f_1) \circ (G_2, f_2)$ the graph G obtained by removing the leaves f_1, f_2 from G_1 and G_2 and by identifying v_1 and v_2.

Example 2.2.30 *In Figure 2.6, it is displayed the graph obtained by gluing F_3 and F_4 via the operation $\circ$.*

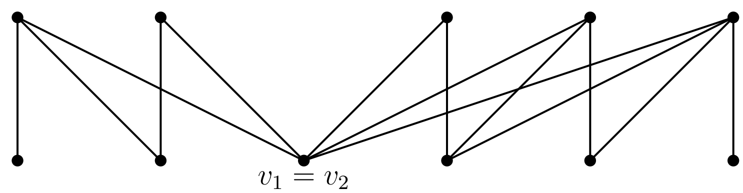


Figure 2.6: The graph $F_3 \circ F_4$.

In $G = (G_1, f_1) \circ (G_2, f_2)$, to refer to the vertex v resulting from the identification of v_1 and v_2 we write $\{v\} = V(G_1) \cap V(G_2)$. For both operations, if it is not important to specify the vertices f_i or it is clear from the context, we simply write $G_1 * G_2$ or $G_1 \circ G_2$.

Let K_m be the complete graph on $[m]$ and $W = \{v_1, \ldots, v_s\} \subseteq [m]$. Let F_m^W be the graph obtained from K_m by attaching, for every $i = 1, \ldots, s$, a complete graph K_{h_i} to K_m in such a way $V(K_m) \cap V(K_{h_i}) = \{v_1, \ldots, v_i\}$, for some $h_i > i$. We say that the graph F_m^W is obtained by adding a *fan* to K_m on the set W. If $h_i = i + 1$ for all $i = 1, \ldots, s$, we say that F_m^W is obtained by adding a *pure fan* to K_m on the set W.

Let $W = W_1 \sqcup \cdots \sqcup W_k$ be a non-trivial partition of a subset $W \subseteq [m]$. Let $F_m^{W,k}$ be the graph obtained from K_m by adding a fan to K_m on each set W_i, for $i = 1, \ldots, k$. The graph $F_m^{W,k}$ is called a k-fan of K_m on the set W. If all the fans are pure, we called it a *k-pure* fan graph of K_m on W.

Example 2.2.31 *The graph showed in Figure 2.7 (a) is a 2-fan graph $F_6^{W,2}$ with $W = \{1, 2, 3\} \sqcup \{4, 5\}$, whereas the one in Figure 2.7 (b) is the 2-pure fan graph $F_6^{W,2}$ with $W = \{1, 2, 3\} \sqcup \{4, 5\}$.*

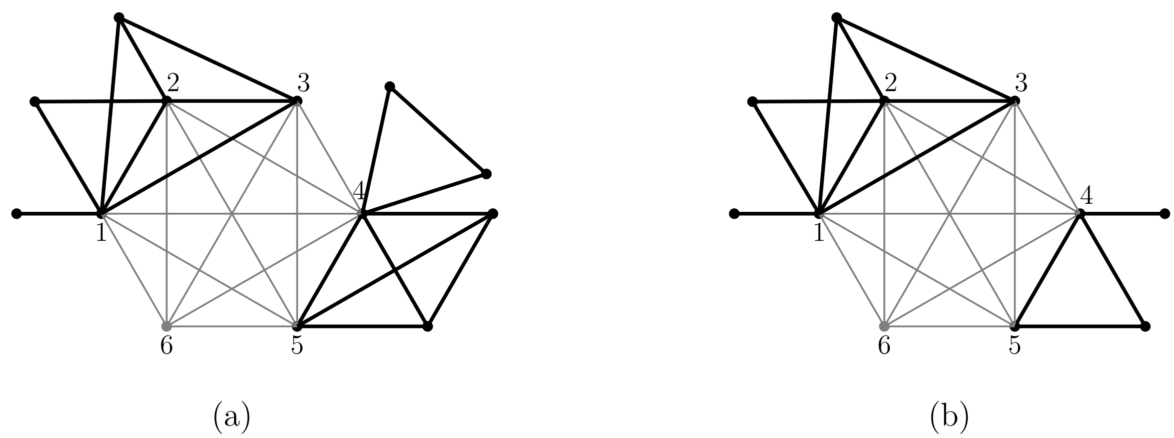


Figure 2.7: Fan graphs $F_m^{W,k}$.

When $k = 1$, we write F_m^W instead of $F_m^{W,1}$. Consider the pure fan graph F_m^W on $W = \{v_1, \dots, v_s\}$. We observe that $F_m^W = \mathrm{cone}(v_1, F_{m-1}^{W'} \sqcup \{w\})$, where $W' = W \setminus \{v_1\}$, w is the leaf of F_m^W, $\{w, v_1\} \in E(F_m^W)$, and $F_{m-1}^{W'}$ is the pure fan graph of K_{n-1} on W'.

Theorem 2.2.32. *[4, Lemma 3.2, Proposition 3.3, Theorem 4.9] Let $G = F_{m_1} \circ \cdots \circ F_{m_t} \circ F$, where F denotes either F_m or a k-pure fan graph $F_m^{W,k}$, with $t \geq 0$, $m \geq 3$, and $m_i \geq 3$ for all $i = 1, \dots, t$. Then J_G is Cohen-Macaulay.*

Theorem 2.2.33. *[4, Theorem 6.1] Let G be a connected bipartite graph. The following properties are equivalent:*

1. *J_G is Cohen-Macaulay;*
2. *$G = A_1 * A_2 * \cdots * A_k$, where, for $i = 1, \dots, k$, either $A_i = F_m$ or $A_i = F_{m_1} \circ \cdots \circ F_{m_t}$, for some $m \geq 1$ and $m_j \geq 3$.*
3. *J_G is unmixed and for any non-empty $C \in \mathcal{C}(G)$, there exists $v \in C$ such that $C \setminus \{v\} \in \mathcal{C}(G)$.*

In [41], the regularity of Cohen-Macaulay binomial edge ideals of fan and bipartite graphs is computed.

Proposition 2.2.34. *[41, Theorem 3.4] Let $G = F_m^{W,k}$ be the k-pure fan graph of K_m on*

W, with $m \geq 2$. Then

$$\text{reg } S/J_G = k + 1.$$

When $G = F_m^{W,k}$ is a k-fan graph, it does not hold the equality in Proposition 2.2.34. On the other hand, if $k \geq 2$, the longest induced path of G has length 3. Then, by Theorem 2.2.19, reg $S/J_G \geq 3$. But, by applying Lemma 2.2.16 and Lemma 2.2.17, one gets an improved lower bound for the regularity of binomial edge ideals of fan graphs, that is reg $S/J_G \geq k + 1$.

Proposition 2.2.35. *[41, Proposition 4.1] For every $m \geq 2$,* reg $S/J_{F_m} = 3$.

Observe that if $G = F_m^W$ is a pure fan graph, the regularity of J_G is equal to 3 for any m and $W \subseteq [m]$, then all of these graphs belong to the class of graphs studied by Madani and Kiani in [63].

Proposition 2.2.36. *[41, Proposition 4.3, Proposition 4.4, Remark 4.5] For $m_1, m_2 \geq 3$, let $G = F_{m_1} \circ F$, where either $F = F_{m_2}$ or F is a k-pure fan graph $F_{m_2}^{W,k}$, with $W = W_1 \sqcup \cdots \sqcup W_k$ and $\{v\} = V(F_{m_1}) \cap V(F)$. Then*

$$\text{reg } S/J_G = \begin{cases} 6, & \textit{if } F = F_{m_2}; \\ k+3, & \textit{if } F = F_{m_2}^{W,k} \textit{ and } |W_i| = 1 \textit{ for all } i; \\ k+4, & \textit{if } F = F_{m_2}^{W,k} \textit{ and } |W_i| \geq 2 \textit{ for some } i \textit{ and } v \in W_i. \end{cases}$$

Proposition 2.2.37. *[41, Theorem 4.6] Let $m_1, \ldots, m_t, m \geq 3$ and $t \geq 2$. Consider $G = F_{m_1} \circ \cdots \circ F_{m_t} \circ F$, where F denotes either F_m or the k-pure fan graph $F_m^{W,k}$ with $W = W_1 \sqcup \cdots \sqcup W_k$ and $|W_i| \geq 2$ for some i. Then*

$$\text{reg } S/J_G = \text{reg } S/J_{F_{m_1-1}} + \text{reg } S/J_{F_{m_2-2}} + \cdots + \text{reg } S/J_{F_{m_t-2}} + \text{reg } S/J_{F \setminus \{v,f\}}$$

where $\{v\} = V(F_{m_1} \circ \cdots \circ F_{m_t}) \cap V(F)$, $v \in W_i$ and f is a leaf such that $\{v, f\} \in E(F)$.

2.3 POLYOMINO IDEALS

Polyomino ideals are ideals generated by the inner 2-minors of a polyomino, and they were first introduced by Qureshi in [56]. This section is devoted to recall definitions and recent results on polyomino ideals. We refer the reader to [31] for a self-contained presentation on polyomino ideals.

Let $a = (i, j)$, $b = (k, \ell) \in \mathbb{N}^2$, with $i \leq k$ and $j \leq \ell$, the set

$$[a, b] = \{(r, s) \in \mathbb{N}^2 : i \leq r \leq k \text{ and } j \leq s \leq \ell\}$$

is called an *interval* of $\mathbb{N}^2$. If $i < k$ and $j < \ell$, $[a, b]$ is called a *proper interval*, and the elements a, b, c, d are called *corners* of $[a, b]$, where $c = (i, \ell)$ and $d = (k, j)$ (see Figure 2.8).

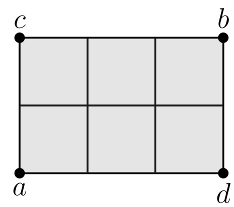


Figure 2.8: A proper interval.

In particular, a, b are called *diagonal corners* and c, d *anti-diagonal corners* of $[a, b]$. The corner a (resp. c) is also called the *left lower* (resp. *upper*) *corner* of $[a, b]$, and d (resp. b) is the *right lower* (resp. *upper*) *corner* of $[a, b]$.

A proper interval of the form $C = [a, a + (1, 1)]$ is called a *cell*. Its vertices $V(C)$ are $a, a+(1, 0), a+(0, 1), a+(1, 1)$ and its edges are $E(C) = \{a, a+(1, 0)\}, \{a, a+(0, 1)\}, \{a+(1, 0), a + (1, 1)\}, \{a + (0, 1), a + (1, 1)\}$.

Let $\mathcal{P}$ be a finite collection of cells of $\mathbb{N}^2$, and let C and D be two cells of $\mathcal{P}$. Then C and D are said to be *connected* if there is a sequence of cells $C = C_1, \ldots, C_m = D$ of $\mathcal{P}$ such that $C_i \cap C_{i+1}$ is an edge of C_i for $i = 1, \ldots, m - 1$. In addition, if $C_i \neq C_j$ for all $i \neq j$, then $C_1, \ldots, C_m$ is called a *path* connecting C and D (see Figure 2.9).

A finite collection of cells $\mathcal{P}$ is called a *polyomino* if any two cells of $\mathcal{P}$ are connected.

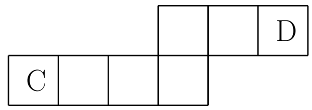


Figure 2.9: A path connecting C and D.

We denote by $V(\mathcal{P}) = \bigcup_{C \in \mathcal{P}} V(C)$ the vertex set of $\mathcal{P}$. The number of cells of $\mathcal{P}$ is called the *rank* of $\mathcal{P}$, and we denote it by rank $(\mathcal{P})$.

Example 2.3.1 *In Figure 2.10, there are displayed two collections of cells. The one on the left is not a polyomino, since, for instance, the cells C and D are not connected. Whereas, the one on the right is a polyomino.*

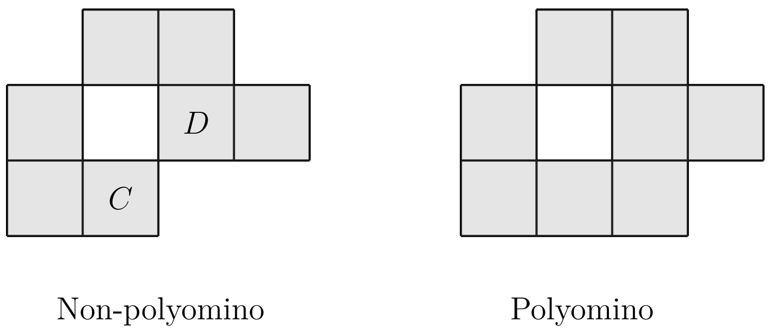


Figure 2.10: Two collections of cells.

Each interval $\mathcal{I} \subset \mathbb{N}^2$ can be regarded as a polyomino in the obvious way. This polyomino is denoted by $\mathcal{P}_{\mathcal{I}}$. A polyomino $\mathcal{Q}$ is said to be a *subpolyomino* of a polyomino $\mathcal{P}$ if each cell belonging to $\mathcal{Q}$ belongs to $\mathcal{P}$, and we write $\mathcal{Q} \subset \mathcal{P}$.

A proper interval $[a, b]$ is called an *inner interval* of $\mathcal{P}$ if all cells of $[a, b]$ belong to $\mathcal{P}$. We say that a polyomino $\mathcal{P}$ is *simple* if for any two cells C and D of $\mathbb{N}^2$ not belonging to $\mathcal{P}$, there exists a path $C = C_1, \ldots, C_m = D$ such that $C_i \notin \mathcal{P}$ for any $i = 1, \ldots, m$. If the polyomino is not simple then it is said *multiply connected*, following the notation used in [25]. A finite collection $\mathcal{H}$ of cells not in $\mathcal{P}$ is called a *hole* of $\mathcal{P}$, if any two cells in $\mathcal{H}$ are connected through a path of cells in $\mathcal{H}$, and $\mathcal{H}$ is maximal with respect to the inclusion. Note that a hole $\mathcal{H}$ of a polyomino $\mathcal{P}$ is itself a simple polyomino.

Example 2.3.2 *In Figure 2.11, there are displayed two polyominoes. The one on the left, (a), is a simple polyomino, whereas the one on the right, (b), is a multiply connected polyomino, with two holes* H_1, H_2.

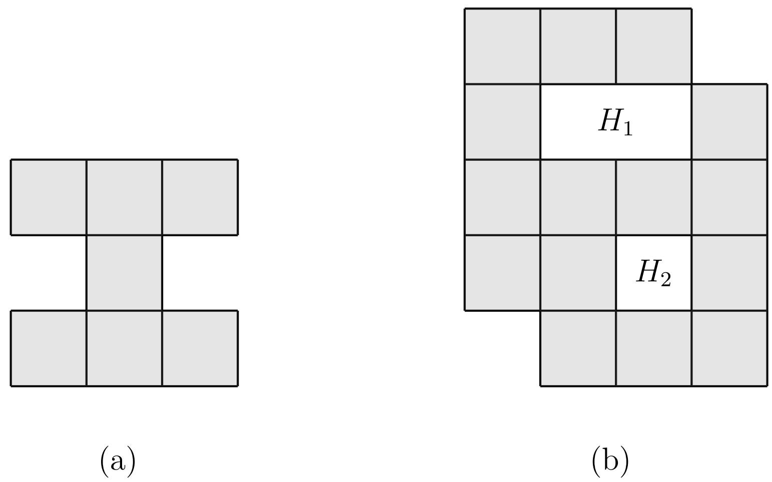


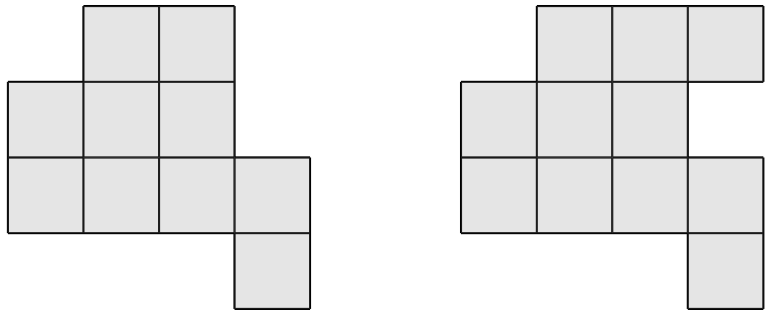
Figure 2.11: Simple and multiply connected polyominoes.

An interval $[a, b]$ with $a = (i, j)$ and $b = (k, \ell)$ is called a *horizontal edge interval* of $\mathcal{P}$ if $j = \ell$ and the sets $\{(r, j), (r + 1, j)\}$ for $r = i, \ldots, k - 1$ are edges of cells of $\mathcal{P}$. If a horizontal edge interval of $\mathcal{P}$ is not strictly contained in any other horizontal edge interval of $\mathcal{P}$, then we call it *maximal horizontal edge interval*. Similarly one defines vertical edge intervals and maximal vertical edge intervals of $\mathcal{P}$.

A polyomino $\mathcal{P}$ is called *row convex* if any two cells $C = [(i, j), (i + 1, j + 1)]$, $D = [(k, j), (k + 1, j + 1)]$ of $\mathcal{P}$ with $i < k$, all cells $[(\ell, j), (\ell + 1, j + 1)] \in \mathcal{P}$ for $i \leq \ell \leq k$. Similarly, one defines *column convex* polyominoes. A polyomino is called *convex* if it is both row and column convex.

Example 2.3.3 *In Figure 2.12, there are displayed two polyominoes. Both of them are row convex. The polyomino on the left, (a), is column convex, as well, but the one on the right, (b), is not.*

Let $a = (a_1, a_2)$ and $b = (b_1, b_2) \in V(\mathcal{P})$, we define on the vertices of $\mathcal{P}$ the following total orders:



(a) (b)

Figure 2.12: Convex and row convex polyominoes.

1. $a <^1 b$ if $a_1 < b_1$ or $a_1 = b_1$ and $a_2 < b_2$;
2. $a <^2 b$ if $b_1 < a_1$ or $a_1 = b_1$ and $a_2 < b_2$.

Let $\mathcal{P}$ be a polyomino, and let $\mathbb{K}$ be a field. We denote by S the polynomial ring over $\mathbb{K}$ with variables x_v, where $v \in V(\mathcal{P})$. The binomial $x_a x_b - x_c x_d \in S$ is called an *inner 2-minor* of $\mathcal{P}$ if $[a, b]$ is an inner interval of $\mathcal{P}$, where c, d are the anti-diagonal corners of $[a, b]$. We denote by $\mathcal{M}$ the set of all inner 2-minors of $\mathcal{P}$. The ideal $I_{\mathcal{P}} \subset S$ generated by $\mathcal{M}$ is called the *polyomino ideal* of $\mathcal{P}$. We also set $\mathbb{K}[\mathcal{P}] = S/I_{\mathcal{P}}$, and we call it call it the *coordinate ring* of $\mathcal{P}$.

2.3.1 Balanced and simple polyominoes

Among the polyominoes, the balanced polyominoes admit coordinate rings with many nice properties. Let $\mathcal{P}$ be a polyomino and let $\mathscr{I}$ be the set of all maximal vertical or horizontal edge intervals. An integer value function $\alpha : V(\mathcal{P}) \to \mathbb{Z}$ is called *admissible*, if for $I \in \mathscr{I}$ one has

$$\sum_{v \in I} \alpha(v) = 0.$$

Example 2.3.4 *In Figure 2.13, an admissible labeling of $\mathcal{P}$ is shown.*

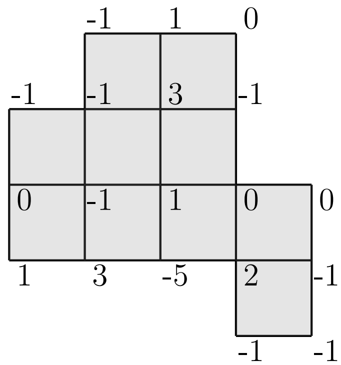


Figure 2.13: An admissible labeling.

Given an admissible labeling α, define the binomial

$$f_\alpha = \prod_{\substack{v \in V(\mathcal{P}) \\ \alpha(v) > 0}} x_v^{\alpha(v)} - \prod_{\substack{v \in V(\mathcal{P}) \\ \alpha(v) < 0}} x_v^{\alpha(v)}.$$

It is immediate to see that $I_\mathcal{P} \subset (f_\alpha \mid \alpha \text{ is an admissible labeling})$. A polyomino $\mathcal{P}$ is said *balanced* if, for any admissible labeling α, the binomial $f_\alpha \in I_\mathcal{P}$, that is $I_\mathcal{P} = (f_\alpha \mid \alpha \text{ is an admissible labeling})$.

Let $\mathcal{P} \subseteq [(1,1),(m,n)]$ be a polyomino. Let

$$\mathcal{B} = \{e_{ij} \ : i \in \{1,\dots,m\},\ j \in \{1,\dots,n\}\}$$

be the canonical basis of $\mathbb{Z}^{m \times n}$ and let $\mathcal{C} = \{C_1,\dots,C_r\}$, where $r = \operatorname{rank} \mathcal{P}$, be the set of all cells of $\mathcal{P}$. Define $c_k = e_{ij} + e_{i+1j+1} - e_{i+1j} - e_{ij+1}$, where (i,j) is the lower left corner of $C_k \in \mathcal{C}$.

Let Λ be the lattice spanned by $\{c_k\}_{k=1,\dots,r}$. It is known from [20] that an ideal generated by any set of adjacent 2-minors of a $m \times n$-matrix is a lattice ideal and that its corresponding lattice is saturated. Hence, the lattice Λ is a saturated lattice of rank equal to rank $\mathcal{P}$, and I_Λ is a prime ideal.

Proposition 2.3.5. *[34, Proposition 2.2, Corollary 2.3] Let $\mathcal{P}$ be a balanced polyomino. Then $I_\mathcal{P} = I_\Lambda$. In particular, $I_\mathcal{P}$ is a prime ideal of height* rank $\mathcal{P}$*.*

In [34], the primitive binomials of $I_\mathcal{P}$ are identified (see [34, Theorem 3.1]). From this result, the authors deduce that for a balanced polyomino $\mathcal{P}$ the ideal $I_\mathcal{P}$ has a squarefree initial ideal for any monomial order. Therefore, the residue class of $I_{\mathcal{P}}$ is a normal Cohen-Macaulay domain.

In [34], it was conjectured that a polyomino is balanced if and only if it is simple. In [33], the conjecture is positively solved. The main consequence of this result is to state that all simple polyominoes have prime polyomino ideal.

Theorem 2.3.6. *[33, Theorem 2.1] A polyomino is simple if and only if it is balanced.*

Corollary 2.3.7. *[33, Corollary 2.2] Let $\mathcal{P}$ be a simple polyomino. Then $K[\mathcal{P}]$ is a Cohen–Macaulay normal domain.*

2.3.2 Multiply connected polyominoes and prime ideals

In [37], a class of prime multiply connected polyominoes is presented. It is shown that the polyomino ideal of the polyomino which is obtained by removing a convex polyomino from its ambient rectangle is prime. This is proved by using a localization argument. In [64], the author gives a toric representation of the quotient rings of the polyomino ideals of this class of multiply connected polyominoes. In this section, we recall definitions and result, mostly following [64].

Let $\mathcal{I} \subseteq \mathbb{N}^2$ be an interval of $\mathbb{N}^2$ and $\mathcal{Q}$ a convex polyomino which is a subpolyomino of $\mathcal{P}_{\mathcal{I}}$. Let $\mathcal{P} = \mathcal{P}_{\mathcal{I}} \setminus \mathcal{Q}$ and suppose that $\mathcal{P}$ is a polyomino. Moreover, we may assume that $\mathcal{P}$ is not simple, indeed, when $\mathcal{P}$ is a simple polyomino, its toric representation is well studied in [57]. Let $e = (i_e, j_e)$ be the left lower corner of $\mathcal{Q}$.

Define the following subset of $V(\mathcal{P})$

$$\mathcal{F} = \{(i,j) \in V(\mathcal{P}) \mid i \leq i_e \text{ and } j \leq j_e\}.$$

Let $\{V_i\}_{i\in I}$ be the set of all the maximal vertical edge intervals of $\mathcal{P}$, and $\{H_j\}_{j\in J}$ be the set of all the maximal horizontal edge intervals of $\mathcal{P}$. Let $\{v_i\}_{i\in I}, \{h_j\}_{j\in J}$, and $\{w\}$ be the three sets of variables associated to $\{V_i\}_{i\in I}, \{H_j\}_{j\in J}$, and $\mathcal{F}$, respectively. We consider the map

$$\alpha : V(\mathcal{P}) \longrightarrow \mathbb{K}[\{h_i, v_j, w\} \mid i \in I, j \in J]$$
$$a \longmapsto \prod_{a\in H_i\cap V_j} h_i v_j \prod_{a\in\mathcal{F}} w.$$

The *toric ring* $T_{\mathcal{P}}$ associated to $\mathcal{P}$ is defined as

$$T_{\mathcal{P}} = \mathbb{K}[\alpha(a) | a \in V(\mathcal{P})] \subset \mathbb{K}[\{h_i, v_j, w\} \mid i \in I, j \in J].$$

The homomorphism

$$\varphi : S \longrightarrow T_{\mathcal{P}}$$
$$x_a \longmapsto \alpha(a)$$

is surjective and the *toric ideal* $J_{\mathcal{P}}$ is the kernel of φ.

In Section 4.1, we will generalize the construction of $T_{\mathcal{P}}$ for any multiply connected polyomino $\mathcal{P}$, proving that the toric ideal $J_{\mathcal{P}}$ contains the polyomino ideal $I_{\mathcal{P}}$ (see Proposition 4.1.1).

Theorem 2.3.8. *[37, Theorem 2.1], [64, Theorem 2.3] Let $\mathcal{I} \subseteq \mathbb{N}^2$ be an interval of $\mathbb{N}^2$ and $\mathcal{Q}$ a convex polyomino which is a subpolyomino of $\mathcal{P}_{\mathcal{I}}$. Let $\mathcal{P} = \mathcal{P}_{\mathcal{I}} \setminus \mathcal{Q}$ and suppose that $\mathcal{P}$ is a polyomino. Then $I_{\mathcal{P}} = J_{\mathcal{P}}$. In particular, the polyomino ideal $I_{\mathcal{P}}$ is a prime ideal.*

Chapter 3

ALGEBRAIC INVARIANTS OF SOME CLASSES OF BINOMIAL EDGE IDEALS

This chapter is devoted to collect all the original results of this thesis regarding binomial edge ideals. In particular, first we focus on binomial edge ideals of block graphs. We give a lower bound for the Castelnuovo-Mumford regularity of binomial edge ideals of block graphs, and we present a linear time algorithm to compute the Castelnuovo-Mumford regularity and Krull dimension of binomial edge ideals of block graphs. Second, we study some classes of Cohen-Macaulay binomial edge ideals, investigating the extremal Betti numbers, regularity and CM-type.

All the results here discussed can be found in [48] and [49].

3.1 BINOMIAL EDGE IDEALS OF BLOCK GRAPHS

In this section, we present some algebraic properties of binomial edge ideals of a class of graphs: block graphs. We present an algorithm (Theorem 3.1.7), that is linear in time and space, to compute the Krull dimension. The idea is to find a minimal prime ideal of minimum height since it induces the Krull dimension of S/J_G. We have implemented the algorithm using CoCoA ([9]), when G is a tree and it is downloadable on [47].

It is still an open problem to determine an explicit formula for the regularity of binomial edge ideals for block graphs in terms of the combinatorics of the graph. Inspired by the results in [35], we define a new class of graphs, namely the flower graphs (see Definition 3.1.11 and Figure 3.2), for which we compute the superextremal Betti numbers (see Theorem 3.1.13) and the regularity (see Corollary 3.1.14). As a consequence we obtain new lower bounds in Theorem 3.1.15 and Corollary 3.1.17 for the regularity of any block graph.

Finally, we state our main result on the regularity, Theorem 3.1.20, that provides an efficient method to compute the Castelnuovo-Mumford regularity of any binomial edge ideal of block graphs, exploiting the notion of end-flowers (see Definition 3.1.19) and by means of an unique block graph traversal.

3.1.1 On the height of minimal prime ideals of J_G and decomposability of block graphs

We denote by $\mathcal{M}(G)$ the minimal prime ideals of J_G, by $\mathcal{M}\text{inh}(G) \subseteq \mathcal{M}(G)$ the minimal prime ideals $P_T(G)$ of minimum height and by $\mathcal{M}\text{axh}(G) \subseteq \mathcal{M}(G)$ the minimal prime ideals $P_T(G)$ of maximum height.

The next proposition collects the results showed in [35] and [58] concerning Krull dimension of S/J_G and height of the ideals $P_T(G)$, when G is decomposable.

Proposition 3.1.1. *Let G be a graph decomposable into G_1 and G_2, with $V(G_1) \cap V(G_2) = \{v\}$. Then*

1. $\dim S/J_G = \dim S_1/J_{G_1} + \dim S_2/J_{G_2} - 2$, *where* $S_i = \mathbb{K}[x_j, y_j]_{j \in V(G_i)}$ *for* $i = 1, 2$;
2. $\text{height } P_T(G) = \text{height } P_{T_1}(G_1) + \text{height } P_{T_2}(G_2)$, *with* $T \in \mathcal{C}(G)$, $T_1 \in \mathcal{C}(G_1)$, *and* $T_2 \in \mathcal{C}(G_2)$ *and either* $T = T_1 \cup T_2$ *or* $T = T_1 \cup T_2 \cup \{v\}$;

For a block graph G, being decomposable can be read from the primary decomposition of J_G, in particular from the ideals in $\mathcal{M}\text{axh}(G)$.

Proposition 3.1.2. *Let G be a block graph. The following are equivalent:*

1. *G is indecomposable;*
2. *if $v \in V(G)$, then* $\text{cdeg}(v) \neq 2$;
3. $\mathcal{M}\text{axh}(G) = \{P_\emptyset(G)\}$.

Proof.

(1) $\Leftrightarrow$ (2) It is trivial.

(2) $\Rightarrow$ (3) Without loss of generality, let G be connected. Since height $P_\emptyset(G) = n - 1$, we want to prove that for any $T \neq \emptyset$, height $P_T(G) < n - 1$. Let $T \in \mathcal{C}(G)$, with height $P_T(G) \geq n - 1$, that is $c(T) - |T| \leq 1$. If $T = \{v\}$, then $c(T) \leq 2$ or equivalently $\text{cdeg}(v) \leq 2$. Since v is a cutpoint, it is not a free vertex, and then $\text{cdeg}(v) = 2$, which is in contradiction to the hypothesis. Let $T = \{v_1, \ldots, v_r\}$, with $r \geq 2$, such that height $P_T(G) \geq n - 1$ and suppose it is minimal with respect to this property. In a block graph, $T_1 = T \setminus \{v_r\}$ is a cutset, too. By definition, $c(T_1) < c(T)$ and $|T_1| = |T| - 1$, then $c(T_1) - |T_1| < 2$. It follows that height $P_{T_1}(G) \geq n - 1$, but it is in contradiction to the hypothesis on the minimality of T.

(3) $\Rightarrow$ (2) Assume that there exists a vertex $v \in V(G)$ such that $\operatorname{cdeg}(v) = 2$. Let $T = \{v\}$, then height $P_T(G) =$ height $P_\emptyset(G) = n - 1$. Hence, $P_T(G) \in \mathcal{M}\mathrm{axh}(G)$, too. The latter is in contradiction to the hypothesis.

□

We observe that for a generic graph G, is not true that if G is indecomposable then $\operatorname{cdeg}(v) \neq 2$ for any $v \in V(G)$. It is sufficient to consider $G = C_4$, with $V(G) = \{1, \ldots, 4\}$ and $E(G) = \{\{i, i+1\} | i = 1, \ldots, 3\} \cup \{1, 4\}$. All its vertices have clique degree equal to 2, but G is indecomposable. Moreover, for a generic graph G being indecomposable is not equivalent to the fact that $P_\emptyset(G)$ is the prime ideal of the maximum height in the primary decomposition of J_G. In fact, consider again $G = C_4$. The subset $T = \{1, 3\}$ is a cutset for G and height $P_T(G) = 4$, whereas height $P_\emptyset(G) = 3$.

3.1.2 Krull dimension of binomial edge ideals of block graphs

For some classes of graphs, there exists an immediate way to compute the Krull dimension. For example, if G is a complete graph or a graph obtained by gluing free vertices of complete graphs and such that any vertex $v \in V(G)$ is either a free vertex or has $\operatorname{cdeg}(v) = 2$, then $\dim S/J_G = n + 1$. For a generic block graph G, we show an algorithm to compute the Krull dimension of S/J_G in linear time.

From now on, we consider only connected block graphs, since the Krull dimension of S/J_G, where G is a graph with c connected components, $G_1, \ldots, G_c$, is given by the sum of the Krull dimension of S_i/J_{G_i}, with $i = 1, \ldots, c$ and $S_i = \mathbb{K}[\{x_j, y_j\}_{j \in V(G_i)}]$. Before showing the aforementioned algorithm, we need some auxiliary results.

Lemma 3.1.3. *Let G be a block graph, $P_T(G) \in \mathcal{M}\mathrm{inh}(G)$, and $v \in V(G)$. If v belongs to*

1. *exactly two endblocks, then $P_{T \cup \{v\}}(G) \in \mathcal{M}\mathrm{inh}(G)$;*
2. *at least three endblocks, then $v \in T$.*

Proof. Let $P_T(G) \in \mathcal{M}\mathrm{inh}(G)$ and let v belong to r endblocks, $B_1, \ldots, B_r$, with $r \geq 2$, and let $G_1, \ldots, G_c$ be the connected components of $G_{[n] \setminus T}$, then height$P_T(G) = n - c + |T|$. Suppose that $v \notin T$. Without loss of generality, we can suppose $v \in G_1$. The connected components induced by $T \cup \{v\}$ are $B'_1, \ldots, B'_r, G'_1, G_2, \ldots, G_c$, where $B'_i = B_i \setminus \{v\}$ for

$i = 1, \ldots, r$ and $G_1' = G_1 \setminus \{B_1, \ldots, B_r\}$. If $r = 2$ and $G_1' = \emptyset$, then height$P_{T\cup\{v\}}(G) =$ height$P_T(G)$, and then also $P_{T\cup\{v\}}(G) \in \mathcal{M}\text{inh}(G)$. If $r \geq 3$ or $r = 2$ and $G_1' \neq \emptyset$, the number of connected components induced by $T \cup \{v\}$ is at least $r + c$ and hence it is greater than or equal to $c + 2$. Thus, height$P_{T\cup\{v\}}(G) \leq n - (c+2) + (|T|+1) <$ height$P_T(G)$, which is in contradiction to the minimality of $P_T(G)$. $\square$

Remark 3.1.4 *Let G be a block graph and $T \in \mathcal{C}(G)$ such that $P_T(G) \in \mathcal{M}\text{inh}(G)$. If $\{v_1, \ldots, v_r\} \subseteq T$ is the set of all the vertices in T with clique degree equal to 2, by Proposition 3.1.1.(2), $P_{T\setminus\{v_1,\ldots,v_r\}}(G) \in \mathcal{M}\text{inh}(G)$.*

Lemma 3.1.5. *Let G be a block graph and $v \in V(G)$ be a cutpoint. If*

1. *v belongs to at least 2 endblocks of an indecomposable component of G,*
2. *$P_{T'}(H) \in \mathcal{M}\text{inh}(H)$, where $T' \in \mathcal{C}(H)$ and H is the graph obtained from G by removing v and the endblocks to which v belongs*

then $P_{T'\cup\{v\}}(G) \in \mathcal{M}\text{inh}(G)$.

Proof. Let $T \in \mathcal{C}(G)$ be such that $P_T(G) \in \mathcal{M}\text{inh}(G)$ and $v \in T$. By Lemma 3.1.3, we know that such T exists. Let $T = T_1 \cup \{v\}$. Let $r \geq 2$ be the number of endblocks to which v belongs, then $c(T) = r + c(T_1)$, where $c(T_1)$ denotes the number of connected components of H induced by T_1. It follows that

$$\begin{aligned}\text{height } P_T(G) &= n - (r + c(T_1)) + (1 + |T_1|)\\ &= n - V(H) - r + 1 + [V(H) - c(T_1) + |T_1|]\\ &= s + \text{height } P_{T_1}(H)\end{aligned}$$

where $s = n - V(H) - r + 1$. Observe that $P_{T_1}(H) \in \mathcal{M}\text{inh}(H)$: if there exists $T_2 \in \mathcal{C}(G)$ such that height $P_{T_2}(H) <$ height $P_{T_1}(H)$, then height $P_{T_2\cup\{v\}}(G)$ is lower than height $P_T(G)$, and this is in contradiction to the minimality of $P_T(G)$. Since, by hypothesis, $P_{T'}(H), P_{T_1}(H)$ have the same height, it follows height $P_T(G) = s +$ height $P_{T'}(H) =$ height $P_{T'\cup\{v\}}(G)$, and $P_{T'\cup\{v\}}(G) \in \mathcal{M}\text{inh}(G)$. $\square$

The following result is the core of the algorithm that allows to compute the Krull dimension of S/J_G.

Theorem 3.1.6. *Let G be a block graph and $T = \{v_1, \ldots, v_t\} \in \mathcal{C}(G)$. We denote by H_0 the graph G and by H_i the graph obtained from H_{i-1} by removing v_i and the endblocks to which v_i belongs, for all $i = 1, \ldots, t$. If*

1. *v_i belongs to at least 2 endblocks of an indecomposable component of H_{i-1}, for all $i = 1, \ldots, t$,*
2. *H_t is decomposable into blocks,*

then $P_T(G) \in \mathcal{M}\text{inh}(G)$.

Proof. We use induction on t. Let $t = 1$. Consider $T = \{v_1\} \in \mathcal{C}(G)$, with $v_1 \in V(G)$ that belongs to at least 2 endblocks of an indecomposable component of G and H_1 is decomposable into blocks. By Lemma 3.1.3 and Remark 3.1.4, there exists a cutset T' that contains v_1 and no vertices of clique degree equal to 2 such that $P_{T'}(G) \in \mathcal{M}\text{inh}(G)$. Since H_1 is decomposable into blocks, all the non-free vertices of H_1 have clique degree equal to 2, then $T' = \{v_1\} = T$ and $P_T(G) \in \mathcal{M}\text{inh}(G)$.
Let $t > 1$. Consider $T = \{v_1, \ldots, v_t\} \in \mathcal{C}(G)$. The vertex v_1 belongs to at least 2 endblocks of an indecomposable component of G and, by induction hypothesis, $P_{T'}(H_1) \in \mathcal{M}\text{inh}(H_1)$, where $T' = \{v_2, \ldots, v_t\}$. By Lemma 3.1.5, $P_{T' \cup \{v_1\}}(G) \in \mathcal{M}\text{inh}(G)$. □

Theorem 3.1.7 (Algorithm: Krull Dimension of binomial edge ideals of block graphs)**.**

- *Input: A connected block graph G over $[n]$.*
- *Output: Krull dimension of S/J_G.*

1. $\dim := n + 1$;
2. $\mathcal{G} := \{G\}$;
3. *for every graph* $H \in \mathcal{G}$
4. $\quad \mathcal{G} := \mathcal{G} \setminus \{H\}$;
5. $\quad$ *decompose H into its indecomposable subgraphs $\mathcal{I} = \{G_1, \ldots, G_r\}$;*
6. $\quad$ *remove from $\mathcal{I}$ the graphs which are blocks;*
7. $\quad$ *for every graph $G_i \in \mathcal{I}$*
8. $\quad\quad$ *take $v \in V(G_i)$ such that v belongs to at least 2 endblocks;*
9. $\quad\quad \dim := \dim + \text{cdeg}(v) - 2$;
10. $\quad\quad \mathcal{G} := \mathcal{G} \cup \{H_v\}$;

where H_v denotes the graph obtained from G_i by removing v and the endblocks to which v belongs.

Proof. The aim of the algorithm is to compute the Krull dimension by finding a cutset T such that $P_T(G) \in \mathcal{M}\text{inh}(G)$. In particular, after a finite number of steps we obtain a cutset $T = \{v_1, \dots, v_t\}$ that fulfils the hypothesis of Theorem 3.1.6, and then $P_T(G) \in \mathcal{M}\text{inh}(G)$. Now we explain in detail the algorithm.

Line 1. We set $\dim = n+1$. This is the case when the graph G is a block or is decomposable into blocks, that is $T = \emptyset$.

Line 2. We denote by $\mathcal{G}$ the set of graphs that are to consider still.

Lines 3-4. We consider each graph $H \in \mathcal{G}$. The algorithm finishes when $\mathcal{G}$ is empty.

Lines 5-6. We decompose H into its indecomposable components $G_1, \dots, G_r$. These subgraphs are the elements of the set $\mathcal{I}$. This is equivalent to do away with the vertices of clique degree 2 (see Remark 3.1.4). Now, by a branch and bound strategy we study each indecomposable subgraphs of H. We discard the blocks since their vertices are free vertices and then they do not belong to T.

Lines 7-8. For every subgraph $G_i \in \mathcal{I}$, since G_i is indecomposable there exists a vertex v that belongs to at least 2 endblocks. By Lemma 3.1.3, we assume $v \in T$.

Line 9. We update the Krull dimension: the number of connected components induced by v in G_i is exactly its clique degree. One of these components has been already considered, when we set $\dim = n+1$ in the Line 1. Therefore, the contribute of v is equal to $\text{cdeg}(v) - 1$ less the cardinality of the cutset, which is 1.

Line 10. We remove from G_i the vertex v and the endblocks which contain v, and we add this new graph H_v in $\mathcal{G}$, the set of graphs to consider still.

The wanted T consists of all the vertices v considered in Line 8. Observe that, by construction, any $v \in T$ satisfies the condition (1) of Theorem 3.1.6, and the condition (2) holds at the end of the algorithm, when $\mathcal{G} = \emptyset$. Moreover, the algorithm finishes after a finite number of steps: in Line 4, we remove a graph H from $\mathcal{G}$ but we add some new graphs H_v in $\mathcal{G}$ in Line 10. For any of these H_v, it holds $|V(H_v)| < |V(H)|$, hence after a finite number of iterations the new graphs in Line 10 will be either blocks, and then they will be discarded in Line 6, or empty graphs. □

We highlight that the above algorithm works also for disconnected graphs: it is sufficient to set dim $:= n + c$ in Line 1, where c is the number of connected components of G.

We are going to show that the Krull dimension of S/J_G can be computed with a unique visit of G by a recursive function, named IsInT(v, G, c_T). The cost of traversing a graph G is $\mathcal{O}(|V(G)| + |E(G)|)$ (see [11, Section 22]). This implies that the algorithm presented in Theorem 3.1.7 can be implemented through a procedure which is linear with respect to the number of vertices and edges of G, without any decomposition. The function IsInT(v, G, c_T) constructs a $T \in \mathcal{C}(G)$ that fulfils the conditions (1) and (2) of Theorem 3.1.6, and then $P_T(G) \in \mathcal{M}\text{inh}(G)$. For the sake of simplicity, in the following let G be a tree.

```
IsInT(v, G, c_T)
if v is a leaf then
    return 0
else
    degree := cdeg(v);
    childrenInT := 0;
    for every children w of v
        childrenInT := childrenInT + IsInT(w, G, c_T);
    degree := degree - chidrenInT;
    if degree > 2 then
        c_T := c_T + degree - 2;
        return 1
    else
        return 0
```

Even if the algorithm works for any undirected tree, we assign an orientation given by the visit of the tree itself: the children of a given vertex are its adjacent vertices that have not been visited yet. The purpose of IsInT(v, G, c_T) is twofold: on one side, starting from any vertex $v \in V(G)$, it checks if v belongs to T and in this case it returns 1, otherwise 0, on the other side it computes $c(T) - |T|$, which is saved in c_T. For a vertex v being in T depends on its children that are in T, and on its degree. The latter is given by the initial degree less the number of children of v that are in T (Line 9). In particular, $v \in T$ if at least 2 of its children are not in T and its degree is greater than 2 (Line 10).

To compute the Krull dimension of S/J_G, it is sufficient to call the function IsInT(v, G, c_T), where v is any vertex of G and c_T is a global variable set to 1, and then $\dim S/J_G = n + c_T$.

We have implemented this procedure for trees using CoCoA version 4.7 and it is freely downloadable on [47].

3.1.3 Regularity bounds for binomial edge ideals of block graphs

The main result of this section is the lower bound for the Castelnuovo-Mumford regularity of binomial edge ideals of block graphs (Theorem 3.1.15).

In [35], the authors compute one of the distinguished Betti numbers of the binomial edge ideal of a block graph, and classify all block graphs admitting precisely one extremal Betti number.

Let G be a graph. We denote by $i(G)$ the number of inner vertices of G and by $f(G)$ the number of free vertices of G.

Theorem 3.1.8. *[35, Theorem 6] Let G be an indecomposable block graph on $[n]$. Furthermore, let $<$ be the lexicographic order induced by $x_1 > x_2 > \cdots > x_n > y_1 > y_2 > \cdots > y_n$. Then $\beta_{n-1,n-1+i(G)+1}(S/J_G)$ and $\beta_{n-1,n-1+i(G)+1}(S/\mathrm{in}_<(J_G))$ are extremal Betti numbers of S/J_G and $S/\mathrm{in}_<(J_G)$, respectively. Moreover,*

$$\beta_{n-1,n-1+i(G)+1}(S/\mathrm{in}_<(J_G)) = \beta_{n-1,n-1+i(G)+1}(S/J_G) = f(G) - 1.$$

Corollary 3.1.9. *[35, Corollary 7] Let G be a block graph for which $G = G_1 \cup \cdots \cup G_s$ is the decomposition of G into indecomposable graphs. Then each G_i is a block graph, $\beta_{n-1,n-1+i(G)+s}(S/J_G)$ is an extremal Betti number, and*

$$\beta_{n-1,n-1+i(G)+s}(S/J_G) = \prod_{i=1}^{s}(f(G_i) - 1).$$

The following theorem classifies all block graphs which admit precisely one extremal Betti number.

Theorem 3.1.10. *[35, Theorem 8] Let G be an indecomposable block graph. Then*

1. reg $S/J_G \geq i(G) + 1$;
2. *the following are equivalent:*
   (a) *S/J_G admits precisely one extremal Betti number;*
   (b) *G does not contain one of the induced subgraphs T_0, T_1, T_2, T_3 of Figure 3.1;*
   (c) *let $P = \{v \in V(G) \mid \deg(v) \neq 1\}$. Then, each cutpoint of $G_{|P}$ belongs to exactly two maximal cliques.*

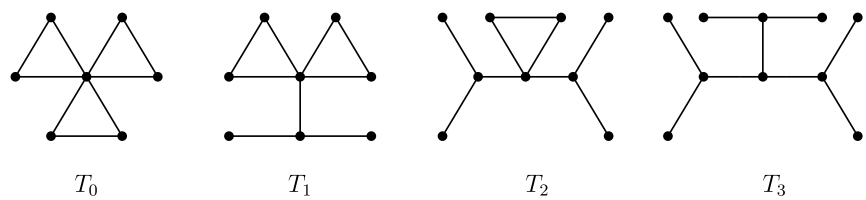


Figure 3.1

Inspired by Theorem 3.1.10, we introduce a new class of block graph, called *flower graphs*, and we compute their regularity and superextremal Betti numbers.

Definition 3.1.11 *A flower graph $F_{h,k}(v)$ is a connected block graph constructed by joining h copies of the cycle graph C_3 and k copies of the bipartite graph $K_{1,3}$ with a common vertex v, where v is one of the free vertices of C_3 and of $K_{1,3}$, and* $\text{cdeg}(v) \geq 3$.

We observe that any flower graph $F_{h,k}(v)$ has $2h+3k+1$ vertices and $3(h+k)$ edges. The clique degree of v is given by $h + k$, and the number of inner vertices is $i(F_{h,k}(v)) = k + 1$ and all of them are cutpoints for $F_{h,k}(v)$. When it is unnecessary to make explicit the parameters h and k, we refer to $F_{h,k}(v)$ as $F(v)$. Moreover, by applying the algorithm in Theorem 3.1.7, one gets a precise formula for the Krull dimension, when G is a flower graph, that is $\dim S/J_G = n + \text{cdeg}(v) - 1$.

Before stating the distinguished extremal Betti numbers of the binomial edge ideal of a flower graph, we need the following remark.

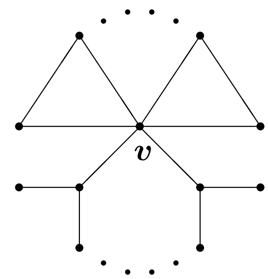


Figure 3.2: A flower graph $F_{h,k}(v)$

Remark 3.1.12 *Let G be a disconnected block graph with $G_1, \dots, G_r$ its connected components. If all the G_j have precisely one extremal Betti number, $\beta_{n_j-1,n_j+i(G_j)}(S_j/J_{G_j})$, for any $j = 1, \dots, r$, with $S_j = \mathbb{K}[x_i, y_i]_{i \in V(G_j)}$ and $n_j = |V(G_j)|$, then S/J_G has precisely one extremal Betti number and it is given by*

$$\beta_{n-r,n+i(G)}(S/J_G) = \prod_{j=1}^{r} \beta_{n_j-1,n_j+i(G_j)}(S_j/J_{G_j}).$$

Theorem 3.1.13. *Let G be a flower graph $F(v)$. The following are extremal Betti numbers of S/J_G:*

1. $\beta_{n-1,n+i(G)}(S/J_G) = f(G) - 1$;
2. $\beta_{n-\mathrm{cdeg}(v)+1,n+i(G)}(S/J_G) = 1$.

In particular, they are the only nonzero superextremal Betti numbers.

Proof. The fact (1) is an immediate consequence of Theorem 3.1.8. As regards (2), we focus on the cutpoint v of G. Consider the short exact sequence (2.4), with $u = v$, where G', G'', and H are as in Set-up 2.2.13. We observe that G' and H are block graphs satisfying equivalent conditions in Theorem 3.1.10, with $i(G') = i(H) = i(G) - 1$, and then $\mathrm{reg}\, S/J_{G'} = \mathrm{reg}\, S/((x_v, y_v) + J_H) = i(G)$. The graph G'' has $\mathrm{cdeg}(v)$ connected components $G_1, \dots, G_{\mathrm{cdeg}(v)}$: all of them are either K_2 or paths of length 2, namely P_2. The latter are decomposable into two K_2 and it holds $\mathrm{reg}\, S'/J_{P_2} = 2 = i(P_2) + 1$, with $S' = \mathbb{K}[x_i, y_i]_{i \in V(P_2)}$. Then, by Theorem 3.1.10 and since the ring $S/((x_v, y_v) + J_{G''})$ is the tensor product of

S_j/J_{G_j}, with $j = 1, \dots, \operatorname{cdeg}(v)$ and $S_j = \mathbb{K}[x_i, y_i]_{i \in V(G_j)}$, we have

$$\operatorname{reg} \frac{S}{(x_v, y_v) + J_{G''}} = \sum_{j=1}^{\operatorname{cdeg}(v)} \operatorname{reg} \frac{S_j}{J_{G_j}} = \sum_{j=1}^{\operatorname{cdeg}(v)} (i(G_j) + 1) = i(G) - 1 + \operatorname{cdeg}(v).$$

We get the following bound on the regularity of S/J_G

$$\begin{aligned} \operatorname{reg} S/J_G &\leq \max\{\operatorname{reg} \frac{S}{J_{G'}}, \operatorname{reg} \frac{S}{(x_v, y_v) + J_{G''}}, \operatorname{reg} \frac{S}{(x_v, y_v) + J_H} + 1\} \\ &= \max\{i(G), i(G) - 1 + \operatorname{cdeg}(v), i(G) + 1\} \\ &= \ i(G) - 1 + \operatorname{cdeg}(v). \end{aligned}$$

By Theorem 2.2.15, the depth of S/J_G for any block graph G over $[n]$ is equal to $n + c$, where c is the number of connected components of G. Since we know the depth of all quotient rings involved in (2.4) and by Auslander-Buchsbaum formula, we get $\operatorname{proj\,dim} S/J_G = \operatorname{proj\,dim} S/J_{G'} = \operatorname{proj\,dim} S/((x_v, y_v) + J_H) - 1 = n - 1$, and $\operatorname{proj\,dim} S/((x_v, y_v) + J_{G''}) = n - \operatorname{cdeg}(v) + 1$.

Let $j > i(G)$, then

$$T_{m,m+j}(S/J_{G'}) = T_{m,m+j}(S/((x_v, y_v) + J_H)) = 0 \qquad \text{for any } m,$$

and

$$T_{m,m+j}(S/((x_v, y_v) + J_{G''})) = 0 \qquad \text{for any } m > n - \operatorname{cdeg}(v) + 1,$$

Of course, all the above Tor modules $T_{m,m+j}(-)$ are zero when $j > i(G) - 1 + \operatorname{cdeg}(v)$.

Therefore, for $m = n - \operatorname{cdeg}(v) + 1$ and $j = i(G) - 1 + \operatorname{cdeg}(v)$ we obtain the following long exact sequence

$$\begin{aligned} \cdots \ &\to T_{m+1,m+1+(j-1)}(S/((x_v, y_v) + J_H)) \to T_{m,m+j}(S/J_G) \to \\ &T_{m,m+j}(S/J_{G'}) \oplus T_{m,m+j}(S/((x_v, y_v) + J_{G''})) \to \\ &T_{m,m+j}(S/((x_v, y_v) + J_H)) \to \cdots \end{aligned}$$

In view of the above, all the functors on the left of $T_{m,m+j}(S/J_G)$ in the long exact sequence are zero, and $T_{m,m+j}(S/J_{G'}) = T_{m,m+j}(S/((x_v, y_v) + J_H)) = 0$ too. It follows

$$T_{m,m+j}(S/J_G) \cong T_{m,m+j}(S/((x_v, y_v) + J_{G''})).$$

It means that

$$\beta_{n-\mathrm{cdeg}(v)+1,n+i(G)}(S/J_G) = \beta_{n-\mathrm{cdeg}(v)+1,n+i(G)}(S/((x_v, y_v) + J_{G''})).$$

We observe that

$$T^S_{m,m+j}(S/((x_v, y_v) + J_{G''})) \cong T^{S''}_{m-2,m-2+j}(S''/J_{G''})$$

where $S'' = S/(x_v, y_v)$. Since all the connected components $G_1, \dots, G_{\mathrm{cdeg}(v)}$ of G'' are either a K_2 or a path of length 2, the quotient rings S_j/J_{G_j} have an unique extremal Betti number $\beta_{n_j-1,n_j+i(G_j)}(S_j/J_{G_j})$, for $j = 1, \dots, \mathrm{cdeg}(v)$ and $n_j = |V(G_j)|$, which is equal to 1. Therefore, by Remark 3.1.12, we have

$$\beta_{m-2,m-2+j}(S''/J_{G''}) = \prod_{j=1}^{\mathrm{cdeg}(v)} \beta_{n_j-1,n_j+i(G_j)}(S_j/J_{G_j}) = 1.$$

Observe that for $m = n-\mathrm{cdeg}(v)+1$ and $j = i(G)-1+\mathrm{cdeg}(v)$ we get that $m+j = n+i(G)$ is the maximal integer such that $\beta_{i,m+j}(S/J_G) \neq 0$ for some i. We want to prove that $\beta_{i,n+i(G)} \neq 0$, only for $i = n - \mathrm{cdeg}(v) + 1$ and $i = n - 1$. Let i be an integer such that $\beta_{i,n+i(G)} \neq 0$. Since $\operatorname{proj\,dim} S/J_G = n - 1$ and $\operatorname{reg} S/J_G \leq i(G) + \mathrm{cdeg}(v) - 1$, we have to examine $n - \mathrm{cdeg}(v) + 1 \leq i \leq n - 1$. Consider the following long exact sequence

$$\begin{aligned} \cdots \quad &\to T_{i+1,n+i(G)}\left(\frac{S}{(x_v, y_v) + J_H}\right) \to T_{i,n+i(G)}\left(\frac{S}{J_G}\right) \to \\ &T_{i,n+i(G)}\left(\frac{S}{J_{G'}}\right) \oplus T_{i,n+i(G)}\left(\frac{S}{(x_v, y_v) + J_{G''}}\right) \to \\ &T_{i,n+i(G)}\left(\frac{S}{(x_v, y_v) + J_H}\right) \to \cdots \end{aligned}$$

If $n - \mathrm{cdeg}(v) + 1 < i < n - 1$, since $i > \operatorname{proj\,dim} S/((x_v, y_v) + J_{G''})$ and $n + i(G) - i > \operatorname{reg} S/J_{G'}, \operatorname{reg} S/((x_v, y_v) + J_H)$, it holds $\mathrm{Tor}_{i,n+i(G)}(M) = 0$, for $M \in \{S/J_{G'}, S/((x_v, y_v) + J_{G''}), S/((x_v, y_v) + J_H)\}$, and then we can conclude that also $\mathrm{Tor}_{i,n+i(G)}(S/J_G) = 0$. □

An immediate consequence of the proof of the Theorem 3.1.13 is the regularity of any flower graphs $F(v)$, that depends only on the clique degree of v and the number of inner vertices of $F(v)$.

Corollary 3.1.14. *Let $F(v)$ be a flower graph, then*

$$\text{reg } S/J_{F(v)} = i(F(v)) + \text{cdeg}(v) - 1.$$

If $F(v)$ is an induced subgraph of a block graph G, we denote by $\text{cdeg}_F(v)$ the clique degree of v in $F(v)$. Note that if $F(v)$ is the maximal flower induced subgraph of G and all the blocks of G containing v are C_3 or $K_{1,3}$, then $\text{cdeg}_F(v) = \text{cdeg}(v)$, otherwise $\text{cdeg}_F(v) < \text{cdeg}(v)$.

Theorem 3.1.15. *Let G be an indecomposable block graph and let $F(v)$ be an induced subgraph of G. Then*

$$\text{reg } S/J_G \geq i(G) + \text{cdeg}_F(v) - 1.$$

Proof. We use induction on the number of blocks of G that are not in $F(v)$. If $G = F(v)$, the statement follows from Corollary 3.1.14. Suppose now G contains properly $F(v)$ as induced subgraph. Since G is connected, there exists an endblock B of G and a subgraph G' of G such that $G = G' \cup B$, G' contains $F(v)$ as induced subgraph, $V(G') \cap V(B) = \{w\}$, and all the blocks containing w are endblocks, except for the one that is in G'. Since G is assumed to be indecomposable, $\text{cdeg}(w) \geq 3$. If $\text{cdeg}(w) = 3$, then G' is decomposable into $G_1 \cup G_2$, and $\text{reg } S/J_{G'} = \text{reg } S/J_{G_1} + \text{reg } S/J_{G_2}$. We may suppose that G_1 contains $F(v)$, and then $i(G_1) = i(G) - 1$, but $\text{cdeg}_F(v)$ is still the same. Whereas, G_2 is a block and $\text{reg } S/J_{G_2} = 1$. Then by using induction, we may assume that $\text{reg } S/J_{G_1} \geq i(G) + \text{cdeg}_F(v) - 2$. Therefore,

$$\text{reg } S/J_{G'} = \text{reg } S/J_{G_1} + \text{reg } S/J_{G_2} \geq i(G) + \text{cdeg}_F(v) - 1.$$

If $\text{cdeg}(w) > 3$, then $i(G') = i(G)$ and $\text{cdeg}_F(v)$ is still the same. Then, by using induction on the number of blocks of G, we may assume $\text{reg } S/J_{G'} \geq i(G) + \text{cdeg}_F(v) - 1$. By Corollary 2.2.18, one has that

$$\text{reg } S/J_G \geq \text{reg } S/J_{G'}.$$

and then $\text{reg } S/J_G \geq i(G) + \text{cdeg}_F(v) - 1$, as desired. □

Definition 3.1.16 *Let G be a block graph. If G has no flower graphs as induced subgraphs then G is called* flower-free.

We are ready to state the following bound for the regularity for any binomial edge ideal of block graphs.

Corollary 3.1.17. *Let G be a connected block graph which is not an isolated vertex.*

1. *If G is a flower-free graph, then* $\operatorname{reg} S/J_G = i(G) + 1$.
2. *If G contains $r \geq 1$ flower graphs $F_1(v_1), \ldots, F_r(v_r)$ as induced subgraphs, then* $\operatorname{reg} S/J_G \geq i(G) + \max_{i=1,\ldots,r}\{\operatorname{cdeg}_{F_i}(v_i)\} - 1$.

Proof. (1) If G is indecomposable, by Theorem 3.1.10, the result follows. Otherwise, suppose G is decomposable into indecomposable graphs $G_1, \ldots, G_r$. Observe that if v is an inner vertex in G then either $\{v\} = G_i \cap G_j$ for some $i \neq j$ and it is a free vertex in G_i and G_j, or it belongs to an unique G_i and it is an inner vertex of G_i. The former are exactly $r-1$. In fact, if we consider the graph T, with vertices $V(T) = \{G_1, \ldots, G_r\}$ and edges $E(T) = \{\{G_i, G_j\} : G_i \cap G_j \neq \emptyset\}$ we observe that T is a tree and $|E(T)| = r-1$. Hence

$$i(G) = r - 1 + \sum_{i=1}^{r} i(G_i).$$

By Proposition 3.1.1 and Theorem 3.1.10, we get

$$\operatorname{reg} S/J_G = \sum_{i=1}^{r} \operatorname{reg} S/J_{G_i} = \sum_{i=1}^{r} (i(G_i) + 1) = i(G) + 1.$$

(2) It is an immediate consequence of Corollary 2.2.18 and Theorem 3.1.15.

□

Example 3.1.18 *Let G be the graph in Figure 3.3. It contains 2 flower graphs as induced subgraphs: $F_{2,1}(v_1)$ and $F_{3,1}(v_2)$. By Corollary 3.1.17, we have* $\operatorname{reg} S/J_G \geq 2 + \max\{3, 4\} - 1 = 5$, *whereas the length of the longest induced path in G is 3 and the number of maximal cliques of G is 6. Also using the upper bound proved in [43], we get* $\operatorname{reg} S/J_G \leq 6$. *By means of a computation in CoCoA,* $\operatorname{reg} S/J_G = 5$, *it means the lower bound given in Corollary 3.1.17 is sharp. We observe that G is the graph with the minimum number of vertices such that S/J_G has 3 nonzero superextremal Betti numbers.*

Example 3.1.18 encourages us to follow up with an algorithm to compute the regularity of binomial edge ideal of block graphs, and it will be the content of the Section 3.1.4.

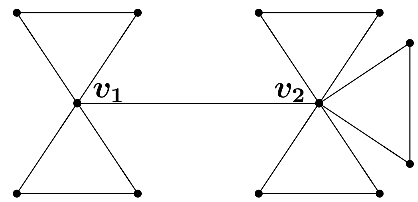


Figure 3.3: A graph G such that reg $S/J_G = i(G) + \max_{i=1,2}\{\mathrm{cdeg}_{F_i}(v_i)\} - 1$.

The bound exhibited in Corollary 3.1.17 can be improved for block graphs with several flowers $F_i(v_i)$ with the vertices v_i far enough from each other. In particular, let H be an induced subgraph of G and suppose H is decomposable into $H_1, \ldots, H_r$ such that any H_i contains a flower graph $F_i(v_i)$ as an induced subgraph for $i = 1, \ldots, r$. Then reg $J_G \geq$ reg $J_H = \sum_{i=1}^{r}$ reg J_{H_i}, which could be better than the one provided in Corollary 3.1.17.

3.1.4 How to compute the Castelnuovo-Mumford regularity of block graphs

In this section we provide an efficient method to compute the Castelnuovo-Mumford regularity for S/J_G when G is a block graph.

Definition 3.1.19 *Let G be a block graph and $F(v)$ be a flower graph that is an induced subgraph of G. $F(v)$ is called an* end-flower *of G if $G = G_1 \cup \ldots \cup G_c$, where $c = \mathrm{cdeg}(v)$, and such that $G_i \cap G_j = \{v\}$, for all $1 \leq i < j \leq c$, and $G_2, \ldots G_c$ are flower-free graphs.*

Theorem 3.1.20. *Let G be a block graph, $v_1, \ldots, v_r \in V(G)$,*

$$H_j = G \setminus \{v_1, \ldots, v_j\}$$

for $j = 1, \ldots, r$, and $H_0 = G$. If

1. *$F(v_j)$ is an end-flower for H_{j-1}, for all $j = 1, \ldots, r$,*
2. *H_r is flower-free,*

then

$$\mathrm{reg}\ S/J_G = \mathrm{reg}\ S/J_{H_r} = c + i(H_r)$$

where c is the number of connected components of H_r which are not isolated vertices.

Proof. First of all, observe that the equality reg $S/J_{H_r} = c + i(H_r)$ in the statement is an immediate consequence of Corollary 3.1.17 (1).

To prove reg $S/J_G =$ reg S/J_{H_r}, we make induction on

$$f = |\{v \in V(G) | F(v) \text{ is an induced subgraph of } G\}|.$$

If $f = 0$, that is G is a flower-free graph, then the assertion follows by Corollary 3.1.17 (1). Let $f = 1$ and v be such that $F(v)$ is an induced subgraph of G. Consider the exact sequence

$$0 \longrightarrow S/J_G \longrightarrow S/J_{G'} \oplus S/((x_v, y_v) + J_{G''}) \longrightarrow S/((x_v, y_v) + J_H) \longrightarrow 0 \qquad (3.1)$$

where G', G'', and H are as in Set-up 2.2.13, with $u = v$. We observe that G', G'', and H are flower-free. Hence

$$\text{reg } S/J_{G'} = \text{reg } S/J_H = i(G') + 1 = i(G) - 1 + 1 = i(G).$$

Moreover, removing the vertex v from G we obtain G'' and reg $S/J_{G''}$ is

$$\sum_{j=1}^{c} \text{reg } S/J_{G_j} = \sum_{j=1}^{c} (i(G_j) + 1) = \sum_{j=1}^{\text{cdeg}_F(v)} (i(G_j) + 1) + \sum_{k=1}^{c'} (i(G_k) + 1)$$

where $G_1, \dots, G_c$ are the connected components of G'', and $\{v, w_k\}$ are maximal cliques in G with w_k a free vertex of G_k, and $|V(G_k)| \geq 2$, for $k = 1, \dots, c'$. Observe that, for $j = 1, \dots, \text{cdeg}_F(v)$, all the inner vertices of G that belong to G_j are inner vertices also in G''. Whereas, for $k = 1, \dots, c'$, the w_k are inner vertices in G but not in G'', and all the other inner vertices of G that belong to G_k are inner vertices also in G''. Hence, removing v from G, we have $c' + 1$ less inner vertices in G'' with respect to G, that are all the w_k and v, but this is compensated by the formula $\sum_{k=1}^{c'} (i(G_k) + 1) = c' + \sum_{k=1}^{c'} i(G_k)$. Hence

$$\text{reg } S/J_{G''} = i(G) + \text{cdeg}_F(v) - 1.$$

Since $\text{cdeg}_F(v) \geq 3$,

$$\text{reg } S/J_{G'}, \text{reg } S/((x_v, y_v) + J_H) < \text{reg } S/((x_v, y_v) + J_{G''})$$

and then reg $S/J_G =$ reg $S/((x_v, y_v) + J_{G''})$.

Let $f > 1$. Let $v_1, \dots, v_r \in V(G)$ be a sequence that fulfills (1) and (2). Consider the exact sequence (3.1), with $v = v_1$. Observe that the sequence $v_2, \dots, v_r$ satisfies (1) and (2) for G', G'', and H and, since they have less than f flower graphs as induced subgraphs, by induction hypothesis their regularity is given by the sum of the regularity of the connected components induced by $v_2, \dots, v_r$.

Let $G_1, \dots, G_m$ be the connected components induced by $v_2, \dots, v_r$ in G. One of them contains v_1, suppose G_1, and then it is not flower-free, whereas the others are flower-free. The connected components induced by $v_2, \dots, v_r$ in G' and H are $G_1', G_2, \dots, G_m$ and $G_1' \setminus \{v_1\}, G_2, \dots, G_m$, respectively, where G_1' denotes the graph obtained from G_1 by connecting all the vertices adjacent to v_1. We get

$$\text{reg } S/J_{G'} = \text{reg } S/J_H = \text{reg } S/J_{G_1'} + \sum_{i=2}^{m} \text{reg } S/J_{G_i}.$$

Whereas, the connected components induced by $v_2, \dots, v_r$ in G'' are the connected components of $G_1 \setminus \{v_1\}$ and $G_2, \dots, G_m$, and then

$$\text{reg } S/J_{G''} = \text{reg } S/J_{G_1 \setminus \{v_1\}} + \sum_{i=2}^{m} \text{reg } S/J_{G_i}.$$

Since

$$\text{reg } S/J_{G_1'} = i(G_1) < i(G_1) + \text{cdeg}_F(v_1) - 1 = \text{reg } S/J_{G_1 \setminus \{v_1\}},$$

where the last equality follows from the same arguments of above and $\text{cdeg}_F(v_1)$ denotes the clique degree of v_1 in $F(v_1)$, with $F(v_1)$ seen as induced subgraph of G_1. Since $F(v_1)$ is an end-flower and $\text{cdeg}_F(v_1) \geq 3$ in G, it follows $\text{cdeg}_F(v_1) \geq 2$ in G_1. Observe that, when $\text{cdeg}_F(v_1) = 2$ in G_1, G_1 is flower-free and it is easy to see that the equality $\text{reg } S/J_{G_1 \setminus \{v_1\}} = i(G_1) + \text{cdeg}_F(v_1) - 1$ is still true. Then

$$\text{reg } S/J_{G'}, \text{reg } S/((x_v, y_v) + J_H) < \text{reg } S/((x_v, y_v) + J_{G''})$$

and the assertion is proved. □

The Theorem 3.1.20 suggests a recursive way to compute the regularity of S/J_G when G is a block graph.

1. ComputeRegularity(G)

```
if G is flower-free and is not an isolated vertex
    return i(G) + 1
else
    reg := 0;
    pinpoint an end-flower F(v) of G;
    remove v from G;
    for every connected component G_i induced by v in G
        reg := reg + ComputeRegularity(G_i);
    return reg
```

By means of an unique block graph traversal, that is linear with respect to the number of vertices and edges of G (see [11, Section 22]), one gets the regularity of S/J_G. This allows to compute the regularity of S/J_G also for those block graphs with a large number of vertices, and then for those binomial edge ideals with a large number of variables, for which the algebraic softwares, as CoCoA [9] and Macaulay2 [27], fail.

3.2 BETTI NUMBERS AND COHEN-MACAULAY TYPE OF SOME CLASSES OF COHEN-MACAULAY BINOMIAL EDGE IDEALS

In this section, we show the extremal Betti numbers for binomial edge ideals of some classes of Cohen-Macauly graphs: cones, bipartite and fan graphs. The former were introduced and investigated in [58]. As showed in Subsection 2.2.3, connecting all the vertices of two disjoint Cohen-Macaulay graphs to a new vertex, the resulting graph is Cohen-Macaulay. For these graphs, we compute the regularity and also the Cohen-Macaulay type (see Subsection 3.2.2). The latter two has been deeply studied in [4]. As showed in Subsection 2.2.4, the authors give a complete classification of the bipartite graphs whose binomial edge ideal is Cohen-Macaulay: if G is connected and bipartite, then J_G is Cohen-Macaulay if and only if G can be obtained recursively by gluing a finite number of graphs of the form F_m via two operations. In the same article, they describe the fan graphs, a new family of Cohen-Macaulay binomial edge ideals associated with non-bipartite graphs. For both these families, in [41], Jayanthan and Kumar compute a precise expression for the regularity. Here, we provide the unique extremal Betti number of the binomial edge ideal of these graphs. In addition, we exploit the unique extremal Betti number of S/J_{F_m} to describe completely its Hilbert-Poincaré series.

3.2.1 Betti numbers of binomial edge ideals of disjoint graphs

In this subsection, we introduce some preliminary lemmas that we will be relevant in the next subsections.

Proposition 3.2.1. *[32, Corollary 4.3] For any simple graph G, it holds*

$$\beta_{i,i+1}(S/J_G) = i f_i(\Delta(G))$$

where $\Delta(G)$ is the clique complex of G and $f_i(\Delta(G))$ is the number of faces of $\Delta(G)$ of dimension i.

Lemma 3.2.2. *Let G be a connected graph on $[n]$. Suppose J_G be Cohen-Macaulay, and let $p = \operatorname{proj\,dim}\, S/J_G$. Then*

(i) $\beta_{p,p+1}(S/J_G) \neq 0$ if and only if G is a complete graph on $[n]$.

(ii) If $G = H_1 \sqcup H_2$, where H_1 and H_2 are graphs on disjoint vertex sets, then $\beta_{p,p+2}(S/J_G) \neq 0$ if and only if H_1 and H_2 are complete graphs.

Proof. (i) Since J_G is Cohen-Macaulay, it holds $p = n-1$, and the statement is an immediate consequence of Proposition 3.2.1, with $i = p$.
(ii) Since J_G is generated by homogeneous binomials of degree 2, $\beta_{1,1}(S/J_G) = 0$. This implies that $\beta_{i,i}(S/J_G) = 0$ for all $i \geq 1$. For all $j \geq 1$, we have

$$\beta_{p,p+j}(S/J_G) = \sum_{\substack{1 \leq j_1, j_2 \leq r \\ j_1 + j_2 = j}} \beta_{p_1,p_1+j_1}(S_1/J_{H_1})\beta_{p_2,p_2+j_2}(S_2/J_{H_2}),$$

where $S_i = \mathbb{K}[\{x_j, y_j\}_{j \in V(H_i)}]$ and $p_i = \operatorname{proj\,dim}\, S_i/J_{H_i}$, for $i = 1, 2$. For $j = 2$, we get

$$\beta_{p,p+2}(S/J_G) = \beta_{p_1,p_1+1}(S_1/J_{H_1})\beta_{p_2,p_2+1}(S_2/J_{H_2}). \tag{3.2}$$

By part (i), both the Betti numbers on the right are nonzero if and only if H_1 and H_2 are complete graphs, and the thesis follows. $\square$

Let G be a simple connected graph on $[n]$. We recall that if J_G is Cohen-Macaulay, then $p = \operatorname{proj\,dim}\, S/J_G = n-1$, and it admits a unique extremal Betti number, that is $\beta_{p,p+r}(S/J_G)$, where $r = \operatorname{reg}\, S/J_G$. Hereafter, when S/J_G has a unique extremal Betti number, we denote it by $\widehat{\beta}(S/J_G)$.

Lemma 3.2.3. *Let H_1 and H_2 be connected graphs on disjoint vertex sets and $G = H_1 \sqcup H_2$. Suppose J_{H_1} and J_{H_2} be Cohen-Macaulay binomial edge ideals. Let $S_i = \mathbb{K}[\{x_j, y_j\}_{j \in V(H_i)}]$ for $i = 1, 2$. Then*

(i) CM-type$(S/J_G) =$ CM-type(S_1/J_{H_1}) CM-type(S_2/J_{H_2}).

(ii) $\widehat{\beta}(S/J_G) = \widehat{\beta}(S_1/J_{H_1})\widehat{\beta}(S_2/J_{H_2})$.

Proof. (i) The equality $J_G = J_{H_1} + J_{H_2}$ implies that the minimal graded free resolution of S/J_G is the tensor product of the minimal graded free resolutions of S_1/J_{H_1} and S_2/J_{H_2}, where $S_i = \mathbb{K}[\{x_j, y_j\}_{j \in V(H_i)}]$ for $i = 1, 2$. Then

$$\beta_t(S/J_G) = \sum_{k=0}^{t} \beta_k(S_1/J_{H_1})\beta_{t-k}(S_2/J_{H_2}).$$

Let $p = \operatorname{proj\,dim}\, S/J_G$, that is $p = p_1 + p_2$, where $p_i = \operatorname{proj\,dim}\, S_i/J_{H_i}$ for $i = 1, 2$. Since $\beta_k(S_1/J_{H_1}) = 0$ for all $k > p_1$ and $\beta_{p-k}(S_2/J_{H_2}) = 0$ for all $k < p_1$, it follows

$$\beta_p(S/J_G) = \beta_{p_1}(S_1/J_{H_1})\beta_{p_2}(S_2/J_{H_2}).$$

(ii) Let $r = \operatorname{reg}\, S/J_G$. Consider

$$\beta_{p,p+r}(S/J_G) = \sum_{\substack{1 \le j_1, j_2 \le r \\ j_1 + j_2 = r}} \beta_{p_1,p_1+j_1}(S_1/J_{H_1})\beta_{p_2,p_2+j_2}(S_2/J_{H_2}).$$

Since $\beta_{p_i,p_i+j_i}(S_i/J_{H_i}) = 0$ for all $j_i > r_i$, where $r_i = \operatorname{reg}\, S_i/J_{H_i}$ for $i = 1, 2$, and $r = r_1 + r_2$, it follows

$$\beta_{p,p+r}(S/J_G) = \beta_{p_1,p_1+r_1}(S_1/J_{H_1})\beta_{p_2,p_2+r_2}(S_2/J_{H_2}).$$

□

3.2.2 Regularity and Cohen-Macaulay type of cones

Lemma 3.2.4. *Let* $G = \mathrm{cone}(v, H_1 \sqcup \cdots \sqcup H_s)$*, with* $s \geq 2$*. Then*

$$\mathrm{reg}\ S/J_G = \max\left\{\sum_{i=1}^{s} \mathrm{reg}\ S/J_{H_i}, 2\right\}.$$

Proof. Consider the short exact sequence (2.4), with $G = \mathrm{cone}(v, H_1 \sqcup \cdots \sqcup H_s)$ and $u = v$, then $G' = K_n$, the complete graph on $[n]$, $G'' = H_1 \sqcup \cdots \sqcup H_s$, and $H = K_{n-1}$, where $n = |V(G)|$. Since G' and H are complete graphs, the regularity of $S/J_{G'}$ and $S/((x_u, y_u) + J_H)$ is 1. Whereas the regularity of $S/((x_u, y_u) + J_{G''})$ is given by $\mathrm{reg}\ S/J_{H_1} + \cdots + \mathrm{reg}\ S/J_{H_s}$. We get the following bound on the regularity of S/J_G

$$\begin{aligned}\mathrm{reg}\ S/J_G &\leq \max\left\{\mathrm{reg}\ \frac{S}{J_{G'}}, \mathrm{reg}\ \frac{S}{((x_u, y_u) + J_{G''})}, \mathrm{reg}\ \frac{S}{((x_u, y_u) + J_H)} + 1\right\} \\ &= \max\left\{1, \sum_{i=1}^{s} \mathrm{reg}\ S/J_{H_i}, 2\right\}.\end{aligned}$$

Suppose $\sum_{i=1}^{s} \mathrm{reg}\ S/J_{H_i} \geq 2$, hence $\mathrm{reg}\ S/J_G \leq \sum_{i=1}^{s} \mathrm{reg}\ S/J_{H_i}$. Since $H_1 \sqcup \cdots \sqcup H_s$ is an induced subgraph of G, by [52, Corollary 2.2] of Matsuda and Murai we have

$$\mathrm{reg}\ S/J_G \geq \mathrm{reg}\ S/J_{H_1 \sqcup \cdots \sqcup H_s} = \sum_{i=1}^{s} \mathrm{reg}\ S/J_{H_i}.$$

Suppose now $\sum_{i=1}^{s} \mathrm{reg}\ S/J_{H_i} < 2$, hence $\mathrm{reg}\ S/J_G \leq 2$. Since G is not a complete graph, $\mathrm{reg}\ S/J_G \geq 2$, and the statement follows.

□

Observe that it happens $\mathrm{reg}\ S/J_G = 2$, for $G = \mathrm{cone}(v, H_1 \sqcup \cdots \sqcup H_s)$, with $s \geq 2$, if and only if all the H_i are isolated vertices except for at most two which are complete graphs.

We are going to give a description of the Cohen-Macaulay type and some Betti numbers of S/J_G when S/J_G is Cohen-Macaulay, and G is a cone, namely $G = \mathrm{cone}(v, H)$. By Lemma 2.2.25, Lemma 2.2.26, and Theorem 2.2.27, it is necessary for G being Cohen-Macaulay that H has exactly two connected components and both are Cohen-Macaulay.

Proposition 3.2.5. *Let $G = \operatorname{cone}(v, H_1 \sqcup H_2)$ on $[n]$, with J_{H_1} and J_{H_2} Cohen-Macaulay binomial edge ideals. Then*

$$\textit{CM-type}(S/J_G) = n - 2 + \textit{CM-type}(S/J_{H_1})\textit{CM-type}(S/J_{H_2}).$$

In particular, the unique extremal Betti number of S/J_G is given by

$$\widehat{\beta}(S/J_G) = \begin{cases} \widehat{\beta}(S_1/J_{H_1})\widehat{\beta}(S_2/J_{H_2}) & \textit{if } r > 2 \\ n - 2 + \widehat{\beta}(S_1/J_{H_1})\widehat{\beta}(S_2/J_{H_2}) & \textit{if } r = 2 \end{cases}$$

where $r = \operatorname{reg} S/J_G$ and $S_i = \mathbb{K}[\{x_j, y_j\}_{j \in V(H_i)}]$ for $i = 1, 2$. In addition, if $r > 2$, it holds

$$\beta_{p,p+2}(S/J_G) = n - 2.$$

Proof. Consider the short exact sequence (2.4), with $u = v$, then we have $G' = K_n$, $G'' = H_1 \sqcup H_2$, and $H = K_{n-1}$. It holds

$$\begin{aligned} r = \operatorname{reg} S/J_G &= \max\{\operatorname{reg} S/J_{H_1} + \operatorname{reg} S/J_{H_2}, 2\}, \\ \operatorname{reg} S/((x_u, y_u) + J_{G''}) &= \operatorname{reg} S/J_{H_1} + \operatorname{reg} S/J_{H_2}, \\ \operatorname{reg} S/J_{G'} &= \operatorname{reg} S/((x_u, y_u) + J_H) = 1, \end{aligned} \tag{3.3}$$

and

$$p = \operatorname{proj\,dim} S/J_G = \operatorname{proj\,dim} S/J_{G'} = \operatorname{proj\,dim} S/((x_u, y_u) + J_{G''}) = n - 1,$$

$$\operatorname{proj\,dim} S/((x_u, y_u) + J_H) = n.$$

Consider the long exact sequence (2.5) with $i = p$. By (3.3), we have

$$\beta_{p,p+j}(S/J_{G'}) = \beta_{p,p+j}(S/((x_u, y_u) + J_H)) = 0 \text{ for all } j \geq 2$$

and

$$\beta_{p+1,p+1+(j-1)}(S/((x_u, y_u) + J_H)) \neq 0 \text{ only for } j = 2.$$

By Lemma 3.2.2 and Lemma 3.2.3 (i), it follows that

$$\begin{aligned} \text{CM-type}(S/J_G) &= \sum_{j=0}^{r} \beta_{p,p+j}(S/J_G) = \sum_{j=2}^{r} \beta_{p,p+j}(S/J_G) \\ &= \beta_{p-1,p-2+2}(S/J_H) + \sum_{j=2}^{r} \beta_{p-2,p-2+j}(S/J_{G''}) \\ &= n - 2 + \text{CM-type}(S/J_{G''}) \\ &= n - 2 + \text{CM-type}(S/J_{H_1})\text{CM-type}(S/J_{H_2}). \end{aligned}$$

If $r = 2$,

$$\begin{aligned} \text{CM-type}(S/J_G) &= \beta_{p,p+2}(S/J_G) \\ &= \beta_{p-1,p-2+2}(S/J_H) + \beta_{p-2,p-2+2}(S/J_{G''}) \\ &= n - 2 + \widehat{\beta}(S_1/J_{H_1})\widehat{\beta}(S_2/J_{H_2}), \end{aligned}$$

where the last equality follows from Equation (3.2).

If $r > 2$, it means that H_1 and H_2 are not both complete graphs, and then, by Lemma 3.2.2 (ii), $\beta_{p-2,p-2+2}(S/J_{G''}) = 0$, then $\beta_{p,p+2}(S/J_G) = n - 2$, and $\widehat{\beta}(S/J_G) = \widehat{\beta}(S_1/J_{H_1})\widehat{\beta}(S_2/J_{H_2})$.

□

3.2.3 Extremal Betti numbers of some classes of Cohen-Macaulay binomial edge ideals

In this subsection, we exhibit the extremal Betti numbers of Cohen-Macaulay binomial edge ideals of bipartite graphs and fan graphs. We conclude providing the Hilbert-Poincaré series of S/J_G, when $G = F_m$.

Exploiting Proposition 3.2.5, we get hold a formula for the CM-type of any $G = F_m^W$ pure fan graph.

Proposition 3.2.6. *Let $m \geq 2$, and $G = F_m^W$ a pure fan graph, with $|W| \geq 1$. Then*

$$\textit{CM-type}(S/J_G) = \widehat{\beta}(S/J_G) = (m-1)|W|. \tag{3.4}$$

Proof. We use induction on $n = m + |W|$, the number of vertices of G. If $n = 3$, that is $m = 2$ and $|W| = 1$, G is decomposable into two K_2's and it is straightforward to check that (3.4) holds. Let $n > 3$ and suppose the thesis true for all the pure graphs with at most $n - 1$ vertices. We have $G = \text{cone}(v_1, H_1 \sqcup H_2)$, where $W = \{v_1, \ldots, v_s\}$, $H_1 = F_{m-1}^{W'}$ is the pure graph of K_{m-1} on W', with $W' = W \setminus \{v_1\}$, w is the leaf of G, $\{w, v_1\} \in E(G)$, and $H_2 = \{w\}$. By induction hypothesis CM-type$(S/J_{H_1}) = (m-2)(|W|-1)$, and CM-type$(S/J_{H_2}) = 1$, then using Proposition 3.2.5, it follows

$$\begin{aligned} \text{CM-type}(S/J_G) &= |V(G)| - 2 + \text{CM-type}(S/J_{H_1})\text{CM-type}(S/J_{H_2}) \\ &= (m + |W| - 2) + (m-2)(|W|-1) = (m-1)|W|. \end{aligned}$$

Since $|W| \geq 1$, the graph F_m^W is not a complete graph, then $\beta_{p,p+1}(S/J_G) = 0$, where $p = \operatorname{proj\,dim} S/J_G$. Due to $\operatorname{reg} S/J_G = 2$, the CM-type$(S/J_G)$ coincides with the unique extremal Betti number of S/J_G, that is $\beta_{p,p+2}$. □

In the following result we provide the unique extremal Betti number of any k-pure fan graph.

Proposition 3.2.7. *Let $G = F_m^{W,k}$ be a k-pure fan graph, where $m \geq 2$ and $W = W_1 \sqcup \cdots \sqcup W_k \subseteq [m]$ is a non-trivial partition of W. Then*

$$\widehat{\beta}(S/J_G) = (m-1)\prod_{i=1}^{k} |W_i|. \tag{3.5}$$

Proof. Let $|W_i| = \ell_i$, for $i = 1, \dots k$. First of all, we observe that if $\ell_i = 1$ for all $i = 1, \dots, k$, that is $W_i = \{v_i\}$, then G is decomposable into $G_1 \cup \cdots \cup G_{k+1}$, where $G_1 = K_m$, $G_j = K_2$ and $G_1 \cap G_j = \{v_j\}$, for all $j = 2, \dots, k+1$. This implies

$$\widehat{\beta}(S/J_G) = \prod_{j=1}^{k+1} \widehat{\beta}(S/J_{G_j}) = m - 1$$

where the last equality is due to the fact $\widehat{\beta}(S/J_{K_m}) = m-1$ for any complete graph K_m, with $m \geq 2$. Without loss of generality, we suppose $\ell_1 \geq 2$.
We are ready to prove the statement on induction on n, the number of vertices of $G = F_m^{W,k}$, that is $n = m + \sum_{i=1}^{k} \ell_i$. Let $n = 4$, then G is a pure fan graph F_2^W, with $|W| = 2$, satisfying Proposition 3.2.6 and it holds (3.4). Let $n > 4$. Pick $v \in W_1$ such that $\{v, w\} \in E(G)$, with w a leaf of G. Consider the short exact sequence (2.4), with $u = v$, $G' = F_{m+\ell_1}^{W',k-1}$ the $(k-1)$-pure fan graph of $K_{m+\ell_1}$ on $W' = W_2 \sqcup \cdots \sqcup W_k$, $G'' = F_{m-1}^{W'',k} \sqcup \{w\}$ the disjoint union of the isolated vertex w and the k-pure fan graph of K_{m-1} on $W'' = W \setminus \{v\}$, and $H = F_{m+\ell_1-1}^{W',k-1}$. For the quotient rings involved in (2.4), from Proposition 2.2.34, we have

$$\begin{aligned} r = &\operatorname{reg} S/J_G \ = \operatorname{reg} S/((x_u, y_u) + J_{G''}) = 1 + k, \\ &\operatorname{reg} S/J_{G'} = \operatorname{reg} S/((x_u, y_u) + J_H) = k. \end{aligned}$$

As regard the projective dimensions, we have

$$\begin{aligned} p &= \operatorname{proj\,dim} S/J_G = \operatorname{proj\,dim} S/J_{G'} = \operatorname{proj\,dim} S/((x_u, y_u) + J_{G''}) \\ &= \operatorname{proj\,dim} S/((x_u, y_u) + J_H) - 1 = m + \sum_{i=1}^{k} \ell_i - 1. \end{aligned}$$

Fix $i = p$ and $j = r$ in the long exact sequence (2.5). The Tor modules $T_{p+1,p+1+(r-1)}(S/((x_u, y_u) + J_H))$ and $T_{p,p+r}(S/((x_u, y_u) + J_{G''}))$ are the only nonzeroes. It follows

$$\begin{aligned}\beta_{p,p+r}(S/J_G) &= \beta_{p-1,p+r-2}(S/J_H) + \beta_{p-2,p+r-2}(S/J_{G''}) \\ &= \widehat{\beta}(S/J_H) + \widehat{\beta}(S/J_{F_{m-1}^{W'',k}}).\end{aligned}$$

Both $F_{m-1}^{W'',k}$ and H fulfil the hypothesis of the proposition and they have less than n vertices, then by induction hypothesis

$$\begin{aligned}\widehat{\beta}(S/J_H) &= (m + \ell_1 - 2) \prod_{s=2}^{k} \ell_s, \\ \widehat{\beta}(S/J_{F_{m-1}^{W'',k}}) &= (m-2)(\ell_1 - 1) \prod_{s=2}^{k} \ell_s.\end{aligned}$$

Adding these extremal Betti numbers, the thesis is proved. □

Proposition 3.2.8. *Let $m \geq 2$. The unique extremal Betti number of the bipartite graph F_m is given by*

$$\widehat{\beta}(S/J_{F_m}) = \sum_{k=1}^{m-1} k^2.$$

Proof. We use induction on m. If $m = 2$, then $F_2 = K_2$ and it is well known that $\widehat{\beta}(S/J_{F_m}) = 1$. Suppose $m > 2$. Consider the short exact sequence (2.4), with $G = F_m$ and $u = 2m - 1$, with respect to the labelling introduced at the begin of this section. The graphs involved in (2.4) are $G' = F_{m+1}^{W}$, that is the pure fan graph of K_{m+1}, with $V(K_{m+1}) = \{u\} \cup \{2i | i = 1, \dots, m\}$, on $W = \{2i - 1 | i = 1, \dots, m-1\}$, $G'' = F_{m-1} \sqcup \{2m\}$, and the pure fan graph $H = F_m^W$. By Proposition 2.2.34 and Proposition 2.2.35, we have

$$\begin{aligned}r =& \text{reg } S/J_G = \text{reg } S/((x_u, y_u) + J_{G''}) = 3 \\ &\text{reg } S/J_{G'} = \text{reg } S/((x_u, y_u) + J_H) = 2.\end{aligned}$$

As regards the projective dimension of the quotient rings involved in (2.4), it is equal to $p = 2m-1$ for all, except for $S/((x_u, y_u) + J_H)$ whose projective dimension is $2m$. Consider the long exact sequence (2.5), with $i = p$ and $j = r$. In view of the above, $T_{p,p+r}(S/J_{G'})$,

$T_{p,p+r}(S/((x_u,y_u)+J_H))$, and all the Tor modules on the left of $T_{p+1,p+1+(r-1)}(S/((x_u,y_u)+J_H))$ in (2.5) are zero. It follows that

$$T_{p,p+r}(S/J_G) \cong T_{p+1,p+1+(r-1)}(S/((x_u,y_u)+J_H)) \oplus T_{p,p+r}(S/((x_u,y_u)+J_{G''})).$$

Then, using Proposition 3.2.7 and induction hypothesis, we obtain

$$\begin{aligned} \beta_{p,p+r}(S/J_G) &= \beta_{p-1,p+r-2}(S/J_H) + \beta_{p-2,p+r-2}(S/J_{G''}) \\ &= \widehat{\beta}(S/J_H) + \widehat{\beta}(S/J_{G''}) \\ &= (m-1)^2 + \sum_{k=1}^{m-2} k^2 = \sum_{k=1}^{m-1} k^2. \end{aligned}$$

□

Question 3.2.9 *Based on explicit calculations we believe that S/J_G is a level ring when either $G = F_m$ or $G = F_m^{W,k}$ for $m \geq 2$, that is $\beta_{p,p+j}(S/J_G) = 0$ and $\beta_{p,p+r}(S/J_G) =$ CM-type(S/J_G) for all $j = 0, \ldots, r-1$, and $p = \operatorname{proj\,dim} S/J_G$ and $r = \operatorname{reg} S/J_G$.*

Let $G = G_1 * \cdots * G_t$, for $t \geq 1$. Observe that G is decomposable into $G_1 \cup \cdots \cup G_t$, with $G_i \cap G_{i+1} = \{f_i\}$, for $i = 1, \ldots, t-1$, where f_i is the leaf of G_i and G_{i+1} which has been identified in $G_i * G_{i+1}$ and $G_i \cap G_j = \emptyset$, for $1 \leq i < j \leq t$.

Lemma 3.2.10. *[35, Corollary 1.4] Let G be decomposable into $G_1, \ldots, G_t$ and suppose S/J_G and S/J_{G_i}, with $i = 1, \ldots, t$, admit only one extremal Betti number. Then,*

$$\widehat{\beta}(S/J_G) = \prod_{i=1}^{t} \widehat{\beta}(S/J_{G_i}).$$

In light of the above, we will focus on graphs of the form $G = F_{m_1} \circ \cdots \circ F_{m_t}$, with $m_i \geq 3$, $i = 1, \ldots, t$.

Lemma 3.2.11. *Let $m_1, m_2 \geq 3$ and $G = F_{m_1} \circ F$, where F is either F_{m_2} or a k-pure fan graph $F_{m_2}^{W,k}$, with $W = W_1 \sqcup \cdots \sqcup W_k$ and $|W_i| \geq 2$ for some i. Let $\{v\} = V(F_{m_1}) \cap V(F)$ and suppose $v \in W_i$. Let G'' be as in Set-up 2.2.13, with $u = v$. Then the unique extremal Betti number of S/J_G is given by*

$$\widehat{\beta}(S/J_G) = \widehat{\beta}(S/J_{G''}).$$

In particular,

$$\widehat{\beta}(S/J_G) = \begin{cases} \widehat{\beta}(S/J_{F_{m_1-1}})\widehat{\beta}(S/J_{F_{m_2-1}}) & \text{if } F = F_{m_2} \\ \widehat{\beta}(S/J_{F_{m_1-1}})\widehat{\beta}(S/J_{F^{W',k}_{m_2-1}}) & \text{if } F = F^{W,k}_{m_2} \end{cases}$$

where $W' = W \setminus \{v\}$.

Proof. Consider the short exact sequence (2.4), with $G = F_{m_1} \circ F$ and $u = v$.
If $F = F_{m_2}$, then the graphs involved in (2.4) are: $G' = F^{W,2}_m$, $G'' = F_{m_1-1} \sqcup F_{m_2-1}$, and $H = F^{W,2}_{m-1}$, where $m = m_1 + m_2 - 1$, $W = W_1 \sqcup W_2$ with $|W_i| = m_i - 1$ for $i = 1, 2$, and G' and H are 2-pure fan graphs. By Proposition 2.2.34 and Proposition 2.2.35, we have the following values for the regularity

$$\begin{aligned} r = &\text{reg } S/J_G = \text{reg } S/((x_u, y_u) + J_{G''}) = 6 \\ &\text{reg } S/J_{G'} = \text{reg } S/((x_u, y_u) + J_H) = 3. \end{aligned}$$

In the matter of projective dimension, it is equal to $p = n - 1$ for all the quotient rings involved in (2.4), except for $S/((x_u, y_u) + J_H)$, for which it is n. Considering the long exact sequence (2.5) with $i = p$ and $j = r$, it holds

$$\beta_{p,p+r}(S/J_G) = \beta_{p,p+r}(S/((x_u, y_u) + J_{G''}))$$

and by Lemma 3.2.3 (ii) the second part of thesis follows.
The case $F = F^{W,k}_{m_2}$ follows by similar arguments. Indeed, suppose $|W_1| \geq 2$ and $v \in W_1$. The graphs involved in (2.4) are: $G' = F^{W',k}_m$, $G'' = F_{m_1-1} \sqcup F^{W'',k}_{m_2-1}$, and $H = F^{W',k}_{m-1}$, where $m = m_1 + m_2 + |W_1| - 2$, all the fan graphs are k-pure, $W' = W_1' \sqcup W_2 \sqcup \cdots \sqcup W_k$, with $|W_1'| = m_1 - 1$, whereas $W'' = W \setminus \{v\}$. Fixing $r = \text{reg } S/J_G = \text{reg } S/((x_u, y_u) + J_{G''}) = k + 4$, since $\text{reg } S/J_{G'} = \text{reg } S/((x_u, y_u) + J_H) = k + 1$, and the projective dimension of all the quotient rings involved in (2.4) is $p = n - 1$, except for $S/((x_u, y_u) + J_H)$, for which it is n, it follows

$$\beta_{p,p+r}(S/J_G) = \beta_{p,p+r}(S/((x_u, y_u) + J_{G''}))$$

and by Lemma 3.2.3 (ii) the second part of the thesis follows. □

Theorem 3.2.12. *Let $t \geq 2$, $m \geq 3$, and $m_i \geq 3$ for all $i = 1, \dots, t$. Let $G = F_{m_1} \circ \cdots \circ F_{m_t} \circ F$, where F denotes either F_m or a k-pure fan graph $F_m^{W,k}$ with $W = W_1 \sqcup \cdots \sqcup W_k$. Let $\{v\} = V(F_{m_1} \circ \cdots \circ F_{m_t}) \cap V(F)$ and, if $F = F_m^{W,k}$, assume $|W_1| \geq 2$ and $v \in W_1$. Let G'' and H be as in Set-up 2.2.13, with $u = v$. Then the unique extremal Betti number of S/J_G is given by*

$$\widehat{\beta}(S/J_G) = \widehat{\beta}(S/J_{G''}) + \begin{cases} \widehat{\beta}(S/J_H) & \text{if } m_t = 3 \\ 0 & \text{if } m_t > 3 \end{cases}$$

In particular, if $F = F_m$, it is given by

$$\widehat{\beta}(S/J_G) = \widehat{\beta}(S/J_{F_{m_1}\circ\cdots\circ F_{m_t-1}})\widehat{\beta}(S/J_{F_{m-1}}) + \begin{cases} \widehat{\beta}(S/J_H) & \text{if } m_t = 3 \\ 0 & \text{if } m_t > 3 \end{cases}$$

where $H = F_{m_1} \circ \cdots \circ F_{m_{t-1}} \circ F^{W',2}_{m+m_t-2}$, and $F^{W',2}_{m+m_t-2}$ is a 2-pure fan graph of K_{m+m_t-2} on $W' = W'_1 \sqcup W'_2$, with $|W'_1| = m_t - 1$ and $|W'_2| = m - 1$.
If $F = F_m^{W,k}$, it is given by

$$\widehat{\beta}(S/J_G) = \widehat{\beta}(S/J_{F_{m_1}\circ\cdots\circ F_{m_t-1}})\widehat{\beta}(S/J_{F^{W'',k}_{m-1}}) + \begin{cases} \widehat{\beta}(S/J_H) & \text{if } m_t = 3 \\ 0 & \text{if } m_t > 3 \end{cases}$$

where $W'' = W \setminus \{v\}$, $H = F_{m_1} \circ \cdots \circ F_{m_{t-1}} \circ F^{W''',k}_{m'}$, with $m' = m + m_t + |W_1| - 2$, $W''' = W''_1 \sqcup W_2 \sqcup \cdots \sqcup W_k$, and $|W''_1| = m_t - 1$.

Proof. If $F = F_m$, we have $G' = F_{m_1} \circ \cdots \circ F_{m_{t-1}} \circ F^{W',2}_{m+m_t-1}$, $G'' = F_{m_1} \circ \cdots \circ F_{m_t-1} \sqcup F_{m-1}$, and $H = F_{m_1} \circ \cdots \circ F_{m_{t-1}} \circ F^{W',2}_{m+m_t-2}$, where $W' = W'_1 \sqcup W'_2$, with $|W'_1| = m_t - 1$ and $|W'_2| = m - 1$. As regard the regularity of these quotient rings, we have

$$\begin{aligned} r = \operatorname{reg} S/J_G &= \operatorname{reg} S/((x_u, y_u) + J_{G''}) \\ &= \operatorname{reg} S/J_{F_{m_1-1}} + \operatorname{reg} S/J_{F_{m_2-2}} + \cdots + \operatorname{reg} S/J_{F_{m_t-2}} + \operatorname{reg} S/J_{F_{m-1}} \end{aligned}$$

and both reg $S/J_{G'}$ and reg $S/((x_u, y_u) + J_H)$ are equal to

$$\operatorname{reg} S/J_{F_{m_1-1}} + \operatorname{reg} S/J_{F_{m_2-2}} + \cdots + \operatorname{reg} S/J_{F_{m_{t-1}-2}} + \operatorname{reg} S/J_{F^{W',2}_{m+m_t-1}}.$$

Since reg $S/J_{F_{m-1}} = $ reg $S/J_{F^{W',2}_{m+m_t-1}} = 3$, whereas if $m_t = 3$, reg $S/J_{F_{m_t-2}} = 1$, otherwise reg $S/J_{F_{m_t-2}} = 3$, it follows that

$$\text{reg } S/J_{G'} = \text{reg } S/((x_u, y_u) + J_H) = \begin{cases} r-1 & \text{if } m_t = 3 \\ r-3 & \text{if } m_t > 3 \end{cases}$$

For the projective dimensions, we have

$$\begin{aligned} p &= \operatorname{proj\,dim} S/J_G = \operatorname{proj\,dim} S/((x_u, y_u) + J_{G''}) \\ &= \operatorname{proj\,dim} S/J_{G'} = \operatorname{proj\,dim} S/((x_u, y_u) + J_H) - 1 = n - 1. \end{aligned}$$

Passing through the long exact sequence (2.5) of Tor modules, we obtain, if $m_t = 3$

$$\beta_{p,p+r}(S/J_G) = \beta_{p,p+r}(S/((x_u, y_u) + J_{G''})) + \beta_{p+1,(p+1)+(r-1)}(S/((x_u, y_u) + J_H))$$

and, if $m_t > 3$

$$\beta_{p,p+r}(S/J_G) = \beta_{p,p+r}(S/((x_u, y_u) + J_{G''})).$$

The case $F = F^{W,k}_m$ follows by similar arguments. Indeed, the involved graphs are: $G' = F_{m_1} \circ \cdots \circ F_{m_{t-1}} \circ F^{W''',k}_{m'}$, $G'' = F_{m_1} \circ \cdots \circ F_{m_t-1} \sqcup F^{W'',k}_{m-1}$, and $H = F_{m_1} \circ \cdots \circ F_{m_{t-1}} \circ F^{W''',k}_{m'-1}$, where all the fan graphs are k-pure, $W'' = W \setminus \{v\}$, $m' = m + m_t + |W_1| - 1$, $W''' = W''_1 \sqcup W_2 \sqcup \cdots \sqcup W_k$, and $|W''_1| = m_t - 1$. Fixing $r = $ reg S/J_G, we get reg $S/((x_u, y_u) + J_{G''}) = r$, whereas

$$\text{reg } S/J_{G'} = \text{reg } S/((x_u, y_u) + J_H) = \begin{cases} r-1 & \text{if } m_t = 3 \\ r-3 & \text{if } m_t > 3 \end{cases}$$

The projective dimension of all the quotient rings involved is $p = n - 1$, except for $S/((x_u, y_u) + J_H)$, for which it is n. Passing through the long exact sequence (2.5) of Tor modules, the thesis follows. □

Corollary 3.2.13. *Let $t \geq 2$, $m, m_1 \geq 3$, and $m_i \geq 4$ for all $i = 2, \ldots, t$. Let $G = F_{m_1} \circ \cdots \circ F_{m_t} \circ F$, where F denotes either F_m or a k-pure fan graph $F^{W,k}_m$ with $W = W_1 \sqcup \cdots \sqcup W_k$. Let $\{v\} = V(F_{m_1} \circ \cdots \circ F_{m_t}) \cap V(F)$ and, when $F = F^{W,k}_m$, assume $|W_1| \geq 2$ and $v \in W_1$. Then the unique extremal Betti number of S/J_G is given by*

$$\widehat{\beta}(S/J_G) = \begin{cases} \widehat{\beta}(S/J_{F_{m_1-1}}) \prod_{i=2}^{t} \widehat{\beta}(S/J_{F_{m_i-2}}) \widehat{\beta}(S/J_{F_{m-1}}) & \text{if } F = F_m \\ \widehat{\beta}(S/J_{F_{m_1-1}}) \prod_{i=2}^{t} \widehat{\beta}(S/J_{F_{m_i-2}}) \widehat{\beta}(S/J_{F^{W',k}_{m-1}}) & \text{if } F = F^{W,k}_m \end{cases}$$

where $W' = W \setminus \{v\}$.

Proof. By Theorem 3.2.12 and by hypothesis on the m_i's, we get

$$\widehat{\beta}(S/J_G) = \widehat{\beta}(S/J_{F_{m_1}} \circ \cdots \circ F_{m_{t-1}})\widehat{\beta}(S/J_{F_{m-1}}).$$

Repeating the same argument for computing the extremal Betti number of $S/J_{F_{m_1}\circ\cdots\circ F_{m_{t-1}}}$, and by Lemma 3.2.11, we have done. □

Remark 3.2.14 *Contrary to what we believe for bipartite graphs* F_m *and* k*-pure fan graphs* $F_m^{W,k}$ *(see Question 3.2.9), in general for a Cohen-Macaulay bipartite graph* $G = F_{m_1} \circ \cdots \circ F_{m_t}$*, with* $t \geq 2$*, the unique extremal Betti number of* S/J_G *does not coincide with the Cohen-Macaulay type of* S/J_G*, for example for* $G = F_4 \circ F_3$*, we have* $5 = \widehat{\beta}(S/J_G) \neq$ *CM-type*$(S/J_G) = 29$.

In the last part of this section, we completely describe the Hilbert-Poincaré series HS of S/J_G, when G is a bipartite graph F_m. In particular, we are interested in computing the h-vector of S/J_G.

For any graph G on $[n]$, let

$$\mathrm{HS}_{S/J_G}(t) = \frac{p(t)}{(1-t)^{2n}} = \frac{h(t)}{(1-t)^d}$$

be the Hilbert-Poincaré series of S/J_G, where $p(t), h(t) \in \mathbb{Z}[t]$ and $d = \dim S/J_G$. Recall that the polynomial $p(t)$ is related to the graded Betti numbers of S/J_G in the following way

$$p(t) = \sum_{i,j} (-1)^i \beta_{i,j}(S/J_G) t^j. \tag{3.6}$$

Lemma 3.2.15. *Let* G *be a graph on* $[n]$*, and suppose* S/J_G *has a unique extremal Betti number, then the last nonnegative entry in the* h*-vector is* $(-1)^{p+d}\beta_{p,p+r}$*, where* $p = \operatorname{proj\,dim} S/J_G$ *and* $r = \operatorname{reg} S/J_G$.

Proof. The unique extremal Betti number of S/J_G is $\beta_{p,p+r}(S/J_G)$. Since $p(t) = h(t)(1-t)^{2n-d}$, then $\text{lc}(p(t)) = (-1)^d \text{lc}(h(t))$, where lc denotes the leading coefficient of a polynomial. By Equation (3.6), the leading coefficient of $p(t)$ is the coefficient of t^j for $j = p+r$. Since $\beta_{i,p+r} = 0$ for all $i < p$, $\text{lc}(p(t)) = (-1)^p \beta_{p,p+r}$, and the thesis follows. $\square$

Let Δ be a simplicial complex with vertex set $V = \{v_1, \dots, v_n\}$. In the polynomial ring $\mathbb{K}[v_1, \dots, v_n]$, one can associate to Δ the following monomial ideal

$$I_\Delta = (v_{i_1} \cdots v_{i_r} \mid \{v_{i_1}, \dots, v_{i_r}\} \notin \Delta).$$

I_Δ is called the *Stanley-Reisner ideal* of Δ.

Proposition 3.2.16. *Let $G = F_m$, with $m \geq 2$, then the Hilbert-Poincaré series of S/J_G is given by*

$$\text{HS}_{S/J_G}(t) = \frac{h_0 + h_1 t + h_2 t^2 + h_3 t^3}{(1-t)^{2m+1}}$$

where

$$h_0 = 1, \qquad h_1 = 2m-1, \qquad h_2 = \frac{3m^2 - 3m}{2}, \quad \text{and} \quad h_3 = \sum_{k=1}^{m-1} k^2.$$

Proof. By Proposition 2.2.35, $\deg h(t) = \text{reg } S/J_G = 3$. Let $\text{in}(J_G) = I_\Delta$, for some simplicial complex Δ. Let f_i be the number of faces of Δ of dimension i with the convention that $f_{-1} = 1$. Then

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{k-i} f_{i-1}. \tag{3.7}$$

Exploiting the Equation (3.7) we get

$$h_1 = f_0 - d = 4m - (2m+1) = 2m - 1$$

To obtain h_2 we need first to compute f_1, that is the number of edges in Δ: they are all the possible edges, except for those that appear in $(I_\Delta)_2$, which are the number of edges in G. So

$$f_1 = \binom{4m}{2} - \frac{m(m+1)}{2} = \frac{15m^2 - 5m}{2}.$$

And then we have

$$h_2 = \binom{2m+1}{2} f_{-1} - \binom{2m}{1} f_0 + \binom{2m-1}{0} f_1 = \frac{3m^2 - 3m}{2}.$$

By Lemma 3.2.15, and since $p = 2m - 1$ and $d = 2m + 1$,

$$h_3 = (-1)^{4m} \beta_{p,p+r}(S/J_G) = \sum_{k=1}^{m-1} k^2$$

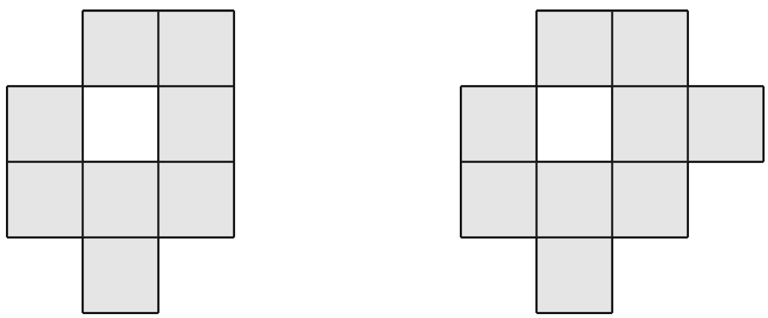
where the last equality follows from Proposition 3.2.8. $\square$

Chapter 4

ON THE PRIMALITY OF SOME POLYOMINO IDEALS

The polyomino ideals have been introduced, in this dissertation, in Section 2.3. From now on, a polyomino is briefly called *prime* if its polyomino ideal is prime. In Subsection 2.3.1 and Subsection 2.3.2, we have discussed the known results concerning the primality of polyomino ideals, showing that simple polyominoes and a particular family of multiply connected polyominoes are prime.

Aim of this section is mostly to investigate primality of any multiply connected polyominoes. Consider the polyominoes in Figure 4.1. They look alike except for one cell. Anyway, there exists a great difference: $\mathcal{P}_1$ is prime, but $\mathcal{P}_2$ is not.



(a) $\mathcal{P}_1$ (b) $\mathcal{P}_2$

Figure 4.1: Prime and non-prime polyominoes

Motivated by this simple but relevant example, we have followed up on the geometric representation of the polyominoes, until we have found a necessary condition for the primality of the polyomino ideal. This condition is related to a sequence of inner intervals contained in the polyomino, called a *zig-zag walk* (see Definition 4.1.2), whose existence determines the non-primality of the polyomino ideal (see Proposition 4.1.5 and Corollary 4.1.6).

We have implemented an algorithm, described in [50], to compute all the polyominoes, and their ideal, with rank less than or equal to 14. We get that if $\mathcal{P}$ is a polyomino with rank $(\mathcal{P}) \leq 14$, then $\mathcal{P}$ is prime if and only if $\mathcal{P}$ contains no zig-zag walk (see Theorem 4.1.9). Finally, in Subsection 4.2, we define a new infinite family of polyominoes, called *grid polyominoes*, that are obtained by removing inner intervals from a given rectangle in a way that avoids the existence of zig-zag walks. By using a Gröbner basis technique and lattice ideals, we prove that grid polyominoes are prime.

Therefore, the natural conjecture arises:

Conjecture 4.0.1 *Let $\mathcal{P}$ be a polyomino. The following conditions are equivalent:*

(i) the polyomino ideal $I_{\mathcal{P}}$ is prime;

(ii) $\mathcal{P}$ contains no zig-zag walk.

All the results presented in this section are contained in [51].

4.1 THE TORIC RING OF POLYOMINOES AND ZIG-ZAG WALKS

In this section, we first present a toric ideal associated to a polyomino, generalizing Shikama's construction (see Subsection 2.3.2). We will prove that this toric ideal contains the polyomino ideal. Secondly, we define the zig-zag walks, a sequence of inner intervals that are the key idea to give the necessary condition for having prime ideals. Moreover, we state that if the polyomino contains a zig-zag walk, the binomial associated to the zig-zag walk belongs to the toric ideal and, therefore, the above inclusion is strict.

Let $\mathcal{P}$ be a polyomino. Let $S = \mathbb{K}[x_v | v \in V(\mathcal{P})]$ and $I_{\mathcal{P}} \subset S$ the polyomino ideal associated to $\mathcal{P}$. Let $\mathcal{H}_1, \dots, \mathcal{H}_r$ be holes of $\mathcal{P}$. For $k = 1, \dots, r$, we denote by $e_k = (i_k, j_k)$ the lower left corner of $\mathcal{H}_k$. For $k \in K = \{1, \dots, r\}$, we define the following subset of $V(\mathcal{P})$

$$\mathcal{F}_k = \{(i, j) \in V(\mathcal{P}) \ | \ i \leq i_k \text{ and } j \leq j_k\}.$$

Let $\{V_i\}_{i \in I}$ be the set of all the maximal vertical edge intervals of $\mathcal{P}$, and $\{H_j\}_{j \in J}$ be the set of all the maximal horizontal edge intervals of $\mathcal{P}$. Let $\{v_i\}_{i \in I}, \{h_j\}_{j \in J}$, and $\{w_k\}_{w \in K}$ be three sets of variables associated to $\{V_i\}_{i \in I}, \{H_j\}_{j \in J}$, and $\{\mathcal{F}_k\}_{k \in K}$, respectively. We consider the map

$$\begin{aligned}\alpha : V(\mathcal{P}) &\longrightarrow \mathbb{K}[\{h_i, v_j, w_k\} \mid i \in I, j \in J, k \in K]\\ a &\longmapsto \prod_{a \in H_i \cap V_j} h_i v_j \prod_{a \in \mathcal{F}_k} w_k\end{aligned}$$

The *toric ring* $T_{\mathcal{P}}$ associated to $\mathcal{P}$ is defined as $T_{\mathcal{P}} = \mathbb{K}[\alpha(a) | a \in V(\mathcal{P})] \subset \mathbb{K}[\{h_i, v_j, w_k\} \mid i \in I, j \in J, k \in K]$. The homomorphism

$$\begin{aligned}\varphi : S &\longrightarrow T_{\mathcal{P}}\\ x_a &\longmapsto \alpha(a)\end{aligned}$$

is surjective and the *toric ideal* $J_{\mathcal{P}}$ is the kernel of φ.

Proposition 4.1.1. *Let $\mathcal{P}$ be a polyomino and $(J_{\mathcal{P}})_2$ the homogeneous part of degree 2 of $J_{\mathcal{P}}$. Then $I_{\mathcal{P}} = (J_{\mathcal{P}})_2$.*

Proof. First of all we show that $I_{\mathcal{P}} \subseteq (J_{\mathcal{P}})_2$. Let $f \in \mathcal{M}$, with $f = x_a x_b - x_c x_d$. Since $[a, b]$ is an inner interval of $\mathcal{P}$, the corners a and d (resp. b and c) lie on the same horizontal edge interval H_i (resp. H_j). In the same way, it holds that a and c (resp. b and d) lie on the same vertical edge interval V_ℓ (resp. V_m). Therefore,

$$\varphi(x_a x_b) = h_i h_j v_\ell v_m \prod_{k=1,\dots,r} w_k^{p_k} \tag{4.1}$$

and

$$\varphi(x_c x_d) = h_i h_j v_\ell v_m \prod_{k=1,\dots,r} w_k^{n_k} \tag{4.2}$$

for some $p_k, n_k \in \{0, 1, 2\}$. We have to show that for any $k \in \{1, \dots, r\}$ $p_k = n_k$. If $\mathcal{P}$ has not holes, then $n_k = p_k = 0$ and $\varphi(x_a x_b) = \varphi(x_c x_d)$, that is $f \in J_{\mathcal{P}}$. Suppose that $\mathcal{H}_1, \dots, \mathcal{H}_r$ are holes of $\mathcal{P}$ and consider $\mathcal{H}_k$ for $k = 1, \dots, r$. Observe that the left lower corner e_k of $\mathcal{H}_k$ satisfies one of the following

(I) $e_k < a$;

(II) $a \leq e_k \leq d$;

(III) $d < e_k$,

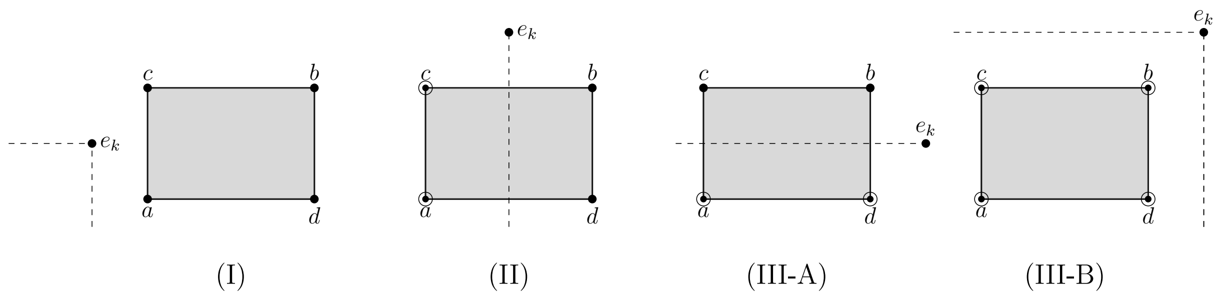


Figure 4.2: Some positions of e_k. The circled vertices v induce the variable w_k in the image $\phi(x_v)$.

where $<$ stands for $<^1$.

Case (I). w_k does not divide neither $\phi(x_a x_b)$ nor $\phi(x_c x_d)$ (see Figure 4.2(I)).

Case (II). w_k divides either both $\varphi(x_a)$ and $\varphi(x_c)$ (see Figure 4.2(II)) or it does not divide neither $\varphi(x_a x_b)$ nor $\varphi(x_c x_d)$.

Case (III). w_k divides either $\varphi(x_a)$ and $\varphi(x_d)$ (see Figure 4.2(III-A)) or all $\varphi(x_a), \varphi(x_b), \varphi(x_c)$ and $\varphi(x_d)$ (see Figure 4.2(III-B)) or w_k does not divide neither $\varphi(x_a x_b)$ nor $\varphi(x_c x_d)$.

Therefore $n_k = p_k$, and it holds for any $k = 1, \dots, r$. It follows $\varphi(x_a x_b) = \varphi(x_c x_d)$, and $f \in \ker \varphi = J_{\mathcal{P}}$. Since all generators of $I_{\mathcal{P}}$ belong to $J_{\mathcal{P}}$, the inclusion $I_{\mathcal{P}} \subseteq (J_{\mathcal{P}})_2$ is proved.

We are going to prove the other inclusion, namely $(J_{\mathcal{P}})_2 \subseteq I_{\mathcal{P}}$. Let $f \in J_{\mathcal{P}}$, $f = x_a x_b - x_c x_d$. We start observing that if $a = b$ or $a \in \{c, d\}$ we obtain that f is null. Hence we assume without loss of generality $a < b$ and $c < d$. Since $\varphi(x_a x_b) = \varphi(x_c x_d)$, by (4.1) and (4.2) the vertices a and d (resp. b and c) lie on the same horizontal edge interval of $\mathcal{P}$, and a and c (resp. b and d) lie on the same vertical edge interval of $\mathcal{P}$, and all the vertices of these edge intervals belong to $\mathcal{P}$. Therefore, the vertices a, b, c, and d are the corners of the interval $[a, b]$. By contradiction, we assume that $[a, b]$ is not an inner interval of $\mathcal{P}$, namely exists a set $\mathcal{C}$ of cells of $[a, b]$ that do not belong to $\mathcal{P}$. Since $[a, d]$, $[a, c]$, $[b, c]$ and $[b, d]$ are edge intervals in $\mathcal{P}$, $\mathcal{C}$ is a set of holes of $\mathcal{P}$ which are properly contained in $[a, b]$. Let $\mathcal{H}_1$ be a hole of $\mathcal{P}$ in $[a, b]$ with lower left corner $e = (i, j)$. Let $\mathcal{F}_1 = \{(m, n) \in V(\mathcal{P}) \mid m \leq i \text{ and } n \leq j\}$, then a is the unique vertex in $\{a, b, c, d\}$ such that $a \in \mathcal{F}_1$, namely $w_1 | \varphi(x_a x_b)$ but $w_1 \nmid \varphi(x_c x_d)$, and $f \notin J_{\mathcal{P}}$. The assertion follows. $\square$

Describing completely the elements of $J_{\mathcal{P}} \setminus I_{\mathcal{P}}$ is not an easy task. However, if the polyomino has a particular collection of inner intervals, then we have some partial information on the elements of $J_{\mathcal{P}} \setminus I_{\mathcal{P}}$.

Definition 4.1.2 *Let $\mathcal{P}$ be a polyomino. A sequence of distinct inner intervals $\mathcal{W}$: $I_1, \dots, I_\ell$ of $\mathcal{P}$ such that v_i, z_i are diagonal (resp. anti-diagonal) corners and u_i, v_{i+1} the anti-diagonal (resp. diagonal) corners of I_i, for $i = 1, \dots, \ell$, is a* zig-zag walk *of $\mathcal{P}$, if*

(Z1) $|I_1 \cap I_\ell| = \{v_1 = v_{\ell+1}\}$ *and* $I_i \cap I_{i+1} = \{v_{i+1}\}$, *for* $i = 1, \dots, \ell - 1$,

(Z2) v_i *and* v_{i+1} *are on a same edge interval of* $\mathcal{P}$, *for* $i = 1, \dots, \ell$,

(Z3) for any $i, j \in \{1, \dots, \ell\}$, *with* $i \neq j$, *does not exist an inner interval* J *of* $\mathcal{P}$ *such that* $z_i, z_j \in J$.

Remark 4.1.3 *Let $\mathcal{W} : I_1 \dots, I_\ell$ be a zig-zag walk of $\mathcal{P}$. Then*

(i) if v_i is a diagonal vertex of I_i then v_{i+1} is an anti-diagonal vertex of I_{i+1};

(ii) ℓ is even.

Proof. (1) Assume that v_k, with $k \in \{1, \dots, \ell-1\}$ is a diagonal corner of I_k. From condition (Z2), v_{k+1} lies on the same edge interval of v_k, say E, and is an anti-diagonal corner of I_k. The line containing E divides $\mathbb{N}^2$ in two semi-planes. From condition (Z1), we have $I_k \cap I_{k+1} = \{v_{k+1}\}$, hence I_k and I_{k+1} do not lie on the same semi-plane. Therefore, v_{k+1} is anti-diagonal corner of I_{k+1}, as well. Observe that the latter justifies the name "zig-zag".
(2) Assume that the starting point v_1 is a diagonal corner of I_1. From (1) it follows that the vertex v_k is a diagonal corner of I_k if and only if k is odd (resp. anti-diagonal corner if and only if k is even). Since $v_{\ell+1} = v_1$, $\ell + 1$ is odd. □

Remark 4.1.4 *Let $\mathcal{P}$ be a polyomino and $I_{\mathcal{P}} \subset S$ the polyomino ideal associated to $\mathcal{P}$. If $f \in I_{\mathcal{P}}$, then*

$$f = \sum f_{I_j} h_j = \sum x_{a_j} x_{b_j} h_j - \sum x_{c_j} x_{d_j} h_j$$

where $f_{I_j} = x_{a_j} x_{b_j} - x_{c_j} x_{d_j} \in \mathcal{M}$, hence for every m, monomial of f, there are two variables in m that are (anti-)diagonal corners of an inner interval of $\mathcal{P}$.

Proposition 4.1.5. *Let $\mathcal{P}$ be a polyomino and $I_{\mathcal{P}}$ the polyomino ideal associated to $\mathcal{P}$. If* *there exists a zig-zag walk* $\mathcal{W}: I_1, \dots, I_\ell$ *in* $\mathcal{P}$ *then*

$$x_{v_1}, \dots, x_{v_\ell} \text{ and } f_{\mathcal{W}} = \prod_{k=1,\dots,\ell} x_{z_k} - \prod_{j=1,\dots,\ell} x_{u_j}$$

are zero divisors of $K[\mathcal{P}]$ *with* $x_{v_i} f_{\mathcal{W}} \in I_{\mathcal{P}}$ *for* $i = 1, \dots, \ell$.

Proof. For any vertex v_j in $v_1, \dots, v_\ell$, after relabelling, we may assume $j = 1$ and, without loss of generality, that v_1 is a diagonal corner of I_1. Let $f_{I_i} \in \mathcal{M}$ be associated to the inner interval I_i.

We define the following polynomial

$$\tilde{f} = \omega_1 f_{I_1} + \cdots + (-1)^{i+1} \omega_i f_{I_i} + \cdots + (-1)^{\ell+1} \omega_\ell f_{I_\ell},$$

where, for $i = 1, \dots, \ell$,

$$\omega_i = \prod_{j<i} x_{u_j} \prod_{k>i} x_{z_k}.$$

Let $i = 1, \dots, \ell-1$. Suppose that v_i is a diagonal corner of I_i, hence v_{i+1} is an anti-diagonal corner of I_{i+1}. It holds

$$\omega_i f_{I_i} - \omega_{i+1} f_{I_{i+1}} \tag{4.3}$$

is

$$\omega_i (x_{v_i} x_{z_i} - x_{v_{i+1}} x_{u_i}) - \omega_{i+1} (x_{v_{i+2}} x_{u_{i+1}} - x_{v_{i+1}} x_{z_{i+1}}),$$

where

$$\omega_i x_{u_i} = \omega_{i+1} x_{z_{i+1}} \text{ for all } i. \tag{4.4}$$

That is (4.3) becomes

$$(\omega_i x_{z_i}) x_{v_i} - (\omega_{i+1} x_{u_{i+1}}) x_{v_{i+2}}.$$

Due to the alternation of the signs in $\tilde{f}$, Remark 4.1.3, Equation (4.4) and since $v_1 = v_{\ell+1}$, it follows that

$$\tilde{f} = \left(\prod_{k=1,\dots,\ell} x_{z_k} \right) x_{v_1} - \left(\prod_{j=1,\dots,\ell} x_{u_j} \right) x_{v_{\ell+1}} = x_{v_1} f_{\mathcal{W}}.$$

Since $\tilde{f}$ is sum of polynomials in $I_{\mathcal{P}}$, then $\tilde{f} \in I_{\mathcal{P}}$. Observe that, by hypothesis, for $i \neq j$, z_i, z_j do not belong to the same inner interval of $\mathcal{P}$, and the same fact holds for u_i and u_j, with $i \neq j$. Due to this fact and by Remark 4.1.4, $f_{\mathcal{W}} \notin I_{\mathcal{P}}$. Therefore, x_{v_1} and $f_{\mathcal{W}}$ are zero divisors of $K[\mathcal{P}]$. □

Corollary 4.1.6. *Let $\mathcal{P}$ be a polyomino and $I_{\mathcal{P}}$ the polyomino ideal associated to $\mathcal{P}$. If there exists a zig-zag walk in $\mathcal{P}$, then $I_{\mathcal{P}}$ is not prime.*

Remark 4.1.7 *Let $\mathcal{W} : I_1, \dots, I_\ell$ be a zig-zag walk of $\mathcal{P}$ and let $f_{\mathcal{W}} = \prod_{k=1,\dots,\ell} x_{z_k} - \prod_{j=1,\dots,\ell} x_{u_j}$ be its associated binomial. The ideal $J_{\mathcal{P}}$ contains the binomials associated to zig-zag walks. Indeed, by Proposition 4.1.5, it arises that*

$$x_{v_1} f_{\mathcal{W}} \in I_{\mathcal{P}} \subseteq J_{\mathcal{P}}$$

and, due to primality of $J_{\mathcal{P}}$, it follows $f_{\mathcal{W}} \in J_{\mathcal{P}}$.

We give some examples to better understand the structure of $J_{\mathcal{P}}$.

Example 4.1.8 *We consider the polyomino $\mathcal{P} \subset [(1,1),(8,4)]$ in Figure 4.3.*

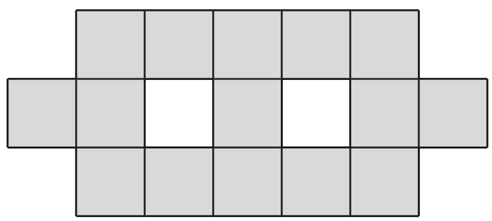


Figure 4.3

By using `Macaulay2` *[27], we computed the ideal $J_{\mathcal{P}}$ associated to $\mathcal{P}$. $J_{\mathcal{P}}$ has 50 generators, 46 having degree 2, corresponding to the inner 2-minors of $\mathcal{P}$, and 4 having degree 4 that do not belong to $I_{\mathcal{P}}$. The latter are:*

$$f_1 = x_{(1,3)}x_{(3,1)}x_{(7,4)}x_{(8,2)} - x_{(1,2)}x_{(3,4)}x_{(7,1)}x_{(8,3)},$$

$$f_2 = x_{(1,3)}x_{(2,1)}x_{(7,4)}x_{(8,2)} - x_{(1,2)}x_{(2,4)}x_{(7,1)}x_{(8,3)},$$

$$f_3 = x_{(1,3)}x_{(3,1)}x_{(6,4)}x_{(8,2)} - x_{(1,2)}x_{(3,4)}x_{(6,1)}x_{(8,3)},$$

$$f_4 = x_{(1,3)}x_{(2,1)}x_{(6,4)}x_{(8,2)} - x_{(1,2)}x_{(2,4)}x_{(6,1)}x_{(8,3)}.$$

The four binomials above correspond to the four zig-zag walks drawn in Figure 4.4.

Figure 4.4: The zig-zag walks related to $f_1, \dots, f_4$.

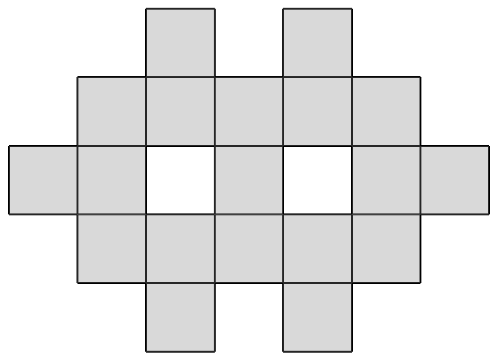


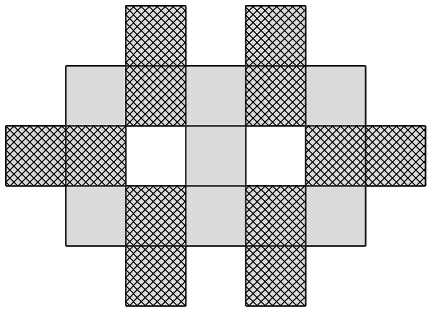
Figure 4.5

In this case, the generators of $J_\mathcal{P}$ in $J_\mathcal{P} \setminus I_\mathcal{P}$ are all related to zig-zag walks. However, we computed $J_\mathcal{P}$ for the polyomino $\mathcal{P} \subset [(1,1),(8,6)]$ in Figure 4.5, and we found that there are generators of degree 6 that are not related to zig-zag walks, for example

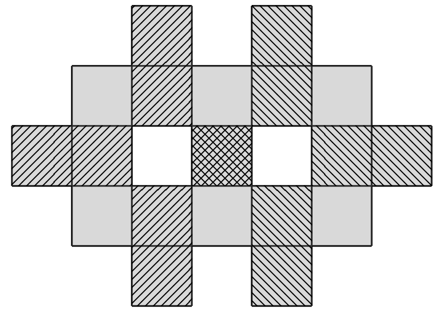
$$g = x_{(1,4)}x_{(3,1)}x_{(4,6)}x_{(5,1)}x_{(6,6)}x_{(8,3)} - x_{(1,3)}x_{(3,6)}x_{(4,1)}x_{(5,6)}x_{(6,1)}x_{(8,4)}.$$

In Figure 4.6(A), we highlight the intervals related to g. On the other hand, there are two zig-zag walks that arises from g, as in Figure 4.6(B).

Verifying that the non-existence of zig-zag walks is a sufficient condition for the primality of $I_\mathcal{P}$, for any multiply connected polyomino $\mathcal{P}$ of rank ≤ 14, is not an easy task. In fact, the cardinality of the set of polyominoes grows exponentially with respect to the rank. In Table 4.1, we show the numbers of distinct free multiply connected polyominoes, the ones there are not a translation, rotation, reflection or glide reflection of another poly-



(A) g is not related to a zig-zag walk...



(B) ...but there are two zig-zag walks.

Figure 4.6

omino, of rank ≤ 14, obtained by the implementation in [50] (see also [25, Chapter 6]).

Rank	7	8	9	10	11	12	13	14
Free multiply connected polyominoes	1	6	37	195	979	4663	21474	96496

Table 4.1: Numbers of distinct free multiply connected polyominoes.

Theorem 4.1.9. *Let $\mathcal{P}$ be a polyomino with* rank $(\mathcal{P}) \leq 14$. *The following conditions are equivalent:*

1. *the polyomino ideal $I_{\mathcal{P}}$ is prime;*
2. *$\mathcal{P}$ contains no zig-zag walk.*

Proof. (1) $\Rightarrow$ (2) It is an immediate consequence of Corollary 4.1.6.
(2) $\Rightarrow$ (1) By Corollary 4.1.6, simple polyominoes have no zig-zag walk, since they are prime. Therefore, we have to consider only multiply connected polyominoes. We prove that if $\mathcal{P}$ is a non-prime multiply connected polyomino with rank $(\mathcal{P}) \leq 14$, then $\mathcal{P}$ has a zig-zag walk. To this aim, we implemented a computer program that performs the following 3 steps:

(S1) Compute the set of all multiply connected polyominoes with rank ≤ 14, namely P.

(S2) Compute the set of polyominoes $\mathrm{NP} \subset P$ whose polyomino ideal is not prime. We used a routine developed in `Macaulay2` [27].

(S3) Verify that all polyominoes in NP have at least one zig-zag walk.

We refer to [50] for a complete description of the algorithm that we used.

□

4.2 GRID POLYOMINOES

From a view point of finding a new class of prime polyomino ideals, due to Corollary 4.1.6, it is reasonable to consider multiply connected polyominoes without zig-zag walks. In this section, we consider polyominoes obtained by subtracting some inner intervals by a given interval of $\mathbb{N}^2$, similarly as done in [37] and [64]. However, if the cells are removed without a specific pattern, one can easily obtain a zig-zag walk in this case, too (see Figure 4.7(B)). Hence we define an infinite family of polyominoes with no zig-zag walks by their intrinsic shape: the grid polyominoes. We prove that their polyomino ideal is prime by using Gröbner basis technique and lattice ideals. To this aim, we define the following monomial orders.

The total orders $<^1$ and $<^2$ on the vertices of $\mathcal{P}$ induce in a natural way the following monomial orders on $S = \mathbb{K}[x_v | v \in V(\mathcal{P})]$, respectively:

1. $x_a <^1_{\text{lex}} x_b$ if $a <^1 b$;
2. $x_a <^2_{\text{lex}} x_b$ if $a <^2 b$.

In [56], the author provides a necessary and sufficient condition for the set $\mathcal{M}$ of inner 2-minors to be a reduced Gröbner basis of $I_{\mathcal{P}}$, where $\mathcal{P}$ is a collection of cells of $\mathbb{N}^2$. In the following, we state the result when $\mathcal{P}$ is a polyomino.

Proposition 4.2.1. *Let $\mathcal{P}$ be a polyomino. $\mathcal{M}$ forms a reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to $<^1_{\text{lex}}$ if and only if for any two intervals $[a, b]$ and $[b, e]$ of $\mathcal{P}$, at least one between $[a, f]$ and $[a, g]$ is an inner interval of $\mathcal{P}$, where f and g are the anti-diagonal corners of $[b, e]$. Similarly, $\mathcal{M}$ forms a reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to $<^2_{\text{lex}}$ if and only if for any two inner intervals $[a, b]$ and $[e, f]$ of $\mathcal{P}$, with d anti-diagonal corner of both the inner intervals, either a, e or b, f are anti-diagonal corners of an inner interval of $\mathcal{P}$.*

Let $V(\mathcal{P}) = \{v_1, \dots, v_n\}$. Given a monomial order $<$ such that

$$x_{v_1} < x_{v_2} < \cdots < x_{v_n},$$

we define $<_v$, with $v = v_k \in V(\mathcal{P})$, as the following monomial order:

$$x_{v_k} < x_{v_{k+1}} < \cdots < x_{v_n} < x_{v_1} < x_{v_2} < \cdots < x_{v_{k-1}}.$$

From now on, we will respectively denote $(<^1_{\text{lex}})_v$ and $(<^2_{\text{lex}})_v$ by $<^1_v$ and $<^2_v$.

Definition 4.2.2 *Let $\mathcal{P} \subseteq I := [(1,1),(m,n)]$ be a polyomino such that*

$$\mathcal{P} = I \setminus \{\mathcal{H}_{ij} : i \in \{1,\dots,r\},\ j \in \{1,\dots,s\}\},$$

where $\mathcal{H}_{ij} = [a_{ij}, b_{ij}]$, with $a_{ij} = ((a_{ij})_1, (a_{ij})_2)$, $b_{ij} = ((b_{ij})_1, (b_{ij})_2)$, $1 < (a_{ij})_1 < (b_{ij})_1 < m$, $1 < (a_{ij})_2 < (b_{ij})_2 < n$, and

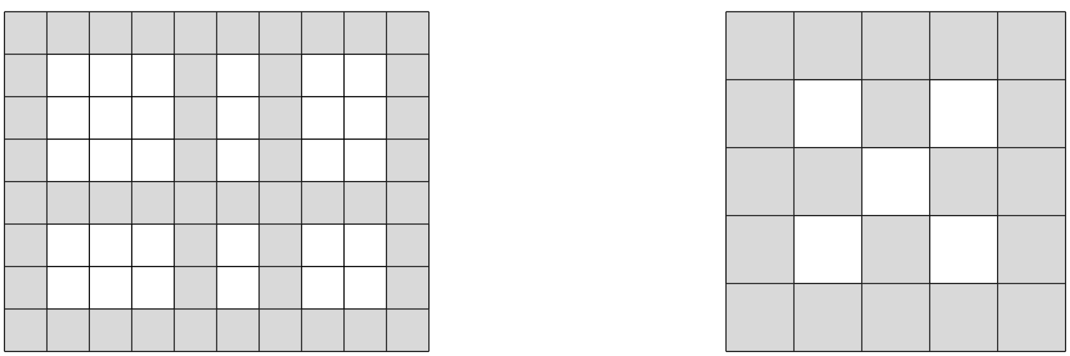
(i) for any $1 \leq i \leq r$ and $1 \leq \ell, k \leq s$, $(a_{i\ell})_1 = (a_{ik})_1$ and $(b_{i\ell})_1 = (b_{ik})_1$;

(ii) for any $1 \leq j \leq s$ and $1 \leq \ell, k \leq r$, $(a_{\ell j})_2 = (a_{kj})_2$ and $(b_{\ell j})_2 = (b_{kj})_2$;

(iii) for any $1 \leq i \leq r-1$ and $1 \leq j \leq s-1$, $(a_{i+1j})_1 = (b_{ij})_1 + 1$ and $(a_{ij+1})_2 = (b_{ij})_2 + 1$.

We call $\mathcal{P}$ a grid polyomino*.*

Example 4.2.3 *In Figure 4.7, two polyominoes are displayed: the one on the left (A) is a grid polyomino, while the one on the right (B) is not.*



(A) A grid polyomino. (B) A non-grid polyomino.

Figure 4.7

Definition 4.2.4 *Let $\mathcal{P}$ be a polyomino and let $v \in V(\mathcal{P})$. We say that v satisfies the condition* (Π) *if it fulfils at least one of the following:*

(I) there exist two inner intervals $I = [a, b]$ and $K = [b, e]$ of $\mathcal{P}$, with c upper left corner of I, d lower right corner of I, v upper left corner of K, and g lower right corner of K, such that $J = [c, v]$ is inner interval of $\mathcal{P}$, whereas $L = [d, g]$ is not (see Figure 4.8 Case (I)).

(II) There exist two inner intervals $J = [a, b]$ and $L = [e, f]$ of $\mathcal{P}$, with d lower right corner of J and upper left corner of L, such that the interval $K = [d, v]$ having a and e as anti-diagonal corners is inner interval of $\mathcal{P}$, whereas the interval I having a and e as anti-diagonal corners is not (see Figure 4.8 Case (II)).

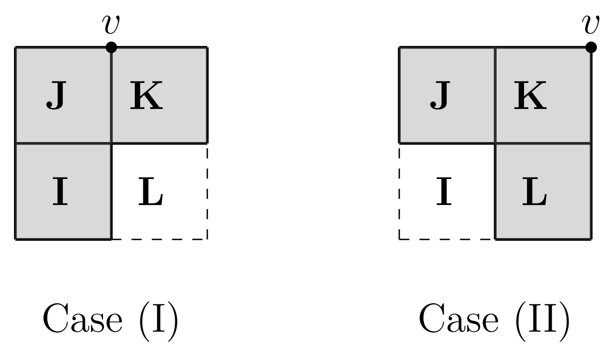


Figure 4.8: Condition (Π)

Proposition 4.2.5. *Let $\mathcal{P}$ be a grid polyomino. For all $v \in V(\mathcal{P})$, $\mathcal{M}$ forms a reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to either $<_v^1$ or $<_v^2$.*

Proof. Let $\mathcal{P}$ be a grid polyomino. We observe that $\mathcal{M}$ forms a Gröbner basis of $I_{\mathcal{P}}$ with respect to $<_{\text{lex}}^1$ or $<_{\text{lex}}^2$, since $\mathcal{P}$ satisfies the conditions of Proposition 4.2.1. Let $f, g \in \mathcal{M}$, where $f = x_a x_b - x_c x_d$ is associated to the inner interval $[a, b]$ of $\mathcal{P}$ and $g = x_p x_q - x_r x_s$ is associated to the inner interval $[p, q]$ of $\mathcal{P}$. Let $v \in V(\mathcal{P})$. We have to show that for each pair of inner 2-minors, f and g, the corresponding S-polynomial reduces to 0 with respect to a fixed monomial order $<_v \in \{<_v^1, <_v^2\}$. In the following, we denote by S the S-polynomial between f and g, by $\text{in}(h)$ the leading monomial of a polynomial h with respect to $<_v$, and by $f_{m,n}$ the inner 2-minor associated to the inner interval $[m, n]$ of $\mathcal{P}$.

We leave to the reader the trivial cases $\{a, b, c, d\} \cap \{p, q, r, s\} = \varnothing$, and $|\{a, b, c, d\} \cap \{p, q, r, s\}| = 2$ where S reduces to 0 since the polyomino ideal is generated by all inner 2-minors of $\mathcal{P}$.

Note that if for all vertices $w \in \{a,b,c,d,p,q,r,s\}$ and a monomial order $< \in \{<^1_{\text{lex}}, <^2_{\text{lex}}\}$, it holds $x_w <_v x_v$ or $x_v <_v x_w$, then S reduces to 0 with respect to $<_v$, since it reduces by 0 with respect to $<$. Therefore, we consider the cases

$$u <^1 v \leq^1 w \text{ for } u, w \in \{a,b,c,d,p,q,r,s\} \text{ with } |\{a,b,c,d\} \cap \{p,q,r,s\}| = 1.$$

If one of the inner intervals, namely $[a,b]$, is contained in the second one, namely $[p,q]$, S reduces to 0 since the polyomino ideal is generated by all inner 2-minors of $\mathcal{P}$. Without loss of generality, let $a \leq p$. The possible situations are:

$$a = p, \qquad b, d \in \{p,q,r,s\}, \qquad c \in \{p,r\}.$$

If v does not satisfy the condition (Π), we fix the monomial order $<^1_v$. Otherwise, we fix $<^2_v$. Assume v does not satisfy (Π). In the following cases, denote by $<$ the total order $<^1$ on the vertices of $\mathcal{P}$.

Let $a = p$, that is $f = x_a x_b - x_c x_d$ and $g = x_a x_q - x_r x_s$, and assume $a < r < c < d < b < s < q$ as in Figure 4.9.

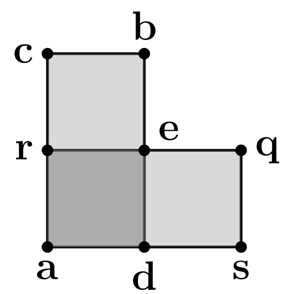


Figure 4.9: Case $a = p$.

We start observing that if $r < v \leq q$, then $\gcd(\text{in}(f), \text{in}(g)) = 1$. If $v \in \{a, r\}$, then $S = x_r x_s x_b - x_c x_d x_q$ and $\text{in}(S) = x_q x_c x_d$. Therefore,

$$S = -x_c(x_q x_d - x_e x_s) + x_s(x_b x_r - x_c x_e),$$

that is S reduces to 0 with respect to $\mathcal{M}$.

Let $b = p$, then $a < c < d < b < r < s < q$, as in Figure 4.10.

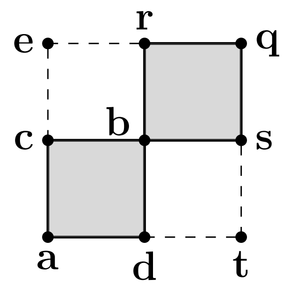


Figure 4.10: Case $b = p$.

If either $c < v \leq b$ or $r < v \leq q$, then $\gcd(\mathrm{in}(f), \mathrm{in}(g)) = 1$. In the other cases, namely $a \leq v \leq c$ and $b < v \leq r$, we have $S = x_a x_r x_s - x_q x_c x_d$. If $v = a$, then $\mathrm{in}(S) = x_q x_c x_d$. By hypothesis, there exist the inner interval $[c, q]$ or $[d, q]$, with $\mathrm{in}(f_{c,q}) = x_c x_q$ and $\mathrm{in}(f_{d,q}) = x_d x_q$, and then

$$S = -x_d(x_c x_q - x_s x_e) + x_s(x_a x_r - x_d x_e)$$

or

$$S = -x_c(x_d x_q - x_r x_t) + x_r(x_a x_s - x_c x_t),$$

that is S reduces to 0 with respect to the inner 2-minors $f_{c,q}$ and $f_{a,r}$ or $f_{d,q}$ and $f_{a,s}$. If $a < v \leq c$, then $\mathrm{in}(S) = x_a x_r x_s$. By hypothesis, there exists the inner interval $[a, r]$ or $[a, s]$, with $\mathrm{in}(f_{a,r}) = x_a x_r$ and $\mathrm{in}(f_{a,s}) = x_a x_s$. Similarly, one shows that S reduces to 0. If $b < v \leq r$, since v does not satisfy the condition (Π), $[d, s]$ is an inner interval of $\mathcal{P}$, whereas $[c, v]$ is not. Therefore, $[d, q]$ is an inner interval of $\mathcal{P}$. Since $\mathrm{in}(S) = x_q x_c x_d$, and

$$S = x_c(x_d x_q - x_r x_t) - x_r(-x_c x_t + x_a x_s)$$

it follows that S reduces to 0.

Note that when $b < v \leq r$, if there exists the inner interval $[c, v]$ but $[d, s]$ does not, then $v = r$, since $\mathcal{P}$ is a grid polyomino. Therefore, v satisfies condition (Π) and S does not reduce to 0 with respect to $\mathcal{M}$ and $<^1_v$. In fact, $\mathrm{in}(S) = x_q x_c x_d$, but the monomials $x_c x_d$, $x_c x_q$, and $x_d x_q$ are not leading monomials of any inner 2-minors of $\mathcal{P}$. This situation justifies the hypothesis v not satisfying the condition (Π), and in particular the case (I) in Definition 4.2.2.

Let $b = r$. We have to distinguish two different situations: $d < p$ (see Figure 4.11(A)) or $p < d$ (see Figure 4.11(B)).

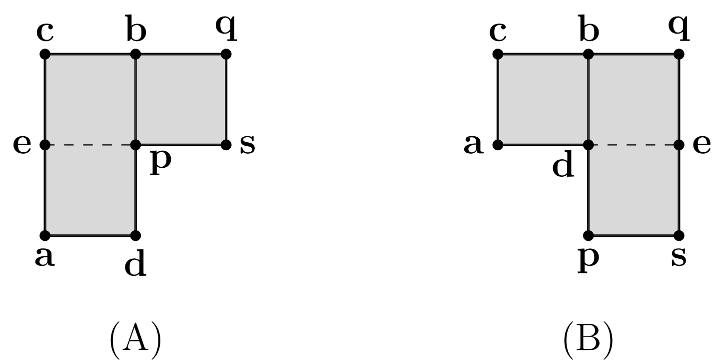


Figure 4.11: Case $b = r$.

Assume $d < p$. If $a \leq v \leq b$, then $\gcd(\text{in}(f), \text{in}(g)) = 1$. In the other cases, namely $b < v \leq q$, $S = x_a x_p x_q - x_c x_d x_s$. When $b < v \leq s$, $\text{in}(S) = x_a x_p x_q$, whereas, when $s < v \leq q$, that is $v = q$, $\text{in}(S) = x_c x_d x_s$. We consider the inner intervals $[e, q]$ and $[a, p]$. In both cases, $\text{in}(f_{e,q}) = x_c x_s$ and $\text{in}(f_{a,p}) = x_a x_p$, and we have

$$S = x_d(-x_c x_s + x_q x_e) + x_q(x_a x_p - x_e x_d),$$

that is S reduces to 0.

Assume $p < d$. If $a \leq v \leq b$, then $\gcd(\text{in}(f), \text{in}(g)) = 1$. Otherwise, $S = x_a x_p x_q - x_c x_d x_s$. Let $b < v \leq q$. First of all, note that since $\mathcal{P}$ is a grid polyomino, and since v does not satisfy (Π), by hypothesis, then the interval with anti-diagonal corners a and p is not an inner interval of $\mathcal{P}$ and $v \neq q$. Therefore, let $b < v < q$, and, in particular, $b < v \leq e$. In this case, $\text{in}(S) = x_c x_d x_s$. Since $\text{in}(f_{p,e}) = x_d x_s$, we have

$$S = x_c(x_p x_e - x_d x_s) + x_p(x_a x_q - x_c x_e),$$

that S reduces to 0.

Note that $v = q$ satisfies condition (Π) and S does not reduces to 0, since neither $x_c x_d$, nor $x_c x_s$, nor $x_d x_s$ are leading monomials of any inner 2-minor of $\mathcal{P}$. This situation justifies the hypothesis v not satisfying condition (Π), and in particular the case (II) in Definition 4.2.2.

Let $d = q$, with $a < c < p < r < s < d < b$ (see Figure 4.12). If either $a \le v \le c$ or $r < v \le d$, then $\gcd(\text{in}(f), \text{in}(g)) = 1$. Otherwise, $S = x_a x_b x_p - x_c x_r x_s$. If $c < v \le p$, then $\text{in}(S) = x_c x_r x_s$.

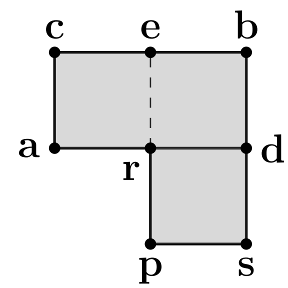


Figure 4.12: Case $d = q$.

Since $\text{in}(f_{a,e}) = x_c x_r$, we have

$$S = x_s(x_a x_e - x_c x_r) + x_a(x_p x_b - x_e x_s),$$

that is S reduces to 0. If $p < v \le r$, then $\text{in}(S) = x_a x_b x_p$. Since $\text{in}(f_{p,b}) = x_p x_b$, we have

$$S = x_a(x_p x_b - x_e x_s) - x_s(x_a x_e - x_c x_r),$$

that is S reduces to 0. Let $d < v \le b$, that is $v = b$. Since b satisfies the condition (Π), we do not consider this case.

Let $c = r$, with $a < p < c < d < b < s < q$ (see Figure 4.13). If either $a \le v \le c$ or $b < v \le q$, then $\gcd(\text{in}(f), \text{in}(g)) = 1$. Otherwise, namely $c < v \le b$, we have $S = x_a x_b x_s - x_d x_p x_q$ and $\text{in}(S) = x_d x_p x_q$.

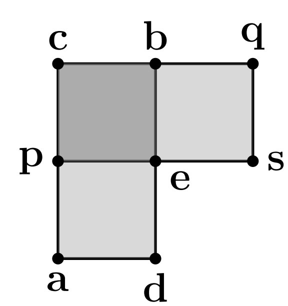


Figure 4.13: Case $c = r$.

Due to v does not satisfy the condition (Π), $v \neq b$, that is $c < v \le e$. Since $a < p < c$, then

$\mathrm{in}(f_{a,e}) = x_p x_d$, we have

$$S = x_q(x_a x_e - x_p x_d) - x_a(x_e x_q - x_b x_s),$$

We leave to the reader to check, in a similar way, that if $b \in \{q, s\}$, $d \in \{p, r, s\}$, and $c = p$, then all the S-polynomials reduce to 0, and no one of the corners in these cases satisfy the condition (Π).

Note that the orders $<^1_{\mathrm{lex}}$ and $<^2_{\mathrm{lex}}$ are symmetric, i.e. if $[a, b]$ is an inner interval of $\mathcal{P}$ with anti-diagonal corners c and d, then $\mathrm{in}_{<^1_{\mathrm{lex}}}(f_{a,b}) = x_a x_b$ and $\mathrm{in}_{<^2_{\mathrm{lex}}}(f_{a,b}) = x_c x_d$. This property reflects naturally on the orders $<^1_v$ and $<^2_v$. Hence, it is possible to verify that all S-polynomials of inner 2-minors of $\mathcal{P}$ reduce to 0 with respect to $\mathcal{M}$ and $<^2_v$, except for these four cases:

A) when $b = q$ and $v = r$ (see Figure 4.14(A)), analogous to the case $c = r$ and $v = b$ treated above;

B) when $b = r$ and $v = c$ (see Figure 4.14(B)) analogous to the case $b = r$ and $v = q$ treated above;

C) when $c = p$ and $v = r$ (see Figure 4.14(C)) analogous to the case $d = q$ and $v = b$ treated above;

D) when $d = r$ and $v = b$, and the interval of $\mathcal{P}$ with anti-diagonal corners v, q is an inner interval of $\mathcal{P}$, but the one with anti-diagonal corners a and p is not (see Figure 4.14(D)), analogous to the case $b = p$ and $v = r$ treated above.

Since $\mathcal{P}$ is a grid polyomino, it does not happen simultaneously that v satisfies the condition (Π) and one of the above situations A)–D). This implies that if v satisfies the condition (Π) and we fix the monomial order $<^2_v$, all S-polynomials reduce to 0 and $\mathcal{M}$ is a reduced Gröbner basis of $I_{\mathcal{P}}$.

□

Let I_Λ be the lattice ideal defined in Subsection 2.3.1.

Lemma 4.2.6. *Let $\mathcal{P}$ be a collection of cells of $\mathbb{N}^2$, let S be the polynomial ring associated to $\mathcal{P}$. Then, there exists a monomial $u \in S$ such that*

$$I_\Lambda = (I_{\mathcal{P}} : u).$$

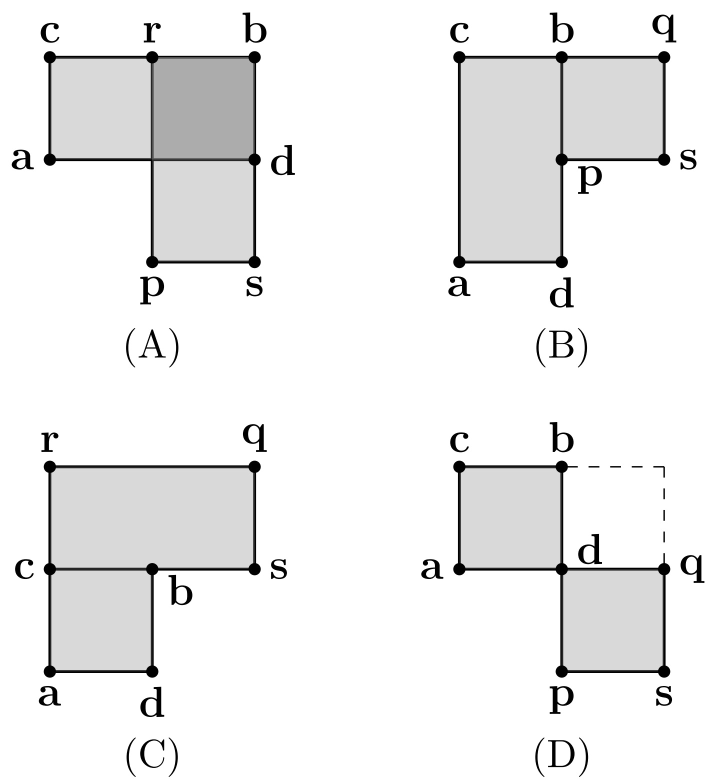


Figure 4.14

Proof. $\supseteq$). It holds for any monomial $u \in S$, since $I_{\mathcal{P}} \subseteq I_\Lambda$ and I_Λ is a prime ideal.
$\subseteq$). Let $f_E = x^{E^+} - x^{E^-}$ be a generator of I_Λ, with

$$E = E^+ - E^- = \sum_{k=1}^{r} \lambda_k c_k = \sum_{k=1}^{r} \Big((\lambda_k c_k)^+ - (\lambda_k c_k)^- \Big) \in \Lambda,$$

where v^+ denotes the vector obtained from $v \in \mathbb{Z}^{m \times n}$ by replacing all negative components of v by zero, and $v^- = -(v - v^+)$.

Let $\lambda = \sum_{k=1}^{r} (\lambda_k c_k)^+ - E^+ = \sum_{k=1}^{r} (\lambda_k c_k)^- - E^-$. We have that all the components of λ are nonnegative, as for any $k \in \{1, \dots, r\}$ one has $(c_k^+)_{ij} \geq (c_k)_{ij}$, for all $1 \leq i \leq m$ and $1 \leq j \leq n$. This implies that the monomial $x^\lambda \in S$ is such that

$$x^\lambda (x^{E^+} - x^{E^-}) = \prod x^{(\lambda_k c_k)^+} - \prod x^{(\lambda_k c_k)^-} = \sum_{k=1}^{r} \mu_k (x^{c_k^+} - x^{c_k^-}) \in I_{\mathcal{P}}.$$

If we set u as the least common multiple of the elements x^λ induced by all the generators f_E of I_Λ the assertion follows. $\square$

An immediate consequence is the following

Corollary 4.2.7. *Let $\mathcal{P}$ be a polyomino. Then $I_\Lambda \subseteq J_\mathcal{P}$.*

Proof. Since $J_\mathcal{P}$ is a prime ideal and $I_\mathcal{P} \subseteq J_\mathcal{P}$, then for any monomial $u \in S$, we have

$$(I_\mathcal{P} : u) \subseteq J_P.$$

From Lemma 4.2.6, the assertion follows. □

We do not know anything about the inclusion $J_\mathcal{P} \subseteq I_\Lambda$, that leads to the following

Question 4.2.8 *Let $\mathcal{P}$ be a polyomino. $I_\mathcal{P}$ is prime if and only if $I_\mathcal{P} = J_\mathcal{P}$?*

We now prove the main theorem of this section.

Theorem 4.2.9. *Let $\mathcal{P}$ be a grid polyomino. Then $I_\mathcal{P} = I_\Lambda$.*

Proof. By Proposition 4.2.5, for all $v \in V(\mathcal{P})$, there exists a monomial order $<_v$ such that x_v is the smallest variable with respect to $<_v$ and $\mathcal{M}$ forms a reduced Gröbner basis of $I_\mathcal{P}$ with respect to $<_v$. Fix $v \in V(\mathcal{P})$. By [67, Lemma 12.1], the reduced Gröbner basis of $(I_\mathcal{P} : x_v)$ with respect to $<_v$ is given by

$$\{f \in \mathcal{M} \mid x_v \text{ does not divide } f\} \cup \{f/x_v \mid f \in \mathcal{M} \text{ and } x_v \text{ divides } f\}.$$

Since no $f \in \mathcal{M}$ can be divided by x_v, the reduced Gröbner basis of $(I_\mathcal{P} : x_v)$ with respect to $<_v$ is $\mathcal{M}$. Therefore $(I_\mathcal{P} : x_v) = I_\mathcal{P}$, for all $x_v \in V(\mathcal{P})$. It follows that $(I_\mathcal{P} : u) = I_\mathcal{P}$ for any monomial $u \in S$. By Lemma 4.2.6, we have that there exists a monomial $u \in S$ such that $I_\Lambda = (I_\mathcal{P} : u)$. Then

$$I_\Lambda = (I_\mathcal{P} : u) = I_\mathcal{P}.$$

□

Corollary 4.2.10. *Let $\mathcal{P}$ be a grid polyomino. Then $I_\mathcal{P}$ is prime.*

From the main results of this chapter, namely Corollary 4.1.6, Theorem 4.1.9 and Corollary 4.2.10, it arises naturally the following:

Conjecture 4.2.11 *Let $\mathcal{P}$ be a polyomino. The following conditions are equivalent:*

(i) *the polyomino ideal $I_{\mathcal{P}}$ is prime;*

(ii) *$\mathcal{P}$ contains no zig-zag walk.*

CONCLUSION

In this thesis, we have studied two classes of ideal generated by 2-minors, the binomial edge ideals and the polyomino ideals. As showed, many results on this topic have been obtained. Nevertheless, there are several open problems which we have mentioned regarding their algebraic and homological characterization, such as a precise formula for the regularity of the binomial edge ideals or a complete classification of the Cohen-Macaulay ones, and a complete classification of the polyominoes $\mathcal{P}$ such that $\mathbb{K}[\mathcal{P}]$ is a domain.

The study of ideals generated by a subset of k-minors of an $m \times n$-matrix of indeterminates is itself a fascinating area of research. A lot of works devoted to describe the graded minimal free resolution and the relation with Eagon-Northcott resolution of some families of such ideals have been produced. But, not all of these families have been investigated yet. It could be interesting to research in this direction in further works.

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