

UNIQUE DECOMPOSITION FOR A POLYNOMIAL OF LOW RANK

EDOARDO BALLICO AND ALESSANDRA BERNARDI

ABSTRACT. Let F be a homogeneous polynomial of degree d in $m + 1$ variables defined over an algebraically closed field of characteristic 0 and suppose that F belongs to the s -th secant variety of the d -uple Veronese embedding of \mathbb{P}^m into $\mathbb{P}^{\binom{m+d}{d}-1}$ but that its minimal decomposition as a sum of d -th powers of linear forms requires more than s addenda. We show that if $s \leq d$ then F can be uniquely written as $F = M_1^d + \cdots + M_t^d + Q$, where M_1, \dots, M_t are linear forms with $t \leq (d-1)/2$, and Q a binary form such that $Q = \sum_{i=1}^q l_i^{d-d_i} m_i$ with l_i 's linear forms and m_i 's forms of degree d_i such that $\sum (d_i + 1) = s - t$.

INTRODUCTION

In this paper we will always work with an algebraically closed field K of characteristic 0. Let $X_{m,d} \subset \mathbb{P}^N$, with $m \geq 1$, $d \geq 2$ and $N := \binom{m+d}{m} - 1$, be the classical Veronese variety obtained as the image of the d -uple Veronese embedding $\nu_d : \mathbb{P}^m \rightarrow \mathbb{P}^N$. The s -th secant variety $\sigma_s(X_{m,d})$ of Veronese variety $X_{m,d}$ is the Zariski closure in \mathbb{P}^N of the union of all linear spans $\langle P_1, \dots, P_s \rangle$ with $P_1, \dots, P_s \in X_{m,d}$. For any point $P \in \mathbb{P}^N$, we indicate with $\text{sbr}(P)$ the minimum integer s such that $P \in \sigma_s(X_{m,d})$. This integer is called the *symmetric border rank* of P .

Since $\mathbb{P}^m \simeq \mathbb{P}(K[x_0, \dots, x_m]_1) \simeq \mathbb{P}(V^*)$, with V an $(m+1)$ -dimensional vector space over K , the generic element belonging to $\sigma_s(X_{m,d})$ is the projective class of a form (a symmetric tensor) of type:

$$(1) \quad F = L_1^d + \cdots + L_r^d, \quad (T = v_1^{\otimes d} + \cdots + v_r^{\otimes d}).$$

The minimum $r \in \mathbb{N}$ such that F can be written as in (1) is the *symmetric rank* of F and we denote it $\text{sr}(F)$ ($\text{sr}(T)$, if we replace F with T).

The decomposition of a homogeneous polynomial that combines a minimum number of terms and that involves a minimum number of variables is a problem that is having a great deal of attentions not only from classical

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Algebraic Geometry ([2], [9], [7], [8], [11]), but also from applications like Computational Complexity ([10]) and Signal Processing ([12]).

At the Workshop on Tensor Decompositions and Applications (September 13–17, 2010, Monopoli, Bari, Italy), A. Bernardi presented a work in collaboration with E. Ballico where a possible structure of small rank homogeneous polynomials with border rank smaller than the rank was characterized (see [3]). It is well known that, if a homogeneous polynomial F is such that $\text{sbr}(F) < \text{sr}(F)$, then there are infinitely many decompositions of F as in (1). Our purpose in [3] was to find, among all the possible decompositions of F , a “best” one in terms of number of variables. Namely: Does there exist a canonical choice of two variables such that most of the terms involved in the decomposition (1) of F depend only on those two variables? The precise statement of that result is the following:

[3], Corollary 1) Let $F \in K[x_0, \dots, x_m]_d$ be such that $\text{sbr}(F) + \text{sr}(F) \leq 2d + 1$ and $\text{sbr}(F) < \text{sr}(F)$. Then there are an integer $t \geq 0$, linear forms $L_1, L_2, M_1, \dots, M_t \in K[x_0, \dots, x_m]_1$, and a form $Q \in K[L_1, L_2]_d$ such that $F = Q + M_1^d + \dots + M_t^d$, $t \leq \text{sbr}(F) + \text{sr}(F) - d - 2$, and $\text{sr}(F) = \text{sr}(Q) + t$. Moreover t , M_1, \dots, M_t and the linear span of L_1, L_2 are uniquely determined by F .

In terms of tensors it can be translated as follows:

[3], Corollary 2) Let $T \in S^d V^*$ be such that $\text{sbr}(T) + \text{sr}(T) \leq 2d + 1$ and $\text{sbr}(T) < \text{sr}(T)$. Then there are an integer $t \geq 0$, vectors $v_1, v_2, w_1, \dots, w_t \in S^1 V^*$, and a symmetric tensor $v \in S^d(\langle v_1, v_2 \rangle)$ such that $T = v + w_1^{\otimes d} + \dots + w_t^{\otimes d}$, $t \leq \text{sbr}(T) + \text{sr}(T) - d - 2$, and $\text{sr}(T) = \text{sr}(v) + t$. Moreover t , w_1, \dots, w_t and $\langle v_1, v_2 \rangle$ are uniquely determined by T .

The natural questions that arose from applied people at the workshop in Monopoli mentioned above, were about the possible uniqueness of the binary form Q in [3], Corollary 1 (i.e. the vector v in [3], Corollary 2) and a bound on the number t of linear forms (i.e. rank 1 symmetric tensors). We are finally able to give an answer as complete as possible to these questions. The main result of the present paper is the following:

Theorem 1. *Let $P \in \mathbb{P}^N$ with $N = \binom{m+d}{d} - 1$. Suppose that:*

$$\begin{aligned} &\text{sbr}(P) < \text{sr}(P) \text{ and} \\ &\text{sbr}(P) + \text{sr}(P) \leq 2d + 1. \end{aligned}$$

Let $\mathcal{S} \subset X_{m,d}$ be a 0-dimensional reduced subscheme that realizes the symmetric rank of P , and let $\mathcal{Z} \subset X_{m,d}$ be a 0-dimensional non-reduced subscheme such that $P \in \langle \mathcal{Z} \rangle$ and $\deg \mathcal{Z} \leq \text{sbr}(P)$. There is a unique rational normal curve $C_d \subset X_{m,d}$ such that $\deg(C_d \cap (\mathcal{S} \cup \mathcal{Z})) \geq d + 2$. Then for all points $P \in \mathbb{P}^N$ as above we have that:

$$\mathcal{S} = \mathcal{S}_1 \sqcup \mathcal{S}_2, \quad \mathcal{Z} = \mathcal{Z}_1 \sqcup \mathcal{S}_2,$$

where $\mathcal{S}_1 = \mathcal{S} \cap C_d$, $\mathcal{Z}_1 = \mathcal{Z} \cap C_d$ and $\mathcal{S}_2 = (\mathcal{S} \cap \mathcal{Z}) \setminus \mathcal{S}_1$.

Moreover C_d , \mathcal{S}_2 and \mathcal{Z} are unique, $\deg(\mathcal{Z}) = \text{sbr}(P)$, $\deg(\mathcal{Z}_1) + \deg(\mathcal{S}_1) = d + 2$, $\mathcal{Z}_1 \cap \mathcal{S}_1 = \emptyset$ and \mathcal{Z} is the unique zero-dimensional subscheme N of $X_{m,d}$ such that $\deg(N) \leq \text{sbr}(P)$ and $P \in \langle N \rangle$.

In the language of polynomials, Theorem 1 can be rephrased as follows.

Corollary 1. *Let $F \in K[x_0, \dots, x_m]_d$ be such that $\text{sbr}(F) + \text{sr}(F) \leq 2d + 1$ and $\text{sbr}(F) < \text{sr}(F)$. Then there are an integer $0 \leq t \leq (d - 1)/2$, linear forms $L_1, L_2, M_1, \dots, M_t \in K[x_0, \dots, x_m]_1$, and a form $Q \in K[L_1, L_2]_d$ such that $F = Q + M_1^d + \dots + M_t^d$, $t \leq \text{sbr}(F) + \text{sr}(F) - d - 2$, and $\text{sr}(F) = \text{sr}(Q) + t$. Moreover the line $\langle L_1, L_2 \rangle$, the forms M_1, \dots, M_t and Q such that $Q = \sum_{i=1}^q l_i^{d-d_i} m_i$ with l_i 's linear forms and m_i 's forms of degree d_i such that $\sum (d_i + 1) = s - t$, are uniquely determined by F .*

An analogous corollary can be stated for symmetric tensors.

Corollary 2. *Let $T \in S^d V^*$ be such that $\text{sbr}(T) + \text{sr}(T) \leq 2d + 1$ and $\text{sbr}(T) < \text{sr}(T)$. Then there are an integer $0 \leq t \leq (d - 1)/2$, vectors $v_1, v_2, w_1, \dots, w_t \in S^1 V^*$, and a symmetric tensor $v \in S^d(\langle v_1, v_2 \rangle)$ such that $T = v + w_1^{\otimes d} + \dots + w_t^{\otimes d}$, $t \leq \text{sbr}(T) + \text{sr}(T) - d - 2$, and $\text{sr}(T) = \text{sr}(v) + t$. Moreover the line $\langle v_1, v_2 \rangle$, the vectors v_1, \dots, v_t and the tensor v such that $v = \sum_{i=1}^q u_i^{\otimes (d-d_i)} \otimes z_i$ with $u_i \in \langle v_1, v_2 \rangle$ and $z_i \in S^{d_i}(\langle v_1, v_2 \rangle)$ such that $\sum (d_i + 1) = s - t$, are uniquely determined by T .*

Moreover, by introducing the notion of linearly general position of a scheme (Definition 1), we can perform a finer geometric description of the condition for the uniqueness of the scheme \mathcal{Z} of Theorem 1. This is the main purpose of Theorem 2 and Corollary 4. In terms of homogeneous polynomials and symmetric tensors, they can be phrased as follows:

Corollary 3. *Fix integers $m \geq 2$ and $d \geq 4$. Fix a degree d homogeneous polynomial F in $m + 1$ variables (resp. $T \in S^d V$) such that $\text{sbr}(F) \leq d$ (resp. $\text{sbr}(T) \leq d$). Let $Z \subset \mathbb{P}^m$ be any smoothable zero-dimensional scheme such that $\nu_d(Z)$ evinces $\text{sbr}(F)$ (resp. $\text{sbr}(T)$). Assume that Z is in linearly*

general position. Then Z is the unique scheme which evinces $\text{sbr}(F)$ (resp. $\text{sbr}(T)$).

1. PROOFS

The existence of a scheme \mathcal{Z} as in Theorem 1 was known from [4] and [5] (see Remark 1 of [3]).

Lemma 1. *Fix integers $m \geq 2$ and $d \geq 2$, a line $\ell \subset \mathbb{P}^m$ and any finite set $E \subset \mathbb{P}^m \setminus \ell$ such that $\sharp(E) \leq d$. Then $\dim(\langle \nu_d(E) \rangle) = \sharp(E) - 1$ and $\langle \nu_d(\ell) \rangle \cap \langle \nu_d(E) \rangle = \emptyset$.*

Proof. Since $h^0(\ell \cup E, \mathcal{O}_{\ell \cup E}(d)) = d + 1 + \sharp(E)$, to get both statements it is sufficient to prove $h^1(\mathcal{I}_{\ell \cup E}(d)) = 0$. Let $H \subset \mathbb{P}^m$ be a general hyperplane containing ℓ . Since E is finite and H is general, we have $H \cap E = \emptyset$. Hence the residual exact sequence of the scheme $\ell \cup E$ with respect to the hyperplane H is the following exact sequence on \mathbb{P}^m :

$$(2) \quad 0 \rightarrow \mathcal{I}_E(d-1) \rightarrow \mathcal{I}_{\ell \cup E}(d) \rightarrow \mathcal{I}_{\ell, H}(d) \rightarrow 0.$$

Since $h^1(\mathcal{I}_E(d-1)) = h^1(H, \mathcal{I}_{\ell, H}(d)) = 0$, we get the lemma. \square

Proof of Theorem 1. All the statements are contained in [3], Theorem 1, except the uniqueness of \mathcal{Z} , the fact that $\deg(\mathcal{Z}_1) + \deg(\mathcal{S}_1) = d + 2$ and $\mathcal{Z}_1 \cap \mathcal{S}_1 = \emptyset$. Let $\ell \subset \mathbb{P}^m$ be the line such that $\nu_d(\ell) = C_d$. Take $Z, S, Z_1, S_1, S_2 \subset \mathbb{P}^m$, such that $\nu_d(Z) = \mathcal{Z}$, $\nu_d(S) = \mathcal{S}$, $\nu_d(Z_1) = \mathcal{Z}_1$, and $\nu_d(S_i) = \mathcal{S}_i$ for $i = 1, 2$. Assume the existence of another subscheme $\mathcal{Z}' \subset X_{m,d}$ such that $P \in \langle \mathcal{Z}' \rangle$ and $\deg(\mathcal{Z}') \leq \text{sbr}(P)$. Set $\mathcal{Z}'_1 := \mathcal{Z}' \cap C_d$. The fact that $\mathcal{Z}' = \mathcal{Z}'_1 \sqcup \mathcal{S}_2$, is actually the proof of [3], Theorem 1 (parts (b), (c) and (d)). At the end of step (a) (last five lines) of proof of [3], Theorem 1, there is a description of the next steps (b), (c) and (d) needed to prove that $\mathcal{Z} = (\mathcal{Z} \cap C_d) \sqcup \mathcal{S}_2$ for a certain scheme \mathcal{Z} . The role played by \mathcal{Z} in [3], Theorem 1, is the same that \mathcal{Z}' plays here, hence the same steps (b), (c) and (d) give $\mathcal{Z}' = \mathcal{Z}'_1 \sqcup \mathcal{S}_2$ as we want here (one just needs to write \mathcal{Z}' instead of \mathcal{Z}).

Since C_d is a smooth curve, $\mathcal{Z}_1 \cup \mathcal{Z}'_1 \subset C_d$, $\mathcal{S}_2 \cap C_d = \emptyset$, and $\mathcal{Z} \cup \mathcal{Z}' = (\mathcal{Z}_1 \cup \mathcal{Z}'_1) \sqcup \mathcal{S}_2$, the schemes \mathcal{Z} and \mathcal{Z}' are curvilinear. Hence all subschemes of \mathcal{Z} and \mathcal{Z}' are smoothable. Hence any subscheme of either \mathcal{Z} or \mathcal{Z}' may be used to compute the border rank of some point of \mathbb{P}^N . Since $\deg(\ell \cap (Z \cup S)) \geq d + 2$, $\nu_d((Z \cup S) \cap \ell)$ spans $\langle C_d \rangle$. Lemma 1 implies $\langle C_d \rangle \cap \langle \mathcal{S}_2 \rangle = \emptyset$. Since $P \in \langle \mathcal{S}_1 \cup \mathcal{S}_2 \rangle$ and $\sharp(S) = \text{sr}(P)$, we have $P \notin \langle \mathcal{A} \rangle$ for any $\mathcal{A} \subsetneq \mathcal{S}$. Therefore we get that $\langle \{P\} \cup \mathcal{S}_2 \rangle \cap \langle \mathcal{S}_1 \rangle$ is a unique point. Call P_1 this point. Similarly, $\langle \mathcal{Z}_1 \rangle \cap \langle \mathcal{S}_2 \rangle$ is a unique point

and we call it P_2 . Similarly, $\langle \mathcal{Z}'_1 \rangle \cap \langle \mathcal{S}_2 \rangle$ is a unique point and we call it P_3 . Since $\langle C_d \rangle \cap \langle \mathcal{S}_2 \rangle = \emptyset$, the set $\langle C_d \rangle \cap \langle \{P\} \cup \mathcal{S}_2 \rangle$ is at most one point. Since $P_i \in \langle C_d \rangle \cap \langle \{P\} \cup \mathcal{S}_2 \rangle$, $i = 1, 2, 3$, we have $P_1 = P_2 = P_3$ and $\{P_1\} = \langle C_d \rangle \cap \langle \{P\} \cup \mathcal{S}_2 \rangle$. Since $P_1 = P_3$, we have $P_1 \in \langle \mathcal{Z}'_1 \rangle \cap \langle \mathcal{S}_1 \rangle$. Take any $E \subseteq \mathcal{Z}_1$ such that $P_1 \in \langle E \rangle$. Since $P \in \langle \{P_1\} \cup \mathcal{S}_2 \rangle \subseteq \langle E \cup \mathcal{S}_2 \rangle$ and $P \notin \langle \mathcal{U} \rangle$ for any $\mathcal{U} \subsetneq \mathcal{Z}$, we get $E \cup \mathcal{S}_2 = \mathcal{Z}$. Hence $E = \mathcal{Z}_1$. Therefore \mathcal{Z}_1 computes $\text{sbr}(P_1)$ with respect to C_d . Similarly, \mathcal{Z}'_1 computes $\text{sbr}(P_2)$ with respect to the same rational normal curve C_d . For any $Q \in \langle C_d \rangle$ with $\text{sbr}(Q) < (d+2)/2$ (equivalently $\text{sbr}(Q) \neq (d+2)/2$), there is a unique zero-dimensional subscheme of $\langle C_d \rangle$ which evinces $\text{sbr}(Q)$ ([9], Proposition 1.36; in [9], Definition 1.37, this scheme is called the canonical form of the polynomial associated to P). Since $P_1 = P_2$, we have $\mathcal{Z}'_1 = \mathcal{Z}_1$. \square

Definition 1. A scheme $Z \subset \mathbb{P}^m$ is said to be in *linearly general position* if for every linear subspace $R \subsetneq \mathbb{P}^m$ we have $\deg(R \cap Z) \leq \dim(R) + 1$.

Notice that the next theorem is false if either $d = 2$ or $m = 1$. Moreover if $d = 3$ and $m > 1$, then it essentially says that a point in the tangential variety of a Veronese variety belongs to a unique tangent line. This is a consequence of the well known Sylvester's theorem on the decompositions of binary forms ([4], [11]).

Theorem 2. Fix integers $m \geq 2$ and $d \geq 4$. Fix $P \in \mathbb{P}^N$. Let $Z \subset \mathbb{P}^m$ be any smoothable zero-dimensional scheme such that $P \in \langle \nu_d(Z) \rangle$ and $P \notin \langle \nu_d(\overline{Z}) \rangle$ for any $\overline{Z} \subsetneq Z$. Assume $\deg(Z) \leq d$ and that Z is in linearly general position. Then Z is the unique scheme $Z' \subset \mathbb{P}^m$ such that $\deg(Z') \leq d$ and $P \in \langle \nu_d(Z') \rangle$. Moreover $\nu_d(Z)$ evinces $\text{sbr}(P)$.

Proof. Since $\deg(Z) \leq d$ and Z is smoothable, [4], Proposition 11 (last sentence), gives $\text{sbr}(P) \leq d$. Hence there is a scheme which evinces $\text{sbr}(P)$ ([3], Remark 3). The existence of such a scheme follows from [3], Remark 1, and the inequality $\text{sbr}(P) \leq d$. Fix any scheme $Z' \subset \mathbb{P}^m$ such that $Z' \neq Z$, $\deg(Z') \leq d$, $P \in \langle \nu_d(Z') \rangle$, and $P \notin \langle \nu_d(Z'') \rangle$ for any $Z'' \subsetneq Z'$. Since $\deg(Z \cup Z') \leq 2d + 1$ and $h^1(\mathbb{P}^m, \mathcal{I}_{Z \cup Z'}(d)) > 0$ ([3], Lemma 1), there is a line $D \subset \mathbb{P}^m$ such that $\deg(D \cap (Z \cup Z')) \geq d + 2$ ([4], Lemma 34). Since Z is in linearly general position and $m \geq 2$, we have $\deg(Z \cap D) \leq 2$. Hence $\deg(Z' \cap D) \geq d$. Hence $\deg(Z') = d$. Since $\deg(Z') = d$, we get $Z' \subset D$. Hence $P \in \langle \nu_d(D) \rangle$. Hence $\text{sbr}(P) = d$. The secant varieties of any non-degenerate curve have the expected dimension ([1], Remark 1.6). Hence $\text{sbr}(P) \leq \lfloor (d+2)/2 \rfloor$. Since $\deg(Z') = d$, we assumed $\deg(Z') \leq \text{sbr}(P)$, contradicting the assumption $d \geq 4$. \square

Corollary 4. *Fix integers $m \geq 2$ and $d \geq 4$. Fix $P \in \mathbb{P}^N$ such that $\text{sbr}(P) \leq d$. Let $Z \subset \mathbb{P}^m$ be any smoothable zero-dimensional scheme such that $\nu_d(Z)$ evinces $\text{sbr}(P)$. Assume that Z is in linearly general position. Then Z is the unique scheme which evinces $\text{sbr}(P)$.*

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DEPT. OF MATHEMATICS, UNIVERSITY OF TRENTO, 38123 POVO (TN), ITALY
E-mail address: edoardo.ballico@unitn.it

DIPARTIMENTO DI MATEMATICA “GIUSEPPE PEANO”, UNIVERSITÀ DEGLI STUDI
 DI TORINO, VIA CARLO ALBERTO 10, I-10123 TORINO, ITALY .
E-mail address: alessandra.bernardi@unito.it