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DOCTORAL THESIS

Geometric applications of Linear and Nonlinear Potential Theory

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Quis custodiet ipsos custodes?

Decimus Iunius Iuvenalis

UNIVERSITY OF TRENTO

Abstract

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Geometric applications of Linear and Nonlinear Potential Theory

by Mattia FOGAGNOLO

We provide geometric inequalities on $\mathbb{R}^n$ and on general manifolds with nonnegative Ricci curvature by employing suitable monotone quantities along the flow of capacitary and p-capacitary potentials, as well as through related boundary value problems. Among the main achievements, we cite

(i) a Willmore-type inequality on manifolds with nonnegative Ricci curvature leading in turn to the sharp Isoperimetric Inequality on 3-manifolds with nonnegative Ricci curvature ;

(ii) enhanced Kasue/Croke-Kleiner splitting theorems ;

(iii) a generalised Minkowski-type inequality in $\mathbb{R}^n$ holding with no assumptions on the boundary of the domain considered except for smoothness ;

(iv) a complete discussion of maximal volume solutions to the least area problem with obstacle on Riemannian manifolds and its relation with the variational p-capacity.

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Contents

Notation and conventions

We briefly discuss some of the conventions adopted. Most of them will be readily recalled along the thesis.

Riemannian manifolds and submanifolds

Riemannian manifolds will be usually denoted by (M, g). We address the reader to [Pet06] for the Riemannian geometric notions we employ. We denote the Levi-Civita connection by D, the Riemann curvature by Riem and the Ricci curvature by Ric. Let $N \hookrightarrow M$ be a codimension-1 submanifold. In the present work, submanifolds will mostly appear as boundaries of open subsets of Ω or smooth level sets of functions. We denote by h its second fundamental form, and by H its mean curvature, that we define just as the trace of h. No normalising constants are considered in the definition. Sometimes, in order to highlight the submanifold involved, we will add its name as a subscript. For example, H_N denotes the mean curvature of N.

We largely use the following computational convention. Let N be a codimension-1 submanifold of M, and let f be a smooth function on an open neighbourhood of $x \in N$. Submanifolds are always endowed with the Riemannian metric inherited by the ambient metric$g$, and we denote it by g_N or by $g^\top$, where in the last notation the submanifold is understood. Then, we denote the normal derivative of f with respect to N on x as

$$\mathrm{D}^\perp f(x) = \langle \mathrm{D} f \,|\, \nu \rangle\,(x)\nu(x),$$

where $\nu(x)$ is a unit normal vector to N in x, and where by $\langle X \,|\, Y \rangle$ we are denoting $g(X, Y)$. The orientation chosen for ν is always the one pointing towards the (usually unique) unbounded component of $M \setminus N$. The dependence on x is usually omitted. Accordingly, we define the tangential derivative of f with respect to N as

$$\mathrm{D}^\top f = \mathrm{D} f - \mathrm{D}^\perp f,$$

where we omitted the point where the computation is performed. It is straightforward to observe that

$$|\mathrm{D} f|^2 \; = \; |\mathrm{D}^\top f|^2 + |\mathrm{D}^\perp f|^2.$$

With $|T|$, we denote the norm of the tensor T with respect to the underlying metric g. When different metrics are considered, a suitable subscript appears. The submanifold N the above formalism refers to is often understood, the typical situation being the following. Let f be a function of the norm of some derivative tensor of a function u, such as the gradient $\mathrm{D}u$ or the Hessian tensor $\mathrm{DD}u$. In this case, when no other information is added, the tangential and normal computations at a point x take place with respect to the level set of u the point x belongs to.

Volumes, areas and perimeter

We are denoting by $\mathrm{d}\mu$ the volume element of the ambient metric g, and, given a submanifold N of M, by $\mathrm{d}\sigma$ the volume element of the metric g_N induced by M on N. We are referring to it as area element induced on N. Given a measurable set $\Omega \subseteq M$, we denote by $|\Omega|$ its volume measure, that we are going to confuse with the Lebesgue measure induced by g. Similarly, for a smooth submanifold N we let

$$|N| = \int_N d\sigma.$$

Occasionally, in Chapters 1 and 2, when N is not smooth, but still has finite $n-1$-dimensional Hausdorff measure $\mathscr{H}^{n-1}$, we let $|N| = \mathscr{H}^{n-1}(N)$.

Sets with nonsmooth boundary are important mostly in Chapter 3, where we are using the theory of sets with finite perimeter. We decided to collect definitions and notions employed in Subsection 3.2.1. Here, we just point that our main source about tools of Geometric Measure Theory is [Mag12].

Dimension

Except when differently indicated, the dimensions of the ambient metric this thesis consider are the following. In Chapter 1, the dimension of the underlying metric is $n \geq 3$. Chapter 2, that takes place in the ambient $\mathbb{R}^n$, is instead valid in dimension $n \geq 2$. However, we remark that the conclusions for $n = 2$ are somehow different, and we are never going to consider them. Chapter 3 involves Riemannian manifolds of dimension $n \geq 2$. Appendix A, being related to Chapter 1 holds for $n \geq 3$ while Appendix B, that we apply in Chapter 2 and 3 makes sense for $n \geq 2$.

Introduction

Geometric inequalities are from long ago thoroughly studied in both Analysis and Geometry, with also ground-breaking incursions in General Relativity. To name the maybe most famous one, let us mention the Isoperimetric Inequality in $\mathbb{R}^n$, the ultimate version of which, holding true for sets with finite perimeter, was obtained, according to [Mir97], by adding Federer's insights of [Fed58] to De Giorgi's [DG58]. In curved Riemannian settings, we cite the celebrated version on compact Riemannian manifolds with positive Ricci curvature known as Levý-Gromov Isoperimetric Inequality, obtained in [Gro80] building on [Lev22], and the one on noncompact Cartan-Hadamard manifolds of dimension 3 and 4 due respectively to Kleiner [Kle92] and Croke [Cro84]. Let us briefly point out that the general case, known in literature as *Cartan-Hadamard conjecture* is still object of ongoing research, and that a maybe decisive attempt at its solution has been proposed in [GS19]. Among the countless generalisations and applications of the Isoperimetric Inequality developed through more of a century of Mathematics, let us highlight now the family of inequalities known in literature as *Alexandrov-Fenchel Inequalities*, involving quantities called *quermassintegrals*, that are suitable integrals of elementary functions of the principal curvature of the boundary $\partial\Omega$ of an open bounded sets with smooth boundary Ω. These inequalities originated in [Ale37; Ale38; Fen29], where they were shown to hold for convex sets Ω. The simplest inequality in this family is often referred to as Minkowski Inequality, and was first obtained for convex sets in [Min03], and can be written as follows

$$\left(\frac{|\partial\Omega|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}} \leq \frac{1}{|\mathbb{S}^{n-1}|} \int\limits_{\partial\Omega} \frac{\mathrm{H}}{n-1} \, \mathrm{d}\sigma \,, \tag{1}$$

where by $\mathbb{S}^{n-1}$ we denote the unit sphere of $\mathbb{R}^n$. In recent years, effort has been put in order to go beyond the convex settings, as well as to establish this inequality on relevant Riemannian manifolds. We are mentioning some of these extensions available in literature later on in this Introduction. For now, let us just point out that some motivations behind this inequality are the challenge this problem offers and its importance in General Relativity. For what it concerns the latter point, we refer to the thorough survey [Mar09], where, precisely in Subsection 7.1, the author clearly explains how the Penrose inequality for shells of matter collapsing at the speed of light in flat Minkowski space, conjectured in [Pen73], is related to (1), as first observed by Gibbons in his Ph. D. thesis [Gib73].

We focus now on a mathematical technique that has been recently employed in the context of the Minkowski inequality. In order to best describe the approach adopted in this thesis, we explain how the Minkowski Inequality can be deduced through the Inverse Mean Curvature Flow (IMCF), a geometric evolution equation whose weak formulation led, just to mention the most striking application, to the proof by Huisken and Ilmanen, [HI01], of the Riemannian Penrose Inequality, constituting a fundamental step in the realisation of the conjectures of the aforementioned [Pen73]. Its classical, smooth formulation consists in evolving the immersion $F_0 : \partial\Omega \hookrightarrow \mathbb{R}^n$ by

$$\frac{\partial}{\partial t} F(t,x) = \frac{1}{\mathrm{H}}(t,x)\nu(t,x), \qquad F(0,x) = F_0(x),$$

yielding, at least for a small $T > 0$, a sequence of sets $\{\Omega_t\}_{t\in[0,T)}$ with boundaries $\partial\Omega_t$ given by the immersions $F(t,\cdot) : \partial\Omega \hookrightarrow \mathbb{R}^n$, and where $\mathrm{H}(t,x)$ and $\nu(t,x)$ are respectively the mean curvature and the outward pointing unit normal of $\partial\Omega_t$. Clearly, what just described makes sense for a strictly mean-convex initial hypersurface $\partial\Omega$, that is, with strictly positive mean curvature, an assumption obviously satisfied by convex initial sets. Along this evolution, one can then define the quantity

$$\mathscr{Q}(t) = |\partial\Omega_t|^{-\frac{n-2}{n-1}} \int_{\partial\Omega_t} \mathrm{H}(t,\cdot)\, \mathrm{d}\sigma_t, \tag{2}$$

where $\mathrm{d}\sigma_t$ is the area element induced on $\partial\Omega_t$ by the ambient flat metric. A direct computation then shows that $\mathscr{Q}(t)$ is a monotone nonincreasing function. In particular, if $T = +\infty$ and, as geometrically desirable, $\partial\Omega_t$ converges smoothly enough to expanding spheres so that the scaling invariant $\mathscr{Q}(t)$ converges to its value on spheres, one gets (1). These strong requirements are actually shown to be satisfied in the class of bounded starshaped sets with smooth strictly mean-convex boundary in [Ger90; Urb90], yielding, through the argument just outlined the extension of the classical Minkowski inequality to this broader class of sets. This scheme, together with the derivation of other Alexandrov-Fenchel inequalities, is carried out in details in [GL09]. We point out, on the other hand, that this approach is suitable only for those hypersurfaces that do not change topology along their evolution. For example, in $\mathbb{R}^n$, there is no chance to employ it if the initial boundary does not have spherical topology. These topological restrictions can be overtaken considering the delicate notion of weak solutions introduced by Huisken-Ilmanen in the aforementioned [HI01], that allows, as lectured in [Hui], to prove (1) in the broader class of outward minimising sets with smooth boundary. Leaving the details of this approach to the sequel, where we are drawing a systematic comparison between that and our methods, we just say here that a bounded set E with finite perimeter is outward minimising if $P(E) \leq P(F)$ for any $E \subseteq F$, and that, if the boundary of an outward minimising set is smooth, then it is mean-convex, as immediately seen through the standard variational argument. The inclusion of the class of strictly starshaped sets with smooth strictly mean-convex boundary into this one substantially follows from the combination of [Ger90; Urb90] and [HI01], and we refer the reader to Proposition 3.25 for a complete proof of (a generalised version of) this result.

Finally reaching the starting point of this work, we do now recall another celebrated geometric inequality, that along this work will be referred to as Willmore-type inequality, asserting that a bounded set $\Omega \subset \mathbb{R}^n$ with smooth boundary satisfies

$$\left|\mathbb{S}^{n-1}\right| \leq \int_{\partial\Omega} \left|\frac{\mathrm{H}}{n-1}\right|^{n-1} \mathrm{d}\sigma. \tag{3}$$

This inequality was first obtained for surfaces inside $\mathbb{R}^3$ in [Wil68], where it is usually called Willmore inequality, and later extended to all dimensions in [Che71]. We remark that the Willmore energy, that is the right hand side of (3) for $n = 3$, arises also in the context of cell membranes, as a first approximation of the Helfrich energy introduced in [Hel73]. Despite, on those sets satisfying the Minkowski Inequality, the latter implies (3) as a straightforward application of the Hölder inequality immediately shows, the Willmore-type inequality holds true with no assumptions on Ω other than smoothness of its boundary. Moreover, equality is achieved in (3) only on balls. Observe that an Inverse Mean Curvature Flow-proof is viable also for (3), considering, along the evolution of a set with strictly mean-convex boundary, the function

$$\mathscr{W}(t) = \int_{\partial\Omega_t} \mathrm{H}^{n-1}(t,x)\, d\sigma_t \tag{4}$$

and finally obtaining (3) as a consequence of its monotonicity. As before, this scheme allows to prove the Willmore-type inequality for starshaped sets with smooth strictly mean-convex boundary, by appealing to [Ger90; Urb90], and most likely also for outward minimising sets by combining the deep weak notion of IMCF of [HI01] with the observations performed in [Hui].

A different approach to (3) was introduced some years ago by Agostiniani and Mazzieri in [AM20]. They considered the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n \setminus \overline{\Omega} \\ u = 1 & \text{on } \partial\Omega \\ u(y) \to 0 & \text{as } |y| \to +\infty, \end{cases} \tag{5}$$

and showed that the the function

$$U(t) \;=\; t^{-(n-1)} \int\limits_{\{u=t\}} |\mathrm{D}u|^{n-1}\, d\sigma, \qquad\qquad t \in (0,1] \tag{6}$$

is monotone nondecreasing, and in particular by coupling

$$U'(1) \;=\; (n-2) \int\limits_{\partial\Omega} |\mathrm{D}u|^{n-2} \left[\mathrm{H} - \left(\tfrac{n-1}{n-2}\right) |\mathrm{D}u| \right] d\sigma \;\geq\; 0$$

with

$$|\mathbb{S}^{n-1}| \;=\; \lim_{t\to 0^+} U(t) \leq U(1), \tag{7}$$

they showed, with the aid of the Hölder inequality, (3). Moreover, since the monotonicity is strict unless Ω is a ball, this also shows that equality in (3) is achieved just on balls, recovering the full statement of [Che71]. This approach immediately shows an analogy with the IMCF's one, since it is based on a monotonic quantity along a suitable evolution of the set considered, but it also displays the power arising from the simplicity of (5). Indeed, being the solution of this exterior boundary value problem well known to exist, and to be smooth, thus classical, both long-time existence results as that of [Ger90; Urb90] for the IMCF, or the necessity of a delicate notion of weak solution as that of Huisken-Ilmanen's weak IMCF are immediately by-passed. Moreover, no starshapedness or outward minimising assumptions are necessary, nor mean-convexity of the boundary. Let us also observe that also (7) follows from very classical asymptotic expansions at spatial infinity of the solution to (5).

The starting point of the present thesis is to generalise and better understand this new approach to geometric inequalities. This is done essentially in two directions. The first one concerns the application of this method to complete noncompact manifolds with nonnegative Ricci curvature. The second one, concerning instead flat $\mathbb{R}^n$, regards the nonlinear generalisation of it, that is, considering the evolution given by the level sets of the solution of the nonlinear version of (5), where the laplacian is replaced by the p-laplacian. The following two Sections are devoted to illustrate the main results arising from these lines of research. They will be developed in Chapter 1 and Chapter 2.

Sharp geometric inequalities in nonnegative Ricci curvature

Looking at [AM20], one can realise that the main inequalities, actually based on the Bochner identity, *formally* hold true even if the ambient is a nonnegatively Ricci curved complete noncompact Riemannian manifold (M, g). We do actually show that this is truly the case, and prove that the function U defined as in (6) is monotone nondecreasing, everytime a solution to

$$\begin{cases} \Delta u = 0 & \text{in } M \setminus \overline{\Omega} \\ u = 1 & \text{on } \partial\Omega \\ u(y) \to 0 & \text{as } d(O, y) \to +\infty, \end{cases} \tag{8}$$

with O being a fixed point in Ω and d the distance on M , exists. Clearly, the definition of U is given in terms of such solution. It is known, and re-proved in Theorem 1.14, that a solution to (8) exists if and only if (M, g) is *nonparabolic*, that is, if it admits a positive Green's function. As in [AM20] we show that the monotonicity property of U is in fact shared by a family of functions $U_\beta : (0, 1] \to \mathbb{R}$ defined by

$$U_\beta(t) \;=\; t^{-\beta\left(\frac{n-1}{n-2}\right)} \int\limits_{\{u=t\}} |\mathrm{D}u|^{\beta+1}\, \mathrm{d}\sigma,$$

if $\beta \geq (n-2)/(n-1)$. If, for β in this range, $U'_\beta(t_0) = 0$ for some $t_0 \in (0, 1]$, we do show that $(\{u \leq t_0\}, g)$ must be isometric to a truncated cone. The statement of this Monotonicity-Rigidity Theorem for U_β in nonparabolic manifolds with nonnegative Ricci curvature is to be found in Theorem 1.19.

Recalling that the monotonicity of U_β, and precisely that of $U_{n-2} = U$, constituted the main ingredient in the recent potential-theoretic proof of the Willmore-type inequality outlined above, we are committed to apply that in order to derive a Willmore-type inequality for manifolds with nonnegative Ricci curvature. However, a serious issue arises in the characterisation of the limit of U_β as $t \to 0^+$, since, as we detail in Remark 1.39, there is no hope to derive pointwise Euclidean-like asymptotic expansions of the gradient in our general setting, ultimately due to the possibility of infinite topology. On the other hand, we overcome this problem by working out suitable delicate integral asymptotic expansions, in the spirit of [CM97b]. This is an important technical difference from the Euclidean case. This will be carried out in Section 1.4, that culminates in the computation

$$\lim_{t \to 0^+} U_\beta(t) \;=\; \mathrm{Cap}(\Omega)^{1-\beta/(n-2)} \mathrm{AVR}(g)^{\beta/(n-2)} (n-2)^{\beta+1} |\mathbb{S}^{n-1}|, \tag{9}$$

see Proposition 1.40 and (1.4.28). With $\mathrm{AVR}(g)$, appeared above, we denote the Asymptotic Volume Ratio of g, defined as

$$\mathrm{AVR}(g) = \lim_{r \to +\infty} \Theta(r)$$

and where

$$(0, +\infty) \ni r \longmapsto \Theta(r) = \frac{|B(x, r)|}{|\mathbb{B}^n| r^n},$$

with $\mathbb{B}^n$ denoting the unit ball of $\mathbb{R}^n$. With $B(x, r)$ we indicate the geodesic ball of radius r centered at $x \in M$. The classical Bishop-Gromov Theorem [Bis63; Gro81] ensures that Θ is monotone nonincreasing and thus admitting a limit as $r \to +\infty$, that actually does not depend on the point x. Such limit $\mathrm{AVR}(g)$ is then included in $[0, 1]$, and is 1 only on flat

$\mathbb{R}^n$. We say that (M, g) has Euclidean volume growth if $\mathrm{AVR}(g) > 0$. The limit in (9) actually tells that a nontrivial Willmore-type inequality is likely to be deduced only if (M, g) has Euclidean volume growth. We notice that by a known characterisation, see [Var82], a complete noncompact manifold with nonnegative Ricci curvature is in particular nonparabolic. In fact, such characterisation states that a complete noncompact Riemannian manifold with $\mathrm{Ric} \geq 0$ is nonparabolic if and only if, roughly speaking, the volume of geodesic balls grows faster than quadratically, see Theorem 1.4.

The first main result of this thesis asserts that on a complete noncompact Riemannian manifold (M, g) with $\mathrm{Ric} \geq 0$ and Euclidean volume growth it holds

$$\mathrm{AVR}(g)|\mathbb{S}^{n-1}| \leq \int_{\partial\Omega} \left|\frac{\mathrm{H}}{n-1}\right|^{n-1} \mathrm{d}\sigma \,, \tag{10}$$

with equality achieved only if $(M \setminus \Omega, g)$ is isometric to a truncated cone. This is the content of Theorem 1.44. This result encompasses a variety of manifolds of interest. For example, for $n \geq 4$, there exists an important class of complete noncompact Ricci flat Riemannian manifolds with $0 < \mathrm{AVR}(g) < 1$, that is the class of Ricci flat *Asymptotically Locally Euclidean* (ALE for short) manifolds. We refer the reader to Definition 1.48 for the precise notion. For the time being, we just recall that an n-dimensional Riemannian manifold is ALE if it is asymptotic to $\big((\mathbb{R}^n \setminus \{0\})/\Gamma\,,\, g_{\mathbb{R}^n}\big)$, where Γ is a finite subgroup of $\mathrm{SO}(n)$ acting freely on $\mathbb{R}^n \setminus \{0\}$. This family of Riemannian manifolds is widely studied. In this regard, we first mention that in [BKN89] it is proved that any Ricci flat manifold with Euclidean volume growth and strictly faster than quadratic curvature decay is actually ALE. Moreover, we point out that 4-dimensional Ricci flat ALE manifolds appear as important examples of *gravitational instantons*, that are noncompact hyperkähler 4-manifolds with decaying curvature at infinity, introduced by Hawking in [Haw77] in the framework of his Euclidean quantum gravity theory. An explicit example is given by the famous Eguchi-Hanson metric, introduced in [EH79], where $n = 4$, $\mathrm{Ric} = 0$ and $\Gamma = \mathbb{Z}_2$. We remark that ALE gravitational instantons are completely classified in [Kro89b] and [Kro89a]. Concerning the general class of gravitational instantons, let us cite, after the important works of Minerbe [Min09a; Min10; Min11], the recent Ph.D. thesis [Che17], where gravitational instantons with strictly faster than quadratic curvature decay are classified. We refer the reader to the latter work and to the references therein for a more complete picture on this subject. Our Willmore-type inequality actually applies to ALE manifolds with nonnegative Ricci curvature, where it improves to

$$\inf \left\{ \int_{\partial\Omega} \left|\frac{\mathrm{H}}{n-1}\right|^{n-1} \mathrm{d}\sigma \;\middle|\; \Omega \subset M \text{ bounded with smooth boundary} \right\} = \frac{|\mathbb{S}^{n-1}|}{\operatorname{card}\Gamma}\,, \tag{11}$$

with equality achieved on truncated cones with link homothetic to $(\mathbb{S}^{n-1}/\Gamma, g_{\mathbb{S}^{n-1}/\Gamma})$. Notice in particular that if Γ is trivial one recovers the classical Willmore-type inequality described above with its rigidity statement. The lower bound of the Willmore-type functional shown in (10) being actually an infimum is a fact that holds true for a larger class of manifolds. Indeed, as proved in Theorem 1.47, it is sufficient to assume the hypotheses of Theorem 1.44 together with a quadratic curvature decay condition. This will immediately yield (11) as shown in Corollary 1.49. Understanding metric and topological consequences of curvature decay conditions is a very interesting and widely studied problem in geometric analysis. Dropping any attempt of being complete, we refer the interested reader to the aforementioned [BKN89], to the seminal [CGT82], to [Rei15], where the case $n = 3$ is considered, and to [Yeg09] and the references therein.

To make the picture more complete, let us also mention that Willmore-type inequalities are proven in [AM17] for asymptotically flat (AE) static metrics in the framework of General Relativity, and in [Sch18] for integral 2-varifolds in Cartan-Hadamard manifolds.

As for the Euclidean case, Theorem 1.19 will actually be proved working in the manifold $(M \setminus \Omega, \tilde{g})$ where $\tilde{g}$ is conformally related to g by

$$\tilde{g} = u^{\frac{1}{n-2}} g,$$

where u is a solution to (8). In this setting, integral identities and splitting techniques are employed to infer the monotonicity of (the conformal analogue of) U_β and the related rigidity. We point out that these techniques can easily produce a more general version of Theorem 1.19 for nonparabolic ends with $\mathrm{Ric} \geq 0$ of a noncompact Riemannian manifold. This *conformal splitting method* was introduced in [AM15] and has proved to be fruitful in various other situations, such as [AM17; BM18] and [BM17], where it has been applied to the relativistic setting. A different approach to exterior problems, but still relying on a conformal change of metric, has been introduced in [BMM19]. As a consequence of that approach, the authors were able to prove that in $\mathbb{R}^n$, suitable, new pinching conditions on the mean curvature of $\partial\Omega$ or on the normal derivative of u at $\partial\Omega$ force Ω to be a ball. Here, we show that those results can be achieved also in nonparabolic manifolds with nonnegative Ricci curvature. To do so, we propose an alternative, easier tool, based on a maximum principle for $|\mathrm{D}u|/u^{(n-1)/(n-2)}$ and the Hopf boundary lemma, that we interpret as a Monotonicity-Rigidity Theorem for the function

$$U_\infty(t) = \sup_{\{u=t\}} \frac{|\mathrm{D}u|}{u^{\frac{n-1}{n-2}}}$$

Two main consequences of the monotonicity-rigidity properties of $U_\infty$are described in Section 1.5, together with some other consequences of the monotonicity of U_β , and, as just hinted, constitute the analogues of the main results of [BMM19] in our general Riemannian setting. As for the Willmore-type inequality, the main difficulty in reaching them lies in the asymptotic behaviour of U_∞ as $t \to 0^+$, described in Proposition 1.43.

So far, we have considered *nonparabolic* Riemannian manifolds with $\mathrm{Ric} \geq 0$. We now turn our attention to *parabolic* Riemannian manifolds with nonnegative Ricci curvature. As we will see in Section 1.2, for this class of manifolds, problem (8) does not admit a solution, while the following problem does

$$\begin{cases} \Delta\psi = 0 & \text{in } M \setminus \overline{\Omega} \\ \psi = 0 & \text{on } \partial\Omega \\ \psi(y) \to +\infty & \text{as } d(O, y) \to +\infty\,, \end{cases} \tag{12}$$

where $\Omega \subset M$ is any bounded and open subset with smooth boundary. Inspired by the fact that problem (12) presents strong formal analogies with the conformal reformulation of problem (8) in terms of $\tilde{g}$ (see problem (1.3.13) below), we also provide a Monotonicity-Rigidity Theorem for parabolic manifolds with $\mathrm{Ric} \geq 0$ involving ψ in place of u. For $\beta \geq 0$, we define the function $\Psi_\beta : [0, \infty) \to \mathbb{R}$ as

$$\Psi_\beta(s) = \int_{\{\psi=s\}} |\mathrm{D}\psi|^{\beta+1} \, \mathrm{d}\sigma\,,$$

and we prove that it is monotone nonincreasing if $\beta \geq (n-2)/(n-1)$, with derivative vanishing at $s_0 \in [0, \infty)$ only if $(\psi \geq s_0, g)$ is isometric to a truncated cylinder, see Theorem 1.54. In complete analogy with the nonparabolic case, we may also define the function

$$\Psi_\infty(s) = \sup_{\{\psi = s\}} |D\psi|, \tag{13}$$

and prove that it is nonincreasing, as stated in Theorem 1.55. The coupling of the Monotonicity - Rigidity Theorems for nonparabolic and parabolic manifolds of nonnegative Ricci curvature yields as a straightforward consequence an enhanced version of a theorem of Kasue, [Kas83, Theorem C (2)], see also [CK92], asserting that if a smooth boundary $\partial\Omega \subset M$ has mean curvature $\mathrm{H} \leq 0$ on the whole $\partial\Omega$, then $\mathrm{H} = 0$ on $\partial\Omega$ and $M \setminus \Omega$ is isometric to a half cylinder. Our result actually gives precise lower bounds for the supremum of H in terms of our monotone quantities and their derivatives, see Theorem 1.56. We point out that Kasue/Croke-Kleiner result holds true also in a compact version with a two-connected-components boundary. We recover, through our techniques, also (a slightly weaker form of) such statement in Theorem 1.59. We inform the reader that similar splitting theorems have been recently provided in dimension 3 under a mild scalar curvature lower bound in [GJ19].

Finally, we combine our sharp Willmore-type inequality (10) with curvature flow techniques along the lines of an argument presented by Huisken in [Hui]. We obtain a characterisation of the infimum of the Willmore functional in terms of the isoperimetric ratio of complete noncompact 3-manifolds with nonnegative Ricci curvature, refining the analogous result stated in the aforementioned contribution. We get in fact

$$\inf \frac{|\partial\Omega|^3}{36\pi|\Omega|^2} = \inf \frac{\int_{\partial\Omega} \mathrm{H}^2 \, d\sigma}{16\pi} = \mathrm{AVR}(g), \tag{14}$$

on any complete noncompact Riemannian manifold (M, g) with $\mathrm{Ric} \geq 0$ and Euclidean volume growth, yielding in turn

$$\frac{|\partial\Omega|^3}{|\Omega|^2} \geq 36\pi \, \mathrm{AVR}(g). \tag{15}$$

The key argument lectured by [Hui] involves the computation of the evolution along the Mean Curvature Flow of a quantity we call Isoperimetric Difference, namely

$$D(t) = |\partial\Omega_t|^{3/2} - \mathrm{C}|\Omega_t|,$$

with $\Omega_0 = \Omega$ and some positive constant C. Such quantity is indeed shown to be monotone if along the evolution some lower bound on the Willmore energy is preserved, and if the flow vanishes as $t \to T^-$ for some $T > 0$. In this case, consequently, an Isoperimetric Inequality is deduced with constant related to such lower bound. So, we plug our new Willmore inequality (10) for $n = 3$ into this argument, combine it with known important results in the weak Mean Curvature Flow of mean-convex sets and with the Kasue/Croke-Kleiner Theorem we have re-discovered to get (14). We are also able to show that equality holds in (15) if and only if Ω is isometric to a ball of flat $\mathbb{R}^n$. Beside the characterisation of the isoperimetric constant in terms of the Asymptotic Volume Ratio, the novelties with respect to [Hui] lie in the rigidity statement and in the fact that the infimum of the Willmore functional is taken over the whole class of bounded open subsets Ω with smooth boundary, and not just over *outward minimising* subsets. All of these improvements substantially come from our optimal Willmore-type inequality (10). We wish to remark that the structure of complete noncompact 3-manifolds with nonnegative Ricci curvature is well understood, see [Liu13], although this knowledge never enters in the argument employed. Actually, as pointed out in [Heb99, Theorem 8.4], a positive nonsharp isoperimetric constant for complete noncompact Riemannian manifolds with $\mathrm{Ric} \geq 0$ and Euclidean volume growth is known to exist. This was realised for the first time most likely in [Var95], and can be achieved rigorously through the methods employed in [Car94].

The Isoperimetric Inequality is well known to be related to a huge variety of issues. We discuss some of them in Subsection 1.7.4. Briefly, we consider the Sobolev Inequality and the Faber-Krahn Inequality, both achieving optimal constants deduced from (23) on three-dimensional ambient manifolds, we state and comment the natural higher dimensional conjecture and compare the Isoperimetric Inequality with the Isoperimetric problem.

Let us finally observe that relations between isoperimetry and mean curvature functionals date back to Almgren [Alm86], while a first derivation of isoperimetric inequalities through a curvature flow has been obtained by Topping in the case of curves [Top98]. Isoperimetric inequalities in $\mathbb{R}^n$ and in Cartan-Hadamard manifolds through curvature flows have been established by Schulze in [Sch08] and [Sch18], while the application to manifolds with nonnegative Ricci curvature is suggested in the already mentioned [Hui]. The techniques lectured in [Hui] have interesting applications also in connection with the relativistic ADM mass, see [JL17] for the details.

Geometric aspects of p-capacitary potentials

The natural nonlinear generalisation of problem (8) is

$$\begin{cases} \Delta_p u = 0 & \text{in } \mathbb{R}^n \setminus \overline{\Omega} \\ u = 1 & \text{on } \partial\Omega \\ u(y) \to 0 & \text{as } |y| \to +\infty\,, \end{cases} \tag{16}$$

for $\Omega \subset \mathbb{R}^n$ a bounded set with smooth boundary and where the p-laplacian operator Δ_p acts on smooth functions f as $\Delta_p f = \mathrm{div}(|\mathrm{D}f|^{p-2}\mathrm{D}f)$. In general, a Sobolev-type notion of p-harmonic functions is available, and it allows to show existence of a function $u \in W^{1,p}$ weakly solving (16), and where the boundary value is reached smoothly. In Chapter 2, we are committed to study monotonicity formulas along the level set flow of the solution u of the problem above, that we refer to as the p-capacitary potential of Ω. However, after guessing the correct quantities that should be formally smooth, a major problem appears at once. Indeed, the lack of smoothness for solutions to (16), the optimal regularity being mild $\mathscr{C}^{1,\alpha}$, invalidates most of the arguments employed in the linear case to infer monotonicity, the most dramatic issue being the lack of a satisfactory Sard-type property. On the other hand, it is evident that all of the consequences we were able to draw from the monotonicity formulas in the linear setting just followed from $U'_\beta(1) \geq 0$ and $\lim_{t\to 0^+} U_\beta(t) \leq U_\beta(1)$. Some reason for optimism can then arise, since Tolksdorf's Hopf lemma [Tol83] for p-harmonic functions and Kichenassamy-Veron's [KV86] asymptotic expansion ensures that the solution to (16) is actually smooth in a neighbourhood of Ω as well as sufficiently far away. What we succeed to prove in Theorem 2.7 is in fact an *effective monotonicity* property of the function

$$U_\beta^p(t) \;=\; t^{-\beta(p-1)\frac{(n-1)}{(n-p)}} \int\limits_{\{u=t\}} |\mathrm{D}u|^{(\beta+1)(p-1)}\,\mathrm{d}\sigma,$$

defined only on regular values of u, for $\beta \geq (n-p)/[(p-1)(n-1)]$. By effective monotonicity, we substantially mean that $(U_\beta^p)'(1) \geq 0$ and $\lim_{t\to 0^+} U_\beta^p(t) \leq U_\beta^p(1)$. Besides their geometric implications, these formulas have a technical relevance on their own, as they persist through all the possible singularities of the flow. It is worth noticing that, in the present framework, the flow singularities correspond to the critical points of u, and these might in principle be arranged in sets with full measure.

The main geometric implication of such effective monotonicity is an extension of the Minkowski Inequality (1), holding for every bounded and smooth subset of $\mathbb{R}^n$, in which the total mean curvature of the boundary is replaced by the L^1-norm of the mean curvature, whereas the perimeter of the set Ω is replaced by the one of its *strictly outward minimising hull* Ω^*. For the reader's convenience we briefly recall that a set is called *outward minimising* if it minimises the perimeter among all the sets containing it; moreover, an outward minimising set is called *strictly outward minimising* if it coincides almost everywhere with any outward minimising set containing it and having the same perimeter. Loosely speaking, Ω^* is, up to negligible components, the smallest strictly outward minimising set that contains Ω, and it happens to be the set containing Ω with smallest perimeter. We are actually dedicating Chapter 3 to describe various aspects of the strictly outward minimising hull in Riemannian geometry, and then we address the reader to its presentation below for further discussions. Such Extended Minkowski Inequality, content of Theorem 2.25, reads

$$\left(\frac{|\partial\Omega^*|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}} \leq \frac{1}{|\mathbb{S}^{n-1}|}\int\limits_{\partial\Omega}\left|\frac{\mathrm{H}}{n-1}\right|\,\mathrm{d}\sigma\,. \tag{17}$$

Let us point out that (17) leaves the long-standing question about the validity of the Minkowski Inequality (1) under the sole assumpion of mean-convexity for $\partial\Omega$ still open. To this end, a straightforward consequence of the celebrated Michael-Simon Sobolev inequality [MS73] yields the existence of a constant $\mathrm{C}(n)$ depending only on the dimension such that

$$\mathrm{C}(n) \;\leq\; |\partial\Omega|^{-\frac{n-2}{n-1}}\int\limits_{\partial\Omega}|\mathrm{H}|\,\mathrm{d}\sigma \tag{18}$$

is true for any bounded set with smooth boundary Ω. The question, then, can be extended in terms of the optimal constant in (18), at least under a mean-convexity assumption. We observe that an attempt to answer in the positive, that is, to show that one can take as $\mathrm{C}(n)$ the one given by (17) replacing $\partial\Omega^*$ with $\partial\Omega$ when the boundary is just mean-convex was made by Trudinger in [Tru94]. However, such proof is acknowledged to be incomplete in [Gua+10]. On the other hand, in [Dal+16] it is shown that (18) holds true with such conjectured constant in the class of axisymmetric sets with $\mathscr{C}^{1,1}$ boundary regularity of $\mathbb{R}^3$.

As a matter of fact, the above Extended Minkowski Inequality is deduced as the limit, for $p \to 1^+$, of the following geometric p-capacitary inequality, which we believe of independent interest

$$\mathrm{C}_p(\Omega)^{\frac{n-p-1}{n-p}} \leq \frac{1}{|\mathbb{S}^{n-1}|} \int\limits_{\partial\Omega} \left|\frac{\mathrm{H}}{n-1}\right|^p \mathrm{d}\sigma\,, \tag{19}$$

stated in Theorem 2.24. Here, by $\mathrm{C}_p(\Omega)$ we denote a normalised version of the classical variational p-capacity of Ω denoted by $\mathrm{Cap}_p(\Omega)$. Briefly we remark that (19) follows from the effective monotonicity of U^p_β with value $\beta = 1/(p-1)$ with a derivation very similar to that of the Willmore-type inequality from the monotonicity of U_β with $p = 2$.

On the other hand, in order to deduce (17) from (19), one needs to compute the limit of the p-capacity of a bounded set with smooth boundary as $p \to 1^+$. We have established that

$$\lim_{p\to 1^+} \mathrm{C}_p(\Omega) = \frac{|\partial\Omega^*|}{|\mathbb{S}^{n-1}|}. \tag{20}$$

This relation, that we observe at once to be equivalent to $\lim_{p\to 1^+} \mathrm{Cap}_p(\Omega) = |\partial\Omega^*|$, to the author's knowledge has never been explicitly considered in literature. Actually, it happens to be intimately related to the geometry and to the potential theoretic properties of the underlying ambient, and this is the reason we prove it in a general setting in Theorem 3.1 of Chapter 3.

As an immediate corollary of (17), we recover the Minkowski Inequality for *outward minimising* sets, since for every Ω in this class it holds $|\partial\Omega| = |\partial\Omega^*|$ (see Remark 3.16) and $\mathrm{H} \geq 0$, as a standard variational computation readily shows. Such inequality was originally conceived by Huisken in [Hui], exploiting the theory of weak solutions to the IMCF, previously developed in [HI01] (see also [FS14, Theorem 2–(b)] for a published version of the argument in the case of outward minimising sets with strictly mean-convex boundary, or [Wei18] in the more general Schwarzschild setting). We remark here that the class of outward minimising sets contain that of strictly starsharped sets with smooth strictly mean-convex boundary. This substantially follows from [Ger90; Urb90] together with the area minimising properties of the Inverse Mean Curvature Flow observed and exploited with dramatic success in [HI01]. We give a self contained proof and slightly improve this fact in Subsection 3.3.2.

It is worth pointing out that an approximation argument through Mean Curvature Flow, pointed out in [HI01] and refined in [HI08] allows to deduce the Extended Minkowski Inequality from its version for outward minimising sets if $n \leq 7$, as we show in Subsection 2.4.1. On the other hand, this argument breaks down in higher dimension due to minimal surfaces regularity issues, and then in higher dimensions inequality (17) seems to be actually stronger.

A simple and very nice application of inequality (17) is a nearly umbilical estimate for *outward minimising* surfaces in $\mathbb{R}^3$ with *optimal constant*. The relation between the Minkowski Inequality and the nearly umbilical estimates was suggested by Huisken in [Hui]. Here, for the sake of reference, we included a proof of this fact in Section 2.5 (see Theorem 2.28). The general nearly umbilical estimate for surfaces in $\mathbb{R}^3$ with an implicit dimensional constant is a very remarkable theorem, proved in [DLM05] by De Lellis and Müller. We refer the reader to the original paper [DLM05] as well as to the Ph.D. thesis [Per11] and the references therein for a complete account about the geometric features and implications of such a deep result.

Another application of the Extended Minkowski Inequality is the following inequality that we call Volumetric Minkowski Inequality,

$$\left(\frac{|\Omega|}{|\mathbb{B}^n|}\right)^{\frac{n-2}{n}} \leq \frac{1}{|\mathbb{S}^{n-1}|} \int_{\partial\Omega} \left|\frac{\mathrm{H}}{n-1}\right| \, \mathrm{d}\sigma \,, \tag{21}$$

again holding true for any bounded open set $\Omega \subset \mathbb{R}^n$ with smooth boundary. Inequality (21) follows from (17) through a direct application of the Isoperimetric Inequality to Ω^*, that actually also yields the isometry of $\partial\Omega$ with a sphere if equality is achieved in (21). To the authors' knowledge, the Volumetric Minkowski Inequality was previously known to hold for domains with a strictly mean-convex boundary of positive scalar curvature (for short $\partial\Omega \in \Gamma_2^+$). On this regard, we refer the reader to the paper [CW13] and the subsequent [Qiu15], where the inequality was proved with methods based on Optimal Transport.

To complete the analogy with the linear case treated in Chapter 1, we also prove in Theorem 2.8 that the function

$$U_\infty^p(t) \;=\; \sup_{\{u=t\}} \frac{|\mathrm{D}u|}{u^{\frac{n-1}{n-p}}},$$

is, in a suitable sense, effectively monotone. Together with some nonlinear generalisations of results appeared in [AM20] and [BMM19], we observe in Section 2.5 how the monotonicity of U_∞^p is related to modern techniques in overdetermined boundary value problems, with particular mention for [GS99] and [Pog18].

Before going on, let us recall, as anticipated before, that Minkowski-type inequalities as well as many other related inequalities, like for example Penrose-type inequalities, are provided in literature by using the Inverse Mean Curvature Flow or suitable generalisations of it. Without any attempt to be complete, in [GL09; LW17; BHW16; GWW13; GWW14; Ge+15; LG16; MS16] suitable nontrivial generalisations of the long time existence result of [Ger90; Urb90] under more or less restrictive requirements on the initial sets are derived and applied to get the desired geometric applications. On the other hand, in [BM08; FS14; Wei18; McC17; LN15] the Huisken-Ilmanen's weak notion of Inverse Mean Curvature Flow is employed for these purposes. We then find convenient to yield some other insights on the relations between the method employed here and that of the IMCF.

Inverse Mean Curvature Flow VS Nonlinear Potential Theory

The relation with the Inverse Mean Curvature Flow, in particular to its weak formulation, is quite transparent when dealing with the nonlinear approach to the Minkowski Inequality developed in Chapter 2. Actually, in light of the discussion that follows, the success of the linear potential theoretic argument of [AM20] and of the extensions worked out in Chapter 1 will appear somewhat surprising.

The key point in our approach is to replace the delicate elliptic regularisation procedure of Huisken-Ilmanen with a novel analysis of a very natural family of approximate solutions, namely the p-capacitary potentials of Ω, with $p \to 1^+$. In fact, a well known result due to Moser [Mos07], subsequently extended by Kotschwar-Ni [KN09] and recently in [MRS19], says that if u_p is a weak solution to problem

$$\begin{cases} \Delta_p u_p = 0 & \text{in} \quad \mathbb{R}^n \setminus \overline{\Omega}\,, \\ u_p = 1 & \text{on} \quad \partial\Omega\,, \\ u_p(y) \to 0 & \text{as} \quad |y| \to \infty\,, \end{cases}$$

then, as $p \to 1^+$, the functions $w_p = -(p-1)\log u_p$ converge locally uniformly in $\mathbb{R}^n \setminus \Omega$ to a weak solution of the IMCF, the definition of which is recalled in Section 3.3.2. It must be noted that w_p satisfies the identity

$$\Delta_p w_p \;=\; |\mathrm{D} w_p|^p\,,$$

which is formally converging to

$$\mathrm{div}\left(\frac{\mathrm{D}w}{|\mathrm{D}w|}\right) = |\mathrm{D}w|\,, \tag{22}$$

and the latter equation is known to rule the level sets formulation of the IMCF. Albeit its simplicity, this very clean approximation scheme has found applications so far only to the existence theory for the weak IMCF. Very roughly, what we discover in Chapter 2 is that the p-harmonic approximators u_p possess self consistent monotonicity formulas, interpolating the linear ones of [AM15] and Chapter 1 with that of the function $\mathscr{Q}$ defined in (2) considered above. In light of this achievement the techniques worked out in [HI01] allowing, as discussed in [Hui; FS14; Wei18], to extend the validity of the Minkowski Inequality to outward minimising sets are fully replaced by new ones in the context of nonlinear potential theory. In order to understand the relation between the monotonicity of U^p_β and the one of $\mathscr{Q}$ defined by (2), we proceed formally. Setting as above $w_p = -(p-1)\log u_p$ and $t = -(p-1)\log\tau$, the monotonicity of the functions $U^p_\beta(\tau)$ with $\beta = 1/(p-1)$ is equivalent to say that for every $1 < p < n$ the function

$$[0,+\infty) \ni t \longmapsto \mathrm{e}^{-\frac{n-p-1}{n-p}\,t} \int\limits_{\{w_p = t\}} |\mathrm{D}w_p|^p \,\mathrm{d}\sigma$$

is nonincreasing. Taking the formal limit as $p \to 1^+$, one would get the same monotonicity statement for the function

$$[0,+\infty) \ni t \longmapsto \mathrm{e}^{-\frac{n-2}{n-1}\,t} \int\limits_{\{w = t\}} |\mathrm{D}w| \,\mathrm{d}\sigma\,,$$

where w solves (22), and thus $|\mathrm{D}w|(x)$ coincides with the mean curvature of the level set passing through x. Recalling that $|\{w = t\}| = |\partial\Omega_t| = |\partial\Omega|\,\mathrm{e}^t$ along the IMCF, it is easy to realise that the latter monotonicity is equivalent to the one in (2). We stress that, at least with the technology available so far, such a computation is purely formal, since w_p is converging to w only locally uniformly and w itself is nothing more than a weak solution to the IMCF. This is the best result available so far also under assumptions on Ω ensuring both w_p and w to be smooth, such as convexity. In particular, we emphasise that the techniques employed in Chapters 1 and 2 are fully self contained and free from any argument involving the IMCF or its weak formulation. The approximation of the IMCF through the p-capacitary potentials is somehow replaced by the simple (20), the proof of which, in $\mathbb{R}^n$, still will not use any of the IMFC approximation results of [Mos07; KN09; MRS19].

Coming back for a while to the linear setting, the above formal computations show that the achievement of the Willmore-type inequality (3) through the monotonicity of U_β along the level set flow of the solution u to (5), carried out in [AM20], as well as, consequently, the extension performed in Chapter 1, is quite surprising. Indeed, proceeding formally as above, the monotonicity of U_β with $\beta = (n-2)$ is equivalent to that of

$$t \to \int_{\{w=t\}} |\mathrm{D}w|^{n-1}\,\mathrm{d}\sigma,$$

where $w = -\log u$. On the other hand, the monotonicity of $\mathscr{W}$ defined in (4) along the IMCF would be formally recovered by letting $p \to 1^+$ in the monotonicity of U^p_β along the solution u_p to (16) with $\beta = (n-p)/(p-1)$ that is equivalent to the monotonicity of

$$t \to \int_{\{w_p=t\}} |\mathrm{D}w_p|^{n-1}\,\mathrm{d}\sigma,$$

where $w_p = -(p-1)\log u_p$, so that $w = w_2$. This sort of uniformity in p, that makes a limit as $p \to 1^+$ useless when looking after the Willmore-type inequality, is rigorously partially explained in Subsection 2.5.1, where we show, among some other things, that the Willmore-type inequality in $\mathbb{R}^n$ can actually be deduced from *any* value of p, the value $p = 2$ remaining thus the most convenient.

In Chapter 1 and 2, we are using the notion of strictly outward minimising hull outlined above twice. In the latter, we have already explicitly illustrated how it (actually, its area) will appear in the limit of the p-capacity of a bounded set with smooth boundary Ω. In Chapter 1, it will allow us to reduce the proof of the Isoperimetric Inequality (15) to the mean-convex sets. Chapter 3 is devoted to discuss this notion in considerable generality, in particular justifying the previous applications.

The Strictly Outward Minimising Hull in Riemannian manifolds

The main property one would like the strictly outward minimising hull Ω^* of a bounded set Ω to satisfy is that of being a bounded set minimising the perimeter among all of those sets containing Ω. This is actually the property needed in the argument of the proof of the Isoperimetric Inequality (15). However, easy examples show that in a general Riemannian ambient this is not always quite so, since Ω^*, roughly defined as the intersection of all the strictly outward minimising sets containing Ω, can in general happen to be the empty set. Indeed, manifolds with cuspidal or cylindrical ends, discussed in more details in Examples 3.8 and 3.9 provide examples of spaces containing bounded sets Ω that are not contained in any strictly outward minimising set, yielding no admissible sets in the intersection defining Ω^*. Moreover, cuspidal manifolds do not even admit solution to the least area problem with obstacle Ω, for any bounded Ω with finite perimeter contained in it, no matter the regularity of the boundary. On the other hand, the cylindrical manifold of Example 3.9 do admit solutions for the least area problem with obstacle Ω, but it happens that one cannot select, among these solutions, a bounded one with maximal volume. We realise in fact, in Theorem 3.13, that the well posedness of Ω^* is equivalent to the solvability of the maximum volume-least area problem with obstacle, and that this is in turn equivalent to the existence of an exhausting sequence of bounded strictly outward minimising sets.

If, on the one end, the condition of having such an exhausting sequence is not difficult to check on explicit metrics, such as warped products , on the other hand this general criterion does not seem easy to be directly applied to relevant classes of manifolds, for example, it is still not clear whether in complete noncompact manifolds with nonnegative Ricci curvature with Euclidean volume growth we can define the strictly outward minimising hull. The first achievement of Theorem 3.1 is that Ω^* is an open bounded maximal volume solution to the least area problem with obstacle a bounded set with finite perimeter Ω if one of the following two conditions are satisfied.

(i) The ambient manifold satisfies an Euclidean-like Isoperimetric Inequality, namely, there exists $C_{\rm iso} > 0$ such that

$$\frac{|\partial\Omega|^n}{|\Omega|^{n-1}} \geq C_{\rm iso} \tag{23}$$

for any bounded set Ω with smooth boundary;

(ii) the ambient manifold has nonnegative Ricci curvature and the *superlinear uniform volume growth condition* holds

$$C_{\rm vol}^{-1}\, r^b \leq |B(O,r)| \leq C_{\rm vol}\, r^b \tag{24}$$

for some $b > 1$ and $C_{\rm vol} > 0$, for any $r \geq R$ for some $R > 0$.

As we observe in Examples 3.8 and 3.9, the cylindrical and cuspidal manifolds described above coherently do not satisfy (i) nor (ii). However, the above assumptions are satisfied by a great variety of manifolds. First of all, let us remark that complete noncompact manifolds with nonnegative Ricci curvature satisfy condition (23) as pointed out above (see [Heb99, Theorem 8.4]), as well as Cartan-Hadamard manifolds, as a consequence of [HS74] (see [Heb99, Theorem 8.3] for an explicit derivation). We observe that the strictly outward minimising hull in Cartan-Hadamard manifolds was considered, for reasons that were analogue to ours in Chapter 1, in [Sch08]. Condition (23) is verified also on Asymptotically Flat ambient metrics, object of [HI01], and Asymptotically Locally Hyperbolic Riemannian manifolds, considered in [LN15]. This is a consequence of the fact that by the aforementioned [HS74] and the assumed asymptotic expansion of the metric (23) is valid outside a compact set, and thus, by [PST14, Theorem 3.2], this is true on the whole manifold. For what it concerns the assumptions in (ii) observe that they are naturally satisfied by complete noncompact Riemannian manifolds with nonnegative Ricci cuvature that are (locally) asymptotic to warped products $d\rho \otimes d\rho + \rho^{2\alpha} g_N$ for some smooth $n-1$ dimensional manifold N. In particular (ii) encompasses the case of the so called ALF and ALG manifolds, arising in the classification issue of gravitational instantons already hinted above. Moreover, the above warped products arise as models in the celebrated [CC96]. Concerning the relation between (i) and (ii), we point out that if $b < n$ (observe that by Bishop-Gromov $b \leq n$) they are mutually exclusive. Indeed, if the Isoperimetric Inequality holds, then it is known that so does the related Sobolev Inequality and thus by the arguments of [Car95] one can conclude that the volume of geodesic balls $|B(x,r)|$ grow at least as r^n, see [PST14, Proposition 3.1] for a self contained proof.

We emphasise that also the description of Ω^*, although very briefly outlined in [HI01, Section 1], was not, to the author's knowledge, rigorously carried out in literature even in the easier case of flat $\mathbb{R}^n$. In this framework, the most close study is that performed in [BT84], where similar properties were derived for a related notion the authors refer to as *minimal hull*. This notion differs subtly but fundamentally from that of the strictly outward minimising hull, since, in the present terminology, it coincides with the intersection of all the outward minimising sets containing Ω, not just the strictly outward minimising ones. Despite, as it will be clear by the arguments used in Chapter 3, the area of the resulting set will be the same, the volume in general differs, since Bassanezi-Tamanini's minimal hull will not in general yield a *maximal volume* solution to the least area problem with obstacle Ω. However, we acknowledge that some of the techniques presented [BT84] will be very useful in our arguments.

The proof differs sensibly when dealing with assumption (i) or (ii). Indeed, in the first case, we prove through the Direct Method in Calculus of Variations that the least area problem with obstacle Ω admits a maximal volume solution, and then obtaining, through the general criterion given in Theorem 3.13, the desired properties of Ω^*. The assumed Isoperimetric Inequality enters in an ODE argument [1] that shows uniform boundedness of the minimising sequence considered. On the other hand, the proof in case (ii) is an application of the existence theorem for weak IMCF developed in [MRS19, Theorem 1.8]. Indeed, once existence is established, the Huisken-Ilmanen's theory applies and show that level sets of this flow are strictly outward minimising, easily yielding the exhausting sequence of strictly outward minimising sets required by Theorem 3.13. It is important to notice that, anyway, in the case of nonnegative Ricci curvature with Euclidean volume growth, being (23) in force, the weak IMCF still does not enter the game, leaving the proof of the Isoperimetric Inequality free from such an expanding flow.

The last issue we face in Chapter 3 is that of the convergence as $p \to 1^+$ of the variational p-capacity of a bounded set with smooth boundary Ω to the area of its strictly outward minimising hull. In general this is not quite so. Indeed, we observe in Example 3.29 that the natural higher dimensional version of the 2-dimensional famous Hamilton's cigar [Ham88] admits a satisfactory notion of strictly outward minimising hull in the variational sense explained above for any bounded set Ω, but on the other hand the variational p-capacity of Ω is zero for any $p > 1$. The manifold in this example has nonnegative Ricci curvature and *linear* volume growth. In particular, it does not satisfy (i), and neither, sharply, (ii), since b in (24) is not allowed to be 1, that corresponds to linear volume growth. On the other hand we prove in Theorem 3.1 that if (i) or (ii) is satisfied then the claimed convergence takes place for any bounded set with smooth boundary. Again, the techniques in the most delicate part of the proof, that is $|\partial\Omega^*| \leq \liminf_{p\to 1^+} \mathrm{Cap}_p(\Omega)$, differs in relation to the assumption satisfied. If the ambient metric comes with an Isoperimetric Inequality, then an argument inspired by [Xu96] involving the Sobolev Inequality following from the Isoperimetric one does the job, yielding a self-contained proof. Assumption (i) is obviously satisfied in $\mathbb{R}^n$, that is what we needed when passing to the limit as $p \to 1^+$ in Chapter 2 in reaching the Extended Minkowski Inequality (17). If, on the other hand, we are assuming (ii), then we mostly rely on the decay estimates of the p-Green's function provided in [MRS19], in turn leading to the existence of the weak IMCF.

The present work ends with two appendices. Briefly, in Appendix A we systematically compare the monotonicity formulas for nonparabolic manifolds with nonnegative Ricci curvature of Chapter 1 with those for the Green's function obtained by Colding in [Col12] and subsequently extended in [CM14b]. In particular, in light of this, observe that Chapter 2 can be interpreted as a bridge between such monotonicity formulas, that according to [Col12] and [CM14a] can be understood as a sort of regularised version of the Bishop-Gromov monotonicity, and that of the function $\mathscr{Q}$ defined in (2) leading to the Minkowski inequality through Inverse Mean Curvature Flow.

In Appendix B, we give a proof of the existence of the p-capacitary potential under the sole hypothesis of p-nonparabolicity and decay at infinity of the p-Green's function,

[1]We thank Prof. G. P. Leonardi for having outlined this idea.

together with basic properties of the p-capacity. We are applying this result in Chapter 2 and 3.

Most of the achievements of Chapter 1 are contained in [AFM18], those of Chapter 2 are obtained through the combination of [FMP19] with [AFM19], while the main result of Chapter 3 is not yet available in literature. It will be the content of a forthcoming joint paper with L. Mazzieri.

Chapter 1

Geometric inequalities in nonnegative Ricci curvature via Linear Potential theory

1.1 Structure of the chapter

In Section 1.2, we review, for ease of the reader, the theory of harmonic functions on Riemannian manifolds with nonnegative Ricci curvature we are going to employ along this chapter. The most important results are the existence Theorems for exterior boundary value problems characterising respectively nonparabolic and parabolic manifolds with nonnegative Ricci curvature, namely Theorems 1.14 and 1.17. Although Theorem B.1 in Appendix B fully encompasses Theorem 1.14, we propose here a proof that in some sense is best suited for $\mathrm{Ric} \geq 0$. In Section 1.3 we introduce the conformal formulation of problem (1.2.15) and we prove, in this setting, (the conformal version of) Theorems 1.19 and 1.21, that are the Monotonicity-Rigidity Theorems in nonparabolic manifolds with nonnegative Ricci respectively for the function U_β and U_∞ already encountered in the Introduction. In Section 1.4 we work out the integral asymptotic estimates for the electrostatic potential on manifolds with nonnegative Ricci curvature. With these estimates at hand, we conclude the proof of Theorem 1.44, that is the Willmore-type inequality for manifolds with nonnegative Ricci curvature and Euclidean volume growth, and discuss its Corollary 1.49 for ALE manifolds. In Section 1.5, we consider some other consequences of the Monotonity-Rigidity Theorems, generalising in the present general setting results of [AM20] and [BMM19]. In Section 1.6 we turn our attention to parabolic manifolds, and prove a suitable Monotonicity-Rigidity Theorem in this setting. We then combine it with that for nonparabolic manifolds to get enhanced versions of the noncompact splitting theorem by Kasue/Croke-Kleiner, see Theorem 1.56. At the end of the section, we also show how to recover its compact version, see Theorem 1.59. Finally, in Section 1.7, we prove the Isoperimetric Inequality on complete noncompact Riemannian 3-manifolds with $\mathrm{Ric} \geq 0$. Finally, we give some perspectives and applications of such result, already mentioned in the Introduction.

1.2 Harmonic functions in exterior domains

In this section we are mainly concerned with characterising Riemannian manifolds for which problems (5) and (12) admit a solution. We are going to see that complete noncompact nonnegatively Ricci curved manifolds for which a solution to (5) exists are the *nonparabolic* ones, namely, manifolds admitting a positive Green's function, while those admitting a solution to (12) are the *parabolic* ones.

Nothing substantially new appears in this section. We are just collecting, re-arranging and applying classical results contained in [LT87; LT92; LT95; LY86; Yau75], and [Var82]. The interested reader might also refer to the nice survey [Gri99], where the relation with the Brownian motion on manifolds is also explored, or, for a more general account on the vast subject of harmonic functions on manifolds, to the lecture notes [Li] and the references therein. Other important works in this field will be readily cited along the chapter. Before starting, let us mention that the results gathered in this preliminary section are spread in a huge literature, and frequently they do not appear exactly in the form we need, or comes without a with a detailed proof. For this reason, we include the most relevant ones.

1.2.1 Green's functions and parabolicity

Let us begin with the definition of Green's functions on Riemannian manifolds.

Definition 1.1 (Green's function). *A smooth function*

$$G : (M \times M) \setminus \mathrm{Diag}(M) \to \mathbb{R}\,,$$

where $\mathrm{Diag}(M) = \{(O,O), O \in M\}$, *is said to be a Green's function for the Riemannian manifold* (M,g) *if the following requirements are satisfied.*

(i) $G(x,y) = G(y,x)$ *for any* $x, y \in M$, $x \neq y$.

(ii) $\Delta G(O,\cdot) = 0$ *on* $M \setminus \{O\}$, *for any* $O \in M$.

(iii) The following asymptotic expansion holds for $x \to O$*:*

$$G(O,x) \,=\, \big(1 + o(1)\big)\, d^{2-n}(O,x)\,. \tag{1.2.1}$$

It is well known that on a complete noncompact Riemannian manifold there always exists a Green's function. This result has been obtained for the first time by Malgrange in [Mal55], while a constructive proof, best suited for applications, has been given by Li-Tam in [LT87]. Complete noncompact Riemannian manifolds are then divided into two classes.

Definition 1.2 (Parabolicity). *Complete noncompact Riemannian manifolds which support a* positive *Green's function are called* nonparabolic. *Otherwise they are called* parabolic.

A by-product of Li-Tam's construction of Green's function gives the following very useful characterisation of parabolicity, see for example [Li, Theorem 2.3] for a proof.

Theorem 1.3 (Li-Tam). *Let* (M,g) *be a complete noncompact Riemannian manifold. Then, it is nonparabolic if and only if there exists a positive super-harmonic function* f *defined on the complement of a geodesic ball* $B(O,R)$ *such that*

$$\liminf_{d(O,x)\to\infty} f(x) < \inf_{\partial B(O,R)} f. \tag{1.2.2}$$

Notice that if (M,g) is a nonparabolic Riemannian manifold then a barrier function f as in Theorem 1.3 is just the function $G_{|M\setminus B(p,R)}$. A positive Green's function G is called *minimal* if

$$G(p,q) \leq \tilde{G}(p,q)$$

for any other positive Green's function $\tilde{G}$. The construction of the Green's function in [LT87] actually provides the minimal one.

The following theorem is a fundamental characterisation of parabolicity for manifolds with $\mathrm{Ric} \geq 0$ in terms of the volume growth of geodesic balls, first appeared in a complete version in [Var82].

Theorem 1.4 (Varopoulos). *Let (M, g) be a complete noncompact Riemannian manifold with $\mathrm{Ric} \geq 0$. Then (M, g) is nonparabolic if and only if*

$$\int_1^{+\infty} \frac{r}{|B(O,r)|}\, \mathrm{d}r < +\infty, \tag{1.2.3}$$

for any $O \in M$, where $B(O,r)$ is a geodesic ball centered at O with radius $r \geq 0$.

The above characterisation roughly says that on nonparabolic manifolds volumes are growing faster than quadratically, while on the parabolic ones they grow at most quadratically. On the other hand, on a complete noncompact Riemannian manifold with $\mathrm{Ric} \geq 0$ Bishop-Gromov's Theorem and a result of Yau [Yau76] respectively show that the growth of volumes of geodesic balls $B(p,r)$ is controlled from above by r^n and from below by r.

From now on we focus our discussion only on complete noncompact manifolds with nonnegative Ricci curvature. In the following two subsections we collect some basic though fundamental facts in this context, for the ease of references.

1.2.2 Harmonic functions on manifolds with nonnegative Ricci curvature.

A basic tool in the study of the potential theory on Riemannian manifolds is the following celebrated gradient estimate, first provided by Yau in [Yau75] (see also the nice presentation given in [SY94]).

Theorem 1.5 (Yau's Gradient Estimate). *Let (M, g) be a complete noncompact Riemannian manifold with $\mathrm{Ric} \geq 0$. Let u be a positive harmonic function defined on a geodesic ball $B(O, 2R)$ of center $O \in M$ and radius $2R$. Then, there exists a constant $\mathrm{C} = \mathrm{C}(n) > 0$ such that*

$$\sup_{x \in B(O,R)} \frac{|\mathrm{D}u|}{u} \leq \frac{C}{R}. \tag{1.2.4}$$

We now apply the above inequality to a harmonic function v defined in a geodesic annulus $B(O, R_1) \setminus \overline{B(O, R_0)}$. We obtain a decay estimate on the gradient of u that we will employ several times along this paper.

Proposition 1.6. *Let (M, g) be a complete noncompact Riemannian manifold with $\mathrm{Ric} \geq 0$, and let u be a positive harmonic function defined in a geodesic annulus $B(O, R_1) \setminus \overline{B(O, R_0)}$, with $R_1 > 3R_0$. Then, there exists a constant $\mathrm{C} = \mathrm{C}(n)$ such that*

$$|\mathrm{D}u|(x) \leq \mathrm{C}\, \frac{u(x)}{d(O,x)}, \tag{1.2.5}$$

for any point q such that $2R_0 \leq d(O,x) < \dfrac{R_1 + R_0}{2}$. In particular, if u is a harmonic function defined in $M \setminus \overline{B(O, R_0)}$, then

$$|\mathrm{D}u|(x) \leq C \frac{u(x)}{d(O,x)} \tag{1.2.6}$$

for any point x with $2R_0 \leq d(O,x)$.

Proof. Let q be such that $2R_0 \leq d(O,x) < \dfrac{R_1+R_0}{2}$. Then the ball $B\big(x,\, d(O,x)-r_0\big)$ is all contained in the annulus $B(O,R_1) \setminus \overline{B(O,R_0)}$. In particular, by Yau's inequality (1.2.4) we have

$$|\mathrm{D}u|(x) \leq \mathrm{C}\frac{u(x)}{d(O,x)-r_0} \leq 2\,\mathrm{C}\frac{u(x)}{d(O,x)}.$$

Letting $R_1 \to \infty$, we get also (1.2.6). □

A fundamental and classical application of Yau's gradient inequality is the Harnack's inequality, that we are going to apply several times in the sequel. The explicit dependencies of the Harnack constant on large annuli will often play an important role. As usual we state and prove it in the form and in the setting we are interested, mainly following [Li]

Proposition 1.7 (Harnack's inequality)**.** *Let (M,g) be a complete noncompact manifold with nonnegative Ricci curvature, and let $K \subset M$ be a* connected *compact subset, and let it be covered by a finite number of geodesic balls $B(x_i,R_i)$ with $x_i \in M$. Then, for u a harmonic function defined on an open neighbourhood of K, we have*

$$u(x) \;\leq\; e^{\,\mathrm{C_{har}}\,|\gamma_{x,y}|}\, u(y) \tag{1.2.7}$$

for any $x,y \in K$, where $\gamma_{x,y}$ is a curve joining x and y fully contained in K, and where the constant $\mathrm{C_{har}}$ satisfies

$$\mathrm{C_{har}} = \sup_i \mathrm{C}_i,$$

with C_i being such that $|\mathrm{D}\log u| \leq \mathrm{C}_i$ on $B(x_i,R_i)$.

Proof. The nonnegative Ricci curvature, ensures, by Yau's Theorem 1.5, the existence of C_i as in the statement. Let then $x,y \in K$, and let $\gamma_{x,y}$ be a curve joining x and y fully contained in K (it obviously exists by connectedness). Then, we have

$$\log u(x) - \log u(y) = \int_{\gamma_{x,y}} \mathrm{D}\log u \,\mathrm{d}s \leq \mathrm{C_{har}}|\gamma_{x,y}|.$$

Exponentiating, we obtain (1.2.7). □

An immediate application of the above Harnack inequality yields at once the following compactness theorem for sequences of harmonic functions. Again, the source of the proof is [Li]

Lemma 1.8. *Let (M,g) be a complete Riemannian manifold with* $\mathrm{Ric} \geq 0$*, and let $D \subset M$ be an open connected subset. Let $\{f_i\}$ be a sequence of positive harmonic functions defined on D, and suppose there exists a constant* C *such that $f_i(x) \leq \mathrm{C}$ at some point $x \in D$ for any $i \in \mathbb{N}$. Then, there exists a subsequence $\{f_{i_j}\}$ converging to a positive harmonic function f uniformly on any compact set $K \subset U$.*

Proof. It clearly suffices to prove the statement for connected compact $K \subset D$ containing x. By Proposition 1.7, we have

$$f_i(y) \leq f_i(x)e^{\mathrm{C}_1|\gamma|} \tag{1.2.8}$$

for some constant C_1 not depending on i. By the uniform bound on $f_i(x)$ and the compactness of K we then deduce that the sequence f_i is uniformly bounded in K. Combining (1.2.8) with the Yau's inequality (1.2.4) we also get that $|\mathrm{D}f_i|$ are uniformly bounded in K, and then by Ascoli-Arzelà a subsequence f_{i_j} converges uniformly on K to a continuous function f.

Let now $\psi \in \mathscr{C}_c^\infty(D)$ be a test function. Then, by harmonicity of f_{i_j} and uniform convergence on compact sets,

$$0 = \lim_{j\to\infty} \int_D f_{i_j} \Delta\psi \, d\mu = \int_D f \Delta\psi \, d\mu,$$

that is, f is harmonic in the sense of distribution. Standard regularity theory then implies that f is classically harmonic, completing the proof. □

1.2.3 Ends of manifolds with nonnegative Ricci curvature

It is a well-known and largely exploited fact that a complete noncompact Riemannian manifold (M, g) with nonnegative Ricci curvature that is not a Riemannian cylinder has just one end. However, since a complete proof of this fact is hard to find in standard literature, we discuss the details below.

We employ the following definition of end, that is, the one used in the works by Li and Tam, see for example [LT92, Definition 0.4 and discussion thereafter].

Definition 1.9 (Ends of Riemannian manifolds)**.** *An end of a Riemannian manifold (M, g) with respect to a compact subset $K \subset M$ is an unbounded connected component of $M \setminus K$. We say that (M, g) has a finite number of ends if the number of ends with respect to any compact subset $K \subset M$ is bounded by a natural number k independent of K. In this case, we say that (M, g) has k ends if it has k ends with respect to a compact subset $K \subset M$ and to any other compact subsets of M containing K.*

To state the result, we quickly recall some terminology. A *line* in (M, g) is a curve $\gamma : \mathbb{R} \to M$ which is a minimal geodesic between any two points lying on it. A *ray* is half a line. We also recall that a complete Riemannian manifold (M, g) is called a *Riemannian cylinder* if it is isometric to the Riemannian product $(\mathbb{R} \times N^{n-1}, dt \otimes dt + g_{N^{n-1}})$, where N^{n-1} is a compact manifold.

Remark 1.10. Using the above definition and terminology it is clear that if a Riemannian manifold has at least two ends, then it contains a line.

Proposition 1.11. *Let (M, g) be a complete noncompact Riemannian manifold with* Ric ≥ 0. *If (M, g) is not a Riemannian cylinder, then it has just one* end.

Proof. Since (M^n, g) has nonnegative Ricci curvature by hypothesis, then the Cheeger-Gromoll Splitting Theorem [CG71] implies that

$$(M, g) \text{ is isometric to } \left(\mathbb{R}^m \times N^{n-m}, g_{\mathbb{R}^k} + g_{N^{n-m}}\right), \tag{1.2.9}$$

for some $m \in \{0, \dots, n\}$, where the manifold $(N^{n-m}, g_{N^{n-m}})$ has nonnegative Ricci curvature and does not contain any line. The Riemannian manifold (M, g) is a Riemannian cylinder if $m = 1$ and N^{n-1} is compact. Let us then suppose that (M, g) is not a Riemannian cylinder. We consider the cases $m = 0$, $m = 1$ and $m \geq 2$.

Case $m = 0$. In this case, (M, g) does not contain any line. Then there is no more than one end, in view of Remark 1.10.

Case $m = 1$. Since (M, g) is not a cylinder, we have that N^{n-1} is a noncompact Riemannian manifold that contains no lines. Then again by Remark 1.10, it has at most one end. Thus, also $\mathbb{R} \times N^{n-1}$ has at most one end.

Case $m \geq 2$. We show that

$$M \setminus K \text{ is connected for every compact } K \subset M. \tag{1.2.10}$$

In view of Definition 1.9, this readily implies that M has at most one end. Now, to check (1.2.10) in view of (1.2.9), it is sufficient to check that for every compact $Q \in \mathbb{R}^m$ and every compact $P \in N^{n-m}$, we have that $(\mathbb{R}^k \times N^{n-m}) \setminus (Q \times P)$ is connected. Let then (x, q), $(y, p) \in (\mathbb{R}^m \times N^{n-m}) \setminus (Q \times P)$ and suppose for the moment that $x, y \in Q$, so that, in turn, $q, p \notin P$. Choose $z \in \mathbb{R}^m \setminus Q$ and define the curves

$$\alpha(t) = \big(tx + (1-t)z, q\big), \qquad \beta(t) = \big(ty + (1-t)z, p\big),$$

$t \in [0, 1]$, connecting (x, q) to (z, q) and (y, p) to (z, p), respectively. Note that $\alpha(t), \beta(t) \in (\mathbb{R}^m \times N^{n-m}) \setminus (Q \times P)$ for every $t \in [0, 1]$, because $q, p \notin Q$. Now, let $\gamma(t)$ be a continuous curve in N^{n-m} connecting q and p. Then the curve $\big(z, \gamma(t)\big) \in \big((\mathbb{R}^m \setminus Q) \times N^{n-m}\big)$ connects (z, q) to (z, p). Gluing together the curves α, β, and (z, γ), we obtain a continuous path lying in $(\mathbb{R}^m \times N^{n-m}) \setminus (Q \times P)$ and connecting (x, q) to (y, p). Obtaining such a curve in the case where either $x \notin Q$ or $y \notin Q$ requires a similar simpler construction. We have thus proved that $(\mathbb{R}^m \times N^{n-m}) \setminus (Q \times P)$ is path-connected, hence connected.

We proved that (M, g) has at most one end, if it is not a cylinder. Then, it has exactly one end, because it is noncompact. □

Let us now describe separately some aspects of harmonic functions on nonparabolic and parabolic manifolds. In particular, we are going to characterise these two classes of manifolds through a couple of existence results for solutions of suitable boundary value problems in exterior domains (see Theorems 1.14 and 1.17 below). The monotone quantities analysed in Theorems 1.19 and 1.54 are defined along the level sets of these solutions.

1.2.4 The exterior problem on nonparabolic manifolds

The following is a fundamental estimate proved by Li-Yau in [LY86].

Theorem 1.12 (Li-Yau). *Let (M, g) be a nonparabolic Riemannian manifold with* Ric ≥ 0. *Then, its minimal Green's function G satisfies*

$$\mathrm{C}^{-1} \int\limits_{d(O,x)}^{+\infty} \frac{r}{|B(O,r)|}\, \mathrm{d}t \;\leq\; G(O,x) \;\leq\; \mathrm{C} \int\limits_{d(O,x)}^{+\infty} \frac{r}{|B(O,r)|}\, \mathrm{d}t\,, \tag{1.2.11}$$

for some $\mathrm{C} = \mathrm{C}(n) > 0$.

Combining (1.2.3) with (1.2.11), we get that the minimal Green's function goes to 0 at infinity, i.e. for any fixed O in M

$$\lim_{d(O,y)\to\infty} G(O,y) = 0\,. \tag{1.2.12}$$

An easy application of Laplace Comparison Theorem then gives the following well known fact.

Lemma 1.13. *Let (M, g) be a nonparabolic Riemannian manifold with* Ric ≥ 0, *and let G be its minimal Green's function. Then, for any fixed pole $O \in M$ we have*

$$d^{2-n}(O,x) \leq G(O,x). \tag{1.2.13}$$

for any $x \neq O$ in M.

Proof. Let r be the function mapping a point x in M to $d(O,x)$. By the Laplacian Comparison Theorem, we have

$$\Delta r \;\leq\; \frac{n-1}{r}$$

in the sense of distributions (see e.g. [CLN06, Theorem 1.128]). Therefore, we have, in the sense of distributions,

$$\Delta r^{2-n} \;=\; (n-2)\left[(n-1)r^{-n} - r^{1-n}\Delta r\right] \;\geq\; 0\,, \tag{1.2.14}$$

and then the function $r^{2-n} - G(O,\cdot)$ is sub-harmonic. By the maximum principle, for any $\varepsilon > 0$ and $R > \varepsilon$

$$\max_{\overline{B(O,R)}\setminus B(O,\varepsilon)} \left(r^{2-n} - G(O,\cdot)\right) \;=\; \max_{\partial B(O,R)\cup\partial B(O,\varepsilon)} \left(r^{2-n} - G(O,\cdot)\right).$$

We conclude by passing to the limit as $\varepsilon \to 0$ and $R \to \infty$, taking into account the asymptotic behaviour at the pole O given by (1.2.1) and that $G \to 0$ at infinity, as observed in (1.2.12). ☐

Now, we characterise the existence of a solution to problem (8) with the nonparabolicity of the ambient manifold. Let us recall that, with respect to a bounded open subset $\Omega \subset M$ with smooth boundary, and denoting by O a generic reference point taken inside Ω, we consider

$$\begin{cases} \Delta u = 0 & \text{in } M \setminus \overline{\Omega} \\ u = 1 & \text{on } \partial\Omega \\ u(y) \to 0 & \text{as } d(O,y) \to +\infty, \end{cases} \tag{1.2.15}$$

The proof we give below uses some aspects of the nonnegative Ricci curvature, that we believe of independent interest. We refer the reader to Theorem B.1 for a more general argument involving boundary regularity estimates for solutions of elliptic equations.

Theorem 1.14. *Let (M,g) be a complete noncompact Riemannian manifold with* Ric ≥ 0*, and let $\Omega \subset M$ be a bounded open subset with smooth boundary. Then, there exists an unique solution to problem* (1.2.15) *if and only if (M,g) is nonparabolic.*

Proof. Most of the arguments here are suitable adaptation of some already presented in [LT92]. By Theorem 1.3 the existence of a solution to problem (1.2.15) implies nonparabolicity of M, since the restriction of u to $M \setminus B(O,R)$ with $\Omega \subset B(O,R)$ clearly satisfies condition (1.2.2).

Conversely, assume that M is nonparabolic, and consider an increasing sequence of radii $\{R_i\}_{i\in\mathbb{N}}$ such that $\Omega \subset B(O,R_1)$ and $R_i \to \infty$. Let, for any $i \in \mathbb{N}$, u_i be the solution to the following problem:

$$\begin{cases} \Delta u = 0 & \text{in } B(O,R_i) \setminus \overline{\Omega} \\ u = 1 & \text{on } \partial\Omega \\ u = 0 & \text{on } \partial B(O,R_i). \end{cases} \tag{1.2.16}$$

Let now G be the minimal positive Green's function, and consider the function $G(O,\cdot)$. Due to the Maximum Principle for harmonic functions and the boundary conditions in problem (1.2.16) we have that

$$0 \leq u_{Ri}(x) \leq \frac{G(O,x)}{\min_{\partial\Omega} G(O,\cdot)}\,, \tag{1.2.17}$$

for $x \in B(O, R_i)$. Let then K be a compact set contained in $M \setminus \overline{\Omega}$. We can clearly suppose without loss of generality that K is contained in $B(O, R_i) \setminus \overline{\Omega}$ for any i. Then, (1.2.17) and Lemma 1.8 give that u_i converges up to a subsequence to a harmonic function u on K. We now need to check the convergence of (a subsequence of) u_i on compact subsets K of $M \setminus \Omega$, that, is possibly containing portions of $\partial\Omega$. To see this, we show that $|\mathrm{D}u_i|$ is bounded uniformly in i also on these compact subsets. To see this, consider the P-function (see [PP79])

$$f_i = \frac{|\mathrm{D}u_i|}{u_i^{\frac{(n-1)}{(n-2)}}}.$$

It can be directly checked that the Bochner formula, and it is actually the content of (1.3.35) written in terms of the current metric g, that f_i is subsolution of an elliptic equation. In particular, once restricted on the annulus $B(O, R) \setminus \overline{\Omega}$ for some fixed $R > 0$ such that the ball contains $\overline{\Omega}$, we deduce from the Maximum Principle that the maximum value of f_i is assumed, for i big enough, on $\partial\Omega \cup B(O, R)$. Then, if K is a compact set of $M \setminus \Omega$, it suffices to consider separately the case where the maximum values of (a subsequence of) f_i are assumed on interior points x_i of $M \setminus \Omega$ *uniformly away* from $\partial\Omega$ and the case where the points x_i lie on $\partial\Omega$. In the first case, $\mathrm{D}u_i$ is uniformly bounded on K as an immediate consequence of Yau's (1.2.4) together with the uniform convergence of u_i on the compact subsets of $M \setminus \overline{\Omega}$. Assume then that the points x_i lie on $\partial\Omega$. Then, letting $\nu = -\mathrm{D}u_i/|\mathrm{D}u_i|$ the unit normal to $\partial\Omega$, we have $(\partial f_i/\partial\nu)(x_i) \leq 0$. On the other hand, recalling that $u_i = 1$ on $\partial\Omega$, it holds

$$\frac{\partial f_i}{\partial \nu} = 2\left[\frac{n-1}{n-2}|\mathrm{D}u_i|^3 - 2\mathrm{H}|\mathrm{D}u_i|^2\right],$$

on x_i, where we wrote the mean curvature H of $\partial\Omega$ in terms of the Hessian of u_i exactly as in (1.3.17). Using the Hopf's lemma, that ensures that $|\mathrm{D}u_i|$ does not vanish on $\partial\Omega$, we then get

$$\sup_K |\mathrm{D}u_i| \leq \frac{n-1}{n-2} \sup_{\partial\Omega} \mathrm{H},$$

that, is $|\mathrm{D}u_i|$ is uniformly bounded also in this case.

We have built a harmonic function defined on $M \setminus \Omega$ taking the value 1 on $\partial\Omega$. Finally, by (1.2.17), the uniform convergence of u_i to u and (1.2.12), we get $u(y) \to 0$ as $d(O, y) \to +\infty$, completing the proof of existence.

Uniqueness is checked as usual, considering for another solution v the harmonic function $u - v$, and satisfying $u - v \leq \varepsilon$ on (a smoothed out approximation of) $\partial\Omega \cup \partial B(O, R_\varepsilon)$ for R_ε big enough, for any $\varepsilon > 0$. The Maximum Principle then yields $u \leq v + \varepsilon$ on $B(O, R_\varepsilon) \setminus \overline{\Omega}$, that implies $u \leq v$ letting $\varepsilon \to 0$. Reversing the roles of u and v yields uniqueness. □

We conclude this section with the following easy lemma, which shows that we can control function u by the minimal Green's function G.

Lemma 1.15. *Let (M, g) be a nonparabolic Riemannian manifold with* Ric ≥ 0*, and let G be its minimal Green's function. Let u be a solution to* (1.2.15) *for some open and bounded set Ω with smooth boundary, and let $O \in \Omega$. Then, there exist constants $\mathrm{C}_1 = \mathrm{C}_1(\Omega) > 0$ and $\mathrm{C}_2 = \mathrm{C}_2(\Omega) > 0$ such that*

$$\mathrm{C}_1\, G(O, y) \leq u(y) \leq \mathrm{C}_2\, G(O, y) \tag{1.2.18}$$

on $M \setminus \Omega$. In particular

$$C_1\, d(O,y)^{2-n} \le u(y). \tag{1.2.19}$$

on $M \setminus \Omega$.

Proof. Just set $0 < C_1 < 1/\max_{\partial\Omega} G(O,\cdot)$, and $C_2 > 1/\min_{\partial\Omega} G(O,\cdot)$. The claim follows from the Maximum principle and the observation that both u and G are vanishing at infinity. The inequality (1.2.19) is obtained combining the lower estimate on u by (1.2.18) with (1.2.13). □

1.2.5 The exterior problem on parabolic manifolds

The following inequalities, proved in [LT95, Theorem 2.6], can be interpreted as a version for parabolic manifolds of the Li-Yau inequalities recalled in Theorem 1.12. We point out that we are always dealing with Green's functions obtained by the Li-Tam's construction.

Theorem 1.16. *Let (M,g) be a parabolic manifold with $\mathrm{Ric} \ge 0$, and let $O \in M$. Let G be a Green's function. Then, for any fixed $r_0 > 0$ and for any x with $d(O,x) > 2r_0$ there holds*

$$-\,G(O,x) \;\le\; C_1 \int_{r_0}^{d(O,x)} \frac{r}{|B(O,r)|}\, \mathrm{d}r \,+\, C_2\,, \tag{1.2.20}$$

for some constants C_1 and C_2 depending only on n, r_0 and the choice of G. Moreover, for any $R > r_0$, there holds

$$C_3 \int_{r_0}^{R} \frac{r}{|B(O,r)|}\, \mathrm{d}r \,+\, C_4 \;\le\; \sup_{\partial B(O,R)} -G(O,\cdot)\,, \tag{1.2.21}$$

for some constants C_3 and C_4 depending only on n, r_0 and the choice of G.

When (M,g) is parabolic, Li-Tam substantially proved in [LT92, Lemma 1.2] that the exterior problem

$$\begin{cases} \Delta\psi = 0 & \text{in } M \setminus \overline{\Omega} \\ \psi = 0 & \text{on } \partial\Omega \\ \psi(y) \to +\infty & \text{as } d(O,y) \to +\infty\,, \end{cases} \tag{1.2.22}$$

admits a solution. The construction of such a solution ψ, combined with Yau's inequality and Theorem 1.16, readily implies a uniform gradient bound on ψ.

Theorem 1.17. *Let (M,g) be a parabolic Riemannian manifold with $\mathrm{Ric} \ge 0$, and let $\Omega \subset M$ be a bounded and open subset with smooth boundary. Then, there exists a solution to problem (1.2.22). Moreover, $|\mathrm{D}\psi|$ is uniformly bounded in $M \setminus \Omega$.*

Remark 1.18. Recall from Subsection 1.2.3 that if (M,g) is a *Riemannian cylinder*, then $M \setminus \Omega$ might have two connected components. If this is the case, it will be understood that we consider problem (1.2.22) on a connected component of $M \setminus \Omega$. All the proofs work unchanged in this case.

Proof of Theorem 1.17. Let $O \in \Omega$, let $U \subset \Omega$ be an open neighbourhood of O and let K be the compact set defined by $K = \overline{\Omega} \setminus U$. Consider, for a sequence $B(O,R_i)$ of geodesic balls with increasing radii containing Ω, a corresponding sequence of positive Green's functions $G_i(O,\cdot)$ of $B(O,R_i)$ with pole in O such that $G_i(O,x)=0$ for $x\in\partial B(O,R_i)$. We then consider the sequence of functions defined in $B(O,R_i)\setminus\{O\}$ by

$$h_i(y) \;=\; \sup_{x\in K} G_i(O,x) - G_i(O,y).$$

The construction in [LT87] implies that there exists a Green's function G on M such that h_i converges to $-G(O,\cdot)$ uniformly on compact subsets of $M\setminus\{O\}$ (compare with the discussions around Lemma 1.2 in [LT92]). Observing that $h_i=\sup_K G_i(O,\cdot)$ on $\partial B(O,R_i)$, we set

$$a_i \;=\; \sup_{x\in K} G_i(O,x)$$

and consider the solution ψ_i to the problem

$$\begin{cases} \Delta\psi = 0 & \text{in } B(O,R_i)\setminus\overline{\Omega} \\ \psi = 0 & \text{on } \partial\Omega \\ \psi = a_i & \text{on } \partial B(O,R_i). \end{cases}$$

Since $\sup_K G_i(O,\cdot)\geq \sup_{\partial\Omega} G_i(O,\cdot)$, the Maximum Principle immediately gives

$$h_i - \sup_{\partial\Omega} h_i \;\leq\; \psi_i \;\leq\; h_i \tag{1.2.23}$$

on $B(O,R_i)\setminus\Omega$. Since the sequence h_i is converging (uniformly on compact sets) to the Li-Tam Green's function, the second inequality in (1.2.23) combined with Lemma 1.8 shows that ψ_i converges uniformly on the compact subsets of $M\setminus\overline{\Omega}$ to a harmonic function ψ. Moreover, since for every $y\in M\setminus\{O\}$ the sequence $h_i(y)$ converges to $-G(O,y)$ and since by (1.2.21) we have that $-G(O,y_j)\to+\infty$ along a sequence of points y_j such that $d(O,y_j)\to+\infty$, we use the first inequality in (1.2.23), to deduce that $\psi(y_j)\to+\infty$, as $j\to+\infty$. In particular, since by [CM97b, Lemma 3.40] ψ must admit a limit at infinity, we infer that $\psi(y)\to+\infty$, as $d(O,y)\to+\infty$ As in the proof of Theorem 1.14, we need to check the convergence of ψ_i also on compact $K\subset M\setminus\Omega$ possibly containing portions of the boundary $\partial\Omega$, by showing that the sequence of $|\mathrm{D}\psi_i|$ is uniformly bounded on such compact sets. To see this, let $B(O,R)$ big enough so that it contains $K\cup\Omega$, and consider the harmonic functions

$$v_i = 1-\frac{\psi_i}{\sup_{B(O,R)}\psi_i+1},$$

and define, as in the proof of Theorem 1.14, the P-functions

$$f_i=\frac{|\mathrm{D}v_i|}{v_i^{\frac{n-1}{n-2}}}.$$

Then, again, f_i is subsolution of an elliptic equation (that again, corresponds exactly to (1.3.35) in the original metric), and thus the maximum values of f_i are achieved on $x_i\in\partial\Omega\cup\partial B(O,R)$. If $x_i\in\partial B(O,R)$, we easily see that $|\mathrm{D}\psi_i|$ is uniformly bounded by Yau's inequality (1.2.4) and the uniform convergence of ψ_i on compact subsets of $M\setminus\overline{\Omega}$. If, on the other hand, $x_i\in\partial\Omega$, the very same computation performed in the proof of Theorem 1.14 involving the Hopf's lemma shows that

$$|\mathrm{D}\psi_i|\leq \mathrm{H}\left(1+\sup_{B(O,R)}\psi_i\right). \tag{1.2.24}$$

Since $\psi\to\infty$ at infinity, we can infer from the uniform convergence of ψ_i to ψ on compact subsets of $M \setminus \overline{\Omega}$ that the points where the functions ψ_i achieve the maximum value inside $B(O,R)$ are located, for i big enough, in a compact subset K' away from $\partial\Omega$, and then, again by the uniform convergence of ψ_i on K', (1.2.24) allows to conclude that $|D\psi_i|$ is uniformly bounded in this case too.

We are left to show that $|D\psi|$ is uniformly bounded. Observe that, again by (1.2.23), $\psi \leq -G(O,\cdot)$. Inequality (1.2.6) then yields

$$|D\psi|(x) \,\leq\, C\frac{\psi(x)}{d(O,x)} \,\leq\, C\frac{-G(O,x)}{d(O,x)} \tag{1.2.25}$$

for some constant C and any x outside some big geodesic ball $B(O,r_0)$. Combining now (1.2.20) with Yau's lower bound on the growth of geodesic balls, saying that $|B(O,r)| \geq C\,r$ for any $r \geq 1$ and for some constant C, we also have

$$\frac{-G(O,x)}{d(O,x)} \leq C_1\frac{d(O,x)+C_2}{d(O,x)}$$

for x with $d(O,x) > 2r_0$ and constants C_1 and C_2. Plugging it in (1.2.25), this shows that $|D\psi|$ is uniformly bounded, as claimed. □

1.3 The Monotonicity-Rigidity Theorems for nonparabolic manifolds

The goal of this section is to prove the monotonicity of the function $U_\beta : (0,1] \to \mathbb{R}$ for $\beta \geq (n-2)/(n-1)$ and $U_\infty : (0,1] \to \mathbb{R}$ already presented in the Introduction, defined respectively by

$$U_\beta(t) \,=\, t^{-\beta\left(\frac{n-1}{n-2}\right)} \int\limits_{\{u=t\}} |Du|^{\beta+1}\, d\sigma \tag{1.3.1}$$

and

$$U_\infty(t) = \sup_{\{u=t\}} \frac{|Du|}{u^{\frac{n-1}{n-2}}}. \tag{1.3.2}$$

The following is the statement of the Monotonicity-Rigidity Theorem for U_β.

Theorem 1.19 (Monotonicity-Rigidity Theorem for nonparabolic manifolds)**.** *Let (M,g) be a nonparabolic Riemannian manifold with $\mathrm{Ric} \geq 0$. Given a bounded and open subset $\Omega \subset M$ with smooth boundary, let u be the solution to problem (1.2.15) and let $U_\beta : (0,1] \to \mathbb{R}$ be the function defined in (1.3.1). Then, for every $\beta \geq (n-2)/(n-1)$, the function U_β is differentiable, with derivative*

$$\frac{dU_\beta}{dt}(t) \,=\, \beta\, t^{-\beta\left(\frac{n-1}{n-2}\right)} \int\limits_{\{u=t\}} |Du|^\beta \left[H - \left(\tfrac{n-1}{n-2}\right) |D\log u| \right] d\sigma, \tag{1.3.3}$$

where H *is the mean curvature of the level set* $\{u = t\}$ *computed with respect to the unit normal vector field* $\nu = -\mathrm{D}u/|\mathrm{D}u|$. *The derivative of* U_β *fulfils*

$$\begin{aligned} \frac{\mathrm{d}U_\beta}{\mathrm{d}t}(t) \;=\; \frac{\beta}{t^2}\int_{\{u<t\}} u^{2-\beta\left(\frac{n-1}{n-2}\right)}\,|\mathrm{D}u|^{\beta-2}\Big\{\; & \mathrm{Ric}(\mathrm{D}u,\mathrm{D}u) \\ & + |\mathrm{D}u|^2\left|\mathrm{h} - \frac{\mathrm{H}}{n-1}\,g^\top\right|^2 \\ & + \beta\,\big|\mathrm{D}^\top|\mathrm{D}u|\big|^2 \\ & + \big(\beta - \frac{n-2}{n-1}\big)\;|\mathrm{D}u|^2\left[\mathrm{H} - \left(\tfrac{n-1}{n-2}\right)|\mathrm{D}\log u|\,\right]^2 \Big\}\; \mathrm{d}\mu\,, \end{aligned} \tag{1.3.4}$$

where H *is the mean curvature of the level sets of* u *computed with respect to the unit normal vector field* ν. *In particular,* U_β *is nondecreasing. Moreover,* $(\mathrm{d}U_\beta/\mathrm{d}t)(t_0) = 0$ *for some* $t_0 \leq 1$ *and some* $\beta \geq (n-2)/(n-1)$ *if and only if* (M, g) *has Euclidean volume growth and* $\big(\{u \leq t_0\}, g\big)$ *is isometric to*

$$\left([r_0, +\infty) \times \{u = t_0\}\,,\ \mathrm{d}r \otimes \mathrm{d}r + \left(\frac{r}{r_0}\right)^2 g_{\{u=t_0\}}\right), \qquad \textit{with} \quad r_0 = \left(\frac{|\{u = t_0\}|}{\mathrm{AVR}(g)|\mathbb{S}^{n-1}|}\right)^{\frac{1}{n-1}}. \tag{1.3.5}$$

In this case, in particular, $\{u = t_0\}$ *is a connected totally umbilic submanifold with constant mean curvature.*

Remark 1.20 (Meaning of (1.3.3) at singular values). It is important to point out that when t is a singular value for u, (1.3.3) has to be interpreted as the (unique) extension of such quantity on regular values. This will turn to be allowable by Sard Theorem, and it is an important technical point of the proof. This agreement will be assumed throughout the whole chapter.

Analogously, the following is the Monotonicity-Rigidity Theorem for U_∞.

Theorem 1.21 (Monotonicity-Rigidity Theorem for U_∞). *Let* (M, g) *be a complete non-parabolic Riemannian manifold with* Ric ≥ 0. *Then, the function* $U_\infty : (0, 1] \to \mathbb{R}$ *defined by* (1.3.2) *is monotone nondecreasing. Moreover, if* $x_t \in \{u = t\}$ *is such that*

$$\frac{|\mathrm{D}u|}{u^{\frac{n-1}{n-2}}}(x_t) = \sup_{\{u=t\}} \frac{|\mathrm{D}u|}{u^{\frac{n-1}{n-2}}},$$

then

$$\mathrm{H} - \frac{n-1}{n-2}|\mathrm{D}\log u|(x_t) = -\frac{\partial}{\partial \nu_t}\log\frac{|\mathrm{D}u|}{u^{\frac{n-1}{n-2}}}(x_t) \geq 0, \tag{1.3.6}$$

where ν_t *is the unit normal to* $\{u = t\}$ *given by* $\nu_t = -\mathrm{D}u/|\mathrm{D}u|$. *Moreover,* $U_\infty(T) = U_\infty(t_0)$ *for some* $T < t_0$ *or inequality* (1.3.6) *holds with equality sign a* $t = t_0$ *with* $t_0 \in (0, 1]$ *if only if* (M, g) *has Euclidean volume growth and and* $\big(\{u \leq t_0\}, g\big)$ *is isometric to*

$$\left([r_0, +\infty) \times \{u = t_0\}\,,\ \mathrm{d}r \otimes \mathrm{d}r + \left(\frac{r}{r_0}\right)^2 g_{\{u=t_0\}}\right), \qquad \textit{with} \quad r_0 = \left(\frac{|\{u = t_0\}|}{\mathrm{AVR}(g)|\mathbb{S}^{n-1}|}\right)^{\frac{1}{n-1}}.$$

In this case, in particular, $\{u = t_0\}$ *is a connected totally umbilic submanifold with constant mean curvature.*

1.3.1 Preparation of the proof of the Monotonicity-Rigidity-Theorems

Before providing the proof of Theorems 1.19 and 1.21, we give a sharp and complete statement the refined Kato's inequality for harmonic functions, that is one of the main responsible of the monotonicity, and describe the conformal setting where we are actually going to work out the proofs.

A Kato-type identity and a related splitting principle

The classical Kato inequality for functions on a Riemannian manifold asserts that if f is a smooth function defined somewhere on a Riemannian manifold, then

$$\Big|\mathrm{D}|\mathrm{D}f|\Big|^2 \leq |\mathrm{DD}f|^2.$$

When, in addition, f satisfies some differential equation, the above inequality can be crucially improved. In the case of harmonic functions, the refined Kato inequality says that

$$\Big|\mathrm{D}|\mathrm{D}f|\Big|^2 \leq \left(\frac{n-1}{n}\right)|\mathrm{DD}f|^2.$$

Other refined Kato inequalities, suited for various elliptic equations, have been derived and applied successfully in the literature. We address the reader to [Her00] for an account on this subject.

In this work, and dramatically in the next chapter, we will not only use the inequality, but also the deficit from being an inequality will be fundamental. The following is the Kato-type identity for harmonic functions in Riemannian manifolds.

Proposition 1.22 (Kato-type identity for harmonic function)**.** *Let (M, g) be a Riemannian manifold , and let f be a harmonic function defined on some subset of M. Then, in an open neighbourhood of point where $|\nabla v| \neq 0$, the following identity holds true*

$$|\mathrm{DD}f|^2 - \left(\frac{n}{n-1}\right)\Big|\mathrm{D}|\mathrm{D}f|\Big|^2 = |\mathrm{D}f|^2\left|\mathrm{h} - \frac{\mathrm{H}}{n-1}\,g^{\top}\right|^2 + \left(\frac{n-2}{n-1}\right)\Big|\mathrm{D}^{\top}|\mathrm{D}f|\Big|^2. \tag{1.3.7}$$

Moreover, if f is nonconstant and the right hand side of (1.3.7) vanishes in $\{\rho_0 \leq f \leq \rho_1\}$ for some $\rho_0 < \rho_1$ with ρ_1 possibly infinite, then the Riemannian manifold $(\{\rho_0 \leq f \leq \rho_1\}, g)$ is isometric to the warped product $([\rho_0, \rho_1] \times \{f = \rho_0\}, \mathrm{d}\rho \otimes \mathrm{d}\rho + \eta^2(\rho) g_{\{f=\rho_0\}})$, where f, η and the coordinate ρ are related as

$$\eta(\rho) = \left(\frac{f'(\rho_0)}{f'(\rho)}\right)^{\frac{1}{n-1}} \tag{1.3.8}$$

We will not prove Proposition 1.22 here, since a more general statement will be proved in the next Chapter, see Proposition 2.5. It is anyway substantially known, and can be deduced for example by tracking the proof of [BC12, Proposition 5.1]. The splitting result arising from the vanishing of the right hand side of (1.3.7) is actually an interesting generalisation of a well known splitting result asserting that a function with null hessian in some region of a Riemannian manifold forces such a region to be a cylinder. This fact can be deduced from the more general principle discussed in [CC96, Section 1] and [CMM12, Theorem 1.1] or it can be directly proved as done in [AM15, Theorem 4.1 (i)]. Here, we derive it from Proposition 1.22.

Corollary 1.23. *Let (M, g) be a Riemannian manifold, and let f be a function defined on some subset of M that is nonconstant and satisfies $|\mathrm{DD}f| = 0$ on $\{\rho_0 \leq f \leq \rho_1\}$ for some $\rho_0 < \rho_1$*

with ρ_1 possibly infinite. Then, the Riemannian manifold $(\{\rho_0 \le f \le \rho_1\}, g)$ is isometric to the cylinder $([\rho_0, \rho_1] \times (\{f = \rho_0\}, \mathrm{d}\rho \otimes \mathrm{d}\rho + g_{|\{f=\rho_0\}})$ and, in this case, f is an affine function of ρ.

Proof. Clearly the function f is harmonic. Since $|\mathrm{D}|\mathrm{D}f|| \le |\mathrm{DD}f|$, the function $|\mathrm{D}f|$ is constant in $\{\rho_0 \le f \le \rho_1\}$, and this constant must be positive by assumption. Then, (1.3.7) holds, and since the left hand side is zero so is the right hand one, and in particular the rigidity statement of Proposition 1.22 applies. Moreover, deriving (1.3.8) and taking into account that $f''(\rho) = 0$, we find that $\eta'(\rho) = 0$. Since $\eta(\rho_0) = 1$, we conclude that η is the constant 1, that means cylindrical splitting. Moreover, again by (1.3.8), this implies that $f'(\rho) = f'(\rho_0)$, implying that f is an affine function of ρ. □

The conformal setting

Let (M, g) be a nonparabolic Riemannian manifold with nonnegative Ricci curvature. Let $\Omega \subset M$ be a bounded and open set with smooth boundary, and let u be the solution to problem (1.2.15). We introduce, in $M \setminus \Omega$ the metric

$$\tilde{g} = u^{\frac{2}{n-2}} g. \tag{1.3.9}$$

The expression for $\tilde{g}$ is formally the same as in [AM15] and [AM20]. Let us explain why such a conformal change of metric is natural also in the current setting. Our model geometry is that of a truncated cone

$$(M \setminus \Omega, g) \cong \left(\left[r_0, +\infty\right) \times \partial\Omega\,,\ \mathrm{d}r \otimes \mathrm{d}r + \left(\frac{r}{r_0}\right)^2 g_{\partial\Omega} \right), \tag{1.3.10}$$

for some positive constant r_0, and where $g_{\partial\Omega}$ is the metric induced by g on $\partial\Omega$. We also assume that $\partial\Omega$ is a smooth closed sub-manifold with $\mathrm{Ric}_{\partial\Omega} \ge (n-2) g_{\partial\Omega}$. Such a curvature assumption on $\partial\Omega$ is equivalent to suppose that the conical region in (1.3.10) has nonnegative Ricci curvature. In this model setting, the solution to problem (1.2.15) is $u(r) = (r/r_0)^{2-n}$. With this specific u, the metric $\tilde{g}$ becomes

$$\tilde{g} = \mathrm{d}\rho \otimes \mathrm{d}\rho + g_{\partial\Omega}\,,$$

where $\rho = \log r$. In other words $\tilde{g}$ is a (half) Riemannian cylinder over $(\partial\Omega, g_{\partial\Omega})$. In parallel, as the rigidity statement in Theorem 1.19 gives a characterisation of the truncated cone metrics (1.3.10), so its conformal version in Theorem 1.25 characterises cylindrical metrics.

Having this in mind, we are now going to describe the general features of $(M \setminus \Omega, \tilde{g})$ in more details. Letting

$$\varphi = -\log u\,, \tag{1.3.11}$$

we have that $\tilde{g} = \mathrm{e}^{-\frac{2\varphi}{n-2}} g$.

We denote by ∇, the Levi-Civita connection of the metric $\tilde{g}$, by $\nabla\nabla$ its Hessian, and we put the subscript $\tilde{g}$ on any other quantity induced by $\tilde{g}$. We have, for a smooth function w

$$\nabla_\alpha \nabla_\beta w = \mathrm{D}_\alpha \mathrm{D}_\beta w + \frac{1}{n-2}\Big(\partial_\alpha w \partial_\beta \varphi + \partial_\beta w \partial_\alpha \varphi - \langle \mathrm{D}w \,|\, \mathrm{D}\varphi \rangle g_{\alpha\beta}\Big),$$

where by $\langle \cdot, \cdot \rangle$ we denote the scalar product induced by g. In particular,

$$\Delta_{\tilde{g}} \varphi = 0\,. \tag{1.3.12}$$

Moreover, the Ricci tensor $\mathrm{Ric}_{\tilde g}$ of $\tilde g$ and the Ricci tensor Ric of g satisfy

$$\mathrm{Ric}_{\tilde g} \;=\; \mathrm{Ric} + \nabla_\alpha \nabla_\beta \varphi - \frac{d\varphi \otimes d\varphi}{n-2} + \frac{|\nabla \varphi|^2_{\tilde g}}{n-2}\, \tilde g\,. \tag{1.3.13}$$

Finally, by (1.3.12) and (1.3.13) problem (1.2.15) becomes

$$\begin{cases} \Delta_{\tilde g} \varphi = 0 & \text{in } M \setminus \overline{\Omega} \\ \mathrm{Ric}_{\tilde g} - \nabla\nabla\varphi + \dfrac{d\varphi \otimes d\varphi}{n-2} = \dfrac{|\nabla\varphi|^2_{\tilde g}}{n-2}\tilde g + \mathrm{Ric} & \text{in } M \setminus \overline{\Omega} \\ \varphi = 0 & \text{on } \partial\Omega \\ \varphi(x) \to +\infty & \text{as } d(O,x) \to +\infty. \end{cases} \tag{1.3.14}$$

The classical Bochner identity applied to φ in $(M \setminus \Omega, \tilde g)$, combined with the first two equations of the above system, immediately yields the following identity

$$\Delta_{\tilde g} |\nabla \varphi|^2_{\tilde g} - \langle \nabla |\nabla \varphi|^2_{\tilde g} \,|\, \nabla \varphi \rangle_{\tilde g} = 2 \left[\mathrm{Ric}(\nabla\varphi, \nabla\varphi) + |\nabla\nabla\varphi|^2_{\tilde g} \right], \tag{1.3.15}$$

where Ric is the Ricci tensor of the background metric g. Such a relation is at the heart of this work. As a first application, we state the following slight generalisation of Corollary 1.23 for solutions of (1.3.14) arising from solutions of (1.2.15).

Lemma 1.24. *Let (M, g) be a nonparabolic Riemannian manifold with $\mathrm{Ric} \geq 0$, let $\Omega \subset M$ be a bounded open subset with smooth boundary. Assume that $\nabla |\nabla \varphi|_{\tilde g} = 0$ on $\{\varphi \geq s_0\}$ for some $s_0 \in [0, +\infty)$. Then the Riemannian manifold $(\{\varphi \geq s_0\}, \tilde g)$ is isometric to the Riemannian product $\big([s_0, +\infty) \times \{\varphi = s_0\}, d\rho \otimes d\rho + \tilde g_{\{|\varphi = s_0\}}\big)$. In particular, $\partial\Omega$ is a connected totally geodesic submanifold inside $(M \setminus \Omega, \tilde g)$.*

Proof. It suffices to observe that plugging $\nabla |\nabla \varphi|_{\tilde g} = 0$ in (1.3.15) readily implies, since $\mathrm{Ric} \geq 0$, that

$$\nabla\nabla\varphi \;\equiv\; 0 \qquad \text{in} \quad \{\varphi \geq s_0\}.$$

The isometry of $(M \setminus \Omega, \tilde g)$ with the claimed Riemannian product then follows from Corollary 1.23.

□

We now briefly record some of the main relations among geometric quantities induced by the two metrics. We omit the computations, since they are straightforward and completely analogous to those carried out in [AM15] and [AM20]. First, observe that

$$|\nabla \varphi|_{\tilde g} = \frac{|\mathrm{D}u|}{u^{\frac{n-1}{n-2}}}\,. \tag{1.3.16}$$

Let H and $\mathrm{H}_{\tilde g}$ be the mean curvatures of the level sets of u, that coincide with those of φ, respectively in the Riemannian manifold $(M \setminus \Omega, g)$ and in $(M \setminus \Omega, \tilde g)$. They are computed using the unit normal vectors $-\mathrm{D}u/|\mathrm{D}u|$ and $\nabla\varphi/|\nabla\varphi|_{\tilde g}$, respectively. Exploiting the g-harmonicity and $\tilde g$-harmonicity of u and φ, we obtain that

$$\mathrm{H} \;=\; \frac{\mathrm{DD}u(\mathrm{D}u, \mathrm{D}u)}{|\mathrm{D}u|^3}\,, \qquad \mathrm{H}_{\tilde g} \;=\; -\frac{\nabla\nabla\varphi(\nabla\varphi, \nabla\varphi)}{|\nabla\varphi|^3_{\tilde g}}\,. \tag{1.3.17}$$

These quantities are related as follows

$$\mathrm{H}_{\tilde{g}} \;=\; u^{-\frac{1}{n-2}}\left[\mathrm{H} - \left(\frac{n-1}{n-2}\right)\frac{|\mathrm{D}u|}{u}\right]. \tag{1.3.18}$$

Letting $\mathrm{d}\sigma_{\tilde{g}}$ and $\mathrm{d}\mu_{\tilde{g}}$ denote respectively the area and the volume elements naturally induced by $\tilde{g}$ on $M \setminus \Omega$, we have

$$\mathrm{d}\sigma_{\tilde{g}} \;=\; u^{\frac{n-1}{n-2}}\,\mathrm{d}\sigma\,, \qquad \mathrm{d}\mu_{\tilde{g}} \;=\; u^{\frac{n}{n-2}}\,\mathrm{d}\mu\,. \tag{1.3.19}$$

Clearly, the first identity in (1.3.19) has to be understood with respect to a hypersurface N in M. Finally, for every $\beta \geq 0$, define the conformal analogue of the U_β as the function $\Phi_\beta : [0,+\infty) \longrightarrow \mathbb{R}$ mapping

$$\Phi_\beta(s) \;=\; \int\limits_{\{\varphi=s\}} |\nabla\varphi|_{\tilde{g}}^{\beta+1}\,\mathrm{d}\sigma_{\tilde{g}}. \tag{1.3.20}$$

The functions U_β and Φ_β, and their derivatives are related to each other as follows

$$\begin{aligned} U_\beta &= \Phi_\beta(-\log t), \\ -tU'_\beta(t) &= \Phi'_\beta(-\log t)\,, \end{aligned} \tag{1.3.21}$$

for $0 < t \leq 1$. The following theorems are the conformal versions of the Monotonicity-Rigidity Theorem 1.19.

Theorem 1.25. *Let (M,g) be a nonparabolic Riemmanian manifold with $\mathrm{Ric} \geq 0$. Let $\Omega \subset M$ be a bounded and open subset with smooth boundary, and let Φ_β be defined as in (1.3.20). Then, for every $\beta \geq (n-2)/(n-1)$, the function Φ_β is differentiable with derivative*

$$\frac{\mathrm{d}\Phi_\beta}{\mathrm{d}s}(s) \;=\; -\,\beta\int\limits_{\{\varphi=s\}} |\nabla\varphi|_{\tilde{g}}^{\beta}\,\mathrm{H}_{\tilde{g}}\,\mathrm{d}\sigma_g\,, \tag{1.3.22}$$

where H_g is the mean curvature of the level set $\{\varphi = s\}$ computed with respect to the unit normal vector field $\nu_{\tilde{g}} = \nabla\varphi/|\nabla\varphi|_{\tilde{g}}$. Moreover, for every $s \geq 0$, the derivative fulfils

$$\begin{aligned} \frac{\mathrm{d}\Phi_\beta}{\mathrm{d}s}(s) &= -\,\beta\,\mathrm{e}^{s}\int\limits_{\{\varphi\geq s\}} \frac{|\nabla\varphi|_{\tilde{g}}^{\beta-2}\Big\{\mathrm{Ric}(\nabla\varphi,\nabla\varphi) + |\nabla\nabla\varphi|_{\tilde{g}}^{2} + (\beta-2)\,\big|\nabla|\nabla\varphi|_{\tilde{g}}\big|_{\tilde{g}}^{2}\Big\}}{\mathrm{e}^{\varphi}}\,\mathrm{d}\mu_{\tilde{g}} \\ &= -\,\beta\,\mathrm{e}^{s}\int\limits_{\{\varphi\geq s\}} \mathrm{e}^{-\varphi}\,|\nabla\varphi|_{\tilde{g}}^{\beta-2}\Big\{\mathrm{Ric}(\nabla\varphi,\nabla\varphi) + \left(\beta - \frac{n-2}{n-1}\right)\big|\nabla|\nabla\varphi|_{\tilde{g}}\big|_{\tilde{g}}^{2} \\ &\qquad + |\nabla\varphi|^2\left|\mathrm{h}_{\tilde{g}} - \frac{\mathrm{H}}{n-1}\tilde{g}^{\top}\right|^2 + \left(\frac{n-2}{n-1}\right)\left|\nabla^{\top}|\nabla\varphi|\right|^2\Big)\Big\}\,\mathrm{d}\mu_{\tilde{g}}, \end{aligned} \tag{1.3.23}$$

where the tangential elements are referred to the level sets of φ. In particular, $\mathrm{d}\Phi_\beta/\,\mathrm{d}s$ is always nonpositive. Moreover, $(\mathrm{d}\Phi_\beta/\,\mathrm{d}s)(s_0) = 0$ for some $s_0 \geq 0$ and some $\beta \geq (n-2)(n-1)$ if and only if $(\{\varphi \geq s_0\},\tilde{g})$ is isometric to the Riemannian product $\Big([s_0,+\infty) \times \{\varphi = s_0\},\, \mathrm{d}\rho \otimes \mathrm{d}\rho + \tilde{g}_{\{|\varphi=s_0\}}\Big)$. In particular, $\{\varphi = s_0\}$ is a connected totally geodesic submanifold.

On the other hand, letting $\Phi_\infty : [0, +\infty) \to \mathbb{R}$ be defined by

$$\Phi_\infty(s) = \sup_{\{\varphi = s\}} |\nabla \varphi|, \tag{1.3.24}$$

the conformal version of Theorem 1.21 reads as follows.

Theorem 1.26. *Let (M, g) be a nonparabolic Riemmanian manifold with $\mathrm{Ric} \geq 0$. Let $\Omega \subset M$ be a bounded and open subset with smooth boundary. Then, the function $\Phi_\infty : [0, \infty) \to \mathbb{R}$ defined by (1.3.24) is monotone nonincreasing. Moreover, if $x_s \in \{\varphi = s\}$ is such that*

$$|\nabla \varphi|(x_s) = \sup_{\{\varphi = s\}} |\nabla \varphi|,$$

then

$$\mathrm{H}(x_s) = -\frac{\partial}{\partial \nu_s} \log |\nabla \varphi|(x_s) \geq 0, \tag{1.3.25}$$

where ν_s is the unit normal to $\{\varphi = s\}$ given by $\nu_s = \mathrm{D}\varphi / |\mathrm{D}\varphi|$. Moreover, equality holds in the inequality of (1.3.25) for some $s = s_0 \in [0, \infty)$ or $\Phi_\infty(S) = \Phi_\infty(s_0)$ for some $S > s_0$ if and only if $(\{\varphi \geq s_0\}, \tilde{g})$ is isometric to the Riemannian product $([s_0, +\infty) \times \{\varphi = s_0\}, \mathrm{d}\rho \otimes \mathrm{d}\rho + \tilde{g}_{\{\varphi = s_0\}})$. In this case, in particular, $\partial\Omega$ is a connected totally geodesic submanifold.

We would like to highlight the full and striking formal analogy between the two statements above and Theorems 1.54 and 1.55, that are the Monotonicity-Rigidity Theorems for parabolic manifolds. As we are going to see later, the proofs also will follow the very same scheme.

Proofs of Theorems 1.19 and 1.21 after Theorems 1.25 and 1.26. Deducing formulas (1.3.3) and (1.3.4) from formulas (1.3.22) and (1.3.23) is just a matter of lengthy but straightforward computations carried out using the relations between g and $\tilde{g}$ recalled above. We sketch the main steps. First, compute

$$|\nabla\nabla\varphi|^2_{\tilde{g}} = u^{-\frac{4}{n-2}} \left\{ \left| \frac{\mathrm{DD}u}{u} \right|^2 + \frac{n(n-1)}{(n-2)^2} \left| \frac{\mathrm{D}u}{u} \right|^4 - \frac{2n}{n-2} \left| \frac{\mathrm{D}u}{u} \right|^3 \mathrm{H} \right\}, \tag{1.3.26}$$

where H is defined as in (1.3.17), and

$$|\nabla|\nabla\varphi|_{\tilde{g}}|^2_{\tilde{g}} = u^{-\frac{4}{n-2}} \left\{ \left| \frac{\mathrm{D}|\mathrm{D}u|}{u} \right|^2 + \left(\frac{n-1}{n-2} \right)^2 \left| \frac{\mathrm{D}u}{u} \right|^4 - 2 \left(\frac{n-1}{n-2} \right) \left| \frac{\mathrm{D}u}{u} \right|^3 \mathrm{H} \right\}. \tag{1.3.27}$$

By (1.3.26) and (1.3.27), we can write

$$\begin{aligned} |\nabla\nabla\varphi|^2_{\tilde{g}} + (\beta - 2)|\nabla|\nabla\varphi|_{\tilde{g}}|^2_{\tilde{g}} = u^{-\frac{4}{n-2}} \Bigg\{ & \frac{|\mathrm{DD}u|^2 - (\frac{n}{n-1})|\mathrm{D}|\mathrm{D}u||^2}{u^2} + \\ & + \left(\beta - \frac{n-1}{n-2} \right) \left[\left| \frac{\mathrm{D}|\mathrm{D}u|}{u} \right|^2 + \left(\frac{n-1}{n-2} \right)^2 \left| \frac{\mathrm{D}u}{u} \right|^4 - 2 \left(\frac{n-1}{n-2} \right) \left| \frac{\mathrm{D}u}{u} \right|^3 \mathrm{H} \right] \Bigg\}. \end{aligned} \tag{1.3.28}$$

Now, considering a orthonormal frame as $\{e_1, \ldots, e_{n-1}, \mathrm{D}u/|\mathrm{D}u|\}$, where the first $n-1$ vectors are tangent to the level sets of u, we can decompose

$$\left| \frac{\mathrm{D}|\mathrm{D}u|}{u} \right|^2 = \left| \frac{\mathrm{D}u}{u} \right|^2 \mathrm{H}^2 + \sum_{j=1}^{n-1} \left\langle \frac{\mathrm{D}|\mathrm{D}u|}{u} \middle| e_j \right\rangle^2. \tag{1.3.29}$$

Plugging the above decomposition into (1.3.28), we obtain, with the aid of some algebra,

$$
\begin{aligned}
|\nabla\nabla\varphi|^2_{\tilde g} + (\beta-2)\big|\nabla|\nabla\varphi|_{\tilde g}\big|^2_{\tilde g} \;=\; u^{-\frac{4}{n-2}} \Big\{ & \Big[\, \big|\mathrm{D}\mathrm{D}u\big|^2 - \big(\tfrac{n}{n-1}\big)\big|\mathrm{D}|\mathrm{D}u|\big|^2 \,\Big] \\
& + \big(\beta - \tfrac{n-2}{n-1}\big)\; \big|\mathrm{D}^\top|\mathrm{D}u|\big|^2 \\
& + \big(\beta - \tfrac{n-2}{n-1}\big)\; |\mathrm{D}u|^2 \big[\, \mathrm{H} - \big(\tfrac{n-1}{n-2}\big)|\mathrm{D}\log u|\,\big]^2 \Big\},
\end{aligned}
\tag{1.3.30}
$$

where

$$
\big|\mathrm{D}^\top|\mathrm{D}u|\big|^2 \;=\; \sum_{j=1}^{n-1} \left\langle \frac{\mathrm{D}|\mathrm{D}u|}{u} \,\middle|\, e_j \right\rangle^2 .
$$

The monotonicity formula (1.3.4) now follows easily by pluggin (1.3.7) into the right hand side of (1.3.30). On the other hand, (1.3.3) follows from (1.3.22) by an easy computation using directly (1.3.16), (1.3.18) and (1.3.19).

With regard to the relation between the monotonicity U_∞ and Ψ_∞, they are obviously equivalent since $|\nabla\varphi|_{\tilde g} = |\mathrm{D}u|/u^{(n-1)/(n-2)}$. On the other hand, (1.3.6) follows from (1.3.25) using directly (1.3.16) and (1.3.18).

Assume now that $U'_\beta(t_0) = 0$ or the inequality in (1.3.6) holds with equality sign at x_{t_0} for some $t_0 \in (0,1]$ and some $\beta \geq (n-2)/(n-1)$. Then, by (1.3.21) or by (1.3.16) respectively, we deduce that $\Phi'_\beta(-\log t_0) = 0$ or equality holds in the inequality of (1.3.25) respectively. Similarly, if $U_\infty(t_1) = U_\infty(t_0)$ then $\Phi_\infty(-\log t_0) = \Phi_\infty(-\log t_1)$. In all of these cases, the cylindrical rigidity statements of Theorems 1.25 and 1.26 are in force, and we are thus left to show that they imply the conical splitting of Theorems 1.19 and 1.21. To see this, observe first that that φ is an affine function of the coordinate s, and we write it as $\varphi = \mathrm{a}\, s + \mathrm{b}$, for some constants a and b. Observe that $a \neq 0$ due to the nonconstancy of φ (that follows from (1.3.14)). By the definition of $\tilde g$ given by (1.3.9), we have that the metric g on $\{u \leq t_0\}$, with $t_0 = \mathrm{e}^{-s_0}$ is

$$
g \;=\; \mathrm{A}\, \mathrm{e}^{\frac{2s}{n-2}} \left(\mathrm{d}s \otimes \mathrm{d}s + \tilde g_{|\{\varphi = s_0\}} \right),
$$

for some positive constant A. Define now the coordinate r by

$$
\mathrm{d}r \;=\; \sqrt{\mathrm{A}}\, \mathrm{e}^{\frac{s}{n-2}}\, \mathrm{d}s,
$$

so that

$$
g \;=\; \mathrm{d}r \otimes \mathrm{d}r + \left(\frac{r}{r_0}\right)^2 g_{\{u=t_0\}}, \tag{1.3.31}
$$

with $\{u \leq t_0\} = \{r \geq r_0\}$, so that $\{r = r_0\} = \{u = t_0\}$, where we also used that by the constancy of the conformal factor in the definition of $\tilde g$ on the level set of φ, the metric $\tilde g$ and g are proportional when restricted to a level set of φ. The structure of g displayed in (1.3.31) is that of a (truncated) cone with cross section given by $\{u = t_0\}$, that in particular has Euclidean volume growth. We are then left to show that r_0 assumes the value claimed in (1.3.5). To compute it, observe that for $r_0 \leq r \leq R$, the set $\{r_0 \leq r \leq R\}$ is a geodesic annulus, and so we can treat $\{r = R\}$ as a geodesic sphere $\partial B(O, R)$ centered at some $O \in M$. We have

$$
|\partial B(O,R)| \;=\; \int\limits_{\partial B(O,R)} \mathrm{d}\sigma \;=\; \int\limits_{\{u=t_0\}} \left(\frac{R}{r_0}\right)^{n-1} \sqrt{\det g^{\Sigma}_{ij}}\; \mathrm{d}\vartheta^1 \dots \mathrm{d}\vartheta^{n-1} \;=\; \left(\frac{R}{r_0}\right)^{n-1} |\{u = t_0\}|,
$$

so that we can explicitly express r_0 in terms of $\mathrm{AVR}(g)$ as in (1.3.5) by

$$\mathrm{AVR}(g) = \lim_{R\to+\infty} \frac{|\partial B(O,R)|}{R^{n-1}|\mathbb{S}^{n-1}|} = \frac{|\{u = t_0\}|}{r_0^{n-1}|\mathbb{S}^{n-1}|},$$

where the first identity is a well known characterisation of $\mathrm{AVR}(g)$ in terms of areas of geodesic spheres instead of volumes of geodesic balls (see Subsection 1.4.1 , it follows easily by the standard proof of the Bishop-Gromov Theorem (see e.g. [Pet06]). The proof of the second claim is completed. □

The computations we performed above show that the derivative of U_β (1.3.4) can also be written displaying the quantity on the right hand side of (1.3.30), that manifests the role of the Kato inequality in the monotonicity. This is actually how the monotonicity formula was presented in [AFM18]. Here, we preferred to plug (1.3.7) in order to ease the comparison with the (effective) monotonicity of the nonlinear counterpart worked out in Chapter 2. See in particular Theorem 2.23.

1.3.2 Proof of Theorems 1.25 and 1.26

In what follows, we are always referring to a background nonparabolic Riemannian manifold (M, g) with $\mathrm{Ric} \geq 0$. Outside a bounded and open subset $\Omega \subset M$ with smooth boundary, we define the conformal metric $\tilde{g}$ as in (1.3.9). In particular, with the same notations as in the previous subsection, $(M \setminus \Omega, \tilde{g}, \varphi)$ is a solution to (1.3.14).

We first prove (the conformal version of) Theorem 1.21, that is the Monotonicity-Rigidity Theorem for Φ_∞, since it is considerably easier and gives as a corollary the uniform boundedness of $|\nabla\varphi|$, a property that will be important also in the proof of (the conformal version of) Theorem 1.19.

Proof of Theorems 1.26. We first show that there exists a constant $\mathrm{C} = \mathrm{C}(g, \Omega) > 0$ such that $|\nabla\varphi|_{\tilde{g}} \leq \mathrm{C}$ in $M \setminus \Omega$. Fixing a reference point O inside Ω, and letting $d(O, \cdot)$ be the distance from this point with respect to the metric g, we have, by (1.2.6), that

$$|\mathrm{D}u|(y) \leq \mathrm{C}\,\frac{u(y)}{d(O,y)}$$

outside some ball containing Ω. Then,

$$|\nabla\varphi|_{\tilde{g}}(y) = \frac{|\mathrm{D}u|}{u^{\frac{n-1}{n-2}}}(y) \leq \mathrm{C}\,\frac{u^{-\frac{1}{n-2}}(y)}{d(O,y)} \leq \mathrm{C}\,\mathrm{C}_1^{-\frac{1}{n-2}},$$

where in the last inequality we used (1.2.19). Consider now, for a given constant $\alpha > 0$, the auxiliary function

$$w_\alpha = |\nabla\varphi|^2_{\tilde{g}}\mathrm{e}^{-\alpha\varphi}.$$

Observe that by the just proved upper bound on $|\nabla\varphi|_{\tilde{g}}$ we have $w_\alpha(q) \to 0$ as $d(O,q) \to +\infty$ for any $\alpha > 0$. Moreover, a direct computation combined with (1.3.15) shows that w_α satisfies the identity

$$\Delta_{\tilde{g}} w_\alpha - (1-2\alpha)\,\langle \nabla w_\alpha \,|\, \nabla\varphi\rangle_{\tilde{g}} = 2\,\mathrm{e}^{-\alpha\varphi}\left[\mathrm{Ric}(\nabla\varphi, \nabla\varphi) + |\nabla\nabla\varphi|^2_{\tilde{g}}\right] + \alpha(1-\alpha)|\nabla\varphi|^2_{\tilde{g}} w_\alpha. \tag{1.3.32}$$

In particular, for any $\alpha \in (0,1)$ the right hand side above is nonnegative.

For the time being, we let s be a regular value. Then, applying the Maximum Principle to w_α, in force by (1.3.32), one gets that for every $y \in \{\varphi \geq s\}$ it holds

$$w_\alpha(y) \leq \sup_{\{\varphi = s\}} |\nabla \varphi|_{\tilde{g}} \,.$$

Letting $\alpha \to 0^+$ in the above inequality yields

$$|\nabla \varphi|_{\tilde{g}}(y) \leq \sup_{\{\varphi = s\}} |\nabla \varphi|_{\tilde{g}},$$

for every $y \in \{\varphi \geq s\}$. that in particular implies the monotonicity of Φ_∞. It is now easy to realise that

$$|\nabla \varphi|^2_{\tilde{g}} \mathrm{H}(x_s) = -\frac{1}{2}\frac{\partial}{\partial \nu_s}|\nabla \varphi|^2_{\tilde{g}} \geq 0, \tag{1.3.33}$$

where $x_s \in \{\varphi = s\}$ is the maximum point of $|\nabla \varphi|_{\tilde{g}}$ in $\{\varphi \geq s\}$. Indeed, the equality in (1.3.33) follows immediately from a direct computation with the second identity of (1.3.17), while the inequality immediately follows from the definition of x_s to be a maximum point.

Let us consider now the rigidity statements. Assume that $\Phi_\infty(S) = \Phi_\infty(s_0)$. Then, the maximum value of $|\nabla \varphi|_{\tilde{g}}$ in $\{\varphi \geq s_0\}$ is attained at some interior point $y \in \{\varphi \geq s_0\}$. Then, by the Strong Maximum Principle , $|\nabla \varphi|^2_{\tilde{g}}$ must be constant, since it satisfies

$$\Delta_{\tilde{g}} |\nabla \varphi|^2_{\tilde{g}} - \left\langle \nabla |\nabla \varphi|^2_{\tilde{g}} \,|\nabla \varphi \right\rangle_{\tilde{g}} \geq 0\,. \tag{1.3.34}$$

Lemma 1.24 then yields the desired conclusion.

On the other hand, if $|\nabla \varphi|^2_{\tilde{g}}$ is not constant in $\{\varphi \geq s\}$, the above strong maximum principle argument implies that x_s is a point of strict global maximum, and then the Hopf lemma (again in force due to (1.3.34)) immediately yields that (1.3.33) holds with a strict inequality sign.

We are left to discuss the case of singular level sets, that is, admitting a subset $\mathrm{Crit}_s \subset \{\varphi = s\}$ (by [HS89] of Hausdorff dimension at most $n-2$, compare also with [CNV15]) where $|\nabla \varphi|_{\tilde{g}} = 0$. This situation is easily controlled by Sard's Theorem, ensuring that this can occur only for a set of values of measure zero. Indeed, assume that s is a critical value, that is, $\{\varphi = s\}$ is a critical level set, and let $y \in \{\varphi > s\}$. Then, by the Sard's Theorem recalled above, there exists a sequence of smooth level sets $\{\varphi = s_j\}$ with $s_j \to s$ as $j \to \infty$ satisfying $y \in \{\varphi > s_j\}$. By the proof given above for smooth level sets, we infer

$$|\nabla \varphi|^2_{\tilde{g}}(y) \leq \sup_{\{\varphi = s_j\}} |\nabla \varphi|^2_{\tilde{g}},$$

that through passing to the limit yields the monotonicity of Φ_∞ also through singular level sets. If $\Phi_\infty(s_0) = \Phi_\infty(s_1)$ for some $s_1 > s_0$, with s_0 a possibly singular value, the monotonicity yields that $\Phi_\infty(s_0) = \Phi_\infty(s_j)$ for a sequence of $s_1 < s_j < s_0$ as above, implying that $|\nabla \varphi|_{\tilde{g}}$ is a positive constant on $\{\varphi \geq s_j\}$. In particular, by the continuity of $|\nabla \varphi|_{\tilde{g}}$, the level set $\{\varphi = s_0\}$ is actually not critical, and the splitting principle applies. Finally, observe that even for critical level sets $\{\varphi = s\}$, the point x_s lies outside Crit_s, and, in particular, (1.3.33) and the the Hopf Lemma argument employed above for the rigidity case work substantially unchanged in this case. □

A direct consequence is that the functions Φ_β's are bounded.

Corollary 1.27. *For every $\beta \geq 0$ the function $\Phi_\beta : [0, +\infty) \to \mathbb{R}$ defined in (1.3.20) is bounded as*

$$\Phi_\beta(s) \;\leq\; \sup_{\partial\Omega} |\nabla\varphi|^{\beta}_{\tilde{g}} \int_{\partial\Omega} |\nabla\varphi|_{\tilde{g}} \, \mathrm{d}\sigma_{\tilde{g}} \, .$$

for every $s \in [0, +\infty)$.

Proof. Just observe that a simple application of the Divergence Theorem combined with the $\tilde{g}$-harmonicity of φ gives the constancy in s of the function

$$s \longmapsto \int_{\{\varphi = s\}} |\nabla\varphi|_{\tilde{g}} \, \mathrm{d}\sigma_{\tilde{g}} \, .$$

Combining this observation with the boundon $|\nabla\varphi|_{\tilde{g}}$ following from Theorem 1.26, we have

$$\Phi_\beta(s) \;=\int_{\{\varphi=s\}} |\nabla\varphi|^{\beta}_{\tilde{g}} \, |\nabla\varphi|_{\tilde{g}} \, \mathrm{d}\sigma_{\tilde{g}} \;\leq\; \sup_{\partial\Omega} |\nabla\varphi|^{\beta}_{\tilde{g}} \int_{\partial\Omega} |\nabla\varphi|_{\tilde{g}} \, \mathrm{d}\sigma_{\tilde{g}} \, ,$$

for every $s \in [0, +\infty)$, as claimed. □

A fundamental tool in the forthcoming computations, leading to the expression of the derivative of Φ_β in terms of a nonpositive integral (as in (1.3.23)), is the following Bochner-type identity.

Lemma 1.28 (Bochner-type identity)**.** *At every point where $|\nabla\varphi|_{\tilde{g}} \neq 0$, the following identity holds for every $\beta \geq 0$*

$$\Delta_{\tilde{g}} |\nabla\varphi|^{\beta}_{\tilde{g}} - \left\langle \nabla |\nabla\varphi|^{\beta}_{\tilde{g}} \,\middle|\, \nabla\varphi \right\rangle_{\tilde{g}} \;=\; \beta \, |\nabla\varphi|^{\beta-2}_{\tilde{g}} \Big[\mathrm{Ric}(\nabla\varphi, \nabla\varphi) + |\nabla\nabla\varphi|^2_{\tilde{g}} + (\beta - 2) \big|\nabla|\nabla\varphi|_{\tilde{g}}\big|^2_{\tilde{g}} \Big], \tag{1.3.35}$$

where the Ric *is the Ricci tensor of the background metric g.*

Proof. By a direct computation one gets

$$\Delta_{\tilde{g}} |\nabla\varphi|^{\beta}_{\tilde{g}} \;=\; |\nabla\varphi|^{\beta-2}_{\tilde{g}} \left[\frac{\beta}{2} \Delta_{\tilde{g}} |\nabla\varphi|^2_{\tilde{g}} \,+\, \beta(\beta-2) \big|\nabla|\nabla\varphi|_{\tilde{g}}\big|^2_{\tilde{g}} \right],$$

that, combined with (1.3.15), leads to (1.3.35). □

We prove now an integral identity that will enable us to link the derivative of Φ_β to the Bochner-type formula above. Although, differently from the Euclidean case where u, and thus φ, is analytic, we cannot rely on the discreteness of the set of singular values of φ, the proof of such fundamental relation does not importantly differ from that of [AM20, Lemma 3.4]. We report it for the reader's sake. We also point out that a different proof was proposed in [AFM18], building on the fine estimates on the Minkowski content of the critical set provided in [CNV15]. However, that proof just recovered the monotonicity of U_β for $\beta \geq 1$, that was, anyway, still sufficient to provide the Willmore-type inequalities. In what follows, we let $\mathrm{Crit}\,\varphi = \{x \in M \setminus \Omega \,|\, |\nabla\varphi| = 0\}$.

Lemma 1.29 (Fundamental Integral Identity for regular values)**.** *Let $0 \leq s < S < +\infty$ be regular values, that is,* $\mathrm{Crit}\,\varphi \cap \{\varphi = s\} = \mathrm{Crit}\,\varphi \cap \{\varphi = S\} = \emptyset$*. Then, the following identity* *holds*

$$\int\limits_{\{\varphi=S\}} \frac{\left\langle \nabla|\nabla\varphi|^{\beta}_{\tilde g} \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|_{\tilde g}}\right\rangle}{e^S}\, d\sigma_{\tilde g} - \int\limits_{\{\varphi=s\}} \frac{\left\langle \nabla|\nabla\varphi|^{\beta}_{\tilde g} \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|_{\tilde g}}\right\rangle}{e^s}\, d\sigma_{\tilde g} = \\ = \beta \int\limits_{\{s\leq\varphi\leq S\}} \frac{|\nabla\varphi|^{\beta-2}_{\tilde g}\left[\mathrm{Ric}(\nabla\varphi,\nabla\varphi) + |\nabla\nabla\varphi|^2_{\tilde g} + (\beta-2)\big|\nabla|\nabla\varphi|_{\tilde g}\big|^2_{\tilde g}\right]}{e^{\varphi}}\, d\mu_{\tilde g}\,, \tag{1.3.36}$$

where the Ricci tensor is referred to the background metric g.

Proof. Since all the quantities that appear in this proof are referred to the metric $\tilde g$, except for Ric, which is the Ricci tensor of the background metric g, we shorten the notation, dropping the subscript $\tilde g$. We consider the vector field

$$X = \frac{\nabla|\nabla\varphi|^{\beta}}{e^{\varphi}},$$

that is well defined at every point where $|\nabla\varphi| \neq 0$. By (1.3.35), we have that wherever $|\nabla\varphi| \neq 0$ it holds

$$\begin{aligned} \mathrm{div} X &= \frac{\Delta|\nabla\varphi|^{\beta} - \langle\nabla|\nabla\varphi|^{\beta}, \nabla\varphi\rangle}{e^{\varphi}} \\ &= \beta\, \frac{|\nabla\varphi|^{\beta-2}\left[\mathrm{Ric}(\nabla\varphi,\nabla\varphi) + |\nabla\nabla\varphi|^2 + (\beta-2)\big|\nabla|\nabla\varphi|\big|^2\right]}{e^{\varphi}}. \end{aligned} \tag{1.3.37}$$

Let s and S as in the statement, and consider, for $\varepsilon > 0$, the cut-off function χ_ε defined by

$$\begin{cases} \chi(t) = 0 & \text{in } t < \frac{1}{2}\varepsilon\,, \\ 0 < \dot\chi(t) \leq 2\varepsilon^{-1} & \text{in } \frac{1}{2}\varepsilon \leq t \leq \frac{3}{2}\varepsilon\,, \\ \chi(t) = 1 & \text{in } t > \frac{3}{2}\varepsilon\,, \end{cases} \tag{1.3.38}$$

and define accordingly the auxiliary vector field $X_\varepsilon = \chi_\varepsilon(|\nabla\varphi|^2)X$. Let also $N_\delta = \{|\nabla\varphi|^2 \leq \delta\}$ be the tubular neighbourhood of the critical set introduced in [AM20]. By the continuity of $|\nabla\varphi|^2$, for any ε small enough we have $\{\varphi = s\} \cap N_\varepsilon = \{\varphi = S\} \cap N_{3\varepsilon/2} = \emptyset$. Observing that $X_\varepsilon = X$ on $\{\varphi = s\}$ and $\{\varphi = S\}$ we get, using the Divergence Theorem

$$\int\limits_{\{\varphi=S\}} \frac{\left\langle \nabla|\nabla\varphi|^{\beta} \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle}{e^S}\, d\sigma - \int\limits_{\{\varphi=s\}} \frac{\left\langle \nabla|\nabla\varphi|^{\beta} \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle}{e^s}\, d\sigma = \int\limits_{\{s\leq\varphi\leq S\}} \mathrm{div} X_\varepsilon\, d\mu \\ = \int\limits_{\{s\leq\varphi\leq S\}} \chi_\varepsilon\, \mathrm{div} X\, d\mu + \int\limits_{N_{3\varepsilon/2}\setminus N_{\varepsilon/2}} \langle \nabla\chi_\varepsilon \,|\, X\rangle\, d\mu \\ = \int\limits_{\{s\leq\varphi\leq S\}} \chi_\varepsilon\, \mathrm{div} X\, d\mu + \beta \int\limits_{N_{3\varepsilon/2}\setminus N_{\varepsilon/2}} \dot\chi_\varepsilon(|\nabla\varphi|^2)\, \frac{|\nabla\varphi|^{\beta-2}|\nabla|\nabla\varphi|^2|^2}{2e^{\varphi}}\, d\mu. \tag{1.3.39}$$

By (1.3.37) and (1.3.7), the integrand in the first term of the rightmost hand side is non-negative, and thus, by Monotone Convergence Theorem, its integral converges to the right hand side of (1.3.36). One is thus left to show that the second term in the rightmost hand side of (1.3.39) vanishes as $\varepsilon \to 0^+$. To see this, observe that by the coarea formula and the second property in (1.3.38) of χ_ε

$$\int\limits_{N_{3\varepsilon/2}\setminus N_{\varepsilon/2}} \dot\chi_\varepsilon(|\nabla\varphi|^2)\frac{|\nabla\varphi|^{\beta-2}|\nabla|\nabla\varphi|^2|^2}{2e^\varphi}\,\mathrm{d}\mu \le \frac{1}{\varepsilon}\int\limits_{\varepsilon/2}^{3\varepsilon/2} s^{(\beta-2)/2}\int\limits_{\{|\nabla\varphi|^2=s\}}\frac{|\nabla|\nabla\varphi|^2|}{\mathrm{e}^\varphi}\,\mathrm{d}\sigma\,\mathrm{d}s.$$

By means of the Mean Value Theorem, it then suffices to show that the function $r^{(\beta-2)/2}F(r)$, with

$$F(r) = \int\limits_{\{|\nabla\varphi|^2=r\}}\frac{|\nabla|\nabla\varphi|^2|}{\mathrm{e}^\varphi}\,\mathrm{d}\sigma,$$

vanishes as $r \to 0^+$. We prove that this is actually the case if $\beta > (n-2)/(n-1)$. Indeed, observe that

$$\begin{aligned}F(r) &= \int\limits_{\partial N_r}\frac{\langle\nabla|\nabla\varphi|^2\rangle}{e^\varphi}\,\mathrm{d}\sigma = \int\limits_{N_r}\mathrm{div}\left(\frac{\nabla|\nabla\varphi|^2}{e^\varphi}\right)\mathrm{d}\mu = 2\int\limits_{N_r}\frac{|\nabla\nabla\varphi|^2}{e^\varphi}\,\mathrm{d}\mu\\ &= \int\limits_0^r\int_{\{|\nabla\varphi|^2=s\}}\frac{|\nabla\nabla\varphi|^2}{e^\varphi|\nabla|\nabla\varphi|^2|}\,\mathrm{d}\sigma\,\mathrm{d}s.\end{aligned}$$

The above relation implies in particular that F is absolutely continuous. Then, taking derivatives, and exploiting the refined Kato's inequality provided by (1.3.7), we get

$$F'(r) = 2\int\limits_{\{|\nabla\varphi|^2=r\}}\frac{|\nabla\nabla\varphi|^2}{e^\varphi|\nabla|\nabla\varphi|^2|}\,\mathrm{d}\sigma \ge 2\left(\frac{n}{n-1}\right)\int\limits_{\{|\nabla\varphi|^2=r\}}\frac{|\nabla|\nabla\varphi||}{e^\varphi|\nabla|\nabla\varphi|^2|}\,\mathrm{d}\sigma = \frac{1}{2}\frac{n}{n-1}\frac{F(r)}{r}.$$

for almost any $r > 0$. Integrating, we get, for $R > r$,

$$\frac{F(r)}{r^{\frac{n}{2(n-1)}}} \le \frac{F(R)}{R^{\frac{n}{2(n-1)}}}.$$

Then, we deduce that for $\beta > (n-2)/(n-1)$ the function $r^{(\beta-2)/2}F(r)$ vanishes as $r \to 0^+$. To complete the proof of the present Lemma also for $\beta = (n-2)/(n-1)$, it suffices to pass to the limit as $\beta \to (n-2)/(n-1)$ from above in (1.3.36), employing the Dominated Convergence Theorem on the one side and the Monotone Convergence Theorem on the other.

□

Building on the Fundamental Integral Identity for regular values proved in Lemma 1.29, we are now ready to deduce the following monotonicity result, holding up also at critical level sets. Here, it becomes transparent the key role played by the density of regular values, ensured by the Sard Theorem, a property that we are going to miss in the nonlinear case treated in the sequel of this thesis.

Theorem 1.30 (Monotonicity of $e^{-s}\Phi'_\beta(s)$)**.** *Let $\beta \geq (n-2)/(n-1)$. The function Φ_β defined in (1.3.20) is differentiable for any $s \geq 0$, and its derivative satisfies*

$$\Phi'_\beta(s) \;=\; \int\limits_{\{\varphi=s\}} \left\langle \nabla|\nabla\varphi|^\beta_{\tilde g} \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|_{\tilde g}} \right\rangle \mathrm{d}\sigma_{\tilde g} \;=\; -\beta \int\limits_{\{\varphi=s\}} |\nabla\varphi|^\beta_{\tilde g} \mathrm{H}_{\tilde g}\, \mathrm{d}\sigma_{\tilde g}\,. \tag{1.3.40}$$

In particular, for any $0 \leq s < S < \infty$, we have

$$\begin{aligned} e^{-S}\Phi'_\beta(S) - e^{-s}\Phi'_\beta(s) \;=\; \beta \int\limits_{\{s\leq\varphi\leq S\}} |\nabla\varphi|^{\beta-2}_{\tilde g} \Big[&\mathrm{Ric}(\nabla\varphi,\nabla\varphi) + |\nabla\nabla\varphi|^2_{\tilde g} \\ &+ (\beta-2)\,\big|\nabla|\nabla\varphi|_{\tilde g}\big|^2_{\tilde g} \Big] e^{-\varphi}\, \mathrm{d}\mu_{\tilde g}\,, \end{aligned} \tag{1.3.41}$$

where the Ricci tensor is referred to the background metric g.

Proof. Let us drop the subscript $\tilde g$. Let $s_0 \in [0,+\infty)$, and $s \geq s_0$ be possibly singular values of φ. We first claim that

$$\Phi_\beta(s) - \Phi_\beta(s_0) \;=\; \int\limits_{\{s_0\leq\varphi\leq s\}} \mathrm{div}(|\nabla\varphi|^\beta\nabla\varphi)\,\mathrm{d}\mu \;=\; \int\limits_{\{s_0\leq\varphi\leq s\}} \left\langle \nabla|\nabla\varphi|^\beta \,\middle|\, \nabla\varphi \right\rangle \mathrm{d}\mu\,. \tag{1.3.42}$$

To see this, consider again the tubular neighbourhoods N_ε considered in the proof of Lemma 1.29 above. Applying the Divergence Theorem to the vector field $Y_\beta = |\nabla\varphi|^\beta\nabla\varphi$ in the set $\{s_0 \leq \varphi \leq s\} \setminus N_\varepsilon$ we get

$$\begin{aligned} \int\limits_{\{s_0\leq\varphi\leq s\}\setminus N_\varepsilon} \mathrm{div}\left(|\nabla\varphi|^\beta\nabla\varphi\right)\mathrm{d}\mu = \int\limits_{\{\varphi=s\}\setminus N_\varepsilon} |\nabla\varphi|^{\beta+1}\,\mathrm{d}\sigma \;-\; \int\limits_{\{\varphi=s_0\}\setminus N_\varepsilon} |\nabla\varphi|^{\beta+1}\,\mathrm{d}\sigma \\ + \int\limits_{\partial N_\varepsilon\cap\{s_0\leq\varphi\leq s\}} |\nabla\varphi|^\beta \left\langle \nabla\varphi \,\middle|\, \frac{\nabla|\nabla\varphi|^2}{|\nabla|\nabla\varphi|^2|} \right\rangle \mathrm{d}\mu. \end{aligned} \tag{1.3.43}$$

Observing now that

$$\mathrm{div}(|\nabla\varphi|^\beta\nabla\varphi) = \left\langle \nabla|\nabla\varphi|^\beta \,\middle|\, \nabla\varphi \right\rangle = \beta|\nabla\varphi|^\beta\nabla\nabla\varphi\left(\frac{\nabla\varphi}{|\nabla\varphi|}, \frac{\nabla\varphi}{|\nabla\varphi|}\right),$$

we deduce that the integrand in the left hand side of (1.3.43) is uniformly bounded, and thus by Dominated Convergence Theorem we deduce that its integral converges as $\varepsilon \to 0^+$ to the right hand side of (1.3.42). By Monotone Convergence Theorem , it is also easy to see that the first two terms in the right hand side of (1.3.43) converges to the left hand side of (1.3.42). We are thus left to show that the last term in (1.3.43) vanishes as $\varepsilon \to 0$. In fact, since $|\nabla\varphi|^2 = \varepsilon$ on ∂N_ε, we have

$$\int\limits_{\partial N_\varepsilon\cap\{s_0\leq\varphi\leq s\}} |\nabla\varphi|^\beta \left\langle \nabla\varphi \,\middle|\, \frac{\nabla|\nabla\varphi|^2}{|\nabla|\nabla\varphi|^2|} \right\rangle \mathrm{d}\mu \leq \varepsilon^{(\beta+1)/2}|\partial N_\varepsilon \cap \{s_0 \leq \varphi \leq s\}|,$$

that tends to 0 as $\varepsilon \to 0^+$, as desired.

We now aim at showing that the function mapping

$$s \to I(s) = e^{-s} \int\limits_{\varphi = s} \left\langle \nabla |\nabla \varphi|^{\beta} \middle| \frac{\nabla \varphi}{|\nabla \varphi|} \right\rangle \mathrm{d}\sigma,$$

a priori defined just for almost any $s \in [0, \infty)$, can be extended to a continuous function defined on the whole $[0, +\infty)$. Letting D be the set of regular values of φ, it is well known that it suffices to show that I is *uniformly* continuous on $D \cap [s, S]$ for $0 \leq s < S < +\infty$ to ensure the existence of such an extension in $[s, S]$. To see this, let $s_0 \in D \cap [s, S]$, and let $s_j \in D \cap [s, S]$ be a sequence of regular values converging to s_0. Let us assume for simplicity that $s_j > s_0$ for any j, the general case displaying no additional difficulties. By (1.3.36), we have

$$\frac{|\nabla \varphi|^{\beta-2} \left[\mathrm{Ric}(\nabla \varphi, \nabla \varphi) + |\nabla\nabla \varphi|^2 + (\beta - 2) \big|\nabla |\nabla \varphi|\big|^2 \right]}{e^{\varphi}} \in L^1(\{s \leq \varphi \leq S\}).$$

In particular, for any $\varepsilon > 0$ and $\tilde{s} \in D \cap [s, S]$, there exists $\delta = \delta(\varepsilon, \tilde{s})$ such that

$$\int\limits_{\{\tilde{s} - \delta \leq \varphi \leq \tilde{s} + \delta\}} \frac{|\nabla \varphi|^{\beta-2} \left[\mathrm{Ric}(\nabla \varphi, \nabla \varphi) + |\nabla\nabla \varphi|^2 + (\beta - 2) \big|\nabla |\nabla \varphi|\big|^2 \right]}{e^{\varphi}} \, \mathrm{d}\mu \leq \varepsilon.$$

By the density of regular values, the union of open intervals $(\tilde{s} - \delta, \tilde{s} + \delta)$ as $\tilde{s}$ ranges in $D \cap [s, S]$, covers the whole compact $[s, S]$, and thus we can extract a *finite* number of intervals $(\tilde{s}_i - \delta_i, \tilde{s}_i + \delta_i)$. Let then $\delta = \min_i \delta_i > 0$, and let us come back to our sequence of regular values s_j converging to s_0. It is now clear that, if $s_j \in (s_0, s_0 + \delta)$, by the above discussion and Lemma 1.29,

$$I(s_j) - I(s_0) = \int\limits_{\{s_0 \leq \varphi \leq s_0 + \delta\}} \frac{|\nabla \varphi|^{\beta-2} \left[\mathrm{Ric}(\nabla \varphi, \nabla \varphi) + |\nabla\nabla \varphi|^2 + (\beta - 2) \big|\nabla |\nabla \varphi|\big|^2 \right]}{e^{\varphi}} \, \mathrm{d}\mu \leq \varepsilon.$$

The radius δ being independent of x_0 shows that I is uniformly continuous on $D \cap [s, S]$. Since this holds for any interval $[s, S]$, we can extend I to a continuous function on $[0, +\infty)$, that we are still calling $I(s)$. Clearly, by (1.3.36), this function satisfies

$$I(S) - I(s) = \int\limits_{\{s \leq \varphi \leq S\}} \frac{|\nabla \varphi|^{\beta-2} \left[\mathrm{Ric}(\nabla \varphi, \nabla \varphi) + |\nabla\nabla \varphi|^2 + (\beta - 2) \big|\nabla |\nabla \varphi|\big|^2 \right]}{e^{\varphi}} \, \mathrm{d}\mu$$

for *any* $0 \leq s \leq S < \infty$. Moreover, with this terminology, the function $e^{(\cdot)} I(\cdot)$ is a continuous function on $[0, \infty)$ such that

$$e^s I(s) = \int\limits_{\{\varphi = s\}} \left\langle \nabla |\nabla \varphi|^{\beta} \middle| \frac{\nabla \varphi}{|\nabla \varphi|} \right\rangle \mathrm{d}\sigma$$

everytime s is a regular value for φ.

Finally, apply the coarea formula in (1.3.42), to get

$$\Phi_\beta(s) - \Phi_\beta(s_0) = \int_{s_0}^{s} d\tau \int_{\{\varphi=\tau\}} \left\langle \nabla|\nabla\varphi|^\beta \,\Big|\, \frac{\nabla\varphi}{|\nabla\varphi|} \right\rangle d\sigma\,.$$

By the Fundamental Theorem of Calculus, the continuity of (the extension of) the mapping

$$\tau \longmapsto e^\tau I(\tau) = \int_{\{\varphi=\tau\}} \left\langle \nabla|\nabla\varphi|^\beta \,\Big|\, \frac{\nabla\varphi}{|\nabla\varphi|} \right\rangle d\sigma$$

proved above implies the differentiability of Φ_β, together with the first identity in (1.3.40). The second one follows from the first just by a direct computation involving (1.3.17). The proof is completed. □

We are now in a position to complete the proof of Theorem 1.25.

Conclusion of the proof of Theorem 1.25. We are going to pass to the limit as $S \to +\infty$ in (1.3.41). The following argument is due to Colding and Minicozzi, [CM14b]. We first prove that the derivative of Φ_β has a sign. By (1.3.41) we have that, for every $\beta \geq 1$ and every $0 \leq s < S < +\infty$,

$$\Phi'_\beta(S) \geq e^{(S-s)}\Phi'_\beta(s)\,.$$

Integrating the above differential inequality, we get

$$\Phi_\beta(S) \geq \left(e^{(S-s)} - 1\right)\Phi'_\beta(s) + \Phi_\beta(s)\,. \tag{1.3.44}$$

for every $0 \leq s < S < +\infty$. Assume now, by contradiction, that $\Phi'_\beta(s) > 0$ for some $s \in [0, +\infty)$. Then, passing to the limit as $S \to +\infty$ in (1.3.44), we would get $\Phi_\beta(S) \to +\infty$, against the boundedness of Φ_β provided in Corollary 1.27. Thus, $\Phi'_\beta(s) \geq 0$ for every $s \in [0, +\infty)$. Therefore Φ_β is a nonincreasing, differentiable bounded function on $[0, +\infty)$, and in particular, $\Phi'_\beta(S) \to 0$ as $S \to +\infty$, possibly along a subsequence. Passing to the limit as $S \to +\infty$ in (1.3.41) finally gives the monotonicity formula (1.3.23).

For the rigidity statement, assume now that $\Phi'_\beta(s_0) = 0$ for some $s_0 \in [0, +\infty)$. Then, if, $\beta > (n-2)/(n-1)$ we see from (1.3.23) that $\left|\nabla|\nabla\varphi|_{\tilde g}\right|_{\tilde g} = 0$, and we can thus conclude by Lemma 1.24. If on the other hand $\beta = (n-2)/(n-1)$, (1.3.23) still shows that the terms on the right hand side of (1.3.7) vanish, and so the related rigidity statement ensures that $(\{\varphi \geq s_0\}, \tilde g)$ is a warped product. In particular, the mean curvature of $\{\varphi = s\}$ depends only on s for any $s \geq s_0$. Consequently, (1.3.22) shows us that they all have zero mean curvature. By (1.3.17), we deduce in particular that $|\nabla^\perp|\nabla\varphi|_{\tilde g}|_{\tilde g} = 0$, that, coupled with $\left|\nabla^\top|\nabla\varphi|_{\tilde g}\right|_{\tilde g} = 0$ following from the vanishing of (1.3.23), implies that $\left|\nabla|\nabla\varphi|_{\tilde g}\right|_{\tilde g} = 0$, and we can conclude as before appealing to Lemma 1.24. □

1.4 Long time behaviour of the electrostatic potential

In this section, we are going to prove the Willmore inequality on manifolds with nonnegative Ricci curvature and Euclidean volume growth, using the geometric features of the capacitary potential u of a given bounded domain with smooth boundary Ω. For the reader's convenience we recall that u is a solution to the following problem

$$\begin{cases} \Delta u = 0 & \text{in } M \setminus \overline{\Omega} \\ u = 1 & \text{on } \partial\Omega \\ u(y) \to 0 & \text{as } d(O,y) \to +\infty\,, \end{cases}$$

whose existence in the present context follows from Theorem 1.14. As sketched in the Introduction, the proof makes use of the global features of the Monotonicity-Rigidity Theorem, that is, it compares the behaviour of U_β in the large with $U_\beta(1)$. The well known asymptotics of u and of its gradient in the Euclidean case were the crucial tool to compute the limits of U_β in [AM15], that consequently gave the sharp lower bound on the Willmore-type functional. On manifolds with nonnegative Ricci curvature and Euclidean volume growth, the work of Colding-Minicozzi [CM97b] actually implies that the asymptotic behaviour of the potential is completely analogous to that in $\mathbb{R}^n$. However, as we will clarify in Remark 1.39, there is no hope to get an Euclidean-like pointwise behaviour of $\mathrm{D}u$ in the general case. Nevertheless, using techniques developed in the celebrated [CC96], we are able to achieve asymptotic integral estimates for the gradient that in turn will let us conclude the proof of the Willmore-type inequalities (10).

We introduce the (normalised) capacity of Ω, that is

$$\mathrm{Cap}(\Omega) \;=\; \inf\left\{ \frac{1}{(n-2)|\mathbb{S}^{n-1}|} \int_{\mathbb{R}^n} |\mathrm{D}f|^2 \,\mathrm{d}\mu \;\middle|\; f \in \mathscr{C}^\infty_c(\mathbb{R}^n), f \geq 1 \text{ on } \Omega \right\}. \tag{1.4.1}$$

Such notion will arise here in the following form

$$\mathrm{Cap}(\Omega) \;=\; \frac{\displaystyle\int_{\partial\Omega} |\mathrm{D}u| \,\mathrm{d}\sigma}{(n-2)|\mathbb{S}^{n-1}|}\,, \tag{1.4.2}$$

written in terms of the capacitary potential u of Ω. We address the reader to the proof of Theorem B.1 for the equivalence between (1.4.1) and (1.4.2).

1.4.1 Manifolds with Euclidean volume growth

A foundational result in comparison geometry, that we did not explicitly employed yet, is the Bishop-Gromov Theorem. In order to ease the reader with the conventions we adopt, we recall the classical statement in the case of nonnegative Ricci curvature.

Theorem 1.31 (Bishop-Gromov Theorem)**.** *Let (M,g) be a complete noncompact Riemannian manifold with $\mathrm{Ric} \geq 0$. Then, for any point $x \in M$, the function $\Theta_x : (0,+\infty) \to \mathbb{R}$ defined by*

$$\Theta_x(r) = \frac{|B(x,r)|}{|\mathbb{B}^n| r^n}\,,$$

is monotone nonincreasing. Moreover, the limit as $r \to +\infty$ of Θ_x is independent of x, and it satisfies

$$\lim_{r\to\infty} \Theta_x(r) = 1$$

for some $x \in M$ if and only if (M,g) is isometric to $\mathbb{R}^n$ with the standard metric.

The above Theorem justifies the following commonly used Definition.

Definition 1.32 (Asymptotic Volume Ratio and Euclidean volume growth)**.** *Let (M,g) be a complete noncompact Riemannian manifold with $\mathrm{Ric} \geq 0$, and let $O \in M$. Then we define its*

Asymptotic Volume Ratio $\mathrm{AVR}(g) \in [0,1]$ *as*

$$\mathrm{AVR}(g) = \lim_{r\to\infty} \frac{|B(O,r)|}{|\mathbb{R}^n| r^n}.$$

We say that (M,g) *has Euclidean volume growth if* $\mathrm{AVR}(g) > 0$.

If follows at once that manifolds with Euclidean volume growth satisfy (1.2.3), and thus from Varopoulos' characterisation *manifolds with Euclidean volume growth are nonparabolic*.

Let us also recall the well known fact that in a noncompact complete Riemannian manifold (M,g) with $\mathrm{Ric} \geq 0$ also the function

$$(0,+\infty) \ni r \longmapsto \vartheta_x(r) = \frac{|\partial B(O,r)|}{r^{n-1}|\mathbb{S}^{n-1}|}$$

is monotone nonincreasing for any $x \in M$, and the asymptotic volume ratio satisfies

$$\mathrm{AVR}(g) = \lim_{r\to+\infty} \frac{|\partial B(O,r)|}{r^{n-1}|\mathbb{S}^{n-1}|}. \tag{1.4.3}$$

The proof of the monotonicity of $\vartheta_x(r)$ is actually the main step in the classical proof of the Bishop-Gromov Theorem, see e.g. the proof given for [CLN06, Theorem 1.132] or that proposed for Theorem 3.1 in the survey [Zhu97], while the validity of (1.4.3) is easily checked through an application of de l'Hôpital rule.

Let us point out an obvious but very important feature of the Bishop-Gromov Theorem and of the Euclidean volume growth assumption, that is

$$\mathrm{AVR}(g)|\mathbb{B}^n|r^n \leq |B(x,r)| \leq |\mathbb{B}^n|r^n \quad \text{and} \quad \mathrm{AVR}(g)|\mathbb{S}^{n-1}|r^{n-1} \leq |\partial B(x,r)| \leq |\mathbb{S}^{n-1}|r^{n-1}. \tag{1.4.4}$$

The importance of the above bounds lie also in the fact that the constants in front of r^n do not depend on the base point x. We are explicitly making use of this fact in the computation of the limit of $U_\infty(t)$ as $t \to 0^+$, see the proof of Proposition 1.43.

In the remainder of this section we are repeatedly going to use both the area and the volume formulations of the Bishop-Gromov Theorem without always mentioning them.

Rough estimates for the electrostatic potential

Let us now derive some rough estimates for the electrostatic potential u and its gradient $\mathrm{D}u$ holding on manifolds with nonnegative Ricci curvature and Euclidean volume growth, to be refined later. Here, and in the sequel, we set $r(x) = d(O,x)$, where $O \in \Omega$.

Proposition 1.33. *Let* (M,g) *be a complete noncompact Riemannian manifold with* $\mathrm{Ric} \geq 0$ *and Euclidean volume growth. Then, it is nonparabolic and the solution* u *to problem* (1.2.15) *for some bounded and open* Ω *with smooth boundary satisfies*

$$\mathrm{C}_1\, r^{2-n}(x) \;\leq\; u(x) \;\leq\; \mathrm{C}_2\, r^{2-n}(x) \tag{1.4.5}$$

on $M \setminus \Omega$ *for some positive constants* C_1 *and* C_2 *depending on* M *and* Ω. *Moreover, if* $\Omega \subset B(O,R)$, *it holds*

$$|\mathrm{D}u|(x) \;\leq\; \mathrm{C}_3\, r^{1-n}(x)\,, \tag{1.4.6}$$

on $M \setminus B(O,2R)$ *with* $\mathrm{C}_3 = \mathrm{C}_3(M,\Omega) > 0$.

Proof. The first inequality in (1.4.5) is just (1.2.19). To obtain the second one, recall from (1.4.4) that

$$|B(O,r)| \geq \Big(n|\mathbb{S}^{n-1}|\mathrm{AVR}(g)\Big)\, r^n .$$

Then, the second inequality in the Li-Yau estimate (1.12), combined with the second inequality in (1.2.18) completes the proof of (1.4.5). Finally, inequality (1.4.6) is achieved just by plugging the upper estimate on u given by (1.4.5) into (1.2.5). □

1.4.2 Asymptotics for u and its gradient

The behaviour at infinity of u can be deduced along the path indicated in [CM97b]. However, we prefer to present here a simplified version of that proof, taking advantage of some of the refinements provided in [LTW97]. Let us first recall the following asymptotic behaviour of G (see [CM97b, Theorem 0.1], or [LTW97, Theorem 1.1] for a completely different proof),

$$\lim_{r(x)\to+\infty} \frac{G(O,x)}{r(x)^{2-n}} \;=\; \frac{1}{\mathrm{AVR}(g)} . \tag{1.4.7}$$

Building on this fact, we deduce precise asymptotics for the capacitary potential of Ω.

Lemma 1.34. *Let (M,g) be a complete noncompact Riemannian manifold with $\mathrm{Ric} \geq 0$ and Euclidean volume growth, and let u be a solution to problem (1.2.15). Then*

$$\lim_{r(x)\to+\infty} \frac{u(x)}{r(x)^{2-n}} \;=\; \frac{\mathrm{Cap}(\Omega)}{\mathrm{AVR}(g)} . \tag{1.4.8}$$

Proof. By [LTW97, Theorem 1.2], that slightly extends [CM97b, Theorem 0.3], we have that outside some large ball $B(O,R)$ containing Ω

$$u \;=\; -\,\frac{G}{(n-2)|\mathbb{S}^{n-1}|}\int\limits_{\partial B(O,R)} \frac{\partial u}{\partial \nu}\, \mathrm{d}\sigma \,+\, v\,, \tag{1.4.9}$$

where G is the Green's function with pole in O, ν is the exterior unit normal to the boundary of $B(O,R)$ and v is a harmonic function defined in $M \setminus B(O,R)$ satisfying

$$|v| \;\leq C\,\frac{G}{r} \tag{1.4.10}$$

for some constant $C > 0$. We point out that the Green's function considered in [LTW97] is, in our notation, $G/\big((n-2)|\mathbb{S}^{n-1}|\big)$. By the Divergence Theorem and the harmonicity of u, we infer that

$$\int\limits_{\partial B(O,R)} \frac{\partial u}{\partial \nu}\, \mathrm{d}\sigma + \int\limits_{\partial\Omega} \frac{\partial u}{\partial \nu}\, \mathrm{d}\sigma \;=\; \int\limits_{B(O,R)\setminus\Omega} \Delta u \, \mathrm{d}\mu \;=\; 0\,,$$

where we denote by ν the exterior unit normal to the boundary of $B(O,R) \setminus \overline{\Omega}$. Since, on $\partial\Omega$, $\nu = \mathrm{D}u/|\mathrm{D}u|$, we get, by the above identity, that

$$-\,\frac{\displaystyle\int_{\partial B(O,R)} \frac{\partial u}{\partial \nu}\, \mathrm{d}\sigma}{(n-2)|\mathbb{S}^{n-1}|} \;=\; \frac{\displaystyle\int_{\partial\Omega} |\mathrm{D}u|\, \mathrm{d}\sigma}{(n-2)|\mathbb{S}^{n-1}|} \;=\; \mathrm{Cap}(\Omega)\,. \tag{1.4.11}$$

Dividing both sides of (1.4.9) by r^{2-n} and passing to the limit as $r \to +\infty$ taking into account (1.4.7), (1.4.11) and (1.4.10), we get the claim. □

As a straightforward corollary of the above lemma, we compute the rescaled area of large geodesic spheres in M.

Corollary 1.35. *Let (M, g) be a complete noncompact Riemannian manifold with $\mathrm{Ric} \geq 0$ and Euclidean volume growth, and let u be a solution to (1.2.15). Then*

$$\lim_{R_i \to +\infty} \int\limits_{\partial B(O, R_i)} u^{\frac{n-1}{n-2}} d\sigma = |\mathbb{S}^{n-1}| \,\mathrm{AVR}(g) \left(\frac{\mathrm{Cap}(\Omega)}{\mathrm{AVR}(g)} \right)^{\frac{n-1}{n-2}} \tag{1.4.12}$$

Proof. It follows from Lemma 1.34 and (1.4.3). □

Building on the above formula (1.4.8) for the asymptotics of u, we now derive integral asymptotics for $\mathrm{D}u$. They are achieved through an adaptation of the methods used in [CC96, Section 4]. Similar estimates have been widely considered in literature (see e.g. [Col12] [CM97b], [CM97a], [CM14a]). We are providing here a complete and self-contained proof.

In the computations below, we are going to consider first and second derivatives of the distance function $x \mapsto d(O, x)$. This can be readily justified by approximating the distance function by convolution, by performing the computations below for the approximating sequence and finally passing to the limit. A scheme like this, often implicit in the aforementioned literature, has been carried out in details for example in [LTW97]. We state this underlying approximation lemma, a complete proof of which can be obtained following a standard regularisation argument as in [Col97, Lemma 1.4].

Lemma 1.36 (Smooth approximation of the distance function)**.** *Let (M, g) be a complete Riemannian manifold. Let $O \in M$, and let $K \subset M$ be a compact set such that $O \in M \setminus K$. Let $r(x) = d(O, x)$, and let $f : (0, +\infty) \to \mathbb{R}$ be a smooth function. Moreover, assume that $\Delta(f \circ r) \leq h \circ r$ in the sense of distributions for some smooth function $h : (0, +\infty) \to \mathbb{R}$. Then, there exists a sequence of smooth functions $\{f_j\}_{j \in \mathbb{N}}$ such that for any $\varepsilon > 0$ there exists a $j_0 = j_0(K, h)$ such that*

(i) for any $j \geq j_0$ we have

$$|f_j(x) - f(r(x))| \leq \varepsilon$$

for any $x \in K$,

(ii) for any $j \geq j_0$ we have

$$|\mathrm{D}f_j(x)| \leq |\mathrm{D}f(r(x))| + \varepsilon$$

for any $x \in K$ and $Df_j(x) \to \mathrm{D}f(r(x))$ for almost any $x \in K$,

(iii) for any $j \geq j_0$ we have

$$\Delta f_j(x) \leq h(r(x)) + \varepsilon$$

for any $x \in K$.

The above lemma will be applied mostly with $f(t) = -t^{2-n}$ and $h(t) = 0$, for $t \in (0, +\infty)$, so that (iii) approximates the well known relation $\Delta r^{2-n} \geq 0$ in the sense of distribution, already observed in (1.2.14). Observe also that (ii) easily allows to pass to the limit under the integral sign when approximating the gradient of functions of the distance. Moreover, we are also going to repeatedly apply the Divergence Theorem to vector fields depending on (approximations of) the distance function on annuli

$B(O,kR) \setminus B(O,R)$ with $k > 1$. We can do this since by the coarea formula the intersection of the cut locus with geodesic spheres is $(n-1)$-negligible for almost any value of the radius. We are actually applying the Divergence Theorem to annuli bounded by such geodesic spheres (we implicitly used this observation also in the proof of Lemma 1.34). Alternatively, we can refer to the much more general weak Sard-type property of Lipschitz functions object of [ABC11].

In order to keep the presentation as transparent as possible, we are going to work directly with derivatives of the distance function, having in mind a repeated use of Lemma 1.36 as hinted above, and on geodesic annuli allowing us to apply the Divergence Theorem as just illustrated.

Proposition 1.37 (Integral Asymptotics for Du)**.** *Let (M, g) be a complete noncompact Riemannian manifold with* Ric ≥ 0 *and Euclidean volume growth, and let u be a solution to problem* (1.2.15)*. Then, for every $k > 1$, it holds*

$$\lim_{R\to+\infty} \frac{R^{2n-2}}{|A_{R,kR}|} \int\limits_{A_{R,kR}} \left| \mathrm{D}u - \frac{\mathrm{Cap}(\Omega)}{\mathrm{AVR}(g)} \mathrm{D}r^{2-n} \right|^2 \mathrm{d}\mu \;=\; 0\,, \tag{1.4.13}$$

where, for $R > 0$, we set $A_{R,kR} = B(O,kR) \setminus \overline{B(O,R)}$.

Proof. A simple integration by parts combined with the harmonicity of u leads to the following identity

$$\begin{aligned}\int\limits_{A_{R,kR}} \left| \mathrm{D}u - \frac{\mathrm{Cap}(\Omega)}{\mathrm{AVR}(g)} \mathrm{D}r^{2-n} \right|^2 \mathrm{d}\mu \;=\; &-\int\limits_{A_{R,kR}} \left(\frac{\mathrm{Cap}(\Omega)}{\mathrm{AVR}(g)} \Delta r^{2-n} \right) \left(u - \frac{\mathrm{Cap}(\Omega)}{\mathrm{AVR}(g)} r^{2-n} \right) \mathrm{d}\mu \;+ \\ &+ \int\limits_{\partial A_{R,kR}} \left(u - \frac{\mathrm{Cap}(\Omega)}{\mathrm{AVR}(g)} r^{2-n} \right) \left\langle \mathrm{D}u - \frac{\mathrm{Cap}(\Omega)}{\mathrm{AVR}(g)} \mathrm{D}r^{2-n} \,\middle|\, \nu \right\rangle \mathrm{d}\sigma\,.\end{aligned} \tag{1.4.14}$$

Let us estimate separately the integrals on the right hand side of the above identity. Let $\varepsilon > 0$. Then, by (1.4.8),

$$\left| \frac{u}{r^{2-n}} - \frac{\mathrm{Cap}(\Omega)}{\mathrm{AVR}(g)} \right| < \varepsilon.$$

for r big enough. We have, for R big enough

$$\begin{aligned}\left| \int\limits_{A_{R,kR}} \left(\frac{\mathrm{Cap}(\Omega)}{\mathrm{AVR}(g)} \Delta r^{2-n} \right) \left(u - \frac{\mathrm{Cap}(\Omega)}{\mathrm{AVR}(g)} r^{2-n} \right) \mathrm{d}\mu \right| \leq \\ \leq \int\limits_{A_{R,kR}} \left| \frac{\mathrm{Cap}(\Omega)}{\mathrm{AVR}(g)} \Delta r^{2-n} \right| \left| \frac{u}{r^{2-n}} - \frac{\mathrm{Cap}(\Omega)}{\mathrm{AVR}(g)} \right| r^{2-n} \,\mathrm{d}\mu \\ \leq \varepsilon\, R^{2-n} \int\limits_{A_{R,kR}} \left| \frac{\mathrm{Cap}(\Omega)}{\mathrm{AVR}(g)} \Delta r^{2-n} \right| \mathrm{d}\mu \\ = \varepsilon\, R^{2-n} \int\limits_{A_{R,kR}} \frac{\mathrm{Cap}(\Omega)}{\mathrm{AVR}(g)} \Delta r^{2-n} \,\mathrm{d}\mu\,,\end{aligned} \tag{1.4.15}$$

where in the last identity we used $\Delta r^{2-n} \geq 0$ in the sense of distributions, already discussed above. Integrating by parts Δr^{2-n} we obtain

$$\int\limits_{A_{R,kR}} \Delta r^{2-n}\, \mathrm{d}\mu \;=\; (2-n)\left[(kR)^{1-n}|\{r=kR\}| - R^{1-n}|\{r=R\}|\right].$$

In particular, by the assumption on the Euclidean volume growth, the above quantity is uniformly bounded in R. We have thus proved that the first summand on the right hand side of (1.4.14), for R large enough, is bounded as follows

$$\left|\int\limits_{A_{R,kR}} \left(\frac{\mathrm{Cap}(\Omega)}{\mathrm{AVR}(g)}\Delta r^{2-n}\right)\left(u - \frac{\mathrm{Cap}(\Omega)}{\mathrm{AVR}(g)} r^{2-n}\right) \mathrm{d}\mu\right| \;\leq\; C_1 \varepsilon R^{2-n}. \tag{1.4.16}$$

Let us turn our attention to the second integral in the right hand side of (1.4.14). Proceeding as in (1.4.15), we have

$$\begin{aligned}\int\limits_{\partial A_{R,kR}} \left|\left(u - \frac{\mathrm{Cap}(\Omega)}{\mathrm{AVR}(g)} r^{2-n}\right)\left\langle \mathrm{D}u - \frac{\mathrm{Cap}(\Omega)}{\mathrm{AVR}(g)}\mathrm{D}r^{2-n} \,\middle|\, \nu\right\rangle\right| \mathrm{d}\sigma \;\leq\; \\ \leq\; \varepsilon R^{2-n} \int\limits_{\partial A_{R,kR}} \left[|\mathrm{D}u| + (n-2)\frac{\mathrm{Cap}(\Omega)}{\mathrm{AVR}(g)} r^{1-n}\right] \mathrm{d}\sigma.\end{aligned} \tag{1.4.17}$$

Recall now that in Proposition 1.33 we proved that

$$|\mathrm{D}u| \;\leq\; \mathrm{C}_2\, r^{1-n},$$

for some positive constant C_2 independent of r. Thus, by the Euclidean volume growth, the integral on the right hand side of (1.4.17) is uniformly bounded in R and so

$$\left|\int\limits_{\partial A_{R,kR}} \left(u - \frac{\mathrm{Cap}(\Omega)}{\mathrm{AVR}(g)} r^{2-n}\right)\left\langle \mathrm{D}u - \frac{\mathrm{Cap}(\Omega)}{\mathrm{AVR}(g)}\mathrm{D}r^{2-n} \,\middle|\, \nu\right\rangle \mathrm{d}\sigma\right| \;\leq\; \mathrm{C}_3\, \varepsilon\, R^{2-n} \tag{1.4.18}$$

for some C_3 independent of R. Finally, by (1.4.14), (1.4.16) and (1.4.18), we obtain, for R big enough, the estimate

$$\frac{1}{|A_{R,kR}|}\int\limits_{A_{R,kR}} \left|\mathrm{D}u - \frac{\mathrm{Cap}(\Omega)}{\mathrm{AVR}(g)}\mathrm{D}r^{2-n}\right|^2 \mathrm{d}\mu \;\leq\; \mathrm{C}_4\, \varepsilon\, \frac{R^{2-n}}{|A_{R,kR}|} \;\leq\; \mathrm{C}_5\, \varepsilon\, R^{2-2n},$$

for some positive constants C_4 and C_5 independent of R. In the last inequality we used the Euclidean volume growth assumption. Our claim (1.4.13) is thus proved. □

From the above Proposition we easily deduce the integral asymptotic behaviour of $|\mathrm{D}u|$ on geodesic spheres.

Corollary 1.38. *Let (M,g) be a complete noncompact Riemannian manifold with* $\mathrm{Ric} \geq 0$ *and Euclidean volume growth, and let u be a solution to problem* (1.2.15). *Then, there exists a sequence* *of positive real numbers R_k with $R_k \to +\infty$ such that*

$$\lim_{R_k \to +\infty} \int\limits_{\{r=R_k\}} \left| |\mathrm{D}u| - (n-2)\frac{\mathrm{Cap}(\Omega)}{\mathrm{AVR}(g)} r^{1-n} \right| \mathrm{d}\sigma \;=\; 0\,. \tag{1.4.19}$$

Proof. Let us first observe that, by means of Hölder inequality, we can deduce from the L^2 asymptotics (1.4.13) an analogous L^1 behaviour. Namely, for any $\varepsilon > 0$ we have

$$\frac{\int_{A_{R,kR}} \left| \mathrm{D}u - \frac{\mathrm{Cap}(\Omega)}{\mathrm{AVR}(g)} \mathrm{D}r^{2-n} \right| \mathrm{d}\mu}{R^{1-n}|A_{R,kR}|} \;\leq\; \left(\frac{\int_{A_{R,kR}} \left| \mathrm{D}u - \frac{\mathrm{Cap}(\Omega)}{\mathrm{AVR}(g)} \mathrm{D}r^{2-n} \right|^2 \mathrm{d}\mu}{R^{2-2n}|A_{R,kR}|} \right)^{1/2} \;\leq\; \varepsilon$$

for any R large enough. By the Coarea Formula, the above L^1 estimate gives, for R large enough,

$$\frac{\int\limits_R^{kR} \mathrm{d}s \int\limits_{\{r=s\}} \left| \mathrm{D}u - \frac{\mathrm{Cap}(\Omega)}{\mathrm{AVR}(g)} \mathrm{D}r^{2-n} \right| \mathrm{d}\sigma}{R^{1-n}|A_{R,kR}|} \;\leq\; \varepsilon.$$

Thus, by the Mean Value Theorem, there exists $R_k \in (R, kR)$ such that

$$\frac{\int\limits_{\{r=R_k\}} \left| \mathrm{D}u - \frac{\mathrm{Cap}(\Omega)}{\mathrm{AVR}(g)} \mathrm{D}r^{2-n} \right| \mathrm{d}\sigma}{R^{-n}|A_{R,kR}|} \;\leq \mathrm{C}\,\varepsilon\,,$$

for some constant C independent of R. The Euclidean volume growth of the annulus $A_{R,kR}$ as R increases then implies the existence of a sequence $R_k \to +\infty$ as in the statement. □

Remark 1.39. The integral asymptotic for $|\mathrm{D}u|$ given by Corollary 1.38 cannot, in general, be improved to a pointwise asymptotic expansion at infinity on a manifold with nonnegative Ricci curvature and Euclidean volume growth. Indeed, the validity of such a formula would imply $|\mathrm{D}u| \neq 0$ outside some big ball $B(O,R)$, and, in turn, $M \setminus B(O,R)$ would be diffeomorphic to $\partial B(O,R) \times [R, +\infty)$. This would imply that M has finite topological type. However, Menguy provided in [Men00] examples of manifolds of dimension $n \geq 4$ with $\mathrm{Ric} \geq 0$ and $\mathrm{AVR}(g) > 0$ with infinite topological type. On the other hand, we notice that in dimension 3 topological obstructions like these cannot occur. This is due to the structure result obtained in [Liu13], where it is proved that if a 3-dimensional complete noncompact Riemannian manifold (M,g) with $\mathrm{Ric} \geq 0$ is not diffeomorphic to $\mathbb{R}^3$, then its universal cover isometrically splits a product $N \times \mathbb{R}$ where N is compact Riemannian manifold with nonnegative sectional curvature. In particular, in this case, (M,g) has nonnegative sectional curvature, and the finite topology immediately follows from the celebrated Soul Theorem proved in [CG72].

1.4.3 Limits of the monotone quantities.

Recalling that

$$U_\beta(t) \;=\; t^{-\beta\left(\frac{n-1}{n-2}\right)} \int\limits_{\{u=t\}} |\mathrm{D}u|^{1+\beta}\, \mathrm{d}\sigma\,,$$

it is easy to realise that we have now at hand all the elements to compute the limit as $t \to 0^+$.

Proposition 1.40. *Let (M, g) be a complete noncompact Riemannian manifold with $\mathrm{Ric} \geq 0$ and Euclidean volume growth. Then, for every $\beta \geq 0$, we have that*

$$\lim_{t\to 0^+} U_\beta(t) \;=\; \mathrm{Cap}(\Omega)^{1-\frac{\beta}{n-2}}\, \mathrm{AVR}(g)^{\frac{\beta}{n-2}}\, (n-2)^{\beta+1}\, |\mathbb{S}^{n-1}|\,. \tag{1.4.20}$$

Proof. We multiply and divide inside the integral in (1.4.19) by $u^{(n-1)/(n-2)}$. Using the asymptotics of u obtained in Lemma 1.34, we obtain

$$\lim_{R_i\to+\infty} \int\limits_{\{r=R_i\}} \left| \left| \frac{\mathrm{D}u}{u^{\frac{n-1}{n-2}}} \right| - (n-2)\left(\frac{\mathrm{AVR}(g)}{\mathrm{Cap}(\Omega)}\right)^{\frac{1}{n-2}} \right| u^{\frac{n-1}{n-2}}\, \mathrm{d}\sigma \;=\; 0\,.$$

Let us now recall the following basic interpolation inequality

$$\|f\|_{L^p(X)} \;\leq\; \|f\|^{\vartheta}_{L^1(X)} \|f\|^{1-\vartheta}_{L^q(X)},$$

holding for any $f \in L^1(X) \cap L^q(X)$, where X is a measure space and the numbers p and q satisfy $1 < p < q < +\infty$ and $1/p = \vartheta + (1-\vartheta)/q$. We apply such an estimate with

$$f \;=\; \left| \left| \frac{\mathrm{D}u}{u^{\frac{n-1}{n-2}}} \right| - (n-2)\left(\frac{\mathrm{AVR}(g)}{\mathrm{Cap}(\Omega)}\right)^{\frac{1}{n-2}} \right|$$

$p = 1+\beta$, $q = 2+\beta$, so that $\vartheta = 1/(1+\beta)^2$, and with respect to the measure $u^{(n-1)/(n-2)}\,\mathrm{d}\sigma$. We get

$$\begin{aligned}
&\left(\int\limits_{\{r=R_i\}} \left| \left| \frac{\mathrm{D}u}{u^{\frac{n-1}{n-2}}} \right| - (n-2)\left(\frac{\mathrm{AVR}(g)}{\mathrm{Cap}(\Omega)}\right)^{\frac{1}{n-2}} \right|^{1+\beta} u^{\frac{n-1}{n-2}}\, \mathrm{d}\sigma \right)^{1/(1+\beta)} \leq \\
&\quad\leq \left(\int\limits_{\{r=R_i\}} \left| \left| \frac{\mathrm{D}u}{u^{\frac{n-1}{n-2}}} \right| - (n-2)\left(\frac{\mathrm{AVR}(g)}{\mathrm{Cap}(\Omega)}\right)^{\frac{1}{n-2}} \right| u^{\frac{n-1}{n-2}}\, \mathrm{d}\sigma \right)^{1/(1+\beta)^2} \times \\
&\quad\times \left(\int\limits_{\{r=R_i\}} \left| \left| \frac{\mathrm{D}u}{u^{\frac{n-1}{n-2}}} \right| - (n-2)\left(\frac{\mathrm{AVR}(g)}{\mathrm{Cap}(\Omega)}\right)^{\frac{1}{n-2}} \right|^{2+\beta} u^{\frac{n-1}{n-2}}\, \mathrm{d}\sigma \right)^{\beta/(1+\beta)^2}.
\end{aligned} \tag{1.4.21}$$

Due to both the uniform bounds on $|\mathrm{D}u|/u^{(n-1)/(n-2)} = |\nabla\varphi|_{\tilde{g}}$ following from Theorem 1.21 and the bound on $\int_{\{r=R_i\}} u^{\frac{n-1}{n-2}}\,\mathrm{d}\sigma$, that follows from Corollary 1.35, it is easy to see that the second integral on the right hand side of (1.4.21) is bounded in R_i. Thus, by (1.4.21) and (1.4.19), we deduce that

$$\lim_{R_i\to\infty} \int\limits_{\{r=R_i\}} \left| \left| \frac{\mathrm{D}u}{u^{\frac{n-1}{n-2}}} \right| - (n-2)\left(\frac{\mathrm{AVR}(g)}{\mathrm{Cap}(\Omega)}\right)^{\frac{1}{n-2}} \right|^{1+\beta} u^{\frac{n-1}{n-2}}\, \mathrm{d}\sigma \;=\; 0\,.$$

The above limit in particular implies that

$$\lim_{R_i\to+\infty}\int\limits_{\{r=R_i\}}\left|\frac{\mathrm{D}u}{u^{\frac{n-1}{n-2}}}\right|^{1+\beta} u^{\frac{n-1}{n-2}}\,\mathrm{d}\sigma \;=\; (n-2)^{1+\beta}\left(\frac{\mathrm{AVR}(g)}{\mathrm{Cap}(\Omega)}\right)^{\frac{1+\beta}{n-2}}\lim_{R_i\to\infty}\int\limits_{\{r=R_i\}} u^{\frac{n-1}{n-2}}\,\mathrm{d}\sigma\,,$$

that, combined with (1.4.12), gives

$$\lim_{R_i\to+\infty}\int\limits_{\{r=R_i\}}\left|\frac{\mathrm{D}u}{u^{\frac{n-1}{n-2}}}\right|^{1+\beta} u^{\frac{n-1}{n-2}}\,\mathrm{d}\sigma \;=\; (n-2)^{1+\beta}\,\mathrm{Cap}(\Omega)^{1-\frac{\beta}{n-2}}\,\mathrm{AVR}(g)^{\frac{\beta}{n-2}}\,|\mathbb{S}^{n-1}|\,.$$

Moreover, by the asymptotic behaviour of u, we have

$$\lim_{R_i\to+\infty}\int\limits_{\{r=R_i\}}\left|\frac{\mathrm{D}u}{u^{\frac{n-1}{n-2}}}\right|^{1+\beta} u^{\frac{n-1}{n-2}}\,\mathrm{d}\sigma = \lim_{t_i\to0^+}\int\limits_{\{u=t_i\}}\left|\frac{\mathrm{D}u}{u^{\frac{n-1}{n-2}}}\right|^{1+\beta} u^{\frac{n-1}{n-2}}\,\mathrm{d}\sigma \;=\; \lim_{t_i\to0^+} U_\beta(t_i)\,,$$

and we have thus proved our claim for some sequence $t_i\to0^+$. However, by the boundedness and monotonicity of U_β the whole limit as $t\to0^+$ exists, and it coincides with the just computed one. □

The computation of the limit of $U_\infty(t)$ as $t\to0^+$ is more subtle, but it can still be returned to Proposition 1.37, thanks to a clever trick sketched by Colding in the proof of [Col12, Theorem 3.5]. The argument is based on a very nice mean value inequality for positive super-harmonic functions in manifolds with nonnegative Ricci curvature, first provided in [CCM95] and stated and proved separately in [Col12, Lemma 3.6].

Lemma 1.41 (Mean value inequality for positive super-harmonic functions)**.** *Let (M,g) be a complete noncompact Riemannian manifold with nonnegative Ricci curvature and let u be a positive super-harmonic function defined in an open set $D\subseteq M$. Then, for any geodesic ball $B(x,r)\Subset D$ we have*

$$\frac{1}{|\partial B(x,r)|}\int\limits_{\partial B(x,r)} u\,\mathrm{d}\sigma\le \mathrm{C}\;u(x)$$

for some $\mathrm{C}=\mathrm{C}(n)$.

Remark 1.42. The correspondent statement for positive sub-harmonic function is classical and is proved in the celebrated [LS84], see Theorem 2.1.

We can finally compute the limit of U_∞ at infinity.

Proposition 1.43. *Let (M,g) be a complete noncompact Riemannian manifold with* $\mathrm{Ric}\ge0$ *and Euclidean volume growth. Then*

$$\lim_{t\to0^+}U_\infty(t)=(n-2)\left[\frac{\mathrm{AVR}(g)}{\mathrm{Cap}(\Omega)}\right]^{1/(n-2)}. \tag{1.4.22}$$

Proof. Let us first observe that proving

$$\lim_{j\to\infty}\ \sup_{M\setminus B(O,R_j)}\frac{|\mathrm{D}u|^2}{u^{2\frac{n-1}{n-2}}}=(n-2)^2\left[\frac{\mathrm{AVR}(g)}{\mathrm{Cap}(\Omega)}\right]^{2/(n-2)} \tag{1.4.23}$$

for some sequence $\{R_j\}_{j\in\mathbb{N}}$ such that R_j diverges to infinity as $j\to\infty$ suffices to get (1.4.22). Indeed, if (1.4.23) holds, it suffices to take a sequence x_{R_j} realising the supremum of $|\mathrm{D}u|^2/u^{2(n-1)/(n-2)}$ in $M \setminus \overline{B(O,R_j)}$. In fact, x_R is easily seen by the same argument as in the proof of Theorem 1.21 to belong to $\partial B(O,R_j)$. Up to possibly consider a subsequence of the R_j's, let t_j such that $B(O,R_j) \subseteq M \setminus \{u \leq t_j\} \subseteq B(O,R_{j+1})$. Then, we have

$$\sup_{M\setminus \overline{B(O,R_{j+1})}} \frac{|\mathrm{D}u|^2}{u^{2\frac{n-1}{n-2}}} \leq \sup_{M\setminus\{u\leq t_j\}} \frac{|\mathrm{D}u|^2}{u^{2\frac{n-1}{n-2}}} \leq \sup_{M\setminus \overline{B(O,R_j)}} \frac{|\mathrm{D}u|^2}{u^{2\frac{n-1}{n-2}}}. \tag{1.4.24}$$

In particular, since

$$\sup_{M\setminus\{u\leq t_j\}} \frac{|\mathrm{D}u|^2}{u^{2\frac{n-1}{n-2}}} = \sup_{\{u=t_j\}} \frac{|\mathrm{D}u|^2}{u^{2\frac{n-1}{n-2}}}$$

as the argument in the proof of Theorem 1.21 showed, we get, passing to the limit as $j \to +\infty$ in (1.4.24) and using (1.4.23), that

$$\lim_{t_j\to 0^+} \sup_{\{u=t_j\}} \frac{|\mathrm{D}u|^2}{u^{2\frac{n-1}{n-2}}} = (n-2)^2 \left[\frac{\mathrm{AVR}(g)}{\mathrm{Cap}(\Omega)}\right]^{2/(n-2)},$$

that is obviously equivalent to (1.4.22) for the subsequence t_j. However, the monotonicity of U_∞ provided by Theorem 1.21 yields the desired full limit.

We thus devote ourselves to prove (1.4.23). To this aim, let, for $R > 0$ big enough,

$$L_R = \sup_{M\setminus B(O,R)} \frac{|\mathrm{D}u|^2}{u^{2\frac{n-1}{(n-2)}}},$$

and let $x_R \in \partial B(O,R)$, as above, to satisfy $L_R = |\mathrm{D}u|^2/u^{2(n-1)/(n-2)}(x_R)$. Consider now on $M \setminus B(O,R)$ the *positive* function

$$f = u\left(L_R - \frac{|\mathrm{D}u|^2}{u^{2\frac{n-1}{n-2}}}\right).$$

A direct computation shows that

$$\Delta f = -u\left[\Delta\frac{|\mathrm{D}u|^2}{u^{2\frac{n-1}{n-2}}} + 2\left\langle \mathrm{D}u \,\middle|\, \mathrm{D}\left(\frac{|\mathrm{D}u|^2}{u^{2\frac{n-1}{n-2}}}\right)\right\rangle\right],$$

that amounts, by the formulas recalled in Subsection 1.3.1, to

$$\Delta f = -\mathrm{e}^{-\frac{n}{n-2}\varphi}\left(\Delta_{\tilde g}|\nabla\varphi|^2_{\tilde g} - \langle\nabla|\nabla\varphi|^2\,|\nabla\varphi\rangle\right) \leq 0,$$

where the nonpositivity is due to (1.3.15). In particular, the function f is a super-harmonic positive function, and so it satisfies the assumptions of Lemma 1.41. We thus have

$$\frac{1}{|\partial B(x_{4R},\tilde R)|}\int_{\partial B(x_4R,\tilde R)} f\,\mathrm{d}\sigma \leq \mathrm{C}\,f(x_{4R}) = \mathrm{C}\,u(x_{4R})\,(L_R - L_{4R})$$

for any $R < \tilde R < 3R$ and some $\mathrm{C} = \mathrm{C}(n)$. In particular, we get

$$\frac{\inf_{\partial B(x_{4R},\tilde R)} u}{u(x_{4R})}\,\frac{1}{|\partial B(x_{4R},\tilde R)|}\int_{\partial B(x_{4R},\tilde R)} \left(L_R - \frac{|\mathrm{D}u|^2}{u^{2\frac{n-1}{n-2}}}(x_{4R})\right)\mathrm{d}\sigma \leq \mathrm{C}\,(L_R - L_{4R}). \tag{1.4.25}$$

A direct application of the Harnack's inequality (1.2.7) combined with Yau's gradient inequality (1.2.4) shows that the ratio of on the left hand side of (1.4.25) is bounded from below from a constant $C = C(n)$, and then we are left with

$$\frac{1}{R^{n-1}} \int_{\partial B(x_{4R},\tilde{R})} \left(L_R - \frac{|Du|^2}{u^{2\frac{n-1}{n-2}}}(x_{4R}) \right) d\sigma \leq C \left(L_R - L_{4R} \right), \tag{1.4.26}$$

for some constant $C = C(\mathrm{AVR}(g))$, where we also used the assumption of Euclidean volume growth in the form of (1.4.4). Observe now that since, as observed above, L_R is nonincreasing as $R \to +\infty$, it admits a finite limit and in particular the right hand side of (1.4.26) tends to 0 as $R \to +\infty$, forcing the right hand side to do so too. To complete the proof, we are going to show that also

$$\frac{1}{R^{n-1}} \int_{\partial B(x_{4R},\tilde{R})} \left[(n-2)^2 \left(\frac{\mathrm{AVR}(g)}{\mathrm{Cap}(\Omega)} \right)^{2/(n-2)} - \frac{|Du|^2}{u^{2\frac{n-1}{n-2}}}(x_{4R}) \right] d\sigma \to 0 \tag{1.4.27}$$

as $R \to +\infty$. Indeed, since obviously

$$\begin{aligned}\frac{1}{R^{n-1}} \int_{\partial B(x_{4R},\tilde{R})} \left(L_R - \frac{|Du|^2}{u^{2\frac{n-1}{n-2}}}(x_{4R}) \right) d\sigma &= \frac{1}{R^{n-1}} \int_{\partial B(x_{4R},\tilde{R})} \left[L_R - (n-2)^2 \left(\frac{\mathrm{AVR}(g)}{\mathrm{Cap}(\Omega)} \right)^{2/(n-2)} \right] d\sigma \\ &+ \frac{1}{R^{n-1}} \int_{\partial B(x_{4R},\tilde{R})} \left[(n-2)^2 \left(\frac{\mathrm{AVR}(g)}{\mathrm{Cap}(\Omega)} \right)^{2/(n-2)} - \frac{|Du|^2}{u^{2\frac{n-1}{n-2}}}(x_{4R}) \right] d\sigma,\end{aligned}$$

if (1.4.27) holds, we deduce by coupling it with (1.4.26) that

$$\lim_{R \to +\infty} L_R = (n-2)^2 \left[\frac{\mathrm{AVR}(g)}{\mathrm{Cap}(\Omega)} \right]^{2/(n-2)},$$

that is (1.4.23).

To finally prove (1.4.27), consider the annulus $A_{R,3R}(x_{4R}) = B(x_{4R}, 3R) \setminus \overline{B(x_{4R}, R)}$ centered at x_{4R}, and observe that it is contained in the annulus $A_{R,7R}$ centered at the reference point O. By Proposition 1.37, for $\varepsilon > 0$ it holds

$$\frac{R^{2n-2}}{|A_{R,7R}|} \int_{A_{R,7R}} \left| Du - \frac{\mathrm{Cap}(\Omega)}{\mathrm{AVR}(g)} Dr^{2-n} \right|^2 d\mu \leq \varepsilon$$

if R is big enough. Multiplying and dividing for $u^{2(n-1)/(n-2)}$, and employing Lemma 1.34, we deduce that

$$\frac{1}{R^n} \int_{A_{R,7R}} \left| \frac{Du}{u^{\frac{n-1}{n-2}}} - (2-n) \left[\frac{\mathrm{AVR}(g)}{\mathrm{Cap}(\Omega)} \right]^{1/(n-2)} Dr \right|^2 d\mu \leq C_1 \, \varepsilon,$$

for some C_1 independent of R, where we also used the Euclidean volume growth assumption to control the volume of the annulus. Since as just observed $A_{R,3R}(x_{4R}) \subset A_{R,7R}$, we also get

$$\frac{1}{R^n} \int_{A_{R,3R}(x_{4R})} \left| \frac{Du}{u^{\frac{n-1}{n-2}}} - (2-n) \left[\frac{\mathrm{AVR}(g)}{\mathrm{Cap}(\Omega)} \right]^{1/(n-2)} Dr \right|^2 d\mu \leq C_2 \, \varepsilon,$$

for some C_2 independent of R. By the coarea formula and the integral mean value theorem we actually find $\tilde{R} \in (R, 3R)$ such that

$$\frac{1}{R^{n-1}} \int\limits_{\partial B(x_{4R},\tilde{R})} \left| \frac{\mathrm{D}u}{u^{\frac{n-1}{n-2}}} - (2-n) \left[\frac{\mathrm{AVR}(g)}{\mathrm{Cap}(\Omega)}\right]^{1/(n-2)} \mathrm{D}r \right|^2 \mathrm{d}\mu \leq \mathrm{C}_3\, \varepsilon.$$

for any R big enough and some constant C_3 independent of R, that is, we proved (1.4.27). ☐

Limits of the monotone quantities in the sub-Euclidean volume growth case

As happens also for the monotone quantities in [Col12], U_β and U_∞ vanish as $t \to 0^+$ if (M, g) has Euclidean volume growth. In this case, we have

$$\lim_{r\to+\infty} \frac{|B(x,r)|}{r^n} \;=\; 0$$

for any $x \in M$. Then, by (1.2.18), (1.12) and (1.2.6), one easily obtains

$$\frac{|\mathrm{D}u|}{u^{\frac{n-1}{n-2}}} \;\leq \mathrm{C}\; \left[\frac{|B(p,r)|}{r^n}\right]^{\frac{1}{n-2}}$$

outside some ball containing Ω for some $\mathrm{C} = \mathrm{C}(M, \Omega)$. This implies that

$$\lim_{t\to 0^+} U_\beta(t) \;=\; \lim_{t\to 0^+} \int\limits_{\{u=t\}} \left|\frac{\mathrm{D}u}{u^{\frac{n-1}{n-2}}}\right|^\beta |\mathrm{D}u|\, \mathrm{d}\sigma \;=\; 0, \tag{1.4.28}$$

where we used the constancy of $t \mapsto \int_{\{u=t\}} |\mathrm{D}u|\, \mathrm{d}\sigma$. Similarly, one realises that $\lim_{t\to 0^+} U_\infty(t) = 0$. This computation clearly shows that U_β cannot be employed to deduce a Willmore inequality on manifolds with sub-Euclidean volume growth, and it supports the perception that the infimum of the Willmore-type functional is zero on these manifolds. This actually happens for example on noncompact Riemannian manifolds with a metric which is asymptotic to a warped product metric of the following type

$$g \;=\; \mathrm{d}\rho \otimes \mathrm{d}\rho + \mathrm{C}\,\rho^{2\alpha} g_\Sigma\,,$$

where Σ is a closed hypersurface, $\mathrm{C} > 0$ and $0 < \alpha < 1$. Indeed, this is readily checked by computing the Willmore-type functional on large level sets $\{\rho = r\}$. Actually, Theorem 1.68 below, following from the arguments leading to the Isoperimetric Inequality, shows that this happens also on complete noncompact 3-manifolds with $\mathrm{Ric} \geq 0$ admitting a uniform superlinear volume growth condition.

1.4.4 The Willmore-type inequalities for nonnegative Ricci

We are finally in position to state and prove one the first main results of this work, a Willmore-type inequality on complete noncompact Riemannian manifolds with nonnegative Ricci curvature.

Theorem 1.44 (Willmore-type inequality)**.** *Let (M, g) be a complete noncompact Riemannian manifold with* $\mathrm{Ric} \geq 0$ *and Euclidean volume growth. If $\Omega \subset M$ is a bounded and open subset* *with smooth boundary, then*

$$\mathrm{AVR}(g)|\mathbb{S}^{n-1}| \leq \int\limits_{\partial\Omega} \left|\frac{\mathrm{H}}{n-1}\right|^{n-1} \mathrm{d}\sigma\,, \tag{1.4.29}$$

where $\mathrm{AVR}(g) \in (0,1]$ *is the asymptotic volume ratio of* (M,g)*. Moreover, the equality holds if and only if* $(M \setminus \Omega, g)$ *is isometric to*

$$\Big([r_0, +\infty) \times \partial\Omega\,,\ \mathrm{d}r \otimes \mathrm{d}r + (r/r_0)^2 g_{\partial\Omega}\Big), \qquad \text{with} \quad r_0 \;=\; \left(\frac{|\partial\Omega|}{\mathrm{AVR}(g)|\mathbb{S}^{n-1}|}\right)^{\frac{1}{n-1}}.$$

In particular, $\partial\Omega$ *is a connected totally umbilic submanifold with constant mean curvature.*

Proof of Theorem 1.44. With Theorem 1.19 and Proposition 1.40 at hand, Theorem 1.44 follows exactly as in the Euclidean proof recalled in the Introduction. Precisely, let $\beta = n-2$. Then, (1.4.20) reads

$$\lim_{t\to 0^+} U_{n-2}(t) \;=\; \mathrm{AVR}(g)\,(n-2)^{n-1}\,|\mathbb{S}^{n-1}|.$$

Moreover, by the nonnegativity of expression (1.3.3) proved in Theorem 1.19, and the Hölder inequality,

$$\begin{aligned}\frac{(n-1)}{(n-2)}\int_{\partial\Omega} |\mathrm{D}u|^{n-1}\,\mathrm{d}\sigma \;&\leq\; \int_{\partial\Omega} |\mathrm{D}u|^{n-2}\,\mathrm{H}\,\mathrm{d}\sigma \\ &\leq\; \left(\int_{\partial\Omega} |\mathrm{D}u|^{n-1}\,\mathrm{d}\sigma\right)^{(n-2)/(n-1)} \left(\int_{\partial\Omega} \mathrm{H}^{n-1}\,\mathrm{d}\sigma\right)^{1/(n-1)},\end{aligned}$$

that gives

$$\int_{\partial\Omega} |\mathrm{D}u|^{n-1}\,\mathrm{d}\sigma \;\leq\; (n-2)^{n-1}\int_{\partial\Omega}\left|\frac{\mathrm{H}}{n-1}\right|^{n-1}\mathrm{d}\sigma. \tag{1.4.30}$$

Finally, one has

$$\begin{aligned}\mathrm{AVR}(g)\,(n-2)^{n-1}\,|\mathbb{S}^{n-1}| \;=\; \lim_{t\to 0^+} U_{n-2}(t) \leq U_{n-2}(1) = \int_{\partial\Omega} |\mathrm{D}u|^{n-1}\,\mathrm{d}\sigma \;\leq \\ \leq (n-2)^{n-1}\int_{\partial\Omega}\left|\frac{\mathrm{H}}{n-1}\right|^{n-1}\mathrm{d}\sigma\,.\end{aligned} \tag{1.4.31}$$

This completes the proof of the Willmore-type inequality. The rigidity statement when equality is attained follows straightforwardly from the rigidity part of Theorem 1.19. □

Application to ALE manifolds

We can improve our Willmore-type inequality if (M,g) satisfies a *quadratic curvature decay* condition, showing that, in this case, the lower bound $\mathrm{AVR}(g)\big||\mathbb{S}^{n-1}|$ on the Willmore functional is actually an infimum. Let us recall the following well known definition.

Definition 1.45 (Quadratic curvature decay)**.** *A complete noncompact Riemannian manifold* (M,g) *has* quadratic curvature decay *if there exists a point* $x \in M$ *and a constant* $\mathrm{C} = \mathrm{C}(M,x)$ *such that*

$$|\mathrm{Riem}|(y) \;\leq\; \mathrm{C}\,d(x,y)^2,$$

where by Riem *we denote the Riemann curvature tensor of* (M,g)*.*

When this assumption is added on a Riemannian manifold with $\mathrm{Ric} \geq 0$ and Euclidean volume growth (M, g), [CM97b, Proposition 4.1] gives the following asymptotic behaviour of the gradient and the Hessian of the minimal Green's function G.

Theorem 1.46. *Let (M, g) be a complete noncompact Riemannian manifold with $\mathrm{Ric} \geq 0$, Euclidean volume growth and quadratic curvature decay, and let G be its minimal Green's function. Then*

$$\lim_{d(O,x)\to+\infty} \frac{|\mathrm{D}_x G|(O,x)}{d^{1-n}(p,x)} = \frac{(n-2)}{\mathrm{AVR}(g)} \tag{1.4.32}$$

$$\lim_{d(O,x)\to+\infty} \left| \mathrm{DD}_x \left(G^{2/(2-n)} \right)(O,x) - 2 \left(\frac{1}{\mathrm{AVR}(g)} \right)^{\frac{2}{2-n}} g(x) \right| = 0 . \tag{1.4.33}$$

Observe that arguing as in Remark 1.39, one realises that (1.4.32) actually implies that Riemannian manifolds satisfying the assumptions of the above Theorem have finite topological type.

Theorem 1.46 enables us to prove that the Willmore-type functional of large level sets of the Green's function approach $\mathrm{AVR}(g)|\mathbb{S}^{n-1}|$. This fact, combined with our Willmore-type inequality (1.4.29), easily yields the following refinement.

Theorem 1.47. *Let (M, g) be a complete noncompact Riemannian manifold with $\mathrm{Ric} \geq 0$, Euclidean volume growth and quadratic curvature decay. Then,*

$$\inf \left\{ \int_{\partial\Omega} \left| \frac{\mathrm{H}}{n-1} \right|^{n-1} \mathrm{d}\sigma \;\middle|\; \Omega \subset M \text{ bounded and smooth} \right\} = \mathrm{AVR}(g)\, |\mathbb{S}^{n-1}| . \tag{1.4.34}$$

Moreover, the infimum is attained at some bounded and smooth $\Omega \subset M$ if and only if $(M \setminus \Omega, g)$ is isometric to

$$\left([r_0, +\infty) \times \partial\Omega \,,\ \mathrm{d}r \otimes \mathrm{d}r + (r/r_0)^2 g_{\partial\Omega} \right), \qquad \textit{with} \quad r_0 = \left(\frac{|\partial\Omega|}{\mathrm{AVR}(g)|\mathbb{S}^{n-1}|} \right)^{\frac{1}{n-1}} .$$

Proof. Let $p \in M$ be fixed, and let G be the minimal Green's function of (M, g). Let us denote again by G the function $x \mapsto G(O, x)$. In light of Theorem 1.44, it suffices to prove that

$$\lim_{t\to+\infty} \int_{\{G^{2/(2-n)}=t\}} \left| \frac{\mathrm{H}}{n-1} \right|^{n-1} \mathrm{d}\sigma = \mathrm{AVR}(g)|\mathbb{S}^{n-1}| . \tag{1.4.35}$$

To see this, consider, at a point x, orthonormal vectors $\{e_1, \ldots, e_{n-1}\}$ tangent to the level set $\{G = G(x)\}$. Then, letting $r(x) = d(O, x)$, we have, by (1.4.33),

$$\lim_{r(x)\to\infty} \sum_{i=1}^{n-1} \mathrm{DD}\left(G^{\frac{2}{2-n}} \right)(e_i, e_i)(x) = 2(n-1) \left(\frac{1}{\mathrm{AVR}(g)} \right)^{\frac{1}{2-n}} . \tag{1.4.36}$$

The mean curvature of the level sets of $G^{2/(2-n)}$ is easily computed as

$$\mathrm{H}_{G^{2/(2-n)}} = \frac{\sum_{i=1}^{n-1} \mathrm{DD}\left(G^{\frac{2}{2-n}} \right)(e_i, e_i)}{|\mathrm{D}G^{2/(2-n)}|} ,$$

that, combined with (1.4.36), (1.4.32) and (1.4.7) gives

$$\lim_{r(x)\to\infty} r(x)\,\mathrm{H}_{G^{2/(2-n)}}(x) \;=\; (n-1)\,.$$

Combining it again with (1.4.7) and Euclidean volume growth gives (1.4.35), in fact completing the proof. □

We now particularise Theorem 1.47 to *Asymptotically Locally Euclidean* (ALE) manifolds, proving Corollary 1.49. We adopt the following definition, that is a sort of extension of the one considered in the celebrated [BKN89] – where striking relations between curvature decay conditions and behaviour at infinity of manifolds are drawn – and sensibly weaker than the one used by Joyce in the classical reference [Joy00].

Definition 1.48 (ALE manifolds)**.** *We say that a complete noncompact Riemannian manifold (M,g) is ALE (of order τ) if there exist a compact set $K\subset M$, a ball $B\subset\mathbb{R}^n$, a diffemorphism $F: M\setminus K\to\mathbb{R}^n\setminus B$, a subgroup $\Gamma<SO(n)$ acting freely on $\mathbb{R}^n\setminus B$ and a number $\tau>0$ such that*

$$(F^{-1}\circ\pi)^* g(z) \;=\; g_{\mathbb{R}^n} + O(|z|)^{-\tau} \tag{1.4.37}$$

$$\Big|\partial_i[(F^{-1}\circ\pi)^* g]\Big|(z) \;=\; O(|z|)^{-\tau-1} \tag{1.4.38}$$

$$\Big|\partial_i\partial_j[(F^{-1}\circ\pi)^* g]\Big|(z) \;=\; O(|z|)^{-\tau-2}, \tag{1.4.39}$$

where π is the natural projection $\mathbb{R}^n\to\mathbb{R}^n/\Gamma$, $z\in\mathbb{R}^n\setminus B$ and $i,j=1,\dots,n$.

The following is the version of Theorem 1.47 sharpened for ALE manifolds.

Corollary 1.49. *Let (M,g) be an ALE Riemannian manifold with* $\mathrm{Ric}\geq 0$*. Then,*

$$\inf\left\{\int_{\partial\Omega}\left|\frac{\mathrm{H}}{n-1}\right|^{n-1} d\sigma \;\middle|\; \Omega\subset M \text{ bounded and smooth}\right\} \;=\; \frac{|\mathbb{S}^{n-1}|}{\mathrm{card}\,\Gamma}\,. \tag{1.4.40}$$

Moreover, if the infimum is attained by some Ω, then $M\setminus\Omega$ is isometric to

$$\Big([r_0,+\infty)\times(\mathbb{S}^{n-1}/\Gamma)\,,\; dr\otimes dr + r^2 g_{\mathbb{S}^{n-1}/\Gamma}\Big), \qquad \textit{with}\quad r_0 \;=\; \left(\frac{\mathrm{card}\,\Gamma|\partial\Omega|}{|\mathbb{S}^{n-1}|}\right)^{\frac{1}{n-1}}, \tag{1.4.41}$$

for some $r_0>0$ and some finite subgroup Γ of $\mathrm{SO}(n)$. In particular, $(\partial\Omega, g_{\partial\Omega})$ is homothetic to $(\mathbb{S}^{n-1}/\Gamma, g_{\mathbb{S}^{n-1}/\Gamma})$.

Proof. Conditions (1.4.37), (1.4.38), (1.4.39) readily imply that ALE manifolds have Euclidean volume growth and quadratic curvature decay. Moreover, condition (1.4.37) and a direct computation give that

$$\mathrm{AVR}(g) \;=\; \frac{|\mathbb{S}^{n-1}/\Gamma|}{|\mathbb{S}^{n-1}|} \;=\; \frac{1}{\mathrm{card}\,\Gamma}\,. \tag{1.4.42}$$

The characterisation (1.4.40) then follows immediately from (1.4.34).

Assume now that the infimum of the Willmore functional is attained at some $\Omega\subset M$. Then, by the rigidity part in Theorem 1.47, $M\setminus\Omega$ is isometric to a truncated cone over $\partial\Omega$. However, by (1.4.37), (M,g) is also C^0-close at infinity to a metric cone with link $\mathbb{S}^{n-1}/\Gamma$. Since the cross sections of a cone are all homothetic to each other, $\partial\Omega$ is homothetic to

$\mathbb{S}^{n-1}/\Gamma$, that is, they are diffeomorphic and $g_{\partial\Omega} = \lambda^2 g_{\mathbb{S}^{n-1}/\Gamma}$ for some positive constant λ. This fact, together with (1.4.42) in the rigidity part of Theorem 1.44 imply that

$$\Big(\big[r_0, +\infty\big) \times \partial\Omega\,,\ \mathrm{d}r \otimes \mathrm{d}r + (\lambda r/r_0)^2 g_{\mathbb{S}^{n-1}/\Gamma} \Big), \qquad \text{with} \quad r_0 \ = \ \left(\frac{|\partial\Omega|}{|\mathbb{S}^{n-1}/\Gamma|} \right)^{\frac{1}{n-1}}.$$

In particular, one has

$$|\partial B(O,R)| \ = \ \left(\frac{R\lambda}{r_0}\right)^{n-1} |\mathbb{S}^{n-1}/\Gamma|.$$

Combining this with (1.4.42) we conclude that $\lambda = r_0$, proving the isometry with (1.4.41) and completing the proof. □

1.5 Other consequences of the Monotonicity-Rigidity Theorem for nonparabolic manifolds

In this short section we draw some other consequences of Theorem 1.19, in particular of the monotonicity of U_∞, that we did not apply yet.

1.5.1 Capacity estimates

The following capacity estimate hold on any nonparabolic manifold with $\mathrm{Ric} \geq 0$, and, at this point not surprisingly, holds as equality only on truncated cones.

Proposition 1.50 (Capacity estimates for nonparabolic manifolds)**.** *Let (M, g) be a complete noncompact Riemannian manifold with nonnegative Ricci curvature, and let $\Omega \subset M$ be a bounded set with smooth boundary. Then, for any $\beta \geq (n-2)/(n-1)$, we have*

$$\mathrm{Cap}(\Omega) \leq \frac{|\partial\Omega|}{|\mathbb{S}^{n-1}|} \left(\fint_{\partial\Omega} \left| \frac{\mathrm{H}}{n-1} \right|^{1+\beta} \mathrm{d}\sigma \right)^{1/(1+\beta)}. \tag{1.5.1}$$

Moreover, equality is achieved if and only if $(M \setminus \Omega, g)$ is isometric to

$$\Big(\big[r_0, +\infty\big) \times \partial\Omega\,,\ \mathrm{d}r \otimes \mathrm{d}r + (r/r_0)^2 g_{\partial\Omega} \Big), \qquad \textit{with} \quad r_0 \ = \ \left(\frac{|\partial\Omega|}{\mathrm{AVR}(g)|\mathbb{S}^{n-1}|} \right)^{\frac{1}{n-1}}.$$

Proof. By the nonnegativity of expression (1.3.3) for $t = 1$, we get, as for (1.4.30), that

$$\int_{\partial\Omega} |\mathrm{D}u|^{\beta+1}\, \mathrm{d}\sigma \leq (n-2)^{1+\beta} \int_{\partial\Omega} \left| \frac{\mathrm{H}}{n-1} \right|^{1+\beta} \mathrm{d}\sigma. \tag{1.5.2}$$

Recalling that

$$\mathrm{Cap}(\Omega) \ = \ \frac{\displaystyle\int_{\partial\Omega} |\mathrm{D}u|\, \mathrm{d}\sigma}{(n-2)|\mathbb{S}^{n-1}|}$$

we easily obtain (1.5.1) by applying the Hölder inequality in the above definition of capacity and combining with (1.5.2). See the analogous proof of the nonlinear version Theorem 2.5.1 in $\mathbb{R}^n$ for additional details. The rigidity statement follow immediately from the equality case of Theorem 1.19. □

We would like to highlight, among the various inequalities provided in (1.5.1), the one obtained setting $\beta = n-2$, reading

$$\mathrm{Cap}(\Omega) \leq \frac{|\partial\Omega|^{\frac{(n-2)}{(n-1)}}}{|\mathbb{S}^{n-1}|} \left(\int_{\partial\Omega} \left| \frac{\mathrm{H}}{n-1} \right|^{n-1} \mathrm{d}\sigma \right)^{1/(n-1)},$$

since it combines the Willmore-type functional and the capacity of Ω. In the next chapter we will show how in $\mathbb{R}^n$ the nonlinear version of it, that is (2.5.4), involving the p-capacity of Ω, actually leads to the Willmore inequality in the limit as $p \to n$.

In the the above inequality we just used $U_\beta{}'(0) \geq 0$, and this is why it is meaningful on any nonparabolic manifold with nonnegative Ricci curvature. The following one, on the other hand, will employ $\lim_{t\to 0^+} U_\beta(t) \leq U_\beta(1)$, and this is where, by Proposition 1.40, the Asymptotic Volume Ratio and thus the Euclidean volume growth assumption enters.

Proposition 1.51 (Capacity estimates for manifolds with Euclidean volume growth)**.** *Let (M, g) be a complete noncompact Riemannian manifold with nonnegative Ricci curvature and Euclidean volume growth, and let $\Omega \subset M$ be a bounded set with smooth boundary. Then, for every $\beta \geq (n-2)/(n-1)$ we have*

$$\mathrm{AVR}(g)^{\beta/(n-2)} \frac{|\mathbb{S}^{n-1}|}{|\partial\Omega|} \leq [\mathrm{Cap}(\Omega)]^{\frac{\beta-(n-2)}{n-2}} \fint_{\partial\Omega} \left| \frac{\mathrm{H}}{n-1} \right|^{1+\beta} \mathrm{d}\sigma.$$

Moreover, equality is achieved if and only if $(M \setminus \Omega, g)$ is isometric to

$$\Big([r_0, +\infty) \times \partial\Omega \,,\; \mathrm{d}r \otimes \mathrm{d}r + (r/r_0)^2 g_{\partial\Omega} \Big), \qquad \text{with} \quad r_0 = \left(\frac{|\partial\Omega|}{\mathrm{AVR}(g)|\mathbb{S}^{n-1}|} \right)^{\frac{1}{n-1}}.$$

Proof. This is substantially (1.4.31) with a generic $\beta \geq (n-2)/(n-1)$, obtained combining (1.4.20) with (1.5.2) in the proof above, and suitably rearranging the terms. The rigidity statement follows as usual from the rigidity statement of Theorem 1.19. □

As before, we would like to point out a particularly relevant inequality given by the above result, that is, the inequality obtained through setting $\beta = 1$, reading

$$\Big(\mathrm{AVR}(g)\mathrm{Cap}^{n-3}(\Omega) \Big)^{1/(n-2)} \leq \frac{1}{|\mathbb{S}^{n-1}|} \int_{\partial\Omega} \left| \frac{\mathrm{H}}{n-1} \right|^2 \mathrm{d}\sigma.$$

This is, with the terminology of Theorem 2.24 in the next chapter, the L^2-Minkowski inequality in the general setting of manifolds with nonnegative Ricci curvature and Euclidean volume growth. The L^p version of it, indeed, will be seen to lead, in $\mathbb{R}^n$, to the (extended) Minkowski inequality.

1.5.2 Rigidity Theorems under pinching conditions

Observe that, passing to the limit as $\beta \to \infty$ in (1.5.1) after taking the $1+\beta$-power on both sides, one gets

$$\left[\frac{\mathrm{AVR}(g)}{\mathrm{Cap}(\Omega)} \right]^{\frac{1}{n-2}} \leq \sup_{\partial\Omega} \left| \frac{\mathrm{H}}{n-1} \right|.$$

However, with this procedure one cannot apply the rigidity statement of Theorem 1.19, that regards finite values of β, in order to prove that equality in the above inequality characterises truncated cones. On the other hand, this can be deduced from the monotonicity property of U_∞ and the value of its limit (1.4.22).

Theorem 1.52. *Let (M, g) be a complete noncompact Riemannian manifold with nonnegative Ricci curvature and Euclidean volume growth. If there exists a bounded set $\Omega \subset M$ with smooth boundary satisfying*

$$-\left[\frac{\mathrm{AVR}(g)}{\mathrm{Cap}(\Omega)}\right]^{\frac{1}{n-2}} \leq \frac{\mathrm{H}}{n-1} \leq \left[\frac{\mathrm{AVR}(g)}{\mathrm{Cap}(\Omega)}\right]^{\frac{1}{n-2}} \tag{1.5.3}$$

on every point of $\partial\Omega$, then $(M \setminus \Omega, g)$ is isometric to

$$\Big([r_0, +\infty) \times \partial\Omega\,,\ dr \otimes dr + (r/r_0)^2 g_{\partial\Omega}\Big), \qquad \text{with} \quad r_0 \;=\; \left(\frac{|\partial\Omega|}{\mathrm{AVR}(g)|\mathbb{S}^{n-1}|}\right)^{\frac{1}{n-1}}.$$

Proof. By Theorem 1.21 and Proposition 1.43, we immediately get

$$\lim_{t\to 0^+} U_\infty(t) = (n-2)\left[\frac{\mathrm{AVR}(g)}{\mathrm{Cap}(\Omega)}\right]^{1/(n-2)} \leq \sup_{\partial\Omega}|\mathrm{D}u| \leq (n-2)\sup_{\partial\Omega}\left|\frac{\mathrm{H}}{n-1}\right|,$$

where the equality is achieved only if $(M \setminus \Omega, g)$ splits as in the statement. In particular, this happens if (1.5.3) holds. □

The above result can be interpreted as a rigidity theorem under a pinching condition on the mean curvature of $\partial\Omega$ with respect to the capacity of Ω, first observed for $\mathbb{R}^n$ in [BMM19], where it can be interpreted as a sphere theorem. An analogous statement is immediately deduced from the proof above for $|\mathrm{D}u|$, stating that

$$\left[\frac{\mathrm{AVR}(g)}{\mathrm{Cap}(\Omega)}\right]^{1/(n-2)} \leq \sup_{\partial\Omega}\left|\frac{\mathrm{D}u}{n-2}\right|, \tag{1.5.4}$$

with equality achieved only on truncated metric cones. An application of the above sharp inequality yields the following analogue of [BMM19, Corollary 1.4], that, in our case, gives a sufficient condition to infer that the whole manifold is actually $\mathbb{R}^n$ with Euclidean metric and Ω is a ball.

Theorem 1.53. *Let (M, g) be a complete noncompact Riemannian manifold with $\mathrm{Ric} \geq 0$ and Euclidean volume growth, and let $\Omega \subset M$ be bounded set with smooth boundary. Let u be the solution to (1.2.15), and assume that*

$$\sup_{\partial\Omega}\left|\frac{\mathrm{D}u}{n-2}\right| \leq \mathrm{AVR}(g)\left(\frac{|\mathbb{S}^{n-1}|}{|\partial\Omega|}\right)^{\frac{1}{n-1}}. \tag{1.5.5}$$

Then (M, g) is isometric to $\mathbb{R}^n$ with the Euclidean metric and Ω is a ball.

Proof. Under the validity of (1.5.5), we get

$$\mathrm{Cap}(\Omega) \;=\; \frac{\displaystyle\int_{\partial\Omega}|\mathrm{D}u|\, d\sigma}{(n-2)|\mathbb{S}^{n-1}|} \leq \mathrm{AVR}(g)\left(\frac{|\mathbb{S}^{n-1}|}{|\partial\Omega|}\right)^{-\frac{n-2}{n-1}},$$

that in particular yields

$$\left(\frac{|\mathbb{S}^{n-1}|}{|\partial\Omega|}\right)^{\frac{n-2}{n-1}} \leq \frac{\mathrm{AVR}(g)}{\mathrm{Cap}(\Omega)} \leq \sup_{\partial\Omega}\left|\frac{\mathrm{D}u}{n-2}\right|^{n-2} \leq \mathrm{AVR}(g)^{n-2}\left(\frac{|\mathbb{S}^{n-1}|}{|\partial\Omega|}\right)^{\frac{n-2}{n-1}}, \tag{1.5.6}$$

where we used (1.5.4) and again (1.5.5) respectively in the second and third inequality. In particular, the chain above implies that $\mathrm{AVR}(g) = 1$, that implies, by the Bishop-Gromov

Theorem, that (M, g) is actually isometric to $(\mathbb{R}^n, g_{\mathbb{R}^n})$. Moreover, since the second inequality in (1.5.6) becomes an equality, we infer from Theorem 1.21 that $\partial\Omega$ is a compact connected totally umbilical submanifold, that is, Ω is a ball. □

The strong rigidity statement proved above is the same that happens in the equality case of the Isoperimetric Inequality for manifolds with $\mathrm{Ric} \geq 0$, that is Theorem 1.60.

1.6 Monotonicity-Rigidity Theorems for parabolic manifolds and the Enhanced Kasue's Theorem

For $\beta \geq 0$, we recall the definition of the function $\Psi_\beta : [0, +\infty) \to \mathbb{R}$ satisfying

$$\Psi_\beta(s) = \int_{\{\psi = s\}} |D\psi|^{\beta+1}\, d\sigma\,,$$

and that of $\Psi_\infty : [0, +\infty) \to \mathbb{R}$

$$\Psi_\infty(s) = \sup_{\{\psi = s\}} |D\psi|\,,$$

where ψ is a solution to problem (1.2.22) for some bounded $\Omega \subset M$ with smooth boundary and $\beta \geq 0$. The following is the statement of the monotonicity-rigidity properties of Ψ_β.

Theorem 1.54 (Monotonicity-Rigidity Theorem for parabolic manifolds)**.** *Let (M, g) be a parabolic manifold with $\mathrm{Ric} \geq 0$. Let $\Omega \subset M$ be a bounded and open subset with smooth boundary, and let ψ be a solution to problem (1.2.22). Then, for every $\beta \geq (n-2)/(n-1)$, the function Ψ_β is differentiable with derivative*

$$\frac{d\Psi_\beta}{ds}(s) = -\beta \int_{\{\psi = s\}} |D\psi|^{\beta}\, \mathrm{H}\, d\sigma\,,$$

where H is the mean curvature of the level set $\{\psi = s\}$ computed with respect to the unit normal vector field $\nu = D\psi/|D\psi|$. Moreover, for every $s \geq 0$, the derivative fulfils

$$\begin{aligned}\frac{d\Psi_\beta}{ds}(s) = -\,\beta \int_{\{\psi \geq s\}} |D\psi|^{\beta-2}\Big\{ \mathrm{Ric}(D\psi, D\psi) + \left(\beta - \frac{n-2}{n-1}\right) \big|D|D\psi|\big|^2 \\ + |D\psi|^2 \left|\mathrm{h} - \frac{\mathrm{H}}{n-1}\right|^2 + \left(\frac{n-2}{n-1}\right) \big|D^\top |D\psi|\big|^2 \Big)\Big\}\, d\mu\,.\end{aligned} \tag{1.6.1}$$

In particular, $d\Psi_\beta/ds$ is always nonpositive. Moreover, $(d\Psi_\beta/ds)(s_0) = 0$ for some $s_0 \geq 1$ and some $\beta \geq (n-2)/(n-1)$ if and only if $\big(\{\psi \geq s_0\}, g\big)$ is isometric to the Riemannian product $\big([s_0, +\infty) \times \{\psi = s_0\}, d\rho \otimes d\rho + g_{\{\psi = s_0\}}\big)$. In this case, in particular, $\partial\Omega$ is a connected totally geodesic submanifold.

Analogously, these properties are shared by Ψ_∞.

Theorem 1.55 (Monotonicity-Rigidity Theorem for Ψ_∞)**.** *Let (M, g) be a complete parabolic manifold with $\mathrm{Ric} \geq 0$. Then, the function $\Psi_\infty : [0, \infty) \to \mathbb{R}$ defined by (13) is monotone* *nonincreasing. Moreover, if $x_s \in \{\psi = s\}$ is such that*

$$|D\psi|(x_s) = \sup_{\{\psi=s\}} |D\psi|,$$

then

$$H(x_s) = -\frac{\partial}{\partial \nu_s} \log|D\psi| \geq 0, \tag{1.6.2}$$

where ν_s is the unit normal to $\{\psi = s\}$ given by $\nu = D\psi/|D\psi|$. Moreover, equality holds in the inequality of (1.6.2) for some $s = s_0 \in [0, \infty)$ or $\Psi_\infty(s_1) = \Psi_\infty(s_0)$ for some $s_1 > s_0$ if and only if $(\{\psi \geq s_0\}, g)$ is isometric to the Riemannian product $([s_0, +\infty) \times \{\psi = s_0\}, d\rho \otimes d\rho + g_{\{\psi=s_0\}})$. In this case, in particular, $\partial\Omega$ is a connected totally geodesic submanifold.

As for the proofs of Theorems 1.25 and 1.26, the triggering factor implying the monotonicity of the above functions is a Bochner-type identity. In the parabolic case, it reads, for $\beta \geq 0$,

$$\Delta|D\psi|^\beta = \beta|D\psi|^{\beta-2}\left[|DD\psi|^2 + (\beta-2)\big|D|D\psi|\big|^2 + \mathrm{Ric}(D\psi, D\psi)\right]. \tag{1.6.3}$$

The details of the above basic computation are analogous to those giving the Bochner-type identity (1.3.35).

As done in Section 1.3, we first show the monotonicity of Ψ_∞, together with the related rigidity statements.

Proof of Theorem 1.55. Define now the auxiliary function w_α

$$w_\alpha = |D\psi|^2(\psi+1)^{-\alpha}.$$

Applying (1.6.3) with $\beta = 2$, a straightforward computation gives

$$\Delta w_\alpha + 2\alpha \left\langle Dw_\alpha \,\middle|\, D\log(\psi+1)\right\rangle = 2\left[|DD\psi|^2 + \mathrm{Ric}(D\psi, D\psi)\right] + \alpha(1-\alpha)|D\psi^2|w_\alpha.$$

If $\alpha \in (0,1)$, we see that the right hand side of the equation above is nonnegative, and in particular, by the Maximum Principle applied in $\{s \leq \psi \leq S\}$ we find that that the maximum of w_α in such a set is achieved on $\{\psi = s\} \cup \{\psi = S\}$. Since by Theorem 1.17 the function$|D\psi|^2$ is uniformly bounded, and then w_α vanishes at infinity, we infer that $\sup_{\{\varphi \geq s\}} w_\alpha = \sup_{\{\varphi = s\}} w_\alpha$ for any $\alpha \in (0,1)$. Letting $\alpha \to 0$, we get the desired upper bound for $|D\psi|^2$, giving as a consequence the monotonicity of Ψ_∞. The inequality (1.6.2) follows as in the proof of Theorem 1.26 by the property of x_s to be a maximum point.

Once the monotonicity and (1.6.2) are established, the rigidity statements are obtained exactly as in the proof of Theorem 1.26, using that, by (1.6.3)

$$\Delta|D\psi|^2 \geq 0$$

in place of (1.3.34). The case of singular level sets is treated as there too. □

Now, we prove the Monotonicity-Rigidity Theorem for parabolic manifolds with nonnegative Ricci curvature.

Proof of Theorem 1.54. Observe that the right hand side of (1.6.3) is nonnegative if $\beta \geq (n-2)/(n-1)$, by means of the refined Kato's inequality (1.3.7) for harmonic functions. Analogously to (1.3.42), we first prove, applying the Divergence Theorem to the vector field $Z = |D\psi|^\beta D\psi$ in the set $\{s_0 \le \psi \le s\}$, that

$$\Psi_\beta(s) - \Psi_\beta(s_0) \;=\; \int\limits_{\{s_0 \le \varphi \le s\}} \operatorname{div}\left(|D\psi|^\beta D\psi\right) d\mu \;=\; \int\limits_{\{\sigma_0 \le \varphi \le s\}} \left\langle D|D\psi|^\beta \,\middle|\, \frac{D\psi}{|D\psi|} \right\rangle d\mu\,. \tag{1.6.4}$$

Applying the Divergence Theorem on to the vector field $W_\varepsilon = \chi_\varepsilon(|D\psi|^2)W$, with $W = D|D\psi|^\beta$ and χ_ε the same cut-off function as in the proof of Lemma 1.29, we can show that

$$\begin{aligned} \int\limits_{\{\psi=S\}} \left\langle D|D\psi|^\beta \,\middle|\, \frac{D\psi}{|D\psi|} \right\rangle d\sigma \;-\; \int\limits_{\{\psi=s\}} \left\langle D|D\psi|^\beta \,\middle|\, \frac{D\psi}{|D\psi|} \right\rangle d\sigma \;=& \\ =\; \beta \int\limits_{\{s \le \psi \le S\}} |D\psi|^{\beta-2} \left[|DD\psi|^2 + (\beta-2)\big|D|D\psi|\big|^2 + \operatorname{Ric}(D\psi, D\psi)\right] d\mu & \end{aligned} \tag{1.6.5}$$

everytime $s < S$ are regular values. Sard's Theorem then allows to extend the function

$$s \to \int\limits_{\{\psi=s\}} \left\langle D|D\psi|^\beta \,\middle|\, \frac{D\psi}{|D\psi|} \right\rangle d\sigma$$

to a continuous function on $[0, \infty)$, and finally, by applying the coarea formula to (1.6.4), this permits to apply the Fundamental Theorem of Calculus in order to get

$$\Psi'_\beta(s) \;=\; -\beta \int\limits_{\{\psi=s\}} \left\langle D|D\psi|^\beta \,\middle|\, \frac{D\psi}{|D\psi|} \right\rangle d\sigma,$$

where the above right hand side is defined by density also on singular values. In particular, with this convention, (1.6.5) yields

$$\Psi'_\beta(S) - \Psi'_\beta(s) \;=\; \beta \int\limits_{\{s \le \psi \le S\}} |D\psi|^{\beta-2}\Big(\operatorname{Ric}(D\psi, D\psi) + \big|DD\psi\big|^2 + (\beta-2)\,\big|D|D\psi|\big|^2\Big)\, d\mu \;\ge\; 0\,. \tag{1.6.6}$$

Combining the uniform bound on $|D\psi|$ given in Theorem 1.17 with the constancy of

$$s \to \int\limits_{\{\psi=s\}} |D\psi|\, d\sigma,$$

we obtain, as in Corollary 1.27, that Ψ_β is uniformly bounded in s for any $\beta \ge 0$. We can thus argue as in the conclusion of the proof of Theorem 1.25 to pass to the limit as $S \to +\infty$ in (1.6.6) and obtain, plugging also the Kato-type identity (1.22), the monotonicity formula (1.6.1). The rigidity part of the statement is obtained exactly as for that of Theorem 1.25. □

1.6.1 The Enhanced Kasue's Theorem

The Enhanced Kasue's Theorem 1.56 now follows at once.

Theorem 1.56 (Enhanced Kasue's Theorem)**.** *Let (M, g) be a complete noncompact Riemannian manifold with* $\operatorname{Ric} \ge 0$*, and let $\Omega \subset M$ be a bounded and open subset with smooth boundary. Then, the following assertions hold true.*

(i) If (M, g) is nonparabolic, then, for every $\beta \geq (n-2)/(n-1)$,

$$\sup_{\partial\Omega} \mathrm{H} \;\geq\; \frac{1}{\int_{\partial\Omega} |\mathrm{D}u|^{\beta}\, \mathrm{d}\sigma} \left[\left(\frac{n-1}{n-2}\right) U_{\beta}(0) + \frac{1}{\beta} \frac{\mathrm{d}U_{\beta}}{\mathrm{d}t}(0) \right] > 0\,, \tag{1.6.7}$$

and

$$\sup_{\partial\Omega} \mathrm{H} \;\geq\; \left(\frac{n-1}{n-2}\right) |\mathrm{D}u|(x_1) - \frac{\partial}{\partial\nu} \log \frac{|\mathrm{D}u|}{u^{\frac{n-1}{n-2}}}(x_1) \geq \left(\frac{n-1}{n-2}\right) \sup_{\partial\Omega} |\mathrm{D}u| > 0, \tag{1.6.8}$$

where H is the mean curvature of $\partial\Omega$, $\nu = -\mathrm{D}u/|\mathrm{D}u|$ and x_1 is defined in Theorem 1.21.

(ii) If (M, g) is parabolic, then, for every $\beta \geq (n-2)/(n-1)$,

$$\sup_{\partial\Omega} \mathrm{H} \;\geq\; -\frac{1}{\int_{\partial\Omega} |\mathrm{D}\psi|^{\beta}\, \mathrm{d}\sigma} \left[\frac{1}{\beta} \frac{\mathrm{d}\Psi_{\beta}}{\mathrm{d}s}(0) \right] \geq 0\,, \tag{1.6.9}$$

and

$$\sup_{\partial\Omega} \mathrm{H} \;\geq\; -\frac{\partial}{\partial\nu} \log |\mathrm{D}\psi|(x_0) \geq 0 \tag{1.6.10}$$

where H is the mean curvature of $\partial\Omega$, $\nu = \mathrm{D}\psi/|\mathrm{D}\psi|$ and x_0 is defined in Theorem 1.55. Moreover, the second inequality in (1.6.9) and the second inequality in (1.6.10) hold with the equality sign if and only if $(M \setminus \Omega, g)$ is isometric to the Riemannian product $\big([0, +\infty) \times \partial\Omega,\ \mathrm{d}\rho \otimes \mathrm{d}\rho + g_{\partial\Omega}\big)$ and $\partial\Omega$ is a totally geodesic connected submanifold of (M, g).

Kasue's Theorem follows immediately as a corollary.

Corollary 1.57 (Kasue's Theorem)**.** *Let (M, g) be a complete noncompact Riemannian manifold with $\mathrm{Ric} \geq 0$ and let $\Omega \subset M$ be a bounded and open subset with smooth boundary such that $\mathrm{H}_{\partial\Omega} \leq 0$ on $\partial\Omega$. Then $(M \setminus \Omega, g)$ is isometric to the Riemannian product $\big([0, +\infty) \times \partial\Omega,\ \mathrm{d}r \otimes \mathrm{d}r + g_{\partial\Omega}\big)$ and $\partial\Omega$ is a totally geodesic connected submanifold of (M, g).*

Proof of Theorem 1.56 and Corollary 1.57. Assume first that (M, g) is nonparabolic. Then, it is sufficient to use (1.3.3) and (1.3.4) at $t = 1$ to get

$$\left(\frac{n-1}{n-2}\right) U_{\beta}(0) + \frac{1}{\beta} \frac{\mathrm{d}U_{\beta}}{\mathrm{d}t}(0) \;=\; \int_{\partial\Omega} \mathrm{H}\, |\mathrm{D}u|^{\beta}\, \mathrm{d}\sigma \;\leq\; \sup_{\partial\Omega} \mathrm{H}_{\partial\Omega} \int_{\partial\Omega} |\mathrm{D}u|^{\beta}\, \mathrm{d}\sigma\,,$$

that is (1.6.7). If (M, g) is parabolic, one can prove inequality (1.6.9) in a completely analogous fashion. The inequalities (1.6.8) and (1.6.10) on the other hand follow immediately from (1.3.6) and (1.6.2). Since $U_{\beta} > 0$, it is easy to deduce from (1.6.7) and (1.6.9) that $\mathrm{H}_{\partial\Omega} \leq 0$ on $\partial\Omega$ if and only if (M, g) is parabolic and $\mathrm{d}\Psi/\mathrm{d}s_{\beta}(0) = 0$. The rigidity statements of Theorems 1.54 and 1.55 imply at once the rigidity statement in Theorem 1.56, that in particular implies Corollary 1.57. □

Corollary 1.57 can also be interpreted as a rigidity statement when the equality is attained in (1.4.29) if $\mathrm{AVR}(g) = 0$. The following is then a direct consequence of Theorem 1.44 and Corollary 1.57.

Corollary 1.58. *Let (M, g) be a complete noncompact Riemannian manifold with $\mathrm{Ric} \geq 0$. If $\Omega \subset M$ is a bounded subset with smooth boundary, then*

$$\mathrm{AVR}(g)|\mathbb{S}^{n-1}| \;\leq\; \int_{\partial\Omega} \left| \frac{\mathrm{H}}{n-1} \right|^{n-1} \mathrm{d}\sigma\,. \tag{1.6.11}$$

If $\mathrm{AVR}(g) > 0$, *then equality in* (1.6.11) *holds if and only if* $(M \setminus \Omega, g)$ *is isometric to*

$$\Big([r_0, +\infty) \times \partial\Omega\,,\ \mathrm{d}r \otimes \mathrm{d}r + (r/r_0)^2 g_{\partial\Omega} \Big), \qquad \textit{with} \quad r_0 = \left(\frac{|\partial\Omega|}{\mathrm{AVR}(g)\, |\mathbb{S}^{n-1}|} \right)^{\frac{1}{n-1}}.$$

In particular, $\partial\Omega$ *is a connected submanifold with constant mean curvature. If* $\mathrm{AVR}(g) = 0$, *equality holds in* (1.6.11) *if and only if* $(M \setminus \Omega, g)$ *is isometric to a Riemannian product* $([0, +\infty) \times \partial\Omega, \mathrm{d}r \otimes \mathrm{d}r + g_{\partial\Omega})$. *In particular,* $\partial\Omega$ *is a connected totally geodesic submanifold of* (M, g).

1.6.2 A Maximum Principle/Splitting Theorem for manifolds with two boundary components

The Splitting Theorem by Kasue, that we rediscovered in Corollary 1.57, is known to be true also in a compact versions. Namely, it is proved in [Kas83, Theorem B 1] or in [CK92, Theorem 1] that if a compact manifold (M, g) has nonnegative Ricci curvature and boundary $\partial M = N \sqcup N_1$, with N a compact connected submanifold, such that ∂M has nonpositive mean curvature , then it splits a compact cylinder $([0, T] \times N, \mathrm{d}t \otimes \mathrm{d}t + g_N)$. We show that we can recover also this result by means of a suitable boundary value problem. As in Theorem 1.56, this new approach actually bounds from below the supremum of the mean curvature of ∂M in terms of the nonnegative normal derivative on the boundary of a suitable sub-harmonic function, that vanishes exactly when the splitting occurs. In spite of this enhancement, we point out that our assumptions are more restrictive than those of [Kas83, Theorem B 1], in that we also ask for N_1 to be a compact smooth submanifold. The reason for this extra condition is in the definition of a boundary value problem with Dirichlet data on ∂M. This seems to be a delicate issue in case of noncompact boundary, and we are not intentioned to address it here.

Theorem 1.59. *Let* (M, g) *be a compact Riemannian manifold with* $\mathrm{Ric} \geq 0$ *and nonempty boundary* $\partial M = N \sqcup N_1$, *where* N *and* N_1 *are complete connected smooth manifolds of dimension* $n - 1$. *Let* v *the solution to the boundary value problem*

$$\begin{cases} \Delta v = 0 & \textit{in } M \\ v = 0 & \textit{on } N \\ v = 1 & \textit{on } N_1. \end{cases} \tag{1.6.12}$$

Let $x \in \partial M$ *realise* $|\mathrm{D}v|(x) = \sup_{\partial M} |\mathrm{D}v|$. *Then, we have*

$$\sup_{\partial M} \mathrm{H} \geq -\frac{\partial}{\partial \nu} \log |\mathrm{D}v|(x) \geq 0, \tag{1.6.13}$$

where H *is the mean curvature of* ∂M *with respect to the inward pointing unit normal* ν. *Moreover, equality holds in the second inequality above if and only if* (M, g) *splits as* $([0, T] \times N, \mathrm{d}t \otimes \mathrm{d}t + g_N)$. *In particular, this happens if and only* $\mathrm{H} = 0$ *on the whole* ∂M.

Proof. Since v satisfies (1.6.12), the Bochner inequality yields

$$\Delta |\mathrm{D}v|^2 = 2\left[|\mathrm{D}\mathrm{D}v|^2 + \mathrm{Ric}(\mathrm{D}v, \mathrm{D}v) \right]. \tag{1.6.14}$$

In particular, by the Maximum Principle

$$\sup_M |\mathrm{D}v|^2 = \sup_{\partial M} |\mathrm{D}v|^2$$

Let then x as in the statement be the point on the boundary of N where the maximum of $|\mathrm{D}v|^2$ is achieved. Letting ν the inward normal, it is immediately checked, as in the proof of Theorems 1.3 and 1.55 that

$$\frac{\partial}{\partial \nu}|\mathrm{D}v|^2(x) = -2\mathrm{H}|\mathrm{D}v|^2(x) \leq 0, \tag{1.6.15}$$

where the sign is due to the property of x to be a maximum point. This already proves (1.6.13). Assume now that equality holds in the second inequality of (1.6.13), or equivalently in the inequality of (1.6.15). By the Hopf Lemma, this can happen only if $|\mathrm{D}v|^2$ is constant in M. By (1.6.14) and the assumption of nonnegative Ricci curvature, this implies that the Hessian of v vanishes, and this yields, by Corollary 1.23, the claimed splitting. □

1.7 The isoperimetric inequality for 3-manifolds

As already discussed in the Introduction, we show here how to use our Willmore inequality (1.4.29) to improve a result stated by Huisken in [Hui], in which the infimum of the Willmore energy is characterised in terms of the infimum of the isoperimetric ratio on 3-manifolds with nonnegative Ricci curvature.

Theorem 1.60 (AVR(g) & Isoperimetric Constant)**.** *Let (M, g) be a complete noncompact 3-manifold with* $\mathrm{Ric} \geq 0$ *and Euclidean Volume Growth. Then,*

$$\inf \frac{|\partial\Omega|^3}{36\pi|\Omega|^2} = \inf \frac{\displaystyle\int_{\partial\Omega} \mathrm{H}^2 \, \mathrm{d}\sigma}{16\pi} = \mathrm{AVR}(g), \tag{1.7.1}$$

where the infima are taken over bounded and open subsets $\Omega \subset M$ with smooth boundary. In particular, the following isoperimetric inequality holds for any bounded and open $\Omega \subset M$ with smooth boundary

$$\frac{|\partial\Omega|^3}{|\Omega|^2} \geq 36\pi \, \mathrm{AVR}(g). \tag{1.7.2}$$

Moreover, equality is attained in (1.7.2) if and only if $M = \mathbb{R}^3$ and Ω is a ball.

We point out that in the 3-dimensional case Theorem 1.60 extends Theorem 1.47 to any complete noncompact manifold with $\mathrm{Ric} \geq 0$ and Euclidean volume growth, with no curvature assumptions at infinity. Actually, we can even relax the volume growth assumption, see Theorem 1.68.

1.7.1 Huisken's argument.

Let us briefly present Huisken's heuristic argument to deduce an isoperimetric inequality from Willmore's through the *mean curvature flow*. We first recall that a sequence of orientable hypersurfaces $F_t(x) : \Sigma \to M$ immersed in a Riemann manifold (M, g), evolves through the Mean Curvature Flow if

$$\frac{\mathrm{d}}{\mathrm{d}t} F_t(x) = -\mathrm{H}(t,x)\, \nu(t,x)\,,$$

where H is the mean curvature of $\Sigma_t = F_t(\Sigma)$ and ν is its (exterior, in the case where Σ_t is the boundary of a domain) unit normal. Accordingly, we say that $\{\Omega_t\}$ is a mean curvature flow if the boundaries are evolving through mean curvature flow in the sense explained above. Let then Ω be an open bounded set with smooth boundary and let $\{\Omega_t\}$, with $t \in [0,T)$, be a smooth mean curvature flow starting from Ω. Suppose, in addition, that

$$\lim_{t\to T^-} |\Omega_t| = 0. \tag{1.7.3}$$

Consider, for some constant $C > 0$ to be defined later, the *isoperimetric difference*

$$D(t) = |\partial\Omega_t|^{3/2} - C\,|\Omega_t|\,. \tag{1.7.4}$$

Taking derivatives in t, and using standard formulas (see for example [HP99, Theorem 3.2]), one finds

$$\frac{\mathrm{d}}{\mathrm{d}t}D(t) = -\frac{3}{2}\,|\partial\Omega_t|^{1/2}\int_{\partial\Omega_t} \mathrm{H}^2\,\mathrm{d}\sigma + C\int_{\partial\Omega_t} \mathrm{H}\,\mathrm{d}\sigma\,,$$

that, through Hölder inequality, gives

$$\frac{\mathrm{d}}{\mathrm{d}t}D(t) \leq \left(|\partial\Omega_t|\int_{\partial\Omega_t} \mathrm{H}^2\,\mathrm{d}\sigma\right)^{1/2}\left[C - \frac{3}{2}\left(\int_{\partial\Omega_t} \mathrm{H}^2\,\mathrm{d}\sigma\right)^{1/2}\right].$$

Thus, if we choose C such that

$$C \leq \frac{3}{2}\left(\int_{\partial\Omega} \mathrm{H}^2\,\mathrm{d}\sigma\right)^{1/2} \tag{1.7.5}$$

for any bounded and smooth $\Omega \subset M$, $t \mapsto D(t)$ is nonincreasing. This implies that

$$D(0) = |\partial\Omega|^{3/2} - C|\Omega| \geq \lim_{t\to T^-} D(t) \geq 0\,,$$

where we have also used (1.7.3). The above comparison in particular gives the (possibly non sharp) isoperimetric inequality

$$\frac{|\partial\Omega|^{3/2}}{|\Omega|} \geq C\,.$$

In [Hui], the constant C is chosen to be the infimum of the right hand side of (1.7.5), when Ω varies in the class of outward minimising sets.

1.7.2 Tools from the Mean Curvature Flow of mean-convex domains.

We are first concerned with the accurate justification of the above computations. This will be accomplished with the help of a couple of important results due to Schulze and White, respectively. In the first part of our treatment we assume that the boundary $\partial\Omega$ of the bounded set Ω is smooth and mean-convex, that we understand as $\mathrm{H} > 0$. We will see later how to deal with the general cases.

Since the Mean Curvature Flow (MCF for short) is likely to develop singularities, one needs to consider an appropriate weak notion in order to state the following useful results. In particular, we consider the Weak Mean Curvature Flow in the sense defined in [ES91]. A special case of the regularity theorem [Whi00, Theorem 1.1] gives

Theorem 1.61 (White's Regularity Theorem)**.** *Let (M,g) be a complete noncompact 3-dimensional Riemannian manifold, let $\Omega \subset M$ be a bounded set with smooth mean-convex boundary and let*

$\{\Omega_t\}_{t\in[0,T)}$ be its Weak Mean Curvature Flow. Then, the boundary of Ω_t is smooth for almost every $t \in [0, T)$.

We point out that the maximal time T might *a priori* be infinite on a general Riemannian manifold. We are going to combine the above regularity result with the following special case of [Sch08, Proposition 7.2], that is a weak version of the monotonicity of the isoperimetric ratio. It can be checked, indeed, that the computations performed to obtain such a result do not involve the geometry of the underlying manifold.

Theorem 1.62 (Schulze)**.** *Let (M, g) be a 3-dimensional Riemannian manifold, let $\Omega \subset M$ be a bounded set with smooth mean-convex boundary and let $\{\Omega_t\}_{t\in[0,T)}$ be its Weak Mean Curvature Flow. Assume there exists a universal constant $C \geq 0$ such that*

$$C \leq \frac{3}{2}\left(\int_{\partial\Omega_t} H^2 \, d\sigma\right)^{1/2}$$

for almost every $t \in [0, T)$. Then, the isoperimetric difference $t \mapsto D(t)$ defined as in (1.7.4) using the constant C, is nonincreasing for every $t \in [0, T)$.

Remark 1.63. A different tool that might be used to deal with the singularities would be the theory of the *Mean Curvature Flow with surgery*, recently developed by Brendle and Huisken in [BH16] and [BH18]. On this regard, one should first make clear whether the monotonicity of the isoperimetric difference survives the surgeries.

The following theorem provides a complete description of the long time behaviour of the Weak MCF of a surface moving inside a 3-dimensional Riemannian manifold, and it can be readily deduced from [Whi00, Theorem 11.1].

Theorem 1.64 (Long time behaviour of MCF)**.** *Let (M, g) be a 3-dimensional Riemannian manifold, let $\Omega \subset M$ be a bounded set with smooth mean-convex boundary and let $\{\Omega_t\}_{t\in[0,T)}$ be its Weak Mean Curvature Flow. If $|\Omega_t|$ and $|\partial\Omega_t|$ do not vanish at finite time as $t \to T^-$, then Ω_t converges smoothly to a subset K, and the boundary of any connected component of K is a stable minimal submanifold.*

As a consequence, if (M, g) contains no bounded subsets with minimal boundary, the Weak MCF of a bounded set with mean-convex boundary is going to vanish. In particular, combining Corollary 1.57 with Theorem 1.64, one gets the following corollary.

Corollary 1.65. *Let (M, g) be a complete, noncompact, 3-dimensional Riemannian manifold with $\mathrm{Ric} \geq 0$ and no cylindrical ends, let $\Omega \subset M$ be a bounded set with smooth mean-convex boundary and let $\{\Omega_t\}_{t\in[0,T)}$ be its Weak Mean Curvature Flow. Then, T is finite and $|\Omega_t|$ and $|\partial\Omega_t|$ tend to 0 as $t \to T^-$.*

1.7.3 Some properties of the strictly outward minimising hull

In order to prove the isoperimetric inequality for any set Ω with smooth and possibly not mean-convex boundary, we are going to consider the strictly outward minimising hull Ω^*. This notion, with roots in [BT84], manifested its importance in the celebrated construction of weak solutions the Inverse Mean Curvature Flow of [HI01], that led to the proof of the Riemannian Penrose Inequality. We are going to extensively discuss this notion in Chapter 3. For the time being, we limit ourselves to define briefly the relevant notions and state the fundamental facts needed in the proof. We first recall the notion of *strictly outward minising sets*, a variational property of bounded sets with finite perimeter that played a key role in the foundation of the Weak Inverse Mean Curvature Flow worked out in [HI01]. Simply, we say that a bounded set with finite perimeter $E \subset \mathbb{R}^n$ is *outward minimising* if for any bounded set with finite perimeter $F \supset E$ with finite perimeter and $|F \setminus E| > 0$ it holds $P(F) \geq P(E)$. We say that it is *strictly* outward minimising if for any such set F we have in fact $P(F) > P(E)$. We can thus define the *strictly outward minimising hull* Ω^* of an open bounded set Ω as the intersection of all the strictly outward minimising sets containing Ω. What mostly matters to us now, is that on any complete noncompact Riemannian manifold with nonnegative Ricci curvature and Euclidean volume growth Ω^* is an open bounded strictly outward minimising set with finite perimeter containing Ω and satisfying

$$P(\Omega^*) = \inf\{P(E) \mid \Omega \subseteq E\}.$$

In particular, if Ω is outward minimising then Ω^* satisfies $P(\Omega) = P(\Omega^*)$. This is (part of) the content of Theorem 3.1, that is the main result of Chapter 3. Let us just briefly remark that complete noncompact Riemannian manifolds with nonnegative Ricci curvature satisfy (i) in that statement, compare with the discussion in the Introduction. By [SZW91], $\partial\Omega^*$ enjoys $\mathscr{C}^{1,1}$-regularity where it touches the obstacle Ω, and it is a locally area minimising hypersurface where it does not. See Theorem 3.14 for the complete statement. In particular, if Ω has smooth boundary, the strictly outward minimising hull also enjoys $P(\Omega^*) = |\partial\Omega^*|$, see Remark 3.16. By minimal surfaces regularity, up to ambient dimension 7, $\partial\Omega$ is then a $\mathscr{C}^{1,1}$-hypersurface. These last deep and celebrated results on the regularity of locally area minimising hypersurfaces were pioneered in [DG61] and later completed in [Alm66] and [Sim67]. See also the comprehensive [HI01, Regularity Theorem 1.3] for an account on the regularity of solutions to more general obstacle problems. We complete this overview on the properties of $\partial\Omega$ pointing out that, by the outward minimising property, the weak mean curvature $\mathrm{H}_{\partial\Omega^*}$ is nonnegative. This readily follows from the standard first variation argument.

We will actually flow Ω^* by mean curvature. To this aim, we will invoke [HI01, Lemma 5.6], where the authors show that bounded set E with $\mathscr{C}^{1,1}$-boundary having nonnegative variational mean curvature can be approximated in $\mathscr{C}^1$ by smooth hypersurfaces with strictly positive mean-curvature. These approximating hypersurfaces happen be boundaries of sets E_ε that are strictly outward minimising if E was. Moreover, the Willmore energy of the approximators converges to the Willmore energy (in terms of the variational mean curvature) of ∂E. Interestingly, the approximating sequence is built through an appropriate notion of mean curvature flow starting from such a $\mathscr{C}^{1,1}$-hypersurface. Such a result has found many other applications in literature, see for example [LW17, Lemma 4.2] in the ambient setting of a Kottler-Schwarzschild manifold, and the proof of [Sch08, Corollary 1.2]. Moreover, it has been generalised to $\mathscr{C}^1$ boundaries in [HI08, Lemma 2.6] admitting nonnegative weak mean curvature, and the approximation is also shown to take actually place in $W^{2,q}$. We include here a general statement that combines [HI01, Lemma 5.6] and [HI08, Lemma 2.6].

Lemma 1.66 (Huisken-Ilmanen's approximation lemma)**.** *Let (M, g) be a complete Riemannian manifold, let $E \subset M$ be a bounded set with $\mathscr{C}^1$-boundary and let $F_0 : \partial E \hookrightarrow M$ be its immersion . Assume that ∂E has nonnegative weak mean curvature, that is, it admits a nonnegative function $\mathrm{H} \in L^1_{\mathrm{loc}}(\Sigma)$ such that*

$$\int_{\partial E} \mathrm{div}_\Sigma X \, \mathrm{d}\sigma = \int_{\partial E} \mathrm{H} \langle X, \nu \rangle \, \mathrm{d}\sigma \tag{1.7.6}$$

for any compactly supported vector field X of M. Assume also that ∂E is not minimal, that is, there exist a subset $K \subset \Sigma$ of positive measure such that $\mathrm{H} > 0$ on K. Then, there exists a family of bounded open sets E_ε with smooth boundaries ∂E_ε given by the immersions $F(\varepsilon, \cdot) : \partial E \hookrightarrow M$, with $\varepsilon \in (0, \varepsilon_0]$, such that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} F(\varepsilon, x) \;=\; -\mathrm{H}(\varepsilon, x)\nu(\varepsilon, x)$$

for any $\varepsilon \in (0, \varepsilon_0]$, where $\mathrm{H}(\varepsilon, x)$ is the mean curvature of the immersion $F(\varepsilon, \cdot)$ at point x, $\nu(\varepsilon, x)$ its outer unit normal at point x, and we have

$$\lim_{\varepsilon \to 0^+} F(\varepsilon, \cdot) \;=\; F_0(\cdot)$$

in $\mathscr{C}^{1,\alpha} \cap W^{2,q}$ for any $0 < \alpha < 1$ and $q \geq 1$. Moreover, $\mathrm{H}_{\partial E_\varepsilon} > 0$ for any $\varepsilon \in (0, \varepsilon_0]$, and if E is strictly outward minimising then so is E_ε.

We point out that the weak mean curvature of a closed hypersurface Σ is easily verified to exist if Σ is of class $\mathscr{C}^{1,1}$. In the following remark we show how through a parabolic maximum principle the mean curvature flow naturally approximates mean-convex hypersurface with strictly mean-convex ones. This behaviour lies at the core of the proof of Lemma 1.66 given in [HI01], together with fine higher derivative estimates of [EH91].

Remark 1.67. Observe that if Σ is a (non minimal) closed $\mathscr{C}^2$ hypersurface with $\mathrm{H}_\Sigma \geq 0$, then the approximation of Σ by means of a family of smooth mean-convex hypersurfaces $\{\Sigma_\varepsilon\}_{\varepsilon > 0}$ is a straightforward procedure. Indeed, it is sufficient to run the MCF starting at Σ for short time (see [Man11] for an account about the classical existence theory), say until some time $\varepsilon_0 > 0$. This provides a family of hypersurfaces $\{\Sigma_\varepsilon\}_{\varepsilon \in (0, \varepsilon_0]}$, whose mean curvatures satisfy (see e.g. [HP99, Theorem 3.2]) the following reaction-diffusion equation,

$$\frac{\partial}{\partial \varepsilon} \mathrm{H} \;=\; \Delta \mathrm{H} + \mathrm{H}\left(|\mathrm{h}|^2 + \mathrm{Ric}\,(\nu, \nu)\right),$$

where h is the second fundamental form of the evolving hypersurface and Ric is the Ricci tensor of the ambient manifold. Then, a standard maximum principle for parabolic equations (see e.g. Theorem 7 and subsequent remarks in [PW84]) shows that $\mathrm{H}_{\Sigma_\varepsilon} > 0$ for every $\varepsilon \in (0, \varepsilon_0]$, unless H_Σ is constantly null. The latter case is excluded by the non minimality of Σ.

Finally, we have at hand all the ingredients that allow to completely justify the computations of Subsection 1.7.1 and in turn to prove Theorem 1.60.

Proof of Theorem 1.60. The following argument follows the lines of the proof of [Sch08, Corollary 1.2]. Let us first suppose that the boundary of $\partial\Omega$ is strictly mean-convex, that is, $\mathrm{H}_{\partial\Omega} > 0$. Let $\{\Omega_t\}_{t \in [0,T)}$ be a mean curvature flow starting from Ω. Then, by Theorem 1.61, for almost any $t \in [0, T)$ the boundary $\partial\Omega_t$ is a smooth submanifold. Let C be defined as

$$\mathrm{C} \;=\; \inf\left\{ \frac{3}{2}\left(\int_{\partial\Omega} \mathrm{H}^2 \,\mathrm{d}\sigma\right)^{1/2} \,\middle|\, \Omega \subset M \text{ bounded set with smooth boundary} \right\}. \tag{1.7.7}$$

Then, Theorem 1.62 guarantees that, with the above choice of C, the isoperimetric difference $t \mapsto D(t)$ defined in (1.7.4) is nonincreasing for $t \in [0, T)$. Moreover, by Corollary 1.65 $D(t)$ tends to 0 as $t \to T^-$. This implies the inequality

$$\frac{|\partial\Omega|^{3/2}}{|\Omega|} \geq \mathrm{C}$$

for any Ω with smooth mean-convex boundary. If this is not the case, take the minimising hull Ω^* of Ω (see [HI01, Section 1] for details). As discussed above $\partial\Omega^*$ is a $\mathscr{C}^{1,1}$ hypersurface. Observe that, by the minimising property, $|\partial\Omega^*| \leq |\partial\Omega|$, while trivially $|\Omega^*| \geq |\Omega|$. Hence, proving a lower bound on the isoperimetric ratio for Ω^* readily implies that the same lower bound holds for Ω. Moreover, again by the minimising property, we have that $\mathrm{H}_{\partial\Omega^*} \geq 0$ (see also [HI01, (1.15)]). Also notice that $\partial\Omega^*$ cannot be minimal, for otherwise (M, g) will have a cylindrical end, in virtue of Corollary 1.57, against the Euclidean volume growth assumption. By Lemma 1.66, we find a sequence bounded sets E_ε with smooth and strictly mean-convex boundary approximating $\partial\Omega^*$ in $\mathscr{C}^1$. Arguing as above, we thus obtain the isoperimetric inequality

$$\frac{|\partial E_\varepsilon|^{3/2}}{|E_\varepsilon|} \geq \mathrm{C},$$

that, through letting $\varepsilon \to 0^+$, gives the isoperimetric inequality for Ω^*, and, in turn, for any bounded Ω with smooth boundary. Combining it with our Willmore inequality (1.4.29), we get

$$\inf \frac{|\partial\Omega|^3}{36\pi|\Omega|^2} \geq \inf \frac{\left(\displaystyle\int_{\partial\Omega} \mathrm{H}^2 \, d\sigma\right)}{16\pi} \geq \mathrm{AVR}(g), \tag{1.7.8}$$

where the infima are taken over any bounded Ω with smooth boundary. We now want to prove that the equality sign hold in both the above inequalities, as stated in (1.7.1). To do so, we fix a point $O \in M$ and we observe that by the Bishop-Gromov Theorem, we can find, for every $\delta > 0$, a radius R_δ such that

$$\frac{|\partial B(O, R_\delta)|^3}{36\pi|B(O, R_\delta)|^2} \leq \mathrm{AVR}(g) + \delta.$$

Observe that we can suppose $\partial B(O, R_\delta)$ to be smooth. Otherwise, it suffices to consider in place of $B(O, R_\delta)$ a smooth set whose perimeter and volume approximate $|\partial B(O, R_\delta)|$ and $|B(O, R_\delta)|$, respectively (this can be done by standard tools, see e.g. [Mag12, Remark 13.2]). This proves that

$$\inf \left\{ \frac{|\partial\Omega|^3}{36\pi|\Omega|^2} \;\middle|\; \Omega \subset M \text{ bounded and smooth} \right\} \leq \mathrm{AVR}(g).$$

Combining the above inequality with (1.7.8), gives (1.7.1).

To prove the rigidity statement, we assume now that (1.7.2) holds with the equality sign for a smooth and bounded $\Omega \subset M$. In virtue of (1.7.7) and of (1.7.1), we have that

$$\mathrm{C} = \sqrt{36\pi \mathrm{AVR}(g)},$$

By the minimising property, Ω^* satisfies the same equality (recall that we actually proved the isoperimetric inequality for minimising hulls). We claim that it also holds for any approximating E_ε as above. Indeed, by Lemma 1.66, this family is a smooth mean curvature flow, and then, by the monotonicity of the isoperimetric difference, for any fixed $\varepsilon_1 \in (0, \varepsilon_0]$ we have

$$D(\varepsilon) \,\geq\, D(\varepsilon_1) \,\geq\, 0$$

for every $\varepsilon \in (0, \varepsilon_1)$. Since ∂E_ε converges strongly enough to $\partial\Omega^*$ as $\varepsilon \to 0^+$, and on Ω^* the isoperimetric difference is 0, $D(\varepsilon) \to 0^+$ as $\varepsilon \to 0^+$, and thus $D(\varepsilon_1) = 0$ as well. Since ε_1 was arbitrarily chosen, E_ε satisfies the equality in the isoperimetric inequality for $\varepsilon \in (0, \varepsilon_0]$, as claimed. In particular, from the same computations as in Subsection 1.7.1, for any fixed $\varepsilon \in (0, \varepsilon_0]$, we have that

$$0 \,=\, \frac{\mathrm{d}D}{\mathrm{d}\varepsilon}(\varepsilon) \,\leq\, \left(|\partial E_\varepsilon| \int_{\partial E_\varepsilon} \mathrm{H}^2\, \mathrm{d}\sigma\right)^{1/2} \left[\sqrt{36\pi \mathrm{AVR}(g)} \,-\, \frac{3}{2}\left(\int_{\partial E_\varepsilon} \mathrm{H}^2\, \mathrm{d}\sigma\right)^{1/2}\right] \,\leq\, 0\,.$$

This implies that the equality sign holds in the Willmore inequality for Σ_ε, and thus, by the rigidity statement in Theorem 1.44, $(M \setminus E_\varepsilon, g)$ is isometric to the truncated cone

$$\Big(\,[r_\varepsilon, +\infty) \times \partial E_\varepsilon\,,\; \mathrm{d}r \otimes \mathrm{d}r + (r/r_\varepsilon)^2 g_{\partial E_\varepsilon}\Big), \qquad \text{where} \quad r_\varepsilon \,=\, \left(\frac{|\partial E_\varepsilon|}{4\pi \mathrm{AVR}(g)}\right)^{1/2}.$$

Hence, it is easily seen that the MCF $\{\partial E_\varepsilon\}_{\varepsilon \in (0, \varepsilon_0]}$ is given by totally umbilic hypersurfaces coinciding with the cross sections of the cone

$$(M \setminus E_\varepsilon, g) \cong \Big(\,[r_\varepsilon, +\infty) \times \partial\Omega^*\,,\; \mathrm{d}r \otimes \mathrm{d}r + \frac{4\pi \mathrm{AVR}(g)}{|\partial\Omega^*|} r^2 g_{\partial\Omega^*}\Big)\,. \tag{1.7.9}$$

We now claim that the MCF $\{\partial E_\varepsilon\}_{\varepsilon > 0}$ does not develop singularities before the extinction time ε^*. Letting $(0, \varepsilon_*)$ be the maximal interval of existence of the smooth MCF, we claim that $\varepsilon_* = \varepsilon^*$. In fact, from (1.7.9) one can easily see that the mean curvature of Σ_ε is given by $(n-1)/r_\varepsilon$, and in turn the squared norm of its second fundamental form is equal to $(n-1)/r_\varepsilon^2$. It follows then by [Hui86, Theorem 7.1] that ε_* is such that $r_{\varepsilon_*} = 0$, and thus coincides with the extinction time of the flow, i.e., $\varepsilon^* = \varepsilon_*$. We have hence deduced the isometry

$$(M \setminus \{O\}, g) \cong \Big(\,(0, +\infty) \times \partial\Omega^*\,,\; \mathrm{d}r \otimes \mathrm{d}r + \frac{4\pi \mathrm{AVR}(g)}{|\partial\Omega^*|} r^2 g_{\partial\Omega^*}\Big),$$

for some $O \in M$. In particular, the surface area of the geodesic balls centered at O decays as $4\pi r^2 \mathrm{AVR}(g)$, and, since g is smooth at O, this implies that $\mathrm{AVR}(g) = 1$. By Bishop-Gromov, we infer that (M, g) is isometric to $(\mathbb{R}^3, g_{\mathbb{R}^3})$ and $\partial\Omega^*$ is isometric to a sphere. This implies that $\Omega = \Omega^*$, since, otherwise, the mean curvature of $\partial\Omega^*$ would be null on the points not belonging to $\partial\Omega$ (compare with Theorem 3.14), leading to a contradiction. We have thus shown that Ω is a ball, completing the proof. □

Observe that the proof illustrated above works with no modification also without the Euclidean volume growth assumption, that is with $\mathrm{AVR}(g) = 0$, if one were able to ensure that a bounded solution to the least area problem with a bounded open set with smooth boundary exists anyway. Even more weakly, it would suffice to show that there exists an exhausting sequence of sets with smooth and mean-convex boundaries. The existence of such a sequence does not seem evident. However, in Chapter 3, Theorem 3.1, we prove that if the volume growth is uniformly superlinear in the sense of (1.7.10) below, Ω^* is a well defined set with the properties needed above. Thus, we have actually proved also the following result.

Theorem 1.68. *Let (M, g) be a complete noncompact Riemannian manifold with nonnegative Ricci curvature. Assume also that there exists $O \in M$ and $1 < b < n$ and a constant* C *such that*

$$C^{-1} r^b \leq |B(O,r)| \leq C r^b \tag{1.7.10}$$

for any $r \geq r_0$ for some fixed $r_0 > 0$. Then,

$$\inf \frac{|\partial\Omega|^3}{|\Omega|^2} = \inf \int_{\partial\Omega} H^2 \, d\sigma = 0. \tag{1.7.11}$$

.

Observe that the interest in the above result lies in the second equality of (1.7.11), that generalises, in the 3-dimensional case, the examples provided at the end of Subsection 1.4.3.

1.7.4 Perspectives and applications of the Isoperimetric Inequalities

We discuss here some topics related to the Isoperimetric Inequality for 3-manifolds with nonnegative Ricci curvature. Namely, we state it in the equivalent form of Sobolev Inequality, we show how it yields the sharp constant in the Faber-Krahn Inequality, we discuss its natural extension to higher dimensions and we compare it with the existence of isoperimetric sets.

The Sobolev Inequality

Let us first observe how it translates in terms of sharp Sobolev Inequality.

Theorem 1.69 (AVR(g) & Sobolev Constant)**.** *Let (M, g) be a complete noncompact 3-manifold with* Ric ≥ 0 *and Euclidean Volume Growth. Then*

$$\inf_{f \in W_0^{1,1}(M)} \frac{\displaystyle\int_M |Df| \, d\sigma}{\left(\displaystyle\int_M |f|^{3/2} \, d\sigma\right)^{2/3}} = \sqrt[3]{36\pi \, AVR(g)}\,.$$

The equivalence between the Sobolev and the Isoperimetric Inequality is very well known, and seemingly it was first displayed along [FF60, Remark 6.6], where the argument given actually does not depend on the ambient manifold. We refer also to [Cha84, Chapter IV, Theorem 4] for a presentation.

Faber-Krahn Inequality

A very famous application of the Euclidean Isoperimetric Inequality is the celebrated Faber-Krahn inequality, stating that for any bounded $\Omega \subset \mathbb{R}^n$ with smooth boundary there holds

$$\lambda_1(\Omega) \geq \lambda_1(B_\Omega), \tag{1.7.12}$$

where B_Ω is the ball with $|\Omega| = |B_\Omega|$, and $\lambda_1(\Omega)$ is the first Dirichlet eigenvalue of Ω, that is the lowest and strictly greater than zero number λ such that

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.7.13}$$

admits a solution. Moreover, a rigidity statement characterises the equality case in (1.7.12), stating that equality occurs only if $B_\Omega = \Omega$.

It is also well known that the Faber-Krahn inequality (1.7.12) actually *just* depends on the Euclidean Isoperimetric Inequality, and in particular it holds on any Riemannian manifold supporting this structural property. This is carried out in details in the book [Cha84], see Theorem 2 in Chapter 4. As a consequence, (1.7.12) holds on Cartan-Hadamard manifolds of dimension 3 and 4, since with this assumptions the Euclidean Isoperimetric Inequality is known to be true, as proved in [Kle92] and [Cro84]. A simple generalisation of the argument in the proof of the Faber-Krahn Inequality actually allowed to provide it in closed Riemannian manifolds with Ricci curvature bounded below by a positive number, by using, in place of the Euclidean Isoperimetric Inequality, the classical Levy-Gromov Isoperimetric Inequality (see for example [GHL04, Theorem 4.6]). The details of this version of the Faber-Krahn Inequality in closed manifolds are worked out in [BM82].

Here, we point out that the Isoperimetric Inequality for 3-manifolds with nonnegative Ricci curvature (1.7.2) yields a Faber-Krahn Inequality on these manifolds. Since the isoperimetric constant is not the Euclidean one, but it depends on the Asymptotic Volume Ratio, the same is happening for the constant in the eigenvalue comparison.

Theorem 1.70 (Faber-Krahn inequality for 3-manifolds with nonnegative Ricci curvature and Euclidean Volume Growth)**.** *Let (M, g) be a complete noncompact Riemannian 3-manifold with nonnegative Ricci curvature and Euclidean Volume Growth. Let $\Omega \subset M$ be a bounded set with smooth boundary, and let $B_\Omega \subset \mathbb{R}^n$ satisfy $|B_\Omega|_{\mathbb{R}^n} = |\Omega|$. Then, we have*

$$\lambda_1(\Omega) \geq \mathrm{AVR}(g)^{2/3} \lambda_1(B_\Omega), \tag{1.7.14}$$

where $\lambda_1(\Omega)$ is the first Dirichlet eigenvalue of Ω in M with respect to the metric g and $\lambda_1(B_\Omega)$ is the first Dirichlet eigenvalue of B_Ω in $\mathbb{R}^n$. Moreover, equality holds in (1.7.14) if and only if (M, g) is isometric to $(\mathbb{R}^n, g_{\mathbb{R}^n})$ and Ω is isometric to B_Ω.

It is not surprising, in light of the discussion above, that the above result follows from the Isoperimetric Inequality Theorem 1.60 by adapting the classical argument. We provide here the details, mainly to clarify how the constant $\mathrm{AVR}(g)$ arises in (1.7.14). It follows from the following fact, that is a slight generalisation of what is known in literature as Pólia-Szegö Principle. The argument is exactly that of [Cha84]. We sketch it, in order to clarify the role played by the isoperimetric constant.

Proposition 1.71 (Generalised Pólia-Szegö Principle)**.** *Let (M, g) be a Riemannian manifold, and assume that for any bounded $\Omega \subset M$ with smooth boundary there holds*

$$\frac{|\partial\Omega|^n}{|\Omega|^{n-1}} \geq n^n |\mathbb{B}^n| \mathrm{C}_{\mathrm{iso}} \tag{1.7.15}$$

for some $\mathrm{C}_{\mathrm{iso}}$ independent of Ω.

Let, for $\Omega \subset M$ a bounded set with smooth boundary, $f \in \mathscr{C}^\infty(\overline{\Omega})$ be a positive function with $f = 0$ on $\partial\Omega$. Then, there exists a function $F \in W^{1,2}_0(B_\Omega)$, where $B_\Omega \subset \mathbb{R}^n$ is a ball with $|\Omega| = |B_\Omega|_{\mathbb{R}^n}$ such that

$$\int_\Omega |\mathrm{D} f|^2 \, \mathrm{d}\mu \geq \mathrm{C}_{\mathrm{iso}}^{\frac{2}{n}} \int_{B_\Omega} |\mathrm{D} F|^2_{\mathbb{R}^n} \, \mathrm{d}\mu_{\mathbb{R}^n} \qquad \int_\Omega f^2 \, \mathrm{d}\mu = \int_{B_\Omega} F^2 \, \mathrm{d}\mu_{\mathbb{R}^n}. \tag{1.7.16}$$

Moreover, equality holds in the inequality above if and only Ω satisfies equality in (1.7.15).

Proof. Let $V : [0, T] \to \mathbb{R}$, where $T = \max_{\overline{\Omega}} f$, the function defined by $V(t) = |\{f \geq t\}|$. This function is easily seen to be continuous and actually $\mathscr{C}^\infty$ on regular values of f.

Consider $B_t = B_{\{f\geq t\}}$, that is, the ball in $\mathbb{R}^n$ with (Euclidean) volume equal to $V(t)$. This defines a bijective function $\rho : [0,T] \to [0,\rho(0)]$ such that $V(t) = |B(\rho(t))|_{\mathbb{R}^n}$, where $B(\rho(t))$ is any ball of radius $\rho(t)$. Define finally $F : \overline{B_\Omega} \to \mathbb{R}$ by $F = \rho^{-1} \circ r$, where r is the distance from the center of B_Ω. It is shown exactly as in [Cha84] that with this definition the identity in (1.7.16) is satisfied. By coarea formula, we have

$$\int_\Omega |\mathrm{D}f|^2 \,\mathrm{d}\mu = \int_0^T \int_{\{f=t\}} |\mathrm{D}f| \,\mathrm{d}\sigma \,\mathrm{d}t \geq \int_0^T |\{f=t\}|^2 \int_{\{f=t\}} \frac{1}{|\mathrm{D}f|} \,\mathrm{d}\sigma \,\mathrm{d}t \tag{1.7.17}$$

On the other hand, by (1.7.15),

$$|\{f=t\}|^2 \;\geq\; |\{f\geq t\}|^{2\frac{n-1}{n}} n^2 |\mathbb{B}^n|^{\frac{2}{n}} \mathrm{C}_{\mathrm{iso}}^{\frac{2}{n}} \;=\; |B_t|_{\mathbb{R}^n}^{2\frac{n-1}{n}} \; n^2\, |\mathbb{B}^n|^{\frac{2}{n}} \mathrm{C}_{\mathrm{iso}}^{\frac{2}{n}} \;=\; |\partial B_t|^2_{\mathbb{R}^n} \mathrm{C}_{\mathrm{iso}}^{2/n}, \tag{1.7.18}$$

where in the last equality we used the fact that balls in $\mathbb{R}^n$ satisfy equality in the Isoperimetric Inequality. Using that

$$\rho'(t) = -\frac{1}{|\partial B_t|_{\mathbb{R}^n}} \int_{\{f=t\}} \frac{1}{|\mathrm{D}f|} \,\mathrm{d}\sigma \qquad \left(\rho^{-1}\right)'(\rho(t)) = \frac{1}{\rho'(t)},$$

we get, plugging the outcome of (1.7.18) in (1.7.17) and using again the coarea formula we obtain

$$\int_\Omega |\mathrm{D}f|^2 \,\mathrm{d}\mu \;\geq\; \mathrm{C}_{\mathrm{iso}}^{2/n} \int_0^T \left[\left(\rho^{-1}\right)'\right]^2 |\partial B_t|_{\mathbb{R}^n} \rho'(t) \,\mathrm{d}t \;=\; \mathrm{C}_{\mathrm{iso}}^{2/n} \int_{B_\Omega} |\mathrm{D}F|^2_{\mathbb{R}^n} \,\mathrm{d}\mu_{\mathbb{R}^n}, \tag{1.7.19}$$

as desired.

Assume now that equality holds in the inequality of (1.7.16). Then, we deduce from (1.7.19) and (1.7.18) that

$$\int_0^T |\{f=t\}|^2 - |\partial B_t|^2_{\mathbb{R}^n} \mathrm{C}_{\mathrm{iso}}^{\frac{2}{n}} \,\mathrm{d}t = 0,$$

that, by the nonnegativity of the integrand, implies that the sets $\{f \geq t\}$ satisfy equality in the Isoperimetric Inequality of (M,g) for any regular value t. Letting $t \to 0^+$, that is possible by Sard's Theorem, we conclude that Ω satisfy equality in (1.7.15). □

The Faber-Krahn inequality for 3-manifolds with nonnegative Ricci curvature of Theorem 1.70 now follows as a direct consequence of the Isoperimetric Inequality 1.60 and the above general principle

Proof of Theorem 1.70. Just recalling that the first Dirichlet eigenvalue minimises the Rayleigh quotient, we get, applying (1.7.16) to the solution of (1.7.13) with $\lambda = \lambda_1(\Omega)$, and observing that by the sharp Isoperimetric inequality (1.7.2) we have $\mathrm{AVR}(g) = \mathrm{C}_{\mathrm{iso}}$, we get

$$\lambda_1(\Omega) = \frac{\int_\Omega |\mathrm{D}f|^2 \,\mathrm{d}\mu}{\int_\Omega f^2 \,\mathrm{d}\mu} \geq \mathrm{AVR}(g)^{2/3} \frac{\int_{B_\Omega} |\mathrm{D}F|^2_{\mathbb{R}^3} \,\mathrm{d}\mu_{\mathbb{R}^3}}{\int_{B_\Omega} F^2 \,\mathrm{d}\mu_{\mathbb{R}^3}} \geq \mathrm{AVR}(g)^{2/3} \lambda_1(B_\Omega), \tag{1.7.20}$$

that is (1.7.14). If equality holds in (1.7.14), then, equalities hold in (1.7.20), that implies equality holds in the inequality of (1.7.16). The rigidity statement of Proposition 1.71 then shows that Ω satisfies equality in the Isoperimetric Inequality of (M,g), and then the rigidity statement of Theorem 1.60 applies, yielding the isometry of (M, g) with $(\mathbb{R}^3, g_{\mathbb{R}^3})$ and the isometry of Ω with B_Ω. $\square$

Expected Isoperimetric Inequality in higher dimensions

The Isoperimetric Inequality for complete noncompact 3-manifolds with nonnegative Ricci curvature (1.7.2) asserts that the sharp isoperimetric constant is the limit of the isoperimetric quotient of geodesic balls as the radius tends to infinity. It becomes then absolutely natural to conjecture that on manifolds with nonnegative Ricci curvature there holds

$$\frac{|\partial\Omega|^n}{|\Omega|^{n-1}} \geq n^n |\mathbb{B}^n| \mathrm{AVR}(g).$$

It should also happen that equality is achieved only on metric balls of flat $\mathbb{R}^n$. Needless to say that with such an equality at hand, the Faber-Krahn equality

$$\lambda_1(\Omega) \geq \mathrm{AVR}(g)^{2/n} \lambda_1(B_\Omega)$$

would follow by Proposition 1.71 exactly as shown for its 3-dimensional version.

We find interesting to observe that this topic is somewhat parallel to the Cartan-Hadamard conjecture, stating that the isoperimetric constant of a complete noncompact simply connected Riemannian manifolds with nonpostive sectional curvature is that of flat $\mathbb{R}^n$. As already remarked, this conjecture is proved only in dimensions 3 and 4. Let us mention however that [GS19] should give the crucial insights for the resolution of the general case. It is worth pointing out that a Mean Curvature Flow proof of the 3-dimensional version of the Cartan-Hadamard conjecture is carried out in [Sch08].

The Isoperimetric Inequality and isoperimetric sets

The very strong rigidity statement of Theorem 1.60 in particular implies that in any 3-manifold with nonnegative Ricci curvature different from flat $\mathbb{R}^3$ there are no bounded sets with smooth boundary attaining the equality in the Isoperimetric Inequality. One could then wonder whether there exists isoperimetric sets, that is, sets minimising the perimeter under a volume constraint. Actually, the existence of isoperimetric sets in manifolds with nonnegative Ricci curvature is proved in [MN16] if the manifold considered is also $\mathscr{C}^0$-asymptotically locally Euclidean, that is, satisfying (1.4.37). In particular, these isoperimetric sets, in dimension 3, do not satisfy equality in our sharp Isoperimetric Inequality (1.7.2). We bring to the reader's attention also [CEV17], where a more detailed description of isoperimetric sets is obtained under the additional assumption of $\mathscr{C}^{2,\alpha}$-asymptotic conicality. To the author's knowledge, existence result for isoperimetric sets on manifolds with nonnegative Ricci curvature without similar additional assumptions on the structure at infinity are not yet available in literature.

Chapter 2

Minkowski Inequalities and related results via Nonlinear Potential Theory

2.1 Structure of the chapter

In the following Section 2.2, we collect some preliminary results, define the relevant quantities and carry out the computations picturing the conformal setting where the core analysis is realised. Here, we do prove the sharp Kato-type identity for p-harmonic functions in Proposition 2.5, yielding as a corollary the one for harmonic functions employed in the previous chapter. In Section 2.3 we prove the effective monotonicity of (the conformal version of) U^p_β and U^p_∞ discussed in the Introduction, also showing how it improves to a full monotonicity when no critical points are developed. In Section 2.4 we derive the Extended Minkowski Inequality and discuss the application to Volumetric Minkowski Inequality and to the nearly umbilical estimates. In the last Section 2.5, we draw some additional consequences of the effective monotonicity formulas, such as capacity estimates and sphere theorems, yielding the nonlinear version of results from [AM20] and [BMM19] and improving inequalities from [Xia17]. In the end of this last section we also show how to relate with some techniques in overdetermined boundary value problems.

2.2 Effective Monotonicity. Statements and preparatory material.

Before giving the precise statements of the Effective Monotonicity Theorems along the level set flow of the p-capacitary potential, let us give some precise background.

2.2.1 Preliminaries on p-capacitary potentials

We recall the well known notion of p-capacity, introducing at the same time a normalised version of it that is suitable for our applications.

Definition 2.1 (p-capacity & normalised p-capacity)**.** *Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary.*

- *The p-capacity of Ω is defined as*

$$\mathrm{Cap}_p(\Omega) \;=\; \inf\left\{ \int_{\mathbb{R}^n} |\mathrm{D} f|^p \,\mathrm{d}\mu \;\middle|\; f \in \mathscr{C}^\infty_c(\mathbb{R}^n),\ f \geq 1 \text{ on } \Omega \right\}.$$

- *The normalised p-capacity of* Ω *is defined as*

$$\mathrm{C}_p(\Omega) \;=\; \inf\left\{ \left(\frac{p-1}{n-p}\right)^{p-1} \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{R}^n} |\mathrm{D}f|^p \, \mathrm{d}\mu \;\middle|\; f \in \mathscr{C}^\infty_c(\mathbb{R}^n),\ f \geq 1 \text{ on } \Omega \right\}. \tag{2.2.1}$$

Through this chapter, we are anyway mostly referring to $\mathrm{C}_(\Omega)$ simply as p-capacity of Ω.

The variational structure of the above definition leads naturally to the formulation of the following problem

$$\begin{cases} \Delta_p u = 0 & \text{in} \quad \mathbb{R}^n \setminus \overline{\Omega}\,, \\ u = 1 & \text{on} \quad \partial\Omega\,, \\ u(x) \to 0 & \text{as} \quad |x| \to \infty\,. \end{cases} \tag{2.2.2}$$

It is well known that, for every bounded open set Ω with smooth boundary and every $1 < p < n$, problem (2.2.2) admits a unique weak solution. Such a solution is called the p-capacitary potential associated with Ω. For the reader's convenience, we recall that a function v is a weak solution of $\Delta_p v = 0$ in an open set V if $v \in W^{1,p}_{loc}(V)$ and

$$\int_V \left\langle |\mathrm{D}v|^{p-2}\mathrm{D}v \;\middle|\; \mathrm{D}\psi \right\rangle \mathrm{d}\mu = 0$$

for any test function $\psi \in \mathscr{C}^\infty_c(V)$. By the important contributions [DiB83; Eva82; Lew83] and [Ura68], we know that weakly p-harmonic functions are $\mathscr{C}^{1,\alpha}_{loc}$.

Note that the uniqueness of the solution to problem (2.2.2) can be easily proved by suitably applying the Comparison Theorem for weakly p-harmonic functionsfirst provided in [Tol83] on large balls of radius R, and letting then $R \to +\infty$. With the same argument one can also show that the solution u to problem (2.2.2) is such that $0 < u(x) < 1$ for every $x \in \mathbb{R}^n \setminus \overline{\Omega}$. Finally, we recall that such a solution realises the infimum in (2.2.1). This can be proved using a standard exhaustion scheme (for example the one proposed in [CS03a]) and invoking the $\mathscr{C}^{1,\alpha}_{loc}$ regularity to guarantee the convergence of the scheme itself. Let us summarise some of these facts in the following statement, a generalised version of which is proved in Appendix B. For most of the basics on p-harmonic functions we employ in this chapter, we refer the reader to the nice lecture notes [Lin17].

Theorem 2.2 (Existence and regularity of p-capacitary potentials)**.** *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *with smooth boundary, and let* $1 < p < n$. *Then, there exists a unique weak solution* $u \in \mathscr{C}^{1,\alpha}_{loc}(\mathbb{R}^n \setminus \overline{\Omega}) \cap \mathscr{C}(\mathbb{R}^n \setminus \Omega)$ *to problem* (2.2.2)*, and it fulfils*

$$\mathrm{C}_p(\Omega) \;=\; \left(\frac{p-1}{n-p}\right)^{p-1} \frac{1}{|\mathbb{S}^{n-1}|} \int\limits_{\mathbb{R}^n \setminus \overline{\Omega}} |\mathrm{D}u|^p \, \mathrm{d}\mu\,,$$

where $\mathrm{C}_p(\Omega)$ *is the normalised p-capacity of* Ω *defined in* (2.2.1).

The following important result yields the precise asymptotic behaviour of u and $|\mathrm{D}u|$ at infinity.

Lemma 2.3 (Asymptotic expansions of u and $|\mathrm{D}u|$)**.** *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *with smooth boundary, and let* $1 < p < n$. *Then, the solution* u *to* (2.2.2) *satisfies*

(i) $\lim_{|x| \to +\infty} u(x)\, |x|^{\frac{n-p}{p-1}} \;=\; \mathrm{C}_p(\Omega)^{\frac{1}{p-1}}\,,$

(ii) $\lim_{|x| \to +\infty} |\mathrm{D}u(x)|\, |x|^{\frac{n-1}{p-1}} \;=\; \left(\frac{n-p}{p-1}\right) \mathrm{C}_p(\Omega)^{\frac{1}{p-1}}\,,$

where $\mathrm{C}_p(\Omega)$ is the normalised p-capacity of Ω defined in (2.2.1). In particular, $\mathrm{Crit}\, u$ *is a compact subset of $\mathbb{R}^n \setminus \overline{\Omega}$, possibly with full measure.*

For the proof of this lemma we refer the reader to [KV86] (see also the more recent [Pog18, Lemma 2.3 and (2.2)] for a precise statement). It is also worth mentioning [GS99], where similar expansions are employed to infer rotational symmetry of star-shaped domains supporting a solution to problem (2.2.2) with constant normal derivative on the boundary.

On the other hand, the classical regularity theory for quasilinear nondegenerate elliptic equations (see e.g. [LU68]) ensures that they are analytic around the points where the gradient does not vanish. Note that since $\partial\Omega$ is assumed to be smooth, by the Hopf Lemma for p-harmonic functions (see [Tol83, Proposition 3.2.1]), we have that $|\mathrm{D}u| \neq 0$ in a neighbourhood of this hypersurface. In particular, u is analytic in such a neighbourhood, and it smoothly extends to $\partial\Omega$. Coupling this with the asymptotic expansion of the gradient given in Lemma 2.3 implies that $\mathrm{Crit}\, u = \{x \in \mathbb{R}^n \setminus \overline{\Omega} \mid \mathrm{D}u(x) = 0\}$ is a compact subset of $\mathbb{R}^n \setminus \overline{\Omega}$ (generically depending on p), and in turn that u is analytic outside this set. Finally, it is worth recalling that for $p \neq 2$, the set $\mathrm{Crit}\, u$ is *a priori* allowed to have full measure. The following characterisation of the p-capacity of Ω is widely used in the literature and it is also very useful for our purposes. We include a proof, well suited in the general framework of p-nonparabolic Riemannian manifolds in Appendix B.

Lemma 2.4. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary, and let $1 < p < n$. Then, the solution u to (2.2.2) satisfies*

$$\mathrm{C}_p(\Omega) \;=\; \left(\frac{p-1}{n-p}\right)^{p-1} \frac{1}{|\mathbb{S}^{n-1}|} \int\limits_{\partial\Omega} |\mathrm{D}u|^{p-1}\, \mathrm{d}\sigma \;=\; \left(\frac{p-1}{n-p}\right)^{p-1} \frac{1}{|\mathbb{S}^{n-1}|} \int\limits_{\{u=t\}} |\mathrm{D}u|^{p-1}\, \mathrm{d}\sigma, \tag{2.2.3}$$

where $\mathrm{C}_p(\Omega)$ is the normalised p-capacity of Ω defined in (2.2.1) for almost any $t \in (0,1]$ including any regular value t.

In the remaining part of the chapter we will always assume that $1 < p < n$, unless otherwise stated.

2.2.2 A Kato-type identity for p-harmonic functions

We provide the Kato-type identity for p-harmonic functions on a general Riemannian manifolds, that in particular implies the one stated in Proposition 2.5 for harmonic functions. Similar identities, but without any refinement coming from being solution of an equation, were important also in [FMV13], see Proposition 1.8 there.

Proposition 2.5 (Kato-type identity for p-harmonic functions). *Let (M, g) be a Riemannian manifold, and let f be p-harmonic function defined on some subset of M, for $p > 1$. Then, in a neighbourhood of any point where f is such that $|\nabla f|(x) > 0$ there holds*

$$\begin{aligned} |\nabla\nabla f|^2 - \left(1 + \frac{(p-1)^2}{n-1}\right)\left|\nabla |\nabla f|\right|^2 \;=\; & |\mathrm{D}f|^2 \left|\mathrm{h} - \frac{\mathrm{H}}{n-1}\, g^{\top}\right|^2 \\ & + \left(1 - \frac{(p-1)^2}{n-1}\right)\left|\nabla^{\top} |\nabla f|\right|^2, \end{aligned} \tag{2.2.4}$$

where the tangential elements are referred to the level sets of f. Moreover, if

$$\left|\mathrm{h} - \frac{\mathrm{H}}{n-1} g^{\top}\right|^2 (x) = 0 \tag{2.2.5}$$

$$\left|\mathrm{D}^{\top} |\mathrm{D} f|\right|^2 (x) = 0 \tag{2.2.6}$$

for any x in $\{\rho_0 \leq f \leq \rho_1\}$ and $|\mathrm{D} f| > 0$ in this set, then the Riemannian manifold $(\{\rho_0 \leq f \leq \rho_1\}, g)$ is isometric to the warped product $([\rho_0, +\rho_1] \times \{f = \rho_0\}, \mathrm{d}\rho \otimes \mathrm{d}\rho + \eta^2(\rho)\, g_{|\{f=\rho_0\}})$, where η, f and ρ are related as

$$\eta(\rho) = \left(\frac{f'(\rho_0)}{f'(\rho)}\right)^{\frac{p-1}{n-1}}.$$

Proof. We consider on x an orthonormal frame $\{e_1, \ldots, e_{n-1}, e_n = \mathrm{D} f / |\mathrm{D} f|\}$. We introduce the notation $\mathrm{DD}^{\top} f$ to denote the tensor

$$\left(\mathrm{DD}^{\top} f\right)_{ij} = \mathrm{DD} f(e_i, e_j) \qquad i, j = 1, \ldots, n-1.$$

Observe that $\mathrm{DD}^{\top} f = |\mathrm{D} f| \mathrm{h}$. We can write the norm of $\mathrm{DD} f$ as follows

$$|\mathrm{DD} f|^2 = |\mathrm{DD}^{\top} f|^2 + 2 \sum_{j}^{n-1} |\mathrm{DD} f(e_n, e_j)|^2 + |\mathrm{DD} f(e_n, e_n)|^2. \tag{2.2.7}$$

Moreover, we can write the first term in the right hand side of (2.2.7) as follows

$$|\mathrm{DD}^{\top} f|^2 = \frac{(\Delta^{\top} f)^2}{n-1} + \left|\mathrm{DD}^{\top} f - \frac{\Delta^{\top} f}{n-1} g^{\top}\right|^2, \tag{2.2.8}$$

where we denote by $\Delta^{\top} f$ the trace of $\mathrm{DD}^{\top}$. We now exploit the p-harmonicity of f. By

$$\Delta_p f = |\mathrm{D} f|^{p-2} \left(\Delta f + (p-2) \nabla^2 f \left(\frac{\mathrm{D} f}{|\mathrm{D} f|}, \frac{\mathrm{D} f}{|\mathrm{D} f|}\right)\right) = 0$$

and $|\mathrm{D} f|(x) \neq 0$, we have

$$\Delta f = -(p-2) \mathrm{DD} f \left(\frac{\mathrm{D} f}{|\mathrm{D} f|}, \frac{\mathrm{D} f}{|\mathrm{D} f|}\right).$$

The above identity implies

$$\Delta^{\top} f = \Delta f - \mathrm{DD} f(e_n, e_n) = -(p-1) \mathrm{DD} f \left(\frac{\mathrm{D} f}{|\mathrm{D} f|}, \frac{\mathrm{D} f}{|\mathrm{D} f|}\right), \tag{2.2.9}$$

that, plugged into (2.2.8), gives

$$|\mathrm{DD}^{\top} f|^2 = \frac{(p-1)^2}{n-1} \left|\mathrm{DD} f \left(\frac{\mathrm{D} f}{|\mathrm{D} f|} \Big| \frac{\mathrm{D} f}{|\mathrm{D} f|}\right)\right|^2 + \left|\mathrm{DD}^{\top} f - \frac{\Delta^{\top} f}{n-1} g^{\top}\right|^2. \tag{2.2.10}$$

We now turn our attention to the second and the third term in (2.2.7). An easy computation shows

$$\mathrm{DD} f(e_n, e_j) = \langle \mathrm{D} |\mathrm{D} f| | e_j \rangle$$

for any $j = 1, \dots, n-1$, and thus we have

$$\sum_{j=1}^{n-1} |\mathrm{DD}f(e_n, e_j)|^2 = \left|\mathrm{D}^\top |\mathrm{D}f|\right|^2 \tag{2.2.11}$$

and

$$\left|\mathrm{DD}f(e_n, e_n)\right|^2 = \left\langle \mathrm{D}|\mathrm{D}f| \,\middle|\, \frac{\mathrm{D}f}{|\mathrm{D}f|}\right\rangle^2. \tag{2.2.12}$$

Finally, plugging (2.2.10), (2.2.11) and (2.2.12) into (2.2.7) we obtain (2.2.4).

Let us now assume $|\mathrm{D}f| > 0$ on $\{\rho_0 \leq f \leq \rho_1\}$ and conditions (2.2.5)-(2.2.6) hold on this set. Then, by the first assumption, we deduce by standard results in differential geometry (see e.g. [Hir76, Theorem 2.2]) that $\{\rho_0 \leq f \leq \rho_1\}$ is diffeomorphic to $[\rho_0, \rho_1) \times \{f = \rho_0\}$, and in particular there exist new coordinates $\{f, \vartheta^1, \dots, \vartheta^{n-1}\}$ on $\{f \geq \rho_0\}$ such that

$$g = \frac{\mathrm{d}f \otimes \mathrm{d}f}{|\mathrm{D}f|^2} + g_{ij}(f, x)\, \mathrm{d}\vartheta^i \otimes \vartheta^j x^j,$$

where, i, j range in $1, \dots, n-1$ and $\{\vartheta^i\}_{i=1}^{n-1}$ are coordinates on $\{f = \rho_0\}$. Observe now that by (2.2.5), the function $|\mathrm{D}f|$ is constant on each level set of f. In other words, it is a function of f alone. We can then define a new coordinate by $\mathrm{d}\rho = \mathrm{d}f/|\mathrm{D}f|$ so that the metric becomes

$$g = \mathrm{d}\rho \otimes \mathrm{d}\rho + g_{ij}(\rho, x)\, \mathrm{d}\vartheta^i \otimes \mathrm{d}\vartheta^j,$$

with some abuse of notation. In this coordinates, standard computations show that the Hessian is computed as

$$\mathrm{DD}f = f''\, \mathrm{d}\rho \otimes \mathrm{d}\rho + f' \mathrm{DD}\rho = f''\, \mathrm{d}\rho \otimes \mathrm{d}\rho + \frac{1}{2} f' \partial_\rho g_{ij}\, \mathrm{d}\vartheta^i \otimes \mathrm{d}\vartheta^j, \tag{2.2.13}$$

where by f' and f'' we denote the derivatives of f with respect to ρ.

Let us now consider, for any fixed point $x \in \{\rho_0 \leq \rho \leq \rho_1\}$ the orthonormal frame $\{e_1, \dots, e_{n-1}, e_n = \mathrm{D}f/|\mathrm{D}f| = \mathrm{D}\rho\}$, already used in the first part of this proof. Then we have

$$\mathrm{DD}f(e_n, e_j) = \langle \mathrm{D}|\mathrm{D}f| \,|\, e_j\rangle = 0$$

by (2.2.6), and

$$\mathrm{DD}^\top f = -\frac{p-1}{n-1}\, \mathrm{DD}f\left(\frac{\mathrm{D}f}{|\mathrm{D}f|}, \frac{\mathrm{D}f}{|\mathrm{D}f|}\right) g^\top$$

by (2.2.5) combined with (2.2.9). In particular, the Hessian of f can also be computed as

$$\mathrm{DD}f = \mathrm{DD}f\left(\frac{\mathrm{D}f}{|\mathrm{D}f|}, \frac{\mathrm{D}f}{|\mathrm{D}f|}\right) \mathrm{d}\rho \otimes \mathrm{d}\rho - \frac{p-1}{n-1} \mathrm{DD}f\left(\frac{\mathrm{D}f}{|\mathrm{D}f|}, \frac{\mathrm{D}f}{|\mathrm{D}f|}\right) g_{ij}\, \mathrm{d}\vartheta^i \otimes \mathrm{d}\vartheta^j.$$

A comparison with (2.2.13) then gives the system of ordinary differential equations

$$\partial_\rho \log g_{ij}(\rho, \vartheta) = -2\frac{p-1}{n-1} \partial_\rho \log f'(\rho), \tag{2.2.14}$$

that, integrated, yields

$$g_{ij}(\rho, \vartheta) = g_{ij}(\rho_0, \vartheta) \left(\frac{f'(\rho_0)}{f'(\rho)}\right)^{2\frac{p-1}{n-1}},$$

that is, g has the warped product structure claimed. □

As a corollary of the above Proposition, we record the following refined Kato's inequalities for p-harmonic functions, together with a characterisation of the equality case. We will not need this corollary in the sequel, but it actually is of some independent interest. Let us also point out that setting $p = 2$ we recover the well known, and widely used in the last chapter, refined Kato's inequality for harmonic functions, whose associated rigidity statement is discussed in [BC12]. For general p, it is obtained also in [CCW12], but without discussion of related rigidity statements.

Corollary 2.6 (Refined Kato's inequalities for p-harmonic functions)**.** *Let (M, g) be a Riemannian manifold, and let f be a p-harmonic function on some subset of M, with $p \in (1, n)$.*

(i) If $(p-1)^2 < n-1$, then , in a neighbourhood of any x such that $|\mathrm{D}f|(x) > 0$,

$$|\mathrm{DD}f|^2 \geq \left(1 + \frac{(p-1)^2}{n-1}\right) \Big|\mathrm{D}|\mathrm{D}f|\Big|^2.$$

Moreover, if equality is achieved on $\{\rho_0 \leq f \leq \rho_1\}$, and $|\mathrm{D}f| > 0$ in this region, then $(\{\rho_0 \leq f \leq \rho_1\}, g)$ has the same warped product structure as in the rigidity case of Proposition 2.5.

(ii) If $(p-1)^2 > n-1$, then, in a neighbourhood of any x such that $|\mathrm{D}f|(x) > 0$,

$$|\nabla\nabla f|^2 \geq 2\Big|\nabla|\nabla f|\Big|^2. \tag{2.2.15}$$

Moreover, if equality is achieved on $\{\rho_0 \leq f \leq \rho_1\}$ and $|\nabla f| > 0$ in this region, then $(\{\rho_0 \leq f \leq \rho_1\}, g)$ splits as a Riemannian product $([\rho_0, \rho_1] \times g_{\{f=\rho_0\}}, d\rho \otimes d\rho + g_{\{f=\rho_0\}})$ and f is an affine function of ρ.

(iii) If $(p-1)^2 = n-1$, then, in a neighbourhood of any x such that $|\mathrm{D}f|(x) > 0$ it holds (2.2.15). If equality holds on (2.2.15) at some x with $|\mathrm{D}f| > 0$, then x is an umbilical point of $\{f = f(x)\}$, that in a neighbourhood of x is a smooth hypersurface.

Proof. The assertions in (i) follow straightforwardly from Proposition 2.5. Let now $(p-1)^2 \geq n-1$. Plugging

$$\Big|\mathrm{D}^\top|\mathrm{D}f|\Big|^2 = \Big|\mathrm{D}|\mathrm{D}f|\Big|^2 - \left\langle \nabla|\mathrm{D}\frac{\mathrm{D}f}{|\mathrm{D}f|}|\right\rangle^2,$$

into (2.2.4) we obtain (2.2.15), Assume now that equality holds in (2.2.15) for x with $|\mathrm{D}f|(x) \neq 0$. Then, by (2.2.4) and the above identity we obtain

$$|\mathrm{D}f|^2\Big|\mathrm{h} - \frac{\mathrm{H}}{n-1}g^\top\Big|^2 + \left(\frac{(p-1)^2}{n-1} - 1\right)\left\langle \mathrm{D}|\mathrm{D}f|, \frac{\mathrm{D}f}{|\mathrm{D}f|}\right\rangle^2 = 0.$$

If $(p-1)^2 = n-1$, we can only deduce

$$\Big|\mathrm{h} - \frac{\mathrm{H}}{n-1}g^\top\Big|^2 = 0,$$

that is, x is an umbilical point of $\{f = f(x)\}$. This shows (iii). If $(p-1)^2 > n-1$, and equality holds in(2.2.15) on a region $\{\rho_0 \leq f \leq \rho_1\}$ where $|\mathrm{D}f| \neq 0$, then we also get

$$\left\langle \mathrm{D}|\mathrm{D}f| \,\Big|\, \frac{\mathrm{D}f}{|\mathrm{D}f|}\right\rangle^2 = 0.$$

In particular, by the computations performed in the proof of Proposition 2.5, we deduce that $|\mathrm{DD}f| = 0$. The isometry claimed follows by Corollary 1.23. □

Observe that in order to deduce Proposition 2.5 from Proposition 2.5 one should remove, in the case $p = 2$ the assumption of nonvanishing gradient. Such an assumption is added in Proposition 2.5 just to ensure the possibility of taking second derivatives, as we pointed out in the discussion of the regularity of p-harmonic functions above. The smoothness of harmonic functions then allows to remove this assumption.

2.2.3 The effectively monotone quantities

We now rigorously state our Effective Monotonicity-Rigidity Theorems. Let us first recall the definitions of the relevant quantities we just hinted in the Introduction. For $\Omega \subset \mathbb{R}^n$ and $1 < p < n$, consider u the solution of (2.2.2). Let, for $\beta \geq 0$ the function U^p_β : $(0,1] \setminus u(\mathrm{Crit}(u)) \to \mathbb{R}$ be defined by

$$U^p_\beta(t) \; = \; t^{-\beta(p-1)\frac{(n-1)}{(n-p)}} \int\limits_{\{u=t\}} |\mathrm{D}u|^{(\beta+1)(p-1)} \,\mathrm{d}\sigma, \tag{2.2.16}$$

and the function $U^p_\infty : (0,1] \setminus u(\mathrm{Crit}\, u) \to \mathbb{R}$ be defined by

$$U^p_\infty(t) \; = \; \sup_{\{u=t\}} \frac{|\mathrm{D}u|}{u^{\frac{n-1}{n-p}}}. \tag{2.2.17}$$

We stress that the above functions are defined just on regular values of u. The following is the Effective Monotonicity Theorem for U^p_β, when $\beta \geq (n-p)/[(p-1)(n-1)]$.

Theorem 2.7 (Effective Monotonicity of U^p_β). *Let $\Omega \subset M$ be a bounded set with smooth boundary, and let $U^p_\beta : (0,1] \setminus u(\mathrm{Crit}(u)) \to \mathbb{R}$ be the function defined by (2.2.16). Then, if $\beta \geq (n-p)/[(p-1)(n-1)]$ on any regular value $0 < t \leq 1$, we have*

$$(U^p_\beta)'(t) \; = \; \beta\, t^{-\beta(p-1)\left(\frac{n-1}{n-p}\right)} \int\limits_{\{u=t\}} |\mathrm{D}u|^{(\beta+1)(p-1)-1} \left[\mathrm{H} - \frac{(n-1)(p-1)}{(n-p)} |\mathrm{D}\log u| \right] \mathrm{d}\sigma \geq 0 \tag{2.2.18}$$

and

$$\lim_{t\to 0^+} U^p_\beta(t) \leq U^p_\beta(1).$$

Moreover, equality holds in (2.2.18) for some regular $0 < t_0 \leq 1$ if and only if $\{u = t_0\}$ is isometric to a sphere.

On the other hand, the following is the Effective Monotonicity Theorem for U^p_∞.

Theorem 2.8 (Effective Monotonicity of U^p_∞). *Let $\Omega \subset M$ be a bounded set with smooth boundary, and let $U^p_\infty : (0,1] \setminus u(\mathrm{Crit}\, u) \to (0,\infty)$ be the function defined by (2.2.17). Then, on any regular value $0 < t \leq 1$ and any regular $T < t$ we have*

$$U^p_\infty(T) \leq U^p_\infty(t). \tag{2.2.19}$$

Moreover, on any regular $0 < t \leq 1$ we have

$$\left[\mathrm{H} - \frac{(n-1)}{(p-1)(n-p)} |\mathrm{D}\log u| \right](x_t) = -(p-1)\frac{\partial}{\partial \nu_t} \log \frac{|\mathrm{D}u|}{u^{\frac{n-1}{n-p}}}(x_t) \geq 0, \tag{2.2.20}$$

where $x_t \in \{u = t\}$ *is the point where* $\sup_{\{u=t\}}|\mathrm{D}u|/u^{(n-1)/(n-p)}$ *is achieved and* $\nu_t = -\mathrm{D}u/|\mathrm{D}u|$ *is the unit normal to* $\{u = t\}$.

Equality holds in (2.2.19) *for regular* $T < t_0$ *or in* (2.2.20) *for regular* t_0 *if and only if* $\{u = t_0\}$ *is isometric to a sphere.*

Observe that, differently from U^p_β with finite β, the function U^p_∞ is easily defined also on singular values of u, and, as it will be clear from the proof, the value T in (2.2.19) could be taken also singular.

Analogously to what we did Chapter 1 in proving the Willmore-type inequalities, the L^p-Minkowski Inequality as well as some other inequalities will follow combining $(U^p_\beta)'(1) \geq 0$ with $\lim_{t\to 0^+} U^p_\beta(t) \leq U^p_\beta(1)$. The computation of the latter limit is a straightforward consequence of Lemma 2.3, that in fact yields

$$\lim_{t\to 0^+} U^p_\beta(t) = \mathrm{C_p}^{1-\beta\frac{p-1}{n-p}} \left(\frac{n-p}{p-1}\right)^{(\beta+1)(p-1)} |\mathbb{S}^{n-1}|. \tag{2.2.21}$$

Similarly, in the applications of the effective monotonicity of U^p_∞ we are going to use

$$\lim_{t\to 0^+} U^p_\infty(t) = \left(\frac{n-p}{p-1}\right) \mathrm{C}_p(\Omega), \tag{2.2.22}$$

again following from the asymptotic expansion of u and its gradient.

2.2.4 The conformal setting

The heuristics leading to cylindrical conformal change of metric introduced in [AM15] and [AM20] are formally sustainable also starting from problem (2.2.2). Indeed, when Ω is a ball, the solution to (2.2.2) is (proportional to) $r^{-(n-p)/(p-1)}$ where r is the distance from the center of the ball, and we can define a metric g satisfying

$$g = u^{2\frac{p-1}{n-p}} g_{\mathbb{R}^n} = (\mathrm{C}r)^{-2} \left(\mathrm{d}r \otimes \mathrm{d}r + r^2 g_{\mathbb{S}^{n-1}} \right) = \mathrm{d}\rho \otimes \mathrm{d}\rho + g_{\mathbb{S}^{n-1}},$$

where C is a constant depending on the radius of the ball, and ρ is defined by $\mathrm{d}\rho = (\mathrm{C}r)^{-1}\,\mathrm{d}r$. In this case, g is the cylindrical metric on $M \setminus \Omega$, and ρ is harmonic with respect to g. Encouraged by this basic observation we devote this section to fully describe the conformal background where we are going to work.

A conformal reformulation of the boundary value problem

Define, for u a solution to (2.2.2) with $1 < p < n$, the metric

$$g = u^{2\frac{p-1}{n-p}} g_{\mathbb{R}^n}. \tag{2.2.23}$$

Let

$$\varphi = -\frac{n-2}{n-p}(p-1)\log u. \tag{2.2.24}$$

Observe that, in light of the optimal $\mathscr{C}^{1,\alpha}$-regularity of u, the metric g is not a smooth Riemannian metric, but just $\mathscr{C}^1$. For this reason, in the computations that follow those involving just first derivative of the metric (and equivalently, of φ) make sense in the whole of $\mathbb{R}^n \setminus \Omega$, while those involving higher derivatives have to be thought as carried out in the complement of the critical set of u, that is, on $\{|\mathrm{D}u| \neq 0\}$.

Fixing local coordinates $\{x^\alpha\}_{\alpha=1}^n$ in M and using standard formulas (that can be found for example in [Bes08; HE73]) we get

$$\Gamma^\gamma_{\alpha\beta} = -\frac{1}{n-2}\left(\delta^\gamma_\alpha \partial_\beta \varphi + \delta^\gamma_\beta \partial_\alpha \varphi - g^{\mathbb{R}^n}_{\alpha\beta} g^{\gamma\eta}_{\mathbb{R}^n} \partial_\eta \varphi\right) \tag{2.2.25}$$

$$R^g_{\alpha\beta} = \mathrm{D}_\alpha \mathrm{D}_\beta \varphi + \frac{\partial_\alpha \varphi \partial_\beta \varphi}{n-2} - \frac{|\mathrm{D}\varphi|^2 - \Delta\varphi}{n-2} g^{\mathbb{R}^n}_{\alpha\beta}, \tag{2.2.26}$$

where $\Gamma^\gamma_{\alpha\beta}$ are the Christoffel symbols associated to the metric g, $R^g_{\alpha\beta}$ represents the components of the Ricci tensor and ∇_α the covariant derivative. Moreover,

$$\nabla_\alpha \nabla_\beta w = \mathrm{D}_\alpha \mathrm{D}_\beta w + \frac{1}{n-2}\Big(\partial_\alpha w \partial_\beta \varphi + \partial_\alpha \varphi \partial_\beta w - \langle \mathrm{D}w | \mathrm{D}\varphi \rangle\, g^{\mathbb{R}^n}_{\alpha\beta}\Big), \tag{2.2.27}$$

$$\Delta_g w = e^{\frac{2\varphi}{n-2}} \left(\Delta w - \langle \mathrm{D}w \,|\, \mathrm{D}\varphi \rangle\right) \tag{2.2.28}$$

for any $w \in \mathscr{C}^2$, where Δ_g and Δ denote the Laplacian with respect to g and $g_{\mathbb{R}^n}$ respectively. Let T be a vector field. Then,

$$\mathrm{div}_g T = g^{ik}\left(\frac{\partial T_k}{\partial x_i} - \Gamma^l_{ik} T_l\right) = u^{-2\frac{p-1}{n-p}} g^{ik}_{\mathbb{R}^n}\left(\frac{\partial T_k}{\partial x_i} - \Gamma^l_{ik} T_l\right).$$

Using (2.2.25), we get

$$\Gamma^l_{ik} = \frac{p-1}{n-p}\left(\delta^l_i \frac{\partial_k u}{u} + \delta^l_k \frac{\partial_i u}{u} - g^{\mathbb{R}^n}_{ik} \frac{\mathrm{D}^l u}{u}\right)$$

and

$$\begin{aligned} &\mathrm{div}_g(T) \\ &= u^{-2\frac{p-1}{n-p}} g^{ik}_{\mathbb{R}^n}\left(\frac{\partial T_k}{\partial x_i} - \frac{p-1}{n-p} T_i \frac{\partial_k u}{u} - \frac{p-1}{n-p} T_k \frac{\partial_i u}{u} + g^{\mathbb{R}^n}_{ik} \frac{p-1}{n-p}\left\langle \frac{\mathrm{D}u}{u} \,\middle|\, T\right\rangle_{\mathbb{R}^n}\right) \\ &= u^{-2\frac{p-1}{n-p}} \mathrm{div}_{g_{\mathbb{R}^n}} T + \frac{(n-2)(p-1)}{(n-p)}\left\langle \frac{\mathrm{D}u}{u} \,\middle|\, T\right\rangle_g. \end{aligned}$$

where $\langle\cdot|\cdot\rangle_{\mathbb{R}^n}$ and $\langle\cdot|\cdot\rangle_g$ are the scalar products associated to $g_{\mathbb{R}^n}$ and g respectively. Setting $T = |\mathrm{D}u|^{p-2}\mathrm{D}u$ and recalling that $\Delta_p u = 0$ we obtain

$$\begin{aligned} \mathrm{div}_g(|\mathrm{D}u|^{p-2}\mathrm{D}u) &= \frac{(n-2)(p-1)}{n-p}\left\langle \frac{\mathrm{D}u}{u} \,\middle|\, |\mathrm{D}u|^{p-2}\mathrm{D}u\right\rangle_g \\ &= \frac{(n-2)(p-1)}{n-p} \frac{|\mathrm{D}u|^{p-2}}{u} \langle \mathrm{D}u | \mathrm{D}u\rangle_g. \end{aligned} \tag{2.2.29}$$

By the definition of g we immediately get

$$|\mathrm{D}u|^{p-2} = u^{\frac{(p-1)(p-2)}{n-p}} |\nabla u|_g^{p-2}. \tag{2.2.30}$$

Using (2.2.30) in (2.2.29) we get

$$\mathrm{div}_g\left(u^{\frac{(p-1)(p-2)}{n-p}} |\nabla u|_g^{p-2} \nabla u\right) = \frac{(n-2)(p-1)}{n-p} u^{\frac{(p-1)(p-2)}{n-p}-1} |\nabla u|_g^p,$$

and so

$$\Delta_p^g u = \mathrm{div}_g(|\nabla u|_g^{p-2}\nabla u) = (p-1)\frac{|\nabla u|_g^p}{u}. \tag{2.2.31}$$

Lemma 2.9. *Let u be the solution to (2.2.2), let φ be defined in (2.2.24) and g the metric defined by (2.2.23). Then*

$$\Delta_p^g \varphi = 0$$

on $(\mathbb{R}^n \setminus \overline{\Omega}) \setminus \mathrm{Crit}\, u$.

Proof. On $(\mathbb{R}^n \setminus \overline{\Omega}) \setminus \mathrm{Crit}\, u$ the computations performed above are actual. In light of the definition (2.2.24), it obviously suffices to show that $\log u$ is p-harmonic with respect to the metric g. Letting $f = \log u$, we have $\nabla f = \frac{\nabla u}{u}$ and $|\nabla f|_g^{p-2} = \frac{|\nabla u|_g^{p-2}}{u^{p-2}}$. Thus,

$$\Delta_p^g f = \mathrm{div}_g(|\nabla f|_g^{p-2}\nabla f) = \mathrm{div}_g\left(\frac{|\nabla u|_g^{p-2}}{u^{p-2}}\frac{\nabla u}{u}\right).$$

Therefore,

$$\begin{aligned}\Delta_p^g f &= u^{1-p}\Delta_p^g u + \left\langle \nabla u^{1-p} \middle| |\nabla u|_g^{p-2}\nabla u\right\rangle_g \\ &= u^{1-p}\Delta_p^g u + (1-p)u^{-p}|\nabla u|_g^p\end{aligned}$$

and recalling (2.2.31) we get the thesis. □

We now compute the Ricci curvature of g, in order to complete the reformulation of (2.2.2).

Lemma 2.10. *Let u be the solution to (2.2.2), let φ be defined by (2.2.24) and g be the metric defined by (2.2.23). Then,*

$$\mathrm{Ric}_g - \nabla\nabla\varphi + \frac{d\varphi \otimes d\varphi}{n-2} = \left(\frac{|\nabla\varphi|_g^2}{n-2} - \frac{p-2}{n-2}\frac{\nabla\nabla\varphi(\nabla\varphi,\nabla\varphi)}{|\nabla\varphi|_g^2}\right) g \tag{2.2.32}$$

on $(\mathbb{R}^n \setminus \overline{\Omega}) \setminus \mathrm{Crit}\, u$.

Proof. As in the proof of Lemma 2.9 above, let $f = \log u$. Keeping in mind formulas (2.2.26) and (2.2.27), recalling that $\varphi = -\frac{(n-2)(p-1)}{n-p} f$ and $R_{\alpha\beta}^{\mathbb{R}^n} = 0$ we obtain

$$\begin{aligned}R_{\alpha\beta}^g = &-\frac{(n-2)(p-1)}{n-p}\mathrm{D}_\alpha \mathrm{D}_\beta f + \frac{(n-2)(p-1)^2}{(n-p)^2}\partial_\alpha f \partial_\beta f \\ &- \frac{p-1}{n-p}\left(\Delta f + \frac{(n-2)(p-1)}{n-p}|\mathrm{D}f|^2\right)e^{-\frac{2(p-1)f}{n-p}} g_{\alpha\beta}\end{aligned} \tag{2.2.33}$$

and

$$\mathrm{D}_\alpha \mathrm{D}_\beta f = \nabla_\alpha \nabla_\beta f + \frac{p-1}{n-p}\left(2\partial_\alpha f \partial_\beta f - |\nabla f|_g^2 g_{\alpha\beta}\right). \tag{2.2.34}$$

Using (2.2.34) in (2.2.33) we get

$$\begin{aligned} R^g_{\alpha\beta} = &- \frac{(n-2)(p-1)}{n-p}\nabla_\alpha\nabla_\beta f - \frac{(n-2)(p-1)^2}{(n-p)^2}\partial_\alpha f \partial_\beta f + \\ &+ \frac{(n-2)(p-1)^2}{(n-p)^2}|\nabla f|^2_g\, g_{\alpha\beta} \\ &- \frac{p-1}{n-p}\Delta f e^{-\frac{2(p-1)f}{n-p}} g_{\alpha\beta} - \frac{(n-2)(p-1)^2}{(n-p)^2}|\mathrm{D}f|^2 e^{-\frac{2(p-1)f}{n-p}} g_{\alpha\beta} \\ = &\nabla_\alpha\nabla_\beta\varphi - \frac{\partial_\alpha\varphi\partial_\beta\varphi}{n-2} + \frac{|\nabla\varphi|^2_g}{n-2}g_{\alpha\beta} + \frac{1}{n-2}e^{-\frac{2(p-1)f}{n-p}}\left(\Delta\varphi - |\mathrm{D}\varphi|^2\right) g_{\alpha\beta} \\ = &\nabla_\alpha\nabla_\beta\varphi - \frac{\partial_\alpha\varphi\partial_\beta\varphi}{n-2} + \frac{|\nabla\varphi|^2_g}{n-2}g_{\alpha\beta} + \frac{1}{n-2}\Delta_g\varphi\, g_{\alpha\beta} \end{aligned} \tag{2.2.35}$$

where in the last equality we used (2.2.28). Since

$$0 = \Delta^g_p\varphi = |\nabla\varphi|^{p-2}_g\Delta_g\varphi + \left\langle \nabla|\nabla\varphi|^{p-2}_g \,\middle|\, \nabla\varphi\right\rangle_g$$

we obtain

$$\begin{aligned} \Delta_g\varphi &= -(p-2)\frac{\nabla\nabla\varphi(\nabla\varphi,\nabla\varphi)}{|\nabla\varphi|^2_g} \\ &= -\frac{p-2}{2}\frac{\left\langle \nabla|\nabla\varphi|^2_g \,\middle|\, \nabla\varphi\right\rangle_g}{|\nabla\varphi|^2_g}. \end{aligned} \tag{2.2.36}$$

Using (2.2.36) in (2.2.35) we can write

$$R^g_{\alpha\beta} = \nabla_\alpha\nabla_\beta\varphi - \frac{\partial_\alpha\varphi\partial_\beta\varphi}{n-2} + \left(\frac{|\nabla\varphi|^2_g}{n-2} - \frac{p-2}{n-2}\frac{\nabla\nabla\varphi(\nabla\varphi,\nabla\varphi)}{|\nabla\varphi|^2_g}\right) g_{\alpha\beta},$$

that is (2.2.32). □

We are finally in position to reformulate problem (2.2.2) as

$$\begin{cases} \Delta^g_p\varphi = 0 & \text{in} \quad (\mathbb{R}^n\setminus\overline{\Omega})\setminus \mathrm{Crit}\,\varphi \\ \mathrm{Ric}_g - \nabla\nabla\varphi + \dfrac{d\varphi\otimes d\varphi}{n-2} = \left(\dfrac{|\nabla\varphi|^2_g}{n-2} - \dfrac{p-2}{n-2}\dfrac{\nabla\nabla\varphi(\nabla\varphi,\nabla\varphi)}{|\nabla\varphi|^2_g}\right) g & \text{in} \quad (\mathbb{R}^n\setminus\overline{\Omega})\setminus \mathrm{Crit}\,\varphi \\ \varphi = 0 & \text{on} \quad \partial\Omega \\ \varphi(x)\to+\infty & \text{as} \quad |x|\to\infty. \end{cases} \tag{2.2.37}$$

We explicitly observe that if $p = 2$ then (2.2.37) coincides with the problem studied in [AM15]. We conclude this part by isolating from the computations above the useful relation between $|\nabla\varphi|_g$ and $|\mathrm{D}u|$:

$$|\nabla\varphi|_g = \frac{(n-2)(p-1)}{n-p}\frac{|\mathrm{D}u|}{u^{\frac{n-1}{n-p}}} \tag{2.2.38}$$

Observe that involving just u and its gradient, the above is a continuous function of $\mathbb{R}^n\setminus\Omega$.

The geometry of the level sets of u and φ

Let us consider the $g^{\mathbb{R}^n}$−unit vector field

$$\nu := -\mathrm{D}u/|\mathrm{D}u| = \mathrm{D}\varphi/|\mathrm{D}\varphi|$$

and the g−unit vector field

$$\nu_g := -\nabla u/|\nabla u|_g = \nabla\varphi/|\nabla\varphi|_g.$$

Accordingly, we consider the second fundamental forms h and h_g of the level sets of u and φ with respect to the Euclidean metric $g^{\mathbb{R}^n}$ and the conformally related ambient metric g are respectively given by

$$\mathrm{h}_{ij} = -\frac{\mathrm{D}_i\mathrm{D}_j u}{|\mathrm{D}u|} = \frac{\mathrm{D}_i\mathrm{D}_j\varphi}{|\mathrm{D}\varphi|}, \qquad \mathrm{h}^g_{ij} = -\frac{\nabla_i\nabla_j u}{|\nabla u|_g} = \frac{\nabla^2_{ij}\varphi}{|\nabla\varphi|_g} \quad \text{for } i,j = 1,\dots,n-1.$$

Taking the trace of the above expressions with respect to the induced metric we obtain the following expressions for the mean curvatures in the two settings

$$\mathrm{H} = -\frac{\Delta u}{|\mathrm{D}u|} + \frac{\mathrm{D}\mathrm{D}u(\mathrm{D}u,\mathrm{D}u)}{|\mathrm{D}u|^3}, \qquad \mathrm{H}_g = \frac{\Delta_g\varphi}{|\nabla\varphi|_g} - \frac{\nabla\nabla\varphi(\nabla\varphi,\nabla\varphi)}{|\nabla\varphi|_g^3}.$$

Recalling that $\Delta_p u = 0$ and $\Delta_p^g\varphi = 0$ we have

$$\mathrm{H} = \frac{p-1}{p}\frac{\langle \mathrm{D}|\mathrm{D}u|^p|\mathrm{D}u\rangle}{|\mathrm{D}u|^{p+1}} = (p-1)\frac{\mathrm{D}\mathrm{D}u(\mathrm{D}u,\mathrm{D}u)}{|\mathrm{D}u|^3},$$

and

$$\mathrm{H}_g = -\frac{p-1}{p}\frac{\left\langle \nabla|\nabla\varphi|_g^p \,|\, \nabla\varphi\right\rangle_p}{|\nabla\varphi|_g^{p+1}} = -(p-1)\frac{\nabla\nabla\varphi(\nabla\varphi,\nabla\varphi)}{|\nabla\varphi|_g^3}. \tag{2.2.39}$$

The second fundamental forms h and h_g are related by the following formula:

$$\mathrm{h}_g(X,Y) = u^{\frac{p-1}{n-p}}\Big(\mathrm{h}(X,Y) - \frac{p-1}{n-p}\frac{|\mathrm{D}u|}{u}\langle X\,|\,Y\rangle\Big), \tag{2.2.40}$$

for any X, Y tangent vectors to the level sets of u. Tracing the above identity with respect to g we obtain the useful relation between the mean curvatures H and H_g

$$\mathrm{H}_g = u^{-\frac{p-1}{n-p}}\left(\mathrm{H} - \frac{(n-1)(p-1)}{(n-p)}\frac{|\mathrm{D}u|}{u}\right). \tag{2.2.41}$$

Relation between the metric-induced measures We recall the relation between the Lebesgue measure $\mathrm{d}\mu$ and the volume measure $\mathrm{d}\mu_g$ induced by g on M

$$\mathrm{d}\mu_g = u^{(p-1)\frac{n}{n-p}}\,\mathrm{d}\mu \tag{2.2.42}$$

and the relation between the are elements $d\sigma_g$ and $d\sigma$ induced respectively by g and the flat metric on smooth hypersurfaces

$$d\sigma_g = u^{(p-1)\frac{n-1}{n-p}}\, d\sigma. \tag{2.2.43}$$

Bochner formula for solutions of (2.2.37)

As in the last chapter, it is natural to apply the Bochner formula to a solution of (2.2.2). To do so, we start from the version of the Bochner formula suited for the p-Laplacian functions that has been worked out and applied in [Val13], see Proposition 3.1.2 there. Combining it with the first two equations in (2.2.37), we directly get the following relation. We do not provide the details, being straightforward.

Proposition 2.11. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded subset with smooth boundary. Let φ be a solution to (2.2.37). Then, in a neighbourhood of any $x \in \mathbb{R}^n \setminus \overline{\Omega}$ such that $|\nabla\varphi|(x) > 0$, we have*

$$\Delta|\nabla\varphi|^p + (p-2)\frac{\nabla\nabla|\nabla\varphi|^p(\nabla\varphi,\nabla\varphi)}{|\nabla\varphi|^2} - \frac{n-p}{n-2}\left\langle \nabla|\nabla\varphi|^p \,\middle|\, \nabla\varphi\right\rangle = \tag{2.2.44}$$
$$= p|\nabla\varphi|^{p-2}\left(|\nabla\nabla\varphi|^2 + p(p-2)\left\langle \nabla|\nabla\varphi| \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle^2\right).$$

As we are going to see soon, (2.2.44) will be the key ingredient to show the monotonicity of (the conformal version of) U^p_∞. An involved processing of it will instead lead to the vector field with nonnegative divergence encoding the monotonicity of U^p_β. Here, we just point out the enhancement of Corollary 1.23 for solutions of (2.2.37)

Lemma 2.12. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with smooth boundary. Let φ be a solution to (2.2.37). Assume that $|\nabla\varphi|_g > 0$ on $\{s_0 \leq \varphi \leq s_1\}$ for some $s_0 \in [0,\infty)$, and that $\nabla|\nabla\varphi|_g = 0$ on this region . Then, the Riemannian manifold $(\{s_0 \leq \varphi \leq s_1\}, g)$ is isometric to the Riemannian product $\left([s_0,s_1] \times \{\varphi = s_0\}, d\rho\otimes d\rho + g_{\{\varphi=s_0\}}\right)$, and φ is an affine function of ρ.*

Proof. If $\nabla|\nabla\varphi|_g = 0$, then, plugging this information in the relation (2.2.44), we immediately deduce that $|\nabla\nabla\varphi| = 0$ on $\{s_0 \leq \varphi \leq s_1\}$. The result claimed now follows from Corollary 1.23. □

The conformal version of the Effective Monotonicity Theorems

We start introducing the conformal version of the functions U^p_β and U^p_∞ introduced before. Let as always $1 < p < n$, and let $\beta \geq 0$. For φ defined in (2.2.24), let $\Phi^p_\beta : [0,+\infty) \setminus \varphi[\mathrm{Crit}\,\varphi] \to \mathbb{R}$ and $\Phi^p_\infty : [0,+\infty) \setminus \varphi[\mathrm{Crit}\varphi] \to \mathbb{R}$ be defined by

$$\Phi^p_\beta(s) = \int\limits_{\{\varphi=s\}} |\nabla\varphi|_g^{(\beta+1)(p-1)}\, d\sigma_g, \tag{2.2.45}$$

and

$$\Phi^p_\infty(s) = \sup_{\{\varphi=s\}} |\nabla\varphi|_g. \tag{2.2.46}$$

The above functions are defined on regular values of φ.

We are now going to state the conformal version of Theorem 2.7. Observe first that straightforward computations involving (2.2.38) and (2.2.43) show the following relation between the functions U^p_β and Φ^p_β.

$$\begin{aligned} U^p_\beta(t) &= \Phi^p_\beta\Big(-\frac{(n-2)(p-1)}{(n-p)}\log t\Big) \\ -t\frac{\mathrm{d}U^p_\beta}{\mathrm{d}t}(t) &= \frac{\mathrm{d}\Phi^p_\beta}{\mathrm{d}t}\Big(-\frac{(n-2)(p-1)}{(n-p)}\log t\Big) \end{aligned} \tag{2.2.47}$$

and between U^p_∞ and Φ^p_∞

$$U^p_\infty(t) = \Phi^p_\infty\Big(-\frac{(n-2)(p-1)}{(n-p)}\log t\Big).$$

With the above relations, it becomes evident that the effective monotonicity of U^p_β and U^p_∞ is proved if we prove the substantial monotonicity of Φ^p_β and Φ^p_∞. The following statements are the conformal versions of Theorems 2.7 and 2.8.

Theorem 2.13 (Effective monotonicity of Φ^p_β)**.** *Let $\Omega \subset M$ be an open bounded set with smooth boundary and let g, φ and Φ^p_β be defined as in (2.2.23), (2.2.24) and (2.2.45). Then, for any regular value $s \geq 0$, we have*

$$(\Phi^p_\beta)'(s) = -\beta \int_{\{\varphi=s\}} |\nabla\varphi|_g^{(\beta+1)(p-1)-1}\mathrm{H}_g \,\mathrm{d}\sigma_g \leq 0 \tag{2.2.48}$$

and

$$\Phi^p_\beta(s) \leq \Phi^p_\beta(0). \tag{2.2.49}$$

Moreover, equality holds in (2.2.48) for some regular s_0 if and only if $(\{\varphi \geq s_0\}, g)$ splits as $([s_0,+\infty) \times \{\varphi = s_0\}, \mathrm{d}\rho\otimes\mathrm{d}\rho + g_{\{\varphi=s_0\}})$ and φ, when restricted to $\{\varphi \geq s_0\}$, is an affine function of ρ.

On the other hand, the following is the conformal version of Theorem 2.14.

Theorem 2.14 (Effective monotonicity of Φ^p_∞)**.** *Let $\Omega \subset M$ be an open bounded set with smooth boundary and let g, φ and Φ^p_∞ be defined as in (2.2.23), (2.2.24) and (2.2.46). Then, for any regular $s \geq 0$ and any $S > s$ we have*

$$\Phi^p_\infty(S) \leq \Phi^p_\infty(s). \tag{2.2.50}$$

Moreover, for any regular $s \geq 0$, we have

$$\mathrm{H}_g(x_s) = -(p-1)\frac{\partial}{\partial \nu_s}\log|\nabla\varphi|_g \geq 0, \tag{2.2.51}$$

where $x_s \in \{\varphi = s\}$ is the point where $\sup_{\{\varphi=s\}}|\nabla\varphi|_g$ is achieved and ν_s is the unit normal $\nabla\varphi/|\nabla\varphi|_g$ to $\{\varphi = s\}$. Moreover, equality holds in (2.2.50) for $s_0 < S$ regular values or in (2.2.51) at s_0 with s_0 a regular value if and only if $(\{\varphi \geq s_0\}, g)$ splits as $([s_0,+\infty) \times \{\varphi = s_0\}, \mathrm{d}\rho\otimes\mathrm{d}\rho + g_{\{\varphi=s_0\}})$ and φ, when restricted to $\{\varphi \geq s_0\}$, is affine function of ρ.

It is easy to see from formulas (2.2.47), (2.2.38), (2.2.41) and (2.2.42) that the effective monotonicity of Φ^p_β and Φ^p_∞ implies that of U^p_β and U^p_∞, as well as (2.2.48) and (2.2.51) imply (2.2.18) and (2.2.20). It remains to check that the cylindrical splitting of $(\{\varphi \geq$

$s_0\}, g)$ implies that the level set of u corresponding to $\{\varphi = s_0\}$ is isometric to a sphere in $\mathbb{R}^n$.

The cylindrical splitting of g implies rotational symmetry of u. As anticipated above, we just show that the rigidity statements of Theorems 2.13 and 2.14 imply those related to the effective monotonicity of U^p_β and U^p_∞. It suffices to notice that, if φ is an affine function of ρ, as claimed in the conformal versions of the monotonicity theorems, then $|\nabla\nabla\varphi|^2_g = 0$. By means of (2.2.4), this implies that $\mathrm{h}_g = \mathrm{H}_g/(n-1)g^\top$. Since, by (2.2.27) together with (2.2.40) and (2.2.41), we have, after a lengthy computation,

$$|\nabla\varphi|^2_g \left|\mathrm{h}_g - \frac{\mathrm{H}_g}{n-1} g^\top\right|^2_{g^\top} = \left[\frac{(p-1)(n-2)}{(n-p)}\right]^2 u^{-2\frac{n+p-2}{n-p}} |\mathrm{D}u|^2 \left|\mathrm{h} - \frac{\mathrm{H}}{n-1} g^\top_{\mathbb{R}^n}\right|^2_{g^\top_{\mathbb{R}^n}},$$

where the tangential elements of the left hand side are referred to the level set $\{\varphi = s_0\}$, and the tangential elements in the right hand side are referred to correspondent level set of u, we deduce that $\{u = t_0\} = \{\varphi = s_0\}$ is totally umbilical, and thus a sphere. □

2.3 Proof of the Effective Monotonicity Theorems

The aim of this section is to give a complete proof of Theorems 2.13 and 2.14. Being much easier, we will establish first the effective monotonicity of Φ^p_∞.

Since all the computations of this section will be performed in the conformally related setting, the subscript g will be dropped from the notations.

2.3.1 The Effective Monotonicity of Φ^p_∞.

In proving the maximum principle enjoyed by $|\nabla\varphi|$ that as in the last chapter roughly corresponds to the monotonicity of Φ^p_∞, we are not using the same argument used for harmonic functions. Indeed, what we propose is an adaptation to p-capacitary potentials of the argument used for the sharp estimate on the harmonic Green's function on manifolds with nonnegative Ricci curvature provided by Colding in [Col12, Theorem 3.1].

We start by showing that $|\nabla\varphi|^p$ is a subsolution of the operator $\mathscr{L}$ acting on a smooth function f as

$$\mathscr{L}f = \Delta f + (p-2)\nabla\nabla f\left(\frac{\nabla\varphi}{|\nabla\varphi|}, \frac{\nabla\varphi}{|\nabla\varphi|}\right) - \frac{n-p}{n-2}\langle\nabla f\,|\,\nabla\varphi\rangle, \tag{2.3.1}$$

in a neighbourhood of any point $x \in \mathbb{R}^n \setminus \overline{\Omega}$ such that $|\nabla\varphi| > 0$, with φ a solution to (2.2.37). Observe that it appears on the right hand side of (2.2.44) applied to the function $|\nabla\varphi|^p$.

Lemma 2.15. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with smooth boundary. Let $\mathscr{L}$ be the differential operator defined in (2.3.1). Then, a solution φ to (2.2.37) satisfies*

$$\mathscr{L}(|\nabla\psi|^p) \geq 0 \tag{2.3.2}$$

in a neighbourhood of any point $x \in \mathbb{R}^n \setminus \overline{\Omega}$ such that $|\nabla\varphi|(x) > 0$.

Proof. By (2.2.44), we have

$$\mathscr{L}(|\nabla\varphi|^p) = p|\nabla\varphi|^{p-2}\left(|\nabla\nabla\varphi| + p(p-2)\left\langle\nabla|\nabla\varphi|\,\middle|\,\frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle^2\right).$$

By the standard Kato inequality $|\nabla\nabla\varphi|^2 \geq \Big|\nabla|\nabla\varphi|\Big|^2$, we obtain

$$\mathscr{L}(|\nabla\varphi|^p) \geq p|\nabla\varphi|^{p-2}\left(|\nabla|\nabla\varphi||^2 + p(p-2)\left\langle \nabla|\nabla\varphi| \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle^2\right),$$

and, since

$$|\nabla|\nabla\varphi||^2 + p(p-2)\left\langle \nabla|\nabla\varphi| \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle^2 = \Big|\nabla^\top|\nabla\varphi|\Big|^2 + (p-1)^2\left\langle \nabla|\nabla\varphi| \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle^2 \geq 0,$$

we get (2.3.2).

□

The following is just a computational lemma yielding a function lying in the kernel of (2.3.1).

Lemma 2.16. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with smooth boundary. Let $\mathscr{L}$ be the differential operator defined in* (2.3.1)*. Let φ be a solution to* (2.2.37)*. Then*

$$\mathscr{L}\left(\mathrm{e}^{\frac{n-p}{(n-2)(p-1)}\varphi}\right) = 0,$$

in a neighbourhood of any $x \in \mathbb{R}^n \setminus \overline{\Omega}$ with $|\nabla\varphi|(x) > 0$.

Proof. We compute separately the three terms forming $\mathscr{L}(\mathrm{e}^{\frac{n-p}{(n-2)(p-1)}\varphi})$. First, we have

$$\Delta(\mathrm{e}^{\frac{n-p}{(n-2)(p-1)}\varphi}) = \frac{n-p}{(n-2)(p-1)}e^{\frac{n-p}{(n-2)(p-1)}\varphi}\left(\frac{n-p}{(n-2)(p-1)}|\nabla\varphi|^2 + \Delta\varphi\right) \tag{2.3.3}$$

Using

$$\Delta_p\varphi = |\nabla\varphi|^{p-2}\left(\Delta\varphi + (p-2)\nabla\nabla\varphi\left(\frac{\nabla\varphi}{|\nabla\varphi|}, \frac{\nabla\varphi}{|\nabla\varphi|}\right)\right) = 0,$$

where the last equality follows from the p-harmonicity of φ, we obtain, from (2.3.3) and the fact that $|\nabla\varphi| \neq 0$

$$\begin{aligned}\Delta(\mathrm{e}^{\frac{n-p}{(n-2)(p-1)}\varphi}) &= \\ = \frac{n-p}{(n-2)(p-1)}&\mathrm{e}^{\frac{n-p}{(n-2)(p-1)}\varphi}\left[\frac{n-p}{(n-2)(p-1)}|\nabla\varphi|^2 - (p-2)\nabla\nabla\varphi\left(\frac{\nabla\varphi}{|\nabla\varphi|}, \frac{\nabla\varphi}{|\nabla\varphi|}\right)\right].\end{aligned} \tag{2.3.4}$$

The second term to be computed in $\mathscr{L}(\mathrm{e}^{\frac{n-p}{(n-2)(p-1)}\varphi})$ is

$$\begin{aligned}\nabla\nabla(\mathrm{e}^{\frac{n-p}{(n-2)(p-1)}\varphi})\left(\frac{\nabla\varphi}{|\nabla\varphi|}, \frac{\nabla\varphi}{|\nabla\varphi|}\right) = \frac{n-p}{(n-2)(p-1)}\mathrm{e}^{\frac{n-p}{(n-2)(p-1)}\varphi}\Bigg[&\frac{n-p}{(n-2)(p-1)}|\nabla\varphi|^2 \\ &+\nabla\nabla\varphi\left(\frac{\nabla\varphi}{|\nabla\varphi|}, \frac{\nabla\varphi}{|\nabla\varphi|}\right)\Bigg],\end{aligned} \tag{2.3.5}$$

and the last one is

$$\left\langle \nabla\mathrm{e}^{\frac{n-p}{(n-2)(p-1)}\varphi} \,\middle|\, |\nabla\varphi\right\rangle = \frac{n-p}{(n-2)(p-1)}e^{\frac{n-p}{(n-2)(p-1)}\varphi}|\nabla\varphi|^2. \tag{2.3.6}$$

By (2.3.4), (2.3.5) and (2.3.6) we get $\mathscr{L}(e^{\frac{n-p}{(n-2)(p-1)}\varphi}) = 0$, as claimed. □

We now apply the above lemmas to trigger the barrier argument in turn leading to Theorem 2.14.

Proof of Theorem 2.14. We claim that

$$|\nabla\varphi|(x) \leq \sup_{\{\varphi=s\}} |\nabla\varphi| \tag{2.3.7}$$

for any $x \in \{\varphi \geq s\}$ and any noncritical s. This clearly suffices to prove (2.2.50). Indeed, for any $S \geq s$, (2.3.7) implies

$$\sup_{\{\varphi=s\}} |\nabla\varphi| \geq \sup_{\{\varphi\geq s\}} |\nabla\varphi| \geq \sup_{\{\varphi\geq S\}} |\nabla\varphi| \geq \sup_{\{\varphi=S\}} |\nabla\varphi|.$$

Let then s be a noncritical value of φ. By the asymptotic expansions recalled in Proposition 2.3, the relation (2.2.38) and the continuity of $|\nabla\varphi|$ following from the $\mathscr{C}^{1,\alpha}$ regularity of u, it is immediate to deduce that

$$|\nabla\varphi| \leq C$$

uniformly on $\mathbb{R}^n \setminus \Omega$. Consider then, for such a constant C and for an auxiliary $S > s$ that we suppose to be regular (as usual, this is possible by the compactness of $\mathrm{Crit}\varphi$), the function

$$w = |\nabla\varphi|^p - \sup_{\{\varphi=s\}} |\nabla\varphi|^p - C^p e^{\frac{n-p}{(n-2)(p-1)}(\varphi-S)}$$

defined on $\{s \leq \varphi \leq S\}$. Let then $\delta > 0$ such that $N_\delta = \{|\nabla\varphi| \leq \delta\} \subset \{s < \varphi < S\}$. By the smoothness of φ, and in turn of $|\nabla\varphi|$, around the boundary of N_δ, we can suppose, by Sard Theorem, that ∂N_δ is a smooth hypersurface. By definition, we have

$$\sup_{\{\varphi=s\}\cup\{\varphi=S\}\cup\partial N_\delta} w \leq 0.$$

Moreover, by Lemma 2.15 and 2.16, w satisfies $\mathscr{L}w \geq 0$ on $\{s < \varphi < s\} \setminus \overline{N_\delta}$, and since this operator is uniformly elliptic in this bounded set with smooth boundary, the Maximum Principle applies and yields

$$|\nabla\varphi|^p \leq \sup_{\{\varphi=s\}} |\nabla\varphi|^p + C^p e^{\frac{n-p}{(n-2)(p-1)}(\varphi-S)}. \tag{2.3.8}$$

on $\{s \leq \varphi \leq S\} \setminus N_\delta$. On the other hand, upon choosing a smaller δ, we can clearly suppose that (2.3.8) is satisfied also on N_δ, and thus in the whole $\{s \leq \varphi \leq S\}$. By passing to the limit $S \to +\infty$, this proves (2.3.7).

We now turn to prove (2.2.51). Let then, for s nonsingular, $x_s \in \{\varphi = s\}$ be the point where the maximum of $|\nabla\varphi|$ in $\{\varphi = s\}$ is achieved. By (2.3.7), it is also the maximum point of $\{\varphi \geq s\}$, and in particular

$$\frac{\partial}{\partial\nu}|\nabla\varphi|^p(x_s) \leq 0. \tag{2.3.9}$$

Using (2.2.39) we get

$$\left\langle \nabla|\nabla\varphi|^p, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle(x_s) = -\frac{p}{p-1}\left(|\nabla\varphi|^p \mathrm{H}\right)(x_s),$$

where H is the mean curvature of the set $\{\varphi = x_s\}$ and inequality (2.2.51) follows.

Assume now that (2.2.50) holds with equality sign for $s_0 < S$ with s_0 regular. Then, by the effective monotonicity just proved we also have $\Phi^p_\infty(s_0) = \Phi^p_\infty(\tilde{S})$ with $\tilde{S} > s_0$ close enough to s_0 so that $|\nabla\varphi| > 0$ on $\{s_0 \le \varphi \le \tilde{S}\}$. Letting $x_{\tilde{S}}$ the maximum point of $|\nabla\varphi|^p$ on $\{\varphi = \tilde{S}\}$, we get in particular that

$$\sup_{\{\varphi = s_0\}\cup\{\varphi = \tilde{S}+\delta\}} |\nabla\varphi|^p = |\nabla\varphi|^p(x_{\tilde{S}}),$$

for some $\delta > 0$ small enough so that $|\nabla\varphi|^p > 0$ on $\{s_0 \le \varphi \le \tilde{S}+\delta\}$. Since the point $x_{\tilde{S}}$ lies in the interior of $\{s_0 \le \varphi \le \tilde{S}+\delta\}$, the Strong maximum principle, in force because of (2.3.2) ensures that $|\nabla\varphi|$ is a positive constant in $\{s_0 \le \varphi \le \tilde{S}+\delta\}$. This fact, combined with the continuity of $|\nabla\varphi|$, coming from the $\mathscr{C}^{1,\alpha}$-regularity of u, also easily shows that no singular values bigger than s_0 can occur. In particular, δ could be taken arbitrarily big, showing that $|\nabla\varphi|$ is constant on the whole $\{\varphi \ge s_0\}$, and we deduce the claimed splitting principle from Lemma 2.12.

We are left to show that the same splitting happens if (2.2.51) holds with equality sign. In this case, in particular, the normal derivative in (2.3.9) vanishes. But since x_s is a global maximum value for $|\nabla\varphi|^p$ on $\{s \le \varphi \le S\}$ for any $S > s$ such that $|\nabla\varphi| > 0$ on $\{s \le \varphi \le S\}$, and $|\nabla\varphi|^p$ is subsolution of the elliptic equation $\mathscr{L}f = 0$ by (2.3.2), Hopf's lemma implies that $|\nabla\varphi|^p$ is constant on this region. We can deduce arguing as above that there are no singular values bigger than s_0, that then $|\nabla\varphi|$ is constant on the whole $\{\varphi \ge s_0\}$ and conclude with Lemma 2.12. □

2.3.2 The Effective Monotonicity of Φ^p_β.

For a given $1 < p < n$, let us consider the vector field

$$X = \mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}\varphi} |\nabla\varphi|^{p-2}\left(\nabla|\nabla\varphi|^{\beta(p-1)} + (p-2)\nabla^{\perp}|\nabla\varphi|^{\beta(p-1)}\right) \tag{2.3.10}$$

defined around points where $|\nabla\varphi|$ does not vanish. Let us first check the relation between the vector field X and the derivative of Φ^p_β at a regular value s.

Proposition 2.17. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with smooth boundary. Let Φ^p_β be the function defined in (2.2.45) and X the vector field defined in (2.3.10). Then, for any regular value $s \ge 0$, we have*

$$\mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}s}\,(\Phi^p_\beta)'(s) = \frac{1}{p-1}\int\limits_{\{\varphi=s\}}\left\langle X \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle \mathrm{d}\sigma\,. \tag{2.3.11}$$

Proof. It suffices to show that

$$\Phi'_\beta(s) = \int\limits_{\{\varphi=s\}}\left\langle |\nabla\varphi|^{p-2}\nabla|\nabla\varphi|^{\beta(p-1)} \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle \mathrm{d}\sigma. \tag{2.3.12}$$

Indeed, it is immediately checked that

$$e^{-\frac{(n-p)}{(n-2)(p-1)}s}\left\langle |\nabla\varphi|^{p-2}\nabla|\nabla\varphi|^{\beta(p-1)} \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle = \frac{1}{p-1}\left\langle X \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle.$$

Let us then show (2.3.12). Let $\delta > 0$ small enough so that $s+\delta$ is still a regular value. We can write

$$\begin{aligned}\Phi^p_\beta(s+\delta) - \Phi^p_\beta(s) = &\int\limits_{\{\varphi = s+\delta\}} \left\langle \nabla\varphi|\nabla\varphi|^{(\beta+1)(p-1)-1} \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle \mathrm{d}\sigma \\ &- \int\limits_{\{\varphi = s\}} \left\langle \nabla\varphi|\nabla\varphi|^{(\beta+1)(p-1)-1} \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle \mathrm{d}\sigma.\end{aligned}$$

Using the Divergence Theorem, we can write the above quantity as

$$\Phi^p_\beta(s+\delta) - \Phi^p_\beta(s) = \int\limits_{\{s\leq\varphi\leq s+\delta\}} \mathrm{div}\left(\nabla\varphi|\nabla\varphi|^{p-2}|\nabla\varphi|^{\beta(p-1)}\right) \mathrm{d}\mu.$$

Since $\Delta_p\varphi = 0$, we get

$$\mathrm{div}\left(\nabla\varphi|\nabla\varphi|^{p-2}|\nabla\varphi|^{\beta(p-1)}\right) = \left\langle \nabla|\nabla\varphi|^{\beta(p-1)} \,\middle|\, \nabla\varphi\right\rangle |\nabla\varphi|^{p-2},$$

and by the coarea formula we obtain

$$\Phi^p_\beta(s+\delta) - \Phi^p_\beta(s) = \int\limits_s^{s+\delta}\int_{\{\varphi=\tau\}} \left\langle \left(\nabla|\nabla\varphi|^{\beta(p-1)}\right)|\nabla\varphi|^{p-2} \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle \mathrm{d}\sigma\,\mathrm{d}\tau.$$

By dividing both sides of the above equality by δ, and letting $\delta\to 0^+$, we get (2.3.11) by the Fundamental Theorem of Calculus provided that the function I mapping

$$\tau \mapsto \int\limits_{\{\varphi=\tau\}} \left\langle (\nabla|\nabla\varphi|^{\beta(p-1)})|\nabla\varphi|^{p-2} \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle \mathrm{d}\sigma$$

is continuous. In fact, fixed $\tau_0 \geq 0$ we have for any $\tau > \tau_0$ close enough to τ_0, by Divergence Theorem

$$\left|I(\tau) - I(\tau_0)\right| \leq \int\limits_{\{\tau_0\leq\varphi\leq\tau\}} \left|\mathrm{div}\left(\nabla|\nabla\varphi|^{\beta(p-1)})|\nabla\varphi|^{p-2}\right)\right| \mathrm{d}\mu,$$

and the right hand side vanishes in the limit as $\tau\to\tau_0^+$ by Dominated Convergence Theorem. □

In the next fundamental lemma, we compute the divergence of X.

Lemma 2.18 (Divergence of X)**.** *Let X be the vector field defined by (2.3.10). Then, the following identity holds in a neighbourhood of any point $x\in\mathbb{R}^n\setminus\overline{\Omega}$ such that $|\nabla\varphi|(x)\neq 0$*

$$\mathrm{div}X \;=\; e^{-\frac{(n-p)}{(n-2)(p-1)}\varphi}\, Q \;\geq\; 0\,, \tag{2.3.13}$$

where

$$Q = \beta\,(p-1)|\nabla\varphi|^{\beta(p-1)+p-4}\Bigg\{|\nabla\varphi|^2\left|\mathrm{h}-\frac{\mathrm{H}}{n-1}g^{\top}\right|^2 + (p-1)\left[\beta+\left(\frac{p-2}{p-1}\right)\right]\left|\nabla^{\top}|\nabla\varphi|\right|^2 + (p-1)^2\left[\beta-\frac{(n-p)}{(p-1)(n-1)}\right]\left|\nabla^{\perp}|\nabla\varphi|\right|^2\Bigg\}, \tag{2.3.14}$$

where h *and* H *are respectively the second fundamental form and the mean curvature of the level sets of* φ *with respect to the unit normal* $\nabla\varphi/|\nabla\varphi|$*.*

Proof. For the sake of clearness, we write

$$X = \mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}}(W+Z),$$

where

$$W = |\nabla\varphi|^{p-2}\,\nabla\,|\nabla\varphi|^{\beta(p-1)},$$

and

$$Z = (p-2)|\nabla\varphi|^{p-2}\left\langle\nabla\,|\nabla\varphi|^{\beta(p-1)}\,\middle|\,\frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle\frac{\nabla\varphi}{|\nabla\varphi|}.$$

We now compute separately the divergence of W and Z.

The divergence of W*.* Simple computations give

$$\nabla|\nabla\varphi|^p = p|\nabla\varphi|^{p-1}\nabla|\nabla\varphi| \tag{2.3.15}$$

and

$$\nabla\nabla|\nabla\varphi|^p(\nabla\varphi,\nabla\varphi) = p(p-1)|\nabla\varphi|^{p-2}\langle\nabla|\nabla\varphi|\,|\,\nabla\varphi\rangle^2 + p|\nabla\varphi|^{p-1}\nabla\nabla|\nabla\varphi|(\nabla\varphi,\nabla\varphi). \tag{2.3.16}$$

Moreover, we can write

$$W = \frac{\beta(p-1)}{p}|\nabla\varphi|^{\beta(p-1)-2}\nabla|\nabla\varphi|^p. \tag{2.3.17}$$

Using (2.3.15) and (2.3.17) we get

$$\mathrm{div}W = \frac{\beta(p-1)}{p}|\nabla\varphi|^{\beta(p-1)-2}\Delta|\nabla\varphi|^p + \big(\beta(p-1)\big)\big(\beta(p-1)-2\big)|\nabla\varphi|^{\beta(p-1)+p-4}\left|\nabla|\nabla\varphi|\right|^2.$$

Plugging (2.2.44) into the above relation, and using identity (2.3.17), we obtain

$$\begin{aligned}\mathrm{div}W - \frac{n-p}{n-2}\langle W \,|\, \nabla\varphi\rangle = \Big(\beta(p-1)\Big)|\nabla\varphi|^{\beta(p-1)+p-4} \\ \times\Big[|\nabla\nabla\varphi|^2 + \Big(\beta(p-1)-2\Big)\Big|\nabla|\nabla\varphi|\Big|^2 \\ - \frac{(p-2)}{p}\frac{\nabla\nabla|\nabla\varphi|^p(\nabla\varphi,\nabla\varphi)}{|\nabla\varphi|^p} \\ + p(p-2)\Big\langle \nabla|\nabla\varphi| \,\Big|\, \frac{\nabla\varphi}{|\nabla\varphi|}\Big\rangle^2\Big],\end{aligned}$$

that, through (2.3.16), becomes

$$\begin{aligned}\mathrm{div}\, W &= \frac{n-p}{n-2}\,\langle W, \nabla\varphi\rangle \\ &+ \beta\,(p-1)|\nabla\varphi|^{\beta(p-1)+p-4}\Big[|\nabla\nabla\varphi|^2 + \Big[\beta(p-1)-2\Big]\Big|\nabla|\nabla\varphi|\Big|^2 \\ &+ (p-2)\left(\Big\langle \nabla|\nabla\varphi| \,\Big|\, \frac{\nabla\varphi}{|\nabla\varphi|}\Big\rangle^2 - \frac{\nabla\nabla|\nabla\varphi|(\nabla\varphi,\nabla\varphi)}{|\nabla\varphi|}\right)\Big].\end{aligned} \tag{2.3.18}$$

It is time to insert the Kato-type identity (2.2.4) in (2.3.18). By that and the frequently used decomposition

$$\Big|\nabla|\nabla\varphi|\Big|^2 = \Big|\nabla^\top|\nabla\varphi|\Big|^2 + \Big|\nabla^\perp|\nabla\varphi|\Big|^2, \tag{2.3.19}$$

we immediately get

$$\begin{aligned}\mathrm{div}\, W = \frac{n-p}{n-2}\,\langle W \,|\, \nabla\varphi\rangle + \beta\,(p-1)|\nabla\varphi|^{\beta(p-1)+p-4}\Big\{\beta(p-1)\Big|\nabla^\top|\nabla\varphi|\Big|^2 \\ + \Big[\beta(p-1) + \frac{(p-1)^2}{n-1} - 1 + (p-2)\Big]\,\Big|\nabla^\perp|\nabla\varphi|\Big|^2 \\ - (p-2)\frac{\nabla\nabla|\nabla\varphi|(\nabla\varphi,\nabla\varphi)}{|\nabla\varphi|}\Big\}.\end{aligned} \tag{2.3.20}$$

The divergence of Z. Clearly, by the p-harmonicity of φ, we have

$$\mathrm{div}\, Z = (p-2)|\nabla\varphi|^{p-2}\left\langle \nabla\left[\left\langle \nabla\,|\nabla\varphi|^{\beta(p-1)} \,\Big|\, \frac{\nabla\varphi}{|\nabla\varphi|^2}\right\rangle\right] \,\Big|\, \nabla\varphi\right\rangle.$$

We then write

$$\left\langle \nabla\,|\nabla\varphi|^{\beta(p-1)} \,\Big|\, \frac{\nabla\varphi}{|\nabla\varphi|^2}\right\rangle = \beta(p-1)|\nabla\varphi|^{\beta(p-1)-3}\Big\langle \nabla\,|\nabla\varphi| \,\Big|\, \nabla\varphi\Big\rangle,$$

and by a straightforward computation

$$
\begin{aligned}
\operatorname{div} Z = (p-2)\,\beta\,(p-1)|\nabla\varphi|^{\beta(p-1)+p-4}\bigg\{ & \frac{\nabla\nabla|\nabla\varphi|(\nabla\varphi,\nabla\varphi)}{|\nabla\varphi|} \\
& + \left[\beta(p-1)^2 - \beta(p-1) - 2(p-2)\right]\left\langle \nabla|\nabla\varphi|, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle^2 \qquad (2.3.21)\\
& + (p-2)\left|\nabla^{\top}|\nabla\varphi|\right|^2 \bigg\},
\end{aligned}
$$

where we just used the decomposition (2.3.19) and the general fact

$$
\frac{\nabla\nabla\varphi(\nabla|\nabla\varphi|,\nabla\varphi)}{|\nabla\varphi|} = \left|\nabla|\nabla\varphi|\right|^2 .
$$

Completion of the computation. Summing up the expressions (2.3.20) and (2.3.21), and observing that

$$
\langle W \,|\, \nabla\varphi\rangle = \frac{1}{p-1}\,\langle X \,|\, \nabla\varphi\rangle\,,
$$

we finally get

$$
\begin{aligned}
\operatorname{div}(W+Z) - \frac{(n-p)}{(n-1)(p-1)}\langle (W+Z)\,|\,\nabla\varphi\rangle = \beta\,(p-1)|\nabla\varphi|^{\beta(p-1)+p-4}\bigg\{ & |\nabla\varphi|^2\left|\mathrm{h} - \frac{\mathrm{H}}{n-1}\,g^{\top}\right|^2 \\
& + (p-1)\left[\beta + \left(\frac{p-2}{p-1}\right)\right]\left|\nabla^{\top}|\nabla\varphi|\right|^2 \\
& + (p-1)^2\left[\beta - \frac{(n-p)}{(p-1)(n-1)}\right]\left|\nabla^{\perp}|\nabla\varphi|\right|^2\bigg\},
\end{aligned}
$$

that is clearly equivalent to (2.3.13). □

In absence of critical points, the Divergence Theorem applied to the vector field X on the open region $\{s < \varphi < S\}$, with $0 < s < S$, easily yields the inequality

$$
\int_{\{\varphi = s\}} \left\langle X \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle \mathrm{d}\sigma \;\leq\; \int_{\{\varphi = S\}} \left\langle X \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle \mathrm{d}\sigma\,,
$$

and in turns, thanks to (2.3.11), (2.3.22) below. In presence of a possibly wild critical set, this direct argument is no longer working. Fortunately, some of the new ideas introduced in [AM20] to treat the same issues in the case of harmonic functions are exportable to the case of p-harmonic functions, where one does not know *a priori* that the critical set is $(n-1)$-negligible. As a consequence, we are still able to provide an effective version of the considered monotonicity, showing that (2.3.22) is actually in force, provided s is small enough and S is large enough. The desired effective inequality $(\Phi^p_\beta)'(s) \leq 0$ for nonsingular values s, that is (2.2.48), will follow at once. It will also be clear that $(\Phi^p_\beta)'(s_0) = 0$ only if $(\{\varphi \geq s_0\}, g)$ splits a Riemannian product.

Remark 2.19 (Asymptotic boundedness of Φ^p_β). Before proving the above facts, let us briefly observe that the function Φ^p_β are asymptotically bounded for any $\beta \geq 0$. Indeed, this immediately follows from the smoothness of such function at values s big enough and the finite limit it achieves at infinity, coinciding with that of U^p_β computed in (2.2.21).

Theorem 2.20 (First substantial inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded set with smooth boundary, and let Φ^p_β be the function defined in (2.2.45). Then, for all regular values $s < S$ and any $\beta \geq (n-p)/[(n-1)(p-1)]$, the inequality*

$$\frac{(\Phi^p_\beta)'(s)}{e^{\frac{(n-p)}{(n-1)(p-1)}s}} \;\leq\; \frac{(\Phi^p_\beta)'(S)}{e^{\frac{(n-p)}{(n-1)(p-1)}S}} \tag{2.3.22}$$

holds true. In particular, one has that $(\Phi^p_\beta)'(s) \leq 0$ for any nonsingular $s \geq 0$, that is (2.2.48). Moreover, if $(\Phi^p_\beta)'(s_0) = 0$ for some nonsingular $s_0 \geq 0$, then $(\{\varphi \geq s_0\}, g)$ is isometric to the truncated Riemannian cylinder $([s_0, \infty), \mathrm{d}\rho \otimes \mathrm{d}\rho + g_{\{\varphi = s_0\}}$, and φ is an affine function of ρ in $\{\varphi \geq_0\}$.

Proof. Let $\varepsilon > 0$, we consider a smooth nonnegative cut-off-function $\chi : [0, +\infty) \to \mathbb{R}$, such that

$$\begin{cases} \chi(t) = 0 & \text{in } t < \frac{1}{2}\varepsilon, \\ \dot\chi(t) \geq 0 & \text{in } \frac{1}{2}\varepsilon \leq t \leq \frac{3}{2}\varepsilon, \\ \chi(t) = 1 & \text{in } t > \frac{3}{2}\varepsilon. \end{cases} \tag{2.3.23}$$

Since $\chi(|\nabla\varphi|^{\beta(p-1)}) = 0$ on $\mathrm{Crit}\,\varphi \cap \{s < \varphi < S\}$ we can apply the Divergence Theorem to the smooth vector field

$$\widetilde{X} \;=\; \chi\big(|\nabla\varphi|^{\beta(p-1)}\big)\, X$$

in the domain $\{s < \varphi < S\}$. Observe that, choosing ε small enough, we can make sure that $\chi\big(|\nabla\varphi|^{\beta(p-1)}\big) = 1$ on $\{\varphi = s\}$ and $\{\varphi = S\}$. Having this in mind, we compute

$$\begin{aligned} \int\limits_{\{\varphi = S\}} \left\langle X \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|} \right\rangle \mathrm{d}\sigma \;-\; \int\limits_{\{\varphi = s\}} \left\langle X \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|} \right\rangle \mathrm{d}\sigma \;&=\; \int\limits_{\{s<\varphi<S\}} \mathrm{div}\widetilde{X}\, \mathrm{d}\mu \\ &= \int\limits_{\{s<\varphi<S\}\setminus N_{\varepsilon/2}} \chi\big(|\nabla\varphi|^{\beta(p-1)}\big)\, \mathrm{div}X\, \mathrm{d}\mu \;+ \int\limits_{N_{3\varepsilon/2}\setminus N_{\varepsilon/2}} \dot\chi\big(|\nabla\varphi|^{\beta(p-1)}\big) \left\langle X \,\middle|\, \nabla|\nabla\varphi|^{\beta(p-1)} \right\rangle \mathrm{d}\mu\,, \end{aligned} \tag{2.3.24}$$

where in the last identity we have used the tubular neighbourhood of $\mathrm{Crit}\,\varphi \cap \{s < \varphi < S\}$ defined for every $\delta > 0$ as $N_\delta = \{|\nabla\varphi|^{\beta(p-1)} \leq \delta\}$. In view of (2.3.11), (2.3.13), (2.3.23) and (2.3.24), the inequality (2.3.22) is proved if we show that $\left\langle X \,\middle|\, \nabla|\nabla\varphi|^{\beta(p-1)} \right\rangle \geq 0$ on $N_{3\varepsilon/2} \setminus N_{\varepsilon/2}$. On the other hand, a direct computation gives

$$\begin{aligned} \left\langle X \,\middle|\, \nabla|\nabla\varphi|^{\beta(p-1)} \right\rangle \;&=\; e^{-\frac{(n-p)}{(n-2)(p-1)}\varphi}\, |\nabla\varphi|^{p-2} \left[\, \big|\nabla|\nabla\varphi|^{\beta(p-1)}\big|^2 + (p-2)\big|\nabla^\perp|\nabla\varphi|^{\beta(p-1)}\big|^2 \,\right] \\ &=\; e^{-\frac{(n-p)}{(n-2)(p-1)}\varphi}\, |\nabla\varphi|^{p-2} \left[\, \big|\nabla^\top|\nabla\varphi|^{\beta(p-1)}\big|^2 + (p-1)\big|\nabla^\perp|\nabla\varphi|^{\beta(p-1)}\big|^2 \,\right] \\ &\geq\; 0. \end{aligned}$$

This completes the proof of (2.3.22). We show that it implies $(\Phi^p_\beta)'(s) \leq 0$. Recalling that by the asymptotic expansions in Lemma 2.3 any level set $\varphi(S)$ is smooth for S big enough, say for $S \geq S_p$, it follows at once from (2.3.22) that, for every $S \geq S_p$, it holds

$$\mathrm{e}^{\frac{n-p}{(n-1)(p-1)\varphi} S} \, (\Phi^p_\beta)'(s) \;\leq\; \mathrm{e}^{\frac{n-p}{(n-1)(p-1)\varphi} s} (\Phi^p_\beta)'(S) \,.$$

Integrating both sides of the above inequality in the variable S on an interval of the form (S_p, S), we obtain

$$\mathrm{e}^{\frac{(n-p)}{(n-2)(p-1)} S} \, (\Phi^p_\beta)'(s) \,+\, \mathrm{e}^{\frac{n-p}{(n-1)(p-1)\varphi} s} \, \Phi^p_\beta(S_p) \,-\, \mathrm{e}^{\frac{n-p}{(n-2)(p-1)} S_p} \, (\Phi^p_\beta)'(s) \;\leq\; \mathrm{e}^{\frac{n-p}{(n-1)(p-1)\varphi} s} \Phi^p_\beta(S) \,.$$

If by contradiction, $(\Phi^p_\beta)'(s) > 0$, then, letting $S \to +\infty$ in the above identity, we would deduce that $\Phi^p_\beta(S) \to +\infty$, against the asymptotic boundedness of Φ^p_β discussed in Remark 2.19.

We are left to show the rigidity statement occurring when $(\Phi^p_\beta)'(s_0) = 0$ for some $s_0 \geq 0$. Let $S > s_0$ be a regular value close enough to s_0 so that $|\nabla\varphi| > 0$ on $\{s_0 \leq \varphi \leq S\}$. Then, the Divergence Theorem and (2.3.11) show that

$$\begin{aligned} 0 \,\geq\, (p-1)\mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)} S} \, (\Phi^p_\beta)'(S) &= \int\limits_{\{\varphi = S\}} \left\langle X \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|} \right\rangle \mathrm{d}\sigma \;-\; \int\limits_{\{\varphi = s_0\}} \left\langle X \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|} \right\rangle \mathrm{d}\sigma \\ &= \int\limits_{\{s<\varphi<S\}} \mathrm{div} X \, \mathrm{d}\mu \geq 0, \end{aligned} \tag{2.3.25}$$

and thus $\mathrm{div}X$ vanishes on $\{s_0 \leq \varphi \leq S\}$. If $\beta > (n-p)/[(n-1)(p-1)]$, then the expression (2.3.13) implies that $\big|\nabla|\nabla\varphi|\big| = 0$, and thus by Lemma 2.12 $(\{s_0 \leq \varphi \leq S\}, g)$ splits a compact Riemannian product. Arguing as in the proof of the rigidity part of Theorem 2.14, this also implies by the continuity of $|\nabla\varphi|$ that no singular values bigger than s_0 can exist, and the isometry with the Riemannian product extends to the whole $(\{\varphi \geq s_0\}, g)$. If on the other hand $\beta = (n-p)/[(n-1)(p-1)]$, then (2.3.25) coupled with (2.3.13) just implies that (2.2.5) and (2.2.6) holds for φ. The rigidity statement in the Kato-type identity Proposition 2.5 implies that $(\{s_0 \leq \varphi \leq S\}, g)$ splits a warped product, whose cross section are given by level sets of φ. By the vanishing of $(\Phi^p_\beta)'(S)$ deduced from (2.3.25) together with the second expression in (2.3.11), these level sets are actually minimal. By the relation (2.2.39), this implies that also $\nabla^\top|\nabla\varphi| = 0$, and this fact coupled with (2.2.6) for φ implies that $\big|\nabla|\nabla\varphi|\big| = 0$ on $(\{s_0, \leq \varphi \leq S\}, g)$, and we can conclude as before. □

As already observed several times, the presence of critical points and critical values possibly arranged in sets with full measure makes the full monotonicity not expectable in general. In fact, the lack of a sufficiently strong Sard-type property for the p-capacitary potentials prevents any kind of straightforward adaptation of the arguments presented in [AM20]. In other words, there is no hope for deducing the global inequality $\Phi^p_\beta(+\infty) \leq \Phi^p_\beta(0)$ from the pointwise inequality $(\Phi^p_\beta)'(s) \leq 0$ through integration, since the latter inequality may fail to be true – or even well defined – for too many values of $s \in [0, +\infty)$. To face this main difficulty, we craft a new family of auxiliary monotonicity formulas. For a given $1 < p < n$ and a given $0 < \lambda < 1$, we consider the vector field

$$Y_\lambda \;=\; \left(\mathrm{e}^{\frac{(n-p)}{(n-2)(p-1)}\varphi} - \lambda\right) X \,-\, \left(\frac{n-p}{n-2}\right) |\nabla\varphi|^{\beta(p-1)+p-2} \nabla\varphi \,, \tag{2.3.26}$$

where X has been defined in (2.3.10). It is convenient to observe that at a regular value of φ it holds

$$\left(\frac{e^{\frac{(n-p)}{(n-2)(p-1)}s}-\lambda}{e^{\frac{(n-p)}{(n-2)(p-1)}s}}\right)(\Phi^p_\beta)'(s)\;-\;\frac{(n-p)}{(n-2)(p-1)}\Phi^p_\beta(s)\;=\;\frac{1}{(p-1)}\int\limits_{\{\varphi=s\}}\left\langle Y_\lambda\,\middle|\,\frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle d\sigma\,, \tag{2.3.27}$$

by the definitions of the vector fields and the functions involved and Proposition 2.17. In the next lemma, we compute the divergence of Y_λ, that happens to be nonnegative when the so is the divergence of X.

Lemma 2.21 (Divergence of Y_λ). *Let Y_λ be the vector field defined in (2.3.26). Then, the following identity holds in a neighbourhood of any point $x\in\mathbb{R}^n\setminus\overline{\Omega}$ such that $|\nabla\varphi|(x)\neq 0$*

$$\mathrm{div}Y_\lambda\;=\;\left(\frac{e^{\frac{(n-p)}{(n-2)(p-1)}\varphi}-\lambda}{e^{\frac{(n-p)}{(n-2)(p-1)}\varphi}}\right)Q\;\geq\;0\,,$$

where Q is the quantity defined in (2.3.14).

Proof. By the very definition of Y_λ, we have that

$$\begin{aligned}\mathrm{div}Y_\lambda\;=\;&\left(e^{\frac{(n-p)}{(n-2)(p-1)}\varphi}-\lambda\right)\mathrm{div}X\;+\;\frac{(n-p)}{(n-2)(p-1)}\,e^{\frac{(n-p)}{(n-2)(p-1)}\varphi}\,\langle X\,|\,\nabla\varphi\rangle\\ &-\left(\frac{n-p}{n-2}\right)\mathrm{div}\big(|\nabla\varphi|^{\beta(p-1)+p-2}\nabla\varphi\big)\,.\end{aligned}$$

Using the definition (2.3.10) of the vector field X, we compute

$$\langle X\,|\,\nabla\varphi\rangle\;=\;(p-1)\,e^{-\frac{(n-p)}{(n-2)(p-1)}\varphi}\,|\nabla\varphi|^{p-2}\,\langle\nabla|\nabla\varphi|\,|\,\nabla\varphi\rangle\,.$$

Exploiting the p-harmonicity of φ, we get

$$\mathrm{div}\big(|\nabla\varphi|^{\beta(p-1)+p-2}\nabla\varphi\big)\;=\;|\nabla\varphi|^{p-2}\left\langle\nabla|\nabla\varphi|^{\beta(p-1)}\,\middle|\,\nabla\varphi\right\rangle\,.$$

We conclude that

$$\mathrm{div}Y_\lambda\;=\;\left(e^{\frac{(n-p)}{(n-2)(p-1)}\varphi}-\lambda\right)\mathrm{div}X\;=\;\left(\frac{e^{\frac{(n-p)}{(n-2)(p-1)}\varphi}-\lambda}{e^{\frac{(n-p)}{(n-2)(p-1)}\varphi}}\right)Q\,,$$

where in the last equality we made use of the identity (2.3.13). □

Again, in absence of critical points, the Divergence Theorem applied to the vector field Y_λ on the open region $\{s<\varphi<S\}$ easily yields the inequality

$$\int\limits_{\{\varphi=s\}}\left\langle Y_\lambda\,\middle|\,\frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle d\sigma\;\leq\;\int\limits_{\{\varphi=S\}}\left\langle Y_\lambda\,\middle|\,\frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle d\sigma\,,$$

and in turns, thanks to (2.3.27), the inequality (2.3.28) below. As usual, the difficult part is the treatment of the critical points. However, a quite surprising computation in the spirit of Theorem 2.20 shows that it is always possible to deduce the second effective inequality, that is (2.2.49). This fact, together with Theorem 2.22, completes the proof of the Effective Monotonicity-Rigidity Theorem for Φ^p_β.

Theorem 2.22 (Second effective inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with smooth boundary. Then, for every $0 < \lambda < 1$ and any couple nonsingular values $s < S$, the inequality*

$$\left(\frac{\mathrm{e}^{\frac{(n-p)s}{(n-2)(p-1)}} - \lambda}{\mathrm{e}^{\frac{(n-p)s}{(n-2)(p-1)}}} \right) (\Phi^p_\beta)'(s) - \frac{(n-p)\,\Phi^p_\beta(s)}{(n-2)(p-1)} \leq \left(\frac{\mathrm{e}^{\frac{(n-p)S}{(n-2)(p-1)}} - \lambda}{\mathrm{e}^{\frac{(n-p)S}{(n-2)(p-1)}}} \right) (\Phi^p_\beta)'(S) - \frac{(n-p)\,\Phi^p_\beta(S)}{(n-2)(p-1)} \tag{2.3.28}$$

holds true, where Φ^p_β is the function defined in (2.2.24). In particular, one has that $\Phi^p_\beta(S) \leq \Phi^p_\beta(0)$, that is (2.2.49).

Proof. Let $\chi : [0, +\infty) \to \mathbb{R}$ be the same smooth nonnegative cut-off function as in the proof of Theorem 2.20, so that the properties (2.3.23) are in force. To simplify the notation, let us also set

$$\eta_\lambda(\varphi) \; = \; \frac{1}{\mathrm{e}^{\frac{(n-p)}{(n-2)(p-1)}\varphi} - \lambda}\,.$$

Finally, let us consider the smooth vector field

$$\widetilde{Y}_\lambda \; = \; \chi\left(\eta_\lambda(\varphi)\,|\nabla\varphi|^{\beta(p-1)}\right)\, Y_\lambda\,,$$

where Y_λ has been defined in (2.3.26). Again, choosing ε small enough, we can suppose $\widetilde{Y}_\lambda = Y_\lambda$ on $\{\varphi = s\}$ and $\{\varphi = S\}$, with for nonsingular s and S and apply the Divergence Theorem to the smooth vector field $\widetilde{Y}_\lambda$ on the region $\{s < \varphi < S\}$. It yields

$$\int\limits_{\{\varphi=S\}} \left\langle Y_\lambda \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|} \right\rangle \mathrm{d}\sigma \; - \int\limits_{\{\varphi=s\}} \left\langle Y_\lambda \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|} \right\rangle \mathrm{d}\sigma \; = \int\limits_{\{s<\varphi<S\}} \mathrm{div}\widetilde{Y}_\lambda \,\mathrm{d}\mu \; =$$

$$= \int\limits_{\{s<\varphi<S\}\setminus N_{\varepsilon/2}} \chi(\eta_\lambda(\varphi)\,|\nabla\varphi|)\;\mathrm{div}Y_\lambda\,\mathrm{d}\mu \; + \int\limits_{N_{3\varepsilon/2}\setminus N_{\varepsilon/2}} \dot\chi(\eta_\lambda(\varphi)\,|\nabla\varphi|)\;\left\langle Y_\lambda \,\middle|\, \nabla\left(\eta_\lambda(\varphi)|\nabla\varphi|^{\beta(p-1)}\right)\right\rangle \mathrm{d}\mu\,,$$

where this time the tubular neighbourhoods of Crit are defined, for every $\delta > 0$, as $N_\delta = \{\,\eta_\lambda(\varphi)\,|\nabla\varphi|^{\beta(p-1)} \leq \delta\,\}$. Since, as observed in Lemma 2.21, the divergence of Y_λ is nonnegative on $\{s \leq \varphi \leq S\} \setminus N_{\varepsilon/2}$, where clearly $|\nabla\varphi| \neq 0$, in light of (2.3.27) the inequality (2.3.28) is proved if we can show that

$$\left\langle Y_\lambda \,\middle|\, \nabla\left(\eta_\lambda(\varphi)|\nabla\varphi|^{\beta(p-1)}\right)\right\rangle \; \geq \; 0$$

on $N_{3\varepsilon/2} \setminus N_{\varepsilon/2}$. A direct – though not immediately evident – computation, combined with the definition (2.3.26) of Y_λ yields

$$\begin{aligned}
\left\langle Y_\lambda \,\middle|\, \nabla\left(\frac{|\nabla\varphi|}{\mathrm{e}^{\frac{(n-p)}{(n-2)(p-1)}\varphi} - \lambda}\right)\right\rangle &= \mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}\varphi}\,|\nabla\varphi|^{p-2}\,\left|\nabla^\top|\nabla\varphi|^{\beta(p-1)}\right|^2 \\
&\quad + \mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}\varphi}\,|\nabla\varphi|^{p-2}(p-1)\left|\nabla^\perp|\nabla\varphi|^{\beta(p-1)}\right|^2 \\
&\quad - 2\left(\frac{n-p}{n-2}\right)\eta_\lambda(\varphi)\,|\nabla\varphi|^{(\beta+1)(p-1)}\left\langle \nabla|\nabla\varphi|^{\beta(p-1)} \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle \\
&\quad + \left(\frac{n-p}{n-2}\right)^2\eta_\lambda^2(\varphi)\,\frac{\mathrm{e}^{\frac{(n-p)}{(n-2)(p-1)}\varphi}|\nabla\varphi|^{2\beta(p-1)+p}}{p-1} \\
&= \mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}\varphi}\,|\nabla\varphi|^{p-2}\,\left|\nabla^\top|\nabla\varphi|^{\beta(p-1)}\right|^2 \\
&\quad + \left[\left(\frac{n-p}{n-2}\right)\eta_\lambda(\varphi)\left(\frac{\mathrm{e}^{\frac{(n-p)}{(n-2)(p-1)}\varphi}|\nabla\varphi|^{2\beta(p-1)+p}}{(p-1)}\right)^{1/2}\right. \\
&\quad \left. - \left\langle \nabla|\nabla\varphi|^{\beta(p-1)} \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle \left((p-1)\,\mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}\varphi}\,|\nabla\varphi|^{p-2}\right)^{1/2}\right]^2
\end{aligned}$$

This completes the proof of the first part of the statement, since the rightmost hand side is manifestly nonnegative. It remains to show that $\Phi^p_\beta(S) \leq \Phi^p_\beta(0)$ for any regular $S > 0$. Applying the just proved inequality (2.3.28) with $s = 0$, we get

$$\begin{aligned}
\frac{(n-p)}{(n-2)(p-1)}\left(\Phi^p_\beta(S) - \Phi^p_\beta(0)\right) &\leq -(1-\lambda)\,(\Phi^p_\beta)'(0) + \left(\frac{\mathrm{e}^{\frac{(n-p)}{(n-2)(p-1)}S} - \lambda}{\mathrm{e}^{\frac{(n-p)}{(n-2)(p-1)}S}}\right)(\Phi^p_\beta)'(S) \\
&\leq -(1-\lambda)\,(\Phi^p_\beta)'(0)\,,
\end{aligned}$$

where in the last inequality we used $(\Phi^p_\beta)'(S)$ from Theorem 2.20. Letting $\lambda \to 1^-$, we get the claimed inequality. □

2.3.3 Full monotonicity in absence of critical points

It directly follows Theorem 2.20 that if $|\nabla\varphi|_g > 0$ on the whole $\mathbb{R}^n \setminus \Omega$, then Φ^p_β is fully monotone nonincreasing. In particular, in this case, we can infer the full monotonicity of U^p_β. Since, as already recalled, standard elliptic regularity theory ensures in this case that u is analytic, from the vanishing of $(U^p_\beta)'$ at some $0 < t_0 \leq 1$ we can deduce that Ω is a ball. We just write the full statement for U^p_β, leaving the details of its conformal version to the interested reader.

Theorem 2.23 (Full Monotonicity of U^p_β in absence of critical points)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with smooth boundary, and let u be the solution to (2.2.2). Assume that $|\mathrm{D}u| > 0$. Then, the function $U^p_\beta : (0,1] \to \mathbb{R}$ defined by*

$$U^p_\beta(t) = t^{-\beta(p-1)\frac{(n-1)}{n-p}} \int_{\{u=t\}} |\mathrm{D}u|^{(\beta+1)(p-1)}$$

is differentiable and satisfies

$$\begin{aligned}\frac{\mathrm{d}U_\beta^p}{\mathrm{d}t}(t) &= \beta\, t^{-\beta(p-1)\left(\frac{n-1}{n-p}\right)} \int\limits_{\{u=t\}} |\mathrm{D}u|^{(\beta+1)(p-1)-1} \left[\mathrm{H} - \frac{(n-1)(p-1)}{(n-p)} |\mathrm{D} \log u| \right] \mathrm{d}\sigma \\ &= \frac{\beta}{t^2} \int_{\{u\le t\}} u^{2-\beta(p-1)\left(\frac{n-1}{n-p}\right)} |\mathrm{D}u|^{(\beta+1)(p-1)-3} \Big\{ |\mathrm{D}u|^2 \left| \mathrm{h} - \frac{\mathrm{H}}{n-1} g^\top \right|^2 \\ &\quad + \big[\beta(p-1)+p-2\big] \left|\mathrm{D}^\top |\mathrm{D}u|\right|^2 \\ &\quad + (p-1)^2 \left[\beta - \frac{(n-p)}{(n-1)(p-1)}\right] |\mathrm{D}u|^2 \left[\mathrm{H} - \left[\tfrac{(n-1)(p-1)}{n-p}\right] |\mathrm{D}\log u| \right]^2 \Big\} \,\mathrm{d}\mu\,. \end{aligned} \tag{2.3.29}$$

In particular, the derivative of U_β^p is always nonnegative. Moreover $(U_\beta^p)'(t_0) = 0$ for some $t_0 \in (0,1]$ if and only if Ω is a ball and u is rotationally symmetric.

Proof. From the assumptions we have $\mathrm{Crit}\, u = \emptyset$, and thus Theorem 2.2.48 and a direct perusal of the proof of Theorem, 2.20, precisely, by (2.3.24), we get

$$\begin{aligned}(\Phi_\beta^p)'(s) &= -\beta \int\limits_{\{\varphi=s\}} |\nabla\varphi|_g^{(\beta+1)(p-1)-1} \mathrm{H}_g \,\mathrm{d}\sigma_g \\ &= -\beta\, \mathrm{e}^{\frac{n-p}{(n-2)(p-1)}s} \int\limits_{\{\varphi\ge s\}} \mathrm{e}^{-\frac{n-p}{(n-2)(p-1)}\varphi} |\nabla\varphi|_g^{(\beta+1)(p-1)-3} \Big\{ |\nabla\varphi|_g^2 \left| \mathrm{h}_g - \frac{\mathrm{H}_g}{n-1} g^\top \right|^2_{g^\top} \\ &\qquad + \big[\beta(p-1)+p-2\big] \left|\nabla^\top |\nabla\varphi|_g\right|_g^2 \\ &\qquad + (p-1)^2 \left[\beta - \frac{(n-p)}{(n-1)(p-1)}\right] \left|\nabla^\perp |\nabla\varphi|_g\right|_g^2 \Big\} \,\mathrm{d}\mu_g\,. \end{aligned} \tag{2.3.30}$$

The main identities to be used in order to recover (2.3.29) from (2.3.30) are the following.

$$|\nabla\varphi|_g^2 \left| \mathrm{h}_g - \frac{\mathrm{H}_g}{n-1} g^\top \right|^2_{g^\top} = \left[\frac{(p-1)(n-2)}{(n-p)}\right]^2 u^{-2\frac{n+p-2}{n-p}} |\mathrm{D}u|^2 \left| \mathrm{h} - \frac{\mathrm{H}}{n-1} g^\top_{\mathbb{R}^n} \right|^2_{g^\top_{\mathbb{R}^n}},$$

where in the left hand side the second fundamental form and the mean curvature are those of the level sets of φ, in the right hand one they are referred to the corresponding level sets of u. Moreover

$$\left|\nabla^\top |\nabla\varphi|_g\right|_g^2 = \left[\frac{(p-1)(n-2)}{(n-p)}\right]^2 u^{-2\frac{n+p-2}{n-p}} \left|\mathrm{D}^\top |\mathrm{D}u|_{g_{\mathbb{R}^n}}\right|^2_{g_{\mathbb{R}^n}},$$

and

$$\left|\nabla^\perp |\nabla\varphi|_g\right|_g^2 = \left(\frac{n-2}{n-p}\right)^2 u^{-2\frac{n+p-2}{n-p}} |\mathrm{D}u|^2 \left[\mathrm{H} - \frac{(n-1)(p-1)}{(n-p)} \frac{|\mathrm{D}u|}{u}\right]^2.$$

If $(U_\beta^p)'(t_0) = 0$ for some $t_0 \in (0,1]$, then by (2.3.29) all the level sets $\{u = t\}$ for $t \le t_0$ are spheres. Let R_0 such that $\{u = t_0\} = \partial B(O, R_0)$ for some origin $O \in \mathbb{R}^n$. Let then

$$v(x) = t_0 \left(\frac{R_0}{|x|}\right)^{\frac{n-p}{p-1}},$$

that solves

$$\begin{cases} \Delta_p v = 0 & \text{in} \quad \mathbb{R}^n \setminus \overline{B}_{R_0} \\ v = t_0 & \text{on} \quad \partial B_{R_0} \\ v(x) \to 0 & \text{as} \quad |x| \to \infty. \end{cases}$$

Since u solves the same problem then by the uniqueness of solutions we get $u = v$ in $\mathbb{R}^n \setminus B_{R_0}$, and in turn u is rotationally symmetric in this region. Finally, since v can be extended to $\mathbb{R}^n \setminus \{0\}$, we have that u and this extension of v are both analytic function coinciding on an open subset of M, and thus they must coincide on the whole $\mathbb{R}^n \setminus \Omega$. □

It is natural to wonder whether there are geometric conditions on Ω ensuring that the p-capacitary potential u does not develop critical points. To the author's knowledge, the only complete result in this direction available in literature is contained in the main theorem of [Lew77], implying that if Ω is strictly convex then $|\mathrm{D}u| \neq 0$ on $\mathbb{R}^n \setminus \Omega$. Actually, [Lew77] considers the p-capacitary function in convex rings, that in particular entails the case of the p-harmonic function u_R on $B(O,R) \setminus \overline{\Omega}$ with $\Omega \Subset B(O,R)$ such that $u = 1$ on $\partial\Omega$ and $u = 0$ on $\partial B(O,R)$, and proves that $|\mathrm{D}u_R| > 0$. However, it can be checked that the positive lower bound on $|\mathrm{D}u_R|$ does not depend on R (see also the more general [BC18, Lemma B.1] for an explicit computation), and then the limit of u_R as $R \to +\infty$, that produces the p-capacitary function u of Ω, still satisfies $|\mathrm{D}u| > 0$. We refer the reader to the proof of Theorem B.1 in Appendix B for details on the local $\mathscr{C}^1$-convergence of u_R to u on compact sets.

It is worth pointing out that the strict convexity assumption can be relaxed to strict starshapedness in the linear case $p = 2$. This is a well known result, and it can be found in [Eva10, Theorem 1, Section 9.5]. It is based on the harmonicity of the support function $\langle x \,|\, \mathrm{D}u \rangle$ when u is harmonic, and a straightforward maximum principle argument yields the result. Finding a suitable elliptic equation solved by the support function when u is p-harmonic functions does not seem easy, and the validity of this extension to p-capacitary potentials remains to the author's knowledge an open problem. It is worth pointing out, on the other hand, that starshapedness of the level sets of the p-capacitary function of a starshaped set Ω is valid, and it is proved in [Sal05], despite the techniques used do not seem sufficient to infer the *strict* starshapedness, that would imply the absence of critical points.

2.4 Minkowski Inequalities and applications

In the following Subsections, we re going to easily deduce from the Effective Monotonicity Theorem for U^p_β a L^p-version of the Minkowski inequality, that will lead in the limit as $p \to 1^+$ to the Extended Minkowski inequality, and analyse some consequences of our new generalised version of this classical result.

2.4.1 From the L^p-Minkowski inequality to the Extended Minkowski Inequality

The following is the statement of the L^p-Minkowski inequality, together with the related rigidity statement.

Theorem 2.24 (L^p-Minkowski Inequality). *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with smooth boundary. Then, for every $1 < p < n$, the following inequality holds*

$$\mathrm{C}_p(\Omega)^{\frac{n-p-1}{n-p}} \leq \frac{1}{|\mathbb{S}^{n-1}|} \int_{\partial\Omega} \left| \frac{\mathrm{H}}{n-1} \right|^p \mathrm{d}\sigma \,, \tag{2.4.1}$$

where $\mathrm{C}_p(\Omega)$ is the normalised p-capacity of Ω introduced in Definition 2.1. Moreover, equality holds in (2.4.1) if and only if Ω is a ball.

Proof. We let $\beta = 1/(p-1)$, and call for simplicity $U_p = U_p^{1/(p-1)}$. The substantial monotonicity of U_p proved in Theorem 2.13 yields $U_p'(1) \geq 0$ and $U_p(0^+) \leq U_p(1)$, respectively. The first inequality, combined with the expression for the derivative of U_p at regular values of u, given in (2.2.18) implies that

$$\int_{\partial\Omega} \left(\frac{p-1}{n-p}\right) |\mathrm{D}\log u|^p \,\mathrm{d}\sigma \;\leq\; \int_{\partial\Omega} |\mathrm{D}u|^{p-1} \frac{\mathrm{H}}{n-1} \,\mathrm{d}\sigma \,.$$

Applying the Hölder inequality to the above right hand side, with conjugate exponents $a = p/(p-1)$ and $b = p$, one is left with

$$\int_{\partial\Omega} |\mathrm{D}u|^p \,\mathrm{d}\sigma \;\leq\; \left(\frac{n-p}{p-1}\right)^p \int_{\partial\Omega} \left| \frac{\mathrm{H}}{n-1} \right|^p \mathrm{d}\sigma \,. \tag{2.4.2}$$

Using the inequality $U_p(0^+) \leq U_p(1)$ in combination with (2.2.21) we get

$$\left(\frac{n-p}{p-1}\right)^p |\mathbb{S}^{n-1}| \, \mathrm{C}_p(\Omega)^{\frac{n-p-1}{n-p}} \;=\; \lim_{\tau \to 0^+} U_p(\tau) \;\leq\; U_p(1) \;=\; \int_{\partial\Omega} |\mathrm{D}u|^p \,\mathrm{d}\sigma \,,$$

that coupled with (2.4.2) gives the desired

$$\mathrm{C}_p(\Omega)^{\frac{n-p-1}{n-p}} \leq \frac{1}{|\mathbb{S}^{n-1}|} \int_{\partial\Omega} \left| \frac{\mathrm{H}}{n-1} \right|^p \mathrm{d}\sigma \,.$$

Assume now that equality holds in (2.24). Then, equality holds in (2.4.2), and consequently $U_p'(1) = 0$. The isometry of $\partial\Omega$ with a sphere then follows from the rigidity statement in Theorem 2.7. □

We are now interested in passing to the limit as $p \to 1^+$ in the L^p-Minkowski Inequality (2.4.1). The main task here is to compute – and characterise geometrically – the limit of the variational p-capacity of a bounded set with smooth boundary. This topic will be discussed in full details and great generality in the next Chapter. The notion of Ω^* was anyway outlined in Chapter 1 Subsection 1.7.3. Importantly, the following approximation of $|\partial\Omega^*|$ holds true for an open bounded set Ω with smooth boundary

$$\lim_{p \to 1^+} \mathrm{C}_p(\Omega) = \frac{|\partial\Omega^*|}{|\mathbb{S}^{n-1}|}. \tag{2.4.3}$$

We refer the reader to the next Chapter, and in particular to Theorem 3.1, for a rigorous proof of (2.4.3) and of various other aspects of Ω^*.

We can now prove the Extended Minkowski Inequality, together with its Corollary 2.26.

Theorem 2.25 (Extended Minkowski Inequality). *If $\Omega \subset \mathbb{R}^n$ is a bounded open subset with smooth boundary, then*

$$\left(\frac{|\partial\Omega^*|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}} \leq \frac{1}{|\mathbb{S}^{n-1}|}\int\limits_{\partial\Omega}\left|\frac{\mathrm{H}}{n-1}\right| \mathrm{d}\sigma\,, \tag{2.4.4}$$

where Ω^* is the strictly outward minimising hull of Ω defined as in (3.2.17). Moreover, the dimensional constants appearing here are optimal, in the sense that

$$\min\left\{ |\partial\Omega^*|^{-\frac{n-2}{n-1}}\int\limits_{\partial\Omega}|\mathrm{H}|\,\mathrm{d}\sigma \;\middle|\; \Omega \Subset \mathbb{R}^n\,,\ \textit{with } \partial\Omega \textit{ smooth}\right\} = (n-1)\,|\mathbb{S}^{n-1}|^{\frac{1}{n-1}}\,,$$

and the minimum is achieved on spheres.

Proof. It suffices to pass to the limit as $p \to 1^+$ in (2.4.1), that is

$$\mathrm{C}_p(\Omega)^{\frac{n-p-1}{n-p}} \leq \frac{1}{|\mathbb{S}^{n-1}|}\int\limits_{\partial\Omega}\left|\frac{\mathrm{H}}{n-1}\right|^p \mathrm{d}\sigma\,. \tag{2.4.5}$$

Indeed, recalling the relation between p-capacity and normalised p-capacity given in Definition 2.1, the discussion above, or precisely Theorem 3.1, shows that the left hand side of the above inequality satisfies

$$\lim_{p\to 1^+}\mathrm{C}_p(\Omega)^{\frac{n-p-1}{n-p}} = \left(\frac{|\partial\Omega^*|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}}\,,$$

while the right-hand side of (2.4.5) is immediately seen to converge to the right hand side of (2.4.4). Spheres show the optimality of the estimate since their mean curvature is given by $(n-1)/R$, where R is the radius of the ball they enclose. □

The Extended Minkowski inequality readily yields the known version for outward minimising sets. We are spending some lines to prove the rigidity statement when the set considered is actually strictly outward minimising and its boundary is strictly mean-convex. For this last observation we are going to appeal to the Inverse Mean Curvature Flow, since we still do not know how to use our (effective) monotonicity formulas to recover the rigidity statement. The difficulty obviously arises by the need of passing to $p \to 1^+$, that makes impossible to use the effective monotonicity result Theorem 2.7.

Corollary 2.26 (Minkowski Inequality for Outward Minimising Sets). *If $\Omega \subset \mathbb{R}^n$ is a bounded outward minimising open subset with smooth boundary, then*

$$\left(\frac{|\partial\Omega|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}} \leq \frac{1}{|\mathbb{S}^{n-1}|}\int\limits_{\partial\Omega}\frac{\mathrm{H}}{n-1}\, \mathrm{d}\sigma\,. \tag{2.4.6}$$

Moreover, the dimensional constants appearing here are optimal, in the sense that

$$\min\left\{ |\partial\Omega|^{-\frac{n-2}{n-1}}\int\limits_{\partial\Omega}\mathrm{H}\,\mathrm{d}\sigma \;\middle|\; \Omega \Subset \mathbb{R}^n \textit{outward minimising}\,,\ \textit{with } \partial\Omega \textit{ smooth}\right\} = (n-1)\,|\mathbb{S}^{n-1}|^{\frac{1}{n-1}}\,,$$

and the minimum is achieved on spheres. Viceversa, if the equality holds in (2.4.6) for some bounded strictly outward minimising *open subset with smooth and* strictly mean convex *boundary, then Ω is isometric to a round ball.*

Proof. Inequality (2.4.6) immediately follows from the fact that outward minimising sets with smooth boundary satisfy $|\partial\Omega^*| = |\partial\Omega|$ and the mean curvature of their boundaries is nonnegative (see Remark 3.16).

We are left to consider the equality case in (2.4.6) for some strictly outward minimising sets with smooth and strictly mean-convex boundary. To this aim, let $\{\partial\Omega_t\}_{t\in[0,T)}$ be the evolution of $\partial\Omega$ under smooth IMCF, up to some $T > 0$. By [HI01, Lemma 2.4], the sets E_t evolving by weak IMCF starting at $\partial\Omega$, whose definition is recalled in Subsection 3.3.2 in the next chapter, coincide with Ω_t in some time interval $[0, T^*)$, with T^* possibly smaller than T. In particular, by [HI01, Lemma 1.4], Ω_t is strictly outward minimising and strictly mean-convex for every $t \in [0, T^*)$, and then (2.4.6) holds for every $\partial\Omega_t$ with $t \in [0, T^*)$. We can then define, for $t \in [0, T^*)$, the monotonic quantity already discussed in the Introduction

$$\mathscr{Q}(t) \, = \, |\partial\Omega_t|^{-\frac{n-2}{n-1}} \int\limits_{\partial\Omega_t} \mathrm{H} \, \mathrm{d}\sigma \, .$$

Observe that the inequality (2.4.6) is equivalent to $\mathscr{Q}(0) \geq (n-1)|\mathbb{S}^{n-1}|^{1/(n-1)}$, and assuming equality in (2.4.6) is equivalent to $\mathscr{Q}(0) = |\mathbb{S}^{n-1}|^{1/(n-1)}$. By the smoothness of the flow, the function $\mathscr{Q}(t)$ is differentiable for $t \in [0, T)$, and then a straightforward computation involving the standard evolution equations provided e.g. in [HP99, Theorem 3.2] show that

$$\mathscr{Q}'(0) = -|\partial\Omega|^{-\frac{n-2}{n-1}} \int\limits_{\partial\Omega} \frac{|\mathring{\mathrm{h}}|^2}{\mathrm{H}} \, \mathrm{d}\sigma \leq 0 \, . \tag{2.4.7}$$

However, since we assumed $\mathscr{Q}(0) = 0$, $\mathscr{Q}'(0) < 0$ would imply $\mathscr{Q}(t) < 0$ for some $t \in (0, T^*)$ that is equivalent to falsify (2.4.6) for some outward minimising Ω_t with strictly mean-convex boundary. Then $\mathscr{Q}'(0) = 0$, and by formula (2.4.7) this means that $\partial\Omega$ is totally umbilical, and thus a sphere. □

We remark once again that (2.4.6) in particular holds for strictly starshaped sets with mean-convex boundary, see Proposition 3.25.

It is not quite clear, at least at first glance, whether one can in fact deduce the Extended Minkowski Inequality (2.4.4) from its corollary for outward minimising sets. We are able to show that this is the case if $n \leq 7$.

Corollary 2.26 implies Theorem 2.25 if $n \leq 7$. Let $\Omega \subset \mathbb{R}^n$ be a bounded set with smooth boundary, and $n \leq 7$. Then, as recalled in Subsection 1.7.3 in the previous chapter, Ω^* is of class $\mathscr{C}^{1,1}$. We can then apply, as there, Lemma 1.66 to Ω^* to obtain a sequence of strictly outward minimising sets E_ε with smooth and strictly mean-convex boundary approximating Ω^* in $\mathscr{C}^1 \cap W^{1,1}$-topology as $\varepsilon \to 0^+$. In particular, for these sets, (2.4.6) is in force, and then we get, letting $\varepsilon \to 0^+$

$$\left(\frac{|\partial\Omega^*|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}} \leq \frac{1}{|\mathbb{S}^{n-1}|} \int\limits_{\partial\Omega^*} \frac{\mathrm{H}}{n-1} \, \mathrm{d}\sigma \leq \int\limits_{\partial\Omega} \frac{\mathrm{H}}{n-1} \, \mathrm{d}\sigma,$$

where the last inequality is due to the fact that any region of $\partial\Omega$ with negative mean curvature is replaced by regions of $\partial\Omega^*$ that are minimal, as immediately deduced from Theorem 3.14, anticipated by the discussion in Subsection 1.7.3. □

The above argument evidently breaks down when $n \geq 8$, due to area minimising hypersurface regularity issues. Indeed, in this case, nonempty singular sets can appear in $\partial\Omega^*$, preventing this boundary to be $\mathscr{C}^1$ and thus making the application of Lemma 1.66 no more possible. We are not aware of other approximation results, strong enough to let the L^1-norm of the mean-curvature of the approximator converge, holding true for boundaries with (small) singular sets.

2.4.2 Applications of the Extended Minkowski Inequality

The two applications we give consist in a Volumetric Minkowski Inequality, known in literature also as Higher Order Isoperimetric Inequality, and in the De Lellis-Müller nearly umbilical estimates with sharp constant for outward minimising sets.

Volumetric Minkowski Inequality

An application of the Euclidean Isoperimetric Inequality (see e.g. [Mag12, Theorem 14.1]) immediately yields the following.

Theorem 2.27 (Volumetric Minkowski Inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with smooth boundary. Then*

$$\left(\frac{|\Omega|}{|\mathbb{B}^n|}\right)^{\frac{n-2}{n}} \leq \frac{1}{|\mathbb{S}^{n-1}|} \int_{\partial\Omega} \left|\frac{\mathrm{H}}{n-1}\right| \mathrm{d}\sigma\,. \tag{2.4.8}$$

Moreover, equality holds in (2.4.8) *if and only if Ω is a ball.*

Proof. Applying the Euclidean Isoperimetric Inequality to the left hand side of (2.4.4), we get

$$\left(\frac{|\Omega|}{|\mathbb{B}^n|}\right)^{\frac{n-2}{n}} \leq \left(\frac{|\Omega^*|}{|\mathbb{B}^n|}\right)^{\frac{n-2}{n}} \leq \left(\frac{|\partial\Omega^*|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}} \leq \frac{1}{|\mathbb{S}^{n-1}|} \int_{\partial\Omega} \left|\frac{\mathrm{H}}{n-1}\right| \mathrm{d}\sigma\,, \tag{2.4.9}$$

where the first inequality is just due to $\Omega \subseteq \Omega^*$. If equality holds in (2.4.8), then by (2.4.9) Ω^* satisfies equality in the Euclidean Isoperimetric Inequality, and then Ω^* is a ball up to negligible sets. Arguing as in the proof of the Isoperimetric Inequality for complete noncompact 3-manifolds with nonnegative Ricci curvature Theorem 1.7.2, we conclude that Ω is thus a ball. □

To our knowledge, the above inequality was previously known to hold for domains with a stricly mean-convex boundary of positive scalar curvature (for short $\partial\Omega \in \Gamma_2^+$). On this regard, we refer the reader to the paper [CW13] and the subsequent [Qiu15], where the inequality was proved with methods based on Optimal Transport.

Nearly umbilical estimates with optimal constant for outward minimising sets.

This subsection is devoted to the proof of an optimal version of the celebrated De Lellis-Müller nearly umbilical estimates for outward minimising domains.

Theorem 2.28 (Optimal Nearly Umbilical Estimate)**.** *If $\Omega \subset \mathbb{R}^3$ is a bounded outward minimising open domain with smooth boundary, then*

$$\int_{\partial\Omega} \left|\mathrm{h} - \frac{\overline{\mathrm{H}}}{2}\, g_{\partial\Omega}\right|^2 \mathrm{d}\sigma \;\leq\; 2\int_{\partial\Omega} |\mathring{\mathrm{h}}|^2\, \mathrm{d}\sigma\,, \tag{2.4.10}$$

where $g_{\partial\Omega}$ is the metric induced on $\partial\Omega$ by the Euclidean metric of $\mathbb{R}^3$, and

$$\overline{\mathrm{H}} = \fint_{\partial\Omega} \mathrm{H}\,\mathrm{d}\sigma\,, \qquad\qquad \mathring{\mathrm{h}} = \mathrm{h} - \frac{\mathrm{H}}{2} g_{\partial\Omega}\,.$$

Moreover, the equality is achieved in (2.4.10) *by some* strictly mean-convex *and* strictly outward minimising *domain* Ω *if and only if* Ω *is isometric to a round ball.*

A first main tool we are going to use in order to deduce Theorem 2.28 from Theorem 2.25 is the classical Gauss' equation for surfaces in $\mathbb{R}^3$, yielding

$$\mathrm{R}_{\partial\Omega} = \mathrm{H}^2 - |\mathrm{h}|^2, \tag{2.4.11}$$

where $\mathrm{R}_{\partial\Omega}$ is the scalar curvature of $\partial\Omega$ computed with respect to the metric $g_{\partial\Omega}$ induced on it by the Euclidean metric of $\mathbb{R}^3$. A second main tool we need to recall is the famous Gauss-Bonnet formula, stating that

$$\int_{\partial\Omega} \mathrm{R}_{\partial\Omega}\, \mathrm{d}\sigma = 4\pi\chi(\partial\Omega), \tag{2.4.12}$$

for any domain with smooth boundary Ω, where $\chi(\partial\Omega)$ is the Euler characteristic of the surface $\partial\Omega$. The need of a connected boundary in order to apply (2.4.12) is the reason behind the assumption of Ω to be a domain in Theorem 2.28.

We are finally going to show how the Minkowski inequality (2.4.6) for outward minimising sets combined with these basic identities in differential geometry give the optimal nearly umbilical estimate (2.4.10).

Proof of Theorem 2.28. Expanding the squares, it is straightforwardly seen that (2.4.10) is equivalent to

$$\int_{\partial\Omega} \left(|\mathrm{H}|^2 - |\mathrm{h}|^2\right) \mathrm{d}\sigma \leq \frac{\overline{\mathrm{H}}^2}{2}|\partial\Omega|.$$

Invoking Gauss' equation (2.4.11) and Gauss-Bonnet formula (2.4.12), we then obtain that (2.4.10) is equivalent to

$$8\pi\chi(\partial\Omega) = 2\int_{\partial\Omega} \mathrm{R}_{\partial\Omega}\, \mathrm{d}\sigma = 2\int_{\partial\Omega} \left(|\mathrm{H}|^2 - |\mathrm{h}|^2\right) \mathrm{d}\sigma \leq \overline{\mathrm{H}}^2|\partial\Omega|,$$

that is, to

$$\sqrt{2\pi\chi(\partial\Omega)|\partial\Omega|} \leq \int_{\partial\Omega} \frac{\mathrm{H}}{2}\, \mathrm{d}\sigma. \tag{2.4.13}$$

Since obviously $\chi(\partial\Omega) \leq 2$, the inequality (2.4.13) follows from the Minkowski inequality (2.4.6).

Assume now that equality holds for some outward minimising set Ω with smooth and strictly mean-convex boundary. Let $\{\partial\Omega_t\}_{t\in[0,T)}$ be evolving by IMCF with initial datum $\partial\Omega$. By [HI01, Lemma 2.4], the sets E_t evolving by weak IMCF starting at $\partial\Omega$ coincide with Ω_t for $t \in [0, T^*)$, for some T^* possibly smaller than T. In particular, Ω_t is outward minimising and strictly mean-convex for any $t \in [0, T^*)$, for some $T^* > 0$, and then (2.4.10) holds for $\partial\Omega_t$ for any $t \in [0, T^*)$. We can then define, for $T \in [0, T^*)$, the quantity

$$\mathscr{P}(t) = \int_{\partial\Omega_t} |\mathring{\mathrm{h}}|^2\, \mathrm{d}\sigma - \frac{1}{2}\int_{\partial\Omega_t} \left(\mathrm{H} - \frac{1}{|\partial\Omega_t|}\int_{\partial\Omega_t} \mathrm{H}\right)^2 \mathrm{d}\sigma,$$

introduced in [Per11, Chapter 3]. Observe that inequality (2.4.10) is equivalent to $\mathscr{P}(0) \geq 0$, and assuming equality in (2.4.10) is equivalent to $\mathscr{P}(0) = 0$. By the smoothness of the flow, the function $\mathscr{P}(t)$ is differentiable for $t \in [0, T)$, and then [Per11, Lemma 3.4] yields

$$\mathscr{P}'(0) = -\overline{\mathrm{H}} \int_{\partial\Omega} \frac{|\mathring{\mathrm{h}}|^2}{\mathrm{H}} \, d\sigma \leq 0 \tag{2.4.14}$$

However, since we assumed $\mathscr{P}(0) = 0$, $\mathscr{P}'(0) < 0$ would imply $\mathscr{P}(t) < 0$ for some $t \in (0, T^*)$ that is equivalent to falsify (2.4.10) for some outward minimising Ω_t with strictly mean-convex boundary. Then $\mathscr{P}'(0) = 0$, and by formula (2.4.14) this means that $\partial\Omega$ is totally umbilical, thus a sphere. □

Inequality (2.4.13) in the above proof, that we just showed to be equivalent to the nearly umbilical estimate (2.4.10), is actually equivalent to the Minkowski inequality for mean-convex hypersurfaces (2.4.6) if $\chi(\partial\Omega) = 2$, that is, if $\partial\Omega$ is diffeomorphic to a sphere. We want to show, with the following easy proposition, that such a diffeomorphism exists each time the right-hand side of (2.4.10) is smaller than 16π.

Proposition 2.29. *Let $\Omega \subset \mathbb{R}^3$ be a bounded open subset with smooth and* mean-convex *boundary. If*

$$\int_{\partial\Omega} \left| \mathrm{h} - \frac{\mathrm{H}}{2} g_{\partial\Omega} \right|^2 d\sigma \leq 8\pi \tag{2.4.15}$$

then $\partial\Omega$ is diffeomorphic to a sphere. In particular, if (2.4.15) holds, then $\chi(\Omega) = 2$, and in particular the Minkowski inequality (2.4.6) for outward minimising domains is equivalent to the optimal nearly umbilical estimate (2.4.10) for outward minimising domains.

Proof. If (2.4.15) holds, we have, by the classical Willmore inequality [Wil68]

$$\int_{\partial\Omega} \left| \mathrm{h} - \frac{\mathrm{H}}{2} g_{\partial\Omega} \right|^2 d\sigma \leq 8\pi \leq \frac{1}{2} \int \mathrm{H}^2 \, d\sigma,$$

that implies

$$\int_{\partial\Omega} \left(|\mathrm{H}|^2 - |\mathrm{h}|^2 \right) d\sigma \geq 0. \tag{2.4.16}$$

Moreover, since equality is attained in the Willmore inequality if and only if $\partial\Omega$ is isometric to a sphere with the round metric, the same rigidity statement holds if equality is attained in (2.4.16). On the other hand, applying the Gauss' equation (2.4.11) and the Gauss-Bonnet formula (2.4.12), (2.4.16) is equivalent to

$$\chi(\partial\Omega) \geq 0.$$

If $\chi(\partial\Omega) = 0$ then by the characterisation of the equality case in the Willmore inequality $\partial\Omega$ would be even isometric to a sphere, and this is a contradiction. Then $\chi(\partial\Omega) = 2$, as claimed. □

2.5 Other consequences of the Effective Monotonicity Theorems

Here, we illustrate some others consequences and applications of the (effective) monotonicity of U^p_β for various values of β and of U^p_∞. The following two subsections can be compared to Subsections 1.5.1 and Subsection 1.5.2 in the previous chapter.

2.5.1 p-capacitary inequalities and the relations with Willmore-type functionals

The following sharp inequality for the p-capacity of Ω follows just from $(U^p_\beta)'(1) \geq 0$.

Theorem 2.30. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with smooth boundary. Then*

$$\mathrm{C}_p(\Omega) \leq \frac{|\partial\Omega|}{|\mathbb{S}^{n-1}|} \left(\fint_{\partial\Omega} \left| \frac{\mathrm{H}}{n-1} \right|^{(p-1)(1+\beta)} d\sigma \right)^{1/(1+\beta)} . \tag{2.5.1}$$

Moreover, the above inequality holds with equality sign if and only if Ω is a ball.

Proof. By $(U^p_\beta)'(1) \geq 0$, and more precisely by (2.2.18) for $t = 1$, we get

$$\int_{\partial\Omega} \left(\frac{p-1}{n-p}\right) |\mathrm{D}\log u|^{(p-1)(\beta+1)} d\sigma \leq \int_{\partial\Omega} |\mathrm{D}u|^{(p-1)(\beta+1)-1} \frac{\mathrm{H}}{n-1} d\sigma .$$

Applying the Hölder inequality to the right hand side, with conjugate exponents $a = (\beta+1)(p-1)/[(\beta+1)(p-1)-1]$ and $b = (\beta+1)(p-1)$, we get

$$\int_{\partial\Omega} |\mathrm{D}u|^{(\beta+1)(p-1)} d\sigma \leq \left(\frac{n-p}{p-1}\right)^{(\beta+1)(p-1)} \int_{\partial\Omega} \left|\frac{\mathrm{H}}{n-1}\right|^{(\beta+1)(p-1)} d\sigma . \tag{2.5.2}$$

Applying now the Hölder inequality with conjugate exponents $\beta+1$ and $(\beta+1)/\beta$ to $\int_{\partial\Omega} |\mathrm{D}u|^{p-1} d\sigma$, we get

$$\left(\frac{n-p}{p-1}\right)^{p-1} |\mathbb{S}^{n-1}| \mathrm{C}_p(\Omega) = \int_{\partial\Omega} |\mathrm{D}u|^{p-1} d\sigma \leq \left(\int_{\partial\Omega} |\mathrm{D}u|^{(\beta+1)(p-1)} d\sigma \right)^{\frac{1}{\beta+1}} |\partial\Omega|^{\frac{\beta}{\beta+1}}, \tag{2.5.3}$$

where the first identity follows from (2.2.3) for $t = 1$. Coupling (2.5.3) with (2.5.2) we are left with (2.5.1). It is immediately seen that if equality holds in (2.5.1) then $(U^p_\beta)'(1) = 0$, and the rigidity statement follows from that of Theorem 2.7. □

We find particularly interesting in the family of inequalities (2.5.1) the one estimating the Willmore-type functional, largely discussed in a general Riemannian context in the last chapter. It is deduced just by choosing $\beta = (n-p)/(p-1)$.

Corollary 2.31 (p-capacitary estimates of the Willmore-type functional)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with smooth boundary. Then,*

$$\mathrm{C}_p(\Omega)\, |\mathbb{S}^{n-1}|^{\frac{p-1}{n-1}} \leq \frac{|\partial\Omega|}{|\mathbb{S}^{n-1}|}^{\frac{n-p}{n-1}} \left(\int_{\partial\Omega} \left|\frac{\mathrm{H}}{n-1}\right|^{n-1} d\sigma \right)^{\frac{p-1}{n-1}} . \tag{2.5.4}$$

Moreover equality holds if and only if Ω is a ball.

To our knowledge, the above inequality, as well as those in (2.5.1), were known only for strictly mean-convex strictly starshaped domains. It has been provided, together with other inequalities, by J. Xiao in [Xia17, Theorem 3.1] via the Inverse Mean Curvature Flow and relying on Gerhardt's long time existence theory for strictly mean-convex strictly starshaped initial data developed [Ger90].

The Willmore-type inequality through $p \to n^-$

In his just mentioned paper, Xiao also observes that the classical Euclidean Willmore-type inequality can be deduced by letting $p \to n^-$ in (2.5.4). Indeed, letting $B(x_0, r) \subset \Omega \subset B(x_0, R)$ one deduces from the obvious monotonicity property of the p-capacity with respect to set-inclusion and its explicitly computed value on balls that

$$r^{n-p} \leq \mathrm{C}_p(\Omega) \leq R^{n-p}, \tag{2.5.5}$$

yielding

$$\mathrm{C}_p(\Omega) \to 1$$

as $p \to n^-$. Plugging this information in (2.5.4), we are left with the classical Willmore-type inequality. This alternative proof is interesting in relation of our monotonicity formulas, since it just uses $(U^p_\beta)'(1) \geq 0$, that is just a local information. On the other hand, it is heavily bound to the very special geometry of $\mathbb{R}^n$, encoded in the explicit formula for the p-capacity of balls used in (2.5.5).

The Willmore-type inequality from any value of p.

We observe that, algebraically, there was nothing special about the value $p = 2$ for achieving the Willmore-type inequality. Indeed, let as for Corollary 2.31 $\beta = (n-p)/(p-1)$ and observe that from $\lim_{t \to 0^+} U^p_\beta(t) \leq U^p_\beta(1)$ we get

$$\left(\frac{n-p}{p-1}\right)^{n-1} |\mathbb{S}^{n-1}| \leq \int\limits_{\partial\Omega} |\mathrm{D}u|^{n-1} \, \mathrm{d}\sigma,$$

that, coupled with $(U^p_\beta)'(1) \geq 0$ through Hölder inequality as in (2.5.3) leaves us with the classical Willmore-type inequality. In spite of this, $p = 2$ is anyway the most convenient value to use if one is interested in the Willmore-type inequality, both for the easier algebraic computations and primarily for the wonderful regularity properties of harmonic functions, that we employed successfully in the more general geometric framework treated in the last Chapter.

2.5.2 Spherical symmetry under pinching conditions

Now, we finally give some applications of the substantial monotonicity of U^p_∞. They constitute the nonlinear versions of the main results obtained in [BMM19], where a different geometric construction named *spherical ansatz* was employed.

Theorem 2.32. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with smooth boundary. Then, if the mean curvature of $\partial\Omega$ satisfies*

$$-\left(\frac{1}{\mathrm{C}_p(\Omega)}\right)^{\frac{1}{n-p}} \leq \frac{\mathrm{H}}{n-1} \leq \left(\frac{1}{\mathrm{C}_p(\Omega)}\right)^{\frac{1}{n-p}} \tag{2.5.6}$$

on every point of $\partial\Omega$, then Ω is a ball.

Proof. By means of the global aspect of the substantial monotonicity of U^p_∞ (2.2.19), the computation of its limit in (2.2.22) and the local aspect of the substantial monotonicity

(2.2.20) we deduce

$$\lim_{t\to 0^+} U_\infty^p(t) = \left(\frac{n-p}{p-1}\right) \mathrm{C}_p(\Omega)^{-\frac{1}{n-p}} \leq \sup_{\partial\Omega} |\mathrm{D}u| \leq \left(\frac{n-p}{p-1}\right) \sup_{\partial\Omega}\left|\frac{\mathrm{H}}{n-1}\right|,$$

with equality achieved in any of the above inequalities if and only if Ω is a ball, by the rigidity statement in Theorem 2.8. In particular, this happens if (2.5.6) is satisfied. □

As a consequence of the above argument we get

$$\left(\frac{1}{\mathrm{C}_p(\Omega)}\right)^{\frac{1}{n-p}} \leq \sup_{\partial\Omega}\left|\frac{\mathrm{H}}{n-1}\right|. \tag{2.5.7}$$

Letting $p \to 1^+$, we get, by (2.4.3), the following completely geometric consequence.

Corollary 2.33. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with smooth boundary. Then, the following inequality holds*

$$\left(\frac{|\mathbb{S}^{n-1}|}{|\partial\Omega^*|}\right)^{\frac{1}{n-1}} \leq \sup_{\partial\Omega}\left|\frac{\mathrm{H}}{n-1}\right|. \tag{2.5.8}$$

In particular, it holds

$$\left(\frac{|\mathbb{S}^{n-1}|}{|\partial\Omega|}\right)^{\frac{1}{n-1}} \leq \sup_{\partial\Omega}\left|\frac{\mathrm{H}}{n-1}\right|. \tag{2.5.9}$$

If equality in (2.5.8) is achieved on a strictly outward minimising set with strictly mean-convex boundary, then Ω is a ball.

Proof. Letting $p \to 1^+$ in (2.5.7) and using (2.4.3), we get, coupling it with $|\partial\Omega^*| \leq |\partial\Omega|$, we get

$$\left(\frac{|\mathbb{S}^{n-1}|}{|\partial\Omega|}\right)^{\frac{1}{n-1}} \leq \left(\frac{|\mathbb{S}^{n-1}|}{|\partial\Omega^*|}\right)^{\frac{1}{n-1}} \leq \sup_{\partial\Omega}\frac{\mathrm{H}}{n-1}.$$

We now have to justify the rigidity statement. Actually, in the case of outward minimising sets, it is immediately seen that inequality (2.5.8) also follows directly from the Minkowski Inequality (2.4.6), and equality is achieved only if it is achieved in the Minkowski inequality. In particular, if Ω is *strictly* outward minimising with strictly mean-convex boundary, the claimed rigidity statement follows from the rigidity statement of Theorem 2.26. □

We observe that if Ω is not outward minimising, (2.5.8) is strictly stronger than (2.5.9). Moreover, in general (2.5.8) does not seem to be implied by the Extended Minkowski Inequality, if Ω is not outward minimising.

2.5.3 Relations with a classical overdetermined problem

Here, we discuss how the monotonicity of U_∞^p constitutes a new interpretation of a maximum principle for for $|\mathrm{D}u|/u^{(n-1)/(n-p)}$, that is, up to a multiplicative constant, $|\nabla\varphi|_g$, widely used in literature to establish symmetry results for overdetermined boundary value problems. Such a function, known in literature as P-function, together with related maximum principles, was first conceived by Payne-Philippin in [PP79] for harmonic functions. Its generalisation to p-harmonic functions on rings first, and then on exterior domains was treated respectively in [Sar98], that generalised the results and the techniques of [Phi90] for the harmonic case, and [GS99].

The overdetermined boundary value problem considered is the following

$$\begin{cases} \Delta_p u = 0 & \text{in} \quad \mathbb{R}^n \setminus \overline{\Omega}\,, \\ u = 1 & \text{on} \quad \partial\Omega\,, \\ u(x) \to 0 & \text{as} \quad |x| \to \infty \\ |\mathrm{D}u| = c & \text{on} \quad \partial\Omega\,, \end{cases} \tag{2.5.10}$$

for some positive constant $c > 0$. It is well known that the above problem admits a solution if and only if Ω is a ball. This was first provided by Reichel in [Rei96] for bounded sets Ω with $\mathscr{C}^2$-boundary using the moving planes method, and later showed by Garofalo-Sartori in [GS99] through the P-function approach for starshaped sets with no a priori smoothness assumptions. Reichel's result was recently recovered in [Pog18] building on Garofalo-Sartori maximum principle for the P-function but concluding the proof with a very interesting integral formula involving the torsion function of Ω provided in [MP19]. We are coming back to this point in a while.

Our Substantial Monotonicity Theorem for U_∞^p actually yields a new interpretation, with a new proof, of the main maximum principle for the P-function. Indeed, (2.2.20) yields for the p-capacitary potential u of Ω

$$\mathrm{H}(x) \,\geq\, \frac{(n-1)(p-1)}{(n-p)} |\mathrm{D}u|(x), \tag{2.5.11}$$

where $x \in \partial\Omega$ is the point where $|\mathrm{D}u|$ achieves its maximum on $\partial\Omega$, and actually where the P-function $|\mathrm{D}u|/u^{(n-1)/(n-p)}$ achieves its maximum on $\mathbb{R}^n \setminus \Omega$. If u is also assumed to satisfy the additional condition of $|\mathrm{D}u| = c$ on $\partial\Omega$, then in particular (2.5.11) holds for *any* $x \in \partial\Omega$. In particular, since by an easy computation (see [Pog18, Lemma 2.4])

$$c = \frac{(n-p)}{n(p-1)} \frac{|\partial\Omega|}{|\Omega|},$$

we deduce that

$$\mathrm{H} \,\geq\, \frac{(n-1)}{n} \frac{|\partial\Omega|}{|\Omega|} \tag{2.5.12}$$

on $\partial\Omega$. The above estimate was the main outcome of the maximum principle for the P-function worked out in [GS99]. Moreover, in light of the rigidity statement for U_∞^p in Theorem 2.8, equality holds in (2.5.12) at some point of $\partial\Omega$ if and only if Ω is a ball. The symmetry of Ω under the existence of a solution to (2.5.10) is then established if we can prove that equality holds somewhere on $\partial\Omega$. This was accomplished in [GS99] using integral identities valid under the starhapedness of Ω. In [Pog18], this was obtained without the starshapedness assumption by means of the sharp inequality, valid for any bounded Ω with smooth boundary,

$$\int\limits_{\partial\Omega} \left(\mathrm{H} - \frac{(n-1)}{n} \frac{|\partial\Omega|}{|\Omega|} \right) |\mathrm{D}v|^2 \, \mathrm{d}\sigma \leq 0\,, \tag{2.5.13}$$

where v is the torsional function of Ω, that is solving

$$\begin{cases} \Delta v = n & \text{in} \quad \Omega\,, \\ v = 0 & \text{on} \quad \partial\Omega\,. \end{cases}$$

The inequality (2.5.13) is actually a corollary of the remarkable integral identity [MP19, Theorem 2.2]. In particular, the combination of (2.5.12) and (2.5.13) implies that equality is achieved in (2.5.12), and we can thus conclude. We find this interplay between the exterior problem for the p-capacitary potential and the interior problem for the torsion quite intriguing, and somehow reminiscent of the proof of the Isoperimetric Inequality worked out in the previous Chapter, where an exterior boundary value problem was exploited in combination with a shrinking flow.

Chapter 3

On strictly outward minimising hulls and p-capacity in Riemannian manifolds

3.1 Main result and structure of the chapter

The main goal of this chapter is to prove the following result, already vastly discussed in the Introduction.

Theorem 3.1. *Let (M, g) be a complete noncompact Riemannian manifold satisfying at least one of the following two conditions.*

(i) *(M, g) satisfies an Euclidean-like Isoperimetric Inequality, namely, there exists $C_{\mathrm{iso}} > 0$ such that*

$$\frac{|\partial\Omega|^n}{|\Omega|^{n-1}} \geq C_{\mathrm{iso}} \tag{3.1.1}$$

for any bounded set Ω with smooth boundary.

(ii) *(M, g) has nonnegative Ricci curvature and the* superlinear uniform volume growth condition *holds*

$$C_{\mathrm{vol}}^{-1} r^b \leq |B(O, r)| \leq C_{\mathrm{vol}} r^b \tag{3.1.2}$$

for some $b > 1$ and $C_{\mathrm{vol}} > 0$, for any $r \geq R$ for some $R > 0$.

Let $\Omega \subset M$ be an open bounded set with finite perimeter. Then, its strictly outward minimising hull Ω^* is an open bounded maximal volume solution to the least area problem with obstacle Ω.

Moreover, if Ω has smooth boundary, then

$$\lim_{p \to 1^+} \mathrm{Cap}_p(\Omega) = |\partial\Omega^*|. \tag{3.1.3}$$

It is well known that the validity of the Euclidean-like Isoperimetric Inequality (3.1.1) is equivalent to that of the L^1-Sobolev Inequality, that is, (3.1.1) is equivalent to the validity of

$$\left(\int_M f^{n/(n-1)}\, \mathrm{d}\mu\right)^{(n-1)/n} \leq C_{\mathrm{Sob}} \int_M |\nabla f|\, \mathrm{d}\mu$$

for any $f \in C_0^\infty(M)$ and some positive constant C_{Sob}. We briefly point out that condition (3.1.2) can actually be weakened or replaced. We are discussing it in Remark 3.28, after having discussed the difficulties in removing it. Still, we preferred the above presented form for the sake of geometric clarity.

This chapter is structured as follows. Section 3.2, after proving basic properties of (strictly) outward minimising sets, is mostly committed to show the equivalence, in a general complete noncompact Riemannian manifold, among the existence of a maximal volume solution to the least area problem with obstacle, the property of Ω^* to be such a solution and the existence of an exhausting sequence of strictly outward minimising sets. This is summarised in Theorem 3.13. Examples of Riemannian manifolds where these equivalent properties are not satisfied are worked out in Examples 3.8 and 3.9. The last part of the section precisely recalls the regularity properties of Ω^*, that were important also in the previous chapters. Here, these properties are important in order to apply an exterior approximation result due to [Sch15], see Theorem 3.17 below. In Section 3.3, we prove the first part of Theorem 3.1, that is the well-posedness of the strictly outward minimising hull Ω^* under assumptions (i) or (ii). When dealing with (ii) in Subsection 3.3.2, we recall the notion of Weak Inverse Mean Curvature Flow, yielding, in relation with that some insights including a proof of the outward minimising property of starshaped sets with smooth mean-convex boundary, not necessarily strictly mean-convex. Finally, in Section 3.4 we infer (3.1.3) under the assumptions of Theorem 3.1.

3.2 The strictly outward minising hull of a set with finite perimeter

3.2.1 Preliminaries

The main sources are Maggi's book [Mag12]. Miranda's [Mir64] and Caraballo's comprehensive paper [Car11]. Clearly, although these references deal only with the case $M = \mathbb{R}^n$, the notions and results that we recall in this Subsection are adapted with no effort to a Riemannian ambient (M, g).

Let $A \subseteq M$ be an open set. Then, we let $P(E, A)$ denote the De Giorgi's *relative perimeter* of Lebesgue-measurable set E in A. It is defined as

$$P(E, A) = \sup \left\{ \int_E \operatorname{div} T \, d\mu \;\middle|\; T \in \Gamma_c(A),\ \sup_A |T| \leq 1 \right\},$$

where by $\Gamma_c(A)$ we denote the class of vector fields with compact support in A.

When $A = M$, we just talk about the perimeter of E and we denote it by $P(E)$. Given a set E with finite perimeter it is well known [Mag12, Proposition 12.1] the existence of a vector-valued Radon measure μ_E satisfying (and actually defined by)

$$\int_E \operatorname{div} T = \int_M \langle T | \, d\mu_E \rangle$$

for any $T \in \Gamma_c(M)$. Denoting by $|\mu_E|$ the total variation measure of μ_E, one actually has $P(E, A) = |\mu_E|(A)$ for any open set $A \subseteq \mathbb{R}^n$. With the notion of *perimeter measure* μ_E at hand, De Giorgi's *reduced boundary* $\partial^* E$ of a set with finite perimeter E can be defined as

$$\partial^* E = \left\{ x \in \operatorname{supp} \mu_E \;\middle|\; \lim_{r \to 0^+} \frac{\mu_E(B(x,r))}{|\mu_E|(B(x,r))} \text{ exists and belong to } \mathbb{S}^{n-1} \right\}.$$

In fact, one has (see [Mag12, Remark 15.3]) $\operatorname{supp} \mu_E = \overline{\partial^* E}$.

We define now the *measure theoretic interior* of a measurable set E as the points of density one for E, namely

$$\operatorname{Int}(E) = \left\{ x \in M \;\middle|\; \lim_{r \to 0^+} \frac{|E \cap B(x,r)|}{|B(x,r)|} = 1 \right\}. \tag{3.2.1}$$

It follows from Lebesgue Differentiation Theorem (see [Mag12, Theorem 5.16] that

$$|E\Delta\,\mathrm{Int}(E)| = 0. \tag{3.2.2}$$

Importantly, a set with finite perimeter E satisfies,

$$\partial\,\mathrm{Int}(E) = \overline{\partial^* E}, \tag{3.2.3}$$

that is, the topological boundary of the measure theoretic interior of a set with finite perimeter coincides with the closure of its reduced boundary. We address the reader to [Car11, Theorem 10] for a proof of this nice property.

3.2.2 Basic properties of outward minimising sets

We recall, according to [HI01], the notion of *outward* and *strictly outward minimising sets*.

Definition 3.2 (Outward minimising and strictly outward minimising sets)**.** *Let (M, g) be a complete Riemannian manifold. Let $E \subset M$ be a bounded set with finite perimeter. We say that E is* outward minimising *if for any $F \subset M$ with $E \subseteq F$ we have $P(E) \leq P(F)$. We say that E is* strictly outward minimising *if it is outward minimising and any time $P(E) = P(F)$ for some $F \subset M$ with $E \subseteq F$ we have $|F \setminus E| = 0$.*

We observe at once that the notions of outward minimising and strictly outward minimising are stable under zero-measure modification. We give a proof of this basic yet fundamental fact.

Lemma 3.3. *Let (M, g) be a complete Riemannian manifold. Let $N \subset M$ be a null set, that is $|N| = 0$. Then, a bounded set $E \subset M$ with finite perimeter is (strictly) outward minimising if and only if $E \cup N$ is (strictly) outward minimising.*

Proof. We first show that if E is (strictly) outward minimising then $E \cup N$ is strictly outward minimising. Let F be a set with finite perimeter such that $E \cup N \subseteq F$. In particular, $E \subseteq F$. Then one has, by the invariance of the perimeter under measure zero modifications and since E is outward minimising

$$P(E \cup N) = P(E) \leq P(F). \tag{3.2.4}$$

that is, $E \cup N$ is outward minimising. Assume now that E is strictly outward minimising, and assume $P(F) = P(E \cup N)$. Then equalities hold in (3.2.4), and thus $|F \setminus E| = 0$, that obviously implies $|F \setminus (E \cup N)| = 0$, that is, $E \cup N$ is strictly outward minimising.

Assume now that $E \cup N$ is outward minimising, and let F be a set with finite perimeter containing E. Then, since $E \cup N$ is contained in $F \cup N$ we have

$$P(E) = P(E \cup N) \leq P(F \cup N) = P(F), \tag{3.2.5}$$

that is, E is outward minimising. Assume now that $E \cup N$ is strictly outward minimising, and assume that $P(F) = P(E)$. Then, (3.2.5) and $E \cup N$ being strictly outward minimising imply that $|(F \cup N) \setminus (E \cup N)| = 0$, that immediately implies $|F \setminus E| = 0$. □

It is well known that locally area minimising sets satisfy upper and lower density estimates. It is then not surprising to realise that outward minimising sets, that can be thought as one sided minimisers, satisfy upper density estimates. We include a proof, inspired by that of [Mag12, Theorem 16.14].

Lemma 3.4. *Let (M, g) be a complete Riemannian manifold, and let $E \subset M$ be an outward minimising set. Then, there exists $r_0 > 0$ such that for any $0 < r < r_0$ and $x \in \overline{\partial^* E}$ there holds*

$$\frac{|E \cap B(x,r)|}{|B(x,r)|} \leq C$$

for some constant $0 < C < 1$ independent of x

Proof. Consider $E^c = M \setminus E$. Then, for any $F \subset M$ such that $E \subseteq F$ it is immediately seen from the outward minimising property of E that

$$P(E^c) = P(E) \leq P(F) \leq P(F^c). \tag{3.2.6}$$

Let $x \in \overline{\partial^* E}$ and $r > 0$, and choose $F = (E^c \setminus B(x,r))^c \supset E$. Then, for $s > r$, we have, applying (3.2.6)

$$P(E^c, B(x,s)) + P(E^c, M \setminus B(x,s)) = P(E^c) \leq P(F^c) = P(F^c, B(x,s)) + P(F^c, M \setminus B(x,s)). \tag{3.2.7}$$

Since $E^c \setminus F^c \Subset B(x,s)$, we actually have $P(F^c, \mathbb{R}^n \setminus B(x,s)) = P(E^c, \mathbb{R}^n \setminus B(x,s))$, so we deduce from (3.2.7), the definition of F and basic set operations for the perimeter (see e.g. [Mag12, Theorem 16.3])

$$\begin{aligned} P(E^c, B(x,s)) \leq P(F^c, B(x,s)) &= P(E^c \setminus B(x,r), B(x,s)) \\ &= |\mathrm{Int}(E^c) \cap B(x,r)| + P(E^c, B(x,s) \setminus \overline{B(x,r)}), \end{aligned}$$

that, upon letting $s \to r^+$, yields

$$P(E^c, B(x,r)) \leq |\mathrm{Int}(E^c) \cap B(x,r)|. \tag{3.2.8}$$

Recalling again from basic set operations for the perimeter that

$$P(E^c, B(x,r)) + |\mathrm{Int}(E^c) \cap B(x,r)| = P(E^c \cap B(x,r)),$$

we deduce from (3.2.8) the estimate

$$P(E^c \cap B(x,r)) \leq 2|\mathrm{Int}(E^c) \cap B(x,r)|.$$

For r small enough, but uniform as x ranges in the compact $\overline{\partial^* E}$, we can apply an Euclidean-like type inequality to left hand side of the above inequality, leading to

$$C_1(n, r_0) \leq |E^c \cap B(x,r)|^{(n-1)/n} \leq 2|\mathrm{Int}(E^c) \cap B(x,r)| \tag{3.2.9}$$

for any $r \in (0, r_0)$. The constant $C_1(n, r_0)$ can be seen to approach $n|\mathbb{B}^n|$ as $r_0 \to 0^+$, see e.g. [BM82, Appendice C]. Define, for $0 < r < r_0$ the function $m(r) = |E^c \cap B(x,r)|$. Applying the coarea formula (see [Mag12, Theorem 13.1 and Remark 13.4]), one gets that m is an absolutely continuous function satisfying $m'(r) = |\mathrm{Int}(E^c) \cap B(x,r)|$ for almost any $r \in (0, r_0)$, and thus (3.2.9) gives

$$C_1(n, r_0) m(r)^{(n-1)/n} \leq 2m'(r) \tag{3.2.10}$$

for almost any $r \in (0, r_0)$. Moreover, since $x \in \overline{\partial^* E} = \operatorname{supp} \mu_E$, we have, by [Mag12, Proposition 12.19], that $m > 0$ and tends to 0 as $r \to 0^+$. Integrating (3.2.10), and taking into account that for small r we can approximate $|B(x,r)|$ with $|\mathbb{B}^n|r^n$, we then easily get

$$\frac{|E^c \cap B(x,r)|}{|B(x,r)|} \geq C_2(n, r_0),$$

for some constant $0 < C_2(n, r_0) < 1$ that approaches 2^{-n} as $r_0 \to 0^+$. This immediately gives

$$\frac{|E \cap B(x,r)|}{|B(x,r)|} \leq C(n, r_0),$$

with $0 < C(n, r_0) < 1$ for any $0 < r < r_0$, as claimed. □

In [Car11, Theorem 6, (i)], the author showed that if a set with finite perimeter satisfies relaxed density estimates then its measure theoretic interior and exterior are open. We report (a part of) his argument, in order to clarify that an upper density estimate suffices to get the openness of the measure theoretic interior.

Lemma 3.5. *Let (M, g) be a complete Riemannian manifold. Let $E \subset \mathbb{R}^n$ be a set with finite perimeter, and suppose that for any $x \in \overline{\partial^* E}$*

$$\lim_{r \to 0^+} \frac{|E \cap B(x,r)|}{|B(x,r)|} < \delta < 1 \tag{3.2.11}$$

uniformly on $\overline{\partial^* E}$. Then $\mathrm{Int}\,(E)$ is open.

Proof. Let $y \in \mathrm{Int}(E)$. First observe that $y \notin \overline{\partial^* E}$, since otherwise (3.2.11) would contradict the condition (3.2.1) defining $\mathrm{Int}(E)$. We now construct a ball centered at y fully contained in $\mathrm{Int}(E)$. Let $d = \mathrm{dist}(y, \overline{\partial^* E})$. By the definition (3.2.1) of $\mathrm{Int}(E)$,

$$\frac{|E \cap B(y,r)|}{|B(y,r)|} > 0$$

for some $r' \in (0, d)$. Then we can deduce

$$\frac{|(M \setminus E \cap B(y,r')|}{|B(y,r')|} = 0, \tag{3.2.12}$$

since, otherwise, the relative isoperimetric inequality [Mag12, Proposition 12.37] would yield $|\partial^* E \cap B(y,r')| > 0$, in turn resulting in the contradiction $\mathrm{dist}(y, \overline{\partial^* E}) < d$. Observe that such inequality clearly holds for small balls also in Riemannian manifolds, and it can be shown just by performing the computations in a local chart. The above argument in particular works up to eventually choose a suitable, uniform d' smaller than d. By (3.2.12), for any $y \in B(y,r')$, we have

$$\frac{|E \cap B(z,r'')|}{|B(z,r'')|} = 1$$

for any $r'' \in (0, r' - \mathrm{dist}(z,y))$. This clearly implies that $B(y,r') \subset \mathrm{Int}(E)$. □

As a direct consequence of (3.2.2), Lemma 3.4 and Lemma 3.5, we obtain the remarkable property of outward minimising sets to have an open representative.

Proposition 3.6. *Let (M, g) be a complete Riemannian manifold. Let $E \subset M$ be outward minimising. Then $\mathrm{Int}(E)$ is open.*

3.2.3 Maximal volume solutions to the least area problem with obstacle in Riemannian manifolds

Let (M, g) be a complete noncompact Riemannian manifold of dimension $n \geq 2$. We are interested in the existence of a *bounded* open set with finite perimeter E solving the least area problem with obstacle Ω, where $\Omega \subset M$ is a bounded open set with finite perimeter. We say that E solves the *least area problem with obstacle* Ω if it is a bounded set containing Ω satisfying

$$P(E) = \inf\{P(F) \quad | \quad \Omega \subset F, F \,\text{bounded set with finite perimeter}\} \tag{3.2.13}$$

A solution to (3.2.13) could could clearly be non-unique. We can try to select among these solutions a set E'

$$|E'| = \sup\{|E| \quad | \quad E \text{ solves the least area problem with obstacle } \Omega\}. \tag{3.2.14}$$

We refer to a *bounded* set E' with finite perimeter containing Ω, that solves *both* (3.2.13) and (3.2.14) as *maximal volume solutions of the least area problem with obstacle* Ω

Remark 3.7. Observe that bounded solutions to the least area problem with obstacle Ω are outward minimising, as well maximal volume solutions are strictly outward minimising. In particular, outward minimising sets E are characterised by being solutions to the least area problem with obstacle E itself. Interestingly, it is immediately seen that E is strictly outward minimising if and only if it is also a maximal volume solution to the least area problem with obstacle E.

In the Euclidean case, the existence of a maximal volume solution to the least area problem with obstacle Ω is an easy consequence of the Direct Method (compare with [BT84] and [SWZ93]). In a general noncompact Riemannian manifold, minimising sequences could fatally be unbounded, as the following examples show.

Example 3.8 (Cuspidal manifolds). Let (M, g) be a complete noncompact Riemannian manifold whose metric splits as $g = \mathrm{d}\rho \otimes \mathrm{d}\rho + e^{-2\rho} g_{\mathbb{S}^{n-1}}$ on $M \setminus B = [1, +\infty) \times \mathbb{S}^{n-1}$ for some precompact set $B \subset M$. Then, the area of $\partial\{\rho < r\}$ decays as e^{-r}, and in particular enveloping a bounded set Ω with $\{\rho < r\}$, for bigger and bigger r, we see that

$$P(E) = \inf\{P(F) \quad | \quad \Omega \subset F, F \,\text{bounded set with finite perimeter}\} = 0.$$

In particular, there cannot exist a solution to the least area problem with obstacle Ω. Observe also that (M, g) has finite volume, and thus it does not support a L^1-Euclidean-like Sobolev inequality by [PST14, Proposition 3.1] nor the Ricci curvature is nonnegative, and thus coherently the assumptions of Theorem 3.1 are not satisfied.

Example 3.9 (Manifolds with a cylindrical end). Let (M, g) be a complete noncompact Riemannian manifold whose metric splits as $g = d\rho \otimes d\rho + g_{\mathbb{S}^{n-1}}$ on $M \setminus B = [1, +\infty) \times \mathbb{S}^{n-1}$ for some precompact set $B \subset M$. Let Ω be any bounded set with finite perimeter containing B. Then, any set of the form $B \cup \{\rho < r\}$ containing Ω solves the least area problem with obstacle Ω, since the sets $\{\rho = r\}$ are even area minimising, but since $|\{\rho = r_2\}| = |\{\rho = r_1\}|$ and $|B \cup \{\rho < r_2\}| = |B \cup \{\rho < r_1\}| + (r_2 - r_1)$ for any $r_2 \geq r_1 \geq 1$, we see at once that

$$\sup\{|E| \quad | \quad E \text{ solves the least area problem with obstacle } \Omega\} = +\infty.$$

In particular, it cannot exist a *maximal* volume solution to the least area problem with obstacle Ω. Observe that (M, g) has linear volume growth, and therefore by [PST14, Proposition 3.1] the L^1-Euclidean-like Sobolev inequality cannot hold nor (3.1.2) holds.

In the following basic result, we isolate as a necessary and sufficient condition for obtaining a maximal volume solution to the least area problem with obstacle in a complete noncompact Riemannian manifold the existence of an exhausting sequence of bounded strictly outward minimising sets. Observe that, in $\mathbb{R}^n$, such a sequence is trivially given by balls.

Theorem 3.10 (Existence of maximal volume solutions to the least area problem in Riemannian manifolds-abstract criterion)**.** *Let (M, g) be a noncompact Riemannian manifold. Let $O \in M$. Then, for any $R > 0$ there exists a (strictly) outward minimising set S_R with $B(O, R) \Subset S_R$ if and only if there exists a maximal volume solution to the least area problem with obstacle Ω for any bounded $\Omega \subset M$ with finite perimeter.*

Proof. One direction is almost trivial. Indeed if for any bounded $\Omega \subset M$ with finite perimeter there exists a maximal volume solution to the least area problem with obstacle Ω, then, taking a sequence of geodesic balls $B(O, R_j)$ with $R_j \to \infty$ and considering the bounded maximal volume solution to the least area problem with obstacle $B(O, R_j)$ yields the desired sequence of strictly outward minimising sets (compare with Remark 3.7). Observe that $B(O, R_j)$ can be assumed to be of finite perimeter substantially by coarea formula, and that maximal volume solutions are obviously strictly outward minimising.
Let $\Omega \subset M$ be a bounded set with finite perimeter. Let $\{F_j\}_{j \in \mathbb{N}}$ be a minimising sequence for the least area problem with obstacle Ω. Define

$$m = \inf\{P(F) \quad | \quad \Omega \subset F, F \,\text{bounded set with finite perimeter}\}.$$

Then, for any $\varepsilon > 0$, there exists $j_\varepsilon \in \mathbb{N}$ such that

$$m \leq P(F_j) \leq m + \varepsilon$$

for any $j \geq j_\varepsilon$. Let S be an outward minimising set containing Ω, that exists by assumption. By a standard set-theoretical property of the perimeter (see e.g. [Mag12, Lemma 12.22]), we have

$$P(F_j \cap S) \leq P(F_j) + P(S) - P(F_j \cup S).$$

Since S is outward minimising, we have $P(S) \leq P(F_j)$, and then we deduce that

$$P(F_j \cap S) \leq P(F_j). \tag{3.2.15}$$

In particular, since $\Omega \subset (F_j \cap S)$ we have

$$m \leq P(F_j \cap S) \leq m + \varepsilon$$

for any $j \geq j_\varepsilon$, that is, the sets $F_j \cap S$ form an equibounded minimising sequence. Then, the Compactness Theorem [Mag12, Theorem 12.26] yields a bounded set of finite perimeter F such that $\chi_{F_j \cap S} \to \chi_F$ in L^1, possibly along a subsequence. Moreover, by the lower semicontinuity of the perimeter [Mag12, Proposition 12.15], the set F is a minimiser for the minimisation problem (3.2.13). We are left to show that we can modify F in order to obtain a set E containing Ω with $P(E) = P(F)$. Actually, it suffices to define $E = F \cup (\Omega \setminus F)$. Clearly $\Omega \subseteq E$. Moreover, since $\Omega \subset F_j \cap S$ for any j, and $\chi_{F_j \cap S} \to \chi_F$ in L^1, we have $|\Omega \setminus F| = 0$. Since De Giorgi's perimeter is defined up to null sets, we have $P(E) = P(F)$.

Let now $\{E_j\}_{j \in \mathbb{N}}$ be a sequence of bounded sets with finite perimeter containing Ω with

$$|E_j| \to \sup\{|E| \quad | \quad E \text{ solves the least area problem with obstacle } \Omega\} \tag{3.2.16}$$

as $j \to +\infty$. In particular, $P(E_j) = m$ for any j. Let S be a bounded *strictly* outward minimising set containing Ω. Then, as in (3.2.15), we have $P(E_j \cap S) \leq P(E_j)$. We thus have

$$m \leq P(E_j \cap S) \leq P(E_j) = m.$$

Since S is strictly outward minimising, we deduce that $|E_j \setminus (E_j \cap S)| = 0$ for any j. This implies $|E_j| = |E_j \cap S|$ for any j, and then we can consider $E_j \cap S$ in place of E_j in (3.2.16). The Compactness Theorem for sets with finite perimeter then yields a bounded set E' realising the supremum in (3.2.16). Up to a zero-measure modification as before, we can also suppose $\Omega \subseteq E'$. By the lower semicontinuity of the perimeter we also have

$$m = \liminf P(E_j \cap S) \geq P(E') \geq m.$$

We showed that $P(E') = m$ and its volume realises the supremum in (3.2.16), that is, E' is a maximal volume solution to the least area problem with obstacle Ω. □

3.2.4 The strictly outward minimising hull of a set with finite perimeter

Once we know that maximal volume solutions to the least area problem exist, we are interested in a specific representative, that will naturally arise as level set of the Weak Inverse Mean Curvature Flow. We thus define the *strictly outward minimising hull* of a set with finite perimeter.

Definition 3.11 (Strictly outward minimising hull)**.** *Let $\Omega \subset M$ be a bounded open set with finite perimeter. We define the* strictly outward minimising hull *of Ω as*

$$\Omega^* = \mathrm{Int}\left(\bigcap_{E \in \mathrm{SOM}\,(\Omega)} \mathrm{Int}\,(E) \right), \tag{3.2.17}$$

where

$$\mathrm{SOM}\,(\Omega) = \{E \subset M \mid \Omega \subseteq E \text{ and } E \text{ is strictly outward minimising}\}.$$

The following is the main result of this section.

Theorem 3.12. *Let (M, g) be a complete noncompact Riemannian manifold admitting an exhausting sequence of bounded strictly outward minimising sets. Let $\Omega \subset M$ be a bounded set with finite perimeter. Then, Ω^* is an open, bounded, maximal volume solution to the least area problem with obstacle Ω. In particular Ω^* is strictly outward minimising.*

Proof. We split the proof in several steps.

Step 1. We prove that that finite intersections of strictly outward minimising sets are strictly outward minimising. It clearly suffices to prove that the intersection of two strictly outward minimising sets E_1 and E_2 is outward minimising. We do it by first showing that it is outward minimising. This preliminary passage is carried out in the proof of [BT84, Proposition 1.3], but we include the argument, since it will be important also in the sequel. Let $F \subset M$ such that $E_1 \cap E_2 \subset F$, and define $L = F \setminus (E_1 \cap E_2)$. Then, applying the already recalled well known property holding for sets with finite perimeter

$$P(F \cap G) + P(F \cup G) \leq P(F) + P(G)$$

to the couples of sets $F = (E_1 \cap E_2) \cup L$ and $G = E_1$, $F = (E_1 \cap E_2) \cup (L \cap E_1)$ and $G = E_2$, give respectively

$$P\Big((E_1 \cap E_2) \cup (L \cap E_1)\Big) + P(E_1 \cup L) \leq P\Big((E_1 \cap E_2) \cup L\Big) + P(E_1) \tag{3.2.18}$$

and

$$P(E_1 \cap E_2) + P\Big((E_1 \cap L) \cup E_2\Big) \leq P\Big((E_1 \cap E_2) \cup (L \cap E_1)\Big) + P(E_2). \tag{3.2.19}$$

Combining (3.2.19) with (3.2.18) we obtain at once the following chain of inequalities

$$\begin{aligned} P(E_1 \cap E_2) &\leq P\Big((E_1 \cap E_2) \cup (L \cap E_1)\Big) + P(E_2) - P\Big((E_1 \cap L) \cup E_2\Big) \\ &\leq P\Big((E_1 \cap E_2) \cup (L \cap E_1)\Big) \\ &\leq P\Big((E_1 \cap E_2) \cup L\Big) + P(E_1) - P(E_1 \cup L) \\ &\leq P\Big((E_1 \cap E_2) \cup L\Big) = P(F) \end{aligned} \tag{3.2.20}$$

where the second and fourth inequality are due to the property of E_1 and E_2 to be outward minimising. Assume now that that $P(E_1 \cap E_2) = P(F)$ for some $F \supseteq E_1 \cap E_2$. Then, the second and the last inequality in (3.2.20) become equalities, yielding respectively

$$P(E_2) = P\Big((E_1 \cap L) \cup E_2\Big) \tag{3.2.21}$$

$$P(E_1) = P(E_1 \cup L). \tag{3.2.22}$$

Since E_1 is strictly outward minimising, we can deduce from (3.2.22) that

$$|(E_1 \cup L) \setminus E_1| = |L \setminus E_1| = 0. \tag{3.2.23}$$

On the other hand, by (3.2.21) and by E_2 being strictly outward minimising we obtain

$$\left|\Big(E_1 \cap L) \cup E_2\Big) \setminus E_2\right| = 0. \tag{3.2.24}$$

By the definition of L it is easily seen that the sets $(E_1 \cap L) \cup E_2$ and E_2 are disjoint, and then (3.2.24) actually reads

$$|E_1 \cap L| = 0. \tag{3.2.25}$$

Combining (3.2.25) with (3.2.23) we obtain

$$|F \setminus (E_1 \cap E_2)| = |L| = |L \setminus E_1| + |L \cap E_1| = 0,$$

that is, $E_1 \cap E_2$ is strictly outward minimising, as claimed.

Step 2. We prove that Ω^* is strictly outward minimising. Let $\{E_i\}_{i \in \mathbb{N}}$ be a sequence of set realising Ω^* as a countable intersection, that is

$$\Omega^* = \mathrm{Int}\left(\bigcap_{i=1}^{\infty} \mathrm{Int}\,(E_i)\right)$$

By (3.2.2) and Lemma 3.3, it suffices to show that $\widetilde{\Omega} = (\bigcap_{i=1}^{\infty} \mathrm{Int}\,(E_i)$ is strictly outward minimising. First, [BT84, Proposition 1.3] ensures that Ω^*, is outward minimising. Assume then that $P(\widetilde{\Omega}) = P(F)$ for some $F \subset M$ containing $\widetilde{\Omega}$. Clearly, F is outward minimising. Let F' be a maximal volume solution to the least area problem with obstacle F, that exists by Theorem 3.10. Then F' is strictly outward minimising (compare with Remark 3.7) and $P(F') = P(F)$, being F outward minimising. For ease of notation, assume $E_i = \mathrm{Int}(E_i)$ for any $i \in \mathbb{N}$. By (3.2.2) and Lemma 3.3, this will result in no loss of generality. Let $A_j = \bigcap_{k=1}^{j} E_k$. Since F' is strictly outward minimising, by Step 1 $A_j \cap F'$

is strictly outward minimising. In particular, since $A_{j+1} \cap F' \subset A_j \cap F' \subset F'$, we have $P(A_{j+1} \cap F') \leq P(A_j \cap F') \leq P(F')$. Then, by the lower semicontinuity of the perimeter, the definition of $\widetilde{\Omega}$ and the just observed monotonicity of $P(A_j \cap F)$ as j increases we get

$$P(F') = P(\widetilde{\Omega}) \leq \liminf_{j \to \infty} P(A_j \cap F') \leq P(A_k \cap F') \leq P(F')$$

for any $k \in \mathbb{N}$, where the last inequality is again due to the fact that A_k is outward minimising. In particular $P(A_k \cap F') = P(F')$ for any k. Since $A_k \cap F'$ is in fact strictly outward minimising, we deduce that $|F' \setminus (A_k \cap F')| = |F' \setminus A_k| = 0$ for any k. Since we have $|A_k \setminus \widetilde{\Omega}| \to 0$ as $k \to \infty$, we conclude that $|F' \setminus \Omega^*| = \lim_{k \to \infty} |F' \setminus A_k| = 0$. Since $F \subseteq F'$ this trivially implies $|F \setminus \widetilde{\Omega}| = 0$. We have shown that $\widetilde{\Omega}$, and consequently Ω^*, is strictly outward minimising, as claimed.

Step 3. We show that Ω^* is a maximal volume solution to the least area problem with obstacle Ω. Let E' be any solution to (3.2.14). We want to show that $P(\Omega^*) = P(E)$ and $|\Omega^*| = |E'|$. Since Ω^* is an admissible competitor for (3.2.13), we have

$$P(E') \leq P(\Omega^*). \tag{3.2.26}$$

Clearly E' is strictly outward minimising. Letting $\widetilde{\Omega}$ be defined as in Step 2, we thus get by definition $\widetilde{\Omega} \subseteq \mathrm{Int}(E')$, and thus, since $\widetilde{\Omega}$ is outward minimising and differs from Ω^* by a set of measure zero, we get also

$$P(\Omega^*) = P(\widetilde{\Omega}) \leq P(E'),$$

that, combined with (3.2.26), implies $P(\Omega^*) = P(E')$. The set Ω^* becomes in particular a valid competitor in (3.2.14), giving immediately

$$|\widetilde{\Omega}| = |\Omega^*| \leq |E'| = |\mathrm{Int}(E')|. \tag{3.2.27}$$

However, since $P(\widetilde{\Omega}) = P(\mathrm{Int}(E'))$, and $\widetilde{\Omega} \subseteq \mathrm{Int}(E')$, the strict inequality sign in (3.2.27) would contradict the fact that Ω^* is strictly outward minimising. It follows that $|\Omega^*| = |E'|$ proving the claim.

Step 4. We are left to observe that Ω^* is open. Since Ω^* is outward minimising by Step 2, and by definition $\mathrm{Int}(\Omega^*) = \Omega^*$, this follows at once from Proposition 3.6. □

Let us summarise Theorem 3.10 and Theorem 3.12 in the following general statement.

Theorem 3.13. *Let (M, g) be a complete noncompact Riemannian manifold. Then, the following are equivalent.*

(i) (M, g) admits an exhausting sequence of bounded strictly outward minimising sets.

(ii) For any bounded $\Omega \subset M$ with finite perimeter, there exists a bounded maximal volume solution to the least area problem with obstacle Ω.

(iii) Let $\Omega \subset M$ be a bounded set with finite perimeter. Then, Ω^* is an open, bounded, maximal volume solution to the least area problem with obstacle Ω. In particular Ω^* is strictly outward minimising.

Regularity and approximations of the strictly outward minimising hull

We recall a fundamental regularity result for solutions of the least area problem with obstacle, under an additional $\mathscr{C}^2$ assumption on the regularity on $\partial\Omega$. This is the main result of [SZW91].According to the comprehensive [HI01, Theorem 1.3], it applies also to general Riemannian ambient settings. By Theorem 3.13, we can apply it directly to Ω^*.

Theorem 3.14 (Regularity of the strictly outward minimising hull)**.** *Let (M, g) be a complete noncompact Riemannian manifold admitting an exhausting sequence of bounded strictly outward minimising sets. Let $\Omega \subset M$ be a bounded set with $\mathscr{C}^2$ boundary. Then*

(i) $\partial\Omega^*$ is a $\mathscr{C}^{1,1}$ hypersurface in a neighbourhood of any point in $\partial\Omega^* \cap \partial\Omega$.

(ii) $\partial\Omega^*$ is area minimising in $\partial\Omega^* \setminus \partial\Omega$. In particular there exists a singular set $\mathrm{Sing} \subset \partial\Omega^* \setminus \partial\Omega$ with Hausdorff dimension at most $n-8$ such that $\partial\Omega^* \setminus \partial\Omega$ is a real analytic hypersurface in a neighbourhood of any point in $(\partial\Omega^* \setminus \partial\Omega) \setminus \mathrm{Sing}$.

Remark 3.15. In [SZW91], the authors in fact consider a representative E satisfying $\partial E = \overline{\partial^* E}$. However, since, by our definition $\mathrm{Int}(\Omega^*) = \Omega^*$, this condition is automatically satisfied due to (3.2.3).

From now on we are now going to deal just with set with sets with $\mathscr{C}^2$ boundary. In fact, for simplicity, we will always assume $\partial\Omega$ to be a smooth hypersurface.

Let us record, for future references, two direct and easy consequences of the above regularity result.

Remark 3.16. Theorem 3.14 implies $|\partial^*\Omega^*| = |\partial\Omega^*|$. In particular, taking into account Remark 3.7, we can characterise outward minimising sets as those satisfying

$$|\partial\Omega| = |\partial\Omega^*|. \tag{3.2.28}$$

Observe that, by Theorem 3.13, we always have $|\partial\Omega^*| \leq |\partial\Omega|$. Checking condition (3.2.28) then amounts to check that $|\partial\Omega| \leq |\partial\Omega^*|$.

Moreover, let us also point out that, by a very standard variational argument, the weak mean curvature (see (1.7.6)) of $\partial\Omega^*$ is nonnegative.

We finally state the following nice *one sided approximation* result due to Schmidt [Sch15], that we are going to use in the p-capacitary approximation of $|\partial\Omega^*|$. It asserts that a bounded set with finite perimeter admit a one sided approximation in perimeter by bounded sets with smooth boundary if $P(\Omega) = |\partial\Omega|$. This is clearly the case for Ω^*, due to Theorem 3.14. The arguments being purely local, Schmidt's result applies straightforwardly in Riemannian manifolds. Observe also that although in [Sch15] just interior approximation is worked out, an analogous exterior approximation obviously follows as well.

Theorem 3.17 (Exterior approximation of Ω^*)**.** *Let (M, g) be a complete noncompact Riemannian manifold admitting an exhausting sequence of bounded strictly outward minimising sets. Let $\Omega \subset M$ be an open bounded set with smooth boundary, and Ω^* is strictly outward minimising hull. Then, there exists a sequence of bounded sets $\{\Omega_k\}_{k\in\mathbb{N}}$ with smooth boundary such that*

$$\Omega^* \subset \Omega_k, \quad |\partial\Omega_k| \to |\partial\Omega^*|.$$

3.3 Families of manifolds admitting the strictly outward minimising hull

Here, we show that in the assumptions of Theorem 3.1 the notion of strictly outward minimising hull recalled in Definition 3.11 gives a well posed open bounded maximal volume solution to the least area problem with obstacle.

3.3.1 Euclidean-like isoperimetric inequality and strictly outward minimising hull

The following result affirms that a maximal volume solution to the least area problem with obstacle exists on any Riemannian manifold where an Euclidean-like Isoperimetric Inequality is available, that is, satisfying (3.1.1) for any bounded $\Omega \subset M$ with smooth boundary. In particular, by Theorem 3.10 any bounded $\Omega \subset M$ with finite perimeter admits a strictly outward minimising hull. We express our gratitude to Prof. G.P. Leonardi for having outlined the core of the following argument.

Proposition 3.18. *Let (M, g) be a complete noncompact Riemannian manifold satisfying admitting a constant $\mathrm{C}_{\mathrm{iso}} > 0$ such that*

$$\frac{|\partial\Omega|^{n/(n-1)}}{|\Omega|} \geq \mathrm{C}_{\mathrm{iso}} \tag{3.3.1}$$

for any bounded set Ω with smooth boundary. Then, for any bounded $\Omega \subset M$ with finite perimeter there exists a maximal volume solution to the least area problem with obstacle Ω.

Proof. The main goal is showing that the Isoperimetric Inequality (3.3.1) forces a minimising sequence for the least area problem to stay uniformly inside a ball, and then to conclude by compactness and lower semicontinuity as in the proof of Theorem 3.10. Let then F_j be a minimising sequence for the least area problem with obstacle made of bounded sets with finite perimeter. Clearly, we can suppose $P(F_j) \leq P(\Omega)$, and thus the well known *local* compactness-lower semicontinuity properties of the perimeter (namely, the combination of [Mag12, Corollary 12.17] with [Mag12, Proposition 12.15]) yield a set F such that a subsequence (relabeled as usual) of the F_j's locally converges in L^1 to F and

$$P(F) \leq \liminf_{j \to +\infty} P(F_j),$$

so that F realises the infimum in the least area problem with obstacle Ω. We claim that, up to null sets, $F \subset B(O, r)$ for some $r > 0$. Assume then by contradiction $|F \setminus B(O, r_j)| > 0$ for some sequence $r_j \to +\infty$ as $r_j \to +\infty$. In particular, this implies that $|F \setminus B(O, r)| > 0$ for any r big enough. The assumed Isoperimetric Inequality (clearly in force, by approximation, for any set with finite perimeter) yields

$$\mathrm{C}_{\mathrm{iso}}^{\frac{n-1}{n}} |F \setminus B(O, R)|^{\frac{n-1}{n}} \leq P(F \setminus B(O, r)) = P(F, \overline{B(O, r)}^{c}) + |\partial B(O, r) \cap F| \tag{3.3.2}$$

for any r such that $\partial B(O, r)$ has a $(n-1)$-negligible singular set. In particular, (3.3.2) holds for almost any $r > 0$. In the above inequality, as in the following arguments, we are possibly considering a representative for F. Observe that by coarea formula the function $m(r) = |F \setminus B(O, r)|$ is absolutely continuous, with derivative $m'(r) = -|\partial B(O, r) \cap F|$ for almost any $r > 0$. We claim that

$$P(F, \overline{B(O, r)}^{c}) \leq |\partial B(O, r) \cap F|, \tag{3.3.3}$$

in order to deduce from (3.3.2) a differential inequality leading in turn to a contradiction. This is accomplished through an argument similar to the classic one used in the proof of Lemma 3.4, relying on the minimising property of F. Let $L = F \setminus \overline{B(O, r)}^{c}$, and observe that for R big enough we have $\Omega \subset L$. By the minimising property of F, we thus get, for $s < r$

$$P(F, \overline{B(O, s)}^{c}) + P(F, B(O, s)) = P(F) \leq P(L) = P(L, \overline{B(O, s)}^{c}) + P(L, B(O, s)).$$

Since clearly $P(L, B(O,s)) = P(F, B(O,s))$, we deduce

$$P(F, \overline{B(O,s)}^c) \leq P(L, \overline{B(O,s)}^c). \tag{3.3.4}$$

On the other hand, observe that, as usual up to representatives

$$P(L, \overline{B(O,s)}^c) = P(F, B(O,r) \setminus \overline{B(O,s)}) + |\partial B(O,R) \cap F|.$$

Letting $s \to r^-$, the first term in the right hand side above vanishes and thus plugging this information into (3.3.4) leaves us with the claimed (3.3.3). Inserting (3.3.3) into (3.3.2) yields, as explained above, the differential inequality

$$C_{\mathrm{iso}}^{\frac{n-1}{n}} m(r)^{\frac{n-1}{n}} \leq -2m'(r),$$

holding true for almost any r big enough, with $m(r) = |E \setminus B(O,r)|$. Integrating it, we get

$$\frac{1}{2} C_{\mathrm{iso}}^{(n-1)/n}(r_2 - r_1) \leq n \left[m(r_1)^{\frac{1}{n}} - m(r_2)^{\frac{1}{n}} \right]$$

for any $r_2 > r_1$ big enough. Letting $r_2 \to +\infty$ the right hand side converges to $nm(r_1)^{1/n}$, since the other summand vanishes by the property of E to have finite volume, while the left hand side cleary diverges. This contradiction arose from $|E \setminus B(O, r_j)| > 0$ for some diverging sequence of r_j's, and thus we proved that there exists a ball $B(O,R)$ containing, up to a null set, the solution F to the least area problem with obstacle Ω.

It remains to show that that we can find a bounded maximal volume solution to the least area problem. To see this, let E_j be a maximising (for the volume) sequence of bounded solutions to the least area problem with obstacle Ω. Then, by the compactness property used in the first part of this proof, such a sequence converges locally in L^1 to a set with finite perimeter E, that by lower semicontinuity is still a possibly unbounded solution to the least area problem with obstacle Ω. The same argument as above involving the Isoperimetric Inequality then shows that, up a to null sets, E is contained in some ball, completing the proof. □

For future reference, let us state the following immediate Corollary of Proposition 3.18 and Theorem 3.13.

Corollary 3.19. *Let (M, g) be a complete noncompact Riemannian manifold satisfying the assumption* (i) *in the statement of Theorem 3.1. Then, for any bounded $\Omega \subset M$ with finite perimeter, the strictly outward minimising hull Ω^* is an open bounded maximal volume solution to the least area problem with obstacle Ω.*

3.3.2 Weak Inverse Mean Curvature Flow and strictly outward minimising hulls

In this Subsection we show that the existence of a *proper* solution to the Weak Inverse Mean Curvature Flow naturally yields an exhausting sequence of strictly outward minimising sets, and that the strictly outward minimising hull of a bounded set Ω with smooth boundary is simply given by the interior of the the zero-level set of such a solution. This is substantially the content of [HI01, Lemma 1.4], but here we want to emphasise and fully detail the relation between the *existence* of the strictly outward minimising hull and that of the Weak Inverse Mean Curvature Flow.

Let us recall the notion of Inverse Mean Curvature Flow, as well as its weak formulation introduced by Huisken-Ilmanen in [HI01]. Let (M,g) be a complete noncompact Riemannian manifold, and let $\Omega \subset M$ be a bounded subset with smooth boundary given by the immersion $F_0 : \partial\Omega \to \mathbb{R}^n$. Assume in addition that the mean curvature of $\partial\Omega$ is strictly positive. Then, we say that the hypersurfaces $\{\partial\Omega_t\}_{t\in[0,T)}$, for some $T > 0$ are evolving by IMCF with initial datum $\partial\Omega$ if they are given by immersions $F(t,\cdot) : \partial\Omega \to \mathbb{R}^n$ satisfying

$$\frac{\partial}{\partial t}F(t,x) = \frac{1}{\mathrm{H}}(t,x)\nu(t,x), \qquad F(0,x) = F_0(x) \tag{3.3.5}$$

where ν is the exterior unit normal to the hypersurface $\partial\Omega_t$ and H is its related mean curvature. It is well known that the IMCF of a strictly mean-convex hypersurface enjoys existence in some time interval $[0,T)$, see e.g. the comprehensive [HP99, Theorem 3.1].

The weak formulation of the IMCF actually regards the degenerate elliptic problem solved by the function whose level sets evolve by inverse mean curvature. Namely, looking at the evolving hypersurfaces $\partial\Omega_t$ as level sets $\{w = t\}$ of a smooth function, it is easily seen that w must satisfy the equation

$$\operatorname{div}\left(\frac{\mathrm{D}w}{|\mathrm{D}w|}\right) = |\mathrm{D}w| \tag{3.3.6}$$

in the region foliated by the evolving hypersurfaces. In particular, if there exists a smooth solution to (3.3.5) made of embedded closed hypersurfaces, then it is well defined the smooth function w with nonvanishing gradient solving (3.3.6).

A weak solution to (3.3.6) starting from Ω, that we will call Weak Inverse Mean Curvature Flow starting from Ω, will be a function $w \in \mathrm{Lip}_{\mathrm{loc}}(M)$ satisfying the following conditions.

(i) For every $v \in \mathrm{Lip}_{\mathrm{loc}}(M)$ with $\{w \neq v\} \Subset M \setminus \overline{\Omega}$ and any compact set $K \subset M \setminus \Omega$ containing $\{w \neq v\}$,

$$J^K_w(w) \leq J^K_w(v)$$

where

$$J^K_w(v) = \int_K |\mathrm{D}v| + v|\mathrm{D}w| \,\mathrm{d}\mu.$$

(ii) The set Ω is the 0-sublevel set of w, that is

$$\Omega = \{w < 0\}.$$

Remark 3.20 (Properness of the Weak IMCF). We say that the Weak Inverse Mean Curvature Flow is *proper* if w is a proper function. In the rest of this note, we will always consider Weak Inverse Mean Curvature Flows that are proper. Observe that if $w(x) \to +\infty$ as $d(O,x) \to +\infty$, then w is proper. The validity of this condition is assumed as definition of properness in [KN09]. However, there may exist a proper Weak IMCF w such that $w \not\to +\infty$ at infinity (see Example 3.29 below).

It is definitively convenient to rephrase (i) in the definition of Weak Inverse Mean Curvature Flow in terms of the level sets of the solution.

(i-bis) For any $t \geq 0$, the sets $\{w \leq t\}$ satisfies

$$J_w^K(\{w \leq t\}) \leq J_w^K(F)$$

for any $F \subset M$ with locally finite perimeter satisfying $\{w \leq t\}\Delta F \Subset M \setminus \overline{\Omega}$ and any compact $K \subset M \setminus \overline{\Omega}$ containing $\{w \leq t\}\Delta F$, where

$$J_w^K(F) = |\partial^* F \cap K| + \int_{F \cap K} |\mathrm{D}w| \, \mathrm{d}\mu.$$

We refer the reader to [HI01, Lemma 1.1 and Lemma 2.2] and the discussion thereafter for the proof of the equivalence between conditions (i) and (i-bis). One can then deduce from the fundamental Minimizing Hull Property Lemma 1.4 in [HI01] that the sets $\mathrm{Int}\{w \leq t\}$ are strictly outward minimising. In particular, the condition of having an exhausting sequence of strictly outward minimising sets is fulfilled every time there exists the Weak IMCF, and the analysis of the preceding section allows then to define the strictly outward minimising hull of Ω. It finally turns out that $\Omega^* = \mathrm{Int}\{w \leq 0\}$. Let us carefully prove this fact, for the reader's sake.

Proposition 3.21. *Let (M, g) be a complete noncompact Riemannian manifold. Assume that for any bounded $\Omega \subset M$ with smooth boundary there exists a proper weak IMCF w starting from Ω, then the strictly outward minimising hull Ω^* is an open bounded maximal volume solution to the least area problem with obstacle Ω. Moreover, $\mathrm{Int}\{w \leq 0\} = \Omega^*$.*

Proof. Let w be the Weak IMCF emanating from the bounded open set with smooth boundary Ω. By [HI01, Lemma 1, 4, (ii)], the set $\mathrm{Int}\{w \leq 0\}$ is bounded and strictly outward minimising. In particular, if w exists for any such Ω, taking an exhausting sequence of bounded open sets with smooth boundary yields an exhausting sequence of bounded strictly outward minimising sets, that allows us to use Theorem 3.13 to show that Ω^* defined in Definition 3.11 is an open bounded maximal volume solution to the least area problem with obstacle Ω. By Proposition 3.6, the set $\mathrm{Int}\{w \leq 0\}$ is also open. By definition, we have $\Omega^* \subseteq \mathrm{Int}\{w \leq 0\}$. Then, by properties (i-bis) and (ii) in the definition of the Weak IMCF recalled above, we have, up to a suitable choice of K,

$$P(\{w \leq 0\}) \;+\; \int_{\mathrm{Int}\{w \leq 0\} \setminus \Omega} |\mathrm{D}w| \, \mathrm{d}\mu \;\leq\; P(\Omega^*).$$

In particular the set $\mathrm{Int}\{w \leq 0\}$ contains Ω and satisfies $P(\mathrm{Int}\{w \leq 0\}) = P(\{w \leq 0\}) \leq P(\Omega^*)$, where we used (3.2.2). This implies in fact that $P(\mathrm{Int}\{w \leq 0\}) = P(\Omega^*)$, since by Theorem 3.13 the set Ω^* solves the least area problem with obstacle Ω. Since by the same result Ω^* is also strictly outward minimising, we deduce in turn $|(\mathrm{Int}\{w \leq 0\}) \setminus \Omega^*| = 0$. Since, by Theorem 3.14, the volume measure of $\partial\Omega^*$ is zero, we also have that the *open* set $(\mathrm{Int}\{w \leq 0\}) \setminus \overline{\Omega^*}$ is null, but by openness this is possible only if $\mathrm{Int}\{w \leq 0\} = \Omega^*$. □

In [MRS19], the authors showed that under the assumptions of (ii) in Theorem 3.1, any bounded set with smooth boundary can be evolved through a proper weak solution to the Inverse Mean Curvature Flow. In particular from this fact and Proposition 3.21 we get that under these assumptions Ω^* satisfies the desired properties.

Corollary 3.22. *Let (M, g) be a complete noncompact Riemannian manifold satisfying the assumptions of* (ii) *in Theorem 3.1. Then, for any open bounded Ω with smooth boundary, the set Ω^* is an open bounded maximal volume solution to least area problem with obstacle Ω.*

Other aspects of strictly outward minimising sets and IMCF

Here, we observe that a *smooth* solution to the Inverse Mean Curvature Flow of a bounded open set Ω contained in a complete Riemannian manifolds never completely leaves Ω^*, if the latter is an open bounded maximal volume solution to the least area problem with obstacle Ω. In particular, this holds under the assumption of Theorem 3.1. This phenomenon is substantially a direct corollary of the outward minimising property of the hypersurfaces evolving by IMCF, that was observed in [HI01, Smooth Flow Lemma 2.3]. We add a proof of this fact, because it implies, together with the long time existence theory for strictly starshaped sets with smooth and strictly mean-convex boundary, that the latters are strictly outward minimising. A simple approximation argument involving the Mean Curvature Flow will also yield that any mean-convex strictly starshaped set is outward minimising.

Proposition 3.23 (No Escape from Ω^*)**.** *Let (M, g) be a complete noncompact Riemannian manifold satisfying one of the equivalent conditions in Theorem 3.13. Then, for $\Omega \subset M$ a bounded open set with smooth boundary, let $\{\partial\Omega_t\}_{t\in[0,T]}$ be evolving by IMCF with initial datum $\partial\Omega$. Then, if $\Omega^* \Subset \Omega_t$ for some $t \in (0, T]$, then Ω is strictly outward minimising and we have $\Omega = \Omega^*$.*

Proof. Let $w : \Omega_T \to \mathbb{R}$ such that $\{w = t\} = \partial\Omega_t$, so that w classically satisfies the level set equation (3.3.6). Then

$$\begin{aligned} 0 \leq \int_{\Omega^*\setminus\overline{\Omega}} |\mathrm{D}w| \,\mathrm{d}\mu &= \int_{\Omega^*\setminus\overline{\Omega}} \mathrm{div}\left(\frac{\mathrm{D}w}{|\mathrm{D}w|}\right) \mathrm{d}\mu \\ &= \int_{\partial^*\Omega^*} \left\langle \frac{\mathrm{D}w}{|\mathrm{D}w|} \,\middle|\, \nu_{\partial^*\Omega^*} \right\rangle \mathrm{d}\sigma - \int_{\partial\Omega} \left\langle \frac{\mathrm{D}w}{|\mathrm{D}w|} \,\middle|\, \frac{\mathrm{D}w}{|\mathrm{D}w|} \right\rangle \mathrm{d}\sigma \\ &\leq |\partial\Omega^*| - |\partial\Omega| \leq 0, \end{aligned} \tag{3.3.7}$$

where in the second equality $\nu_{\partial^*\Omega^*}$ is the measure theoretic unit normal to the reduced boundary $\partial^*\Omega^*$ and in the last inequality we used the Divergence Theorem for sets of finite perimeter coupled with $|\partial\Omega^*| = |\partial^*\Omega^*|$. In particular, (3.3.7) implies that $|\Omega^* \setminus \overline{\Omega}| = 0$. By openness, that follows from (3.6), this implies that $\Omega = \Omega^*$ and in particular it is strictly outward minimising. □

The celebrated result of Gerhardt [Ger90] and Urbas [Urb90], asserting that strictly starshaped shaped sets of $\mathbb{R}^n$ with smooth and strictly mean-convex boundary admits an immortal solution to their Inverse Mean Curvature Flow immediately combines with Proposition 3.23 to show that these sets are in fact strictly outward minimising. Let us recall that in $\mathbb{R}^n$ a bounded set Ω with smooth boundary is strictly starshaped with respect to some point $x_0 \in \Omega$ if and only

$$\langle x - x_0 \,|\, \nu \rangle > 0$$

for any $x \in \partial\Omega$, where ν is the outward unit normal to the boundary of Ω.

Corollary 3.24. *Let $\Omega \subset \mathbb{R}^n$ be a bounded strictly starshaped set with smooth strictly mean-convex boundary. Then, Ω is strictly outward minimising.*

A simple approximation argument based on the Mean Curvature Flow, that we actually already used in the proof of the Isoperimetric Inequality in Chapter 1, yields that the strict mean-convexity can be relaxed to mean-convexity, that is, the mean curvature H of

$\partial\Omega$ is allowed to satisfy $\mathrm{H} = 0$ on some point of $\partial\Omega$. However, in this case, we just show that Ω is just outward minimising.

Proposition 3.25 (Starshaped mean-convex sets are outward minimising)**.** *Let $\Omega \subset \mathbb{R}^n$ be a strictly* starshaped *bounded open set with smooth* mean-convex *boundary. Then Ω is outward minimising.*

Proof. Let $F_0 : \partial\Omega \to \mathbb{R}^n$ be the immersion of $\partial\Omega$ in $\mathbb{R}^n$. Evolve this hypersurface by Mean Curvature Flow defining time dependent immersions $F : [0,\delta) \times \partial\Omega \to \mathbb{R}$ satisfying

$$\frac{\partial F}{\partial s}(s,x) = -\mathrm{H}(F(s,x))\,\nu(s,x), \qquad F(0,x) = F_0(x), \tag{3.3.8}$$

where ν is the exterior unit normal to the hypersurface $\partial\Omega_s$ given by the immersion $F(s,\cdot) : \partial\Omega \to \mathbb{R}^n$, and H is its related mean curvature. The standard short-time existence theory for geometric evolution equations (see e.g. [HP99, Theorem 3.1]) ensures the existence of a $\delta > 0$ such that a solution F to (3.3.8) is well-defined. In other words, we have defined a sequence of bounded open sets $\{\Omega_s\}_{s\in[0,\delta)}$ with smooth boundary evolving by Mean Curvature Flow (3.3.8). It is well known (see e.g. [HP99, Theorem 3.2]) that the mean curvature of these boundaries evolves by

$$\frac{\partial}{\partial s}\mathrm{H} = \Delta\mathrm{H} + \mathrm{H}|\mathrm{h}|^2,$$

where by h we denote the second fundamental form that, as the other quantities appearing in the equation above, is to be understood with respect to the evolving metric on $\partial\Omega_s$. In particular, the standard Maximum Principle for parabolic equations implies that the mean curvature of $\partial\Omega_s$ for $s \in (0,\delta)$ is strictly positive. Since, by the smoothness of the flow, the sets Ω_s are still strictly starshaped for small s, we can conclude by Corollary 3.24 that these approximating sets Ω_s are strictly outward minimising. Observe now that since the mean curvature H of the initial datum $\partial\Omega$ is nonnegative, the flow (3.3.8) is actually a shrinking flow, and thus $\Omega_s \subseteq \Omega \subseteq \Omega^*$. Then, by the minimising property of Ω_s we have $|\partial\Omega_s| \leq |\partial\Omega^*|$, that, upon letting $s \to 0^+$, implies $|\partial\Omega| \leq |\partial\Omega^*|$. This means, by Remark 3.16, that Ω is outward minimising. □

We conclude by pointing out that in literature there are many generalisations of Gerhardt-Urbas results in non-flat ambient manifolds, for a suitable notion of starshapedness. We mention in particular the work of Brendle-Hung-Wang [BHW16] that covers a considerable variety of warped product ambient manifolds and, outside rotationally symmetric ambients, the work of Pipoli [Pip16] in the Complex Hyperbolic Space. It is easy to see that, thanks to these results, the above argument shows that the theses of Corollary 3.24 and Proposition 3.25 hold in these ambient manifolds too.

3.4 Convergence of p-capacities to $|\partial\Omega^*|$

Aim of this section is to show that in the relevant classes of manifolds satisfying the assumptions of Theorem 3.1 we can recover the value of $|\partial\Omega^*|$ as limit for $p \to 1^+$ of the p-capacity of $\partial\Omega$, thus completing the proof of the latter Theorem. Before doing so, we recall some notation and the basic existence result of p-capacitary potentials.

3.4.1 p-nonparabolicity and p-capacitary potentials

Let (M, g) be a complete noncompact Riemannian manifold, and let $p \geq 1$. We define the variational p-capacity of a bounded set with smooth boundary $\Omega \subset M$ as

$$\mathrm{Cap}_p(\Omega) = \inf\left\{ \int_{\mathbb{R}^n} |\mathrm{D} f|^p \, \mathrm{d}\mu \;\middle|\; f \geq \chi_\Omega,\ f \in \mathscr{C}_0^\infty(\mathbb{R}^n) \right\}. \tag{3.4.1}$$

Let now $p > 1$. Then (M, g) is said to be p-nonparabolic if there exists a *positive* p-Green's function $G : (M \times M) \setminus \mathrm{Diag}(M) \to \mathbb{R}$, that is, satisfying

$$\int_M \left\langle |\mathrm{D}\, G_p(O, \cdot)|^{p-2} \mathrm{D}\, G_p(O, \cdot) \,\middle|\, \mathrm{D}\varphi \right\rangle \mathrm{d}\mu = \varphi(O) \tag{3.4.2}$$

for any $\varphi \in \mathrm{C}_c^\infty(M)$. Relation (3.4.2) is the weak formulation of the equation $-\Delta_p G(O, \cdot) = \delta_O$, where δ_O is the Dirac delta centered at O. Moreover, when referring to the p-Green's function of a p-nonparabolic manifold we mean the minimal one.

One can show that if M is p-nonparabolic and $G_p \to 0$ at the infinity of any end then, for any open bounded $\Omega \subset M$ with smooth boundary there exists a unique weak solution $u_p \in W^{1,p}(M \setminus \overline{\Omega})$ to

$$\begin{cases} \Delta_p u = 0 & \text{in} \quad M \setminus \overline{\Omega}\,, \\ u = 1 & \text{on} \quad \partial\Omega\,, \\ u(x) \to 0 & \text{as} \quad d(O, x) \to \infty\,, \end{cases} \tag{3.4.3}$$

where we can suppose $O \in \Omega$, generalising the correspondent existence theorem for harmonic functions on 2-nonparabolic manifolds Theorem 1.14 that we proved in Chapter 1. Moreover, the integral of $|\mathrm{D} u_p|^p$ on $M \setminus \overline{\Omega}$ realises the p-capacity. Since a complete and self-contained proof of this general result does not seem be easy to find in literature, we included a proof in Appendix B. Here, we just note that the self-contained argument used in Theorem 1.14, cannot evidently be directly exported to our case. Instead, an adaptation of the argument used for [CS03b] together with the deep $\mathscr{C}^{1,\alpha}$-estimates holding true up to the boundary of [Lie88] work also in the general p-nonparabolic case.

Theorem 3.26. *Let (M, g) be a complete noncompact p-nonparabolic Riemannian manifold. Let $\Omega \subset M$ be an open bounded set with smooth boundary, and let $O \in M$. Assume also that the p-Green's function G_p satisfies $G_p(O, x) \to 0$ as $d(O, x) \to \infty$. Then, there exists a weak solution u_p to (3.4.3). Moreover, it holds*

$$\int_{M \setminus \overline{\Omega}} |\mathrm{D} u|^p \, \mathrm{d}\mu = \mathrm{Cap}_p(\Omega). \tag{3.4.4}$$

Remark 3.27. It is worth pointing out that, by [Hol99], the above general result fully describes the nonnegative Ricci curvature case in terms of growth of the volume of geodesic balls. Indeed, we know that if (M, g) has nonnegative Ricci curvature and

$$\int_1^{+\infty} \left(\frac{t}{|B(O, t)|} \right)^{1/(p-1)} \mathrm{d}t < +\infty \tag{3.4.5}$$

for any $O \in M$, then [Hol99, Proposition 5.10] gives the p-nonparabolicity of (M, g), and the decay estimate for the positive p-Green's function G_p allows to conclude that $G_p \to 0$ at infinity. This is observed in [Bia+18, Corollary 2.6].

On the other hand, if the integral in (3.4.5) diverges, by [Hol99, Proposition 1.7] (M, g) is p-parabolic (actually regardless of curvature conditions), this in particular implies that $\mathrm{Cap}_p(\Omega) = 0$ (see e.g. [Hol99, (1.5)]) for any bounded $\Omega \subset M$ with smooth boundary and in particular the thesis of Theorem 3.26 cannot hold true.

We are finally in position to complete the proof of Theorem 3.1. Namely, we are going to show that under the assumptions of this theorem the variational p-capacity of Ω approximates the area of Ω^*. In the most important and original step in the proof (namely, *Step 3* below), the arguments we use are different depending on the validity of assumption (i) or (ii) in Theorem 3.1. In presence of an Euclidean-like Isoperimetric Inequality, an argument inspired by [Xu96] allows to show that

$$\mathrm{Cap}_1(\Omega) \leq \mathrm{C}_{n,p}\,\mathrm{Cap}_p(\Omega),$$

for some constant $\mathrm{C}_{n,p}$ fulfilling $\mathrm{C}_{n,p} \to 1$ as $p \to 1^+$.

On the other hand, if (M, g) is a manifold with nonnegative Ricci curvature satisfying the superlinear uniform volume growth condition, we are still able to prove (3.1.3) by exploiting a decay estimate of the p-Green's function of (M, g) with an explicit dependence on p. This estimate originated in [Hol99, Proposition 5.7], where it was proved for any point on the boundary of an end, and it has been applied in [MRS19] together with the assumed (3.1.2) to obtain an analogous inequality holding true on any point outside some compact set (see [MRS19, Theorem 3.8]). In this regard, we observe that being able to work out Holopainen's inequality without the restriction of lying in the boundary of an end would allow to relax the assumption (3.1.2) in [MRS19], and consequently in the present theory, to

$$\int_1^{+\infty} \frac{t}{|B(O,t)|}\,\mathrm{d}t < +\infty, \tag{3.4.6}$$

that is, roughly speaking, a strictly superlinear volume growth assumption.

Remark 3.28. We point out that in [MRS19] the authors actually assumed

$$\frac{|B(O,t)|}{|B(O,s)|} \geq \mathrm{C}\left(\frac{t}{s}\right)^b \tag{3.4.7}$$

for any $t \geq s > 0$ and some constant $\mathrm{C} > 0$ to provide the inequality described above. In turn, (3.4.7), that is implied by (3.1.2), is used to invoke the technical [Min09b, Proposition 2.8], allowing to control that the size of the bounded components of the complement of $M \setminus B(O, R)$ for big $R > 0$ are not too big. In fact, the desired decay estimate on the p-Green's function, and in turn, the validity of Theorem 3.1 is ensured on any Riemannian manifold with nonnegative Ricci curvature satisfying (3.4.6) such that for any R big enough $M \setminus B(O, R)$ does not have bounded components.

Proof of Theorem 3.1. Let (M, g) be a complete noncompact Riemannian manifold satisfying the lower bound (1.2.19) on the Ricci curvature. Then, if (i) or (ii) hold, then by Corollary 3.19 and Corollary 3.22 respectively, for any bounded Ω with finite perimeter the strictly outward minimising hull Ω^* is an open bounded maximal volume solution to the least area problem with obstacle Ω.

We aim at proving that

$$|\partial\Omega^*| \leq \mathrm{Cap}_1(\Omega) \leq \liminf_{p\to1^+} \mathrm{Cap}_p(\Omega) \leq \limsup_{p\to1^+} \mathrm{Cap}_p(\Omega) \leq |\partial\Omega^*|. \tag{3.4.8}$$

We divide the proofs in several steps.

Step 1. The first and easiest inequality to show in (3.4.8) is

$$|\partial\Omega^*| \leq \mathrm{Cap}_1(\Omega). \tag{3.4.9}$$

It suffices to observe that for any $f \in \mathscr{C}^\infty_c(M)$ with $f \geq \chi_\Omega$ we have, by co-area formula,

$$\int_M |\mathrm{D}f|\,\mathrm{d}\mu \geq \int_0^1 |\{f = t\}|\,\mathrm{d}t \geq \inf\Big\{|\partial E| \,\Big|\, \Omega \subset E,\, \partial E\,\text{smooth}\Big\} \geq |\partial\Omega^*|.$$

In particular, taking the infimum over any such f, we get (3.4.9).

Step 2. Now, we prove

$$\limsup_{p\to 1^+} \mathrm{Cap}_p(\Omega) \leq |\partial\Omega^*|. \tag{3.4.10}$$

Let E be any open and bounded set in M with smooth boundary such that $\Omega \subset E$. Define, for $x \in M$, the function $d_E(x) = \mathrm{dist}(x, E)$. Moreover, let us introduce a smooth cut-off function χ_ε fulfilling

$$\begin{cases} \chi_\varepsilon(t) = 1 & \text{in } t < \varepsilon, \\ -\frac{1}{\varepsilon} < \dot{\chi}(t) < 0 & \text{in } \varepsilon \leq t \leq 2\varepsilon \\ \chi_\varepsilon(t) = 0 & \text{in } t > 2\varepsilon, \end{cases} \tag{3.4.11}$$

and let us set $\eta_\varepsilon(x) = \chi_\varepsilon(d_E(x))$. Choosing ε small enough, it is easily seen, by the regularity of d_E in a neighbourhood of E (first observed for $\mathbb{R}^n$ in [GT01, Lemma 14.6], see [Man, Proposition 5.17] for a self contained proof in a general ambient manifold), that the function η_ε is an admissible competitor in (3.4.1). Then,

$$\mathrm{Cap}_p(\Omega) \leq \int_M |\mathrm{D}\eta_\varepsilon|^p\,\mathrm{d}\mu$$

for any $p \geq 1$. Letting $p \to 1^+$, we get

$$\limsup_{p\to 1^+} \mathrm{Cap}_p(\Omega) \leq \int_M |\mathrm{D}\eta_\varepsilon|\,\mathrm{d}\mu = \int_\varepsilon^{2\varepsilon} |\dot{\chi}_\varepsilon(t)|\,|\{d_E = t\}|\,\mathrm{d}t,$$

where in the last equality we applied the coarea formula combined with the fact that $|\mathrm{D}d_E| = 1$ in a neighbourhood of E. By the Mean Value Theorem, there exist $r_\varepsilon \in (\varepsilon, 2\varepsilon)$ such that the above right hand side satisfies

$$\int_\varepsilon^{2\varepsilon} |\dot{\chi}_\varepsilon(t)|\,|\{d_E = t\}|\,\mathrm{d}t = \varepsilon|\dot{\chi}_\varepsilon(r_\varepsilon)||\{d_E = r_\varepsilon\}| < |\{d_E = r_\varepsilon\}|,$$

where the last inequality is due to the second condition in (3.4.11). Since, as $r_\varepsilon \to 0^+$, we clearly have

$$|\{d_E = r_\varepsilon\}| \to |\partial E|,$$

we conclude that

$$\limsup_{p\to 1^+} \mathrm{Cap}_p(\Omega) \leq |\partial E|$$

for any bounded open set E with smooth boundary containing Ω. In particular, considering a sequence of bounded sets Ω_k with smooth boundary containing Ω^* and with

$|\partial\Omega_k| \to |\partial\Omega^*|$ as $k \to \infty$, provided in Lemma 3.17, we get (3.4.10).

Step 3. The most involved step in the proof of (3.4.8) is inequality

$$\mathrm{Cap}_1(\Omega) \leq \liminf_{p\to 1^+} \mathrm{Cap}_p(\Omega). \tag{3.4.12}$$

To do this, we treat separately the cases when (M,g) satisfies an Euclidean-like Isoperimetric Inequality and the case where (M,g) has nonnegative Ricci curvature and satisfies (3.1.2).

Case 1. (M,g) satisfies an Euclidean-like Isoperimetric Inequality. Let C_{Sob} be the L^1-Sobolev constant, that is

$$\left(\int_M f^{n/(n-1)}\,\mathrm{d}\mu\right)^{(n-1)/n} \leq C_{\mathrm{Sob}} \int_M |\nabla f|\,\mathrm{d}\mu \tag{3.4.13}$$

for any nonnegative $f \in \mathscr{C}_0^\infty(M)$. It is well known and easy to check that applying (3.4.13) to $f^{\frac{n-1}{n-p}p}$ yields the L^p-Sobolev inequality

$$\left(\int_M f^{p^*}\,\mathrm{d}\mu\right)^{(n-1)/n} \leq C_p \int_M |\mathrm{D}f|^p\,\mathrm{d}\mu \tag{3.4.14}$$

with

$$C_{n,p} = \left[\frac{C_{\mathrm{Sob}}(n-1)p}{(n-p)}\right], \tag{3.4.15}$$

and

$$p^* = \frac{np}{n-p}$$

for any $1 < p < n$. In particular, by [PST14, Theorem 3.2], (M,g) is p-nonparabolic for p in this range The argument yielding the key estimate (3.4.18) below is inspired by the proof of [Xu96, Theorem 3.2]. By the definition of 1-capacity and Hölder inequality we get, for any $q > 0$,

$$\mathrm{Cap}_1(\Omega) \leq \int_M |\mathrm{D}f^q|\,\mathrm{d}\mu = q\int_M f^{q-1}|\mathrm{D}f|\,\mathrm{d}\mu \leq q\left(\int_M f^{(q-1)\frac{p}{p-1}}\,\mathrm{d}\mu\right)^{(p-1)/p}\left(\int_M |\mathrm{D}f|^p\,\mathrm{d}\mu\right)^{1/p}. \tag{3.4.16}$$

Choose then

$$q_p = 1 + p^*\frac{(p-1)}{p} \tag{3.4.17}$$

and observe that $q_p > 1$. Then, applying the L^p-Sobolev inequality (3.4.14) we obtain

$$\mathrm{Cap}_1(\Omega) \leq q_p\, C_{n,p}^{(p-1)/p}\left(\int_M |\mathrm{D}f|^p\,\mathrm{d}\mu\right)^{p^*(p-1)/p^2+1/p} = q_p\, {C_{n,p}}^{(p-1)/p}\left(\int_M |\mathrm{D}f|^p\,\mathrm{d}\mu\right)^{(n-1)/(n-p)}.$$

Taking the infimum in the rightmost hand side of the inequality above over any $f \in \mathscr{C}_c^\infty(M)$ we are left with

$$\mathrm{Cap}_1(\Omega) \leq q_p\, {C_{n,p}}^{(p-1)/p}\mathrm{Cap}_p(\Omega), \tag{3.4.18}$$

Letting $p \to 1^+$ in the above inequality, and observing that from the expressions (3.4.15) and (3.4.17) both q_p and $C_{n,p}$ tend to 1, we get (3.4.12) under the validity of an Euclidean-like Isoperimetric Inequality.

Case 2. (M, g) has nonnegative Ricci curvature and satisfies (3.1.2). The superlinear volume growth following from (3.1.2) guarantees that Holopainen's condition

$$\int_1^{+\infty} \frac{t}{|B(O,t)|}\, dt < +\infty$$

is satisfied, and in particular his [Hol99, Theorem 5.10] guarantees that (M, g) is p-nonparabolic for $1 < p < b$. Moreover, [MRS19, Theorem 3.8] gives the following decay estimate for G_p

$$G_p(O,x) \leq \frac{C^{1/(p-1)}}{(p-1)^2} \int_{d(O,x)}^{\infty} \left[\frac{t}{|B(O,t)|}\right]^{1/(p-1)} dt.$$

for any $x \in M \setminus B(O, R_1)$ for some $R_1 > 0$. This estimate in particular allows to apply Theorem 3.26 that yields a p-harmonic function $u_p \in W^{1,p}(\mathbb{R}^n \setminus \overline{\Omega})$ realising (3.4.4). We claim that the same type of estimate holds for u_p. Indeed, choosing $O \in \Omega$, a trivial comparison argument immediately yields $u_p(x) \leq G_p(O,x)/(\inf_{\partial\Omega} G_p)$. Moreover, it follows from the convergence results in [MRS19] that the function $-(p-1)\log G_p(O,\cdot)$ converges locally uniformly to a continuous function (more precisely to a solution of the IMCF with initial condition O) and in particular $-(p-1)\log G_p(O,\cdot)$ is uniformly bounded in the compact set $\partial\Omega$ uniformly in p small enough. This implies that $(\inf_{\partial\Omega} G_p) \geq C_2^{1/(p-1)}$ for some C_2 independent on p. Combining it with the above comparison we thus obtain

$$u_p(x) \leq \frac{{C_3}^{1/(p-1)}}{(p-1)^2} \int_{d(O,x)}^{\infty} \left[\frac{t}{|B(O,t)|}\right]^{1/(p-1)} dt.$$

outside some ball. The volume growth condition $|B(O,t)| \geq C_{\mathrm{vol}}\, t^b$, following from (3.1.2), improves the above estimate to

$$u_p(O,x) \leq \frac{1}{(b-p)} \frac{C_4^{1/(p-1)}}{(p-1)} r(x)^{-(b-p)/(p-1)}, \tag{3.4.19}$$

where we denoted $r(x) = d(O,x)$, outside some big ball and for a positive constants C_4 uniform in p as $p \to 1^+$.

Extend now u_p to be equal to u on $M \setminus \Omega$, and equal to 1 on Ω, for simplicity with the same name. We claim that ${u_p}^{q_p}$ is in $W^{1,1}(M)$, for p close enough to 1, where q_p is defined in (3.4.17). Clearly, (3.4.19) implies that ${u_p}^{q_p} \in L^1(M)$ for p close enough to 1, and also so is u_p. Applying Hölder inequality as in (3.4.16) we also get

$$\begin{aligned}
\int_M |Du_p^{q_p}|\, d\mu &\leq q_p \left(\int_M u_p^{p^*}\, d\mu\right)^{(p-1)/p} \left(\int_{M\setminus\overline{\Omega}} |Du_p|^p\, d\mu\right)^{1/p} \\
&= q_p \left(\int_M u_p^{p^*}\, d\mu\right)^{(p-1)/p} \mathrm{Cap}_p(\Omega)^{1/p} \qquad , \\
&\leq q_p \left(\int_M u_p\, d\mu\right)^{(p-1)/p} \mathrm{Cap}_p(\Omega)^{1/p}
\end{aligned} \tag{3.4.20}$$

where the equality is (3.4.4), and the last inequality is due to $u_p \leq u_p^{p*}$ following from $0 < u \leq 1$ and $p^* > 1$. Since $u_p \in L^1(M)$ uniformly in p close to 1, and so does the p-capacity of Ω as $p \to 1^+$ by (3.4.10) in *Step 2*, (3.4.20) implies that $u_p^{q_p} \in W^{1,1}(M)$

uniformly as $p \to 1^+$. Since the class of competitors for $\mathrm{Cap}_1(\Omega)$ can be easily relaxed to the class of $W^{1,1}(M)$ without changing the infimum (see e.g. [HKM06, Chapter 2]), we can estimate $\mathrm{Cap}_1(\Omega)$ from above with $u_p^{q_p}$ and get, from (3.4.20), that

$$\mathrm{Cap}_1(\Omega) \leq q_p \left(\int_M u_p \,\mathrm{d}\mu \right)^{(p-1)/p} \mathrm{Cap}_p(\Omega)^{1/p}. \tag{3.4.21}$$

We focus on the first round bracket in the rightmost hand side of (3.4.21). Let us decompose M in $\overline{B(O,R)} \cup (M \setminus \overline{B(O,R)})$. We obtain, with the aid of (3.4.19),

$$\int_M u_p \,\mathrm{d}\mu \leq |\overline{B(O,R)}| + \frac{1}{(b-p)} \frac{\mathrm{C}_4^{1/(p-1)}}{(p-1)} \int\limits_{M\setminus\overline{B(O,R)}} r^{-(b-p)/(p-1)} \,\mathrm{d}\mu. \tag{3.4.22}$$

We estimate the integral in the right had side above as follows. We have

$$\begin{aligned}
\int\limits_{M\setminus\overline{B(O,R)}} r^{-(b-p)/(p-1)} \,\mathrm{d}\mu &= \sum_{j=1}^{+\infty} \int\limits_{B(O,2jR)\setminus\overline{B(O,jR)}} r^{-(b-p)/(p-1)} \,\mathrm{d}\mu \\
&\leq \sum_{j=1}^{+\infty} (jR)^{-[(b-p)/(p-1)]} \Big[|B(O,2jR)| - |B(O,jR)| \Big] \\
&\leq |\mathbb{B}^n| 2^n R^{n-(b-p)/(p-1)} \sum_{j=1}^{+\infty} j^{n-[(b-p)/(p-1)]} \leq \mathrm{C}_5 R^{n-[(b-p)/(p-1)]},
\end{aligned} \tag{3.4.23}$$

where we used Bishop-Gromov to estimate $|B(O,2jR)| \leq |\mathbb{B}^n|(2jR)^n$, and, in the last step, the convergence of the series, that holds true for p close to 1 with a value that is uniform in p. Resuming, we have, plugging the outcome of (3.4.23) into (3.4.22),

$$\int_M u_p \,\mathrm{d}\mu \leq \left|\overline{B(O,R)}\right| + R^n \frac{1}{(b-p)(p-1)} \left[\frac{\mathrm{C}_6}{R^{(b-p)}} \right]^{1/(p-1)}$$

for any R big enough. Choose then $R^{b-p} > \mathrm{C}_6$ for any p close enough to 1 to see that, with this choice, the second summand in the above right hand side vanishes in the limit as $p \to 1^+$. In particular, letting $p \to 1^+$ in (3.4.21) we infer

$$\mathrm{Cap}_1(\Omega) \leq \liminf_{p\to 1^+} \mathrm{Cap}_p(\Omega),$$

also under the assumption of nonnegative Ricci curvature with uniform volume growth (3.1.2).

Arranging (3.4.9), (3.4.10) and (3.4.12) into (3.4.8) completes the computation of the limit of $\mathrm{Cap}_p(\Omega)$ as $p \to 1^+$. □

In light of Theorem 3.1, one could wonder whether the existence of a well posed (in the sense of Theorem 3.13) strictly outward minimising hull implies a p-capacitary approximation of $|\partial\Omega^*|$ in the sense of (3.1.3). The following example, inspired in part by [KN09, Section 4], provides a manifold admitting an exhausting sequence of strictly outward minimising sets but such that $\mathrm{Cap}_p(\Omega) = 0$ for any $\Omega \subset M$ with smooth boundary.

Example 3.29. Consider, for $n \geq 2$ the complete noncompact Riemannian manifold (M, g) whose metric splits as $g = \mathrm{d}\rho \otimes \mathrm{d}\rho + \tanh^2(\rho) g_{\mathbb{S}^{n-1}}$ on $[0, +\infty) \times \mathbb{S}^{n-1}$. For $n = 2$, this is the celebrated Hamilton's cigar soliton [Ham88]. The Riemannian manifold (M, g) has linear volume growth and nonnegative Ricci curvature. In particular,

$$\int_1^{+\infty} \frac{t}{|B(O,t)|} \, \mathrm{d}t = +\infty,$$

and by [Hol99, Proposition 1.7] (M, g) is p-parabolic for any $p > 1$. In particular, it is well known (see e.g. [Hol99, (1.5)]) that $\mathrm{Cap}_p(\Omega) = 0$ for any open bounded $\Omega \subset M$ with smooth boundary. However, the level sets of ρ are strictly outward minimising, and so they provide the exhausting sequence (in fact a foliation) required in Theorem 3.13 for the well-posedness of Ω^*. To check this last assertion, we invoke again the level set formulation of the Inverse Mean Curvature Flow, and the minimising properties of such level sets discovered in [HI01]. Let indeed $B = \{\rho < 1\}$, and consider on $M \setminus B$ the function

$$w = (n-1) \log \left(\frac{\tanh \rho}{\tanh 1} \right)$$

extended with continuity at 0 on B. The function w is immediately seen to satisfy the level set equation (3.3.6) in $M \setminus \overline{B}$. Since the level set of w sweep out the whole M as $\rho \to +\infty$, [HI01, Smooth Flow Lemma 2.3] implies that w is also a Weak Inverse Mean Curvature Flow in the sense recalled in Subsection 3.3.2. Observe also that w is proper, although $w \not\to +\infty$ as $\rho \to +\infty$ (compare with Remark 3.20). In particular by [HI01, Minimizing Hull Property 1.4, (ii)], the bounded sets $\mathrm{Int}\{w \leq t\} = \{w < t\}$ are strictly outward minimising for any $t \geq 0$. Theorem 3.13 applies and yields a well posed strictly outward minimising hull to any bounded open set $\Omega \subset M$ with smooth boundary, but its area cannot be recovered by p-capacities, that vanish for any $p > 1$.

Appendix A

Comparison with Colding's monotonicity formulas

In this section, we provide a comparison between our monotonicity formulas and those obtained by Colding and by Colding-Minicozzi in [Col12] and [CM13], respectively. To start with, let u be a solution of (8) in a nonparabolic Riemannian manifold (M, g) with $\mathrm{Ric} \geq 0$, for a bounded subset $\Omega \subset M$ with smooth boundary, and set

$$b = u^{-\frac{1}{n-2}}. \tag{A.1}$$

Note that $b = 1$ on $\partial\Omega$ and that $b \to +\infty$ at infinity. Associated with the level sets of b, consider the family of functions $\{A_\beta\}$, where $A_\beta : [1, +\infty) \to [0, +\infty)$ is defined for every $\beta \geq 0$ as

$$A_\beta(r) = \frac{1}{r^{n-1}} \int\limits_{\{b=r\}} |\mathrm{D}b|^{\beta+1}\, \mathrm{d}\sigma.$$

Now, replacing u be a minimal Green's function $G(O, \cdot)$, for some pole $O \in M$, the above defined A_β is exactly the quantity considered in [CM13, formula (1.1)]. Note that in Colding's setting, the level sets $\{b = r\}$ are considered for every $r > 0$, since $b(q) \to 0$ as $d(O, q) \to 0$.

Our aim is to see how the monotonicity of our functions Φ_β translates in terms of that of A_β. First of all, it is straightforward from (1.3.9)–(1.3.11) and (A.1) that

$$b = \mathrm{e}^{\frac{\varphi}{n-2}}, \qquad |\mathrm{D}b| = \frac{|\nabla\varphi|_{\tilde{g}}}{n-2}, \qquad \mathrm{d}\sigma = b^{n-1}\, \mathrm{d}\sigma_{\tilde{g}}, \tag{A.2}$$

and in turn that

$$\Phi_\beta(s) = (n-2)^{\beta+1} A_\beta\big(\mathrm{e}^{\frac{s}{n-2}}\big), \qquad \text{for every } s \geq 0. \tag{A.3}$$

We look at the derivative (1.3.22) of Φ_β and at its equivalent expression (1.3.23). In particular, the volume integral (1.3.23) contains the following terms.

$$\mathrm{Ric}(\nabla\varphi, \nabla\varphi) = \left(\frac{n-2}{2}\right)^2 \mathrm{Ric}(\mathrm{D}b^2, \mathrm{D}b^2), \tag{A.4}$$

and

$$|\nabla\nabla\varphi|^2_{\tilde g} + (\beta-2)|\nabla|\nabla\varphi|_{\tilde g}|^2_{\tilde g} = \left(\frac{n-2}{2}\right)^2 \Big\{ \Big|\mathrm{DD}\, b^2 - \frac{\Delta b^2}{n}\, g\Big|^2 + (\beta-2)\,\big|\mathrm{D}^\top|\mathrm{D}b|\big|^2 + (\beta-2)\,\big|\mathrm{D}b^2\big|^2\Big[\mathrm{H} - (n-1)\big|\mathrm{D}\log b\big|\Big]^2\Big\} \tag{A.5}$$

which have been expressed in terms of the function b and of the metric g via some computations (compare with the proof of (1.3.4)) . Differentiating both sides of (A.3) and writing expression (1.3.23) in terms of b and g through formulas (A.2), (A.4) and (A.5), we obtain

$$\begin{aligned}\frac{\mathrm{d}A_\beta}{\mathrm{d}r}(r) &= \frac{(n-2)^{-\beta}}{r}\frac{\mathrm{d}\Phi_\beta}{\mathrm{d}s}\big((n-2)\log r\big) \qquad (A.6)\\ &= -\frac{\beta}{4}\, r^{n-3}\int_{\{b>r\}}|\mathrm{D}b|^{\beta-2}\Big\{ \mathrm{Ric}(\mathrm{D}b^2,\mathrm{D}b^2) + \Big|\mathrm{DD}b^2 - \frac{\Delta b^2}{n} g\Big|^2 \\ &\qquad + (\beta-2)\,\big|\mathrm{D}^\top|\mathrm{D}b|\big|^2 \\ &\qquad + (\beta-2)\,\big|\mathrm{D}b^2\big|^2\Big[\mathrm{H} - (n-1)\big|\mathrm{D}\log b\big|\Big]^2\Big\}\, b^{2-2n}\,\mathrm{d}\mu \,\le\, 0\,.\end{aligned}$$

Setting $b = 2$ in the above formula, we obtain exactly the integrand of the right hand side of [Col12, (2.106)], that, arguing as in the conclusion of the present Theorem 1.25, leads to the monotonicity of A_2. For a general $\beta \ge (n-2)/(n-1)$, in [CM14b, Theorem 1.3] the monotonicity of A_β is inferred grouping the terms in (A.6) in a different way. Observe indeed that for $\beta < 2$ the volume integral in (A.6) does not evidently carry a sign. On the other hand, (A.5) combined with Kato's inequality immediately show the nonnegativity of the expression.

We close this appendix by showing how our methods can be applied also to obtain the Monotonicity-Rigidity Theorem for the Green's function, obtaining a new (conformal) proof of Colding-Minicozzi's [CM14b, Theorem 1.3]. Indeed, let (M, g) be a nonparabolic Riemannian manifold with $\mathrm{Ric} \ge 0$, let G be its minimal Green's function and consider the new metric on $M \setminus \{O\}$

$$\tilde g = G(O,\cdot)^{\frac{2}{n-2}} g,$$

for some point $O \in M$. Set

$$\varphi = -\log G(O,\cdot).$$

Then, we have that the triple $M, \tilde g, \varphi$ satisfies the system

$$\begin{cases} \Delta_{\tilde g}\varphi = 0 & \text{in } M\setminus\{O\} \\ \mathrm{Ric}_{\tilde g} - \nabla\nabla\varphi + \dfrac{d\varphi\otimes d\varphi}{n-2} = \dfrac{|\nabla\varphi|^2_{\tilde g}}{n-2}\tilde g + \mathrm{Ric} & \text{in } M\setminus\{O\} \\ \varphi(y) \to +\infty & \text{as } d(O,y) \to +\infty \\ \varphi(y) \to -\infty & \text{as } d(O,y) \to 0. \end{cases}$$

We denote by d the distance with respect to g. Define the function $\Phi_\beta : \mathbb{R} \mapsto \mathbb{R}$ given by

$$\Phi_\beta(s) \;=\; \int\limits_{\{\varphi=s\}} |\nabla\varphi|_{\tilde g}^{\beta+1}\, \mathrm{d}\sigma_{\tilde g}.$$

All the theory developed in Section 3 holds with trivial modification for Φ_β as above, and immediately yields a conformal Monotonicity-Rigidity Theorem for the Green's function.

Theorem A.7. *Let (M, g) be a nonparabolic Riemannian manifold with $\mathrm{Ric} \geq 0$. Let G be its minimal Green's function. Then, with the notations above, we have*

$$\frac{\mathrm{d}\Phi_\beta}{\mathrm{d}s}(s) \;=\; -\,\beta\, \mathrm{e}^s \int\limits_{\{\varphi\geq s\}} \frac{|\nabla\varphi|_{\tilde g}^{\beta-2}\Big(\mathrm{Ric}(\nabla\varphi,\nabla\varphi) + \big|\nabla\nabla\varphi\big|_{\tilde g}^2 + (\beta-2)\,\big|\nabla|\nabla\varphi|_{\tilde g}\big|_{\tilde g}^2\Big)}{\mathrm{e}^{\varphi}}\, \mathrm{d}\mu_{\tilde g}$$

In particular, Φ'_β is alway nonpositive. Moreover, $(\mathrm{d}\Phi_\beta/\,\mathrm{d}s)(s_0) = 0$ for some $s_0 \in \mathbb{R}$ and some $\beta \geq (n-2)/(n-1)$ if and only if $\{\varphi \geq s_0\}$ is isometric to the Riemannian product $\big([s_0, \infty) \times \{\varphi = s_0\},\, \mathrm{d}\rho \otimes \mathrm{d}\rho + \tilde g_{\{|\varphi=s_0\}}\big)$.

The above Theorem clearly translates in terms of (M, g) and G, exactly as Theorem 1.19 was deduced from Theorem 1.25.

Appendix B

Existence of p-capacitary potentials

The following comprehensive statement contains both the thesis of the existence theorem for the p-capacitary potential on p-nonparabolic complete noncompact Riemannian manifolds recalled in Theorem 3.26 as well as the identification of the variational p-capacity of a bounded set with smooth boundary with a boundary integral. These properties hold in particular on $\mathbb{R}^n$ and they were largely employed in Chapter 2. In particular, we furnish a proof of Theorem 3.26 and Lemma 2.4. Despite seemingly well known, it is not easy to find in literature a complete and in particular self-contained proof of these basic facts. Anyway, no original arguments appear below, and we are collecting stuff mainly from [Hol90], [HKM06] and [PST14].

Theorem B.1. *Let (M, g) be a complete noncompact p-nonparabolic Riemannian manifold. Let $\Omega \subset M$ be an open bounded set with smooth boundary, and let $O \in M$. Assume also that the p-Green's function G_p satisfies $G_p(O, x) \to 0$ as $d(O, x) \to \infty$. Then, there exists a unique weak solution u_p to (3.4.3) attaining smoothly the boundary value on $\partial\Omega$. Moreover, it realises*

$$\mathrm{Cap}_p(\Omega) \;=\; \int\limits_{M\setminus\overline{\Omega}} |\mathrm{D}u|^p \,\mathrm{d}\mu \;=\; \int\limits_{\{u_p=t\}} |\mathrm{D}u|^{p-1}\,\mathrm{d}\sigma \tag{B.2}$$

for almost any $t \in (0, 1]$ including any regular value t. In particular, (B.2) holds at $t = 1$, that is

$$\mathrm{Cap}_p(\Omega) \;=\; \int\limits_{\partial\Omega} |\mathrm{D}u|^{p-1}\,\mathrm{d}\sigma \tag{B.3}$$

Proof. Let $B(O, R)$ be a geodesic ball containing Ω. Let $\psi \in \mathscr{C}^\infty_c(B(O, R))$ satisfy $\psi = 1$ on Ω. Then, we can find a solution to the Dirichlet problem for the p-laplacian with boundary values 1 on $\partial\Omega$ and 0 on $\partial B(O, R)$, namely a weakly p-harmonic function u_R on $B(O, R) \setminus \Omega$ satisfying $u - \psi \in W^{1,p}_0(B(O, R) \setminus \overline{\Omega})$. It is immediately deduced from Tolksdorf's Comparison Principle for p-harmonic functions (see for example [HKM06, Lemma 3.18] that if $\tilde{u}$ is another such function relative to another $\tilde{\psi} \in W^{1,p}_0(B(O, R))$ then $u = \tilde{u}$ on $B(O, R) \setminus \overline{\Omega}$. Defining the p-capacity of Ω relative to $B(O, R)$ as

$$\mathrm{Cap}_p(\Omega, B(O, R)) = \inf\left\{ \int_M |\mathrm{D}f|^p \,\mathrm{d}\mu \;\middle|\; f \geq \chi_\Omega,\, f \in \mathscr{C}^\infty_c(B(O, R)) \right\}, \tag{B.4}$$

we immediately see through a very standard argument that u_R can be approximated in $W^{1,p}$ by a sequence of admissible competitors in (B.4), and thus

$$\mathrm{Cap}_p(\Omega, B(O, R)) \leq \int\limits_{B(O,R)\setminus\overline{\Omega}} |\mathrm{D}u_R|^p \,\mathrm{d}\mu. \tag{B.5}$$

On the other hand, observing that in the definition of weakly p-harmonic functions on $B(O,R) \setminus \overline{\Omega}$ the class of tests can easily be relaxed to $W_0^{1,p}(B(O,R) \setminus \overline{\Omega})$, we get

$$\int_{B(O,R)\setminus\overline{\Omega}} |Du_R|^p \, d\mu = \int_{B(O,R)\setminus\overline{\Omega}} \left\langle |Du_R|^{p-2} Du_R \,|\, Du_R \right\rangle d\mu = \int_{B(O,R)\setminus\overline{\Omega}} \left\langle |Du_R|^{p-2} Du_R \,|\, D\psi \right\rangle d\mu, \tag{B.6}$$

where the last equality is deduced from the definition of weak p-harmonicity with $u - \psi \in W_0^{1,p}(B(O,R) \setminus \overline{\Omega})$ as test function. Applying the Hölder inequality to the right hand side of (B.6), we are immediately left with

$$\int_{B(O,R)\setminus\overline{\Omega}} |Du_R|^p \, d\mu \leq \int_{B(O,R)\setminus\overline{\Omega}} |D\psi|^p \, d\mu \tag{B.7}$$

for any $\psi \in \mathscr{C}_c^\infty(B(O,R))$ satisfying $\psi = 1$ on Ω. Since, clearly, a sequence of functions satisfying the same assumptions on ψ suffices to realise the relative p-capacity of Ω (see for example [Maz11, (2.2.1)] for details), we get passing to the limit through such a sequence in (B.7) that we can also take the opposite inequality sign in (B.5), obtaining

$$\operatorname{Cap}_p(\Omega, B(O,R)) = \int_{B(O,R)\setminus\overline{\Omega}} |Du_R|^p \, d\mu. \tag{B.8}$$

Let us finally show that passing to the limit as $R \to +\infty$ yields a solution to (3.4.3). In what follows, we are frequently extending u_R to 0 outside $B(O,R)$ without explicitly mentioning it The deep $\mathscr{C}^{1,\alpha}_{\mathrm{loc}}$-estimates for p-harmonic functions give, for any compact set K of $M \setminus \Omega$, a constant C so that

$$|u_R(x) - u_R(y)| \leq \mathrm{C}\, d(x,y)$$

and

$$\Big| |Du_R|(x) - |Du_R|(y) \Big| \leq \mathrm{C}\, d(x,y).$$

The constant C depends on the dimension, on p, on smooth quantities related to the underlying metric g on K and on the $W^{1,p}$ norm of u_R on K. By the Maximum Principle for p-harmonic functions, $0 \leq u_R \leq 1$ on K. Moreover, by (B.8) and the obvious monotonicity of the $\operatorname{Cap}_p(\Omega, B(O,R))$ as R increases, we have

$$\int_K |Du_R|^p \, d\mu \leq \int_{B(O,R_0)} |Du_{R_0}|^p \, d\mu \tag{B.9}$$

for any $R_0 > 0$ such that $K \Subset B(O,R_0)$ and any $R > R_0$. In particular, the constant C is uniform in R, and by Arzelà-Ascoli applied to the sequence of u_R and the sequence of the gradients we deduce that, up to a subsequence, the sequence u_R converges in $\mathscr{C}^1_{\mathrm{loc}}$ to a $\mathscr{C}^1$ function u. Moreover, as it is immediately seen from the weak formulation of p-harmonicity, such u is p-harmonic. Observe that, since the $\mathscr{C}^{1,\alpha}_{\mathrm{loc}}$-estimates of [Lie88] are valid up to the boundary, K was allowed to contain portions of $\partial\Omega$, and thus the convergence takes place also on $\partial\Omega$, where then the sequence obviously converges to 1. Moreover, since by Tolksdorf's Hopf lemma for p-harmonic functions $|Du| > 0$ on $\partial\Omega$, the continuity up to the boundary of the gradient ensures that $|Du| > 0$ in a neighbourhood of $\partial\Omega$, where thus the solution is a smooth classical solution by quasilinear elliptic regularity, and in particular it attains smoothly the datum on $\partial\Omega$. To complete the existence part of the proof, we just have to check that u vanishes at infinity. This is immediately deduced from the vanishing at infinity of G_p. Indeed, since G_p is positive, a straightforward comparison argument for the approximators u_R yields a constant C independent of R so that $0 < u_R \leq \mathrm{C}\, G_p(O, \cdot)$, that by the convergence shown above, implies, sending $R \to +\infty$,

$$0 < u_R \leq \mathrm{C}\, G_p(O, \cdot). \tag{B.10}$$

This last estimate clearly implies the vanishing at infinity of u.

Briefly, we show uniqueness. Let v be any other solution to (3.4.3), and let $k \in \mathbb{N}$. Let R_k be such that $u \leq v + 1/k$ on (a smoothed out approximation of) $\partial B(O, R_k)$. Such a radius surely exists by the vanishing at infinity of u. Then, since u and v achieve the same value on $\partial\Omega$, the Comparison Principle for p-harmonic functions applied to u and $v + 1/k$ on $B(O, R_k) \setminus \overline{\Omega}$ shows that $u \leq v + 1/k$ on this set. Letting $k \to +\infty$ we obtain $u \leq v$ on $M \setminus \Omega$. Exchanging the roles of u and v gives the opposite inequality, showing that $u = v$, that is uniqueness.

Now, we check that u realises the p-capacity of Ω. This again will come from the properties of the approximators. Since $u_R \to u$ as $R \to \infty$ pointwise, and $\int_K |\mathrm{D}u_R|^p \, \mathrm{d}\mu$ is uniformly bounded in R for any compact $K \Subset M \setminus \Omega$ by (B.9), we can invoke the basic but very useful [HKM06, Lemma 1.33] to infer that $u \in L^p(M \setminus \overline{\Omega})$ and that $\mathrm{D}u_R$ converges weakly to $\mathrm{D}u$. In particular, by lower semicontinuity of the norm we have

$$\int\limits_{M\setminus\overline{\Omega}} |\mathrm{D}u|^p \, \mathrm{d}\mu \leq \liminf_{R\to\infty} \int\limits_{M\setminus\overline{\Omega}} |\mathrm{D}u_R|^p \, \mathrm{d}\mu,$$

and since the righthand side is uniformly bounded again by (B.9), we conclude that $u \in W^{1,p}(M \setminus \overline{\Omega})$. Since it vanishes at infinity, it actually belongs to $W_0^{1,p}(M \setminus \overline{\Omega})$. We can thus argue exactly as done for (B.8) to show that u realises the p-capacity of Ω, that is the first equality in (B.2).

It remains to show the second equality in (B.2). We first show that

$$\int\limits_{\{u=t_2\}} |\mathrm{D}u|^{p-1} \, \mathrm{d}\sigma = \int\limits_{\{u=t_1\}} |\mathrm{D}u|^{p-1} \, \mathrm{d}\sigma \tag{B.11}$$

for almost any $0 < t_1 < t_2 < 1$. To see this, observe first that obviously, by density, if u is p-harmonic we have

$$\int\limits_{M\setminus\overline{\Omega}} \left\langle |\mathrm{D}u|^{p-2}\mathrm{D}u \,\middle|\, \mathrm{D}\psi \right\rangle \mathrm{d}\mu = 0 \tag{B.12}$$

for any $\psi \in \mathrm{Lip}_c(M \setminus \overline{\Omega})$. Consider then $\psi_\varepsilon(u_p)$ with

$$\psi_\varepsilon(s) = \begin{cases} 0 & \text{for } s \in (0, t_1 - \varepsilon) \cup (t_2 + \varepsilon, 1), \\ \dfrac{s - (t_1 - \varepsilon)}{2\varepsilon} & \text{for } s \in [t_1 - \varepsilon, t_1 + \varepsilon) \\ \dfrac{(t_2 + \varepsilon) - s}{2\varepsilon} & \text{for } s \in (t_2 - \varepsilon, t_2 + \varepsilon] \\ 1 & \text{for } s \in [t_1 + \varepsilon, t_2 - \varepsilon]. \end{cases}$$

Then, since the function $u_p \in \mathscr{C}^{1,\alpha}_{\mathrm{loc}}(M \setminus \overline{\Omega})$, we have $\psi_\varepsilon(u_p) \in \mathrm{Lip}_c(M \setminus \overline{\Omega})$. Plugging $\psi_\varepsilon(u_p)$ into (B.12), we get

$$(2\varepsilon)^{-1} \int_{\{t_1-\varepsilon \leq u \leq t_1+\varepsilon\}} |\mathrm{D}u|^p \,\mathrm{d}\sigma - (2\varepsilon)^{-1} \int_{\{t_2-\varepsilon \leq u \leq t_2+\varepsilon\}} |\mathrm{D}u|^p \,\mathrm{d}\sigma = 0.$$

Since u is Lipschitz, the coarea formula applies and yields

$$(2\varepsilon)^{-1} \int\limits_{t_1-\varepsilon}^{t_1+\varepsilon} \int_{\{u=t\}} |\mathrm{D}u|^{p-1} \,\mathrm{d}\sigma \,\mathrm{d}t - (2\varepsilon)^{-1} \int\limits_{t_2-\varepsilon}^{t_2+\varepsilon} \int_{\{u=t\}} |\mathrm{D}u|^{p-1} \,\mathrm{d}\sigma \,\mathrm{d}t = 0.$$

Letting $\varepsilon \to 0^+$ we obtain (B.11) for almost any $0 < t_1 < t_2 < 1$ by Lebesgue's Differentiation Theorem. Observe that, by the continuity of $|\mathrm{D}u|$, this holds for any regular value t_2, where it suffices to apply the Fundamental Theorem of Calculus. In particular, by combinining by Hopf's Lemma for p-harmonic functions with the $\mathscr{C}^{1,\alpha}$ regularity of u up to the boundary as done above, any value close enough to 1 is regular. Passing to the limit as $t \to 1^-$ we get, again by the Hölder gradient boundary regularity of u, that (B.11) holds for $t_2 = 1$. We then get, again by coarea formula,

$$\mathrm{Cap}_p(\Omega) = \int\limits_{M \setminus \overline{\Omega}} |\mathrm{D}u|^p \,\mathrm{d}\mu = \int\limits_0^1 \int_{\{u=t\}} |\mathrm{D}u|^{p-1} \,\mathrm{d}\sigma \,\mathrm{d}t = \int\limits_{\{u=t\}} |\mathrm{D}u|^{p-1} \,\mathrm{d}\sigma,$$

for almost any $0 < t \leq 1$, that is the second equality in (B.2), for any regular value and in particular for $t = 1$, that is (B.3), completing the proof. □

We conclude with one last remark concerning an even more general situation.

Remark B.13. If (M, g) is just assumed to be p-nonparabolic, without knowing that the p-Green's function vanishes at infinity, the very same argument used in the proof above to show existence of a solution to (3.4.3) yields a p-harmonic function attaining smoothly the value 1 on the smooth boundary of an open set Ω together with a sequence of points x_j with $d(O, x_j) \to +\infty$ as $j \to \infty$ such that $\lim_{j \to +\infty} u(x_j) \to 0^+$ as $j \to +\infty$. Everything works unchanged, except for this last point. It comes from the fact that on any p-nonparabolic manifold $G_p(O, \cdot)$ vanishes at infinity along such a sequence x_j. This immediately follows from the construction of G_p carried out in [Hol90], implying, by the basic barrier argument (B.10), that so does u.

On the other hand, we remark how, in spite of the finiteness of the p-Dirichlet energy of u, at least apparently (B.2) does not follow immediately from the arguments above, since $u - \psi$ does not in general belong to $W^{1,p}_0(M \setminus \overline{\Omega})$ if ψ is a test function vanishing at infinity and assuming unit value on Ω.

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