University of Trento

Doctoral Thesis

Feynman path integral for Schrödinger equation with magnetic field

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A thesis submitted in fulfillment of the requirements for the degree of Doctor of Philosophy

in the

Department of Mathematics

Thirty-one years ago, Dick Feynman told me about his "sum over histories" version of quantum mechanics. "The electron does anything it likes," he said. "It just goes in any direction at any speed, forward or backward in time, however it likes, and then you add up the amplitudes and it gives you the wave-function." I said to him, "You're crazy." But he wasn't.

Freeman John Dyson

UNIVERSITY OF TRENTO

Abstract

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Feynman path integrals introduced heuristically in the 1940s are a powerful tool used in many areas of physics, but also an intriguing mathematical challenge. In this work we used techniques of infinite dimensional integration (i.e. the infinite dimensional oscillatory integrals) in two different, but strictly connected, directions. On the one hand we construct a functional integral representation for solutions of a general high-order heat-type equations exploiting a recent generalization of infinite dimensional Fresnel integrals; in this framework we prove a a Girsanov-type formula, which is related, in the case of Schrödinger equation, to the Feynman path integral representation for the solution in presence of a magnetic field; eventually a new phase space path integral solution for higher-order heat-type equations is also presented. On the other hand for the three dimensional Schrödinger equation with magnetic field we provide a rigorous mathematical Feynman path integral formula still in the context of infinite dimensional oscillatory integrals; moreover, the requirement of independence of the integral on the approximation procedure forces the introduction of a counterterm, which has to be added to the classical action functional (this is done by the example of a linear vector potential). Thanks to that, it is possible to give a natural explanation for the appearance of the Stratonovich integral in the path integral formula for both the Schrödinger and the heat equation with magnetic field.

Acknowledgements

I am really grateful to Sonia Mazzucchi for her wise guidance in these three years, but also for her ever present supervision and patience.

Moreover I would like to thank all the professors, who in these years gave a contribution to this work with valuable advices and useful comments, but even with a simple conversation. In particular I'm grating to Sergio Albeverio for the fundamental explanations and the interesting insights about the oscillatory integrals approach.

I am also particularly grateful to my colleagues for the fruitful discussions, which sometimes have led to comical mathematical fights.

I would like to extend a warm thanks to Daniele Taufer for his friendship, for the nights spent together, but also (and maybe especially) for the hated mornings, for *the champions* and in some sense also for the *vegan burgers*.

I didn't forget about Mattia Fogagnolo and his incredible ability to make things harder than already are. Thank you very much.

A special thought is due to Trenitalia for its impeccable service in terms of extreme professionalism and punctuality. It's a joke. I'm joking.

I would like to thank all my friends and my family, who were with me throughout all my years of study.

Finally, I should like to express my heartfelt thanks to my dear love Fernanda Marchiol for her (very huge) tolerance, support, and constant encouragement.

Contents

Introduction

It's many years since R. Feynman introduced in his pioneering paper [40] the concept of *path integral* as different approach to quantum mechanics, and they are still an important challenge in theoretical physics as much as in mathematics. Just to give a brief historical introduction we quote above the two postulates that Feynman gives in [40] to summarize his idea.

> *If an ideal measurement is performed, to determine whether a particle has a path lying in a region of space-time, then the probability that the result will be affirmative is the absolute square of a sum of complex contributions, one from each path in the region.*

With this first postulate Feynman defines, somehow, the space of paths and the rule to determine the probability to find an hypothetical particle in a region of the spacetime. As Feynman himslef noticed, the postulate lacks of a precise mathematical meaning as we shall deepen in the following. With the second postulate, he explains the nature of the contribution of each path relating it with the classical action.

> *The paths contribute equally in magnitude. but the phase of their contribution is the classical action (in units of $\hbar$); i.e., the time integral of the Lagrangian taken along the path.*

These axioms led to the Feynman's heuristic formula

$$\psi(t,x) = C^{-1} \int_\Gamma e^{\frac{i}{\hbar}S(\gamma)} \psi_0(0,\gamma(0)) d\gamma \tag{1}$$

for the solution to the time dependent Schrödinger equation (we fix $m = 1$)

$$i\frac{\partial}{\partial t}\psi(t,x) = -\frac{1}{2}\Delta\psi(t,x) + V(x)\psi(t,x), \qquad x \in \mathbb{R}^d, t \in \mathbb{R}, \tag{2}$$

describing the time evolution of the state $\psi \in L^2(\mathbb{R}^d)$ of a non-relativistic quantum particle moving in the d-dimensional Euclidean space under the action of the force field associated to a real valued potential V. According to Feynman's postulates, the state of the particle should be given by an heuristic integral of the form (1) on the space Γ of continuous paths $\gamma : [0,t] \to \mathbb{R}^d$ with fixed end point $\gamma(t) = x$. The integrand in (1), namely the function

$$S(\gamma) = \int \mathcal{L}(\gamma(\tau), \dot{\gamma}(\tau))d\tau = \int_0^t \left(\frac{|\dot{\gamma}(\tau)|^2}{2} - V(\gamma(\tau)) \right) d\tau,$$

is the classical action functional evaluated along the path γ, where $\mathcal{L}$ denotes the Lagrangian. Here, $\dot{\gamma}(\tau)$ is the derivative of γ at τ, and $|\cdot|$ is the norm in $\mathbb{R}^d$. The symbol $d\gamma$ stands for a heuristic Lebesgue-type measure on Γ and

$$C = \int_\Gamma e^{\frac{i}{2\hbar}\int_0^t |\dot{\gamma}(s)|^2 ds} d\gamma,$$

plays the role of a normalization constant. Formula (1), as it stands, lacks of a well defined mathematical meaning, however it has been widely applied in many areas of quantum physics, providing in fact a quantization procedure and allowing, at least heuristically, to associate a quantum dynamics to any classical Lagrangian. Feynman himself was aware of the lack of a sound mathematical theory for its formula as he says in a comment of his thesis [24].

> *There are very interesting mathematical problems involved in the attempt to avoid the subdivision and limiting processes. Some sort of complex measure is being associated with the space of functions $x(t)$. Finite results can be obtained under unexpected circumstances because the measure is not positive everywhere, but the contributions from most of the paths largely cancel out. These curious mathematical problems are sidestepped by the subdivision process. However, one feels as Cavalieri must have felt calculating the volume of a pyramid before the invention of calculus.*

In fact, neither the normalization constant C, nor the Lebesgue type measure $d\gamma$ are well defined.

In the physical literature, in most cases Feynman's formula is interpreted as the limit of a finite dimensional approximation procedure. Indeed, if we divide the time interval $[0, t]$ into n equal parts of amplitude t/n, and if for any path $\gamma : [0, t] \to \mathbb{R}$ we consider its approximation by means of a broken line path γ_n in $\mathbb{R}^d$ defined as:

$$\gamma_n(s) := x_j + \frac{(x_{j+1} - x_j)}{t/n}(s - jt/n), \quad s \in \left[\frac{jt}{n}, \frac{(j+1)t}{n}\right],$$

where $x_j := \gamma(jt/n)$ and $j = 0, \ldots, n-1$, formula (1) can be interpreted as the limit for $n \to \infty$ of the following approximation:

$$(2\pi\hbar i)^{-nd/2} \int_{\mathbb{R}^{nd}} \psi_0(x_0) e^{\frac{i}{\hbar}\sum_{j=0}^{n-1} \frac{|x_{j+1}-x_j|^2}{2t/n} - \frac{i}{\hbar}\sum_{j=1}^{n} V(x_j)\frac{t}{n}} dx_0 \ldots dx_{n-1}. \tag{3}$$

Indeed, under rather general assumption on the potential V and the initial datum ψ_0, the limit for $n \to \infty$ of the sequence of finite dimensional integrals (3) converges to the solution of (2) (see, e.g., [89, 90, 105, 68, 111, 62] for a discussion of this approach).

A rigorous mathematical construction of an integration theory allowing to realize Feynman's formula in terms of a Lebesgue integral with respect to a well defined (σ-additive) complex measure on the infinite dimensional space of paths Γ, presents severe problems and it is in fact in most cases impossible (see, for instance, [25, 109, 12, 83, 53] for a discussion of this issue). Several approaches have been proposed, relying, e.g., on analytic continuation of Wiener integrals [25, 35], or on an infinite dimensional distribution theory [27, 57], or on suitable approximation procedures [3, 2, 38, 44, 75, 89, 112, 45, 72]. The same techniques have also been applied to the construction of generalized Feynman-Kac formulae for higher-order heat type equations of (which we see below), see, e.g. [19, 60, 73, 59, 47, 81, 78, 76, 22, 21, 99]. We shall focus on the *infinite dimensional oscillatory integral approach*, originally proposed by K. Itô in the 1960s [66, 65] and further developed by S. Albeverio and R. Høegh-Krohn [9, 10] in the 1970s. The main idea is the generalization of the classical theory of oscillatory integrals on finite dimensional vector space due to L. Hörmander [60] and J. J. Duistermaat [36] to the case where the integration is performed on an infinite dimensional real separable Hilbert space [38, 85]. It is important to point out that this approach allows for a systematic implementation of an infinite dimensional version of the stationary phase method and the corresponding application to the study of the semiclassical limit of quantum mechanics [9, 102], that is the analysis of the detailed asymptotic behavior of the solution of the Schrödinger equation when the Planck constant $\hbar$ is regarded mathematically as a small parameter allowed to converge to 0.

The main aim of this work regards the Schrödinger equation for a non-relativistic quantum particle moving under the influence of a magnetic field $\mathbf{B}$ associated to a vector potential $\mathbf{a}$

$$i\hbar\frac{\partial}{\partial t}\psi(t,x) = \frac{1}{2}\left(-i\hbar\nabla - \lambda\mathbf{a}(x)\right)^2\psi(t,x),$$

where $\mathbf{a}(x) \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}$ plays the role of a coupling constant. As we shall deepen in Chpt. 3, the above mentioned theory of infinite dimensional oscillatory integrals is used to construct a rigorous mathematical definition for the corresponding Feynman path integral formula

$$\psi(t,x) = \int_{\gamma(t)=x} e^{\frac{i}{2\hbar}\int_0^t|\dot\gamma(s)|^2ds + \frac{i}{\hbar}\int_0^t \lambda\mathbf{a}(\gamma(s))\cdot\dot\gamma(s)ds}\psi_0(\gamma(0))d\gamma.$$

Let us dwell now on the relation between probability theory, in particular the theory of mathematical stochastic processes, and the study of partial differential equations, which has a fundamental role in the thesis. The most is provided by the

Feynman-Kac formula (4), which gives a representation of the solution of the heat equation with (real-valued) potential V

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\Delta u(t,x) - V(x)u(t,x), & x \in \mathbb{R}^d, t \in [0,+\infty) \\ u(0,x) = u_0(x) \end{cases}$$

for a suitable initial condition u_0, in terms of an integral of the form

$$u(t,x) = \int_{C_t} e^{-\int_0^t V(\omega(s)+x)ds} u_0(\omega(t)+x) d\mathbb{P}(\omega), \tag{4}$$

where $\mathbb{P}$ is the Wiener probability measure on the Borel σ-algebra in the Banach space C_t of continuous paths $\omega : [0,t] \to \mathbb{R}^d$ starting at the origin, endowed with the sup-norm. The process X with paths ω and distribution $\mathbb{P}$ is, as well known, the realization of the (standard) Wiener process (also called mathematical Brownian motion), see, e.g., [70]. In fact, the connection between heat equation and Wiener process is just a particular case of a general theory connecting Markov processes with parabolic equations associated to second order elliptic operators (see, e.g., [34, 42]). This rich and extensively developed theory does not apply to more general PDEs, in particular those which do not satisfy a maximum principle such as, for instance, the Schrödinger equation (2).

However, another interesting example, outside the proper range of the mentioned theory of Markov process, is provided by heat-type equations associated to higher-order differential operators, such as, for instance, the parabolic equation associated to the bilaplacian:

$$\frac{\partial}{\partial t}u(t,x) = -\Delta^2 u(t) - V(x)u(t,x), \tag{5}$$

or, more generally, p-order equations, which we shall study in Chpt 2, of the form:

$$\frac{\partial}{\partial t}u(t,x) = (-i)^p \alpha \frac{\partial^p}{\partial x^p} u(t,x) + V(x)u(t,x), \quad t \in [0,+\infty),\ x \in \mathbb{R}, \tag{6}$$

where $p \in \mathbb{N}$, $p > 2$, $\alpha \in \mathbb{C}$ is a complex constant and $V : \mathbb{R} \to \mathbb{C}$ is a continuous bounded function.

The main reason for the impossibility to construct an analogue of formula (4) for equations (2) and (6) relies in the lack of a maximum principle for the solution of these equations. This fact actually forbids a probabilistic representation formula for the solution of the form

$$\begin{aligned} u(t,x) &= \mathbb{E}^x[e^{-\int_0^t V(X(s))ds} u_0(X(t))] \\ &= \int_\Omega e^{-\int_0^t V(\omega(s))ds} u_0(\omega(t)) dP(\omega), \end{aligned} \tag{7}$$

given in terms of the expectation with respect to a probability measure P on $\Omega :=$

$\mathbb{R}^{[0,t]}$, the probability distribution of a stochastic $X(s)$ (with real valued paths). Indeed, a representation formula of the form (7) would imply a maximum property for the solutions (at least for $V = 0$), which does not hold for Eq. (2) and Eq. (6).

A proof of Feynman-Kac formula (4), as well as a derivation of Feynman heuristic path integral representation seen above can be obtained by an argument relying on the Trotter product formula[1]. We present it shortly here below in a general case which can cover also the case of high order heat-type equations, with the aim of pointing out why the arguments which work in the case of the heat equation fail in the case of equations (2) (and also in the case of higher order heat-type equations). In the following, for notational simplicity, we limit our considerations to the case where $d = 1$.

Let us consider the evolution semigroup $T_t : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ (with $L^2(\mathbb{R})$ the Hilbert space of Lebesgue square integrable complex-valued functions) generated by an operator $A : D(A) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ given on C_0^∞ functions $u \in L^2(\mathbb{R})$ by

$$Au(x) := \alpha(-i)^p \frac{d^p}{dx^p} u(x), \qquad \alpha \in \mathbb{C}, p \in \mathbb{N}, x \in \mathbb{R}, \tag{8}$$

where $\alpha \in \mathbb{C}$ satisfies the condition $Re\,(\alpha y^p) \leq 0$ for all $y \in \mathbb{R}$. Let $K_t(\,\cdot\,,\,\cdot\,)$, $t \geq 0$, denote the kernel of T_t, namely:

$$T_t u(x) = \int_{\mathbb{R}} K_t(x, y) u(y) dy. \tag{9}$$

In fact K_t, for $t > 0$, has the form

$$K_t(x, y) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\kappa(x-y)} e^{\alpha \kappa^p t} d\kappa.$$

In particular, if $p = 2$ and $\alpha = -\frac{1}{2}$, K_t is the fundamental solution of the heat equation:

$$K_t(x, y) = (2\pi t)^{-1/2} \exp\left(-\frac{(x-y)^2}{2t}\right), \qquad t > 0,$$

while if $p = 2$ and $\alpha = -\frac{i\hbar}{2}$, K_t is the fundamental solution of the Schrödinger equation:

$$K_t(x, y) = (2\pi i\hbar t)^{-1/2} \exp\left(i\frac{(x-y)^2}{2\hbar t}\right), \qquad t > 0.$$

[1]It is curious to remark that M. Kac obained is celebrated formula after being isepired by a lecture of R. P. Feynman at Cornell University [69].

Furthermore, by the semigroup property of T_t the Chapman-Kolmogorov equation follows:

$$\int_{\mathbb{R}} K_t(x,y)K_s(y,z)dy = K_{t+s}(x,z),$$

and vice versa. Given a continuous bounded function $V : \mathbb{R} \to \mathbb{R}$, let us also denote by V the associated linear multiplication operator acting on $L^2(\mathbb{R})$ defined on the vectors $u \in C_0^\infty(\mathbb{R})$ by $(Vu)(x) = V(x)u(x)$. Let $\beta \in \mathbb{C}$ and let A as in (8). Consider the operator sum $A + \beta V : D(A) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$. Then let $T_V(t) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the associated semigroup, written formally as $T_V(t) = e^{(A+\beta V)t}$. By the Trotter product formula [110][2], the perturbed semigroup is given by the strong $L^2(\mathbb{R})$-limit

$$e^{(A+\beta V)t}u = \lim_{n\to\infty} \left(e^{A\frac{t}{n}}e^{\beta V\frac{t}{n}}\right)^n u, \qquad u \in C_0^\infty(\mathbb{R}).$$

By taking a subsequence and using (9), the evaluation of both sides at almost every $x \in \mathbb{R}$ yields

$$\begin{aligned} e^{(A+\beta V)t}u(x) &= \lim_{n\to\infty} \left(e^{A\frac{t}{n}}e^{\beta V\frac{t}{n}}\right)^n u(x) \\ &= \lim_{n\to\infty} \int_{\mathbb{R}^n} u(x_0)e^{\beta\frac{t}{n}\sum_{j=1}^n V(x_j)} \prod_{j=0}^{n-1} K_{t/n}(x_{j+1},x_j)dx_j, \end{aligned} \tag{10}$$

where $x_n \equiv x$. In the case where T_t is the heat semigroup and its kernel $K_t(\,\cdot\,,\,\cdot\,)$ is, for $t > 0$, the density of a Gaussian probability measure on $\mathbb{R}$, the limit of the sequence of finite dimensional integrals appearing in Eq. (10) can be interpreted as an integral with respect to the Wiener measure, obtaining, when β is a real negative constant, Eq. (4). In the case of Schrödinger equation (2) or the high order heat type equation (6), the kernel $K_t(\,\cdot\,,\,\cdot\,)$ is no longer real and positive and cannot be interpreted as the density of a probability measure. In particular the Green function $K_t(x,y)$ of the Schrödinger equation is complex, while for the higher-order heat-type equation (5) $K_t(x,y)$ is real and attains both positive and negative values [59]. As a troublesome consequence, the complex (resp. signed) finitely-additive measure μ on $\Omega = \mathbb{R}^{[0,t]}$ defined on the algebra of "cylinder sets"$C \subset \Omega$ of the form

$$C := \{\omega \in \Omega : \omega(t_j) \in I_j, j = 1,\dots k\},$$

for some $k \in \mathbb{N}$, $0 = t_0 < t_1 < \cdots < t_k$ and $I_1,\dots,I_k$ intervals in $\mathbb{R}$, by

$$\mu(C) = \int_{I_1} \cdots \int_{I_k} \prod_{j=0}^{k-1} K_{t_{j+1}-t_j}(x_{j+1},x_j)dx_1\dots dx_k,$$

[2] For generalizations of Trotter product formula, see also, e.g., [89, 105, 88].

does not extend to a corresponding σ-additive measure on the generated σ-algebra. As a matter of fact, if this measure existed, it would have infinite total variation even in "bounded nice regions" [25, 73, 109, 12].

This problem was addressed in 1960 by R. Cameron [25] for the Schrödinger equation and by V. Krylov [73] for Eq. (5). Their results are in fact particular cases of a general theorem later established by E. Thomas [109], which generalizes Kolmogorov existence theorem to the case of projective systems of signed or complex measures, instead of probability ones. These no-go results forbid a functional integral representation of the solution of Eq. (2) or Eq. (5) in terms of a Lebesgue-type (infinite dimensional) integral with respect to a σ-additive complex or signed measure with finite total variation.

The thesis is organized as follows. In Chapter 1 we give an overview about infinite dimensional oscillatory integration with some classical results. In Chapter 2 we apply the techniques of infinite dimensional integration to the construction of a functional integral representation of a general class of high-order heat-type equation. In Chapter 3 we give a rigorous formulation (in terms of infinite dimensional oscillatory integrals) of a Feynman path integral for the Schr̆odinger equation with magnetic field; moreover we prove that the requirement of the independence of the integral on the approximation procedure requires the introduction of a counterterm, which has to be added to the classical action functional. In Chapter 3.3 we conclude with possible future developments and generalizations. The three appendices concern respectively a brief background on the abstract Wiener spaces (A), the Ogawa integral with a generalization on the multidimensional case (B) and the proof on a important stochastic lemma used in Chapter 3 (C).

Chapter 1

General Framework: Infinite Dimensional Fresnel Integrals

In this chapter we introduce a characterization of the infinite dimensional oscillatory integrals. In finite dimension, we consider an object of the following form

$$\int_{\mathbb{R}^n} e^{\frac{i}{\epsilon}\Phi(x)} f(x)dx, \tag{1.1}$$

where $\epsilon \in \mathbb{R} \setminus \{0\}$ is a real parameter, $f : \mathbb{R}^n \to \mathbb{C}$ and $\Phi : \mathbb{R}^n \to \mathbb{R}$ are Borel functions. There are many examples of this kind of integral as the Airy's integrals in the theroy of rainbows [1] and the Fresnel integral introduced in optics and in the theory of wave diffraction [43]. The latter, namely the classical Fresnel integrals have a quadratic form as a phase function Φ: in the simplest case $\Phi(x) = \|x\|^2$, where $\|x\|^2 = \langle x, x\rangle$. This case gives rise to an integral of the form

$$\int_{\mathbb{R}^n} e^{\frac{i}{\epsilon}\|x\|^2} f(x)dx. \tag{1.2}$$

For our purpose we outline that the Fresnel integrals are extensively studied in connection with the theory of Fourier integral operators (see [60]). Moreover, a mathematical particular interest has been devoted to the study of their asymptotic behavior when $\epsilon \downarrow 0$ [36, 60].

If the function f is not summable the integral (1.1) is not defined in Lebesgue sense. In [60], Hörmander proposes and exploits an alternative definition which can handle the case where $f \notin L^1(\mathbb{R}^n)$. We present here a formulation of Hörmander's definition of oscillatory integral, which was applied to the mathematical construction of Feynman path integrals in [38, 3]. We shall emphasise that infinite dimensional integration theory (in particular the functional integration) is a powerful tool in the study of dynamical system [29, 31, 34, 37, 39].

1.1 Definitions and properties

As stated above, the integrals (1.1) can be computed even when the function f is not summable. According with [60], we compute them as the limit of a sequence of regularized integrals.

Definition 1.1. *Let $f : \mathbb{R}^n \to \mathbb{C}$ be a Borel function and $\Phi : \mathbb{R}^n \to \mathbb{R}$ a phase function. If for each Schwartz test function $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\phi(0) = 1$ the integrals*

$$I_\delta(f, \phi) := \int_{\mathbb{R}^n} e^{i\frac{\Phi(x)}{\epsilon}} f(x)\phi(\delta x)dx$$

exist for all $\epsilon > 0$, $\delta > 0$ and $\lim_{\delta\to 0} I_\delta(f, \phi)$ exists and is independent of ϕ, then the limit is called the oscillatory integral of f *with respect to Φ and denoted by*

$$\int^o_{\mathbb{R}^n} e^{i\frac{\Phi(x)}{\epsilon}} f(x)dx \equiv I^{\frac{\Phi}{\epsilon}}(f).$$

We notice that the convergence of the oscillatory integral (1.1) for $f \notin L^1(\mathbb{R}^n)$ can be obtained by exploiting the cancellations due to the oscillatory behavior of the integrand. Moreover, a particular technique of asymptotic analysis, the *stationary phase method*, allows the study of their asymptotic behavior in the limit when the parameter ϵ converges to 0 [36, 60, 87].

As mentioned above, we shall focus on the Fresnel integral, that is the oscillatory integral with a quadratic form as the phase function. Thus we set $\Phi(x) = \frac{\|x\|^2}{2}$.

In the 70s, S. Albeverio and R. Høegh-Krohn introduced in [9] a generalization of this integration technique to the case where $\mathbb{R}^n$ is replaced by a real separable infinite dimensional Hilbert space $(\mathcal{H}, \langle\ ,\ \rangle)$. These studies have been further developed in [38, 3, 10]. For our main purpose, it is convenient to define an *infinite dimensional oscillatory integral* with quadratic phase function (also called *infinite dimensional Fresnel integral*) as the limit of sequences of finite dimensional approximations.

Definition 1.2. *A function $f : \mathcal{H} \to \mathbb{C}$ is said to be* Fresnel integrable *if for any sequence $\{P_n\}_n$ of projectors onto n-dimensional subspaces of $\mathcal{H}$, such that $P_n \le P_{n+1}$ and $P_n \to \mathbb{I}$ strongly as $n \to \infty$ ($\mathbb{I}$ being the identity operator in $\mathcal{H}$), the oscillatory integrals*

$$\int^o_{P_n\mathcal{H}} e^{i\frac{\|P_n x\|^2}{2\epsilon}} f(P_n x)d(P_n x),$$

are well defined (in the sense of Def. 1.1) and the limit

$$\lim_{n\to\infty} (2\pi i\epsilon)^{-n/2} \int^o_{P_n\mathcal{H}} e^{i\frac{\|P_n x\|^2}{2\epsilon}} f(P_n x)d(P_n x) \tag{1.3}$$

exists and is independent of the sequence $\{P_n\}_n$*. In this case the limit is called infinite dimensional oscillatory integral of* f *and is denoted by*

$$\widetilde{\int_{\mathcal{H}}^{o}} e^{i\frac{\|x\|^2}{2\epsilon}} f(x)dx.$$

Let us observe that for Schwartz test functions $f \in \mathcal{S}(\mathbb{R}^n)$, the Fresnel integral (1.2) can be computed in terms of the following Parseval's identity:

$$\int_{\mathbb{R}^n} \frac{e^{\frac{i}{2\epsilon}\|x\|^2}}{(2\pi i\epsilon)^{n/2}} f(x)dx = \int_{\mathbb{R}^n} e^{-\frac{i\epsilon}{2}\|x\|^2} \hat{f}(x)dx, \qquad x \in \mathbb{R}^n, \tag{1.4}$$

$\hat{f}$ being the suitably normalized Fourier transform of f. Let us denote by $\mathcal{F}(\mathbb{R}^n)$ the space of complex-valued functions f, which can be written in the form of Fourier transform of some complex measure μ_f (depending on f) such that

$$f(x) = \int_{\mathbb{R}^n} e^{i\langle x,y\rangle} d\mu_f(y), \qquad x \in \mathbb{R}^n.$$

Then for $f \in \mathcal{F}(\mathbb{R}^n)$, the approximation procedure described in definition (1.1) allows to generalize (1.4) obtaining

$$\int_{\mathbb{R}^n}^{o} \frac{e^{\frac{i}{2\epsilon}\|x\|^2}}{(2\pi i\epsilon)^{n/2}} f(x)dx = \int_{\mathbb{R}^n} e^{-\frac{i\epsilon}{2}\|x\|^2} d\mu_f(x), \quad f(x) = \int_{\mathbb{R}^n} e^{i\langle x,y\rangle} d\mu_f(y). \tag{1.5}$$

Formula (1.5) is the fundamental tool for the definition of Fresnel integrals in the case where $\mathcal{H}$ is a real separable infinite dimensional Hilbert space. Indeed, if the dimension n of $\mathcal{H}$ is no longer finite, the left hand side of (1.5) is meaningless (due to the lack, e.g., of an Lebesgue measure on $\mathcal{H}$) but the right hand side is still meaningful and can be taken as the definition of a linear continuous functional on the space of Fourier transform of complex (bounded) Borel measures on $\mathcal{H}$ [10].

In the following we shall denote by $\mathcal{F}(\mathcal{H})$ the space of functions $f : \mathcal{H} \to \mathbb{C}$ that are Fourier transform of complex (bounded) Borel measures on $\mathcal{H}$, namely functions of the form:

$$f(x) = \int_{\mathcal{H}} e^{i\langle x,y\rangle} d\mu_f(y), \qquad x \in \mathcal{H}, \tag{1.6}$$

where μ_f is a complex bounded Borel measure of finite total absolute variation $|\mu_f|(\mathcal{H}) := \int_{\mathcal{H}} d|\mu_f|$ on $\mathcal{H}$. The space $\mathcal{F}(\mathcal{H})$ is a Banach algebra, where the product is the pointwise one and the norm $\|f\|_{\mathcal{F}(\mathcal{H})}$ of a function f is defined as the total variation of the associate measure μ_f. Indeed, the Banach space $\mathcal{M}(\mathcal{H})$ of complex Borel measures on $\mathcal{H}$ with finite total variation, endowed with the total variation norm (denoted by $\|\mu\|_{\mathcal{M}(\mathcal{H})}$) is a commutative Banach algebra under convolution, where the unit is the δ point measure concentrated at 0. By introducing on $\mathcal{F}(\mathcal{H})$

the norm $\|f\|_{\mathcal{F}(\mathcal{H})} := \|\mu\|_{\mathcal{M}(\mathcal{H})}$, the map

$$f(x) = \int_{\mathcal{H}} e^{\langle x,y\rangle} d\mu(y), \qquad x \in \mathcal{H},$$

becomes an isometry and $\mathcal{F}(\mathcal{H})$ endowed with the norm $\|\cdot\|_{\mathcal{F}(\mathcal{H})}$ becomes a commutative Banach algebra of continuous functions [10].

This construction, based on a Fourier transform approach, is the original proposal to the study of infinite dimensional oscillatory integrals and it is due to K. Itô [66, 65] and S. Albeverio and R. Høegh-Krohn [9, 10] as we also said in the introduction. Now, it seems appropriate to provide the following definition for the functions $f \in \mathcal{F}(\mathcal{H})$ (that it will be used in Chpt. 2).

Definition 1.3. *A function $f \in \mathcal{F}(\mathcal{H})$. The $\mathcal{F}$-*integral *of f is defined as:*

$$\widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\epsilon}\|x\|^2} f(x)dx := \int_{\mathcal{H}} e^{-\frac{i\epsilon}{2}\|x\|^2} d\mu_f(x), \tag{1.7}$$

where $f(x) = \int_{\mathcal{H}} e^{i\langle x,y\rangle} d\mu_f(y)$, $\mu_f \in \mathcal{M}(\mathcal{H})$.

Remark 1.1. The right hand side of (1.7) is a well defined (absolutely convergent) Lebesgue integral. Moreover, by the inequality

$$\left| \widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\epsilon}\|x\|^2} f(x)dx \right| \leq \|\mu_f\|_{\mathcal{M}(\mathcal{H})} = \|f\|_{\mathcal{F}(\mathcal{H})}$$

the application $f \mapsto \widetilde{\int} e^{\frac{i}{2\epsilon}\|x\|^2} f(x)dx$ is a linear continuous functional on $\mathcal{F}(\mathcal{H})$.

The relation between the class of integrable functions in the sense of Def. 1.2 and in the sense of Def. 1.3 is given thanks to the following theorem. In particular, it shows that the class of Fresnel integrable function in the sense of Def. 1.2 includes the class of integrable function in the sense of Def. 1.3, i.e. it includes $\mathcal{F}(\mathcal{H})$. A complete intrinsic characterization of the class of all Fresnel integrable function in the sense of Def. 1.2 constitutes an open problem of harmonic analysis, even in finite dimension.

Theorem 1.1. *Let $L : \mathcal{H} \to \mathcal{H}$ be a self adjoint trace class operator, such that $(I - L)$ is invertible and let $f \in \mathcal{F}(\mathcal{H})$. Then the function $g : \mathcal{H} \to \mathbb{C}$ defined by*

$$g(x) = e^{-\frac{i}{2\epsilon}\langle x,Lx\rangle} f(x), \quad x \in \mathcal{H} \tag{1.8}$$

is Fresnel integrable and its infinite dimensional Fresnel integral is given by the following Parseval-type equality:

$$\widetilde{\int_{\mathcal{H}}^{o}} e^{\frac{i}{2\epsilon}\langle x,(I-L)x\rangle} f(x)dx = (\det(I-L))^{-1/2} \int_{\mathcal{H}} e^{-\frac{i\epsilon}{2}\langle x,(I-L)^{-1}x\rangle} d\mu_f(x) \tag{1.9}$$

where $\det(I-L)$ *is the Fredholm determinant of the operator* $(I-L)$ *(that is the product of the eigenvalues of* $(I-L)$*) and* μ_f *is the complex bounded Borel measure on* $\mathcal{H}$ *related to* f *by* (1.6)*.*

For the proof see Theorem 10.1 in [10].

Remark 1.2. It is interesting to point out that the class of Fresnel integrable functions contains elements different from the ones described by eq (1.8). As remarked in [38], Def. 1.2 allows to handle, e.g., unbounded functions $f : \mathcal{H} \to \mathbb{C}$ of the form:

$$f(\gamma) = e^{i\langle\gamma,v\rangle}\langle\gamma, w_1\rangle \cdots \langle\gamma, w_n\rangle, \qquad \gamma \in \mathcal{H}, \tag{1.10}$$

with $v, w_1, \ldots, w_n \in \mathcal{H}$.

Remark 1.3. Theorem 1.1 has been extended in [4] to the case where the operator L is Hilbert-Schmidt but it is not trace-class. In [84] the definition of infinite dimensional oscillatory integrals has been generalized to polynomial phase functions and applied to the construction of Feynman-Kac type formulae for the high-order heat-type equations (see also [6]).

1.2 Representation for the solution of the Schrödinger equation

The heuristic Feynman path integral representation for the solution of the Schrödinger equation in three dimensions can be rigorously mathematically realized as an infinite dimensional oscillatory integral on a suitable Hilbert space of continuous "paths". Indeed, let us set $\epsilon \equiv \hbar$ and let us consider the so-called Cameron-Martin space $\mathcal{H}_t$, that is the Hilbert space of of absolutely continuous paths $\gamma : [0, t] \to \mathbb{R}^3$ with $\gamma(0) = 0$ and square integrable weak derivative $\int_0^t |\dot{\gamma}(s)|^2 ds < \infty$, endowed with the inner product

$$\langle\gamma_1, \gamma_2\rangle = \int_0^t \dot{\gamma}_1(s) \cdot \dot{\gamma}_2(s) ds.$$

In the case where the potential V in the Schrödinger equation

$$\begin{cases} i\hbar\frac{\partial}{\partial t}\psi(t,x) = -\frac{\hbar^2}{2}\Delta\psi(t,x) + V(x)\psi(t,x), & x \in \mathbb{R}^3, t \in \mathbb{R}, \\ \psi(0,x) = \psi_0(x) \end{cases} \tag{1.11}$$

is the sum of an harmonic oscillator term and a bounded perturbation v belonging to the space $\mathcal{F}(\mathbb{R}^3)$

$$V(x) = \frac{1}{2}x\Omega^2 x + v(x), \qquad x \in \mathbb{R}^3, \tag{1.12}$$

and if the initial datum ψ_0 belongs to $L^2(\mathbb{R}^3) \cap \mathcal{F}(\mathbb{R}^3)$, it has been proved (see, e.g., [10, 38, 84]) that the function $f : \mathcal{H}_t \to \mathbb{C}$ given by

$$f(\gamma) = e^{-\frac{i}{\hbar}\int_0^t V(\gamma(s)+x)ds}\psi_0(\gamma(0)+x), \qquad x \in \mathbb{R}^3, \gamma \in \mathcal{H}_t,$$

is Fresnel integrable. Further, its infinite dimensional oscillatory integral, namely

$$\widetilde{\int_{\mathcal{H}_t}} e^{\frac{i}{2\hbar}\langle\gamma,\gamma\rangle} f(\gamma)d\gamma \equiv \widetilde{\int_{\mathcal{H}_t}} e^{\frac{i}{2\hbar}\int_0^t |\dot\gamma(s)|^2 ds} e^{-\frac{i}{\hbar}\int_0^t V(\gamma(s)+x)ds}\psi_0(\gamma(0)+x)d\gamma,$$

provides a representation for the solution in the (complex) $L^2(\mathbb{R}^3)$ space of the Schrödinger equation (1.11).

Remark 1.4. This result, valid also for $\mathbb{R}^3$ replaced by $\mathbb{R}^d$, for any $d \in \mathbb{N}$, has been generalized in [4] to the case where $d = 2$ and a constant magnetic field is present, providing a formula valid in the Coulomb gauge for the vector potential $\mathbf{a}(x,y) = (-By/2, Bx/2)$, $(x,y) \in \mathbb{R}^2$.

For a detailed proof of these results as well as for their applications to the Feynman path integral representation of the solution of the Schrödinger equation, see, e.g.[38, 3, 10, 85]. For other approaches to the mathematical theory of Feynman path integrals see, e.g., [57, 68, 44, 72].

1.3 The Feynman map

In the context of (1.12) a particular class of finite dimensional approximations plays an important role and has been introduced in an alternative definition of infinite dimensional oscillatory integrals on the Cameron-Martin space. For fixed $n \in \mathbb{N}$, let $H_n \subset \mathcal{H}_t$ be the finite dimensional subspace of piecewise linear paths of the form

$$\gamma(s) = \sum_{k=1}^{n} \chi_{[t_{k-1},t_k)}(s)\left(\gamma(t_{k-1}) + \frac{\gamma(t_k)-\gamma(t_{k-1})}{t_k - t_{k-1}}(s - t_{k-1})\right), \tag{1.13}$$

where $s \in [0,t]$, $t_k = \frac{kt}{n}$ and $k = 0, \ldots, n$. Let $P_n : \mathcal{H}_t \to \mathcal{H}_t$ the projector operator onto H_n, whose action on a generic vector $\gamma \in \mathcal{H}_t$ is given by the right hand side of (1.13). In fact the sequence of operators $\{P_n\}_n$ converges strongly to the identity operator $\mathbb{I}$ as $n \to \infty$ [111].

In this context the definition of *Feynman map* has been proposed [112]. It is defined as a linear functional I_F whose action on functions $f : \mathcal{H}_t \to \mathbb{C}$ is defined by

$$I_F(f) = \lim_{n\to\infty} \frac{\int^o_{P_n\mathcal{H}_t} e^{i\frac{\|P_n\gamma\|^2}{2\hbar}} f(P_n\gamma)d(P_n\gamma)}{\int^o_{P_n\mathcal{H}_t} e^{i\frac{\|P_n\gamma\|^2}{2\hbar}} d(P_n\gamma)}, \tag{1.14}$$

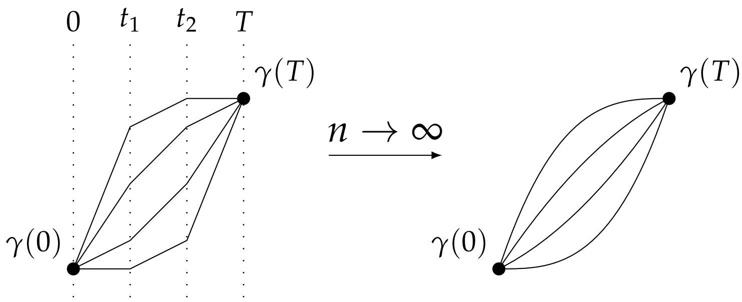
whenever the limit on the right hand side exists.

In the case where $f \in \mathcal{F}(\mathcal{H}_t)$ then its Feynman map is well defined and coincides with its infinite dimensional oscillatory integral:

$$I_F(f) = \widetilde{\int_{\mathcal{H}_t}} e^{\frac{i}{2\hbar}\langle\gamma,\gamma\rangle} f(\gamma) d\gamma.$$

Remark 1.5. In the general case, i.e. for $f \notin \mathcal{F}(\mathcal{H}_t)$, the two alternative definitions of the Feynman integral can yield different results.

It is interesting to outline that the integrability condition required by Def. 1.2 holds if the limit (1.3) is independent of the particular sequence of finite dimensional projection operators $\{P_n\}_n$, whereas the definition of Feynman map (1.14) relies upon the piecewise-linear approximations.



A schematic representation of piecewise linear approximations.

Chapter 2

Generalized Feynman path integrals*

This chapter is devoted to the study of p-order heat-type equations of the form:

$$\frac{\partial}{\partial t}u(t,x) = (-i)^p\alpha\frac{\partial^p}{\partial x^p}u(t,x) + V(x)u(t,x), \quad t \in [0,+\infty),\ x \in \mathbb{R}, \tag{2.1}$$

where $p \in \mathbb{N}$, $p > 2$, $\alpha \in \mathbb{C}$ is a complex constant and $V : \mathbb{R} \to \mathbb{C}$ is a continuous bounded function. For notational convenience we studied the one dimensional case, but the corresponding general case could be treated.

The main idea is the replacement of the concept of integral with the (more general) concept of linear continuous functional on a suitable space of "integrable functions". More precisely, the integral with respect to a σ-additive measure μ has to be replaced by a linear (continuous) functional $L : D(L) \to \mathbb{C}$ defined on a domain $D(L)$ which contains the *cylinder functions*, i.e. the functions $f : \Omega \to \mathbb{C}$ of the form:

$$f(\omega) := F(\omega(t_1), \dots, \omega(t_n)) \tag{2.2}$$

for some $n \in \mathbb{N}$, $0 < t_1 < \dots < t_n < \infty$ and a Borel function F on $\mathbb{R}^n$. The action of the functional L on a function f of the form (2.2) must be given by a (finite dimensional) integral on the space $\mathbb{R}^n$ of the form:

$$\begin{aligned} L(f) = \int_{\mathbb{R}^n} & F(x_n, \dots, x_1)K_{t_n - t_{n-1}}(x_n, x_{n-1}) \dots \\ & \dots K_{t_2 - t_1}(x_2, x_1)K_{t_1}(x_1, x)dx_1 \dots dx_n. \end{aligned} \tag{2.3}$$

with, $x_j \in \mathbb{R}$ for $j = 1, \dots, n$ and where K_t, for $t > 0$, has the form seen in the introduction:

$$K_t(x,y) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\kappa(x-y)}e^{\alpha\kappa^p t}d\kappa,$$

*The results of this chapter are collected in [6].

with $x, y \in \mathbb{R}$ and $\kappa \in R$. In fact this particular view of infinite dimensional integration allows to unify both probabilistic and oscillatory integrals on infinite dimensional spaces, as discussed in [15, 13, 12, 11]. Despite the successful application of these techniques to Feynman integration as well as to the study of the semiclassical asymptotic behavior of quantum mechanics [3, 9, 10, 18] is well known since the 70s, the generalization to a wider class of PDEs such as the higher-order heat-type equations is rather recent [84].
Here we shall use of the theory and applications of *infinite dimensional Fresnel integrals* to the construction of functional integral representation of the solution of PDEs of the form (2.1).

Section 2.1 presents a recent generalization of infinite dimensional Fresnel integrals which covers the case of polynomial phase functions and the application to the representation of the solution of higher-order equation (2.1). In section 2.2 we prove a Girsanov type formula for the higher-order equation (2.1). Eventually section 2.3 presents a *phase space path integral* solution formula for equation (2.1).

2.1 Infinite dimensional Fresnel integrals with polynomial phase function

The Def. 1.3 of $\mathcal{F}$-integral has been recently generalized to cover the case where the quadratic phase function is replaced by an higher-order polynomial [84]. This new functional, named *infinite dimensional Fresnel integral with polynomial phase function* (or shortly *generalized Fresnel integrals*) can be applied to the construction of a functional integral representation for the solution of a general class of higher-order heat-type equations of the form

$$\begin{cases} \frac{\partial}{\partial t} u(t,x) = (-i)^p \alpha \frac{\partial^p}{\partial x^p} u(t,x) + V(x) u(t,x), & t > 0, x \in \mathbb{R}, \\ u(0,x) = u_0(x) \end{cases} \tag{2.4}$$

where $p \in \mathbb{N}$ is a positive integer, $\alpha \in \mathbb{C}$ a complex constant satisfying the following condition

$$\mathrm{Re}(\alpha y^p) \le 0, \qquad \forall y \in \mathbb{R}. \tag{2.5}$$

and V is a complex-valued function. In the case where $p = 2$ and α is purely negative imaginary Eq. (2.4) reduces to the Schrödinger equation with potential iV, while if $p = 4$ and $\alpha = -1$ Eq. (2.4) is the higher-order heat-type equation (5).

The generalization of Def. 1.3 relies upon the replacement of the Hilbert space $\mathcal{H}$ where the integration is performed by a real separable Banach space $\mathcal{B}$ with norm $\|\cdot\|$. Let us denote by $\mathcal{B}^*$ its topological dual. Let $\mathcal{F}(\mathcal{B})$ denote the space of complex valued functions $f : \mathcal{B}^* \to \mathbb{C}$ of the form

$$f(x) = \int_{\mathcal{B}} e^{i\langle x,y\rangle} d\mu_f(y), \qquad x \in \mathcal{B}^* \tag{2.6}$$

for some complex Borel measure μ_f on $\mathcal{B}$. The space $\mathcal{F}(\mathcal{B})$ is a Banach algebra of functions, where the product is the pointwise one $f \cdot g(x) = f(x)g(x)$ and the norm $\|f\|_{\mathcal{F}}$ of an element $f \in \mathcal{F}(\mathcal{B})$ is defined as the total variation $\|\mu_f\|$ of the associated Borel bounded measure μ_f (see Eq (2.6)), namely $\|f\|_{\mathcal{F}} := \|\mu_f\|$, where

$$\|\mu_f\| := \int_{\mathcal{B}} d|\mu_f|(y).$$

Definition 2.1. *Let let $\Phi : \mathcal{B} \to \mathbb{C}$ be a continuous function such that $\mathrm{Re}(\Phi(x)) \leq 0$ for all $x \in \mathcal{B}$.*

The infinite dimensional Fresnel integral on $\mathcal{B}^*$ with phase function Φ is the functional $L_\Phi : \mathcal{F}(\mathcal{B}) \to \mathbb{C}$, given by

$$\begin{aligned} L_\Phi(f) &:= \int_{\mathcal{B}} e^{\Phi(x)} d\mu_f(x), \qquad f \in \mathcal{F}(\mathcal{B}), \quad x \in \mathcal{B}, \\ f(x) &= \int_{\mathcal{B}} e^{i\langle x,y\rangle} d\mu_f(y), \qquad x \in \mathcal{B}^*. \end{aligned} \tag{2.7}$$

By its definition, it is straightforward to see that the functional $L_\Phi : \mathcal{F}(\mathcal{B}) \to \mathbb{C}$ is linear and continuous in the $\mathcal{F}(\mathcal{B})$-norm. Indeed:

$$|L_\Phi(f)| \leq \int_{\mathcal{B}} |e^{\Phi}| d|\mu_f|(x) \leq \|\mu_f\| = \|f\|_{\mathcal{F}}, \qquad f \in \mathcal{F}(\mathcal{B}).$$

Furthermore the functional L is normalized, i.e. its value on the constant function $f(x) = 1$ $\forall x \in \mathcal{B}^*$ is equal to $L_\Phi(1) = 1$ (f being the Fourier transform of the δ point measure at $x = 0$). In particular the functional L_Φ generalizes the infinite dimensional integrals of Def. 1.3 in the sense that if $\mathcal{B} \equiv \mathcal{H}$ and $\Phi(x) = \frac{-i\hbar\|x\|^2}{2}$ then

$$L_\Phi(f) = \widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}\|x\|^2} f(x) dx.$$

Let us consider now a particular example of a functional of the form (2.7) on a suitable Banach space $\mathcal{B}$. Given a positive integer $p \in \mathbb{N}$, with $p \geq 2$, let us consider the Banach space $\mathcal{B}_p$ of absolutely continuous paths $\xi : [0,t] \to \mathbb{R}$ such that $\xi(t) = 0$ and weak derivative $\dot{\xi}$ belonging to $L^p([0,t])$, endowed with the norm

$$\|\xi\|_{\mathcal{B}_p} = \left(\int_0^t |\dot{\xi}(s)|^p ds \right)^{1/p}.$$

The dual space $\mathcal{B}_p^*$ is isomorphic to $\mathcal{B}_q$, with $\frac{1}{p}+\frac{1}{q}=1$, and the pairing between an element $\gamma \in \mathcal{B}_p^*$ and $\xi \in \mathcal{B}_p$ is given by:

$$\langle \gamma, \xi \rangle = \int_0^t \dot{\gamma}(s)\dot{\xi}(s)ds \qquad \gamma \in \mathcal{B}_q, \quad \xi \in \mathcal{B}_p.$$

A function $f : \mathcal{B}_q \to \mathbb{C}$ belonging to the Banach algebra $\mathcal{F}(\mathcal{B}_p)$ has the form

$$f(\gamma) = \int_{\mathcal{B}_p} e^{i\int_0^t \dot{\gamma}(s)\dot{\xi}(s)ds} d\mu_f(\xi), \qquad \gamma \in \mathcal{B}_q \tag{2.8}$$

for some complex Borel measure μ_f on $\mathcal{B}_p$.

Let us introduce a homogeneous phase function $\Phi_p : \mathcal{B}_p \to \mathbb{C}$ of the form

$$\Phi_p(\xi) := (-1)^p \alpha \int_0^t \dot{\xi}(s)^p ds, \qquad \xi \in \mathcal{B}_p$$

where $\alpha \in \mathbb{C}$ is a complex constant such that $\text{Re}(\alpha) \le 0$ if p is even and $\text{Re}(\alpha) = 0$ if p is odd. The corresponding generalized infinite dimensional Fresnel integral L_{Φ_p} : $\mathcal{F}(\mathcal{B}_p) \to \mathbb{C}$ is defined as

$$L_{\Phi_p}(f) = \int_{\mathcal{B}_p} e^{(-1)^p \alpha \int_0^t \dot{\xi}(s)^p ds} d\mu_f(\xi), \tag{2.9}$$

for $f \in \mathcal{F}(\mathcal{B}_p)$ given by Eq. (2.8).

The following lemma is fundamental as it shows that the functional L_{Φ_p} defined above provides a rigorous definition of the "path integral" associated to the higher-order heat-type equation (2.1).

Lemma 2.1. *Let $f : \mathcal{B}_q \to \mathbb{C}$ be a cylindrical function of the following form:*

$$f(\gamma) = F(\gamma(t_1), \gamma(t_2), \dots, \gamma(t_n)), \qquad \gamma \in \mathcal{B}_q,$$

with $0 \le t_1 < t_2 < \dots < t_n < t$ and $F : \mathbb{R}^n \to \mathbb{C}$, $F \in \mathcal{F}(\mathbb{R}^n)$:

$$F(x_1, \dots, x_n) = \int_{\mathbb{R}^n} e^{i\sum_{k=1}^n y_k x_k} d\nu_F(y_1, \dots, y_n),$$

for some bonded complex measure $\mu_{\mathcal{F}}$ with Fourier transform F on $\mathbb{R}^n$. Then $f \in \mathcal{F}(\mathcal{B}_p)$ and its infinite dimensional Fresnel integral with phase function Φ_p is given by

$$L_{\Phi_p}(f) = \int_{\mathbb{R}^n} F(x_1, \dots, x_n) \prod_{k=1}^n K^p_{t_{k+1}-t_k}(x_{k+1}, x_k) dx_1 \dots dx_n, \tag{2.10}$$

where $x_{n+1} \equiv 0$, $t_{n+1} \equiv t$ *and* K_s^p *is the fundamental solution of the high order heat-type equation* $\frac{\partial}{\partial t}u(t,x) = (-i)^p \alpha \frac{\partial^p}{\partial x^p} u(t,x)$*:*

$$K_t^p(x,y) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\kappa(x-y)} e^{\alpha\kappa^p t} d\kappa$$

See [84] for the proof.

According to formula (2.10), the functional L_{Φ_p} provides a mathematical definition of the "path integral" associated to the non positive kernel K_t^p, i.e. it enjoys the property (2.3) described in the introductory section.

A direct consequence of the previous lemma is the proof of the following functional integral representation of the solution to the higher-order heat-type equation (2.4).

Corollary 2.1. *Let* $u_0 \in \mathcal{S}(\mathbb{R})$ *be a Schwartz test function. Then for all* $x \in \mathbb{R}$ *the function* $f : \mathcal{B}_q \to \mathbb{C}$ *defined by* $f(\gamma) = u_0(x + \gamma(t))$ *belongs to* $\mathcal{F}(\mathcal{B}_p)$ *and the function* $u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ *defined by* $u(t,x) = L_{\Phi_p}(f)$ *is a representation for the solution of the Cauchy problem*

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = (-i)^p \alpha \frac{\partial^p}{\partial x^p} u(t,x) \\ u(0,x) = u_0(x), \qquad x \in \mathbb{R}, t \in [0,+\infty) \end{cases}$$

Analogously it is possible to prove (see [84] for the details) the following Feynman-Kac formula.

Theorem 2.1. *Let* $u_0 \in \mathcal{F}(\mathbb{R}) \cap L^2(\mathbb{R})$ *and* $V \in \mathcal{F}(\mathbb{R})$*. In particular let us consider* $u_0(x) = \int_{\mathbb{R}} e^{ixy} d\mu_0(y)$ *and* $V(x) = \int_{\mathbb{R}} e^{ixy} d\nu(y)$*, with* μ_0, ν *bounded complex measures on* $\mathbb{R}$*. Then the functional* $f_{t,x} : \mathcal{B}_q \to \mathbb{C}$ *defined by*

$$f_{t,x}(\gamma) := u_0(x + \gamma(0)) e^{\int_0^t V(x+\gamma(s))ds}, \qquad x \in \mathbb{R}, \gamma \in \mathcal{B}_q,$$

belongs to $\mathcal{F}(\mathcal{B}_p)$ *and its infinite dimensional Fresnel integral with polynomial phase function* Φ_p *provides a representation for the solution of the Cauchy problem* (2.4).

2.2 Girsanov-type formula for generalized Fresnel integrals

In this section we prove a Girsanov-type formula for higher-order heat-type equation of the form (2.4). To the best of our knowledge, in the existing literature on the argument this result is still lacking. We shall show how it can be obtained as a direct consequence of the formalism introduced in the previous section.

Let us consider the p-order heat-type equation

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = \alpha(-i\partial x - a(x))^p u(t,x) \\ u(0,x) = u_0(x), \qquad x \in \mathbb{R}, \quad t \in [0,+\infty) \end{cases} \tag{2.11}$$

where $\alpha \in \mathbb{C}$ and $p \in \mathbb{N}$ satisfy the assumption (2.5) and let $a : \mathbb{R} \to \mathbb{R}$ be a $C^{p-1}(\mathbb{R})$ function. We are going to construct a functional integral representation of the solution of (2.11) by means of the functional L_{Φ_p} introduced in the previous section.

Let us denote by $A : \mathbb{R} \to \mathbb{R}$ a primitive of a, i.e. $a(x) = \frac{d}{dx}A(x)$, $x \in \mathbb{R}$. In the following we shall assume that A belongs to $\mathcal{F}(\mathbb{R})$, i.e. A is the Fourier transform of a complex bounded variation measure ν on $\mathbb{R}$ such that the associated total variation measure has finite p-moments:

$$\begin{aligned} A(x) &= \int_{\mathbb{R}} e^{ixy} d\nu(y), \qquad x \in \mathbb{R}, \\ &\int_{\mathbb{R}} |y|^p d|\nu|(y) < \infty. \end{aligned} \tag{2.12}$$

By (2.12), A is a $C^p(\mathbb{R})$ function. Furthermore, let us consider an initial datum $u_0 \in \mathcal{F}(\mathbb{R})$:

$$u_0(x) = \int_{\mathbb{R}} e^{ixy} d\mu(y), \quad x \in \mathbb{R}.$$

Let us note that $\int_0^t a(\gamma(s)+x)\dot{\gamma}(s)ds$ is well defined for $\gamma \in \mathcal{B}_q$, since $\gamma \mapsto a(\gamma(s)+x)$ is in $\mathcal{F}(\mathcal{B}_p)$.

Theorem 2.2. *Under the assumption above, the map $f : \mathcal{B}_q \to \mathbb{C}$ defined as*

$$f(\gamma) := u_0(\gamma(0)+x)e^{i\int_0^t a(\gamma(s)+x)\dot{\gamma}(s)ds}, \qquad \gamma \in \mathcal{B}_q,$$

belongs to $\mathcal{F}(\mathcal{B}_p)$ and its infinite dimensional Fresnel integral with phase function Φ_p on $\mathcal{B}_q$ (in the sense of Def. 2.1) provides a representation of the solution of the Cauchy problem (2.11):

$$u(t,x) = L_{\Phi_p}(f).$$

Proof. Under the assumption above we have that the maps $g : \mathcal{B}_q \to \mathbb{C}$ and $f_0 : \mathcal{B}_q \to \mathbb{C}$ defined as

$$\begin{aligned} g(\gamma) &= \int_0^t a(\gamma(s)+x)\dot{\gamma}(s)ds - A(x) = -A(\gamma(0)+x) \\ f_0(\gamma) &= u_0(\gamma(0)+x) \end{aligned} \tag{2.13}$$

belong to $\mathcal{F}(\mathcal{B}_p)$. Indeed, denoted by $v_0 \in \mathcal{B}_p$ the vector defined by $v_0(s) = t - s$, $s \in [0, t]$, we have

$$\langle \gamma, v_0 \rangle = \gamma(0), \qquad \forall \gamma \in \mathcal{B}_q.$$

where $\mathcal{H}_t$ was defined in Chpt. 1. Hence:

$$\begin{aligned} A(\gamma(0) + x) &= \int_{\mathbb{R}} e^{iyx} e^{iy\gamma(0)} d\nu(y) \\ &= \int_{\mathbb{R}} e^{iyx} e^{iy\langle \gamma, v_0 \rangle} d\nu(y) \\ &= \int_{\mathbb{R}} e^{iyx} \int_{\mathcal{B}_p} e^{i\langle \gamma, \xi \rangle} \delta_{yv_0}(\gamma) d\nu(y) \\ &= \int_{\mathcal{B}_p} e^{i\langle \gamma, \xi \rangle} d\mu_A(\xi), \end{aligned}$$

where $\mu_A \in \mathcal{M}(\mathcal{B}_p)$ (the space of bounded complex measure on $\mathcal{B}_p$) is defined by

$$\int_{\mathcal{B}_p} F(\xi) d\mu_A(\xi) := \int_{\mathbb{R}} e^{iyx} F(yv_0) d\nu(y), \tag{2.14}$$

for all Borel bounded $F : \mathcal{B}_p \to \mathbb{R}$. Analogously $u_0(\gamma(0) + x) = \int_{\mathcal{B}_p} e^{i\langle \gamma, \xi \rangle} d\mu_0(\xi)$, where $\mu_0 \in \mathcal{M}(\mathcal{B}_p)$ defined by a formula corresponding to (2.14) with ν replaced by μ. By the Banach algebra properties of $\mathcal{F}(\mathcal{B}_p)$, we have that $f_0 \exp(ig) = f_0 \sum_{n=0}^{\infty} \frac{(i)^n g^n}{n!}$ is an element of $\mathcal{F}(\mathcal{B}_p)$. The generalized Fresnel integral of f (cfr. the beginning of this section) is given by

$$L_{\Phi_p}(f) = e^{iA(x)} L_{\Phi_p}(f_0 \exp(ig)) = e^{iA(x)} \sum_{n=0}^{\infty} \frac{i^n}{n!} L_{\Phi_p}(f_0 g^n),$$

where the series is absolutely convergent and we have used that L_{Φ_p} is continuous. By Lemma 2.1 the latter is given by

$$L_{\Phi_p}(f) = e^{iA(x)} \int_{\mathbb{R}} K_t(x, y) u_0(y) e^{-iA(y)} dy. \tag{2.15}$$

Denoted by $u(t,x)$ the right hand side of (2.15), by Corollary 2.1 in Sect. 2.1 we have $u(0,x) = u_0(x)$ and

$$\begin{aligned}
\partial_t u(t,x) &= e^{iA(x)}\partial_t \int_{\mathbb{R}} K_t(x,y)u_0(y)e^{-iA(y)}dy \\
&= (-i^p)\alpha e^{iA(x)}\partial_x^p \int_{\mathbb{R}} K_t(x,y)u_0(y)e^{-iA(y)}dy \\
&= (-i^p)\alpha e^{iA(x)}\partial_x^p (e^{-iA(x)}u(t,x)) \\
&= (-i^p)\alpha(\partial_x - ia(x))^p u(t,x) \\
&= \alpha(-i\partial_x - a(x))^p u(t,x).
\end{aligned}$$

□

Remark 2.1. In the case of the Schrödinger equation, i.e. for $p=2$ and $\alpha = -\frac{i\hbar}{2}$, the generalization of Theorem 2.2 to the case where the space variable x is d-dimensional, with $d=3$, allows to construct a Feynman path integral representation for the solution in the presence of magnetic field (the function a playing the role of the vector potential); for this topic see, for instance, [54, 105]. In this case the technique used in the proof above does no longer work (since equality (2.13) is no longer valid). This issues will be investigated in the next chapter.

2.3 Hamiltonian and Lagrangian path integrals

In its original formulation [40, 41], Feynman's approach to quantum dynamics was essentially of Lagrangian type. On the other hand an Hamiltonian formulation could be preferable from many points of view. For instance the discussion of the approach from quantum mechanics to classical mechanics, i.e the study of the behavior of physical quantities taking into account that $\hbar$ is small, is more natural in an Hamiltonian setting. In other words the "phase space" rather then the "configuration space" is the natural framework for classical mechanics.

Consequently, many attempts to introduce and exploit a "phase space Feynman path integral" formula have been proposed (see, e.g. [28, 104, 32, 20]), namely a representation for the solution of the Schrödinger equation, of the (heuristic) form seen in the introduction

$$u(t,x) = \text{Const} \int_{q(t)=x} e^{\frac{i}{\hbar}S(q,p)}u_0(q(0))dqdp. \tag{2.16}$$

The integral in the heuristic formula (2.16) is meant on the space of paths $q(s), p(s)$, $s \in [0,t]$ in the phase space of the system ($q(s)_{s\in[0,t]}$ is the path in configuration space and $p(s)_{s\in[0,t]}$ is the path in momentum space) and S is the action functional in the

Hamiltonian formulation:

$$S(q,p) = \int_0^t (\dot{q}(s)p(s) - H(q(s),p(s)))ds, \tag{2.17}$$

(H being the classical Hamiltonian of the system). Despite formula (2.16) has been extensively used in the physical literature, from a strictly mathematical point of view just a few results have been obtained [27, 32, 76, 8, 7], providing a rigorous definition of formula (2.16). In particular, $\mathcal{F}$-integrals (Def. 1.3) have been applied to the rigorous mathematical construction of a phase-space Feynman path integral formula for the solution of Schrödinger equation in [8] (see also [7] for an interesting application to the theory of continuous quantum measurements).

The aim of this section is the application of the infinite dimensional Fresnel integral with polynomial phase function (Def. 2.1) to the construction of a phase-space Feynman path integral representation for the solution of higher-order heat-type equations (2.1). For an alternative definition of phase space functional integrals for the solution of (2.1) see [74, 76].

Let us consider the Banach space $\mathcal{B} = \mathcal{B}_q \times L_p$ (where L_p is the space of p-integrable functions) and its dual $\mathcal{B}^* = \mathcal{B}_p \times L_q$. The space $\mathcal{F}(\mathcal{B}_q \times L_p)$ will be the space of complex valued functions of the form

$$f(\gamma,\eta) = \int_{\mathcal{B}_p \times L_q} e^{i\langle \gamma,\xi\rangle + i\langle \eta,\zeta\rangle} d\mu(\xi,\zeta), \quad (\gamma,\eta) \in \mathcal{B}_q \times L_p \tag{2.18}$$

for some complex bounded Borel measure μ on $\mathcal{B}_p \times L_q$.

Let $\Phi : \mathcal{B}_p \times L_q \to \mathbb{C}$ be the phase function defined by

$$\Phi(\xi,\zeta) := (-1)^p \alpha \int_0^t \dot{\xi}(s)^p ds + i \int_0^t \zeta(s)\dot{\xi}(s)ds, \quad (\xi,\zeta) \in \mathcal{B}_p \times L_q.$$

According to Def. 2.1 the infinite dimensional Fresnel integral with polynomial phase Φ is the functional $L_\Phi : \mathcal{F}(\mathcal{B}_q \times L_p) \to \mathbb{C}$ defined as

$$L_\Phi(f) = \int_{\mathcal{B}_p \times L_q} e^{(-1)^p \alpha \int_0^t \dot{\xi}(s)^p ds + i \int_0^t \zeta(s)\dot{\xi}(s)ds} d\mu(\xi,\eta) \tag{2.19}$$

with $f \in \mathcal{F}(\mathcal{B}_q \times L_p)$ of the form (2.18).

Remark 2.2. If $f \in \mathcal{F}(\mathcal{B}_q \times L_p)$ of the form (2.18) does not depend explicitly on the variable $\eta \in L_p$ then $L_\Phi(f)$ reduces to (2.9). In particular, under the assumption of theorem 2.1, the solution of the Cauchy problem associated to the higher-order heat-type equation (2.4) is given by $L_\Phi(f_{t,x})$, where L_Φ is the functional (2.19) and $f_{t,x}$ is the function in $\mathcal{F}(\mathcal{B}_q \times L_p)$ defined by

$$f_{t,x}(\gamma,\eta) := u_0(x+\gamma(0))e^{\int_0^t V(x+\gamma(s))ds}, \quad x \in \mathbb{R}, \quad (\gamma,\eta) \in \mathcal{B}_q \times L_p.$$

Given $0 \leq t_1 < \cdots < t_n < t$, let us consider for any $j = 1, \ldots, n$ the vector $v_{t_j} \in \mathcal{B}_p$ defined as

$$v_{t_j}(s) = (t - t_j)\chi_{[0,t_j]}(s) + (t - s)\chi_{(t_j,t]}(s), \quad s \in [0, t]. \tag{2.20}$$

Analogously, for any $j = 1, \ldots, n$ we shall denote by $w_j \in L_q$ the vector defined by

$$w_j(s) = -(t_{j+1} - t_j)^{-1}\chi_{[t_j,t_{j+1}]}(s), \quad s \in [0, t]. \tag{2.21}$$

Theorem 2.3. *Let $f : \mathcal{B}_q \times L_p \to \mathbb{C}$ be a cylinder function of the following form:*

$$\begin{aligned} f(\gamma, \eta) &= F\left(\gamma(t_1), \gamma(t_2), \ldots, \gamma(t_n), -\frac{\int_{t_1}^{t_2} \eta(s)ds}{t_2 - t_1}, \ldots, -\frac{\int_{t_n}^{t} \eta(s)ds)}{t - t_n}\right) \\ &= F(\langle \gamma, v_{t_1}\rangle, \langle \gamma, v_{t_2}\rangle, \ldots, \langle \gamma, v_{t_n}, \rangle, \langle \eta, w_1\rangle, \ldots, \langle \eta, w_n\rangle) \end{aligned}$$

with $F \in \mathcal{F}(\mathbb{R}^{2n})$, namely

$$F(x_1, \ldots, x_n, p_1, \ldots, p_n) = \int_{\mathbb{R}^{2n}} e^{i\sum_{k=1}^n y_k x_k + i\sum_{k=1}^n z_k p_k} d\nu_F(y_1, \ldots, y_n, z_1, \ldots, z_n),$$

with $\nu_F \in \mathcal{M}(\mathbb{R}^{2n})$. Then $f \in \mathcal{F}(\mathcal{B}_q \times L_p)$ and $L_\Phi(f)$ is given by

$$\begin{aligned} L_\Phi(f) = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^{2n}}^{o} & e^{i\sum_{k=1}^n p_k(x_{k+1}-x_k) + \alpha\sum_{k=1}^n p_k^p(t_{k+1}-t_k)} \\ &\times F(x_1, \ldots, x_n, p_1, \ldots, p_n) dx_1 \ldots dx_n dp_1 \ldots dp_n, \end{aligned} \tag{2.22}$$

where $x_{n+1} \equiv 0$, $t_{n+1} \equiv t$. The integral on the right hand side of (2.10) is an oscillatory integral on $\mathbb{R}^n$ in the sense of Def. 1.1.

Proof. The proof that $f \in \mathcal{F}(\mathcal{B}_q \times L_p)$ follows from the explicit form of the function

$$\begin{aligned} f(\gamma, \eta) &= F(\langle \gamma, v_{t_1}\rangle, \langle \gamma, v_{t_2}\rangle, \ldots, \langle \gamma, v_{t_n}, \rangle, \langle \eta, w_1\rangle, \ldots, \langle \eta, w_n\rangle) \\ &= \int_{\mathbb{R}^{2n}} e^{i\sum_{k=1}^n y_k \langle \gamma, v_{t_k}\rangle + i\sum_{k=1}^n z_k \langle \eta, w_k\rangle} d\nu_F(y_1, \ldots, y_n, z_1 \ldots, z_n) \end{aligned}$$

and the identity

$$e^{iy_k\langle \gamma, v_{t_k}\rangle + iz_k\langle \eta, w_k\rangle} = \int_{\mathcal{B}_p \times L_q} e^{i\langle \gamma, \xi\rangle + i\langle \eta, \zeta\rangle} \delta_{y_k v_{t_k}}(\xi)\delta_{z_k w_k}(\zeta).$$

By the definition of the functional L_Φ we have

$$
\begin{aligned}
L_\Phi(f) &= \int_{\mathbb{R}^{2n}} e^{(-1)^p \alpha \int_0^t \left(\sum_{k=1}^n y_k \dot{v}_{t_k}(\tau)\right)^p d\tau + i \sum_{j,k=1}^n y_j z_k \int_0^t w_k(s)\dot{v}_{t_j}(s)ds} d\nu_F(y_1, \ldots, y_n, z_1, \ldots, z_n) \\
&= \int_{\mathbb{R}^{2n}} e^{\alpha \sum_{k=1}^n (\sum_{j=1}^k y_j)^p (t_{k+1}-t_k)} e^{i \sum_{k=1}^n z_k \sum_{j=1}^k y_j} d\nu_F(y_1, \ldots, y_n, z_1, \ldots, z_n) \qquad (2.23)
\end{aligned}
$$

On the other hand the last line of Eq. (2.23) coincides with the oscillatory integral

$$
\frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}}^{o} F(x_1, \ldots, x_n, p_1, \ldots, p_n) e^{i \sum_{k=1}^n p_k(x_{k+1}-x_k) + \alpha \sum_{k=1}^n p_k^p (t_{k+1}-t_k)} dx_1 \ldots dx_n dp_1 \ldots dp_n
$$

Indeed, taken an arbitrary test function $\phi \in \mathcal{S}(\mathbb{R}^{2n})$ such that $\phi(0) = 1$, by Fubini theorem and a change of variable argument we have, for any $\epsilon > 0$:

$$
\begin{aligned}
&\frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} \phi(\epsilon x_1, \ldots, \epsilon x_n, \epsilon p_1, \ldots, \epsilon p_n) F(x_1, \ldots, x_n, p_1, \ldots, p_n) \\
&\qquad \times e^{i \sum_{k=1}^n p_k(x_{k+1}-x_k) + \alpha \sum_{k=1}^n p_k^p (t_{k+1}-t_k)} dx_1 \ldots dx_n dp_1 \ldots dp_n = \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n} \times \mathbb{R}^{2n}} \phi(\epsilon x_1, \ldots, \epsilon x_n, \epsilon p_1, \ldots, \epsilon p_n) e^{i \sum_{j=1}^n x_j y_j + i \sum_{j=1}^n z_j p_j} \\
&\times e^{i \sum_{k=1}^n p_k(x_{k+1}-x_k) + \alpha \sum_{k=1}^n p_k^p (t_{k+1}-t_k)} dx_1 \ldots dx_n dp_1 \ldots dp_n d\nu_F(y_1, \ldots, y_n, z_1, \ldots, z_n) = \\
&= \frac{1}{\epsilon^n} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^n} \tilde{\phi}\left(\frac{y_1 - p_1}{\epsilon}, \frac{y_2 + p_1 - p_2}{\epsilon}, \ldots, \frac{y_n + p_{n-1} - p_n}{\epsilon}, \epsilon p_1, \ldots, \epsilon p_n\right) \\
&\qquad e^{i \sum_{j=1}^n z_j p_j} e^{\alpha \sum_{k=1}^n p_k^p (t_{k+1}-t_k)} dp_1 \ldots dp_n d\nu_F(y_1, \ldots, y_n, z_1, \ldots, z_n) = \\
&= \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^n} \tilde{\phi}(p_1, p_2, \ldots, p_n, \epsilon(y_1 + \epsilon p_1), \epsilon \sum_{j=1}^2 (y_j + \epsilon p_j), \ldots, \epsilon \sum_{j=1}^n (y_j + \epsilon p_j)) \\
&\qquad \times e^{i \sum_{k=1}^n z_k (\epsilon \sum_{j=1}^k p_j + \sum_{j=1}^k p_j)} e^{\alpha \sum_{k=1}^n (t_{k+1}-t_k)(\epsilon \sum_{j=1}^k p_j + \sum_{j=1}^k p_j)^p} \\
&\qquad dp_1 \ldots dp_n d\nu_F(y_1, \ldots, y_n, z_1, \ldots, z_n),
\end{aligned}
$$

where

$$
\tilde{\phi}(p_1, \ldots, p_n, k_1, \ldots, k_n) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i \sum_{j=1}^n x_j p_j} \phi(x_1, \ldots, x_n, k_1, \ldots, k_n) dx_1 \ldots dx_n.
$$

In particular, since $\int_{\mathbb{R}^n} \tilde{\phi}(p_1, \ldots, p_n, 0, \ldots, 0) dp_1 \ldots dp_n = 1$, by taking the limit for $\epsilon \downarrow 0$ of (2.3) we obtain the last line of (2.23). $\square$

The right hand side of (2.22) admits an Hamiltonian interpretation. Let us denote by $\mathcal{P}$ a partition $t_0 = 0 < t_1 < \cdots < t_n = t$ of the interval $[0, t]$ into n sub-intervals, let us denote by $B_{\mathcal{P}}$ the finite dimensional subspace of $\mathcal{B}_q \times L_p$ of paths (γ, η) that are respectively piecewise linear and piecewise constant on the subintervals $[t_i, t_{i+1})$

of the partition, namely of the form:

$$\begin{aligned}\gamma(s) &= \sum_{j=1}^{n} \chi_{[t_{j-1},t_j)}(s)\left(\gamma(t_{j-1}) + \frac{\gamma(t_j)-\gamma(t_{j-1})}{t_j - t_{j-1}}(s-t_{j-1})\right)\\ \eta(s) &= \sum_{j=1}^{n} \chi_{[t_{j-1},t_j)}(s)\eta(t_{j-1}) \qquad s\in[0,t].\end{aligned} \tag{2.24}$$

Let $S : \mathcal{B}_q \times L_p \to \mathbb{C}$ be the map defined as

$$S(\gamma,\eta) := \int_0^t \eta(s)\dot{\gamma}(s)ds - i\alpha\int_0^t \eta(s)^p ds, \qquad (\gamma,\eta)\in\mathcal{B}_q\times L_p.$$

If the path (γ,η) belongs to $\mathcal{B}_{\mathcal{P}}$ then $S(\gamma,\eta)$ reduces to a finite sum, namely to

$$S(\gamma,\eta) = \sum_{k=1}^{n} p_{k-1}(x_k - x_{k-1}) - i\alpha\sum_{k=1}^{n} p_{k-1}^{p}(t_k - t_{k-1}),$$

where $p_k \equiv \eta(t_k)$ and $x_k \equiv \gamma(t_k)$, $k = 0,\dots,n$, and $x_n \equiv 0$. In particular, for a purely imaginary constant α, i.e. $\alpha = -i\beta$, $\beta\in\mathbb{R}$, the function S can be interpreted as the action functional in the Hamiltonian formulation (2.17) in the case where the classical Hamiltonian is the polynomial function $H(Q,P) = \beta P^p$.

In fact, in analogy with the approach developed in [113, 38] for the Feynman path integrals, it is possible to prove that the functional L_Φ can also be constructed as the the limit of suitable finite dimensional approximations. Let $\mathcal{P}_n$ be a partition $t_0 = 0 < t_1 < \cdots < t_n = t$ of the interval $[0,t]$ into n sub-intervals with $t_j := \frac{jt}{n}$.

Let $\mathcal{B}_n$ be the finite dimensional subspace of $\mathcal{B}_p \times L_q$ of piecewise linear - piecewise constant paths (γ,η) of the form (2.24), with $t_j := \frac{jt}{n}$. Furthermore, let $T_n : \mathcal{B}_p \times L_q \to \mathcal{B}_p \times L_q$ be the linear map defined by

$$T_n(\xi,\zeta) = (\xi_n,\zeta_n)$$

with

$$\begin{aligned}\xi_n(s) &= \sum_{j=1}^{n} \chi_{\left[\frac{(j-1)t}{n},\frac{jt}{n}\right)}(s)\left(\xi\left(\frac{(j-1)t}{n}\right) + \frac{\xi\left(\frac{jt}{n}\right)-\xi\left(\frac{(j-1)t}{n}\right)}{t/n}\left(s-\frac{(j-1)t}{n}\right)\right);\\ \zeta_n(s) &= \sum_{j=1}^{n} \chi_{\left[\frac{(j-1)t}{n},\frac{jt}{n}\right)}(s)\frac{\int_{(j-1)t/n}^{jt/n}\zeta(s)ds}{t/n}, \qquad s\in[0,t].\end{aligned} \tag{2.25}$$

Lemma 2.2. *The linear operators T_n are bounded with operator norm $\|T_n\| = 1$ and for any $(\xi, \zeta) \in \mathcal{B}_p \times L_q$ the following holds:*

$$\lim_{n\to\infty} \|(\xi,\zeta) - T_n(\xi,\zeta)\|_{\mathcal{B}_p\times L_q} = 0$$

PROOF. The boundedness of T_n follows easily by Hölder inequality, indeed:

$$\|T_n(\xi,\zeta)\|_{\mathcal{B}_p\times L_q} =$$

$$= \left(\sum_{j=1}^{n}\left(\frac{\left|\xi\left(\frac{jt}{n}\right)-\xi\left(\frac{(j-1)t}{n}\right)\right|}{t/n}\right)^{p} t/n\right)^{1/p} + \left(\sum_{j=1}^{n}\left(\frac{\left|\int_{(j-1)t/n}^{jt/n}\zeta(s)ds\right|}{t/n}\right)^{q} t/n\right)^{1/q} \leq$$

$$\leq \left(\int_0^t |\dot{\xi}(s)|^p ds\right)^{1/p} + \left(\int_0^t |\zeta(s)|^q ds\right)^{1/q},$$

and $T_n(\xi,\zeta) = (\xi,\zeta)$ for any $(\xi,\zeta) \in \mathcal{B}_p \times L_q$ of the form (2.25), hence $\|T_n\| = 1$. Let us introduce the linear subspace $V \subset \mathcal{B}_p \times L_q$ defined as

$$V := \left\{(\xi,\zeta) \in \mathcal{B}_p \times L_q \colon \lim_{n\to\infty} \|(\xi,\zeta) - T_n(\xi,\zeta)\|_{\mathcal{B}_p\times L_q} = 0\right\}.$$

By direct check, it is simple to prove that V is a closed subspace of $\mathcal{B}_p \times L_q$ and it contains the dense set of trigonometric polynomials, hence V coincides with the whole Banach space $\mathcal{B}_p \times L_q$. □

Remark 2.3. In the case where $p = q = 2$, the operators T_n are orthogonal projections, see [113, 8].

Further, the following approximation theorem holds.

Theorem 2.4. *For any $f \in \mathcal{F}(\mathcal{B}_q \times L_p)$ the Fresnel integral $L_\Phi(f)$ defined by (2.19) is given by:*

$$L_\Phi(f) = \lim_{n\to\infty} \frac{\int^o_{\mathcal{B}_n} e^{iS(\gamma,\eta)} f(\gamma,\eta) d\gamma d\eta}{\int^o_{\mathcal{B}_n} e^{iS(\gamma,\eta)} d\gamma d\eta}, \tag{2.26}$$

where $\mathcal{B}_n$ is the finite dimensional subspace of $\mathcal{B}_p \times L_q$ defined above. The integrals appearing on the right hand side are oscillatory integrals in the sense of Def. 1.1 and

$d\gamma d\eta$ stands for the volume measure associated to the inner product on $\mathcal{B}_n$ defined by

$$\begin{aligned}\langle(\gamma,\eta),(\gamma',\eta')\rangle &= \int_0^t \dot\gamma(s)\dot\gamma'(s)ds + \int_0^t \eta(s)\eta'(s)ds \\ &= \sum_{j=1}^n \frac{(\gamma(t_j)-\gamma(t_{j-1}))(\gamma'(t_j)-\gamma'(t_{j-1}))}{t_j-t_{j-1}} \\ &\quad + \sum_{j=1}^n \eta(t_{j-1})\eta'(t_{j-1})(t_j-t_{j-1}).\end{aligned}$$

PROOF. By definition, for $f \in \mathcal{F}(\mathcal{B}_q \times L_p)$ the functional $L_\Phi(f)$ is equal to

$$L_\Phi(f) = \int_{\mathcal{B}_p\times L_q} e^{(-1)^p\alpha\int_0^t \dot\xi(s)^p ds + i\int_0^t \zeta(s)\dot\xi(s)ds} d\mu(\xi,\zeta)$$

where

$$f(\gamma,\eta) = \int_{\mathcal{B}_p\times L_q} e^{i\langle\gamma,\xi\rangle + i\langle\eta,\zeta\rangle} d\mu(\xi,\zeta), \quad (\gamma,\eta)\in\mathcal{B}_q\times L_p.$$

Furthermore, by Lemma 2.2, the continuity of the function $\Phi : \mathcal{B}_p \times L_q \to \mathbb{C}$ and dominated convergence theorem:

$$L_\Phi(f) = \lim_{n\to\infty}\int_{\mathcal{B}_p\times L_q} e^{\Phi\circ T_n(\xi,\zeta)} d\mu(\xi,\zeta) \tag{2.27}$$

Let us recall the finite dimensional subspace $\mathcal{B}_n \subset \mathcal{B}_p \times L_q$ of paths (ξ,ζ) of the form (2.25) and let us introduce the complex bounded Borel measure μ_n on $\mathcal{B}_n$ defined as the image measure of μ under T_n, namely $\mu_n(I) := \mu(T_n^{-1}(I))$ for any Borel set $I \subset \mathcal{B}_n$. The right hand side of (2.27) is equal to

$$L_\Phi(f) = \lim_{n\to\infty}\int_{\mathcal{B}_n} e^{\Phi|_{\mathcal{B}_n}(\xi,\zeta)} d\mu_n(\xi,\zeta).$$

Let us now consider in the finite dimensional subspace $\mathcal{B}_n$ the coordinates $(y_1,\dots,y_n, z_1,\dots,z_n)$, where $\xi = \sum_{j=1}^n y_j v_{t_j}$, $\zeta = \sum_{j=1}^n z_j w_j$ and the vectors v_{t_j} and w_j, $j = 1,\dots,n$ are defined by (2.20) and (2.21). Let $U_n : \mathcal{B}_n \to \mathbb{R}^{2n}$ be the isomorphism defined as $U_n(\xi,\zeta) = (y_1,\dots,y_n,z_1,\dots,z_n)$ and let ν_n be the Borel measure on $\mathbb{R}^{2n}$ image of μ_n under the map U_n. We obtain:

$$\begin{aligned}L_\Phi(f) &= \lim_{n\to\infty}\int_{\mathbb{R}^{2n}} e^{(-1)^p\alpha\int_0^t\left(\sum_{k=1}^n y_k\dot v_{t_k}(\tau)\right)^p d\tau + i\sum_{j,k=1}^n y_j z_k\int_0^t w_k(s)\dot v_{t_j}(s)ds} \\ &\qquad d\nu_F(y_1,\dots,y_n,z_1,\dots,z_n) \\ &= \lim_{n\to\infty}\int_{\mathbb{R}^{2n}} e^{\alpha\sum_{k=1}^n(\sum_{j=1}^k y_j)^p(t_{k+1}-t_k)} e^{i\sum_{k=1}^n z_k\sum_{j=1}^k y_j} d\nu_F(y_1,\dots,y_n,z_1,\dots,z_n)\end{aligned}$$

Then, by Theorem 2.3 the latter is equal to

$$L_\Phi(f) = \lim_{n\to\infty} \frac{1}{(2\pi)^{2n}} \int^o_{\mathbb{R}^{2n}} e^{i\sum_{k=1}^n p_k(x_{k+1}-x_k)+\alpha\sum_{k=1}^n p_k^p(t_{k+1}-t_k)} \\ \times F(x_1,\ldots,x_n,p_1,\ldots,p_n)dx_1\ldots dx_n dp_1\ldots dp_n,$$

with

$$F(x_1,\ldots,x_n,p_1,\ldots,p_n) = \int_{\mathbb{R}^{2n}} e^{i\sum_{k=1}^n y_k x_k + i\sum_{k=1}^n z_k p_k} d\nu_F(y_1,\ldots,y_n,z_1,\ldots,z_n).$$

On the other hand, by introducing in the finite dimensional subspace $\mathcal{B}_n \subset \mathcal{B}_q \times L_p$ of piecewise linear - piecewise constant paths (γ,η) of the form (2.24), with $t_j := \frac{jt}{n}$, the coordinates $(x_1,\ldots,x_n,p_1,\ldots,p_n)$ with $x_j = \gamma(t_j)$ and $p_j = \frac{\int_{t_{j-1}}^{t_j}\eta(s)ds}{t_j - t_{j-1}}$, we obtain

$$\frac{\int^o_{\mathcal{B}_n} e^{iS(\gamma,\eta)} f(\gamma,\eta)d\gamma d\eta}{\int^o_{\mathcal{B}_n} e^{iS(\gamma,\eta)}d\gamma d\eta} = \\ = \frac{1}{(2\pi)^{2n}} \int^o_{\mathbb{R}^{2n}} e^{i\sum_{k=1}^n p_k(x_{k+1}-x_k)+\alpha\sum_{k=1}^n p_k^p(t_{k+1}-t_k)} \\ f(\gamma_{x_1,\ldots,x_n},\eta_{p_1,\ldots,p_n})dx_1\ldots dx_n dp_1\ldots dp_n,$$

where

$$\gamma_{x_1,\ldots,x_n}(s) = \sum_{j=1}^n \chi_{[t_{j-1},t_j)}(s)\left(x_{j-1} + \frac{x_j - x_{j-1}}{t_j - t_{j-1}}(s - t_{j-1})\right),$$

$$\eta_{p_1,\ldots,p_n}(s) = \sum_{j=1}^n \chi_{[t_{j-1},t_j)}(s)p_j, \qquad s\in[0,t].$$

Eventually, we obtain the final result, namely the Eq. (2.26), by noting that

$$f(\gamma_{x_1,\ldots,x_n},\eta_{p_1,\ldots,p_n}) = F(x_1,\ldots,x_n,p_1,\ldots,p_n).$$

□

Chapter 3

A path integral for the Schrödinger equation with magnetic field*

In this chapter we focus on the Schrödinger equation for a non-relativistic quantum particle moving under the influence of a (rather general) magnetic field **B** associated to a vector potential **a**

$$i\hbar\frac{\partial}{\partial t}\psi(t,x) = \frac{1}{2}\left(-i\hbar\nabla - \lambda\mathbf{a}(x)\right)^2\psi(t,x), \tag{3.1}$$

where $\mathbf{a}(x) \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}$ plays the role of a coupling constant. We construct a rigorous mathematical definition for the corresponding Feynman path integral formula

$$\psi(t,x) = \int_{\gamma(t)=x} e^{\frac{i}{2\hbar}\int_0^t|\dot\gamma(s)|^2ds+\frac{i}{\hbar}\int_0^t\lambda\mathbf{a}(\gamma(s))\cdot\dot\gamma(s)ds}\psi_0(\gamma(0))d\gamma, \tag{3.2}$$

in terms of infinite dimensional oscillatory integrals. In the physical literature [46, 48, 49, 104] the problem of the definition of Feynman path integrals in the presence of magnetic field has been extensively investigated. As we said before, the traditional procedure relies upon a time slicing approximation of the form

$$\int e^{\frac{i}{2\hbar}\sum_i\frac{|\gamma(t_{i+1})-\gamma(t_i)|^2}{t_{i+1}-t_i}+\frac{i}{\hbar}\sum_i\lambda\mathbf{a}(\gamma(\tilde t_i))\cdot(\gamma(t_{i+1})-\gamma(t_i))}\psi_0(\gamma(0))\prod_i\frac{d\gamma(t_i)}{(2\pi i\hbar(t_{i+1}-t_i))^{1/2}}, \tag{3.3}$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_n = t$ and $\tilde t_i \in [t_i, t_{i+1}]$, $i = 0, \dots, n-1$. However this procedure presents some ambiguities since different choices of the point $\tilde t_i \in [t_i, t_{i+1}]$ lead to different results. The correct choice relies on the so-called mid-point rule which requires that in the formula (3.3) the vector potential **a** is evaluated at the point $\tilde t_i \equiv \frac{t_{i+1}+t_i}{2}$. In the Euclidean version of Feynman formula, namely the Feynman-Kac-Itô formula [105] for the solution of the corresponding heat equation

*The results of this chapter are collected in [5].

in a magnetic field, this procedure yields the Wiener integral representation

$$u(t,x) = \mathbb{E}\left[u(x+\omega(t))e^{-i\int_0^t \lambda \mathbf{a}(\omega(s)+x)\circ d\omega(s)}\right], \tag{3.4}$$

where $\circ d\omega(s)$ denotes the Stratonovich stochastic integraland $\mathbb{E}$ is the expectation with respect to the Wiener measure for the standard Brownian motion. In fact other choices for the point $\tilde{t}_i \in [t_i, t_{i+1}]$ in Eq. (3.3) would lead to different stochastic integrals but, as pointed out in [105, 104], the *mid point rule*, or, equivalently, the Stratonovich stochastic integration, is the only one yielding the *gauge invariance* of formulas (3.2) and (3.4). In fact, this particular approximation procedure can be obtained by means of Trotter product formula [49]. Other approaches were proposed in the work by Z. Haba [56], relying in principle on an analytic continuation of Wiener integrals, and by W. Ichinose [63, 61], based on particular time-slicing approximations. Rigorously defined infinite dimensional oscillatory integrals have been applied in [4] to the case of a constant magnetic field in the Coulomb gauge.

Our aim in this chapter is twofold. First of all, for general vector potentials **a**, we prove that the finite dimensional approximation procedure associated to the definition of infinite dimensional oscillatory integrals provides the correct construction of the Feynman integral for the magnetic field *without any additional prescription*. We shall show that the mid-point rule has not to be postulated but it is a direct consequence of our construction (for a discussion of these issues as well as the inclusion of magnetic fields in path integral formulas see, e.g., [44, 17, 114, 71, 80]).

The second result concerns the dependence of the Feynman path integral on the sequence of finite dimensional approximations introduced in the construction. We show that the requirement of the independence of the particular form of the approximation procedure leads to the introduction of a counterterm in the classical action functional. In the case of a constant magnetic field we provide a formula that is gauge-independent, generalizing a similar result proposed in [4] . It is worthwhile to recall that the case of the uniform magnetic was also studied in [64] for Wiener path integrals in relation with the Van Vleck-Pauli formula. Let us also mention that the study of heat semigroup with magnetic field via Feynman-Kac-Itô formula [105, 107] has been extended to the case of fractals, e.g. [58], graphs [55], and manifolds [23, 50, 108, 30].

In Section 3.1 we present some functional analytical results on the Schrödinger equation (3.1) with magnetic field and, under suitable assumptions on the vector potentials **a** and the initial datum ψ_0, we prove that the series expansion of the solution in powers of the coupling constant has a finite radius of convergence. In Section 3.2 we study the Schrödinger equation (3.1) for analytic vector potentials **a** and construct a Feynman path integral representation for its solution in terms of a particular class of infinite dimensional oscillatory integrals. In Section 3.3 we consider the particular case of a constant magnetic field and provide a renormalized Feynman path integral formula which allows to obtain the independence of the construction procedure of the particular choice of finite dimensional approximations.

Let us remark that all our results extend to the case where a potential term V is added on the right hand side of (3.1) (see remarks 3.4, 3.6, and 3.10).

3.1 Schrödinger equation with magnetic field

Let us consider the dynamics of a non-relativistic quantum particle moving in a magnetic field $\mathbf{B} = \operatorname{rot}\mathbf{a}$, where $\mathbf{a}$ is a vector potential associated to $\mathbf{B}$. The quantum Hamiltonian operator for this system is given on a smooth compactly supported vector $\psi \in C_0^\infty(\mathbb{R}^3)$ by

$$H\psi = \frac{1}{2}\left(-i\hbar\nabla - \lambda\mathbf{a}(x)\right)^2\psi, \qquad \psi \in C_0^\infty(\mathbb{R}^3)$$

(where, for notational simplicity, we set equal to 1 the parameters mass m, velocity of light c, and elementary charge e). The parameter $\lambda \in \mathbb{R}$ stands for a coupling constant. In the following we shall assume that the components a_j, $j = 1, 2, 3$, of the vector potential $\mathbf{a}$ are real valued functions and belong to the space $\mathcal{F}_c(\mathbb{R}^3)$ of Fourier transforms of complex Borel measures μ_j on $\mathbb{R}^3$ with compact support, i.e. they are functions of the form

$$a_j(x) = \int_{\mathbb{R}^3} e^{iy\cdot x} d\mu_j(y), \qquad x \in \mathbb{R}^3, \quad j = 1, 2, 3, \tag{3.5}$$

with μ_j of compact support.

Remark 3.1. For later use we point out that $a_j \in \mathcal{F}_c(\mathbb{R}^3)$, $j = 1, 2, 3$ implies that a_j has an analytic continuation to a function on $\mathbb{C}^3$, denoted by the same symbol.

Under this assumption, it is possible to prove (see, e.g. [100, 105, 30, 79, 23]) that H defined on $C_0^\infty(\mathbb{R}^3)$ is positive, symmetric and, $\mathbf{a}$ being bounded, has a unique self-adjoint extension $H : D(H) \subset L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$, with domain

$$D(H) = \left\{\psi \in L^2(\mathbb{R}^3) \colon \int_{\mathbb{R}^3} |y|^4 |\hat{\psi}(y)|^2 dk < \infty\right\}, \tag{3.6}$$

where $\hat{\psi} \in L^2(\mathbb{R}^3)$ denotes the Fourier transform of the vector $\psi \in L^2(\mathbb{R}^3)$ and $|y|$ is the norm of the vector $y \in \mathbb{R}^3$.

By Stone's theorem, H generates a one-parameter group $U(t) = e^{-\frac{i}{\hbar}Ht}$, $t \in \mathbb{R}$, of unitary operators on $L^2(\mathbb{R}^3)$, solving the Schrödinger equation in the following sense:

$$i\hbar\partial_t U(t)\psi_0 = HU(t)\psi_0, \qquad \psi_0 \in D(H), \tag{3.7}$$

(where the derivation on the left is a strong one in $L^2(\mathbb{R}^3)$).

We shall assume, without loss of generality, that the vector potential $\mathbf{a}$ satisfies the Coulomb gauge, namely that $\operatorname{div}\mathbf{a} = 0$. In this case the Hamiltonian operator

H can be written as $H = H_0 + W$, where H_0 is the free Hamiltonian, namely the operator $H_0 = -\frac{\hbar^2}{2}\Delta$ on $D(H_0) = D(H)$ (defined by (3.6)) and $W = \lambda A + \lambda^2 B$, where $A = i\hbar\mathbf{a} \cdot \nabla$ and B is the multiplication operator associated to the function $\frac{1}{2}|\mathbf{a}^2|$ (both well-defined on $D(H_0)$).

Under the assumption that the initial datum $\psi_0 \in L^2(\mathbb{R}^3)$ has a compactly supported Fourier transform, i.e. $\psi_0 \in \mathcal{F}_c(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$, the Dyson series expansion (in powers of the coupling constant λ for the vector $U(t)\psi_0$ has a finite radius of convergence as the following theorem shows. The arguments used in its proof are similar in spirit to the results of the Paley-Wiener theorem [103]. For related results by purely analytic methods see [100]).

Theorem 3.1. *Let us assume that the Fourier transform $\hat{\psi}_0$ of the vector $\psi_0 \in L^2(\mathbb{R}^3)$ has a compact support included in the ball $B_\rho \equiv \{x \subset \mathbb{R}^3 \colon |x| < \rho\}$ and that the component a_j, $j = 1, \dots, 3$, of the vector potentials $\mathbf{a}$ are of the form (3.5), with μ_j, $j = 1, \dots, 3$, bounded Borel measures with support contained in the ball B_R, for some $R \in \mathbb{R}^+$. Then the expansion in powers of the coupling constant λ for the vector $U(t)\psi_0$, namely $U(t)\psi_0 = \sum_m \lambda^m \phi_m(t)$, with*

$$\phi_m(t) = \sum_{(n,k)\in\mathbb{N}^2 \colon 2n-k=m, k\leq n} \left(-\frac{i}{\hbar}\right)^n \phi_{n,k}$$

and $\phi_{n,k}$ given by Eq. (3.10) on page 38, converges in $L^2(\mathbb{R}^3)$ for $|\lambda| < \lambda^*$, with

$$\lambda^* = \left(\frac{2\alpha^2 t}{\hbar}\left(2r^2 t\hbar + 1\right)\right)^{-1/2}, \tag{3.8}$$

where $\alpha = \sup_{x\in\mathbb{R}^3} |\mathbf{a}(x)|$ and $r = \max\{\rho, R\}$.

Remark 3.2. Correspondingly as in remark 3.1, we point out for later use that the assumption on ψ_0 in Theorem 3.1 implies that ψ_0 has a unique extension to an analytic function on $\mathbb{C}^3$, denoted by the same symbol.

At first we prove the following lemma.

Lemma 3.1. *Under the assumptions of Theorem 3.1, the following holds:*

$$\begin{aligned} &\|U_0(t_1)O_1U_0(t_2)O_2 \cdots U_0(t_n)O_nU_0(t_{n+1})\psi_0\| \leq \\ &\leq \hbar^k\alpha^k \left(\frac{\alpha^2}{2}\right)^{n-k} \prod_{j=0}^{k-1}(\rho + 2R(n-k) + jR)\|\psi_0\|, \end{aligned}$$

where $U_0(t) = e^{-\frac{i}{\hbar}H_0t}$, $A = i\hbar\mathbf{a} \cdot \nabla$, $B = \frac{1}{2}|\mathbf{a}|^2$, $O_j \in \{A, B\}$, $j = 1, \dots, n$, $k = \#\{j \colon O_j = A\}$ and $\alpha = \sup_{x\in\mathbb{R}^3} |\mathbf{a}(x)|$.

PROOF. Let $\phi \in L^2(\mathbb{R}^3)$ be a function whose Fourier transform $\hat{\phi}$ has support in a ball B_ρ centered at the origin with radius $\rho \in \mathbb{R}^+$.

Then the following holds.

- For any $t \in \mathbb{R}$, the vector $U_0(t)\phi \in L^2(\mathbb{R}^3)$ has a Fourier transform with support contained in B_ρ. Indeed $\widehat{U_0(t)\phi}$ is simply given by
$$\widehat{U_0(t)\phi}(y) = e^{-\frac{i}{\hbar}|y|^2 t}\hat{\phi}(y), \qquad y \in \mathbb{R}^3.$$
- The vector $B\phi \in L^2(\mathbb{R}^3)$ has a Fourier transform with support contained in $B_{\rho+2R}$. Indeed, under the assumptions on the components a_j of the vector field $\mathbf{a}$, the function $x \mapsto \mathbf{a}^2 \equiv |\mathbf{a}(x)|^2$ is Fourier transform of a Borel measure $\mu_{\mathbf{a}^2}$ on $\mathbb{R}^3$ with support contained in the ball B_{2R}, with $\mu_{\mathbf{a}^2} = \sum_{j=1}^3 \mu_j * \mu_j$, where the symbol $\mu * \nu$ stands for the convolution of the measures μ and ν. It is simple to verify that if the supports of the measures μ_j is contained in B_R, then the support of the convolution $\mu_j * \mu_j$ is contained in B_{2R}. Correspondingly, the Fourier transform of $B\phi$ is given by
$$\widehat{B\phi}(y) = \frac{1}{2}\int_{\mathbb{R}^3} \hat{\phi}(y-y')d\mu_{\mathbf{a}^2}(y'), \quad y \in \mathbb{R}^3,$$
and its support is contained in $B_{\rho+2R}$.
- The norm of the vector $B\phi$ is bounded by
$$\|B\phi\| \leq \frac{1}{2}\|\mathbf{a}^2\|_\infty \|\phi\|,$$
where $\|\mathbf{a}^2\|_\infty = \sup_{x\in\mathbb{R}^3} |\mathbf{a}(x)|^2$, which is finite by the assumptions on the components a_j, $j = 1, \dots, 3$.
- The vector $A\phi \in L^2(\mathbb{R}^3)$ (with A as in Lemma 3.1) has a Fourier transform with support contained in $B_{\rho+R}$, given by
$$\widehat{A\phi}(y) = -\hbar\sum_{j=1}^3 \int_{\mathbb{R}^3} (y_j - y_j')\hat{\phi}(y-y')d\mu_j(y'), \qquad k \in \mathbb{R}^3.$$
Moreover, the norm of $A\phi$ satisfies the following bound
$$\|A\phi\| \leq \hbar\sqrt{\|\mathbf{a}^2\|_\infty}\,\rho\,\|\phi\|.$$

Now it is straightforward to verify that, if $\#\{j\colon O_j = A\} = k$:

$$\begin{aligned}
&\|U_0(t_1)O_1U_0(t_2)O_2\cdots U_0(t_n)O_nU_0(t_{n+1})\psi_0\| \leq \\
&\leq \|U_0(t_1)AU_0(t_2)\cdots U_0(t_k)AU_0(t_{k+1})B\cdots U_0(t_n)BU_0(t_{n+1})\psi_0\| \leq \\
&\leq \hbar^k\alpha^k\left(\frac{\alpha^2}{2}\right)^{n-k}\prod_{j=0}^{k-1}(\rho+2R(n-k)+jR)\|\psi_0\|.
\end{aligned}$$

□

Proof [of Theorem 3.1]. By the classical Dyson-Phillips expansion for the vector $U(t)\psi_0$ [100], we have

$$\begin{aligned}
U(t)\psi_0 &= \sum_{n=0}^{\infty}\left(-\frac{i}{\hbar}\right)^n\int_{\Delta_n(t)} e^{-\frac{i}{\hbar}H_0(t-s_n)}We^{-\frac{i}{\hbar}H_0(s_n-s_{n-1})}\ldots \\
&\quad \cdots We^{-\frac{i}{\hbar}H_0(s_2-s_1)}We^{-\frac{i}{\hbar}H_0s_1}\psi_0 ds_1\ldots ds_n \\
&= \sum_{n=0}^{\infty}\left(-\frac{i}{\hbar}\right)^n\int_{\Delta_n(t)} U_0(t-s_n)(\lambda A+\lambda^2B)U_0(s_n-s_{n-1})\cdots \\
&\quad \cdots(\lambda A+\lambda^2B)U_0(s_1)\psi_0 ds_1\ldots ds_n,
\end{aligned}$$

where $\Delta_n(t)\subset\mathbb{R}^n$ is the n-dimensional simplex defined as $\Delta_n(t) = \{(s_1,\ldots,s_n)\in\mathbb{R}^n\colon 0\leq s_1\leq\cdots\leq s_n\leq t\}$ and $W=\lambda A+\lambda^2B$. The dependence on the coupling constant λ can be made explicit as

$$U(t)\psi_0 = \sum_{n=0}^{\infty}\left(-\frac{i}{\hbar}\right)^n\sum_{k=0}^{n}\lambda^k(\lambda^2)^{n-k}\phi_{n,k}, \tag{3.9}$$

where the term $\phi_{n,k}\in L^2(\mathbb{R}^3)$ is a sum of $\binom{n}{k}$ terms of the form

$$\int_{\Delta_n(t)} U_0(t-s_n)O_1U_0(s_n-s_{n-1})O_2\cdots O_nU_0(s_1)\psi_0 ds_1\ldots ds_n,$$

where we recall that $O_j = A, B$, $j=1,\ldots,n$ and $\#\{j\colon O_j = A\} = k$. More precisely:

$$\phi_{n,k} = \sum_{E\in C_k^n}\int_{\Delta_n(t)} U_0(t-s_n)O_1^EU_0(s_n-s_{n-1})O_2^E\cdots O_n^EU_0(s_1)\psi_0 ds_1\ldots ds_n, \tag{3.10}$$

where the sum is taken over the set C_k^n of all possible subsets $E\subset\{1,\ldots,n\}$ with k elements and the map $O^E:\{1,\ldots,n\}\to\{A,B\}$ is defined as $O_i^E := A$ if $i\in E$ and $O_i^E := B$ if $i\notin E$.

By Lemma 3.1 we have

$$\|\phi_{n,k}\| \leq \binom{n}{k}\frac{t^n}{n!}\hbar^k\alpha^k\left(\frac{\alpha^2}{2}\right)^{n-k}\prod_{j=0}^{k-1}(\rho+2R(n-k)+jR)\|\psi_0\|.$$

In particular, by setting $r := \max\{\rho, R\}$, we obtain:

$$\|\phi_{n,k}\| \leq \frac{t^n}{(n-k)!}\hbar^k\alpha^{2n-k}\left(\frac{1}{2}\right)^{n-k} r^k\binom{2n-k}{k}\|\psi_0\|. \tag{3.11}$$

Now, the sum appearing in (3.9) can be written as:

$$\begin{aligned} U(t)\psi_0 &= \sum_{m=0}^{\infty}\lambda^m \sum_{(n,k)\in\mathbb{N}^2:\, 2n-k=m,\, k\leq n}\left(-\frac{i}{\hbar}\right)^n\phi_{n,k} \\ &= \sum_m \lambda^m\phi_m. \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} \phi_m &= \left(-\frac{i}{\hbar}\right)^{m/2}\sum_{h=0}^{m/2}\left(-\frac{i}{\hbar}\right)^h\phi_{h+\frac{m}{2},2h}, && m \text{ even}; \\ \phi_m &= \left(-\frac{i}{\hbar}\right)^{(m+1)/2}\sum_{h=0}^{(m-1)/2}\left(-\frac{i}{\hbar}\right)^h\phi_{h+\frac{m+1}{2},2h+1}, && m \text{ odd}. \end{aligned} \tag{3.13}$$

By estimate (3.11) we have for m even, namely $m = 2M$, $M \in \mathbb{N}$:

$$\begin{aligned} \|\phi_{2M}\| &\leq \left(\frac{\alpha^2 t}{2\hbar}\right)^M\sum_{h=0}^{M}(2r^2t\hbar)^h\frac{1}{(M-h)!}\binom{2M}{2h}\|\psi_0\| \\ &\leq \left(\frac{\alpha^2 t}{2\hbar}\right)^M(2r^2t\hbar+1)^M\max_{h\in\{0,\ldots,M\}}\frac{h!}{M!}\binom{2M}{2h}\|\psi_0\| \\ &\leq \left(\frac{2\alpha^2 t}{\hbar}(2r^2t\hbar+1)\right)^M\|\psi_0\|; \end{aligned}$$

analogously, for m odd, namely $m = 2M+1$, $M \in \mathbb{N}$ we get

$$\|\phi_{2M+1}\| \leq 2rt\alpha\hbar\left(\frac{2\alpha^2 t}{\hbar}(2r^2t\hbar+1)\right)^M\|\psi_0\|.$$

Hence, the series (3.12) converges in $L^2(\mathbb{R}^d)$ for $|\lambda| < \left(\frac{2\alpha^2 t}{\hbar}(2r^2t\hbar+1)\right)^{-1/2}$.

$\square$

Remark 3.3. Since the Hamiltonian operator H is self-adjoint and positive, as pointed out at the beginning of this section, it generates an analytic semigroup. Hence, for $z \in \mathbb{C}$ belonging to the closure $\bar{D}$ of the open sector $D \subset \mathbb{C}$ of the complex plane defined as

$$D = \{z \in \mathbb{C} \colon Re(z) > 0\},$$

it is possible to define the operator $V(z) = e^{-zH}$ yielding for $z = \frac{i}{\hbar}t$ and $t \in \mathbb{R}$ the Schrödinger group and for $z \in \mathbb{R}^+$ the heat semigroup. In both cases, under the assumptions of Theorem 3.1 the perturbative Dyson expansion for the vector $V(z)\psi_0$ has a positive radius of convergence (depending on $|z|$). Indeed, if $z \in D$, the Dyson expansion can be written as

$$\begin{aligned} e^{-zH}\psi_0 = \sum_{n=0}^{\infty}(-z)^n \int_{\Delta_n} e^{-zH_0(1-s_n)} W e^{-zH_0(s_n-s_{n-1})} \dots \\ \dots W e^{-zH_0(s_2-s_1)} W e^{-zH_0 s_1} \psi_0 ds_1 \dots ds_n, \end{aligned} \tag{3.14}$$

with $\Delta_n \equiv \Delta_n(1) = \{(s_1, \dots, s_n) \in \mathbb{R}^n \colon 0 \le s_1 \le \dots \le s_n \le 1\}$ and $W = \lambda A + \lambda^2 B$. By collecting in the sum (3.14) all the terms associated to the same power of the coupling constant λ, we get

$$e^{-zH}\psi_0 = \sum_m \lambda^m \phi_m(z) \tag{3.15}$$

where $\phi_m(z) = \sum_{(n,k)\in\mathbb{N}^2 \colon 2n-k=m, k\le n} (-z)^n \phi_{n,k}(z)$ with

$$\phi_{n,k}(z) = \sum_{E \in C_k^n} \int_{\Delta_n} e^{-zH_0(t-s_n)} O_1^E e^{-zH_0(s_n-s_{n-1})} O_2^E \cdots O_n^E e^{-zH_0 s_1} \psi_0 ds_1 \dots ds_n, \tag{3.16}$$

where, analogously to Eq. (3.10), the sum is taken over the set C_k^n of all possible subsets $E \subset \{1, \dots, n\}$ with k elements and the map $O^E : \{1, \dots, n\} \to \{A, B\}$ is defined as $O_i^E := A$ if $i \in E$ and $O_i^E := B$ if $i \notin E$.

By repeating the arguments in the proof of Theorem 3.1, it is now easy to verify that the expansion (3.15) converges in $L^2(\mathbb{R}^3)$ for $|\lambda| < \lambda^*(z)$, with

$$\lambda^*(z) = \left(2\alpha^2|z| \left(2r^2\hbar^2|z| + 1\right)\right)^{-1/2}, \qquad z \in D. \tag{3.17}$$

Remark 3.4. The results of Theorem 3.1 and remark 3.3 can easily be extended to the case where a (bounded) scalar potential V is added to the Hamiltonian. Indeed, let us consider the following

$$H\psi(x) = \frac{1}{2}\left(-i\hbar\nabla - \lambda \mathbf{a}(x)\right)^2 \psi(x) + \lambda V(x)\psi(x), \qquad \psi \in C_0^\infty(\mathbb{R}^3), \tag{3.18}$$

where $V \in \mathcal{F}_c(\mathbb{R}^3)$, i.e. $V : \mathbb{R}^3 \to \mathbb{R}$ a function of the form:

$$V(x) = \int e^{ixy} d\mu_V(y), \qquad x \in \mathbb{R}^3, \tag{3.19}$$

with μ_V complex Borel measure with support contained in the ball B_R. Under the assumptions of Theorem 3.1, Lemma 3.1 still holds. In particular, by setting $A = i\hbar\mathbf{a} \cdot \nabla + V$, $B = \frac{1}{2}|\mathbf{a}|^2$, $\alpha = \|\mathbf{a}\|_\infty$ and $\tilde{\alpha} = 2\max\{\hbar\|\mathbf{a}\|_\infty, |V|_\infty\}$, we get:

$$\begin{aligned}
&\|U_0(t_1)O_1U_0(t_2)O_2 \cdots U_0(t_n)O_nU_0(t_{n+1})\psi_0\| \leq \\
&\leq \tilde{\alpha}^k \left(\frac{\alpha^2}{2}\right)^{n-k} \textstyle\prod_{j=0}^{k-1}(\rho + 2R(n-k) + jR)\|\psi_0\|,
\end{aligned} \tag{3.20}$$

where $U_0(t) = e^{-\frac{i}{\hbar}H_0t}$, $O_j \in \{A, B\}$, $j = 1, \dots, n$, $k = \#\{j : O_j = A\}$. By using (3.20) it is now possible to repeat the proof of Theorem 3.1, obtaining the convergence in $L^2(\mathbb{R}^3)$ of the perturbative Dyson expansion for the vector $e^{-\frac{i}{\hbar}Ht}\psi_0$ for $\lambda < \tilde{\lambda}$, where

$$\tilde{\lambda} = \left(\frac{2\tilde{\alpha}^2 t}{\hbar}\left(\frac{2r^2 t}{\hbar} + 1\right)\right)^{-1/2}. \tag{3.21}$$

3.2 Feynman path integral for magnetic field

The present section is devoted to the construction of the Feynman path integral representation of the solution to the time dependent Schrödinger equation (3.7) in terms of Feynman maps on the Cameron-Martin space $\mathcal{H}_t$ defined in Chpt. 1, i.e.

$$\psi(t, x) = \int_{\mathcal{H}_t} e^{\frac{i}{2\hbar}\int_0^t \|\dot{\gamma}(s)\|^2 ds - \frac{i}{\hbar}\int_0^t \lambda\mathbf{a}(\gamma(s)+x)\cdot\dot{\gamma}(s)ds}\psi_0(\gamma(t) + x)d\gamma. \tag{3.22}$$

Remark 3.5. Formula (3.22) differs from (3.2) for the sign in front of the term $\int_0^t \lambda\mathbf{a}(\gamma(s) + x) \cdot \dot{\gamma}(s)ds$. This is due to the fact that in the heuristic Feynman formula (3.2) the paths γ are pointed at the final time (i.e. $\gamma(t) = x$), while in Eq. (3.22) the path $\gamma \in \mathcal{H}_t$ satisfy the condition $\gamma(t) = 0$.

First of all, it is interesting to point out that the existing techniques of infinite dimensional oscillatory integration based on a Parseval-type equality (see Theorem 1.1) do not work in the case where the classical action functional contains the term $\int_0^t \mathbf{a}(\gamma(s)) \cdot \dot{\gamma}(s)ds$ (as the term in the exponent of (3.22)).

In fact the function $f : \mathcal{H}_t \to \mathbb{C}$ defined on vectors γ belonging to the Cameron-Martin space $\mathcal{H}_t$ as

$$f(\gamma) := \int_0^t \mathbf{a}(\gamma(s)) \cdot \dot{\gamma}(s)ds, \tag{3.23}$$

cannot in general belong to the Banach algebra $\mathcal{F}(\mathcal{H}_t)$, even under rather strong assumption on the vector potential $\mathbf{a}$, unless in the trivial case where $\mathbf{a}$ would be a conservative vector field (hence the associated magnetic field $\mathrm{rot}\,\mathbf{a}$ would vanish identically!). In this case indeed it is simple to prove that one has $f \in \mathcal{F}(\mathcal{H}_t)$ and $f(\gamma) := \int_0^t \mathbf{a}(\gamma(s)) \cdot \dot{\gamma}(s)ds = U(\gamma(t)) - U(\gamma(0)))$. This particular case has already been studied in the previous chapter (see also [6]). However, in the physically more interesting case where $\mathrm{rot}\,\mathbf{a} \not\equiv 0$, even if any of the three components a_i, $i = 1, 2, 3$, of the vector potential $\mathbf{a}$ belongs to $\mathcal{F}(\mathbb{R}^3)$, it is not possible to prove that $f \in \mathcal{F}(\mathcal{H}_t)$. In fact, the oscillatory integration of the function f in (3.23) involves most of the problems arising in stochastic integration theory [70]. Indeed, since any $\gamma \in \mathcal{H}_t$ is a bounded variation function, it is easy to show that the function f is the pointwise limit of sequence of cylinder functions of the form

$$f_n(\gamma) := \sum_{j=0}^{n-1} \mathbf{a}(\gamma(t_j)) \cdot (\gamma(t_{j+1}) - \gamma(t_j)), \qquad \gamma \in \mathcal{H}_t$$

or, equivalently, of

$$g_n(\gamma) := \sum_{j=0}^{n-1} \mathbf{a}(\gamma(t_{j+1})) \cdot (\gamma(t_{j+1}) - \gamma(t_j)), \qquad \gamma \in \mathcal{H}_t,$$

where $t_j \equiv jt/n$. Furthermore, if $a_i \in \mathcal{F}(\mathcal{H}_t)$, then the cylinder functions $\{f_n\}_n$ and $\{g_n\}_n$ are Fresnel integrable since they are both finite linear combinations of functions of the form (1.10). Indeed , if for any $i = 1, \dots, 3$ $a_i = \hat{\mu}_i$, with μ_i bounded complex Borel measures on $\mathbb{R}^3$, then

$$f_n(\gamma) := \sum_{j=0}^{n-1} \sum_{\alpha=1}^{3} \int_{\mathbb{R}^3} e^{i\langle \gamma, v_{t_j} k\rangle} \langle \gamma, (v_{t_{j+1}} - v_{t_j})\hat{e}_\alpha \rangle d\mu_\alpha(k)$$

and

$$g_n(\gamma) := \sum_{j=0}^{n-1} \sum_{\alpha=1}^{3} \int_{\mathbb{R}^3} e^{i\langle \gamma, v_{t_{j+1}} k\rangle} \langle \gamma, (v_{t_{j+1}} - v_{t_j})\hat{e}_\alpha \rangle d\mu_\alpha(k),$$

where $\hat{e}_\alpha$, $\alpha = 1, \dots, 3$ are the vectors of the canonical basis of $\mathbb{R}^3$, while for $s \in [0, t]$ the function $v_s : [0, t] \to \mathbb{R}$ is defined by

$$v_s(r) = \chi_{[0,s]}(r) r + \chi_{(s,t]}(r) s, \qquad r \in [0, t].$$

By direct computation, one has $\widetilde{\int_{\mathcal{H}_t}^o} e^{\frac{i}{2\hbar}\langle \gamma, \gamma\rangle} f_n(\gamma) d\gamma = 0$ for all $n \in \mathbb{N}$, while

$$\widetilde{\int_{\mathcal{H}_t}^o} e^{\frac{i}{2\hbar}\langle \gamma, \gamma\rangle} \mathfrak{g}_n(\gamma) d\gamma = -\hbar \sum_{j=0}^{n-1} \sum_{\alpha=1}^{3} \int_{\mathbb{R}^3} e^{-\frac{i\hbar}{2} t_{j+1}|k|^2} (t_{j+1} - t_j) k_\alpha d\mu_\alpha(k),$$

the latter converging, for $n \to \infty$ to $-\hbar \sum_{\alpha=1}^{3} \int_{\mathbb{R}^3} \int_0^t e^{-\frac{i\hbar}{2} s|k|^2} k_\alpha d\mu_\alpha(k) ds$.

Since the Parseval type equality (1.9) cannot be directly applied, we have to implement a different technique, based on analyticity assumptions, in order to show that the limit in definition (1.14) exists, i.e. that the Feynman map of the function f given by (3.23) is well-defined.

In the following we shall denote with $C_t := C([0,t];\mathbb{R}^3)$ the Banach space of continuous paths $\omega : [0,t] \to \mathbb{R}^3$, endowed with the sup-norm $\|\cdot\|_\infty$. Let $\mathbb{P}$ be the Wiener measure on the Borel σ-algebra $\mathcal{B}(C_t)$ of C_t. Since for $\gamma \in \mathcal{H}_t$ we have $\|\gamma\|_\infty \leq \sqrt{t}\cdot\|\gamma\|$, the Cameron-Martin Hilbert space $\mathcal{H}_t$ is densely embedded in C_t. Denoted with C_t^* the topological dual of C_t, we have the following chain of dense inclusions:

$$C_t^* \subset \mathcal{H}_t \subset C_t. \tag{3.24}$$

With an abuse of notation we shall denote $\langle \eta, \omega\rangle$ the dual pairing between two elements $\eta \in C_t^*$ and $\omega \in C_t$. Let μ be the finitely additive standard Gaussian measure defined as

$$\mu(\mathcal{C}_{P_n,D}) = \int_D \frac{e^{-\frac{\|x\|^2}{2}}}{(2\pi)^{n/2}} dx,$$

on the cylinder sets $\mathcal{C}_{P_n,D} \subset \mathcal{H}_t$ of the form

$$\mathcal{C}_{P_n,D} := \{\gamma \in \mathcal{H}_t \colon P_n\gamma \in D\},$$

for some finite dimensional projection operator $P_n : \mathcal{H}_t \to \mathcal{H}_t$ and some Borel set $D \subset \mathcal{H}_t$. The measure μ does not extend to a σ-additive measure on the generated σ-algebra, see e.g. [77]. Defining the cylinder sets in C_t by

$$\tilde{\mathcal{C}}_{\eta_1,\dots,\eta_n;E} := \{\omega \in C_t \colon (\langle\eta_1,\omega\rangle, \dots, \langle\eta_n,\omega\rangle) \in E\},$$

for some $n \in \mathbb{N}$, $\eta_1, \dots, \eta_n \in C_t^*$ and E a Borel set of $\mathbb{R}^3$, we have that the intersection $\tilde{\mathcal{C}}_{\eta_1,\dots,\eta_n;E} \cap \mathcal{H}_t$ is a cylinder set in $\mathcal{H}_t$. According to the fundamental results by L. Gross [51, 52], the finite additive measure $\tilde{\mu}$ defined on the cylinder sets of C_t by

$$\tilde{\mu}(\tilde{\mathcal{C}}_{\eta_1,\dots,\eta_n;E}) := \mu(\tilde{\mathcal{C}}_{\eta_1,\dots,\eta_n;E} \cap \mathcal{H}_t)$$

extends to a σ-additive Borel measure on C_t that coincides with the standard Wiener measure $\mathbb{P}$, in such a way that for any $\gamma \in \mathcal{H}_t$ such that γ is an element of C_t^* the following holds

$$\int e^{i\langle\gamma,\omega\rangle} d\mathbb{P}(\omega) = e^{-\frac{1}{2}\|\gamma\|^2}.$$

Thanks to the results above it is possible to define, for any $\eta \in C_t^*$, a centered Gaussian random variable n_η on $(C_t, \mathcal{B}(C_t), \mathbb{P})$ given by $n_\eta(\omega) := \langle\gamma,\omega\rangle$, $\omega \in C_t$,

$\gamma \in C_t^*$. In particular, for $\eta, \gamma \in C_t^*$, the following holds

$$\mathbb{E}[n_\eta n_\gamma] = \int_0^t \dot{\eta}(s) \cdot \dot{\gamma}(s) ds = \langle \eta, \gamma \rangle,$$

the pairing on the r.h.s. coinciding with the scalar product in $\mathcal{H}_t$. This shows that the map $n : C_t^* \to L^2(C_t, \mathbb{P})$ can be extended, by the density of C_t^* in $\mathcal{H}_t$, to an unitary operator $n : \mathcal{H}_t \to L^2(C_t, \mathbb{P})$. In particular, given a projector operator $P_n : \mathcal{H}_t \to \mathcal{H}_t$ of the form $P_n(\gamma) = \sum_{j=1}^n \langle e_n, \gamma \rangle e_n$, where $\{e_1, \dots, e_n\}$ orthonormal vectors in $\mathcal{H}_t$, it is possible to define the random variable $\tilde{P}_n : C_t \to \mathcal{H}_t$ as

$$\tilde{P}_n(\omega) = \sum_{i=1}^n n_{e_i}(\omega) e_i, \tag{3.25}$$

$n_{e_i} \in L^2(C_t, \mathbb{P})$.

Now we shall show how Feynman maps (defined by Eq. (1.14)) of all powers of the function f defined in (3.23) can be computed in terms of Wiener integrals. For analogous results see [14]. Let us consider now in $\mathcal{H}_t$ the sequence of projection operators $\{P_n\}_n$ onto the subspaces of piecewise linear paths, i.e. for $\gamma \in \mathcal{H}_t$ the vector $P_n(\gamma)$ is defined by the right hand side of (B.4). Let $\{\tilde{P}_n\}_n$ be the corresponding sequence of random variables $\tilde{P}_n : C_t \to \mathcal{H}_t$ given by

$$\tilde{P}_n(\omega)(s) = \sum_{k=1}^n \chi_{[t_{k-1}, t_k)}(s) \left(\omega(t_{k-1}) + \frac{\omega(t_k) - \omega(t_{k-1})}{t_k - t_{k-1}} (s - t_{k-1}) \right), \tag{3.26}$$

with $s \in [0, t]$, $\omega \in C_t$ and $t_k = kt/n$, $k = 1, \dots, n$ as above. Let $\mathbf{a} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field fulfilling the assumptions of Theorem 3.1. Since any component $a_j : \mathbb{R}^3 \to \mathbb{R}$, $j = 1, \dots, 3$, can be written as the Fourier transform of a complex measure μ_j with compact support according to formula (3.5), the map $\mathbf{a}$ can be extended to an holomorphic function on $\mathbb{C}^3$ with components given by

$$\mathbf{a}_j(z) = \int_{\mathbb{R}^3} e^{ikz} d\mu_j(k), \qquad z \in \mathbb{C}^3, \tag{3.27}$$

the integral on the r.h.s. of (3.27) being well-defined and finite since

$$\int_{\mathbb{R}^3} |e^{ikz} d|\mu_j|(k) \leq \int_{\mathbb{R}^3} \prod_{l=1}^3 e^{|k_l||\Im(z_l)|} d|\mu_j|(k) \leq e^{R \sum_{l=1}^3 |\Im(z_l)|},$$

where R denotes the radius of the sphere containing the supports of the measures μ_j. In particular, for $x \in \mathbb{R}^3$ and $z \in \mathbb{C}$ the components of the vector $\mathbf{a}(zx)$ are given by $\mathbf{a}_j(zx) = \int_{\mathbb{R}^3} e^{izkx} d\mu_j(k)$. The following lemma shows the convergence of a particular sequence of random variables defined on the Wiener space.

Lemma 3.2. *Let* $\mathbf{a}$ *be a three dimensional vector field fulfilling the assumptions of Theorem 3.1. Let* $\{f_n\}$ *be the sequence of random variables* $f_n : C_t \to \mathbb{C}$ *defined by*

$$f_n(\omega) = \int_0^t \mathbf{a}\left(\sqrt{i\hbar}\omega_n(s)\right) \cdot \dot{\omega}_n(s) ds,$$

where $\omega_n(s) \equiv P_n(\omega)(s)$ *and* $P_n(\omega)$ *is defined by the right hand side of* (3.26)*. Then for any* $p \in \mathbb{N}$, $1 \leq p \leq \infty$, f_n *converges, as* $n \to \infty$, *in* $L^p(C_t, \mathbb{P})$ *to the random variable* f *defined as the Stratonovich stochastic integral*

$$f(\omega) = \int_0^t \mathbf{a}(\sqrt{i\hbar}\omega(s)) \circ d\omega(s).$$

Proof. We will consider for notational simplicity the 1-dimensional case. The proof in all dimensions, in particular in the 3-dimensional case is analogous. Let us first remark that by remark 3.1 on the analyticity of the extension of $\mathbf{a}(\cdot)$ from $\mathbb{R}$ to $\mathbb{C}$, the integral on the right hand side of f_n is well-defined. Further, by (3.27) the random variables f_n are given by:

$$f_n(\omega(s)) = \sum_{j=0}^{n-1} \int_0^{\frac{t}{n}} \int_{\mathbb{R}} e^{i\sqrt{i\hbar}k\omega(s_j)} e^{i\sqrt{i\hbar}k\frac{(\omega(s_{j+1})-\omega(s_j))s}{t/n}} \cdot \frac{(\omega(s_{j+1}) - \omega(s_j))}{t/n} d\mu(k) ds =$$

$$= \sum_{j=0}^{n-1} \int_{\mathbb{R}} e^{i\sqrt{i\hbar}k\omega(s_j)} \left(e^{i\sqrt{i\hbar}k(\omega(s_{j+1})-\omega(s_j))} - 1\right) \cdot \frac{1}{i\sqrt{i\hbar}k} d\mu(k).$$

with $s_j = \frac{jt}{n}$. By setting $\Delta_j := \omega(s_{j+1}) - \omega(s_j)$ and by a Taylor expansion (to second order with remainder), the last line becomes

$$\sum_{j=0}^{n-1} \int_{\mathbb{R}} e^{i\sqrt{i\hbar}k\omega(s_j)} \left(\Delta_j + \frac{1}{2} i\sqrt{i\hbar}k\Delta_j^2 + \frac{1}{2}(i\sqrt{i\hbar}k)^2 \Delta_j^3 \int_0^1 (1-u)^2 e^{i\sqrt{i\hbar}k\Delta_j u} du\right) d\mu(k).$$

Hence the function f_n can be written as the sum of three contributions, namely $f_n = g_n + h_n + r_n$, where

$$\begin{aligned} g_n(\omega) &= \sum_{j=0}^{n-1} \int_{\mathbb{R}} e^{i\sqrt{i\hbar}k\omega(s_j)}(\omega(s_{j+1}) - \omega(s_j))d\mu(k) = \\ &= \sum_{j=0}^{n-1} a(\sqrt{i\hbar}\omega(s_j))(\omega(s_{j+1}) - \omega(s_j)); \\ h_n(\omega) &= \frac{1}{2}\sum_{j=0}^{n-1} \int_{\mathbb{R}} i\sqrt{i\hbar}ke^{i\sqrt{i\hbar}k\omega(s_j)}(\omega(s_{j+1}) - \omega(s_j))^2 d\mu(k) \\ &= \sum_{j=0}^{n-1} \frac{1}{2} \cdot a'(\sqrt{i\hbar}\omega(s_j))(\omega(s_{j+1}) - \omega(s_j))^2; \\ r_n(\omega) &= \sum_{j=0}^{n-1} \frac{1}{2} \int_0^1 \int_{\mathbb{R}} \left(i\sqrt{i\hbar}\right)^2 k^2 e^{i\sqrt{i\hbar}k(\omega(s_j)+(\omega(s_{j+1})-\omega(s_j))u)} \\ &\quad (\omega(s_{j+1}) - \omega(s_j))^3(1-u)^2 d\mu(k)du, \end{aligned}$$

(a' standing for derivative of a). By computation based on BDG inequalities and Gaussian integration we obtain

$$\begin{aligned} g_n &\xrightarrow{L^p(\Omega,\mathbb{P})} \int_0^t a(\sqrt{i\hbar}\omega(s))d\omega(s); \\ h_n &\xrightarrow{L^p(\Omega,\mathbb{P})} \frac{1}{2}\int_0^t a'(\sqrt{i\hbar}\omega(s))ds; \\ r_n &\xrightarrow{L^p(\Omega,\mathbb{P})} 0, \end{aligned}$$

eventually obtaining:

$$f_n \xrightarrow{L^p(\Omega,\mathbb{P})} \int_0^t a(\sqrt{i\hbar}\omega(s))d\omega(s) + \frac{1}{2}\int_0^t a'(\sqrt{i\hbar}\omega(s))ds = \int_0^t a(\sqrt{i\hbar}\omega(s)) \circ d\omega(s).$$

For further details see Appendix C. □

Theorem 3.2. *Let the vector field* $\mathbf{a}$ *and the function* $\psi_0 \in L^2(\mathbb{R}^3)$ *satisfy the assumptions of Theorem 3.1. Then the Feynman map of the function* $g : \mathcal{H}_t \to \mathbb{C}$ *given by* $g(\gamma) := \psi_0(\gamma(t) + x) \int_0^t \mathbf{a}(\gamma(s) + x) \cdot \dot{\gamma}(s)ds$ *for any* $x \in \mathbb{R}^3$*, is well-defined and equal to the following Wiener integral*

$$I_F(g) = \int_{C_t} \left(\sqrt{i\hbar}\int_0^t \mathbf{a}\left(\sqrt{i\hbar}\omega(s) + x\right) \circ d\omega(s)\right) \psi_0(\sqrt{i\hbar}\omega(t) + x)d\mathbb{P}(\omega).$$

Moreover for any $m \geq 0$, the Feynman map of the function $g^x_m : \mathcal{H}_t \to \mathbb{C}$ defined as

$$g^x_m(\gamma) := \psi_0(\gamma(t)+x)\left(\int_0^t \mathbf{a}(\gamma(s)+x)\cdot\dot\gamma(s)ds\right)^m$$

is given by

$$I_F(g^x_m) = \int_{C_t}\left(\sqrt{i\hbar}\int_0^t \mathbf{a}\left(\sqrt{i\hbar}\omega(s)+x\right)\circ d\omega(s)\right)^m \psi_0\left(\sqrt{i\hbar}\omega(t)+x\right)d\mathbb{P}(\omega),$$

where $\mathbb{P}$ is the Wiener measure on $(C_t, \mathcal{B}(C_t))$.

Proof. For Fixed $n \in \mathbb{N}$ and $m \geq 1$, let us consider the finite dimensional oscillatory integral

$$\left(\int^o_{P_n\mathcal{H}_t} e^{i\frac{\|P_n\gamma\|^2}{2\hbar}}d(P_n\gamma)\right)^{-1}\int^o_{P_n\mathcal{H}_t} e^{i\frac{\|P_n\gamma\|^2}{2\hbar}}g_m(P_n\gamma)d(P_n\gamma) =$$

$$= \int^o_{\mathbb{R}^{3n}}\left(\sum_{j=1}^n \frac{(x_j-x_{j-1})}{t/n}\cdot\int_{t_{j-1}}^{t_j}\mathbf{a}\left(x_{j-1}+\frac{(x_j-x_{j-1})}{t/n}(s-t_{j-1})+x\right)ds\right)^m$$
$$\psi_0(x_n+x)e^{\frac{i}{2\hbar t/n}\sum_{j=1}^n(x_j-x_{j-1})^2}\frac{dx_1\cdots dx_n}{(2\pi i\hbar t/n)^{3n/2}} =$$
$$= \int^o_{\mathbb{R}^{3n}}\left(\sum_{j=1}^n \xi_j\cdot\int_{t_{j-1}}^{t_j}\mathbf{a}\left(x+\left(\sum_{k=1}^{j-1}\xi_k\right)\frac{t}{n}+\xi_j(s-t_{j-1})\right)ds\right)^m\psi_0\left(x+\left(\sum_{j=1}^n\xi_j\right)\frac{t}{n}\right)$$
$$\times e^{\frac{it/n}{2\hbar}\sum_{j=1}^n\xi_j^2}\frac{d\xi_1\cdots d\xi_n}{(2\pi i\hbar(t/n)^{-1})^{3n/2}}.$$

By the stated assumption on $\mathbf{a}$ and ψ_0 the latter is equal to

$$\int^o_{\mathbb{R}^{3n}}\left(\sum_{j=1}^n\sum_{\alpha=1}^3\xi_{j,\alpha}\cdot\int_{t_{j-1}}^{t_j}\int_{\mathbb{R}^d}\exp\left[ik\cdot\left(x+\left(\sum_{l=1}^{j-1}\xi_l\right)\frac{t}{n}+\xi_j(s-t_{j-1})\right)\right]d\mu_\alpha(k)ds\right)^m$$
$$\times\int_{\mathbb{R}^3}e^{ih\cdot\left(x+\sum_{j=1}^n\xi_jt/n\right)}d\mu_0(h)e^{\frac{it/n}{2\hbar}\sum_{j=1}^n\xi_j^2}\frac{d\xi_1\cdots d\xi_n}{(2\pi i\hbar(t/n)^{-1})^{3n/2}}, \tag{3.28}$$

where $\hat\mu_0 = \psi_0$, i.e for any Borel set $I \subset \mathbb{R}^3$, $\mu_0(I) = \frac{1}{2\pi}\int_I\psi_0(x)dx$. Let us consider the open sector $D_{\pi/2} = \{z \in \mathbb{C}\colon z = |z|e^{i\theta}, \theta \in (0,\pi/2)\}$ of the complex plane and function $F : \bar{D}_{\pi/2} \to \mathbb{C}$ defined by

$$F(z) = \int_{\mathbb{R}^{3n}}\left(\sum_{j=1}^n\sum_{\alpha=1}^3 z\xi_{j,\alpha}\cdot\int_{t_{j-1}}^{t_j}\int_{\mathbb{R}^3}\exp\left[ik\cdot\left(x+\left(\sum_{l=1}^{j-1}z\xi_l\right)\frac{t}{n}+z\xi_j(s-t_{j-1})\right)\right]d\mu_\alpha(k)ds\right)^m$$
$$\times\int_{\mathbb{R}^3}e^{ih\cdot(x+z\sum_{j=1}^n\xi_jt/n)}d\mu_0(h)e^{\frac{it/n}{2\hbar}\sum_{j=1}^n z^2\xi_j^2}\frac{d\xi_1\cdots d\xi_n}{(2\pi i\hbar z^{-1}(t/n)^{-1})^{3n/2}},$$

for $z \in \mathbb{R}$, $z > 0$, by the classical change of variable formula, $F(z)$ is a constant function equal to the finite dimensional oscillatory integral (3.28). Further F is analytic on $D_{\pi/2}$, as one can prove by applying Fubini and Morera's theorems. Indeed, for $z \in D_{\pi/2}$, the integral defining $F(z)$ is absolutely convergent since:

$$\int_{\mathbb{R}^{3n}} \left(\sum_{j=1}^{n} \sum_{\alpha=1}^{3} |z\xi_{j,\alpha}| \int_{t_{j-1}}^{t_j} \int_{\mathbb{R}^3} \left| \exp\left[ik \cdot \left(x + \left(\sum_{l=1}^{j-1} z\xi_l \right) \frac{t}{n} + z\xi_j (s - t_{j-1}) \right) \right] \right| d|\mu_\alpha|(k) ds \right)^m$$
$$\int_{\mathbb{R}^3} \left| e^{ih \cdot \left(x + z \sum_{j=1}^n \xi_j \frac{t}{n} \right)} \right| d|\mu_0|(h) e^{\frac{it/n}{2\hbar} \sum_{j=1}^n z^2 \xi_j^2} | \frac{d\xi_1 \cdots d\xi_n}{(2\pi\hbar |z|^{-1} (t/n)^{-1})^{3n/2}} \leq$$
$$\leq \int_{\mathbb{R}^{3n}} \left(\sum_{j=1}^{n} \sum_{\alpha=1}^{3} |z||\xi_{j,\alpha}| \int_0^{t/n} \int_{\mathbb{R}^3} \exp\left[-|z| \sin\theta k \cdot \left(\sum_{l=1}^{j-1} \xi_l \frac{t}{n} + \xi_j s \right) \right] d|\mu_\alpha|(k) ds \right)^m$$
$$\int_{\mathbb{R}^3} \left| e^{-|z| \sin\theta h \cdot \sum_{j=1}^n \xi_j \frac{t}{n}} \right| d|\mu_0|(h) e^{-\frac{|z|^2 \sin(2\theta) t/n}{2\hbar} \sum_{j=1}^n z^2 \xi_j^2} \frac{d\xi_1 \cdots d\xi_n}{(2\pi\hbar |z|^{-1} (t/n)^{-1})^{3n/2}} =$$
$$= \int_{\mathbb{R}^{3n}} \left(\sum_{j=1}^{n} \sum_{\alpha=1}^{3} |\xi_{j,\alpha}| \int_0^{t/n} \int_{\mathbb{R}^3} \exp\left[-\sin\theta k \cdot \left(\sum_{l=1}^{j-1} \xi_l \frac{t}{n} + \xi_j s \right) \right] d|\mu_\alpha|(k) ds \right)^m$$
$$\int_{\mathbb{R}^3} \left| e^{-\sin\theta h \cdot \sum_{j=1}^n \xi_j \frac{t}{n}} \right| d|\mu_0|(h) e^{-\frac{\sin(2\theta) t/n}{2\hbar} \sum_{j=1}^n \xi_j^2} \frac{d\xi_1 \cdots d\xi_n}{(2\pi\hbar (t/n)^{-1})^{3n/2}} \leq$$
$$\leq \left(\int_{\mathbb{R}^{3n}} \left(\sum_{j=1}^{n} \sum_{\alpha=1}^{3} |\xi_{j,\alpha}| \int_0^{t/n} \int_{\mathbb{R}^3} \exp\left[-\sin\theta k \cdot \left(\sum_{l=1}^{j-1} \xi_l \frac{t}{n} + \xi_j s \right) \right] d|\mu_\alpha|(k) ds \right)^{2m} \right.$$
$$\left. e^{-\frac{\sin(2\theta) t/n}{2\hbar} \sum_{j=1}^n \xi_j^2} \frac{d\xi_1 \cdots d\xi_n}{(2\pi\hbar (t/n)^{-1})^{3n/2}} \right)^{1/2} \tag{3.29}$$
$$\left(\int_{\mathbb{R}^{3n}} \left(\int_{\mathbb{R}^3} \left| e^{-\sin\theta h \cdot \sum_{j=1}^n \xi_j \frac{t}{n}} \right| d|\mu_0|(h) \right)^2 e^{-\frac{\sin(2\theta) t/n}{2\hbar} \sum_{j=1}^n \xi_j^2} \frac{d\xi_1 \cdots d\xi_n}{(2\pi\hbar (t/n)^{-1})^{3n/2}} \right)^{1/2}.$$

In the second step above we have got rid of the term $|z|$ in the integral because of classical (finite dimensional) change of variables formula.

For notational simplicity, in the following we shall describe in detail the one dimensional case but similar arguments work also in three dimension. The second factor in the product of integrals above is bounded by

$$\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}} \left| e^{-\sin\theta h \cdot \sum_{j=1}^n \xi_j \frac{t}{n}} \right| d|\mu_0|(h) \right)^2 e^{-\frac{\sin(2\theta) t/n}{2\hbar} \sum_{j=1}^n \xi_j^2} \frac{d\xi_1 \cdots d\xi_n}{(2\pi\hbar (t/n)^{-1})^{n/2}} =$$
$$= (\sin(2\theta))^{-n/2} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{\frac{\hbar t \sin^2\theta (h_1 + h_2)^2}{2\sin(2\theta)}} d|\mu_0|(h_1) d|\mu_0|(h_2) \leq$$
$$\leq (\sin(2\theta))^{-n/2} e^{\frac{2\hbar t \sin^2\theta R^2}{\sin(2\theta)}} |\mu_0|^2,$$

where $R \in \mathbb{R}^+$ is such that the support of μ_0 is contained in $[-R, R]$.
Concerning the first factor on the right hand side of (3.29) we have the following upper bound, again written for simplicity of notations for the 1-dimensional case

$$\int_{\mathbb{R}^n}\left(\sum_{j=1}^n |\xi_j| \int_0^{t/n}\int_{\mathbb{R}} \exp\left[-\sin\theta k\cdot\left(\sum_{l=1}^{j-1}\xi_l\frac{t}{n}+\xi_j s\right)\right] d|\mu|(k)ds\right)^{2m}$$
$$e^{-\frac{\sin(2\theta)t/n}{2\hbar}\sum_{j=1}^n\xi_j^2}\frac{d\xi_1\cdots d\xi_n}{(2\pi\hbar(t/n)^{-1})^{n/2}} =$$
$$= \sum_{j_1,\ldots,j_{2m}=1}^n \int_{\mathbb{R}^n}|\xi_{j_1}\cdots\xi_{j_{2m}}|\int_0^{t/n}\cdots\int_0^{t/n}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}\exp\left[-\sin\theta k_1\cdot\left(\sum_{l_1=1}^{j_1-1}\xi_{l_1}\frac{t}{n}+\xi_{j_1}s_1\right)\right]\cdots$$
$$\cdots\exp\left[-\sin\theta k_{2m}\cdot\left(\sum_{l_{2m}=1}^{j_{2m}-1}\xi_{l_{2m}}\frac{t}{n}+\xi_{j_{2m}}s_{2ms}\right)\right] ds_1\cdots ds_{2m}d|\mu|(k_1)\cdots d|\mu|(k_{2m})$$
$$e^{-\frac{\sin(2\theta)t/n}{2\hbar}\sum_{j=1}^n\xi_j^2}\frac{d\xi_1\cdots d\xi_n}{(2\pi\hbar(t/n)^{-1})^{n/2}} \leq$$
$$\leq \sum_{j_1,\ldots,j_{2m}=1}^n\left(\int_{\mathbb{R}^n}|\xi_{j_1}\cdots\xi_{j_{2m}}|^2 e^{-\frac{\sin(2\theta)t/n}{2\hbar}\sum_{j=1}^n\xi_j^2}\frac{d\xi_1\cdots d\xi_n}{(2\pi\hbar(t/n)^{-1})^{n/2}}\right)^{1/2}$$
$$\left(\int_{\mathbb{R}^n}\left(\int_0^{t/n}\cdots\int_0^{t/n}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}\exp\left[-\sin\theta k_1\cdot\left(\sum_{l_1=1}^{j_1-1}\xi_{l_1}\frac{t}{n}+\xi_{j_1}s_1)\right)\right]\cdots\right.\right.$$
$$\left.\cdots\exp\left[-\sin\theta k_{2m}\cdot\left(\sum_{l_{2m}=1}^{j_{2m}-1}\xi_{l_{2m}}\frac{t}{n}+\xi_{j_{2m}}s_{2ms})\right)\right] ds_1\cdots ds_{2m}d|\mu|(k_1)\cdots d|\mu|(k_{2m})\right)^2$$
$$\left. e^{-\frac{\sin(2\theta)t/n}{2\hbar}\sum_{j=1}^n\xi_j^2}\frac{d\xi_1\cdots d\xi_n}{(2\pi\hbar(t/n)^{-1})^{n/2}}\right)^{-1/2}.$$

The first factor in the sum above is finite since it is equal to the moment of a Gaussian measure, i.e.

$$\int_{\mathbb{R}^n}|\xi_{j_1}\cdots\xi_{j_{2m}}|^2 e^{-\frac{\sin(2\theta)t/n}{2\hbar}\sum_{j=1}^n\xi_j^2}\frac{d\xi_1\cdots d\xi_n}{(2\pi\hbar(t/n)^{-1})^{n/2}} = (\sin(2\theta)^{-n/2})\cdot\mathbb{E}\left[\left|X_{j_1}\cdots X_{j_{2m}}\right|^2\right],$$

where X_j, $j = 1, \ldots, n$ are i.i.d centered Gaussian random variables with covariance $\sigma = \hbar(\sin(2\theta)t/n)^{-1}$. Analogously the second factor is an absolutely convergent integral since it is of the form

$$\int_0^{t/n}\cdots\int_0^{t/n}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}\mathbb{E}\left[e^{\sum_{j=1}^n a_j(k_1,\ldots,k_{4m},s_1,\ldots,s_{4m})X_j}\right] ds_1\cdots ds_{4m}d|\mu|(k_1)\cdots d|\mu|(k_{4m}) =$$
$$= \int_0^{t/n}\cdots\int_0^{t/n}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}} e^{\frac{1}{2\sin(2\theta)t/n}\sum_{j=1}^n\left(a_j(k_1,\ldots,k_{4m},s_1,\ldots,s_{4m})\right)^2} ds_1\cdots ds_{4m}d|\mu|(k_1)\cdots d|\mu|(k_{4m}),$$

where a_j are linear functions of the variables $k_1, \dots, k_{4m}, s_1, \dots, s_{4m}$ and the last integral is finite since μ is by assumption compactly supported. Hence we can conclude that F is analytic on $D_{\pi/2}$. Further, by a classical change of variables formula, it is simple to see that F is constant on any ray of the form $r_\theta := \{z \in \mathbb{C} : z = |z| e^{i\theta}, |z| \in \mathbb{R}^+\}$ with $\theta \in [0, \pi/2]$, hence by analyticity it is constant on $\bar{D}_{\pi/2}$, giving for any $n \in \mathbb{N}$

$$\left(\int^o_{P_n \mathcal{H}} e^{i\frac{\|P_n\gamma\|^2}{2\hbar}} d(P_n\gamma) \right)^{-1} \int^o_{P_n \mathcal{H}_t} e^{i\frac{\|P_n\gamma\|^2}{2\hbar}} g_m(P_n\gamma) d(P_n\gamma) = F(e^{i\pi/4}) =$$

$$= \int_{\mathbb{R}^{3n}} \left(\sum_{j=1}^n \sqrt{i\hbar} \frac{(x_j - x_{j-1})}{t/n} \cdot \int_{t_{j-1}}^{t_j} \mathbf{a} \left(\sqrt{i\hbar}\, x_{j-1} + \sqrt{i\hbar} \frac{(x_j - x_{j-1})}{t/n} (s - t_{j-1}) + x \right) ds \right)^m$$

$$\psi_0 \left(\sqrt{i\hbar}\, x_n + x \right) e^{-\frac{1}{2t/n} \sum_{j=1}^n (x_j - x_{j-1})^2} \frac{dx_1 \cdots dx_n}{(2\pi\hbar t/n)^{3n/2}} =$$

$$= \mathbb{E} \left[\psi_0 \left(\sqrt{i\hbar}\, \omega_n(t) + x \right) \left(\sqrt{i\hbar} \int_0^t \mathbf{a} \left(\sqrt{i\hbar} \omega_n(s) + x \right) \cdot \dot{\omega}_n(s) ds \right)^m \right], \quad (3.30)$$

where ω_n stands for the piecewise linear approximation of Brownian motion defined above, namely:

$$\omega_n(s) = \sum_{k=1}^n \chi_{[t_{k-1}, t_k)}(s) \left(\omega(t_{k-1}) + \frac{\omega(t_k) - \omega(t_{k-1})}{t_k - t_{k-1}} (s - t_{k-1}) \right),$$

with $s \in [0, t]$, $t_k = k/n$. Thanks to the result of Lemma 3.2, the right side of (3.30) converges for $n \to \infty$ to

$$\mathbb{E} \left[\psi_0(\sqrt{i\hbar}\, \omega(t) + x) \left(\sqrt{i\hbar} \int_0^t \mathbf{a} \left(\sqrt{i\hbar} \omega(s) + x \right) \circ d\omega(s) \right)^m \right].$$

$\square$

The last step is the proof that the integral provides a representation of the solution to the Schrödinger equation by the Dyson series expansion.

Theorem 3.3. *Under the assumption of Theorem 3.1, the solution of the Schrödinger equation with magnetic field*

$$i\hbar\partial_t \psi(t) = H\psi(t, x), \quad \psi(0, x) = \psi_0(x), \quad H = \frac{1}{2}(-i\hbar\nabla - \lambda \mathbf{a}(x))^2$$

can be expressed by the perturbative Dyson series expansion as

$$e^{-\frac{i}{\hbar}Ht} \psi_0 = \sum_{m=0}^\infty \lambda^m \psi_m(t),$$

where the vector ψ_m can be expressed by a Feynman map of the form

$$\psi_m(t,x) =$$
$$= \frac{1}{m!}\left(-\frac{i}{\hbar}\right)^m \widetilde{\int_{\mathcal{H}_t}^{o}} \left(\int_0^t \mathbf{a}(\gamma(s)+x)\cdot\dot\gamma(s)ds\right)^m e^{\frac{i}{2\hbar}\int_0^t \|\dot\gamma(s)\|^2 ds}\psi_0(\gamma(t)+x)d\gamma =$$
$$= \frac{1}{m!}\left(-\frac{i}{\hbar}\right)^m \mathbb{E}\left[\left(\sqrt{i\hbar}\int_0^t \mathbf{a}\left(\sqrt{i\hbar}\omega(s)+x\right)\circ d\omega(s)\right)^m \psi_0\left(\sqrt{i\hbar}\omega(t)+x\right)\right]. \quad (3.31)$$

The expansion is convergent in $L^2(\mathbb{R}^3)$ for $\lambda \in \mathbb{C}$, with $|\lambda| < \lambda^*$, λ^* given by (3.8). The integral under the expectation is to be understood as a Stratonovich stochastic integral.

Proof. By Theorem 3.1 for $|\lambda| < \lambda^*$ the vector $\psi(t) = e^{-\frac{i}{\hbar}Ht}\psi_0$ in $L^2(\mathbb{R}^3)$ is given by the convergent power series expansion (3.12). Hence, we are left to prove that for any $m \in \mathbb{N}$ the term ϕ_m in (3.13) is equal to ψ_m as given in (3.31).

By remark 3.3, the Hamiltonian operator H generates an analytic semigroup $e^{-zH} : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$, where $z \in \mathbb{C}$, $Re(z) \geq 0$, with a convergent Dyson expansion of the form $e^{-zH}\psi_0 = \sum_m \lambda^m \phi_m(z)$ with a radius of convergence $\lambda^*(z)$ depending on $|z|$ (see Eq. (3.17)). In particular, for $z \in \mathbb{R}^+$, namely $z = \frac{t}{\hbar}$, the family of operators $T(t) = e^{-\frac{t}{\hbar}H}$, $t \in \mathbb{R}^+$, yields the heat semigroup generated by H (described in Section 3.1). In this case, by Feynman-Kac-Itô formula [105] the vector $e^{-\frac{t}{\hbar}H}\psi_0$ is given by the Wiener integral

$$e^{-\frac{t}{\hbar}H}\psi_0(x) = \mathbb{E}\left[\psi_0(\sqrt{\hbar}\omega(t)+x)e^{-\frac{i\lambda}{\hbar}\int_0^t \mathbf{a}(\sqrt{\hbar}\omega(s)+x)\circ d\omega(s)}\right]. \quad (3.32)$$

For any $\phi \in L^2(\mathbb{R}^3)$ the inner product $\langle\phi, e^{-zH}\psi_0\rangle$ is an analytic function of $z \in D$, $D = \{z \in \mathbb{C} : Re(z) \geq 0\}$, continuous in the closure $\bar{D}$ of D and admitting the expansions

$$\langle\phi, e^{-zH}\psi_0\rangle = \sum_{m=0}^{\infty}\lambda^m\langle\phi,\phi_m(z)\rangle.$$

By formula (3.16) each term $\langle\phi,\phi_m(z)\rangle$ is an analytic function of $z \in D$, continuous in the closure $\bar{D}$ and for $z = t/\hbar$, $t \in \mathbb{R}^+$, by formula (3.32) it is equal to

$$\langle\phi,\phi_m(t/\hbar)\rangle = \left(-\frac{i}{\hbar}\right)^m \int_{\mathbb{R}^3}\bar\phi(x)$$
$$\mathbb{E}\left[\psi_0(\sqrt{\hbar}\omega(t)+x)\left(\sqrt{\hbar}\int_0^t \mathbf{a}(\sqrt{\hbar}\omega(s)+x)\circ d\omega(s)\right)^m\right]dx.$$

By replacing t with $t\xi$, with $\xi \in \mathbb{R}^+$, the expression above assumes the following form:

$$\langle \phi, \phi_m(t\xi/\hbar)\rangle = \left(-\frac{i}{\hbar}\right)^m \int_{\mathbb{R}^3} \bar{\phi}(x)$$
$$\times \mathbb{E}\left[\psi_0(\sqrt{\hbar}\omega(t\xi)+x)\left(\sqrt{\hbar}\int_0^{t\xi} \mathbf{a}(\sqrt{\hbar}\omega(s)+x)\circ d\omega(s)\right)^m\right] dx,$$

and thus

$$\langle \phi, \phi_m(t\xi/\hbar)\rangle = \left(-\frac{i}{\hbar}\right)^m \int_{\mathbb{R}^3} \bar{\phi}(x)$$
$$\times \mathbb{E}\left[\psi_0(\sqrt{\hbar\xi}\omega(t)+x)\left(\sqrt{\xi\hbar}\int_0^{t} \mathbf{a}(\sqrt{\hbar\xi}\omega(s)+x)\circ d\omega(s)\right)^m\right] dx, \tag{3.33}$$

and since by the discussion above, both the right hand side and the left hand side of (3.33) are analytic for $\xi \in D$ and continuous for $\xi \in \bar{D}$, by setting $\xi \equiv i$ we obtain the required equality, namely:

$$\langle \phi, \phi_m(it/\hbar)\rangle = \left(-\frac{i}{\hbar}\right)^m \int_{\mathbb{R}^3} \bar{\phi}(x)$$
$$\times \mathbb{E}\left[\psi_0(\sqrt{i\hbar}\omega(t)+x)\left(\sqrt{i\hbar}\int_0^{t} \mathbf{a}(\sqrt{i\hbar}\omega(s)+x)\circ d\omega(s)\right)^m\right] dx.$$

□

Remark 3.6. Theorem 3.3 can be generalized to the case where a scalar potential V is added to the right hand side of (3.1). Indeed, let us consider an Hamiltonian operator of the form (3.18), with $V \in \mathcal{F}_c(\mathbb{R}^3)$. By remark 3.4, under the assumptions of Theorem 3.3 the vector $e^{-\frac{i}{\hbar}Ht}\psi_0$ admits for $\lambda < \tilde{\lambda}$ ($\tilde{\lambda}$ defined as in (3.21)) a convergent perturbative expansion:

$$e^{-\frac{i}{\hbar}Ht}\psi_0 = \sum_{m=0}^{\infty} \lambda^m \psi_m(t),$$

where the generic vector ψ_m can be expressed by a Feynman map of the form

$$\psi_m(t,x) = \frac{1}{m!}\left(-\frac{i}{\hbar}\right)^m \widetilde{\int_{\mathcal{H}_t}^{o}} \left(\int_0^t \mathbf{a}(\gamma(s)+x)\cdot\dot{\gamma}(s)ds\right.$$
$$\left. + \int_0^t V(\gamma(s)+x)ds\right)^m e^{\frac{i}{2\hbar}\int_0^t \|\dot{\gamma}(s)\|^2 ds}\psi_0(\gamma(t)+x)d\gamma,$$

which can be expressed in terms of the Wiener integral

$$\frac{1}{m!}\left(-\frac{i}{\hbar}\right)^m \mathbb{E}\left[\left(\sqrt{i\hbar}\int_0^t \mathbf{a}\left(\sqrt{i\hbar}\omega(s)+x\right)\circ d\omega(s)\right.\right.$$
$$\left.\left.+\int_0^t V(\sqrt{i\hbar}\omega(s)+x)ds\right)^m \psi_0\left(\sqrt{i\hbar}\omega(t)+x\right)\right].$$

Remark 3.7. All results in Theorems 3.1, 3.2, 3.3 have been formulated and proved for underlying 3-dimensional space, but corresponding results and proofs hold for all space dimensions.

3.3 Independence of the approximation and renormalization term

In the previous section we provided a convergent constructive expansion for the Feynman path integral representation for the solution of the Schrödinger equation with magnetic field. This was made by using a particular class of finite dimensional approximations, namely the ones related to piecewise linear path (see Eq. (1.13)). This last section is devoted to the question, whether the independence of the construction of the Feynman path integral representation is independent of the chosen type of approximation. In particular, in the case of a constant magnetic field, we show that the definition of the Feynman path integral (3.2) in terms of infinite dimensional oscillatory integral (in the sense of Def. 1.2), i.e. requiring the independence of the limit of the sequence of finite dimensional approximations, requires the introduction of a natural renormalization term. This result is a further development of a similar one obtained in [4], the latter being only valid in the Coulomb gauge $\operatorname{div}\mathbf{a} = 0$. On the contrary, our main results (Theorem 3.5 and Corollary 3.1) provide a gauge-independent construction of the renormalization term as well as of the Feynman path integral, yielding a rigorous construction of the solution of the Schrödinger equation with vector potential $\mathbf{a}$.

Let $\mathbf{a}:\mathbb{R}^3\to\mathbb{R}^3$ be a linear vector potential corresponding to a constant magnetic field $\mathbf{B} = \operatorname{rot}\mathbf{a}$. More precisely, we assume that the vector field $\mathbf{a}$ is given by

$$\mathbf{a}(x_1,x_2,x_3) = (\alpha_1^1x_1+\alpha_2^1x_2+\alpha_3^1x_3, \alpha_1^2x_1+\alpha_2^2x_2+\alpha_3^2x_3, \alpha_1^3x_1+\alpha_2^3x_2+\alpha_3^3x_3), \quad (3.34)$$

where $(x_1,x_2,x_3)\in\mathbb{R}^3$ and $\alpha_j^i\in\mathbb{R}$, $i,j=1,\dots,3$ are real constants. We are going to study the Fresnel integrability (in the sense of Def. 1.2) of the function $f:\mathcal{H}_t\to\mathbb{C}$, defined on the Cameron-Martin space $\mathcal{H}_t$ as

$$f(\gamma) := e^{-\frac{i}{\hbar}\int_0^t \mathbf{a}(\gamma(s))\cdot\dot\gamma(s)ds}, \qquad \gamma\in\mathcal{H}_t.$$

For any sequence $\{P_n\}_n$ of projectors onto n-dimensional subspaces of $\mathcal{H}_t$, such that $P_n \leq P_{n+1}$ and $P_n \to \mathbb{I}$ strongly as $n \to \infty$, we have to study the limit of the sequence of finite dimensional oscillatory integrals

$$\lim_{n\to\infty}(2\pi i\hbar)^{-n/2}\int^{o}_{P_n\mathcal{H}_t} e^{i\frac{\|P_n\gamma\|^2}{2\hbar}} f(P_n\gamma)d(P_n\gamma).$$

As we shall see, the limit above cannot be independent of the sequence $\{P_n\}$. In fact it is necessary to renormalize the term $f(P_n\gamma) \equiv e^{-\frac{i}{\hbar}g(P_n\gamma)}$ by replacing the exponent $g(P_n\gamma) = \int_0^t \mathbf{a}(P_n\gamma(s)) \cdot \dot{P}_n\gamma(s)ds$ by $g(P_n\gamma) - r_n$, where r_n is a suitable constant depending on the projector P_n as well as on the magnetic field $\mathbf{B}$.

First of all, let us consider the linear operator $G : \mathcal{H}_t \to \mathcal{H}_t$ defined by

$$G(\gamma)(s) := \int_0^s \mathbf{a}(\gamma(r))dr, \qquad \gamma \in \mathcal{H}_t,\ s \in [0,t], \tag{3.35}$$

in such a way that the function $f : \mathcal{H}_t \to \mathbb{C}$ can be written as $f(\gamma) = e^{-\frac{i}{\hbar}\langle G(\gamma),\gamma\rangle}$, i.e. the function $g : \mathcal{H}_t \to \mathbb{C}$, with $g(\gamma) = \langle G(\gamma), \gamma\rangle$ can be represented as the quadratic form associated to G. Note that G is bounded in $\mathcal{H}_t$ due to our assumptions on $\mathbf{a}$. The following lemma provides some properties of G.

Lemma 3.3. *The operator $G : \mathcal{H}_t \to \mathcal{H}_t$ is Hilbert-Schmidt . The eigenvalues of the positive symmetric operator $G^\dagger G$, are given by $\lambda_{m,j} = \frac{4a_j t^2}{\pi^2(1+2m)^2}$, with $j = 1, 2, 3$, $m \in \mathbb{N}$ and $a_j \in \mathbb{R}^+$ eigenvalues of the matrix (3.37) below.*

Proof. Let us consider the positive symmetric operator $L \equiv G^\dagger G : \mathcal{H}_t \to \mathcal{H}_t$ († standing for the adjoint), whose matrix elements are given by

$$\begin{aligned}\langle \eta, L\gamma\rangle &= \langle G\eta, G\gamma\rangle \\ &= \int_0^t \eta(s)A\gamma(s)^T dt, \qquad \eta,\gamma \in \mathcal{H}_t,\end{aligned} \tag{3.36}$$

where A is the 3×3 symmetric matrix with real elements given by

$$A_{ij} = \sum_{l=1}^{3} \alpha_i^l \alpha_j^l, \qquad i,j = 1,...,3. \tag{3.37}$$

Hence, for $\gamma \in \mathcal{H}_t$ the vector $L(\gamma) \in \mathcal{H}_t$ is given by

$$L(\gamma)(s)^T = -\int_0^s \int_t^r A\gamma(\tau)^T d\tau dr,$$

T stands for transpose. L is a compact operator in $\mathcal{H}_t$ and has a discrete spectrum. Indeed, by introducing in $\mathbb{R}^3$ an orthonormal basis $\{\hat{u}_1, \hat{u}_2, \hat{u}_3\}$ of eigenvectors of the symmetric matrix A, with corresponding eigenvalues $a_1, a_2, a_3 \in \mathbb{R}^+$, the eigenvectors

$\{\gamma_m\}$ of L can be represented as linear combination of $\hat{u}_1$, $\hat{u}_2$ and $\hat{u}_3$ by $\gamma_m = \eta_{m,1}\hat{u}_1 + \eta_{m,2}\hat{u}_2 + \eta_{m,3}\hat{u}_3$, with $\eta_{m,j} : [0,t] \to \mathbb{R}$. Recalling the form of the scalar product in $\mathcal{H}_t$, for the expression (3.36) we get that the components $\{\eta_{m,j}\}$ of the eigenvectors (with eigenvalues $\lambda_{m,j}$) are the solutions of

$$\begin{cases} \lambda_{m,j}\ddot{\eta}_{m,j} + a_j\eta_{m,j} = 0 \\ \dot{\eta}_{m,j}(t) = 0 \\ \eta_{m,j}(0) = 0 \end{cases} \qquad j = 1,2,3,$$

with

$$\lambda_{m,j} = \frac{4a_j t^2}{\pi^2(1+2m)^2}, \qquad m \in \mathbb{N}, \quad j = 1,2,3. \tag{3.38}$$

□

Remark 3.8. From (3.38) we see easily that the operator G is not of trace class on $\mathcal{H}_t$.

Lemma 3.4. *Let* $\mathbf{a}$ *be the linear vector potential* (3.34) *and let* $\psi_0 \in L^2(\mathbb{R}^3)$ *be such that its Fourier transform* $\hat{\psi}_0$ *has compact support. Let* $\{e_j\}_j$ *be an orthonormal basis of the Cameron-Martin space* $\mathcal{H}_t$ *and let* P_n *be the projection operator onto the span of the first* n *vectors. Let* $g^x_{\exp} : \mathcal{H}_t \to \mathbb{C}$ *be the function defined by*

$$g^x_{\exp}(\gamma) = \psi_0(\gamma(t)+x)\exp\left(-\frac{i}{\hbar}\int_0^t \mathbf{a}(\gamma(s)+x)\cdot\dot{\gamma}(s)ds\right), \qquad \gamma \in \mathcal{H}_t,$$

and let $\bar{a} \in \mathbb{R}^+$ *be the constant defined as* $\bar{a} = \max_{j=1,2,3}\{a_j\}$, *where* a_j, $j = 1,2,3$ *are the eigenvalues of the (positive semidefinite) matrix* A *defined in* (3.37)*. Then, for fixed* n *and for*

$$t < t^* := \frac{\pi}{4\sqrt{\bar{a}}}, \tag{3.39}$$

the finite dimensional oscillatory integral

$$(2\pi i\hbar)^{-n/2}\int^o_{P_n\mathcal{H}_t} e^{\frac{i}{2\hbar}\|P_n\gamma\|^2} g^x_{\exp}(P_n\gamma)dP_n\gamma$$

is equal to the Wiener integral:

$$\int^o_{P_n\mathcal{H}_t} e^{\frac{i}{2\hbar}\|\gamma\|^2} g^x_{\exp}(\gamma)d\gamma = \mathbb{E}\left[\psi_0(\sqrt{i\hbar}\omega_n(t)+x)e^{-\frac{i}{\hbar}\sqrt{i\hbar}\int_0^t \mathbf{a}(\sqrt{i\hbar}\omega_n(s)+x)\cdot\dot{\omega}_n(s)ds}\right], \tag{3.40}$$

where $\omega_n := \tilde{P}_n(\omega)$ *is defined by* (3.25)*.*

PROOF. For fixed n, by setting $\gamma_n \equiv P_n\gamma$, we have:

$$(2\pi i\hbar)^{-n/2} \int_{P_n\mathcal{H}_t}^{o} e^{i\frac{\|\gamma_n\|^2}{2\hbar}} g^x_{exp}(\gamma_n) d\gamma_n =$$

$$= (2\pi i\hbar)^{-n/2} \int_{P_n\mathcal{H}_t}^{o} e^{\frac{i}{2\hbar}\|\gamma_n\|^2} e^{-\frac{i}{\hbar}\int_0^t \mathbf{a}(\gamma_n(s)+x)\dot{\gamma}_n ds} \psi_0(\gamma_n(t)+x) d\gamma_n =$$

$$= (2\pi i\hbar)^{-n/2} \int_{P_n\mathcal{H}_t}^{o} e^{\frac{i}{2\hbar}\langle\gamma_n,(\mathbb{I}-2G)\gamma_n\rangle} e^{-\frac{i}{\hbar}\mathbf{a}(x)\cdot\gamma_n(t)} \psi_0(\gamma_n(t)+x) d\gamma_n.$$

Let us consider the function $F : \mathbb{R}^+ \to \mathbb{C}$ defined by

$$F(z) = (2\pi i\hbar)^{-n/2} z^n \int_{P_n\mathcal{H}_t}^{o} e^{\frac{iz^2}{2\hbar}\langle\gamma_n,(\mathbb{I}-2G)\gamma_n\rangle} e^{-z\frac{i}{\hbar}\mathbf{a}(x)\cdot\gamma_n(t)} \psi_0(z\gamma_n(t)+x) d\gamma_n. \tag{3.41}$$

By the classical change of variable formula, for $z \in \mathbb{R}^+$ the function F is a constant equal to the finite dimensional oscillatory integral above. In fact, if $t < t^*$, with t^* given by (3.39), F can be extended to an analytic function defined on the open sector $D_{\pi/2} = \{z \in \mathbb{C} \colon z = |z|e^{i\theta}, \theta \in (0, \pi/2), |z| > 0\}$ of the complex plane. Indeed, for any $\gamma \in \mathcal{H}_t$, if condition (3.39) is fulfilled, we have

$$\langle\gamma, (\mathbb{I} - 2G)\,\gamma\rangle \geq \epsilon\|\gamma\|^2,$$

where $\epsilon > 0$ is given by $\epsilon = 1 - \frac{2t\sqrt{\tilde{a}}}{\pi}$. Indeed:

$$\begin{aligned}\langle\gamma, (\mathbb{I} - 2G)\,\gamma\rangle &= \langle\gamma, \mathbb{I}\gamma\rangle - \langle\gamma, 2G\gamma\rangle \\ &= \langle\gamma, \gamma\rangle - \langle\gamma, 2|G|U\gamma\rangle,\end{aligned}$$

where, by the polar decomposition formula, $G = |G|U$, with $|G| = \sqrt{G^\dagger G}$ and U a unitary operator. Furthermore

$$|\langle\gamma, 2|G|U\gamma\rangle| \leq 2\|\gamma\|\|U\gamma\|\||G|\| \leq 2\|\gamma\|^2 \sup_m \tilde{\lambda}_m,$$

where $\||G|\|$ denotes the operator norm of the positive operator$|G|$, while $\{\tilde{\lambda}_m\}_m$ are its eigenvalues, namely $\tilde{\lambda}_m = \sqrt{\lambda_{m,j}}$, with $\lambda_{m,j}$ given by (3.38). Hence, we get

$$|\langle\gamma, 2|G|U\gamma\rangle| \leq \frac{4t\sqrt{\tilde{a}}}{\pi}\|\gamma\|^2,$$

hence, for $t < t^*$, we have, using the Fourier transform $\hat{\psi}_0$ of ψ, the following bound on the oscillatory integral in (3.41):

$$\int_{P_n\mathcal{H}_t}^{o} \left| e^{z^2 \frac{i}{2\hbar}\langle \gamma_n, (\mathbb{I}-2G)\gamma_n\rangle} e^{-z\frac{i}{\hbar}\mathbf{a}(x)\cdot\gamma_n(t)} \psi_0(z\gamma_n(t)+x)\right| d\gamma_n \leq$$
$$\leq \int_{P_n\mathcal{H}_t}^{o} \int_{\mathbb{R}^3} e^{-\sin(2\theta)\frac{|z|^2}{2\hbar}\epsilon\|\gamma\|^2 + \sin\theta\frac{|z|}{\hbar}\mathbf{a}(x)\cdot\gamma_n(t) - k|z|\gamma_n(t)} \frac{|\hat{\psi}_0(k)|}{(2\pi)^3} dk d\gamma_n < \infty,$$

where the convergence of the integral in the second line is assured by the conditions $\theta \in (0, \pi/2)$, $\epsilon = 1 - \frac{2t\sqrt{\bar{a}}}{\pi} > 0$ and $\hat{\psi}_0$ is compactly supported. Hence, by applying Fubini and Morera's theorems, it is simple to check that the function $F : \bar{D}_{\pi/2} \to \mathbb{C}$ is analytic on $D_{\pi/2}$ and continuous up to $\mathbb{R}^+$. Since by the classical change of variables formula the value of $F(z)$ does not depend on $|z|$, i.e. F is constant along rays $\{z \in D_{\pi/2} \colon z = |z|e^{i\theta}, |z| \in \mathbb{R}^+\}$, by analyticity F is constant on $D_{\pi/2}$ and by continuity up to $\mathbb{R}^+$ we obtain in particular, $F(1) = F(\sqrt{\hbar}e^{i\pi/4})$, namely:

$$(2\pi i\hbar)^{-n/2} \int_{P_n\mathcal{H}_t}^{o} e^{\frac{i}{2\hbar}\|\gamma_n\|^2} e^{-\frac{i}{\hbar}\int_0^t \mathbf{a}(\gamma_n(s)+x)\dot{\gamma}_n ds} \psi_0(\gamma_n(t)+x) d\gamma_n =$$
$$= (2\pi)^{-n/2} \int_{P_n\mathcal{H}_t}^{o} e^{-\frac{1}{2}\|\gamma_n\|^2} e^{-\frac{i\sqrt{i}}{\sqrt{\hbar}}\int_0^t \mathbf{a}(\sqrt{i\hbar}\gamma_n(s)+x)\dot{\gamma}_n ds} \psi_0(\sqrt{i\hbar}\gamma_n(t)+x) d\gamma_n$$

and the last line is equal to the r.h.s. of (3.40), namely to:

$$\mathbb{E}\left[\psi_0\left(\sqrt{i\hbar}\omega_n(t)+x\right) e^{-\frac{i}{\hbar}\sqrt{i\hbar}\int_0^t \mathbf{a}(\sqrt{i\hbar}\omega_n(s)+x)\cdot\dot{\omega}_n(s)ds}\right].$$

□

The next step is the study of the convergence of the Wiener integrals on the r.h.s. of (3.40) which can be written as

$$\mathbb{E}\left[\psi_0\left(\sqrt{i\hbar}\omega_n(t)+x\right) e^{-\frac{i}{\hbar}\sqrt{i\hbar}\mathbf{a}(x)\omega_n(t)} e^{\tilde{\mathfrak{g}}_n(\omega)}\right],$$

where, given an orthonormal basis $\{e_n\}$ of $\mathcal{H}_t$, the random variables $\tilde{\mathfrak{g}}_n : C_t \to \mathbb{C}$ are defined by

$$\tilde{\mathfrak{g}}_n(\omega) := \int_0^t \mathbf{a}(\omega_n(s)) \cdot \dot{\omega}_n(s) ds, \qquad \omega \in C_t, \tag{3.42}$$

with C_t is as in (3.24). Further, let us consider the sequence $\{\mathfrak{g}_n\}$ of real random variables on $(C_t, \mathcal{B}(C_t), \mathbb{P})$ defined as

$$\mathfrak{g}_n(\omega) := \int_0^t \mathbf{a}(\omega(t)) \cdot \dot{\omega}_n(t) dt, \qquad \omega \in C_t, \tag{3.43}$$

where ω_n is given by $\omega_n := \tilde{P}_n(\omega)$ and $\tilde{P}_n$ is defined in (3.25).

Consider the linear operator $\mathfrak{G} : C_t \to \mathcal{H}_t$ defined by

$$\mathfrak{G}(\omega)(s) = \int_0^s \mathbf{a}(\omega(r))dr, \qquad \omega \in C_t,\ s \in [0,t], \tag{3.44}$$

with its help the functions $\{\mathfrak{g}_n\}$ and $\{g_n\}$ can be represented by the inner products:

$$\mathfrak{g}_n(\omega) = \langle \mathfrak{G}(\omega), \tilde{P}_n(\omega)\rangle, \qquad \tilde{\mathfrak{g}}_n(\omega) = \langle \mathfrak{G}(\tilde{P}_n\omega), \tilde{P}_n(\omega)\rangle.$$

For our purpose, it is useful to introduce the definition of $\mathcal{H}$-differentiable function, following, e.g., [101]:

Definition 3.1. *A function $\mathfrak{G} : C_t \to C_t$ with $\mathfrak{G}(C_t) \subset \mathcal{H}_t$ is said to be $\mathcal{H}_t$-*differentiable *if for any $\omega \in C_t$ the function $\mathfrak{G}_\omega : \mathcal{H}_t \to \mathcal{H}_t$ defined as $\mathfrak{G}_\omega(\gamma) = \mathfrak{G}(\omega+\gamma)$, $\gamma \in \mathcal{H}_t$, is Fréchet differentiable at the origin in $\mathcal{H}_t$. Its Fréchet derivative, namely the linear operator $\mathfrak{D}\mathfrak{G}_\omega(0) \in L(\mathcal{H}_t; \mathcal{H}_t)$, will be denoted with the symbol $\mathfrak{D}\mathfrak{G}(\omega)$ and called the $\mathcal{H}_t$-*derivative of $\mathfrak{G}$ at ω.

Lemma 3.5. *Let $\mathfrak{G} : C_t \to \mathcal{H}_t$ be a linear operator such that its restriction $\mathfrak{G}_{\mathcal{H}_t}$ on $\mathcal{H}_t$ is Hilbert-Schmidt. Let $\{P_n\}_n$ be a sequence of finite dimensional projection operators in $\mathcal{H}_t$ converging strongly to the identity. Then the sequences of random variables $\{\mathfrak{g}_n\}$ and $\{\tilde{\mathfrak{g}}_n\}$ on C_t defined as:*

$$\begin{aligned}\mathfrak{g}_n(\omega) &= \langle \mathfrak{G}(\omega), \tilde{P}_n(\omega)\rangle, \qquad \omega \in C_t,\\ \tilde{\mathfrak{g}}_n(\omega) &= \langle \mathfrak{G}(\tilde{P}_n(\omega)), \tilde{P}_n(\omega)\rangle, \qquad \omega \in C_t,\end{aligned}$$

satisfy

$$\lim_{n\to\infty} \mathbb{E}[|\mathfrak{g}_n - \tilde{\mathfrak{g}}_n|^2] = 0. \tag{3.45}$$

PROOF.

$$\begin{aligned}\mathbb{E}[|\mathfrak{g}_n - \tilde{\mathfrak{g}}_n|^2] &= \int |\langle \mathfrak{G}(\omega) - G(\tilde{P}_n(\omega)), \tilde{P}_n(\omega)\rangle|^2 d\mathbb{P}(\omega)\\ &= \int |\langle \mathfrak{G}(\omega - \tilde{P}_n(\omega)), \tilde{P}_n(\omega)\rangle|^2 d\mathbb{P}(\omega)\\ &= \int |\langle \mathfrak{G}(\sum_{j=n+1}^{\infty} e_j n_{e_j}(\omega)), \sum_{i=1}^{n} e_i n_{e_i}(\omega)\rangle|^2 d\mathbb{P}(\omega)\\ &= \sum_{j,j'=n+1}^{\infty} \sum_{i,i'=1}^{n} \langle \mathfrak{G}e_j, e_i\rangle \langle \mathfrak{G}e_{j'}, e_{i'}\rangle \mathbb{E}[n_{e_j} n_{e_{j'}} n_{e_i} n_{e_{i'}}]\\ &= \sum_{j=n+1}^{\infty} \sum_{i=1}^{n} (\langle \mathfrak{G}e_j, e_i\rangle)^2\\ &= \sum_{j=n+1}^{\infty} \langle P_n \mathfrak{G}e_j, P_n \mathfrak{G}e_j\rangle,\end{aligned}$$

where in the third step we have applied Itô-Nisio theorem. By using the assumption that $\mathfrak{G}_{\mathcal{H}_t}$ is an Hilbert-Schmidt operator we obtain (3.45). □

Remark 3.9. The random variables $\mathfrak{g}_n(\omega)$ are a rewriting of the Ogawa integral in the context of abstract Wiener spaces. In general the Ogawa integral (i.e. not necessarily linear) is not useful to describe our problem, in fact Lemma 3.5 holds for $\mathfrak{G}$ linear.

For further details on the Ogawa integral and for a generalization to the multidimensional case, see Appendix B and [26].

In this setting and with $\mathbf{a}$ given by (3.34) the map $\mathfrak{G} : C \to \mathcal{H}_t$, as defined by (3.44), is given by

$$\mathfrak{G}(\omega)(s) = \Big(\alpha_1^1\mathcal{I}(\omega_1) + \alpha_2^1\mathcal{I}(\omega_2) + \alpha_3^1\mathcal{I}(\omega_3), \alpha_1^2\mathcal{I}(\omega_1) + \alpha_2^2\mathcal{I}(\omega_2) + \alpha_3^2\mathcal{I}(\omega_3),$$
$$\alpha_1^3\mathcal{I}(\omega_1) + \alpha_2^3\mathcal{I}(\omega_2) + \alpha_3^3\mathcal{I}(\omega_3)\Big)$$

where

$$\mathcal{I}(\omega_k) = \int_0^s \omega_k(r)dr, \qquad k = 1,2,3,$$

and $\omega = (\omega_1, \omega_2, \omega_3) \in C_t$. Thus, in this case, the $\mathcal{H}_t$-derivative $\mathfrak{D}\mathfrak{G}(\omega)$ for any $\omega \in C_t$ is the linear operator given by $\mathfrak{D}\mathfrak{G}_\omega(\gamma) = G\gamma$, where $G : \mathcal{H}_t \to \mathcal{H}_t$ is defined in (3.35). In particular, according to lemmas 3.5 and 3.3 the sequences of random variables $\{\tilde{\mathfrak{g}}_n\}$ and $\{\mathfrak{g}_n\}$ defined respectively by (3.42) and (3.43) share the same convergence properties in $L^2(C_t, \mathbb{P})$.

The following result is a direct consequence of Lemmas 4.2 and 4.3 in [101].

Theorem 3.4. *Let $\mathfrak{G} : C_t \to C_t$, with $\mathfrak{G}(C_t) \subset \mathcal{H}_t$, be a $\mathcal{H}_t$-differentiable map such that for any $\omega \in C_t$ the $\mathcal{H}_t$-derivative $\mathfrak{D}\mathfrak{G}(\omega) \in L(\mathcal{H}_t, \mathcal{H}_t)$ is an Hilbert-Schmidt operator. Let us assume furthermore that the maps $\|\mathfrak{G}\| : C_t \to \mathbb{R}$ and $\|\mathfrak{D}\mathfrak{G}\|_2 : C_t \to \mathbb{R}$, where $\|\mathfrak{D}\mathfrak{G}(\omega)\|_2$ denotes the Hilbert-Schmidt norm of $\mathfrak{D}\mathfrak{G}(\omega)$, belong to $L^2(C_t, \mathbb{P})$. Let $\{e_i\}$ be an orthonormal basis of $\mathcal{H}_t$ and let $\{P_n\}$ and $\{\tilde{P}_n\}$ be the sequence of finite dimensional projectors on the span of $e_1, \ldots, e_n$ and their stochastic extensions to C_t respectively. Then the sequence of random variables $\{\mathfrak{h}_n\}$ defined as*

$$\mathfrak{h}_n(\omega) := \langle \mathfrak{G}(\omega), \tilde{P}_n(\omega) \rangle - \mathrm{Tr}(P_n \mathfrak{D}\mathfrak{G}(\omega)), \qquad \omega \in C_t,$$

converges in $L^2(C_t, \mathbb{P})$ and the limit does not depend on the basis $\{e_i\}$, $i = 1, \ldots, n$.

The previous theorem applied to our particular case provides actually a no-go result on the convergence of the sequence of random variables $\{\tilde{\mathfrak{g}}_n\}$ given by (3.42). Indeed, since in our particular case $\mathfrak{D}\mathfrak{G}(\omega) = G$ and by remark 3.8 the operator $G : \mathcal{H}_t \to \mathcal{H}_t$ is not trace class, the sequence of real numbers $\mathrm{Tr}(P_n\mathfrak{D}\mathfrak{G}(\omega)) \equiv \mathrm{Tr}(P_nG)$ does not in general converge independently on the choice of finite dimensional approximations $\{P_n\}$. Hence, by Theorem 3.4 and Lemma 3.5 neither the random variables $\{\mathfrak{g}_n\}$ nor $\{\tilde{\mathfrak{g}}_n\}$ admit a well-defined limit in $L^2(C_t, \mathbb{P})$ independent on the sequence $\{P_n\}_n$ of finite dimensional projectors. The following theorem provides a suitable renormalization term, namely a sequence $\{r_n\}$ of real numbers such that the renormalized random variables $h_n : C_t \to \mathbb{R}$ given by $h_n(\omega) := \tilde{\mathfrak{g}}_n(\omega) - r_n$ converge in $L^p(C_t, \mathbb{P})$ for all $p \geq 1$ and in probability to the Stratonovich stochastic integral $h(\omega) = \int_0^t \mathbf{a}(\omega(s)) \circ d\omega(s)$.

Theorem 3.5. *Let $\mathbf{a} : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear vector field, let $\{e_k\}$ be an orthonormal basis of $\mathcal{H}_t$ and let $\{P_n\}_n$ and $\{\tilde{P}_n\}_n$ be the sequence of finite dimensional projectors on the span of $e_1, \ldots, e_n$ in $\mathcal{H}_t$ and their stochastic extensions to C_t respectively. Then by setting*

$$r_n := \mathbf{B} \cdot \frac{1}{2} \sum_{k=1}^{n} \int_0^t e_k(s) \wedge \dot{e}_k(s) ds \tag{3.46}$$

with $\mathbf{B} = \operatorname{rot} \mathbf{a}$, the sequence of random variables $h_n : C_t \to \mathbb{R}$ defined as:

$$h_n(\omega) := \int_0^t \mathbf{a}(\omega_n(s)) \cdot \dot{\omega}_n(s) ds - r_n, \quad \omega \in C_t,$$

where $\omega_n := \tilde{P}_n(\omega)$, converges in $L^2(C_t, \mathbb{P})$, independently of $\{P_n\}_n$, to

$$\int_0^t \mathbf{a}(\omega(s)) \circ d\omega(s).$$

PROOF. Let us set

$$\begin{aligned} X_n(\omega) &= \int_0^t \mathbf{a}(\omega_n(s)) \cdot \dot{\omega}_n(s) ds \\ &= \int_0^t a_1(\omega_n(s))\dot{\omega}_{n,1}(s)\, ds + \int_0^t a_2(\omega_n(s))\dot{\omega}_{n,2}(s) ds + \int_0^t a_3(\omega_n(s))\dot{\omega}_{n,3}(s) ds \end{aligned}$$

where $\omega_n = (\omega_{n,1}, \omega_{n,2}, \omega_{n,3}) \in \mathcal{H}_t$. By Stokes theorem:

$$X_n = \iint_{S_n} \mathbf{B} \cdot \mathbf{n}\, dS - \int_{\Lambda_n} \mathbf{a} \cdot d\mathbf{r}$$

where Λ_n is the (oriented) segment joining $\omega_n(t)$ with 0, while $\int_{\Lambda_n} \mathbf{a} \cdot d\mathbf{r}$ is the line integral of $\mathbf{a}$ along Λ_n. The symbol S_n denotes any regular oriented surface with oriented boundary given by the close path union of ω_n and Λ_n, $\mathbf{n}$ denotes the normal unit vector and $\iint_{S_n} \mathbf{B} \cdot \mathbf{n}\, dS$ is the surface integral of $\mathbf{B}$ on S_n. Our study can be restricted to

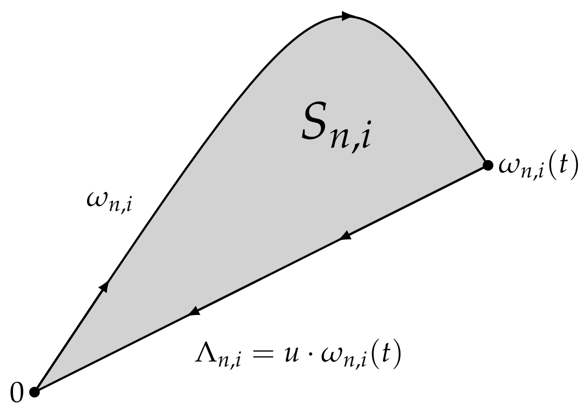
$$\iint_{S_n} \mathbf{B} \cdot \mathbf{n}\, dS,$$

as we can immediately see that the second term converges in $L^2(C_t, \mathbb{P})$ independently of $\{P_n\}_n$. Indeed:

$$\int_{\Lambda_n} \mathbf{a} \cdot d\mathbf{r} = \int_0^1 \mathbf{a}(u\omega_n(t))du \cdot \omega_n(t),$$

and for any sequence of finite dimensional projection operators $\{P_n\}_n$ such that $P_n \to \mathbb{I}$ we have

$$\omega_n(t) \to \omega(t), \qquad \int_0^1 \mathbf{a}(u\,\omega_n(t))du \to \int_0^1 \mathbf{a}(u\omega(t))du, \qquad \forall t \geq 0.$$



Two dimensional projection of stochastic surface S_n.

Let us consider now the surface integral $\iint_{S_n} \mathbf{B} \cdot \mathbf{n}\, dS$. Since by the assumption on $\mathbf{a}$ the magnetic field $\mathbf{B}$ is constant, by the Gauss-Green formula we get

$$\iint_{S_n} \mathbf{B} \cdot \mathbf{n}\, dS = \mathbf{B} \iint_{S_n} \mathbf{n}\; dS = \mathbf{B} \cdot \frac{1}{2} \int_0^t \omega_n(s) \wedge \dot{\omega}_n(s) ds.$$

Let us define for any $i = 1, 2, 3$ the sequence of random variables $h_n^i : C_t \to \mathbb{R}$ by

$$h_n^i(\omega) := \hat{e}_i \cdot \int_0^t \omega_n(s) \wedge \dot{\omega}_n(s)\; ds = \langle H^i(\omega_n), \omega_n \rangle, \qquad \omega \in C_t,$$

where $\hat{e}_i$, $i = 1, 2, 3$, are the vectors of the canonical basis of $\mathbb{R}^3$ and the linear operators $H^i : C_t \to \mathcal{H}_t$ are defined by

$$(H^i(\omega)(s))^T := \int_0^s J^i \omega(u)^T du,$$

with ^{T} denoting the transpose and J^i, $i = 1, 2, 3$, are the matrices:

$$J^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad J^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad J^3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Actually the operators H^i, $i = 1, \dots, 3$ have the form (3.35) and by Lemma 3.3 are Hilbert-Schmidt. Further, by Lemma 3.5 and Theorem 3.4, the renormalized random variables

$$h_n^i(\omega) - r_n^i = \langle H^i(\omega_n), \omega_n \rangle - \mathrm{Tr}[P_n H^i] = \int_0^t (\gamma_n(s) \wedge \dot\gamma_n(s))_i \, ds - \sum_{k=1}^n \int_0^t (e_k(s) \wedge \dot e_k(s))_i \, ds$$

converge in $L^2(C_t, \mathbb{P})$ and the limit does not depend on the sequence $\{P_n\}_n$. By combining these results we obtain the convergence of the sequence

$$\int_0^t \mathbf{a}(\omega_n(s)) \cdot \dot\omega_n(s) ds - \mathbf{B} \cdot \frac{1}{2} \sum_{k=1}^n \int_0^t e_k(s) \wedge \dot e_k(s) ds$$

and the limit is independent of the sequence $\{P_n\}_n$. Eventually, by choosing the sequence $\{P_n\}_n$ of piecewise linear approximations (1.13), where the elements e_n of the corresponding basis $\{e_n\}$ satisfy $\int_0^t e_n(s) \wedge \dot e_n(s) ds = 0$, and by applying Lemma 3.2 we complete the proof. □

Example 3.1. The renormalization term given in Theorem 3.5 contains, besides the magnetic field $\mathbf{B}$, the area integrals of the elements $\{e_n\}$ of the orthonormal basis spanning the finite dimensional Hilbert space $P_n \mathcal{H}_t$. This term is gauge independent. However, for a general orthonormal basis in $\mathcal{H}_t$, it does not converge to a well-defined limit as the following example shows.

Let us fix $t \equiv 1$ and let us consider two sequences of real valued functions $\{u_n\}_{n \geq 0}$, $\{v_n\}_{n \geq 1}$ defined on the interval $[0, 1]$ by $u_0(s) = s$ and for $n \geq 1$

$$u_n(s) = \frac{\cos(2\pi n s)}{2\pi n}, \quad v_n(s) = \frac{\sin(2\pi n s)}{2\pi n}, \qquad s \in [0, 1],$$

and the sequences of vectors in $\mathcal{H}_t$ defined by: $e_{n,1} := (u_n, v_n, 0)$, $e_{n,2} := (u_n, -v_n, 0)$, $e_{n,3} := (v_n, u_n, 0)$, $e_{n,4} := (v_n, -u_n, 0)$, $e_{n,5} := (0, 0, u_n)$, $e_{n,6} := (0, 0, v_n)$, that together with the vectors $e_{0,1} := (u_0, 0, 0)$, $e_{0,2} := (0, u_0, 0)$ and $e_{n,3} := (0.0, u_0)$ provide an orthonormal basis of $\mathcal{H}_t$.

Given the linear the vector field $\mathbf{a} : \mathbb{R}^3 \to \mathbb{R}^3$

$$\mathbf{a}(x, y, z) = \left(\frac{z - y}{2}, \frac{x - z}{2}, \frac{y - x}{2} \right).$$

with $\mathbf{B} = \operatorname{rot} \mathbf{a} = (1,1,1)$, and taking the vectors $e_{k,1}$ and $e_{k,4}$, $k = 1, \dots, n$, we get

$$\mathbf{B} \cdot \frac{1}{2} \sum_{k=1}^{n} \int_0^t e_{k,1} \wedge \dot{e}_{k,1} = \mathbf{B} \cdot \frac{1}{2} \sum_{k=1}^{n} \int_0^t e_{k,4} \wedge \dot{e}_{k,4} = \sum_{k=1}^{n} \frac{1}{4\pi k}.$$

On the other hand, considering the vectors $e_{k,2}$ and $e_{k,3}$, $k = 1, \dots, n$, we have

$$\mathbf{B} \cdot \frac{1}{2} \sum_{k=1}^{n} \int_0^t e_{k,2} \wedge \dot{e}_{k,2} = \mathbf{B} \cdot \frac{1}{2} \sum_{k=1}^{n} \int_0^t e_{k,3} \wedge \dot{e}_{k,3} = -\sum_{k=1}^{n} \frac{1}{4\pi k},$$

while the other vectors of the orthonormal basis give vanishing area integrals. Hence, the renormalization term r_n given by (3.46) is not absolutely convergent as $n \to \infty$.

A direct consequence of Lemma 3.4 and Theorem 3.5 is the following result.

Corollary 3.1. *Under the assumptions of Lemma 3.4, the sequence of finite dimensional renormalized oscillatory integrals*

$$(2\pi i\hbar)^{-n/2} \int_{P_n \mathcal{H}_t}^{o} e^{\frac{i}{2\hbar}\|\gamma_n\|^2} e^{-\frac{i}{\hbar}\left(\int_0^t \mathbf{a}(\gamma_n(s)) \cdot \dot{\gamma}_n(s) ds - r_n\right)} \psi_0(\gamma_n(t) + x) d\gamma_n,$$

with the renormalization term r_n given by (3.46), converges as $n \to \infty$ to the Wiener integral

$$\mathbb{E}\left[\psi_0(\sqrt{i\hbar}\omega(t) + x) e^{-\frac{i}{\hbar}\sqrt{i\hbar}\int_0^t \mathbf{a}(\sqrt{i\hbar}\omega(s)+x)\circ d\omega(s)}\right] \tag{3.47}$$

and the limit is independent of the sequence $\{P_n\}_n$ of finite dimensional approximations. In addition, it provides the solution of the Schrödinger equation with magnetic field

$$\begin{cases} i\hbar \partial_t \psi(t,x) = \frac{1}{2}\left(-i\hbar\nabla - \mathbf{a}\right)^2 \psi(t,x) \\ \psi(0,x) = \psi_0(x) \end{cases}, \qquad t \in \mathbb{R}^+, \quad x \in \mathbb{R}^3. \tag{3.48}$$

Proof. The first part of the theorem follows from Lemma 3.4 and Theorem 3.5. The second part can be proved by using the analyticity properties of the semigroup generated by the quantum Hamiltonian operator $H = \frac{1}{2}(-i\hbar\nabla - \mathbf{a})^2$. More precisely for $t \in \mathbb{R}^+$ the action of the heat semigroup on the vector ψ_0 is given by the Feynman-Kac-Itô formula:

$$e^{-\frac{t}{\hbar}H}\psi_0(x) = \mathbb{E}\left[\psi_0(\sqrt{\hbar}\omega(t) + x) e^{-\frac{i}{\sqrt{\hbar}}\int_0^t \mathbf{a}(\sqrt{\hbar}\omega(s)+x)\circ d\omega(s)}\right].$$

For any $\phi \in L^2(\mathbb{R}^3)$ the inner product $\langle \phi, e^{-z\frac{t}{\hbar}H}\psi_0 \rangle$ is an analytic function of $z \in D$, $D = \{z \in \mathbb{C} : Re(z) \geq 0\}$, continuous on $\bar{D}$, giving for $z = i$ the inner product between $\phi \in L^2(\mathbb{R}^d)$ and the solution of the Schrödinger equation (3.48). For $z \in \mathbb{R}^+$, by the change of variables formula we have

$$\langle \phi, e^{-z\frac{t}{\hbar}H}\psi_0\rangle = \int_{\mathbb{R}^3} \bar{\phi}(x)\mathbb{E}\left[\psi_0(\sqrt{z\hbar}\omega(t)+x)e^{-\frac{i\sqrt{z}}{\sqrt{\hbar}}\int_0^t \mathbf{a}(\sqrt{z\hbar}\omega(s)+x)\circ d\omega(s)}\right]dx.$$

By the assumptions on $t, \mathbf{a}$, and ψ_0, both sides of the equality above are analytic for $z \in D$, continuous in $\bar{D}$ and coincide on $\mathbb{R}^+$. Hence, for $z = i$ we obtain that the solution in $L^2(\mathbb{R}^3)$ of (3.48) is given by (3.47). $\square$

Remark 3.10. The results of Corollary 3.1 can be generalized to the case where a scalar potential $V \in \mathcal{F}_c(\mathbb{R}^3)$ is added to the Hamiltonian, i.e. $H = H_0 + V$ with $H_0 = \frac{1}{2}(-i\hbar\nabla - \mathbf{a})^2$. Indeed, in this case, since the function $V : \mathbb{R}^3 \to \mathbb{R}$ has the form (3.19), it is bounded and can be extended to an analytic function $V : \mathbb{C}^3 \to \mathbb{C}$. It is easy to verify that the multiplication operator associated with V is bounded, i.e. $\|V\psi\| \leq \sup_{x\in\mathbb{R}^3}|V(x)|\|\psi\|_{L^2(\mathbb{R}^3)}$ hence the perturbative Dyson expansion for the vector $e^{-\frac{it}{\hbar}(H_0+V)}\psi_0$ is convergent. We have

$$e^{-\frac{it}{\hbar}(H_0+V)}\psi_0 = \sum_m \left(-\frac{i}{\hbar}\right)^m \phi_m,$$

where

$$\phi_m = \int_{\Delta_m(t)} e^{-\frac{i}{\hbar}H_0(t-s_m)}Ve^{-\frac{i}{\hbar}H_0(s_m-s_{m-1})}\ldots Ve^{-\frac{i}{\hbar}H_0(s_2-s_1)}Ve^{-\frac{i}{\hbar}H_0 s_1}\psi_0 ds_1\ldots ds_m$$

with $\Delta_m(t) = \{(s_1,\ldots,s_m) \in \mathbb{R}^m : 0 \leq s_1 \leq \cdots \leq s_m \leq t\}$.

By exploiting the analyticity for the semigroup generated by H_0, the Dyson expansion for heat semigroup $e^{-\frac{t}{\hbar}(H_0+V)}\psi_0$ as well as the techniques used in the proof of Corollary 3.1, it is simple to prove that for any $m \in \mathbb{N}$ the vector ϕ_m can be represented in terms of the limit of the following sequence of finite dimensional renormalized oscillatory integrals:

$$\phi_m(t,x) = \lim_{n\to\infty}(2\pi i\hbar)^{-n/2}\int^o_{P_n\mathcal{H}_t} e^{\frac{i}{2\hbar}\|\gamma_n\|^2}e^{-\frac{i}{\hbar}\left(\int_0^t \mathbf{a}(\gamma_n(s)\cdot\dot{\gamma}_n(s)ds - r_n\right)}$$
$$\times\left(\int_0^t V(\gamma_n(s)+x)ds\right)^m \psi_0(\gamma_n(t)+x)d\gamma_n.$$

The limit does not depend on the choice of the sequence $\{P_n\}_n$ of finite dimensional approximations and it is equal to the Wiener integral

$$\mathbb{E}\left[\psi_0(\sqrt{i\hbar}\omega(t)+x)\left(\int_0^t V(\sqrt{i\hbar}\omega(s)+x)ds\right)^m e^{-\frac{i}{\hbar}\sqrt{i\hbar}\int_0^t \mathbf{a}(\sqrt{i\hbar}\omega(s)+x)\circ d\omega(s)}\right].$$

Remark 3.11. Similarly as in Remark 3.7 we point out that all the results in Sect. 3.3 can be extended to the case where the space dimension d is arbitrary.

Conclusions

In the foregoing pages, a Feynman path integral formula for the three dimensional Schrödinger equation with magnetic field has been rigorously realized. Once defined the infinite dimensional oscillatory integrals and their context, we used these tools to develop a mathematically consistent approach to that issue. In the previous part we show also as the techniques of finite dimensional integration can be generalized providing a functional integral representation for the solution of a general class of high-ordered heat-type equations. Back to our main aim, it should be noted that the independence of the approximation procedure in the construction of a Feynman path integral for the solution of the Schrödinger equation by adding a renormalization term, is proved in the case of a constant magnetic field. For a generic magnetic field the theory lacks of a closed form for the counterterm, which leads to the Stratonovich integral in the probabilistic representation. Some hypothesis about the form of the cunterterm in the general case could be done. For instance, we can suppose the appearance of the curl of the magnetic field and the stochastic area integrals. In fact we expect that, if one of these two quantities is zero, then the counterterm is null. However, several problems, which can't be solved as in the constant case, arise from the computation. Probably, a different way to complete the theory has to be undertaken. By rewriting the problem through the formalism of differential forms, it is interesting to point out the connection with a further generalization to the case of oriented manifolds. A rigorous mathematical definition of a infinite dimensional integral on manifolds represents an inspiring challenge ahead, as well as the completion of the theory regarding the Schrödinger equation with a generic magnetic field. Nevertheless, this work lays out how the first fundamental step (i.e. the case of a constant magnetic field) can be rigorously done, by using several mathematical techniques.

Appendix A

Abstract Wiener Spaces

In this appendix we give an overview on the theory of abstract Wiener space, which are used in many points of this work. For a more exhaustive study, see [52, 51].

Let $\mathcal{B}$ be a Banach space endowed with the norm $|\cdot|$ and let $\mathcal{B}^*$ be its dual. Moreover, let $\mathcal{H}$ be a real separable Hilbert space endowed with the scalar product $\langle \cdot \rangle$ and the norm $\| \, \|$. In the following we introduce some definitions.

Definition A.1. *A subset $B \subset \mathcal{B}$ is called* cylindric *if it can be described by the following form:*

$$B = \{x \in \mathcal{B} : (\langle y_1, x\rangle, \dots, \langle y_d, x\rangle) \in E\},$$

with $y_1, \dots, y_n \in \mathcal{B}^*$ and E is a Borel set of $\mathbb{R}^d$.

A subset $H \subset \mathcal{H}$ is called cylindric *if it can be described by the following form:*

$$H = \{x \in \mathcal{H} : Px \in F\}$$

with $P : \mathcal{H} \to \mathcal{H}$ is an orthogonal projection onto the finite dimensional subspace PH (such that $PH \equiv \mathbb{R}^d$, for some $d \in \mathbb{R}$) and $F \in \mathcal{B}(P\mathcal{H})$ is a Borel set in $P\mathcal{H}$.

Definition A.2. *A* cylinder measure *on $\mathcal{H}$ is a positive and finitely additive set function defined on the σ-algebra of cylinder sets.*

Let us consider the measure μ on the cylindrical sets of $\mathcal{H}$ given by

$$\mu\left(\{x \in \mathcal{H} | Px \in F\}\right) = \int_F \frac{e^{-\frac{\|x\|^2}{2}}}{(2\pi)^{d/2}} dx, \qquad F \in \mathcal{B}(PH).$$

We shall call μ standard Gaussian measure associated with *$\mathcal{H}$.*

The following theorem holds [77].

Theorem A.1. *Let $\mathcal{H}$ infinite dimensional. Then the measure μ does not extend to a σ-additive measure on the generated σ-algebra.*

Definition A.3. *A* measurable norm *on $\mathcal{H}$ is a norm $|\cdot|$ with the property that for every $\epsilon > 0$ there exists $P_\epsilon : \mathcal{H} \to \mathcal{H}$ finite-dimensional projection such that, for all $P \perp P_\epsilon$, holds*

$$\mu\left(\{x \in \mathcal{H} : |P(x)| > \epsilon\}\right) < \epsilon.$$

Remark A.1. It is easy to verify that $|\cdot|$ is weaker than $\|\cdot\|$. In fact for all $x \in \mathcal{H}$ there exists $k \in \mathbb{R}^+$ such that $|x| \leq k\|x\|$.

Now, denoting by $\mathcal{B}$ the completion of $\mathcal{H}$ respect to the measurable norm $\|\cdot\|$, we can consider the canonical inclusion i of $\mathcal{H}$ into $\mathcal{B}$ that, thanks to Remark A.1, is continuous. Analogously, by duality the inclusion i^* of $\mathcal{B}^*$ into $\mathcal{H}^*$ is continuous (it is given by the restriction $i^*(x) = x|_{\mathcal{H}}$). By the identification $\mathcal{H} \equiv \mathcal{H}^*$we get the following sequence of embedding

$$\mathcal{B}^* \hookrightarrow^{i^*} \mathcal{H} \hookrightarrow^{i} \mathcal{B}.$$

Definition A.4. *The triple $(\mathcal{B}, \mathcal{H}, i)$, with $\mathcal{B}$, $\mathcal{H}$ and i defined as above, is called* abstract Wiener space.

According to by L. Gross [51, 52], we have this fundamental result.

Theorem A.2. *There exists a unique finite additive measure $\tilde{\mu}$ defined on the cylinder sets B of $\mathcal{B}$ by*

$$\tilde{\mu}(B) := \mu(B \cap \mathcal{H}),$$

such that for any $y \in \mathcal{H}$, which is an element of $\mathcal{B}^*$, the following holds

$$\int_{\mathcal{B}} e^{iy(x)} d\tilde{\mu}(x) = e^{-\frac{1}{2}\|y\|^2}.$$

This results allow us to define for any $y \in \mathcal{B}^*$, a centered Gaussian random variable n_y on $(\mathcal{B}, \tilde{\mu})$ such that for any $y_1, y_2 \in \mathcal{B}^*$ we have

$$\int_{\mathcal{B}} n_{y_1} n_{y_2} d\tilde{\mu} = \langle y_1, y_2 \rangle.$$

This shows that the we can extend the map $n : \mathcal{B}^* \to L^2(\mathcal{B}, \tilde{\mu})$ by the density of $\mathcal{B}^*$ in $\mathcal{H}$, to an unitary operator, which, with an abuse of notation, we denote with same symbol $n : \mathcal{H} \to L^2(\mathcal{B}, \tilde{\mu})$.

Let us consider a complete orthonormal system $\{e_i\}$ in $\mathcal{H}$. Then, for any $h \in \mathcal{H}$ the sequence of random variables

$$\sum_{i=1}^{d} \langle e_i, h \rangle n_{e_i}$$

converges in $L^2(\mathcal{B}, \mu)$ to the random variable n_h. In particular, given a projection operator $P : \mathcal{H} \to \mathcal{H}$ of the form

$$P(x) = \sum_{i=1}^{d} \langle e_i, x \rangle e_i,$$

where $\{e_1, \ldots, e_d\}$ orthonormal vectors in $\mathcal{H}$, it is possible to define the random variable $\tilde{P} : \mathcal{B} \to \mathcal{H}$ as

$$\tilde{P}(\cdot) = \sum_{i=1}^{n} n_{e_i}(\cdot) e_i, \qquad n_{e_i} \in L^2(\mathcal{B}, \tilde{\mu}).$$

More generally we have the following definition.

Definition A.5. *A function $F : \mathcal{H} \to \mathcal{E}$ (with $\mathcal{E}$ is a Banach space) is said to admit a* stochastic extension *$\tilde{F} : \mathcal{B} \to \mathcal{E}$ if for any sequence $\{P\}$ of finite dimensional orthogonal projectors $P : \mathcal{H} \to \mathcal{H}$ converging strongly to the identity operator $\mathbb{I}$, the sequence of random variables $\{F \circ \tilde{P}\}$ converges in probability to a random variable $\tilde{F}$ on $\mathcal{B}$ and the limit does not depend on the sequence $\{P\}$.*

Example A.1. The most common example of abstract Wiener space is the space of continuous paths called *classical Wiener space*. In this space we consider the following Sobolev space:

$$\mathcal{H} := \mathcal{W}^{1,2}([0,t], \mathbb{R}^d) = \{\gamma : [0,t] \to \mathbb{R}^d \mid \gamma(0) = 0,\ \gamma \textit{ abs. cont.},\ \dot{\gamma} \in L^2([0,t])\}$$

endowed with the inner product

$$\langle \gamma_1, \gamma_2 \rangle = \int_0^t \dot{\gamma}_1 \dot{\gamma}_2 ds$$

where $\dot{\gamma}$ stands for the distributional derivative of the path γ. It is possible to prove that $\mathcal{H}$ is dense in the Banach space

$$\mathcal{B} := C_0([0,t], \mathbb{R}^d) = \{\omega : [0,t] \to \mathbb{R}^d \mid \omega(0) = 0,\ \omega \textit{ cont.}\}$$

endowed with the supremum norm

$$|\omega| := \sup_{s \in [0,t]} |\omega(s)|.$$

The triple is completed by the inclusion map $i : \mathcal{H} \to \mathcal{B}$.

For further information and examples about abstract Wiener spaces and stochastic extensions see, for instance, [77, 16].

Appendix B

The Ogawa integral*

After the introduction of stochastic integral in the 1940s due to K. Itô the interest to construct a new stochastic theory independently from causality conditions began to take hold. In particular A. Skorokhod defined, in 1970s, the so-called Skorokhod integral [106] and introduced the *anticipative calculus*. A few years later, in 1979, S. Ogawa independently introduced the so-called Ogawa integral and the corresponding *noncausal calculus* [97].

The Ogawa integral was extensively studied also in relation with the Skorohod integral and the Stratonovich integral [92, 93]; its has been extended even to the case of random fields [91, 96, 33]; however a detailed study of the case where the integrand function is d-dimensional (with $d \geq 2$) is still lacking. In this appendix, the definition of Ogawa integral has been reformulated in the framework of abstract Wiener spaces and we show that in the multidimensional case the condition of universal integrability cannot be fulfilled, even in rather simple cases. Nevertheless, that condition can be recovered thanks to the introduction of a renormalization term obtained by exploiting Ramer's functional.

A short survey on Ogawa integral

In the following, we shall adopt Ogawa's recent notations [95, 94]. Let us set a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $(W_t)_{t\in[0,1]}$ be the standard Wiener process with natural filtration $\{\mathcal{F}_t\}$. We define $\mathbf{H}$ as the set of real valued functions $f : [0,1] \times \Omega \to \mathbb{R}$ which are measurable with respect to $B_{[0,1]} \times \mathcal{F}$ and such that the following condition holds:

$$\mathbb{P}\left(\int_0^1 |f(t,\omega)|^2 dt < \infty\right) = 1.$$

*The results of this chapter are collected in [26].

Given an orthonormal basis $\{\phi_n\}$ of the Hilbert space $L^2([0,1], dt)$, let us consider the following formal random series

$$S_\phi(f) \equiv \sum_{n=1}^{\infty} (f, \phi_n)(\phi_n, \dot{W}), \tag{B.1}$$

where $(f, \phi_n) = \int_0^1 f(t)\bar{\phi}_n(t)dt$ is inner product in $L^2([0,1], dt)$ and $(\phi_n, \dot{W}) := \int_0^1 \phi_n(t)dW_t$. Now we can define a noncausal stochastic integral, i.e. the Ogawa integral.

Definition B.1. *A function $f \in \mathbf{H}$ is said to be ϕ-integrable (i.e. integrable with respect to the basis $\{\phi_n\}$) if the random series (B.1) converges in probability. In this case this sum is denoted $\int_0^1 f d_\phi W_t$ and it is called the* Ogawa integral of f with respect to the basis $\{\phi_n\}$. *A function integrable with respect to the basis $\{\phi_n\}$ is called ϕ-integrable.*

In Def. B.1 the orthonormal basis $\{\phi_n\}$ plays an important role. The requirement of the independence of the existence as well as of the value of the sum (B.1) from the basis $\{\phi_n\}$ leads naturally to the definition of *universal integrability*.

Definition B.2. *Let $f \in \mathbf{H}$. If f is integrable in the sense of Def. B.1 with respect to any orthonormal basis and the value of the integral does not depend on the basis, then the function is called* universally integrable *(*u-integrable*).*

A different way to characterize the Ogawa integral, which comes directly from the Itô-Nisio theorem [67], is the following. We can consider the sequence of approximated processes as follows

$$W_n^\phi(t) = \sum_{i=1}^{n} \int_0^t \phi_i(s)ds \int_0^1 \phi_i(s)dW_s.$$

According to the Itô-Nisio theorem we have that the sequence $\{W_n^\phi\}$ converges uniformly in $t \in [0,1]$ to W_t with probability 1. Hence, the Ogawa integral can also be defined as the limit of a sequence of Stieltjes integrals. In fact the following holds.

Proposition B.1. *Let $f \in \mathbf{H}$; then f is ϕ-integrable if and only if the sequence*

$$\int_0^1 f dW_n^\phi(t)$$

of Stieltjes integrals converges in probability. In particular we get

$$\lim_{n\to\infty} \int_0^1 f dW_n^\phi(t) = \int_0^1 f d_\phi W_t.$$

It is important to introduce the definition of regularity of an orthonormal basis.

Definition B.3. *An orthonormal basis* $\{\phi_n\}$ *in* $L^2([0,1], dt)$ *is called* regular *if*

$$\sup_n \|u_n\|_{L^2} < \infty,$$

where

$$u_n(t) = \sum_{i \leq n} \phi_i(t) \int_0^t \phi_i(s) ds.$$

Remark B.1. Two examples of regular basis are trigonometric functions and Haar functions.

Remark B.2. The existence of a non-regular basis was proved by P. Majer and M. E. Mancino in [82].

Remark B.3. The results concerning the integrability with respect regular bases and with respect any orthonormal basis were studied by Ogawa [98] and then, in the context of Malliavin calculus, by D. Nualart and M. Zakai [92].

A renormalization term for multidimensional Ogawa integral

In this section, we are going to present an equivalent definition of Ogawa integral with respect to Wiener process in the framework of abstract Wiener spaces (see Appendix A and [51, 52, 77]).

In fact, in the framework of abstract Wiener spaces, the definition of Ogawa integral can be reformulated. Let us consider the d-dimensional canonical Wiener process, where $(\Omega, \mathcal{F}) = (C, \mathcal{B}(C))$ and $W_t(\omega) = \omega(t)$, $\omega \in C$. Here C denotes the space of continuous paths $\omega : [0,1] \to \mathbb{R}^d$. Let $f : [0,1] \times C \to \mathbb{R}^d$ be a function in $\mathbf{H}$. For any orthonormal basis $\{\phi_n\}$ of $L^2([0,1]; \mathbb{R}^d)$ we can construct a corresponding orthonormal basis $\{e_n\}$ of $\mathcal{H}$ as $e_n(s) = \int_0^s \phi_n(u) du$. In fact the map $U : L^2([0,1]; \mathbb{R}^d) \to \mathcal{H}$ defined by

$$U(\phi)(s) = \int_0^s \phi(u) du, \qquad \phi \in L^2([0,1]; \mathbb{R}^d), \tag{B.2}$$

is unitary with inverse given by $U^{-1}(\gamma) = \dot{\gamma}$, $\gamma \in \mathcal{H}$. The finite dimensional approximations of the formal series (B.1) can be equivalently written as

$$\begin{aligned}
&\sum_{i=1}^{n} \int_0^1 f(t,\omega)\phi_i(t)dt \int_0^1 \phi_i(t)dW_t \\
&= \sum_{i=1}^{n} n_{e_i}(\omega) \int_0^1 f(t,\omega)\dot{e}_i(t)dt \\
&= \int_0^1 f(t,\omega)\cdot\dot{\gamma}_n(\omega)(t)dt \qquad \text{(B.3)}
\end{aligned}$$

where

$$\gamma_n(\omega) := \tilde{P}_n(\omega) = \sum_{i=1}^{n} e_i n_{e_i}(\omega), \qquad \omega \in C. \qquad \text{(B.4)}$$

According to this notation, we can say that f is ϕ-integrable if the sequence (B.3) converges in probability. Analogously f is defined to be universally Ogawa integrable if the limit does not depend on the sequence ϕ_n (or, equivalently, on the sequence $\{e_n\}$).

In the following we shall show that in the case $d \geq 2$ the condition of universal integrability is too strong and cannot be fulfilled even in the simplest cases.

Let us consider a C^1 vector field $\boldsymbol{\alpha} : \mathbb{R}^d \to \mathbb{R}^d$ and let $\mathbf{f} : [0,1] \times C \to \mathbb{R}^d$ defined as $\mathbf{f}(t,\omega) := \boldsymbol{\alpha}(\omega(t))$, $t \in [0,1]$. Given an orthonormal basis $\{e_n\}$ of $\mathcal{H}$, let us consider the sequence $\{g_n\}$ of real random variables on $(C, \mathcal{B}(C), \mathbb{P})$ defined as

$$g_n(\omega) := \int_0^1 \boldsymbol{\alpha}(\omega(t))\cdot\dot{\gamma}_n(\omega)(t)dt, \qquad \omega \in C, \qquad \text{(B.5)}$$

where γ_n is defined in (B.4). Considered the function $G : C \to \mathcal{H}$ defined as

$$G(\omega)(t) = \int_0^t \boldsymbol{\alpha}(\omega(s))ds, \qquad \omega \in C,\ t \in [0,1], \qquad \text{(B.6)}$$

the functions $\{g_n\}$ can be represented by the following inner product

$$g_n(\omega) = \langle G(\omega), \tilde{P}_n(\omega)\rangle. \qquad \text{(B.7)}$$

For $\omega \in C$, let $DG(\omega)$ denote the Fréchet differential of G evaluated in ω, given by:

$$DG(\omega)(\gamma)_j(t) = \int_0^t \nabla\alpha_j(\omega(s))\cdot\gamma(s)ds, \qquad \text{(B.8)}$$

where $\gamma \in \mathcal{H}$, and α_j are the components of $\boldsymbol{\alpha}$, with $j = 1, \ldots, d$.

We require now two more hypothesis on $\boldsymbol{\alpha}$ and $\nabla \alpha_j$ that will be necessary hereinafter:

(H1) $\displaystyle\int_0^1 \int_{R^d} |\boldsymbol{\alpha}(x)|^2 \frac{e^{-\frac{|x|^2}{2t}}}{(2\pi t)^{d/2}} dxdt < \infty;$

(H2) $\displaystyle\int_0^1 \int_{\mathbb{R}^d} |\nabla \alpha_j(x)|^2 \frac{e^{-\frac{|x|^2}{2t}}}{(2\pi t)^{d/2}} dxdt < \infty, \qquad \forall j = 1, \ldots d.$

We can now state the main result.

Theorem B.1. *For any orthonormal basis $\{e_n\}$ of $\mathcal{H}$, the sequence of renormalized finite dimensional approximations of the Ogawa integral, namely the sequence of real random variables $\{h_n\}$ on $(C, \mathcal{B}(C), \mathbb{P})$ defined as*

$$\begin{aligned} h_n(\omega) &= g_n(\omega) - r_n(\omega) \\ &= \langle G(\omega), \tilde{P}_n(\omega) \rangle - \sum_{i=1}^{n} \langle e_i, DG(\omega) e_i \rangle, \end{aligned} \tag{B.9}$$

converges in $L^2(C, \mathbb{P})$ and the limit is independent on the orthonormal basis $\{e_n\}$.

The proof relies upon the following lemmas.

Lemma B.1. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map such that $|f|$ and $\|Jf\|_2$ belong to $L^2(\mathbb{R}^n, \mu)$, with $\|Jf\|_2$ denoting the Hilbert-Schmidt norm of the Jacobian of f and μ is the standard centered Gaussian measure on $\mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} (f(x) \cdot x - \mathrm{Tr}(Jf(x)))^2 \, d\mu(x) \leq \int_{\mathbb{R}^n} \left(|f(x)|^2 + \|Jf(x)\|_2^2\right)^2 d\mu(x),$$

where $\mathrm{Tr}(Jf(x))$ is the trace of the Jacobian of f.

For a detailed proof of Lemma B.1 see [101], where also the following definition is introduced.

Proof [of Theorem B.1]. It is straightforward to verify that the map $G : C \to C$ defined by (B.6) is $\mathcal{H}$-differentiable and its $\mathcal{H}$-derivative DG is given by (B.8). Furthermore, for any $\omega \in C$, the operator $DG(\omega)$ is Hilbert-Schmidt. Indeed $DG(\omega)$: $\mathcal{H} \to \mathcal{H}$ is unitary equivalent to the linear operator $T : L^2([0,1]; \mathbb{R}^d) \to L^2([0,1]; \mathbb{R}^d)$ defined as

$$T = U^{-1} \circ DG(\omega) \circ U, \tag{B.10}$$

where $U : L^2([0,1];\mathbb{R}^d) \to \mathcal{H}$ is the unitary operator defined in (B.2). By direct computation it is simple to see that T is explicitly given in terms of a kernel $K \in L^2([0,1]\times[0,1])$, i.e. for $\phi \in L^2([0,1];\mathbb{R}^d)$ and $t \in [0,1]$,

$$(T\phi)_j(t) = \int_0^1 K_j(t,t')\cdot\phi(t')dt', \quad j=1,\dots,d,$$

where $K_j(t,t') = \nabla\alpha_j(\omega(t))\chi_{[0,t]}(t')$, $t,t' \in [0,1]$. By formula 4.32 in [86], the Hilbert-Schmidt norm of T is equal to:

$$\begin{aligned}\|T\|_2^2 = \int_{[0,1]\times[0,1]} |K(t,t')|^2 dtdt' &= \sum_{j=1}^d \int_0^1\int_0^1 |\nabla\alpha_j(\omega(t))|^2\chi_{[0,t]}(t')dtdt' : \\ &= \sum_{j=1}^d \int_0^1 t|\nabla\alpha_j(\omega(t))|^2 dt \leq \sum_{j=1}^d \int_0^1 |\nabla\alpha_j(\omega(t))|^2 dt < \infty,\end{aligned}$$

where the boundedness of the last expression follows by the continuity of the maps $t \mapsto \nabla\alpha_j(\omega(t))$. By the unitary equivalence of T and $DG(\omega)$, we get

$$\|DG(\omega)\|_2^2 = \sum_{j=1}^d \int_0^1 t|\nabla\alpha_j(\omega(t))|^2 dt < \infty.$$

Moreover, by the hypothesis (H1) and (H2), we have that

$$\begin{aligned}\mathbb{E}[\|G\|^2] &= \int_0^1 \mathbb{E}[|\boldsymbol{\alpha}(\omega(t))|^2]dt = \\ &= \int_0^1\int_{R^d} |\boldsymbol{\alpha}(x)|^2 \frac{e^{-\frac{|x|^2}{2t}}}{(2\pi t)^{d/2}} dxdt < \infty \\ \mathbb{E}[\|DG\|_2^2] &\leq \sum_{j=1}^d \int_0^1 \mathbb{E}[|\nabla\alpha_j(\omega(t))|^2]dt = \\ &= \sum_{j=1}^d \int_0^1\int_{\mathbb{R}^d} |\nabla\alpha_j(x)|^2 \frac{e^{-\frac{|x|^2}{2t}}}{(2\pi t)^{d/2}} dxdt < \infty.\end{aligned}$$

By Theorem 3.4 the sequence of random variables $\{h_n\}$ given by

$$h_n(\omega) = \langle G(\omega), \tilde{P}_n(\omega)\rangle - \mathrm{Tr}(P_n DG(\omega))$$

converges in $L^2(C,\mathbb{P})$ an the limit does not depend on the orthonormal basis $\{e_i\}$. Furthermore, by direct computation, the “renormalization term” $\mathrm{Tr}(P_n DG(\omega))$ is given by

$$\mathrm{Tr}(P_n DG(\omega)) = \sum_{i=1}^{n} \langle e_i, DG(\omega) e_i \rangle = \sum_{i=1}^{n} \int_0^1 \dot{e}_i(t) \cdot (e_i(t) \cdot \nabla) \boldsymbol{\alpha}(\omega(t)) dt.$$

□

Corollary B.1. *For any orthonormal basis $\{e_n\}$ of $\mathcal{H}$, the sequence h_n defined in Theorem B.1 converges in probability and the limit is independent of the basis $\{e_n\}$.*

Examples

According to Theorem B.1, the condition of existence of the limit in probability of the sequence of random variables $\{g_n\}$ defined in (B.5), i.e. the Ogawa integrability of the function $f \in \mathbf{H}$, with $f(t,\omega) := \boldsymbol{\alpha}(\omega(t))$, $t \in [0,1]$, with respect to the orthonormal basis $\{\phi_n\}$ of $L^2([0,1];\mathbb{R}^d)$ (with $\phi_n = \dot{e}_n$) is equivalent to the existence of the limit in probability of the "renormalization term" $r_n(\omega) = \mathrm{Tr}(P_n DG(\omega))$. Analogously, the universal Ogawa integrability of f is equivalent to the convergence in probability of r_n to a limit which does not depend on the basis $\{e_n\}$ of $\mathcal{H}$. In particular, if the linear operator $DG(\omega) \in L(\mathcal{H},\mathcal{H})$ is not trace class, then the convergent of sequence $\mathrm{Tr}(P_n DG(\omega))$ is not guaranteed and, in general, its value depends on the orthonormal basis $\{e_n\}$. We are going to show that this problem occurs even in very simple cases.

We recall the linear case studied in Sect. 3.3, but here with $d = 2$. Thus, let $\boldsymbol{\alpha} : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear vector field of the form

$$\boldsymbol{\alpha}(x,y) = (h_1 x + k_1 y, h_2 x + k_2 y). \tag{B.11}$$

In this case the map $G : C \to \mathcal{H}$ is given by

$$G(\omega)(t) = \left(h_1 \int_0^t \omega_1(s)ds + k_1 \int_0^t \omega_2(s)ds, h_2 \int_0^t \omega_1(s)ds + k_2 \int_0^t \omega_1(s)ds \right),$$

where $\omega = (\omega_1, \omega_2) \in C$. The $\mathcal{H}$-derivative $DG(\omega)$ for any $\omega \in C$ is the linear operator $DG : \mathcal{H} \to \mathcal{H}$ simply given by

$$DG(\gamma)(t) = \left(h_1 \int_0^t \gamma_1(s)ds + k_1 \int_0^t \gamma_2(s)ds, h_2 \int_0^t \gamma_1(s)ds + k_2 \int_0^t \gamma_2(s)ds \right),$$

with $\gamma = (\gamma_1, \gamma_2) \in \mathcal{H}$.

We can compute explicitly the spectrum of the self-adjoint operator $|DG| = \sqrt{DG^* DG}$. Indeed, setting for notational simplicity $L \equiv DG^* DG$ we have, for $\eta, \gamma \in \mathcal{H}$:

$$\langle \eta, L\gamma \rangle = \langle DG\eta, DG\gamma \rangle = \int_0^1 (\eta_1(t), \eta_2(t)) A (\gamma_1(t), \gamma_2(t))^T dt,$$

with

$$A = \begin{pmatrix} h_1^2 + h_2^2 & h_1 k_1 + h_2 k_2 \\ h_1 k_1 + h_2 k_2 & k_1^2 + k_2^2 \end{pmatrix}.$$

Hence, for $\gamma \in \mathcal{H}$ the vector $L(\gamma) \in \mathcal{H}$ is given by

$$L(\gamma)(t)^T = -\int_0^t \int_1^s A\gamma(r)^T drds.$$

L is a compact operator and has a discrete spectrum. By introducing in $\mathbb{R}^2$ an orthonormal basis $\{u_1, u_2\}$ of eigenvectors of the symmetric matrix A, with corresponding eigenvalues $a_1, a_2 \in \mathbb{R}^+$, the eigenvectors $\{\gamma_n\}$ of L can be represented as linear combination of u_1 and u_2, namely $\gamma_n = \eta_{n,1} u_1 + \eta_{n,2} u_2$, with $\eta_{n,j} : [0,1] \to \mathbb{R}$. The components $\{\eta_{n,j}\}$ of the eigenvectors (with eigenvalues λ) are solutions of

$$\begin{cases} \lambda_{n,j} \ddot{\eta}_{n,j} + a_j \eta_{n,j} = 0 \\ \dot{\eta}_{n,j}(1) = 0 \\ \eta_{n,j}(0) = 0 \end{cases} \qquad j = 1, 2,$$

which yields in the non-trivial case where $a_j > 0$ the solutions $\lambda_{n,j} = \frac{4a_j}{\pi^2(1+2n)^2}$, with corresponding eigenvectors $\gamma_{n,j}(t) = \sin\left(\left(\frac{\pi}{2} + n\pi\right) t\right) u_j$, where $j = 1, 2$. Hence, we can conclude that $|DG| = \sqrt{L}$ is not trace class and in general the limit of $r_n = \mathrm{Tr}(P_n DG)$ does not necessary exist and, if it exists, its value depends on the sequence of projectors $\{P_n\}$ or, equivalently, on the choice of the orthonormal basis $\{e_n\}$ of $\mathcal{H}$.

It is interesting to investigate the value that the "renormalization term" assumes for different choices of the orthonormal basis $\{e_n\}$, in order to understand the role it plays in a few particular cases.

Let us consider $L^2([0,1]; \mathbb{R}^2)$ and the following orthonormal basis

$$\begin{aligned} \{\psi_n\} := &\Big\{(1,0), (0,1), \sqrt{2}(\cos(2\pi nt), 0), \sqrt{2}(\sin(2\pi nt), 0), \\ &\sqrt{2}(0, \cos(2\pi nt)), \sqrt{2}(0, \sin(2\pi nt))\Big\} = \\ = &\{\psi_{0,x}, \psi_{0,y}, \psi_{n,1}, \psi_{n,2}, \psi_{n,3}, \psi_{n,4}\}, \end{aligned}$$

with $n \in \mathbb{N} \setminus \{0\}$. Rewriting formula (B) explicitly, we can compute

$$\langle \psi_n, T\psi_n \rangle = \int_0^1 \psi_n(t) \cdot \left(\int_0^t \psi_n(s) ds \cdot \nabla \right) \boldsymbol{\alpha}(\omega(t)) dt,$$

where $\boldsymbol{\alpha} : \mathbb{R}^2 \to \mathbb{R}^2$ is given by (B.11) and $T : L^2([0,1]; \mathbb{R}^2) \to L^2([0,1]; \mathbb{R}^2)$ is defined by (B.10). For the vectors of the form $\psi_{n,j}$ with $j = 1, \dots, 4$ we have:

$$\langle \psi_{n,j}, T\psi_{n,j} \rangle = 0;$$

while for the two constant vectors

$$\langle \psi_{0,x}, T\psi_{0,x} \rangle = \frac{h_1}{2};$$
$$\langle \psi_{0,y}, T\psi_{0,y} \rangle = \frac{k_2}{2}.$$

This gives for the basis $\{\psi_n\}$ the following "renormalization term" (depending on the divergence of $\boldsymbol{\alpha}$):

$$r_n = \mathrm{Tr}(P_n DG) = \sum_{i=1}^{n} \langle \psi_i, T\psi_i \rangle = \frac{1}{2} \nabla \cdot \boldsymbol{\alpha}.$$

Let us now consider a different basis in $L^2([0,1]; \mathbb{R}^2)$:

$$\begin{aligned} \{\xi_n\} := \{&(1,0),(0,1),(\cos(2\pi nt),\sin(2\pi nt)),(\sin(2\pi nt),\cos(2\pi nt)),\\ &(-\cos(2\pi nt),\sin(2\pi nt)),(-\sin(2\pi nt),\cos(2\pi nt))\} = \\ = \{&\xi_{0,x},\xi_{0,y},\xi_{n,1},\xi_{n,2},\xi_{n,3},\xi_{n,4}\}, \end{aligned}$$

with $n \in \mathbb{N} \setminus \{0\}$. We use the same argument as before for the vectors

$$\xi_{n,1} = (\cos(2\pi nt), \sin(2\pi nt)).$$

We obtain:

$$\begin{aligned} \langle \xi_{n,1}, T\xi_{n,1} \rangle = &\int_0^1 \left(k_1 \frac{\sin^2(\pi nt)\cos(2\pi nt)}{\pi n} + h_1 \frac{\sin(2\pi nt)\cos(2\pi nt)}{2\pi n} \right. \\ &\left. + k_2 \frac{\sin(2\pi nt)\sin^2(\pi nt)}{\pi n} + h_2 \frac{\sin^2(2\pi nt)}{2\pi n} \right) dt = \\ = &\frac{h_2 - k_1}{4n\pi} = \frac{\nabla \times \boldsymbol{\alpha}}{4n\pi}. \end{aligned}$$

Analogously

$$\begin{aligned} \langle \xi_{n,2}, T\xi_{n,2} \rangle = &\int_0^1 \left(h_2 \frac{\sin^2(\pi nt)\cos(2\pi nt)}{\pi n} + k_2 \frac{\sin(2\pi nt)\cos(2\pi nt)}{2\pi n} \right. \\ &\left. + h_1 \frac{\sin(2\pi nt)\sin^2(\pi nt)}{\pi n} + k_1 \frac{\sin^2(2\pi nt)}{2\pi n} \right) dt = \\ = &\frac{k_1 - h_2}{4n\pi} = -\frac{\nabla \times \boldsymbol{\alpha}(\omega(t))}{4n\pi}, \end{aligned}$$

and

$$\langle \xi_{n,3}, T\xi_{n,3} \rangle = \frac{k_1 - h_2}{4\pi n} = -\frac{\nabla \times \boldsymbol{\alpha}}{4\pi n}$$
$$\langle \xi_{n,4}, T\xi_{n,4} \rangle = \frac{-k_1 + h_2}{4\pi n} = \frac{\nabla \times \boldsymbol{\alpha}}{4\pi n}$$

In this case the series $\sum_{i=1}^{n}\langle \xi_i, T\xi_i\rangle$ cannot converge absolutely and and the value of the "renormalization term" depends on the order of the terms in the sum.

At last we consider in the Hilbert space $\mathcal{H}$ the sequence of orthogonal projection operators onto the finite dimensional subspaces H_n of piecewise linear paths of the form

$$\gamma(t) = \sum_{i=0}^{n-1} \chi_{\left[\frac{i}{n},\frac{i+1}{n}\right)}(t)\left(\gamma(i/n) + n\left(\gamma(i+1/n) - \gamma(i/n)\right)(t - i/n)\right), \quad \text{(B.12)}$$

with $t \in [0,1]$. An orthonormal basis of H_n is provided, e.g., by the vectors

$$\{(z_{n,i}, 0), (0, z_{n,i})\}_{i=0,\dots,n-1}, \quad \text{(B.13)}$$

where

$$z_{n,i}(t) = \sqrt{n}\chi_{\left[\frac{i}{n},\frac{i+1}{n}\right)}(t)\left(t - \frac{i}{n}\right) + \frac{1}{\sqrt{n}}\chi_{\left[\frac{i+1}{n},1\right)}(t),$$

with $i = 0, \dots, n-1$. We also notice that:

$$\dot{z}_{n,i}(t) = \sqrt{n}\chi_{\left[\frac{i}{n},\frac{i+1}{n}\right)}(t).$$

It is not difficult to compute

$$\langle (z_{n,i}, 0), DG(z_{n,i}, 0)\rangle = \frac{h_1}{2n}, \quad \langle (0, z_{n,i}), DG(0, z_{n,i})\rangle = \frac{k_2}{2n},$$

Thereby we get

$$\lim_{n\to\infty} \mathrm{Tr}(P_n DG) = \frac{1}{2}\nabla \cdot \boldsymbol{\alpha}. \quad \text{(B.14)}$$

This last example is particularly interesting since in the case where $\{P_n\}$ are the projectors on the subspaces of piecewise linear path described above, the limits of the sequences $\{g_n\}$ and $\{r_n\}$ (defined respectively by (B.7) and (B.9)) can be computed explicitly. This provides a possible technique for the computation of the limit of the sequence $\{h_n\}$ for linear vector fields $\boldsymbol{\alpha}$ and, by Theorem (B.1), this limit is independent on the sequence of projectors. We remark that this toy model can be studied also by applying different techniques, such as, for instance, Malliavin calculus [92].

The following lemma provides a useful tool in the proof of theorem B.2, which shows that Ogawa integral with respect the basis (B.13) coincides with Stratonovich Integral.

Theorem B.2. *Let $\boldsymbol{\alpha}$ be the linear vector field given by (B.11) and $G : C \to \mathcal{H}$ the linear operator (B.6). Then the sequence of random variables $\{g_n\}$ defined by*

$$g_n(\omega) = \langle G(\omega), \tilde{P}_n(\omega)\rangle, \qquad \omega \in C,$$

where $\{P_n\}$ is the sequence of orthogonal projectors onto the subspaces H_n of piecewise linear paths (B.12), converges in $L^2(C, \mathbb{P})$ to the Stratonovich integral

$$\int_0^1 \boldsymbol{\alpha}(\omega(t)) \circ d\omega(t).$$

Proof. By lemma 3.5 the sequence $\{g_n\}$ has the same limit of the sequence $\{g_n'\}$, where

$$g_n'(\omega) = \langle G(\tilde{P}_n(\omega)), \tilde{P}_n(\omega)\rangle, \qquad \omega \in C$$

if such a limit exists. Moreover the random variables $\{g_n'\}$ assume the following form

$$g_n'(\omega) = \int_0^1 \boldsymbol{\alpha}(\omega_n(t)) \cdot \dot{\omega}_n(t)dt,$$

where $\omega_n = \tilde{P}_n\omega \in \mathcal{H}$. By Wong-Zakai approximations results [115], in the case where $\{P_n\}$ are projectors on piecewise linear paths, the sequence $\{g_n'\}$ converges in $L^2(C, \mathbb{P})$ to the Stratonovich integral $\int_0^1 \boldsymbol{\alpha}(\omega(t)) \circ d\omega(t)$. □

Theorem B.3. *Let $\boldsymbol{\alpha}$ be the linear vector field given by (B.11) and $G : C \to \mathcal{H}$ the linear operator (B.6). Then the sequence of random variables $\{h_n\}$ defined in Theorem B.1, namely*

$$h_n(\omega) = g_n(\omega) - r_n,$$

with $r_n = \mathrm{Tr}(P_n DG)$, converges to the Itô integral.

$$\int_0^1 \boldsymbol{\alpha}(\omega(t))d\omega(t)$$

and the limit does not depend on the sequence $\{P_n\}$.

Proof. By Theorem B.1 the sequence $\{h_n\}$ converges in $L^2(C, \mathbb{P})$ and the limit is independent of $\{P_n\}$. In the case where $\{P_n\}$ are projectors onto subspaces of piecewise linear paths, we can compute explicitly the limit of both $\{g_n\}$ and $\{r_n\}$. Indeed, by Theorem B.2 and formula (B.14), we obtain

$$\begin{aligned}\lim_{n\to\infty} h_n(\omega) &= \lim_{n\to\infty} g_n(\omega) - \lim_{n\to\infty} r_n \\ &= \int_0^1 \boldsymbol{\alpha}(\omega(t)) \circ d\omega(t) - \frac{1}{2}\nabla \cdot \boldsymbol{\alpha},\end{aligned}$$

where the limits are meant in $L^2(C, \mathbb{P})$. By the conversion formula between Itô and Stratonovich integral we obtain the final result:

$$\int_0^1 \boldsymbol{\alpha}(\omega(t)) \circ d\omega(t) = \int_0^1 \boldsymbol{\alpha}(\omega(t))d\omega(t) + \frac{1}{2}\int_0^1 \nabla \cdot \boldsymbol{\alpha}(\omega(t))dt.$$

$\square$

Appendix C

Proof of Lemma 3.2

In this appendix we give a complete proof of Lemma 3.2 of Chpt. 3 quoted below. We notice that we shall use the same notations as in Chpt. 3

Lemma 3.2. *Let* $\mathbf{a}$ *be a three dimensional vector field fulfilling the assumptions of Theorem 3.1. Let* $\{f_n\}$ *be the sequence of random variables* $f_n : C_t \to \mathbb{C}$ *defined by*

$$f_n(\omega) = \int_0^t \mathbf{a}\left(\sqrt{i\hbar}\omega_n(s)\right) \cdot \dot{\omega}_n(s)ds,$$

where $\omega_n(s) \equiv P_n(\omega)(s)$ *and* $P_n(\omega)$ *is defined by the right hand side of (3.26). Then for any* $p \in \mathbb{N}$, $1 \leq p \leq \infty$, f_n *converges, as* $n \to \infty$, *in* $L^p(C_t, \mathbb{P})$ *to the random variable* f *defined as the Stratonovich stochastic integral*

$$f(\omega) = \int_0^t \mathbf{a}(\sqrt{i\hbar}\omega(s)) \circ d\omega(s).$$

Proof. We will consider for notational simplicity the 1-dimensional case. The proof in all dimensions, in particular in the 3-dimensional case is analogous. By (3.27) the random variables f_n are given by:

$$\begin{aligned} f_n(\omega(s)) &= \sum_{j=0}^{n-1} \int_0^{\frac{t}{n}} \int_{\mathbb{R}} e^{i\sqrt{i\hbar}k\omega(s_j)} e^{i\sqrt{i\hbar}k\frac{(\omega(s_{j+1})-\omega(s_j))s}{t/n}} \cdot \frac{(\omega(s_{j+1}) - \omega(s_j))}{t/n} d\mu(k)ds = \\ &= \sum_{j=0}^{n-1} \int_{\mathbb{R}} e^{i\sqrt{i\hbar}k\omega(s_j)} \left(e^{i\sqrt{i\hbar}k(\omega(s_{j+1})-\omega(s_j))} - 1\right) \cdot \frac{1}{i\sqrt{i\hbar}k} d\mu(k). \end{aligned}$$

By setting $\Delta_j := \omega(s_{j+1}) - \omega(s_j)$ and by a Taylor expansion (to second order with remainder), the last line becomes

$$\sum_{j=0}^{n-1} \int_{\mathbb{R}} e^{i\sqrt{i\hbar}k\omega(s_j)} \left(\Delta_j + \frac{1}{2}i\sqrt{i\hbar}k\Delta_j^2 + \frac{1}{2}(i\sqrt{i\hbar}k)^2\Delta_j^3 \int_0^1 (1-u)^2 e^{i\sqrt{i\hbar}k\Delta_j u} du\right) d\mu(k).$$

Hence the function f_n can be written as the sum of three contributions, namely $f_n = g_n + h_n + r_n$, where

$$\begin{aligned} g_n(\omega) &= \sum_{j=0}^{n-1} \int_{\mathbb{R}} e^{i\sqrt{i\hbar}k\omega(s_j)}(\omega(s_{j+1}) - \omega(s_j))d\mu(k) \\ &= \sum_{j=0}^{n-1} a(\sqrt{i\hbar}\omega(s_j))(\omega(s_{j+1}) - \omega(s_j)); \\ h_n(\omega) &= \frac{1}{2}\sum_{j=0}^{n-1} \int_{\mathbb{R}} i\sqrt{i\hbar}ke^{i\sqrt{i\hbar}k\omega(s_j)}(\omega(s_{j+1}) - \omega(s_j))^2 d\mu(k) \\ &= \sum_{j=0}^{n-1} \frac{1}{2} \cdot a'(\sqrt{i\hbar}\omega(s_j))(\omega(s_{j+1}) - \omega(s_j))^2; \\ r_n(\omega) &= \sum_{j=0}^{n-1} \frac{1}{2} \int_0^1 \int_{\mathbb{R}} \left(i\sqrt{i\hbar}\right)^2 k^2 e^{i\sqrt{i\hbar}k(\omega(s_j)+(\omega(s_{j+1})-\omega(s_j))u)} \\ &\quad \times (\omega(s_{j+1}) - \omega(s_j))^3(1-u)^2 d\mu(k)du, \end{aligned}$$

(a' standing for derivative of a).

Let us consider, at first, the sequence of random variables $\{g_n\}$ defined by

$$g_n(\omega) = \sum_{j=0}^{n-1} a(\sqrt{i\hbar}\omega(s_j)) \left(\omega(s_{j+1}) - \omega(s_j)\right), \qquad \omega \in C_t$$

and the stochastic integral

$$G(\omega) = \int_0^t a(\sqrt{i\hbar}\omega(s))d\omega(s),$$

where $s_j = \frac{jt}{n}$ and a is the Fourier transform of a complex bounded measure on $\mathbb{R}$ with compact support contained in the ball B_R with radius $R \in \mathbb{R}^+$:

$$a(\sqrt{i\hbar}\omega(s)) = \int_{\mathbb{R}} e^{i\sqrt{i\hbar}\xi\omega(s)}d\mu(\xi).$$

Without loss of generality, we can restrict ourselves to prove the convergence of g_n to G in $L^p(C_t, \mathbb{P})$ for p even.

By the BDG inequalities (see, e.g. [70]) we have

$$\mathbb{E}\left[|G - g_n|^{2p}\right] \le C_{2p} \cdot \mathbb{E}\left[\left(\sum_{j=0}^{n-1} \int_{s_j}^{s_{j+1}} \left|a(\sqrt{i\hbar}\omega(s)) - a(\sqrt{i\hbar}\omega(s_j))\right|^2 ds\right)^p\right], \quad \text{(C.1)}$$

with C_{2p} a positive constant. Moreover we have:

$$
\begin{aligned}
&\left|a(\sqrt{i\hbar}\omega(s)) - a(\sqrt{i\hbar}\omega(s_j))\right|^2 = \\
&= (\omega(s)-\omega(s_j))^2 \left| \int_0^1 \sqrt{i\hbar} a' \left(\sqrt{i\hbar} \left(\omega(s_j) + u(\omega(s)-\omega(s_j)) \right) \right) du \right|^2 = \\
&= (\omega(s)-\omega(s_j))^2 \left| i\sqrt{i\hbar} \int_0^1 \int_{\mathbb{R}} \xi e^{i\sqrt{i\hbar}\xi(\omega(s_j)+u(\omega(s)-\omega(s_j)))} d\mu(\xi) du \right|^2 \leq \\
&\leq \hbar(\omega(s)-\omega(s_j))^2 \cdot \mathcal{G}(\omega(s),\omega(s_j)),
\end{aligned}
\tag{C.2}
$$

where

$$
\begin{aligned}
&\mathcal{G}(\omega(s),\omega(s_j)) = \\
&= \int_0^1 \int_0^1 \int_{\mathbb{R}} \int_{\mathbb{R}} |\xi_1||\xi_2| e^{-\frac{\sqrt{2}}{2}\xi_1(\omega(s_j)+u_1(\omega(s)-\omega(s_j)))} \\
&\times e^{-\frac{\sqrt{2}}{2}\xi_1(\omega(s_j)+u_2(\omega(s)-\omega(s_j)))} d|\mu|(\xi_1) d|\mu|(\xi_2) du_1 du_2.
\end{aligned}
\tag{C.3}
$$

Using (C.2), we can rewrite the expectation (C.1) as follows

$$
\begin{aligned}
&\mathbb{E}\left[|G-g_n|^{2p}\right] \leq C_{2p}\hbar^p \mathbb{E}\left(\sum_{j=0}^{n-1} \int_{s_j}^{s_{j+1}} (\omega(s)-\omega(s_j))^2 \mathcal{G}(\omega(s),\omega(s_j)) ds \right)^p = \\
&= C_{2p}\hbar^p \cdot \mathbb{E}\Bigg[\sum_{j_1,\dots,j_p=0}^{n-1} \int_{s_{j_1}}^{s_{j_1+1}} \cdots \int_{s_{jp}}^{s_{jp+1}} (\omega(s_1)-\omega(s_{j_1}))^2 \cdots (\omega(s_p)-\omega(s_{j_p}))^2 \\
&\times \mathcal{G}(\omega(s_1),\omega(s_{j_1})) \cdots \mathcal{G}(\omega(s_p),\omega(s_{j_p})) ds_1 \cdots ds_p \Bigg] \leq \\
&\leq C_{2p}\hbar^p I_n^1 I_n^2,
\end{aligned}
\tag{C.4}
$$

where in the latter inequality we used Schwarz inequality, with

$$
I_n^1 = \sqrt{ \mathbb{E}\left[\sum_{j_1,\dots,j_p=0}^{n-1} \left(\int_{s_{j_1}}^{s_{j_1+1}} \cdots \int_{s_{jp}}^{s_{jp+1}} (\omega(s_1)-\omega(s_{j_1}))^4 \cdots (\omega(s_p)-\omega(s_{j_p}))^4 ds_1 \cdots ds_p \right) \right] }, \tag{C.5}
$$

$$
I_n^2 = \sqrt{ \mathbb{E}\left[\sum_{j_1,\dots,j_p=0}^{n-1} \left(\int_{s_{j_1}}^{s_{j_1+1}} \cdots \int_{s_{jp}}^{s_{jp+1}} \mathcal{G}(\omega(s_1),\omega(s_{j_1}))^2 \cdots \mathcal{G}(\omega(s_p),\omega(s_{j_p}))^2 ds_1 \cdots ds_p \right) \right] }. \tag{C.6}
$$

We will show that $I_n^1 \to 0$ for $n \to \infty$ and that I_n^2 is uniformly bounded for all n.

Let us consider the integral I_n^1 given by (C.5). All the expectations

$$\mathbb{E}\left[(\omega(s_1)-\omega(s_{j_1}))^4\cdots(\omega(s_p)-\omega(s_{j_p}))^4\right]$$

can be computed taking into account the coincidences of the indices j_r, with $r = 1,\dots,p$ in the following way:

$$\begin{aligned}\int_{s_{j_1}}^{s_{j_1+1}}\cdots\int_{s_{j_p}}^{s_{j_p+1}}\mathbb{E}\left[(\omega(s_1)-\omega(s_{j_1}))^4\cdots(\omega(s_p)-\omega(s_{j_p}))^4\right]ds_1\cdots ds_p =\\ = \int_0^{\frac{t}{n}}\cdots\int_0^{\frac{t}{n}}\mathbb{E}\left[\omega(s_1)^4\cdots\omega(s_{p_1})^4\right]ds_1\cdots ds_{p_1}\\ \int_0^{\frac{t}{n}}\cdots\int_0^{\frac{t}{n}}\mathbb{E}\left[\omega(s_{p_1+1})^4\cdots\omega(s_{p_1+p_2})^4\right]ds_{p_1+1}\cdots ds_{p_1+p_2}\cdots\\ \cdots\int_0^{\frac{t}{n}}\cdots\int_0^{\frac{t}{n}}\mathbb{E}\left[\omega(s_{p_{i-1}+1})^4\cdots\omega(s_{p_{i-1}+p_i})^4\right]ds_{p_{i-1}+1}\cdots ds_{p_{i-1}+p_i},\end{aligned} \tag{C.7}$$

with $p_1+p_2+\cdots+p_i = p$ and we have used that in distribution $\omega(s)-\omega(s_j)\sim\omega(s-s_j)$. Further, the generic term containing $\tilde{p}$ factors, for any $\tilde{p} = 1,\dots,p$, can be computed as

$$\begin{aligned}\int_0^{\frac{t}{n}}\cdots\int_0^{\frac{t}{n}}\mathbb{E}\left[\omega(s_1)^4\cdots\omega(s_{\tilde{p}})^4\right]ds_1\cdots ds_{\tilde{p}} =\\ \tilde{p}!\int\cdots\int_{0<s_1<\cdots<s_{\tilde{p}}<t/n}\mathbb{E}\left[\omega(s_1)^4\cdots\omega(s_{\tilde{p}})^4\right]ds_1\cdots ds_{\tilde{p}}.\end{aligned}$$

By a straightforward calculation we can represent $\mathbb{E}\left[\omega(s_1)^4\cdots\omega(s_{\tilde{p}})^4\right]$ as a homogeneous polynomial $P(s_1, s_2-s_1,\dots,s_{\tilde{p}}-s_{\tilde{p}-1})$ with $\deg(P) = 2\tilde{p}$. We can rewrite it as $Q(s_1,s_2,\dots,s_{\tilde{p}})$, with $\deg(Q) = 2\tilde{p}$ (its coefficients depending only on $\tilde{p}$). Thanks to the change of variables $t_i = \frac{s_i}{t/n}$, we have

$$\begin{aligned}\int\cdots\int_{0<s_1<\cdots<s_{\tilde{p}}<t/n} Q(s_1,s_2,\dots,s_{\tilde{p}})ds_1\cdots ds_{\tilde{p}} =\\ = \int\cdots\int_{0<t_1<\cdots<t_{\tilde{p}}<1}\left(\frac{t}{n}\right)^{3\tilde{p}} Q(t_1,t_2,\dots,t_{\tilde{p}})dt_1\cdots dt_{\tilde{p}} = C\cdot\left(\frac{t}{n}\right)^{3\tilde{p}},\end{aligned}$$

with

$$C = \int_{0<t_1<\cdots<t_{\tilde{p}}<1} Q(t_1,t_2,\dots,t_{\tilde{p}})dt_1\cdots dt_{\tilde{p}}.$$

Applying the same argument for all terms in (C.7) we get

$$\int_{s_{j_1}}^{s_{j_1+1}} \cdots \int_{s_{j_p}}^{s_{j_p+1}} \mathbb{E}\left[(\omega(s_1)-\omega(s_{j_1}))^4 \cdots (\omega(s_p)-\omega(s_{j_p}))^4\right] ds_1 \cdots ds_p =$$
$$= C_1 \cdots C_i \cdot \left(\frac{t}{n}\right)^{3(p_1+\cdots+p_i)} = \widetilde{C} \cdot \left(\frac{t}{n}\right)^{3p},$$

with $\tilde{C} = C_1 \cdots C_i$. Thus all the contributions can be estimated by $\widetilde{K}_p \cdot \left(\frac{t}{n}\right)^{3p}$, where $\widetilde{K}_p$ is the maximum of the constants computed as $\widetilde{C}$. Eventually, using $\sum_{j_1,\ldots,j_p=0}^{n-1} 1 = n^p$ we get

$$I_n^1 \leq \sqrt{\sum_{j_1,\ldots,j_p=0}^{n-1} \widetilde{K}_p \cdot \left(\frac{t}{n}\right)^{3p}} = \sqrt{\left(\frac{t}{n}\right)^{3p} \cdot \widetilde{K}_p \cdot n^p} = \widetilde{\mathcal{K}}_p \cdot \frac{t^{\frac{3p}{2}}}{n^p} \xrightarrow{n\to\infty} 0.$$

Concerning I_n^2, recalling the definition (C.3) of $\mathcal{G}$, we have to study

$$I_n^2 = \sqrt{\sum_{j_1,\ldots,j_p=0}^{n-1} \left(\int_{s_{j_1}}^{s_{j_1+1}} \cdots \int_{s_{j_p}}^{s_{j_p+1}} \mathbb{E}\left[\mathcal{G}(\omega(s_1),\omega(s_{j_1}))^2 \cdots \mathcal{G}(\omega(s_p),\omega(s_{j_p}))^2\right] ds_1 \cdots ds_p\right)}.$$

By writing explicitly the functions $\mathcal{G}(\cdot,\cdot)$, we get the following bound:

$$(I_n^2)^2 \leq \sum_{j_1,\ldots,j_p=0}^{n-1} \left(\int_{s_{j_1}}^{s_{j_1+1}} \cdots \int_{s_{j_p}}^{s_{j_p+1}} \int_0^1 \cdots \int_0^1 \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \mathbb{E}\left[\prod_{i=1}^{p} |\xi_i||\tilde{\xi}_i||\zeta_i||\tilde{\zeta}_i|\right.\right.$$
$$e^{-\frac{\sqrt{2}}{2}\xi_i(\omega(s_{j_i}))+u_i(\omega(s_i)-\omega(s_{j_i}))} \cdot e^{-\frac{\sqrt{2}}{2}\tilde{\xi}_i(\omega(s_{j_i}))+\tilde{u}_i(\omega(s_i)-\omega(s_{j_i}))} \cdot e^{-\frac{\sqrt{2}}{2}\zeta_i(\omega(s_{j_i}))+v_i(\omega(s_i)-\omega(s_{j_i}))}$$
$$\left.\left. e^{-\frac{\sqrt{2}}{2}\tilde{\zeta}_i(\omega(s_{j_i}))+\tilde{v}_i(\omega(s_i)-\omega(s_{j_i}))}\right] d|\mu|(\xi_i)d|\mu|(\tilde{\xi}_i)d|\mu|(\zeta_i)d|\mu|(\tilde{\zeta}_i)du_i d\tilde{u}_i dv_i d\tilde{v}_i\right).$$

Since by assumption the support of the measure μ is contained in a ball B_R of radius R, we can bound $|\xi_i||\tilde{\xi}_i||\zeta_i||\tilde{\zeta}_i| \leq R^4$ on the support of μ obtaining :

$$(I_n^2)^2 \leq R^{4p} \sum_{j_1,\ldots,j_p=0}^{n-1} \left(\int_{s_{j_1}}^{s_{j_1+1}} \cdots \int_{s_{j_p}}^{s_{j_p+1}} \int_0^1 \cdots \int_0^1 \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \mathbb{E}\left[\prod_{i=1}^{p} e^{-\frac{\sqrt{2}}{2}\omega(s_i)(\xi_i+\tilde{\xi}_i+\zeta_i+\tilde{\zeta}_i)}\right.\right.$$
$$\left.\left. e^{-\frac{\sqrt{2}}{2}(\omega(s_i)-\omega(s_{j_i}))(\xi_i u_i+\tilde{\xi}_i\tilde{u}_i+\zeta_i v_i+\tilde{\zeta}_i\tilde{v}_i)}\right] d|\mu|(\zeta_i)d|\mu|(\tilde{\zeta}_i)du_i d\tilde{u}_i dv_i d\tilde{v}_i\right).$$

We notice that the term under the expectation can be computed as

$$\exp\left[P(s_1,s_{j_1},\ldots,s_p,s_{j_p},\xi_1,\tilde{\xi}_1,\ldots,v_p,\tilde{v}_p)\right],$$

where P is a polynomial function, which maximum M_P for $s_i, s_{k_i} \in [0,t]$, $u_i, \tilde{u}_i, v_i, \tilde{v}_i \in [0,1]$, and $\xi_i, \tilde{\xi}_i, \zeta_i, \tilde{\zeta}_i \in \mathrm{supp}(\mu)$, for all $i = 1 \dots p$. Finally, by integrating and summing with respects to all variables, we get a finite term of the order $t^p \cdot |\mu|^{4p} \cdot M$, proving a uniform bound for I_n^2. Hence

$$g_n(\omega) \xrightarrow{L^{2p}(C_t,\mathbb{P})} \int_0^t a(\sqrt{i\hbar}\omega(s))d\omega(s), \qquad \omega \in C_t.$$

Let us consider now the sequence of random variables $\{h_n\}$ given by

$$h_n(\omega) = \sum_{j=0}^{n-1} \frac{1}{2} \cdot a'(\sqrt{i\hbar}\omega(s_j))(\omega(s_{j+1}) - \omega(s_j))^2, \qquad \omega \in C_t,$$

and set $a'(\sqrt{i\hbar}\omega(s)) \equiv \phi(\omega(s))$, for any $s \in [0,t]$. Let H be the random variable defined by

$$H(\omega) = \frac{1}{2} \int_0^t \phi(\omega(s))ds, \qquad \omega \in C_t.$$

We have:

$$\begin{aligned} H(\omega) - h_n(\omega) &= \frac{1}{2} \sum_{j=0}^{n-1} \left(\int_{s_j}^{s_{j+1}} \phi(\omega(s))ds - \phi(\omega(s_j)) \left(\omega(s_{j+1}) - \omega(s_j)\right)^2 \right) = \\ &= \frac{1}{2} \sum_{j=0}^{n-1} \Bigg(\int_{s_j}^{s_{j+1}} \phi(\omega(s_j))ds + \int_{s_j}^{s_{j+1}} \phi'(\omega(s_j))(\omega(s) - \omega(s_j))ds \\ &+ \int_{s_j}^{s_{j+1}} \int_0^1 (\omega(s) - \omega(s_j))^2 \phi''(\omega(s_j) + u(\omega(s) - \omega(s_j)))(1-u) \, duds \\ &\qquad - \phi(\omega(s_j)) \left(\omega(s_{j+1}) - \omega(s_j)\right)^2 \Bigg). \end{aligned}$$

Hence

$$\|H - h_n\|_{L^p(C_t,\mathbb{P})} \leq \frac{1}{2} \left(\|J_n^1\|_{L^p(C_t,\mathbb{P})} + \|J_n^2\|_{L^{2p}(C_t,\mathbb{P})} + \|J_n^3\|_{L^p(C_t,\mathbb{P})} \right),$$

where:

$$J_n^1(\omega) = \sum_{j=0}^{n-1} \phi(\omega(s_j)) \left((s_{j+1} - s_j) - (\omega(s_{j+1}) - \omega(s_j))^2\right);$$

$$J_n^2(\omega) = \sum_{j=0}^{n-1} \phi'(\omega(s_j)) \int_{s_j}^{s_{j+1}} (\omega(s) - \omega(s_j))ds;$$

$$J_n^3(\omega) = \sum_{j=0}^{n-1} \int_{s_j}^{s_{j+1}} \int_0^1 \left((\omega(s) - \omega(s_j))^2 \phi''(\omega(s_j) + u(\omega(s) - \omega(s_j)))(1-u)\right) duds.$$

Without loss of generality we can consider the case where the function $\phi : \mathbb{R} \to \mathbb{C}$ is real valued, since the general case follows easily by the inequality $\|J_n^1\|_{L^p} \le \|Re(J_n^1)\|_{L^p} + \|Im(J_n^1)\|_{L^p}$. The L^p norm of the function J_n^1 can be estimated as:

$$\mathbb{E}[|J_n^1|^{2p}] = \sum_{j_1,\ldots,j_{2p}=0}^{n-1} \mathbb{E}\Bigg[\phi(\omega(s_{j_1}))\cdots\phi(\omega(s_{j_{2p}}))\left((s_{j_1+1} - s_{j_1}) - (\omega(s_{j_1+1}) - \omega(s_{j_1}))^2\right)$$
$$\cdots\left((s_{j_{2p}+1} - s_{j_{2p}}) - (\omega(s_{j_{2p}+1}) - \omega(s_{j_{2p}}))^2\right)\Bigg] \le$$
$$\le (2p)! \sum_{0\le j_1 \le \ldots \le j_{2p} \le n-1} \mathbb{E}\Bigg[\phi(\omega(s_{j_1}))\cdots\phi(\omega(s_{j_{2p}}))\left((s_{j_1+1} - s_{j_1}) - (\omega(s_{j_1+1}) - \omega(s_{j_1}))^2\right)$$
$$\cdots\left((s_{j_{2p}+1} - s_{j_{2p}}) - (\omega(s_{j_{2p+1}}) - \omega(s_{j_{2p}}))^2\right)\Bigg].$$

Since $\mathbb{E}[((s_{j+1} - s_j) - (\omega(s_j+1) - \omega(s_j))^2)] = 0$, the sum above contains only the n^{2p-1}terms where $j_1 \le \cdots \le j_{2p-1} = j_{2p}$. Indeed, if $j_1 \le \cdots \le j_{2p-1} < j_{2p}$:

$$\mathbb{E}\left[\prod_{i=1}^{2p} \phi(\omega(s_{j_i}))\left((s_{j_1+1} - s_{j_1}) - (\omega(s_{j_1+1}) - \omega(s_j))^2\right)\right] =$$
$$= \mathbb{E}\left[\prod_{i=1}^{2p-1} \phi(\omega(s_{j_i}))\left((s_{j_1+1} - s_{j_1}) - (\omega(s_{j_1+1}) - \omega(s_j))^2\right)\phi(\omega(s_{j_p}))\right]\cdot$$
$$\mathbb{E}\left[\left((s_{j_{2p}+1} - s_{j_{2p}}) - (\omega(s_{j_{2p+1}}) - \omega(s_{j_{2p}}))^2\right)\right] = 0.$$

Direct computation shows that all the terms in this sum are of order $O((s_{j+1} - s_j)^{2p}) = O(1/n^p)$ or less. Indeed, taking into account the possible coincidences of indexes, all the terms are of the form

$$\mathbb{E}\Big[\phi(\omega(s_{k_1}))^{p_1}\left((s_{k_1+1}-s_{k_1})-(\omega(s_{k_1+1})-\omega(s_{k_1}))^2\right)^{p_1}\cdots$$
$$\cdots\phi(\omega(s_{k_r}))^{p_r}\left((s_{k_r+1}-s_{k_r})-(\omega(s_{k_r+1})-\omega(s_{k_r}))^2\right)^{p_r}\Big], \quad \text{(C.8)}$$

where $p_1+\cdots+p_r=2p$ and $k_1<k_2<\cdots<k_r$. By writing $\phi(x)=\int e^{i\sqrt{i}\xi x}d\nu(\xi)$, $x\in R$ with ν complex Borel measure on $\mathbb{R}$ supported in the ball B_R, the integral (C.8) can be estimates as:

$$\int_{\mathbb{R}^{2p}}\mathbb{E}\left[\left((s_{k_r+1}-s_{k_r})-(\omega(s_{k_r+1})-\omega(s_{k_r}))^2\right)^{p_r}\right]\prod_{\alpha=0}^{r-2}\left(\mathbb{E}\left[e^{i\sqrt{i}(\omega(s_{k_{r-\alpha}})-\omega(s_{k_{r-\alpha-1}+1}))\sum_{l=1}^{\sum_{\beta=0}^{\alpha}p_{r-\beta}}\xi_l}\right]\right)$$
$$\left(\prod_{\alpha=1}^{r-1}\mathbb{E}\left[e^{i\sqrt{i}(\omega(s_{k_{r-\alpha}+1})-\omega(s_{k_{r-\alpha}}))\sum_{l=1}^{\sum_{\beta=0}^{\alpha}p_{r-\beta}}\xi_l}\left((s_{k_{r-\alpha}+1}-s_{k_{r-\alpha}})-(\omega(s_{k_{r-\alpha}+1})-\omega(s_{k_{r-\alpha}}))^2\right)^{p_{r-\alpha}}\right]\right)$$
$$\mathbb{E}\left[e^{i\sqrt{i}\omega(s_{k_1})\sum_{l=1}^{2p}\xi_l}\right]d\nu(\xi_1)\dots d\mu(\xi_{2p}).$$

Now, since $\omega(t_1)-\omega(t_2)$ has the same law as $(t_1-t_2)^{\frac{1}{2}}X$, with X a standard normal random variable and for all $\zeta\in\mathbb{R}$, $0\le t_1\le t_2$, $k\in\mathbb{N}$, we have:

$$\mathbb{E}\left[e^{i\sqrt{i}\zeta(\omega(t_1)-\omega(t_2))}\right]=e^{-\frac{i}{2}(t-s)\xi^2},$$
$$\mathbb{E}\left[e^{i\sqrt{i}\zeta X}X^{2k}\right]=H_{2k}(\sqrt{i}\zeta)e^{-\frac{i}{2}\zeta^2},$$

with H_n denoting the n^{th} Hermite polynomial. Hence:

$$\left|\mathbb{E}\left[e^{i\sqrt{i}(\omega(s_{k_{r-\alpha}})-\omega(s_{k_{r-\alpha-1}+1}))\sum_{l=1}^{\sum_{\beta=0}^{\alpha}p_{r-\beta}}\xi_l}\right]\right|=1$$
$$\left|\mathbb{E}\left[e^{i\sqrt{i}(\omega(s_{k_{r-\alpha}+1})-\omega(s_{k_{r-\alpha}}))\sum_{l=1}^{\sum_{\beta=0}^{\alpha}p_{r-\beta}}\xi_l}\left((s_{k_{r-\alpha}+1}-s_{k_{r-\alpha}})-(\omega(s_{k_{r-\alpha}+1})-\omega(s_{k_{r-\alpha}}))^2\right)^{p_{r-\alpha}}\right]\right|\le$$
$$\le(s_{k_{r-\alpha}+1}-s_{k_{r-\alpha}})^{p_{r-\alpha}}P_{\alpha,p_{r-\alpha}}(\xi_1,\dots,\xi_{2p}),$$

with $P_{\alpha,p_{r-\alpha}}:\mathbb{R}^{2p}\to\mathbb{R}$ suitable polynomial functions. By setting

$$M:=\max_{r,p_1,\dots p_r}\prod_{\alpha=1}^{r-1}\max_{\xi_1,\dots\xi_p\in B_R}|P_{\alpha,p_{r-\alpha}}(\xi_1,\dots,\xi_p)|,$$

we get $\mathbb{E}\left[|J_n^1|^{2p}\right] \leq M\frac{t^{2p}}{n}|\nu(B_R)|^{2p}$, obtaining the required convergence result:

$$\lim_{n\to\infty} \mathbb{E}\left[|J_n^1|^{2p}\right] \to 0.$$

The same argument produces an analogous estimate for $\mathbb{E}[|J_n^2|^{2p}]$. Indeed, always assuming without loss of generality that the function ϕ is real valued, we get:

$$\begin{aligned}
\mathbb{E}[|J_n^2|^{2p}] = &\sum_{j_1,\dots,j_{2p}=0}^{n-1} \mathbb{E}\Bigg[\phi'(\omega(s_{j_1}))\cdots\phi'(\omega(s_{j_{2p}}))\int_{s_{j_1}}^{s_{j_1+1}}(\omega(u_1)-\omega(s_{j_1}))du_1 \\
&\qquad\cdots\int_{s_{j_{2p}}}^{s_{j_{2p}+1}}(\omega(u_{2p})-\omega(s_{j_{2p}}))du_{2p}\Bigg] \\
\leq\ &(2p)! \sum_{0\leq j_1\leq\cdots\leq j_{2p}\leq n-1} \mathbb{E}\Bigg[\phi'(\omega(s_{j_1}))\cdots\phi'(\omega(s_{j_{2p}})) \\
&\int_{s_{j_1}}^{s_{j_1+1}}(\omega(u_1)-\omega(s_{j_1}))du_1\cdots\int_{s_{j_{2p}}}^{s_{j_{2p}+1}}(\omega(u_{2p})-\omega(s_{j_{2p}}))du_{2p}\Bigg].
\end{aligned}$$

Again, since $\mathbb{E}[\int_{s_j}^{s_j+1}(\omega(u)-\omega(s_j))du] = 0$, we can consider only the n^{2p-1} terms with $j_1 \leq \cdots \leq j_{2p-1} = j_{2p}$. All terms have the same structure as the integrals appearing in (C.4) and by using the same arguments applied for the estimates of integrals (C.5) and (C.6), we obtain $\lim_{n\to\infty}\mathbb{E}\left[|J_n^2|^{2p}\right] = 0$. Furthermore, the same argument applies also to the term J_n^3, yielding

$$\lim_{n\to\infty} \mathbb{E}\left[|J_n^3|^{2p}\right] = 0.$$

Thus

$$h_n \xrightarrow{L^p(\Omega,\mathbb{P})} \int_0^t \phi(\omega(s))ds.$$

We estimate the last term r_n by the Cauchy-Schwarz inequality as follows

$$|r_n|^{2p} \leq \mathbb{E}\Bigg[\Bigg(\sum_{j=0}^{n-1}\int_0^1\int_0^1\int_{\mathbb{R}}\int_{\mathbb{R}}|\kappa_1||\kappa_2|e^{-\frac{\sqrt{2}}{2}\kappa_1(\omega(s_j)+(\omega(s_{j+1})-\omega(s_j))u_1)}e^{-\frac{\sqrt{2}}{2}\kappa_2(\omega(s_j)+(\omega(s_{j+1})-\omega(s_j))u_2)}$$

$$(\omega(s_{j+1})-\omega(s_j))^6(1-u_1)^2(1-u_2)^2du_1du_2d|\mu|(\kappa_1)d|\mu|(\kappa_2)\Bigg)^p\Bigg] \leq$$

$$\leq \sqrt{\mathbb{E}\left[\sum_{j_1,\dots,j_p=0}^{n-1}(\omega(s_{j_1+1})-\omega(s_{j_1}))^{12}\cdots(\omega(s_{j_p+1})-\omega(s_{j_p}))^{12}\right]}$$

$$\sqrt{\mathbb{E}\left[\sum_{j_1,\dots,j_p=0}^{n-1}\mathcal{F}(\omega(s_{j_1+1}),\omega(s_{j_1}))^2\cdots\mathcal{F}(\omega(s_{j_p}),\omega(s_{j_p+1}))^2\right]}, \tag{C.9}$$

where

$$\mathcal{F}(\omega(s_j),\omega(s_{j+1})) = \int_0^1\int_0^1\int_{\mathbb{R}}\int_{\mathbb{R}}|\kappa_1||\kappa_2|e^{-\frac{\sqrt{2}}{2}\kappa_1(\omega(s_j)+(\omega(s_{j+1})-\omega(s_j))u_1)}$$
$$e^{-\frac{\sqrt{2}}{2}\kappa_2(\omega(s_j)+(\omega(s_{j+1})-\omega(s_j))u_2)}(1-u_1)^2(1-u_2)^2du_1du_2d|\mu|(\kappa_1)d|\mu|(\kappa_2).$$

Both factors appearing in the last line of (C.9) can be estimated by the same techniques applied in the study of the terms (C.5) and (C.6), obtaining $r_n \xrightarrow{L^p(\Omega,\mathbb{P})} 0$.

Eventually, we conclude that the sequence of random variables f_n defined as

$$f_n(\omega) = \int_0^t \mathbf{a}\left(\sqrt{i\hbar}\omega_n(s)\right)\cdot\dot{\omega}_n(s)ds,$$

converges, as $n\to\infty$, in $L^p(\Omega,\mathbb{P})$ to the random variable f defined as the Stratonovich stochastic integral

$$f(\omega) = \int_0^t \mathbf{a}(\sqrt{i\hbar}\omega(s))\circ d\omega(s).$$

□

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