

A dyadic model on a tree

David Barbato,^{1,a)} Luigi Amedeo Bianchi,^{2,b)} Franco Flandoli,^{3,c)}
 and Francesco Morandin^{4,d)}

¹*Department of Mathematics, University of Padua, Italy*

²*SNS, Pisa, Italy*

³*Department of Mathematics, University of Pisa, Italy*

⁴*DMI, University of Parma, Italy*

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We study an infinite system of nonlinear differential equations coupled in a tree-like structure. This system was previously introduced in the literature and it is the model from which the dyadic shell model of turbulence was derived. It mimics 3D Euler and Navier-Stokes equations in a rough approximation of wavelet decomposition. We prove existence of finite energy solutions, anomalous dissipation in the inviscid unforced case, existence and uniqueness of stationary solutions (either conservative or not) in the forced case. © 2013 American Institute of Physics. [<http://dx.doi.org/10.1063/1.4792488>]

I. INTRODUCTION

A classical scheme used to explain energy cascade in turbulence is based on the picture of the fluid as composed of eddies of various sizes.^{9,12} Larger eddies split into smaller ones because of dynamical instabilities and transfer their kinetic energy from their scale to the one of the smaller eddies. One can think of a tree-like structure where nodes are eddies; any substructure father-offspring, where we denote the father by $j \in J$ (J the set of nodes) and the set of offspring by \mathcal{O}_j , corresponds to an eddy j and the set \mathcal{O}_j of smaller eddies produced by j by instability. In the simplest possible picture, eddies belong to specified discrete levels, *generations*: level 0 is made of the largest eddy, level 1 of the eddies produced by level zero, and so on. The generation of eddy j may be denoted by $|j|$. Denote also the father of eddy j by \bar{j} .

Phenomenologically, we associate to any eddy j a non-negative *intensity* $X_j(t)$, at time t , such that the kinetic energy of eddy j is $X_j^2(t)$. We relate intensities by a differential rule, which prescribes that the intensity of eddy j increases because of a flux of energy from \bar{j} to j and decreases because of a flux of energy from j to its set of offspring \mathcal{O}_j . We choose the rule

$$\frac{d}{dt}X_j = c_j X_{\bar{j}}^2 - \sum_{k \in \mathcal{O}_j} c_k X_j X_k, \quad (1)$$

where the coefficients c_j are positive.

This model has been introduced by Katz and Pavlović¹⁰ as a simplified wavelet description of Euler equations, suitable for understanding the energy cascade. The coefficients $c_j = 2^{\alpha|j|}$ represent in our model the speed of the energy flow from an eddy to its children. The coefficient α is an approximation, averaged in time and space, of the rate of this speed. Regarding solutions of Euler equations in dimension 3, it may happen (usually as a short term phenomenon) that this speed is higher or lower, this is known as intermittency; sometimes that the process itself is reversed, that is the energy flows from the smaller eddies to the bigger ones. Cheskidov, Friedlander and Pavlović^{4,5}

a)Electronic mail: barbato@math.unipd.it.

b)Electronic mail: luigi.bianchi@sns.it.

c)Electronic mail: flandoli@dma.unipi.it.

d)Electronic mail: francesco.morandin@unipr.it.

showed using Bernstein's inequality that the rate β for the dyadic 3D Euler model lies in the interval $[1, 5/2]$ which corresponds to $\alpha \in [5/2, 4]$ for the tree dyadic model. As explained in Sec. IA, the order of magnitude of c_j that correspond to K41 is

$$c_j \sim 2^{\frac{5}{3}|j|}. \quad (2)$$

The tree dyadic model (1) is a more structured version of the so called dyadic model of turbulence. The latter is based on variables Y_n which represent a cumulative intensity of shell n (shell in Fourier or wavelet space) $n = 0, 1, 2, \dots$. Here, on the contrary, shell n is described by a set of variables, all X_j 's with $|j| = n$, the different intensities of eddies of generation n . The equations for Y_n have the form

$$\frac{d}{dt} Y_n = k_n Y_{n-1}^2 - k_{n+1} Y_n Y_{n+1}. \quad (3)$$

Model (1) is thus a little bit more realistic than (3), although it is still extremely idealized with respect to the true Fourier description of Euler equations.

All these models are formally conservative: the global kinetic energy $\mathcal{E}(t) = \sum_j X_j^2(t)$, or $\mathcal{E}(t) = \sum_n Y_n^2(t)$ depending on the case, is formally constant in time; it can be easily seen in both cases, using the telescoping structure of the series $d\mathcal{E}(t)/dt$. However in previous papers^{2,5} it has been shown that the dyadic model (3) is not rigorously conservative: anomalous dissipation occurs. The flux of energy to high values of n becomes so fast after some time of evolution that, in finite time, part of the energy escapes to infinity in n .

The same question for the tree dyadic model (1) is more difficult. Intuitively, it is not clear what to expect. Even if the global flux from a generation to the next one behaves similarly to the shell case (3), energy may split between eddies of the same generation, which increase exponentially in number. Hence there is a lot of "space" (a lot of eddies) to accommodate the large amount of energy which comes from progenitors in the cascade.

The main result of this paper, Theorem 2.1, is the proof of anomalous dissipation also for model (1). To be precise, we have dissipation for a class of coefficients c_j which covers (2). The proof is similar to the one in Barbato, Flandoli and Morandin² but requires new ideas and ingredients.

Apart from anomalous dissipation, we consider also stationary solutions, showing the existence and uniqueness of such solutions in Theorems 2.2 and 2.3. This kind of argument allows and requires a more general model to be studied, namely, one needs to insert a forcing term (to find nontrivial stationary solutions) and we are able to treat also the viscous analogous of the tree dyadic model, adding the viscosity term $-\nu 2^{\gamma|j|} X_j$ to Eq. (1). The most general model that we introduce is thus system (7).

In Sec. II we describe the model and give a short summary of the main results of the paper.

In Sec. III we discuss elementary properties of the model and prove the existence of finite energy solutions.

In Sec. IV we exploit the connection between the "classic" dyadic model on naturals and the tree dyadic model. If the number of children is constant for every node in the tree, then, from each solution of the former, one can build a "lifted" version on the tree which is a solution of the latter.

Sec. V is devoted to the proof of the anomalous dissipation Theorem 2.1 in the inviscid unforced case. Self-similar solutions are also discussed.

In Sec. VI we study the stationary solutions. We prove existence and uniqueness of stationary solutions of classic and tree forced systems (8) and (7) with and without viscosity. Here the positive force f is required because otherwise the unique non-negative stationary solution is the null one.

A. The decay of X_j corresponding to K41 and anomalous dissipation

In the case of the classic dyadic model (3), Kolmogorov inertial range spectrum reads

$$Y_n \sim k_n^{-1/3}.$$

The exponent is intuitive in such case. For the tree dyadic model (1) the correct exponent may look unfamiliar and thus we give a heuristic derivation of it. The result is that Kolmogorov inertial range

spectrum corresponds to

$$X_j \sim 2^{-\frac{11}{6}|j|}. \quad (4)$$

K41 theory states¹² that, if $u(x)$ is the velocity of the turbulent fluid at position x and the expected value E is suitably understood (for instance if we analyze a time-stationary regime), one has

$$E[|u(x) - u(y)|^2] \sim |x - y|^{2/3},$$

when x and y are very close each other (but not too close). Very vaguely this means

$$|u(x) - u(y)| \sim |x - y|^{1/3}.$$

Following Katz and Pavlović,¹⁰ let us think that $u(x)$ may be written in a basis (w_j) (which are norm-one vectors in L^2) as

$$u(x) = \sum_j X_j w_j(x).$$

The vector field $w_j(x)$ corresponds to the velocity field of eddy j . Let us assume that eddy j has a support Q_j of the order of a cube of side $2^{-|j|}$. Given j , take $x, y \in Q_j$. When we compute $u(x) - u(y)$ we use the approximation $u(x) = X_j w_j(x)$, $u(y) = X_j w_j(y)$. Then

$$|u(x) - u(y)| = |X_j| |w_j(x) - w_j(y)|,$$

namely

$$|X_j| |w_j(x) - w_j(y)| \sim |x - y|^{1/3}, \quad x, y \in Q_j.$$

We consider reasonably correct this approximation when $x, y \in Q_j$ have a distance of the order of $2^{-|j|}$, otherwise we should use smaller eddies in this approximation. Thus we have

$$|X_j| |w_j(x) - w_j(y)| \sim 2^{-\frac{1}{3}|j|}, \quad x, y \in Q_j, |x - y| \sim 2^{-|j|}. \quad (5)$$

Moreover, we have

$$|w_j(x) - w_j(y)| = |\nabla w_j(\xi)| |x - y|, \quad (6)$$

for some point ξ between x and y (to be precise, the mean value theorem must be applied to each component of the vector valued function w_j). Recall that $\int w_j(x)^2 dx = 1$, hence the typical size s_j of w_j in Q_j can be guessed from $s_j^2 2^{-3|j|} \sim 1$, namely $s_j \sim 2^{3/2|j|}$. Since w_j has variations of order s_j at distance $2^{-|j|}$, we deduce that the typical values of ∇w_j in Q_j have the order $2^{3/2|j|}/2^{-|j|} = 2^{5/2|j|}$. Thus, from (6),

$$|w_j(x) - w_j(y)| \sim 2^{\frac{5}{2}|j|} 2^{-|j|}.$$

Along with (5) this gives us

$$|X_j| 2^{\frac{5}{2}|j|} 2^{-|j|} \sim 2^{-\frac{1}{3}|j|},$$

namely

$$|X_j| \sim 2^{(-\frac{1}{3}+1-\frac{5}{2})|j|} = 2^{-\frac{11}{6}|j|}.$$

We have established (4), on a heuristic ground of course.

Let us give a heuristic explanation of the fact that, when anomalous dissipation occurs, the decay (4) appears. In a sense, this may be seen as a confirmation that (4) is the correct decay corresponding to K41. Let us start from Eq. (1) with $c_j \sim 2^{5/2|j|}$, the Katz-Pavlović prescription. Let \mathcal{E}_n be the energy up to generation n :

$$\mathcal{E}_n = \sum_{|j| \leq n} X_j^2.$$

Then, as will be seen later with Eq. (9),

$$\frac{d\mathcal{E}_n}{dt} = -2^{\frac{5}{2}(n+1)} \sum_{|k|=n+1} X_k^2 X_k.$$

In order to have anomalous dissipation, we should have

$$\frac{d\mathcal{E}_n}{dt} \stackrel{n}{\sim} -C \neq 0.$$

If we assume a power decay

$$X_j \sim 2^{-\eta|j|}.$$

Then, since the cardinality of $\{Q_j: |j| = n\}$ should be of the order of 2^{3n} ,

$$2^{\frac{5}{2}(n+1)} \sum_{|k|=n+1} X_k^2 X_k \sim 2^{\frac{5}{2}n} 2^{3n} 2^{-3\eta n} = 2^{(\frac{11}{2}-3\eta)n},$$

and thus $\eta = 11/6$.

II. MODEL AND MAIN RESULTS

Let J be the set of nodes. Inside J we identify one special node, called root or ancestor of the tree, which is denoted by 0. For all $j \in J$ we define the generation number $|j| \in \mathbb{N}$ (such that $|0| = 0$), the set of offsprings of j , denoted by $\mathcal{O}_j \subset J$, such that $|k| = |j| + 1$ for all $k \in \mathcal{O}_j$ and a unique parent \bar{j} with $j \in \mathcal{O}_{\bar{j}}$. The root 0 has no parent inside J , but with slight notation abuse we will nevertheless use the symbol $\bar{0}$ when needed.

For sake of simplicity we will suppose throughout the paper that the cardinality of \mathcal{O}_j is constant, $\#\mathcal{O}_j =: N_*$ for all $j \in J$, but some results can be easily generalized at least to the case where $\#\mathcal{O}_j$ is positive and uniformly bounded.

It will turn out to be very important to compare N_* to some coefficients of the model. To this end we set also $\tilde{\alpha} := 1/2 \log_2 N_*$ so that $N_* = 2^{2\tilde{\alpha}}$

The dynamics of the tree dyadic model is described by a family $(X_j)_{j \in J}$ of functions $X_j: [0, \infty) \rightarrow \mathbb{R}$. Its general formulation is described by the equations below. (Notice that $X_{\bar{0}}$ does not belong to the family and merely represents a convenient symbolic alias for the constant forcing term.)

$$\begin{cases} X_{\bar{0}}(t) \equiv f \\ \frac{d}{dt} X_j = -\nu d_j X_j + c_j X_j^2 - \sum_{k \in \mathcal{O}_j} c_k X_j X_k, \quad \forall j \in J. \end{cases} \tag{7}$$

Here we suppose that $f \geq 0$, $\nu \geq 0$, and that the other coefficients have an exponential behavior, namely $c_j = 2^{\alpha|j|}$, $d_j = 2^{\gamma|j|}$ with $\alpha > 0$ and $\gamma > 0$.

If $f = 0$ we call the system *unforced*, if $\nu = 0$ we call it *inviscid*.

This system will usually come with an initial condition which will be denoted by $X^0 = (X_j^0)_{j \in J}$. One natural space for $X(t)$ to live is $l^2(J; \mathbb{R})$, which we will simply denote by l^2 , the setting being understood. The l^2 norm will be simply denoted by $\|\cdot\|$.

Definition 1. Given $X^0 \in \mathbb{R}^J$, we call componentwise solution of system (7) with initial condition X^0 any family $X = (X_j)_{j \in J}$ of continuously differentiable functions $X_j: [0, \infty) \rightarrow \mathbb{R}$ such that $X(0) = X^0$ and all equations in system (7) are satisfied. If moreover $X(t) \in l^2$ for all $t \geq 0$, we call it an *l^2 solution*.

We say that a solution is positive if $X_j(t) \geq 0$ for all $j \in J$ and $t \geq 0$.

Existence of positive l^2 solutions is classical and can be found in Sec. III, while uniqueness is an open problem.

This system of equations is locally conservative, in the sense made rigorous by Proposition 3.2 below, where the following energy balance inequality is proven

$$\|X(t)\|^2 \leq \|X(s)\|^2 + 2f^2 \int_s^t X_0(u)du - 2\nu \sum_{j \in J} d_j \int_s^t X_j^2(u)du.$$

It turns out that in some cases this is in fact an equality and in some cases it is a strict inequality. When the latter happens we say that anomalous dissipation occurs.

The main results of the paper deal with anomalous dissipation and stationary solutions.

A. Anomalous dissipation on the inviscid, unforced tree dyadic model

The proof of the next result is given in Sec. V.

Theorem 2.1. *Let $\#\mathcal{O}_j = 2^{2\tilde{\alpha}}$ for all j . Suppose $\tilde{\alpha} < \alpha$ and $f = \nu = 0$ in Eq. (7). Let X be any positive l^2 solution with initial condition X^0 . Then there exists $C > 0$, depending only on $\|X^0\|$, such that for all $t > 0$*

$$\mathcal{E}(t) := \|X(t)\|^2 := \sum_{j \in J} X_j^2(t) < \frac{C}{t^2}.$$

This theorem holds also if we use the weaker hypothesis $1 \leq \#\mathcal{O}_j \leq 2^{2\tilde{\alpha}}$ for all j . The statement tells us that the energy of the system goes to zero at least as fast as t^{-2} . In Sec. V A we show that for this model there are some self-similar solutions and that their energy goes to zero exactly like t^{-2} . So the estimate of Theorem 2.1 cannot be improved much.

B. Stationary solutions for the forced classic dyadic model

It will be important for our purposes to switch between the tree dyadic model and the classic one, where J is simply the set of non-negative integers with $\mathcal{O}_j := \{j + 1\}$ for all j .

To avoid confusion we will use different symbols for the classic system, whose equations are the following:

$$\begin{cases} Y_{-1}(t) \equiv f \\ \frac{d}{dt} Y_n = -\nu l_n Y_n + k_n Y_{n-1}^2 - k_{n+1} Y_n Y_{n+1}, \quad \forall n \geq 0, \end{cases} \tag{8}$$

with $f \geq 0$, $\nu \geq 0$, $k_n = 2^{\beta n}$, $l_n = 2^{\gamma n}$, $\beta > 0$ and $\gamma > 0$.

When this model is interpreted as a special case of (7) we will have $N_* = 1$, $\tilde{\alpha} = 0$, and $\beta = \alpha$. Observe that the definitions of solutions given on the tree model extend easily to this one, but notice that in this setting l^2 will correspond to the standard space of sequences.

The following theorem deals with stationary solutions, namely solutions constant in time. We do not detail the proof, since, by what we said above, it is a special case of the analogous statement for the tree dyadic model, Theorem 2.3 which is proven in Sec. VI.

Theorem 2.2. *If $f > 0$, then there exists a unique l^2 positive solution Y of system (8) which is stationary. Moreover*

1. if $\nu = 0$ then $Y_n(t) := f 2^{-\beta/3(n+1)}$;
2. if $\nu > 0$ and $3\gamma \geq 2\beta$, the stationary solution is conservative and regular, in that for all real s , $\sum_n [2^{sn} Y_n(t)]^2 < \infty$;
3. if $\nu > 0$ and $3\gamma < 2\beta$, there exists $C > 0$ such that for all $f > C$ the invariant solution of (8) is not regular and exhibits anomalous dissipation.

In the inviscid case, this theorem extends an analogue result of Cheskidov, Friedlander and Pavlovic⁵ where it is proved for $\beta = 5/2$. In the viscous case it extends a result of Cheskidov and Friedlander,⁴ in which existence and uniqueness of stationary solutions are proved for $\gamma = 2$ and $\beta \in (3/2, 5/2]$.

C. Stationary solutions for the forced tree dyadic model

An analogous of Theorem 2.2 holds for the tree dyadic model too. This is proved in Sec. VI.

Theorem 2.3. *Let $\#\mathcal{O}_j = 2^{2\tilde{\alpha}}$ for all j . Suppose $\tilde{\alpha} < \alpha$ and $f > 0$ in Eq. (7). Then there exists a unique l^2 positive solution X which is stationary. Moreover*

1. if $v = 0$ then $X_j(t) := f 2^{-(2\tilde{\alpha} + \alpha)(|j| + 1)/3}$ for all $j \in J$;
2. if $v > 0$ and $0 < \alpha - \tilde{\alpha} \leq \gamma 3/2$, the stationary solution is conservative and regular, in that for all real s , $\sum_{j \in J} [2^{s|j|} X_j(t)]^2 < \infty$;
3. if $v > 0$ and $\alpha - \tilde{\alpha} > \gamma 3/2$, there exists $C > 0$ such that for all $f > C$ the invariant solution of (7) is not regular and exhibits anomalous dissipation.

III. ELEMENTARY PROPERTIES

We will provide, in this section, some basic results on the tree dyadic model. The results are analogous to those provided for the dyadic model,^{1,2,8} but the proofs require some new ideas to cope with the more general structure.

We will suppose throughout the paper that the initial condition X^0 is in l^2 and that $X_j^0 \geq 0$ for all $j \in J$. It will turn out that these two properties hold then for all times.

Definition 2. For $n \geq -1$, we denote by $\mathcal{E}_n(t)$ the total energy on nodes j with $|j| \leq n$ at time t and $\mathcal{E}(t)$ the energy of all nodes at time t (which is possibly infinite):

$$\mathcal{E}_n(t) := \sum_{|j| \leq n} X_j^2(t), \quad \mathcal{E}(t) := \sum_{j \in J} X_j^2(t).$$

Note in particular that $\mathcal{E}_{-1} \equiv 0$.

We will use very often the derivative of \mathcal{E}_n , for $n \geq 0$,

$$\begin{aligned} \frac{d}{dt} \mathcal{E}_n(t) &= 2 \sum_{|j| \leq n} X_j \frac{d}{dt} X_j(t) \\ &= -2v \sum_{|j| \leq n} d_j X_j^2 + 2 \sum_{|j| \leq n} c_j X_j^2 X_j - 2 \sum_{|j| \leq n} \sum_{k \in \mathcal{O}_j} c_k X_j^2 X_k \\ &= -2v \sum_{|j| \leq n} d_j X_j^2 + 2c_0 X_0^2 X_0 - 2 \sum_{|k|=n+1} c_k X_k^2 X_k. \end{aligned}$$

so we get for all $n \geq 0$

$$\frac{d}{dt} \mathcal{E}_n(t) = -2v \sum_{|j| \leq n} d_j X_j^2(t) + 2f^2 X_0(t) - 2 \sum_{|k|=n+1} c_k X_k^2(t) X_k(t). \tag{9}$$

Proposition 3.1. *If $X_j^0 \geq 0$ for all j , then any componentwise solution is positive. If X^0 is in l^2 , any positive componentwise solution is a positive l^2 solution, in particular for all $t \geq 0$,*

$$\mathcal{E}(t) \leq (\mathcal{E}(0) + 1)e^{2f^2 t}. \tag{10}$$

Proof. From the definition of componentwise solution we get that for all $j \in J$

$$X_j(t) = X_j^0 e^{-\int_0^t (vd_j + \sum_k c_k X_k(r)) dr} + \int_0^t c_j X_j^2(s) e^{-\int_s^t (vd_j + \sum_k c_k X_k(r)) dr} ds, \tag{11}$$

yielding $X_j(t) \geq 0$ for all $t > 0$ and all $j \in J$.

Now we turn to the estimates of $\mathcal{E}(t)$. In (9), since $X_k(t) \geq 0$ we have two negative contribution which we drop and we use the bound $X_0(t) \leq X_0^2(t) + 1 \leq \mathcal{E}_n(t) + 1$ to get that for all $n \geq 0$,

$$\frac{d}{dt} \mathcal{E}_n(t) \leq 2f^2(\mathcal{E}_n(t) + 1),$$

so by Gronwall lemma $\mathcal{E}_n(t) + 1 \leq (\mathcal{E}_n(0) + 1)e^{2f^2 t}$. Letting $n \rightarrow \infty$ we obtain (10). □

Proposition 3.2. For any positive l^2 solution X , the following energy balance principle holds, for all $0 \leq s < t$.

$$\begin{aligned} \mathcal{E}(t) = \mathcal{E}(s) + 2f^2 \int_s^t X_0(u)du - 2\nu \sum_{j \in J} d_j \int_s^t X_j^2(u)du \\ - 2 \lim_{n \rightarrow \infty} \int_s^t \sum_{|k|=n} c_k X_k^2(u)X_k(u)du, \end{aligned} \quad (12)$$

where the limit always exists and is non-negative. In particular, for the unforced, inviscid ($f = \nu = 0$) tree dyadic model, \mathcal{E} is non-increasing.

Proof. Let $0 \leq s < t$, then by (9) for all $n \geq 0$,

$$\begin{aligned} \mathcal{E}_n(t) = \mathcal{E}_n(s) - 2\nu \sum_{|j| \leq n} d_j \int_s^t X_j^2(u)du + 2f^2 \int_s^t X_0(u)du \\ - 2 \int_s^t \sum_{|k|=n+1} c_k X_k^2(u)X_k(u)du. \end{aligned}$$

As $n \rightarrow \infty$, since the solution is in l^2 , $\mathcal{E}_n(s) \uparrow \mathcal{E}(s) < \infty$ and the same holds for t . The viscosity term is a non-decreasing sequence bounded by

$$2\nu \sum_{|j| \leq n} d_j \int_s^t X_j^2(u)du \leq \mathcal{E}(s) + 2f^2 \int_s^t X_0(u)du < \infty,$$

so it converges too. Then the border term converges being the sum of converging sequences. □

Definition 3. We say that a positive l^2 solution X is conservative in $[s, t]$ if the limit in (12) is equal to zero that is if

$$\mathcal{E}(t) = \mathcal{E}(s) + 2f^2 \int_s^t X_0(u)du - 2\nu \sum_{j \in J} d_j \int_s^t X_j^2(u)du.$$

Otherwise we say that X has anomalous dissipation in $[s, t]$.

Theorem 3.3. Let $X^0 \in l^2$ with $X_j^0 \geq 0$ for all $j \in J$. Then there exists at least a positive l^2 solution with initial condition X^0 .

Proof. The proof by finite dimensional approximates is completely classic. Fix $N \geq 1$ and consider the finite dimensional system

$$\begin{cases} X_0(t) \equiv f \\ \frac{d}{dt} X_j = -\nu d_j X_j + c_j X_j^2 - \sum_{k \in \mathcal{O}_j} c_k X_j X_k & j \in J, 0 \leq |j| \leq N \\ X_k(t) \equiv 0 & k \in J, |k| = N + 1 \\ X_j(0) = X_j^0 & j \in J, 0 \leq |j| \leq N, \end{cases} \quad (13)$$

for all $t \geq 0$. Notice that Proposition 3.1 is true also for this truncated system (with unchanged proof), so there is a unique global solution. (Local existence and uniqueness follow from the local Lipschitz continuity of the vector field and global existence comes from the bound in (10).) We will denote such unique solution by X^N .

Now fix $j \in J$ and consider on a bounded interval $[0, T]$ the family $(X_j^N)_{N>|j|}$. By (10) we have a strong bound that does not depend on t and N

$$|X_j^N(t)| \leq (\mathcal{E}(0) + 1)^{\frac{1}{2}} e^{\frac{1}{2}TJ^2} \quad \forall N \geq 1 \quad \forall t \in [0, T],$$

thus the family $(X_j^N)_{N>|j|}$ is uniformly bounded, and by applying the same bound to (13), equicontinuous. From Arzelà-Ascoli theorem, for every $j \in J$ there exists a sequence $(N_{j,k})_{k \geq 1}$ such that $(X_j^{N_{j,k}})_k$ converges uniformly to a continuous function X_j . By a diagonal procedure we can modify the extraction procedure and get a single sequence $(N_k)_{k \geq 1}$ such that for all $j \in J$, $X_j^{N_k} \rightarrow X_j$ uniformly. Now we can pass to the limit as $k \rightarrow \infty$ in the equation

$$X_j^{N_k} = X_j^0 + \int_0^t \left[-\nu d_j X_j^{N_k}(r) + c_j (X_j^{N_k}(r))^2 - \sum_{i \in \mathcal{O}_j} c_i X_j^{N_k}(r) X_i^{N_k}(r) \right] dr,$$

and prove that the functions X_j are continuously differentiable and satisfy system (7) with initial condition X_j^0 . Continuation from an arbitrary bounded time interval to all $t \geq 0$ is obvious. Finally, X is a positive l^2 solution by Proposition 3.1. □

We conclude the section on elementary results by collecting a useful estimate on the energy transfer and a statement clarifying that all components are strictly positive for $t > 0$.

Proposition 3.4. The following properties hold:

1. If $f = 0$, for all $n \geq -1$

$$2 \int_0^{+\infty} \sum_{|k|=n+1} c_k X_k^2(s) X_k(s) ds \leq \mathcal{E}_n(0); \tag{14}$$

2. if exists $M \geq 0$ such that $X_j^0 > 0$ for all j with $|j| = M$, then $X_j(t) > 0$ for every j with $|j| \geq M$ and all $t > 0$.

Proof. 1. If $n = -1$ the inequality is trivially true. If $n \geq 0$, by integrating Eq. (9) with $f = 0$, we find that

$$\mathcal{E}_n(t) + 2\nu \int_0^t \sum_{|j| \leq n} d_j X_j^2(s) ds = \mathcal{E}_n(0) - 2 \int_0^t \sum_{|k|=n+1} c_k X_k^2(s) X_k(s) ds.$$

The left hand side is non-negative for all t , so taking the limit for $t \rightarrow \infty$ in the right hand side completes the proof.

2. For $|j| = M$ we have from (11)

$$X_j(t) \geq X_j^0 e^{-\int_0^t (\nu d_j + \sum_k c_k X_k(r)) dr} > 0.$$

Now suppose that for some $j \in J \setminus \{0\}$, $X_j(t) > 0$ for every $t > 0$. Then again by (11),

$$X_j(t) \geq \int_0^t c_j X_j^2(s) e^{-\int_s^t (\nu d_j + \sum_{k \in \mathcal{O}_j} c_k X_k(r)) dr} ds > 0.$$

By induction on $|j| \geq M$ we have our thesis. □

IV. RELATIONSHIP WITH CLASSIC DYADIC MODEL

Recall the differential equations for the tree and classic dyadic models.

$$\begin{cases} X_{\bar{0}}(t) \equiv f \\ \frac{d}{dt} X_j = -\nu d_j X_j + c_j X_j^2 - \sum_{k \in \mathcal{O}_j} c_k X_j X_k, \quad \forall j \in J, \end{cases} \tag{15}$$

$$\begin{cases} Y_{-1}(t) \equiv f \\ \frac{d}{dt} Y_n = -\nu l_n Y_n + k_n Y_{n-1}^2 - k_{n+1} Y_n Y_{n+1}, \quad \forall n \geq 0, \end{cases} \tag{16}$$

where $f \geq 0$, $\nu \geq 0$ and for all $n \in \mathbb{N}$ and $j \in J$,

$$c_j = 2^{\alpha|j|}, \quad k_n = 2^{\beta n}, \quad d_j = 2^{\nu|j|}, \quad l_n = 2^{\gamma n}.$$

Again we assume that $\#\mathcal{O}_j = N_* = 2^{2\tilde{\alpha}}$ for all $j \in J$, but we stress that for this section this is a fundamental hypothesis and not a technical one.

The following proposition shows that examples of solutions of the tree dyadic model (15) can be obtained by lifting the solutions of the classic dyadic model (16).

Proposition 4.1. *If Y is a componentwise (resp. l^2) solution of (16), then $X_j(t) := 2^{-(|j|+2)\tilde{\alpha}} Y_{|j|}(t)$ is a componentwise (resp. l^2) solution of (15) with $\alpha = \beta + \tilde{\alpha}$. If Y is positive, so is X .*

Proof. A direct computation shows that X is a componentwise solution. Then observe that, for any $n \geq 0$,

$$\sum_{|j|=n} X_j^2 = 2^{2\tilde{\alpha}n} X_j^2 = 2^{2\tilde{\alpha}n} 2^{-(2n+4)\tilde{\alpha}} Y_n^2 = 2^{4\tilde{\alpha}} Y_n^2,$$

so

$$\mathcal{E}_n = \sum_{|j| \leq n} X_j^2 = \sum_{k \leq n} 2^{4\tilde{\alpha}} Y_k^2 \leq 2^{4\tilde{\alpha}} \|Y\|^2.$$

Positivity is obvious. □

Remark 1. *If we consider α fixed, since $\beta = \alpha - \tilde{\alpha}$, for small values of N_* we will have larger values of β , and the other way around. That is to say, the less offspring every node has, the faster the dynamics will be.*

Remark 2. *Let us stress that $\beta > 0$ when $N_* < 2^{2\alpha}$. Since the behavior of the solutions of (16) is strongly related to the sign of β , then the behavior of the solutions of (15) is strongly connected to the sign of $\alpha - \tilde{\alpha}$. For example, in the classic dyadic there is anomalous dissipation if and only if $\beta > 0$, and hence in the tree dyadic there will be lifted solutions with anomalous dissipation when $\alpha > \tilde{\alpha}$ and lifted solutions which are conservative when $\alpha \leq \tilde{\alpha}$.*

V. ANOMALOUS DISSIPATION AND SELF-SIMILAR SOLUTIONS IN THE INVISCID AND UNFORCED CASE

Throughout this section we will consider system (7) in its unforced ($f = 0$) and inviscid ($\nu = 0$) version.

$$\begin{cases} X_{\bar{0}}(t) \equiv 0 \\ \frac{d}{dt} X_j = c_j X_j^2 - \sum_{k \in \mathcal{O}_j} c_k X_j X_k, \quad \forall j \in J. \end{cases} \tag{17}$$

Equation (9), that is the derivative of energy up to the n th generation becomes

$$\frac{d}{dt} \mathcal{E}_n(t) = -2 \sum_{|k|=n+1} c_k X_k^2(t) X_k(t), \quad n \geq 0.$$

Since only the border term survives, one would expect it to vanish in the limit $n \rightarrow \infty$. This can be rigorously proven only if the solution lives in a sufficiently regular space, that is to say that X_j^2 goes fast to zero as $|j| \rightarrow \infty$. For the classic dyadic Kiselev and Zlatoš¹¹ proved that solutions that are regular in the beginning, stay regular for some time but then lose regularity in finite time. Thus our analysis is not restricted to regular solutions, and in fact we will prove in this section that for sufficiently large times all solutions dissipate energy.

Let us give some definitions. Let us denote by γ_j the energy at time 0 in the subtree T_j rooted in j plus all the energy flowing in j from the upper generations,

$$\gamma_j := \sum_{k \in T_j} X_k^2(0) + \int_0^\infty 2c_j X_j X_j^2 ds.$$

Let $0 \leq s < t$ and define for all $j \in J$

$$m_j := \inf_{r \in [s,t]} X_j(r).$$

Lemma 5.1. *Let X be a positive l^2 solution of system (17). The following inequalities hold for all $n \geq 0$.*

$$\begin{aligned} \mathcal{E}_n(t) - \mathcal{E}_{n-1}(s) &\leq \sum_{|j|=n} m_j^2 \leq \mathcal{E}(0), \\ \sum_{|j|=n} \gamma_j &\leq \mathcal{E}(0), \\ \sum_{k \in T_j} X_k(r)^2 &\leq \gamma_j, \quad \forall r \geq 0. \end{aligned}$$

Proof. The upper bound is obvious, since

$$\sum_{|j|=n} m_j^2 \leq \sum_{|j|=n} X_j(s)^2 \leq \mathcal{E}_n(s) \leq \mathcal{E}(0),$$

where we used Proposition 3.2. Now let $j \in J$. From (17) we have for the differential of X_j^2

$$\frac{d}{dt} X_j^2 = 2c_j X_j^2 X_j - \sum_{k \in \mathcal{O}_j} 2c_k X_j^2 X_k,$$

Let $r \in [s, t]$ and integrate on $[s, r]$, yielding

$$X_j^2(r) = X_j^2(s) + \int_s^r 2c_j X_j^2(\tau) X_j(\tau) d\tau - \sum_{k \in \mathcal{O}_j} \int_s^r 2c_k X_j^2(\tau) X_k(\tau) d\tau.$$

Choosing now $r \in \operatorname{argmin}_{[s,t]} X_j$, we get

$$m_j^2 \geq X_j^2(s) - \sum_{k \in \mathcal{O}_j} \int_s^t 2c_k X_k^2(\tau) X_k(\tau) d\tau.$$

By summation over all nodes j with $|j| = n$ we have

$$\sum_{|j|=n} m_j^2 \geq \sum_{|j|=n} X_j^2(s) - \int_s^t \sum_{|k|=n+1} 2c_k X_k^2(\tau) X_k(\tau) d\tau.$$

Finally, we apply for $m = n - 1, n$ the following integral form of (9) to get the first part of the thesis. (Even if $n = 0$ and $m = -1$ this is true, trivially.)

$$\mathcal{E}_m(t) - \mathcal{E}_m(s) = - \int_s^t \sum_{|j|=m+1} 2c_j X_j^2(\tau) X_j(\tau) d\tau.$$

We turn to the second part. Sum γ_j on every j with $|j| = n$ to get

$$\sum_{|j|=n} \gamma_j = \sum_{|k| \geq n} X_k^2(0) + \int_0^\infty 2 \sum_{|j|=n} c_j X_j^2 X_j ds,$$

by (14) the integral term is bounded above by $\mathcal{E}_{n-1}(0)$, so

$$\sum_{|j|=n} \gamma_j \leq \sum_{|k|\geq n} X_k^2(0) + \mathcal{E}_{n-1}(0) = \sum_{k \in J} X_k^2(0) = \mathcal{E}(0).$$

Finally, the third part. Let $r \geq 0$. By computing the time derivative of $\sum_{k \in T_j, |k|\leq n} X_k^2$ which is analogous to (9), dropping the border term and integrating on $[0, r]$, we have,

$$\sum_{\substack{k \in T_j \\ |k|\leq n}} X_k(r)^2 \leq \sum_{\substack{k \in T_j \\ |k|\leq n}} X_k(0)^2 + 2 \int_0^r 2c_j X_j X_j^2 du \leq \gamma_j.$$

Now, let $n \rightarrow \infty$ to conclude. □

The following statement will be used in the proof of Lemma 5.3.

Lemma 5.2. For every $h > 0$ and $\lambda > 0$ the following inequality holds:

$$\int_0^h \int_0^s e^{-\lambda(s-r)} dr ds \geq \frac{h}{2\lambda} \left(1 - e^{-\lambda \frac{h}{2}}\right).$$

Proof.

$$\int_0^h \int_0^s e^{-\lambda(s-r)} dr ds \geq \int_{\frac{h}{2}}^h \int_{s-\frac{h}{2}}^s e^{-\lambda(s-r)} dr ds = \frac{h}{2\lambda} \left(1 - e^{-\lambda \frac{h}{2}}\right). \quad \square$$

Lemma 5.3. Assume that $\alpha > \tilde{\alpha}$, where $2^{2\tilde{\alpha}} = N_ = \#\mathcal{O}_j$ is the constant number of children for every node. Let X be a positive l^2 solution of (17). Let $(\delta_n)_{n \geq 0}$ be a sequence of positive numbers such that $\sum_n \delta_n$ and $\sum_n \delta_n^{-2} 2^{-(\alpha-\tilde{\alpha})n}$ are both finite. Then there exists a sequence of positive numbers $(h_n)_{n \geq 0}$ such that $\sum_n h_n < \infty$ and for all $n \geq 0$ for all $t > 0$*

$$\mathcal{E}_n(t + h_n) - \mathcal{E}_{n-1}(t) \leq \delta_n. \tag{18}$$

In particular, for every $M \geq 0$,

$$\mathcal{E} \left(\sum_{n=M}^{\infty} h_n \right) \leq \mathcal{E}_{M-1}(0) + \sum_{n=M}^{\infty} \delta_n. \tag{19}$$

The sequence

$$h_n = \frac{\mathcal{E}(0)^{3/2}}{\delta_n^2} 2^{-(\alpha-\tilde{\alpha})n+3/2}, \tag{20}$$

satisfies (18) and (19).

Proof. Fix $n \geq 0$ and positive real numbers t, h_n . For all j of generation n , let $m_j := \inf_{r \in [t, t+h_n]} X_j(r)$. We claim that if h_n is defined by (20), then $\sum_{|j|=n} m_j^2 \leq \delta_n$, which together with Lemma 5.1 completes the proof of (18).

We prove the claim by contradiction: suppose that $\sum_{|j|=n} m_j^2 > \delta_n$. We will find a contradiction in the estimates on $\mathcal{E}(0)$. By Proposition 3.4

$$\mathcal{E}(0) \geq 2 \int_0^{h_n} \sum_{|j|=n} \sum_{k \in \mathcal{O}_j} c_k X_k(t+s) X_j^2(t+s) ds.$$

We have a lower bound for X_j , namely m_j , but we need one also for X_k .

For all $j \in J$, let $\Gamma_j := \max(\gamma_j, \mathcal{E}(0)N_*^{-|j|})$. From Lemma 5.1 we have $\sum_{|j|=n} \gamma_j \leq \mathcal{E}(0)$ and hence $\sum_{|j|=n} \Gamma_j \leq 2\mathcal{E}(0)$; by the same lemma, for all $i \in T_j$ we have $X_i^2 \leq \gamma_j \leq \Gamma_j$ uniformly in time, so for all $k \in \mathcal{O}_j$,

$$\dot{X}_k = c_k X_j^2 - \sum_{i \in \mathcal{O}_k} c_i X_i X_k \geq c_k m_j^2 - \lambda_j X_k,$$

where $\lambda_j = N_* 2^{n\alpha+2\alpha} \sqrt{\Gamma_j}$. This gives

$$X_k(t+s) \geq c_k m_j^2 \int_0^s e^{-\lambda_j(s-r)} dr.$$

We can write

$$\mathcal{E}(0) \geq 2 \sum_{|j|=n} m_j^4 \int_0^{h_n} \int_0^s e^{-\lambda_j(s-r)} dr ds \sum_{k \in \mathcal{O}_j} c_k^2,$$

and by Lemma 5.2 we have

$$\mathcal{E}(0) \geq 2 \sum_{|j|=n} m_j^4 \frac{h_n}{2\lambda_j} (1 - e^{-\lambda_j h_n/2}) \sum_{k \in \mathcal{O}_j} c_k^2.$$

Let us focus on the exponential. We substitute (20) and make use of the inequality $\Gamma_j \geq \mathcal{E}(0) N_*^{-n} = \mathcal{E}(0) 2^{-2\tilde{\alpha}n}$,

$$\frac{\lambda_j h_n}{2} = N_* 2^{n\alpha+2\alpha} \sqrt{\Gamma_j} \frac{\sqrt{2\mathcal{E}(0)^{3/2}}}{2^{(\alpha-\tilde{\alpha})n} \delta_n^2} \geq \frac{\mathcal{E}(0)^2}{\delta_n^2} \sqrt{2}.$$

By the hypothesis that $\sum_{|j|=n} m_j^2 > \delta_n$ and Lemma 5.1, we know that $\delta_n < \mathcal{E}(0)$ we get $1 - e^{-\lambda_j h_n/2} > 1/2$. We obtain

$$\mathcal{E}(0) > \sum_{|j|=n} m_j^4 \frac{h_n}{2\lambda_j} \sum_{k \in \mathcal{O}_j} c_k^2 = \frac{\sqrt{2}\mathcal{E}(0)^{3/2}}{2^{-\tilde{\alpha}n} \delta_n^2} \sum_{|j|=n} \frac{m_j^4}{\sqrt{\Gamma_j}}. \tag{21}$$

Now we can use Cauchy-Schwarz and the AM-QM inequalities to get

$$\sum_{|j|=n} \frac{m_j^4}{\sqrt{\Gamma_j}} \geq \frac{(\sum_{|j|=n} m_j^2)^2}{\sum_{|j|=n} \sqrt{\Gamma_j}} \geq \frac{(\sum_{|j|=n} m_j^2)^2}{\sqrt{N_*^n \sum_{|j|=n} \Gamma_j}},$$

again by the hypothesis that $\sum_{|j|=n} m_j^2 > \delta_n$ and thanks to $\sum_{|j|=n} \Gamma_j \leq 2\mathcal{E}(0)$,

$$\sum_{|j|=n} \frac{m_j^4}{\sqrt{\Gamma_j}} > \frac{\delta_n^2}{\sqrt{2\mathcal{E}(0) 2^{\tilde{\alpha}n}}},$$

so that the right-hand side of (21) becomes larger than $\mathcal{E}(0)$, which is impossible.

We turn to the second part. Let $M \geq 0$ and define the following sequence $(t_n)_{n \geq M-1}$ by $t_{M-1} = 0$ and $t_n = t_{n-1} + h_n$. By (18) with $t = t_{n-1}$ we get

$$\mathcal{E}_n(t_n) - \mathcal{E}_{n-1}(t_{n-1}) \leq \delta_n.$$

We sum for n from M to N , yielding

$$\mathcal{E}_N(t_N) - \mathcal{E}_{M-1}(0) \leq \sum_{n=M}^N \delta_n,$$

which, due to monotonicity of \mathcal{E}_N , yields

$$\mathcal{E}_N \left(\sum_{n=M}^{\infty} h_n \right) \leq \mathcal{E}_N(t_N) \leq \mathcal{E}_{M-1}(0) + \sum_{n=M}^N \delta_n.$$

Now we let N go to infinity to get the thesis. □

Remark 3. It is easy to prove this result also if relaxing the condition on the number of children from constant number to $1 \leq \#\mathcal{O}_j \leq N_*$. One has to change slightly the definition of h_n , which

becomes

$$h_n = \frac{\mathcal{E}(0)^{3/2}}{\delta_n^2} 2^{-(\alpha-\bar{\alpha})n+2\bar{\alpha}+3/2}.$$

Theorem 5.4. Assume that $\alpha > \bar{\alpha}$, where $2^{2\bar{\alpha}} = N_* = \sharp\mathcal{O}_j$ is the constant number of children for every node. Then for every $\varepsilon > 0$ and $\eta > 0$ there exists some $T > 0$ such that for all positive l^2 solution of (17) with initial energy $\mathcal{E}(0) \leq \eta$ one has $\mathcal{E}(T) \leq \varepsilon$. In particular

$$\lim_{t \rightarrow \infty} \mathcal{E}(t) = 0,$$

i.e., there is anomalous dissipation.

Proof. Given $\varepsilon > 0$ let us take a sequence of positive numbers $(\delta_n)_{n \geq 0}$ such that

$$\sum_{n=0}^{\infty} \delta_n = \varepsilon \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{1}{2^{(\alpha-\bar{\alpha})n} \delta_n^2} < +\infty.$$

This is possible, for example, taking $\delta_n = \varepsilon(1 - 2^{-(\alpha-\bar{\alpha})/3})2^{-(\alpha-\bar{\alpha})n/3}$. Now Lemma 5.3 applies, so by the definition of h_n given in (20)

$$h_n \leq \frac{2\sqrt{2}\eta^{3/2}}{2^{(\alpha-\bar{\alpha})n} \delta_n^2} \quad \text{and} \quad \sum_{n=0}^{\infty} h_n \leq \frac{2\sqrt{2}\eta^{3/2}}{(1 - 2^{-(\alpha-\bar{\alpha})/3})^3} =: T.$$

Take $M = 0$ in (19) and by monotonicity of energy $\mathcal{E}(T) \leq \varepsilon$. \square

We are finally able to prove Theorem 2.1, which is a consequence of Theorem 5.4 with a rescaling argument based on the fact that the nonlinearity is homogeneous of degree two.

Proof of Theorem 2.1. By Theorem 5.4 for every $0 < \rho < 1$ there exists $\tau > 0$ depending only on ρ and $\mathcal{E}(0)$, such that $\mathcal{E}(\tau) \leq \rho^2 \mathcal{E}(0)$. We will apply this bound to many different solutions, all of which have energy at time zero not above $\mathcal{E}(0)$.

Let $\vartheta = 1/\rho > 1$. We can define the sequence

$$X^{(0)} = X$$

$$X^{(n)}(t) = \vartheta X^{(n-1)}(\vartheta t + \tau) = \vartheta^n X\left(\vartheta^n t + \frac{\vartheta^n - 1}{\vartheta - 1} \tau\right), \quad n \geq 1.$$

It is immediate to verify that all of these satisfy the system of Eq. (17), but with possibly different initial conditions. We have

$$\sum_{j \in J} (X_j^{(n)}(0))^2 = \vartheta^2 \sum_{j \in J} (X_j^{(n-1)}(\tau))^2.$$

Recalling the definition of τ , the above equation allows to prove by induction on n that for all $n \geq 0$ one has $\sum_{j \in J} (X_j^{(n)}(0))^2 \leq \mathcal{E}(0)$. For all $n \geq 0$, let

$$t_n = \frac{\vartheta^n - 1}{\vartheta - 1} \tau.$$

Then by the definition of $X^{(n)}$, we have proved $\mathcal{E}(t_n) \leq \vartheta^{-2n} \mathcal{E}(0)$. Since $\vartheta > 1$, $t_n \uparrow \infty$, hence given $t > 0$ there is n such that $t_n \leq t < t_{n+1}$. That means we have by monotonicity

$$\mathcal{E}(t) \leq \vartheta^{-2n} \mathcal{E}(0) \quad \text{and} \quad \frac{1}{t_{n+1}^2} < \frac{1}{t^2}.$$

Finally, by definition $t_{n+1} < \tau \vartheta^{n+1} / (\vartheta - 1) = \vartheta^n \tau / (1 - \rho)$, so for $C = \mathcal{E}(0)(\tau / (1 - \rho))^2$ we get

$$\mathcal{E}(t) \leq \vartheta^{-2n} \mathcal{E}(0) < \frac{C}{t_{n+1}^2} < \frac{C}{t^2}. \quad \square$$

A. Self-similar solutions

We devote the end of this section to prove the existence of self-similar solutions. We call *self-similar* any solution X of system (17) of the form $X_j(t) = a_j \varphi(t)$, for all j and all $t \geq 0$. By substituting this formula inside (17) it is easy to show that any such solution must be of the form

$$X_j(t) = \frac{a_j}{t - t_0},$$

for some $t_0 < 0$. The condition on the coefficients a_j is much more complicated

$$\begin{cases} a_0 = 0 \\ a_j + c_j a_j^2 = \sum_{k \in \mathcal{O}_j} c_k a_j a_k, \quad \forall j \in J, \end{cases}$$

so it is convenient to take advantage of previous results for the classic dyadic, for which existence and some kind of uniqueness of self-similar solution are known.² We obtain the following statement.

Proposition 5.5. *Given $t_0 < 0$ there exists at least one self-similar positive l^2 solution of (17) with $a_0 > 0$.*

Proof. We use Theorem 10 in Barbato, Flandoli and Morandin² which, translated in the notation of this paper, states that there exists a unique sequence of non-negative real numbers $(b_n)_{n \geq 0}$ such that $b_0 > 0$ and $Y_n := b_n/(t - t_0)$ is a positive l^2 solution of the unforced inviscid classic dyadic (8). Thanks to Proposition 4.1 this solution may be lifted to a solution of the inviscid tree dyadic (7) with the required features. \square

Remark 4. *For the tree dyadic model self-similar solutions are many. In the standard dyadic case² it is shown that given $t_0 < 0$ and $n_0 \geq 1$ there is only one l^2 self-similar solution such that n_0 is the index of the first non-zero coefficient. If $n_0 > 1$, this solution can be lifted on the tree to a self-similar solution which is zero on the first $n_0 - 1$ generations. We can then define a new self-similar solution which is equal to this one on one of the subtrees starting at generation n_0 and zero everywhere else. Finally, we can combine many of these solutions, even with different n_0 , as long as t_0 is the same for all and their subtrees do not overlap.*

VI. STATIONARY SOLUTIONS

In this section we will study the stationary solutions for both the classic dyadic model (16) and the tree dyadic one (15). We will in particular restrict ourselves to study positive l^2 solutions which are time independent. Proposition 4.1 allows us to link the two models, in that for any solution of the classic dyadic model one can build a solution of the tree dyadic model. Thus it is enough to prove existence for the classic dyadic and uniqueness for the tree dyadic.

One purpose of this section is to prove the existence and uniqueness of the stationary solution on the tree dyadic model and extend existence and uniqueness results for the dyadic model.^{4,5} Cheskidov, Friedlander and Pavlović proved⁵ that the inviscid dyadic model with $\beta = 5/2$ has a unique stationary solution and that such a solution is a global attractor.⁶ One should notice that these results on the dyadic model are stronger than what can be proven for more realistic shell models such as Sabra (see for example Constantin, Levant and Titi⁷).

For the viscous dyadic model, when $\beta \in (3/2, 5/2]$ the stationary solution is unique and is a global attractor.⁴ Nevertheless it is possible to explicitly provide examples of non-uniqueness of the stationary solution, dropping the $Y_n \geq 0$ condition.³ In this paper we prove the existence and uniqueness of stationary solutions in l^2 for every positive value of the β and γ parameters both in viscous and inviscid dyadic models. This will provide a corresponding result of existence and uniqueness for $\alpha > \tilde{\alpha}$ and $\gamma > 0$ in the tree dyadic model. Furthermore in the inviscid case we will explicitly provide those solutions (Proposition 6.1), while in the viscous case we will prove that the stationary solutions are regular if and only if N_* is big enough, $N_* \geq 2^{2\alpha - 3\gamma}$ or the forcing term f

is small. For $f = 0$ the unique (non-negative) stationary solution is trivially the null one, so in this section we assume $f > 0$.

A. Stationary solutions in the inviscid case: Existence

In the inviscid case, the differential equation is very simple, so it is easy to find stationary solutions in the class of exponential functions. One immediately finds the following result.

Proposition 6.1. Consider the tree dyadic model (15) and the classic dyadic model (16), both inviscid ($\nu = 0$). Let $2^{2\tilde{\alpha}} = N_* = \#\mathcal{O}_j$ be constant for all $j \in J$. Then:

1. the sequence of constant functions $Y_n(t) := f2^{-(n+1)\beta/3}$ is a positive l^2 solution of the system (16).
2. the family of constant functions $X_j(t) := f2^{-(2\tilde{\alpha}+\alpha)(|j|+1)/3}$ for $j \in J$ is a positive componentwise solution of system (15); it is also an l^2 solution iff $\alpha > \tilde{\alpha}$;

Proof. A direct computation shows that X and Y are componentwise solutions. To show that Y is l^2 observe that, since $\beta > 0$, $\|Y\| < \infty$. To check whether X is l^2 compute the energy by generations; we have for $n \geq 0$,

$$\mathcal{E}_n - \mathcal{E}_{n-1} = \sum_{|j|=n} X_j^2 = 2^{2\tilde{\alpha}n} f^2 2^{-\frac{n+1}{3}(4\tilde{\alpha}+2\alpha)} = C 2^{\frac{2}{3}(\tilde{\alpha}-\alpha)n},$$

with C not depending on n . Hence X is l^2 if and only if $\alpha - \tilde{\alpha} > 0$. □

B. Stationary solutions in the viscous case: Existence

In the viscous case, the recurrence relation coming from the definition of stationary solution is more complex, and has no solutions in the class of exponential functions. Anyway, by careful control of the recurrence behavior, we are able to prove that a stationary solution exists, and also to distinguish if it is conservative or has anomalous dissipation.

Definition 4. We say that a stationary positive l^2 solution X is regular if for all $h \in \mathbb{R}$

$$\sum_{j \in J} [2^{h|j|} X_j]^2 < \infty. \tag{22}$$

Theorem 6.2. There exists a stationary positive l^2 solution of the classic dyadic model (16) when $\nu > 0$.

Theorem 6.3. Consider any stationary positive l^2 solution of the classic dyadic model (16) with $\nu > 0$.

1. If $3\gamma \geq 2\beta$ then it is regular and conservative.
2. If $3\gamma < 2\beta$ then there exists some $C > 0$ such that if $f > C$ the stationary solution is not regular and there is anomalous dissipation.

Before we go into the proofs of these theorems, let us introduce a useful change of variables, that will come handy in both proofs. If Y is a stationary solution of (16) then, for every $n \geq 0$, we have

$$-\nu 2^{\gamma n} Y_n + 2^{\beta n} Y_{n-1}^2 - 2^{\beta n + \beta} Y_n Y_{n+1} = 0.$$

This equation can be made into a recurrence, and the change of variables that best simplifies its form is

$$Z_n := \nu^{-1} 2^{\frac{\beta}{3}(n+2)} Y_n. \tag{23}$$

Since the stationary solution in the inviscid case decreases like $2^{-n\beta/3}$, the exponent's rate $n\beta/3$ is in some sense expected. The system of differential equations for Z becomes

$$\begin{cases} Z_{-1} = v^{-1}2^{\frac{\beta}{3}}f =: g \\ Z_{n+1} = \frac{Z_{n-1}^2}{Z_n} - 2^{(\gamma-\frac{2}{3}\beta)n} \end{cases} \quad \forall n \geq 0. \tag{24}$$

Proof of Theorem 6.2. Let us consider the change of variable (23), we have to show that the system (24) has a positive solution for which Y is l^2 . System (24) gives a recursion which, given $Z_{-1} = g$ and Z_0 allows to construct the sequence $(Z_n)_{n \geq -1}$ in a unique way. Any such sequence will give a stationary componentwise solution. What we want to prove is that there is some value of Z_0 such that this turn out to be a positive l^2 solution. Let we exploit the dependence from Z_0 by defining a sequence of real functions

$$\begin{aligned} Z_{-1}(a) &= g, \\ Z_0(a) &= a, \\ Z_{n+1}(a) &= \frac{Z_{n-1}^2(a)}{Z_n(a)} - 2^{(\gamma-\frac{2}{3}\beta)n}, \quad n \geq 0. \end{aligned} \tag{25}$$

Now we construct a descending sequence of open real intervals $(I_n)_{n \geq 0}$ such that $(0, \infty) = I_0 \supset I_1 \supset I_2 \supset \dots$ and such that Z_n is continuous and bijective from I_n to $(0, \infty)$, with Z_n strictly increasing for even n and strictly decreasing for odd n .

Let $I_0 = (0, \infty)$. $Z_0(a)$ is monotone increasing, continuous and bijective from I_0 to $(0, \infty)$.

By (25) we have that $Z_1(a) = g/a^2 - 2^{(\gamma-2\beta/3)}$ is monotone decreasing, continuous and bijective from I_0 to $(-2^{(\gamma-2\beta/3)}, \infty)$ so there exists a limited interval $(b_1, c_1) := I_1 \subset I_0$ such that $Z_1(a)$ is monotone decreasing, continuous and bijective from I_1 to $(0, \infty)$.

Now suppose we already proved for $m \leq n$ that $Z_m(a)$ is continuous and bijective from I_m to $(0, \infty)$, with Z_m strictly increasing for even m and strictly decreasing for odd m .

Suppose that n is odd (resp. even). Then by (25) $Z_{n+1}(a)$ is monotone increasing (resp. decreasing), continuous and bijective from I_n to $(-2^{(\gamma-2\beta/3)n}, \infty)$ so there exists an interval $(b_{n+1}, c_{n+1}) := I_{n+1} \subset I_n$ such that $Z_{n+1}(a)$ is monotone increasing (resp. decreasing), continuous and bijective from I_{n+1} to $(0, \infty)$.

Observe moreover that the borders of these intervals are not definitively constant, since for all n , $b_{n+2} \neq b_n$ and $c_{n+2} \neq c_n$. Hence if we define $b = \lim_n b_n$ and $c = \lim_n c_n$, it is clear that for all n , $b_n < b \leq c < c_n$, that is the closed interval (possibly degenerate) $[b, c]$ is contained in every I_n .

Now we choose any $\bar{a} \in [b, c]$ and we know that the sequence $Z_n(\bar{a})$ is strictly positive. We are left to prove that it is also l^2 . To this end let Y_n be any stationary, positive componentwise solution. Let $\mathcal{E}_n = \sum_{k=0}^n Y_k^2$ in analogy with the definition for the tree model. We compute the derivative

$$0 = \frac{d}{dt} \mathcal{E}_n(t) = -v \sum_{k \leq n} l_k Y_k^2 + f^2 Y_0 - k_{n+1} Y_n^2 Y_{n+1},$$

hence, since $l_k \geq 1$, $\mathcal{E}_n \leq \sum_{k \leq n} l_k Y_k^2 \leq v^{-1} f^2 Y_0$ for all n . □

Proof of Theorem 6.3. Let us consider again system (24) and let $\mu := \gamma - 2\beta/3$. If $\mu > 0$ the corrective term goes to infinity, while if $\mu < 0$ it goes to zero, so we expect two different behaviors in the two cases. We will show that in the first case Z_n goes to zero super-exponentially for $n \rightarrow \infty$, while in the second one $Z_n \downarrow z$ and $z > 0$ if g is large enough.

Case $\mu := \gamma - 2\beta/3 \geq 0$. From (24) we get

$$2^{\mu n} Z_n^2 = Z_{n-1}^2 Z_n - Z_n^2 Z_{n+1}.$$

Sum over n to get

$$\sum_{k \leq n} 2^{\mu k} Z_k^2 = g^2 Z_0 - Z_n^2 Z_{n+1}. \tag{26}$$

Since $\mu \geq 0$, by positivity of Z , we have

$$\lim_{n \rightarrow \infty} Z_n = 0. \tag{27}$$

From (24) and $Z_{n+1} > 0$ we get $Z_n < Z_{n-1}^2$ and since by (27) $Z_{\bar{n}} =: \lambda < 1$ for some \bar{n} , by iterating the above equation we get for all $m \geq 0$

$$Z_{\bar{n}+m} \leq \lambda^{2^m},$$

that is to say that Z_n goes to zero for n going to infinity like the exponential of an exponential, so for every $s > 0$ we have

$$\sum_n (2^{sn} Z_n)^2 < +\infty \quad \text{and} \quad \sum_n (2^{sn} Y_n)^2 < +\infty.$$

It is now clear that $\lim_n k_{n+1} Y_n^2 Y_{n+1} = 0$, so Y is conservative by Definition 3.

Case $\mu := \gamma - 2\beta/3 < 0$. The first step is to prove that Z_n is non-increasing in n . Suppose by contradiction that for some n we have $Z_n/Z_{n-1} = \lambda > 1$, then we claim that $Z_{n+2}/Z_{n+1} > \lambda^4 > 1$ and hence by induction $Z_{n+2m}/Z_{n+2m-1} > \lambda^{4^m}$. By (24) for all $k \geq 0$

$$Z_{k+1} < \frac{Z_{k-1}^2}{Z_k} = \frac{Z_{k-1}}{Z_k} Z_{k-1}.$$

This can be used iteratively together with the claim to show that

$$\begin{aligned} Z_{n+2m+1} &= \frac{Z_{n+2m-1}^2}{Z_{n+2m}} - 2^{\mu(n+2m)} \\ &< \frac{Z_{n+2m-1}}{Z_{n+2m}} \frac{Z_{n+2m-3}}{Z_{n+2m-2}} \dots \frac{Z_{n-1}}{Z_n} Z_{n-1} - 2^{\mu(n+2m)} \\ &< Z_{n-1} \lambda^{-4^m} - 2^{\mu(n+2m)}, \end{aligned}$$

so we get a contradiction because $Z_{n+2m+1} < 0$ for some m .

We prove the claim. Let $x = 2^{\mu n} Z_n / Z_{n-1}^2 = 2^{\mu n} \lambda^2 / Z_n$. Observe that

$$Z_{n+1} = \frac{Z_{n-1}^2}{Z_n} - 2^{\mu n} = \frac{2^{\mu n}}{x} (1 - x). \tag{28}$$

We divide by Z_n (and we notice that $x < 1$),

$$\frac{Z_{n+1}}{Z_n} = \lambda^{-2} (1 - x). \tag{29}$$

Now

$$Z_{n+2} = \frac{Z_n^2}{Z_{n+1}} - 2^{\mu(n+1)} > \frac{Z_n^2}{Z_{n+1}} - 2^{\mu n},$$

so dividing by Z_{n+1} and substituting (29) and (28), we get

$$\frac{Z_{n+2}}{Z_{n+1}} > \lambda^4 (1 - x)^{-2} - \frac{2^{\mu n}}{Z_{n+1}} > \frac{\lambda^4}{1 - x} - \frac{x}{1 - x}.$$

Since $\lambda > 1 > x > 0$, it is now clear that $(\lambda^4 - x)/(1 - x) > \lambda^4$. So we have proven the claim and showed that $\{Z_n\}_{n \geq 0}$ is non-increasing in n .

The last step is to show that for g large enough $Z_n \downarrow z > 0$. By rearranging (26) and recalling what we proved above,

$$Z_n^3 \geq Z_n^2 Z_{n+1} = g^2 Z_0 - \sum_{k=0}^n 2^{\mu k} Z_k^2 \geq g^2 Z_0 - g Z_0 \sum_{k=0}^n 2^{\mu k} > g Z_0 \left(g - \frac{1}{1 - 2^\mu} \right),$$

so if $g > 1/(1 - 2^\mu)$ then Z_n converges to a strictly positive constant z .

To prove anomalous dissipation we compute the limit

$$\lim_{n \rightarrow \infty} k_{n+1} Y_n^2 Y_{n+1} = \lim_{n \rightarrow \infty} 2^{\beta n + \beta} \nu^3 2^{-\beta n - 7\beta/3} Z_n^2 Z_{n+1} = 2^{-4\beta/3} \nu^3 z^3 > 0.$$

So by Definition 3 there is anomalous dissipation. □

C. Stationary solutions in the inviscid and viscous case: Uniqueness

We prove uniqueness in the class of stationary positive l^2 solutions for the tree dyadic model. The result also holds for the classic dyadic, because it is a particular case of the former, or by virtue of the lifting Proposition 4.1.

Theorem 6.4. *Consider the tree dyadic model (7) and assume that $\alpha > \tilde{\alpha}$, where $2^{2\tilde{\alpha}} = N_* = \#\mathcal{O}_j$ is the constant number of children for every node. Then there exists a unique stationary positive l^2 solution.*

Proof. Existence is a consequence of Proposition 6.1 in the inviscid case ($\nu = 0$) and Proposition 4.1 and Theorem 6.2 in the viscous case.

To prove uniqueness we apply a change of variables similar to (23)

$$Z_j := 2^{\frac{(2+j)\alpha}{3}} X_j, \quad \forall j \in J. \tag{30}$$

Then from (7) we have

$$\frac{d}{dt} Z_j = -\nu 2^{\gamma|j|} Z_j + 2^{\frac{2}{3}\alpha|j|} Z_j^2 - \sum_{k \in \mathcal{O}_j} 2^{\frac{2}{3}\alpha|j|} Z_j Z_k, \tag{31}$$

so if X is a stationary solution, Z must satisfy

$$\left\{ \begin{array}{l} Z_0 = f 2^{\alpha/3} \\ \sum_{k \in \mathcal{O}_j} Z_k = \frac{Z_j^2}{Z_j} - \nu 2^{(\gamma - \frac{2}{3}\alpha)|j|}. \end{array} \right. \tag{32}$$

Moreover observe that the condition $X \in l^2$ is equivalent to

$$\sum_{j \in J} (2^{-\frac{\alpha}{3}|j|} Z_j)^2 < \infty. \tag{33}$$

Assume by contradiction that there are two different stationary solutions of (32) which we denote by $W = \{W_j\}_{j \in J}$ and $Z = \{Z_j\}_{j \in J}$. Let n be the smallest integer such that there exist $j_1 \in J$ with $|j_1| = n$ and $W_{j_1} \neq Z_{j_1}$. Without loss of generality we can take $W_{j_1}/Z_{j_1} =: \lambda > 1$.

Let $j_0 = k_0 = \bar{j}_1$ and $k_1 = j_1$. Extend these to two sequences of indices $(j_m)_{m \geq 0}$ and $(k_m)_{m \geq 0}$ with $j_m \in \mathcal{O}_{j_{m-1}}$ and $k_m \in \mathcal{O}_{k_{m-1}}$, picking alternatively among those that maximize or minimize W_{j_m} and Z_{k_m} .

More precisely for $m \geq 2$ choose $j_m \in \mathcal{O}_{j_{m-1}}$ and $k_m \in \mathcal{O}_{k_{m-1}}$ in such a way that if m is even

$$W_{j_m} = \min\{W_i : i \in \mathcal{O}_{j_{m-1}}\} \quad Z_{k_m} = \max\{Z_i : i \in \mathcal{O}_{k_{m-1}}\},$$

and if m is odd

$$W_{j_m} = \max\{W_i : i \in \mathcal{O}_{j_{m-1}}\} \quad Z_{k_m} = \min\{Z_i : i \in \mathcal{O}_{k_{m-1}}\}.$$

The idea supporting the definition of these sequences is to choose the indices so that

$$W_{j_1} < Z_{k_1}, \quad W_{j_2} > Z_{k_2}, \quad W_{j_3} < Z_{k_3}, \quad \dots$$

We will now prove that, with our construction, those inequalities hold and, moreover, the ratio between W_m and Z_m grows according to

$$\frac{Z_{k_m}}{W_{j_m}} \geq \frac{W_{j_{m-1}}}{Z_{k_{m-1}}} \frac{Z_{k_{m-2}}^2}{W_{j_{m-2}}^2} > \lambda^{2^{m-2}} \quad \forall m \geq 2 \text{ even} \tag{34}$$

$$\frac{W_{j_m}}{Z_{k_m}} \geq \frac{Z_{k_{m-1}}}{W_{j_{m-1}}} \frac{W_{j_{m-2}}^2}{Z_{k_{m-2}}^2} > \lambda^{2^{m-2}} \quad \forall m \geq 3 \text{ odd.} \tag{35}$$

We prove inequalities (34) and (35) by induction on $m \geq 2$. First note that for $m = 0$ and $m = 1$,

$$\frac{Z_{k_0}}{W_{j_0}} = 1 \quad \text{and} \quad \frac{W_{j_1}}{Z_{k_1}} = \lambda. \tag{36}$$

Now we proceed by induction. Let $m \geq 2$ even. By the definition of j_m, k_m and by (32) we get

$$\begin{aligned} W_{j_m} &= \min_{i \in \mathcal{O}_{j_{m-1}}} W_i \leq N_*^{-1} \sum_{i \in \mathcal{O}_{j_{m-1}}} W_i = N_*^{-1} \left[\frac{W_{j_{m-2}}^2}{W_{j_{m-1}}} - \nu 2^{(\gamma - \frac{2}{3}\alpha)(n+m-2)} \right], \\ Z_{k_m} &= \max_{i \in \mathcal{O}_{k_{m-1}}} Z_i \geq N_*^{-1} \sum_{i \in \mathcal{O}_{k_{m-1}}} Z_i = N_*^{-1} \left[\frac{Z_{k_{m-2}}^2}{Z_{k_{m-1}}} - \nu 2^{(\gamma - \frac{2}{3}\alpha)(n+m-2)} \right]. \end{aligned} \tag{37}$$

By (36) when $m = 2$ or by inductive hypothesis (34) and (35) when $m \geq 4$,

$$\frac{Z_{k_{m-2}}^2}{Z_{k_{m-1}}} \bigg/ \frac{W_{j_{m-2}}^2}{W_{j_{m-1}}} = \frac{Z_{k_{m-2}}^2}{W_{j_{m-2}}^2} \frac{W_{j_{m-1}}}{Z_{k_{m-1}}} > \begin{cases} \lambda & m = 2 \\ (\lambda^{2^{m-4}})^2 \lambda^{2^{m-3}} = \lambda^{2^{m-2}} & m \geq 4, \end{cases}$$

so in particular the ratio is above 1 and, since for every $a > b > c \geq 0$ we have $(a - c)/(b - c) \geq a/b$, for $m \geq 2$ even

$$\frac{Z_{k_m}}{W_{j_m}} \geq \frac{\frac{Z_{k_{m-2}}^2}{Z_{k_{m-1}}} - \nu 2^{(\gamma - \frac{2}{3}\alpha)(n+m-2)}}{\frac{W_{j_{m-2}}^2}{W_{j_{m-1}}} - \nu 2^{(\gamma - \frac{2}{3}\alpha)(n+m-2)}} \geq \frac{Z_{k_{m-2}}^2}{Z_{k_{m-1}}} \bigg/ \frac{W_{j_{m-2}}^2}{W_{j_{m-1}}} > \lambda^{2^{m-2}}.$$

This concludes the inductive step for m even; for m odd the reasoning is analogous. We now want to use inequalities (34) and (35) to get a contradiction. We will consider separately the cases $\nu > 0$ and $\nu = 0$.

Case $\nu > 0$. Let m be even; by (34)

$$\frac{Z_{k_{m-2}}^2}{Z_{k_{m-1}}} > \lambda^{2^{m-2}} \frac{W_{j_{m-2}}^2}{W_{j_{m-1}}},$$

applying (32) to $W_{j_{m-1}}$ we have

$$\frac{W_{j_{m-2}}^2}{W_{j_{m-1}}} \geq \nu 2^{(\gamma - \frac{2}{3}\alpha)(n+m-2)},$$

so from (37), putting everything together, we get

$$Z_{k_m} \geq N_*^{-1} \left[\frac{Z_{k_{m-2}}^2}{Z_{k_{m-1}}} - \nu 2^{(\gamma - \frac{2}{3}\alpha)(n+m-2)} \right] \geq N_*^{-1} \nu 2^{(\gamma - \frac{2}{3}\alpha)(n+m-2)} (\lambda^{2^{m-2}} - 1).$$

For m even going to infinity we have obviously that Z_{k_m} grows as the exponential of an exponential, which is in contradiction with (33).

Case $\nu = 0$. If $\nu = 0$ we already know one explicit stationary solution, by Proposition 6.1, namely $X_j = f 2^{-(2\tilde{\alpha} + \alpha)(|j|+1)/3}$. By the usual change of variables (30) $V_j = f 2^{2(\alpha - \tilde{\alpha})(|j|-1)/3}$ is a solution of (32) satisfying the regularity condition (33). Without loss of generality we can suppose

that $W_j = V_j$ or $Z_j = V_j$. In the first case, for m even

$$Z_{k_m} > W_{j_m} \lambda^{2^{m-2}} = f 2^{\frac{2}{3}(\alpha - \tilde{\alpha}(n+m-2))} \lambda^{2^{m-2}},$$

in the second case for m odd

$$W_{j_m} > Z_{k_m} \lambda^{2^{m-2}} = f 2^{\frac{2}{3}(\alpha - \tilde{\alpha}(n+m-2))} \lambda^{2^{m-2}}.$$

In both cases the right-hand side grows super-exponentially as $m \rightarrow \infty$ and this is in contradiction with (33). \square

Proof of Theorem 2.3. Existence and uniqueness are given by Theorem 6.4.

If $\nu = 0$ the solution is identified by Proposition 6.1. If $\nu > 0$, by uniqueness, the solution is the lift of the stationary solution of the classic dyadic with $\beta = \alpha - \tilde{\alpha}$, as per Proposition 4.1. Then the two regimes are proven in Theorem 6.3. \square

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