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Existential completion and pseudo-distributive laws: an algebraic approach to the completion of doctrines

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Introduction

Pseudo-monads and universal algebra

By the mid 1960s the categorical understanding of universal algebra was established: Lawvere theories axiomatised the notion of a clone of an equational theory, [34]. Monads, which had arisen in algebraic topology, had been seen to generalise the notion of Lawvere theory.

Monads typically arise from adjoint pairs of functors; and in such a case, the Eilenberg-Moore [30] and Kleisli [14] categories of algebras for the monad provide adjoint pairs which one can regard as approximations to the original adjoint pair.

It is now very well known that the theory of monads and their algebras extends virtually unchanged from the case of ordinary categories to that of categories enriched over a (symmetric monoidal, locally-small, complete and cocomplete) closed category $\mathcal{V}$, see [3]. The cases of interest to us are those where $\mathcal{V}$ is the Cartesian closed category $\mathbf{Cat}$ of small categories.

In this context one can formalize and give a precise definition of what *structure* and *property* should mean.

The notion of *algebraic extra structure* on a category is somewhat wider than that of *algebra* for a 2-monad on $\mathbf{Cat}$.

A monoidal category is an example of a *category with extra structure of an algebraic kind*, in that it is an algebra for a certain 2-monad T on $\mathbf{Cat}$, and is thus given by its (underlying) category $\mathcal{A}$ together with an *action* $a\colon \mathrm{T}\mathcal{A} \longrightarrow \mathcal{A}$ on $\mathcal{A}$ in the usual strict sense. This action encodes the extra (that is, the monoidal) structure given by the tensor product $\otimes$, the unit object, and the various structure-isomorphisms, subject to Mac Lane's coherence conditions. Of course the category $\mathcal{A}$ may admits many such monoidal structures.

A second example of a category with *algebraic extra structure* is given by a category with finite coproducts.

Here the action $a\colon \mathrm{T}\mathcal{A} \longrightarrow \mathcal{A}$ (for a different 2-monad T) encodes the coproduct structure.

However, in contrast to the first example, the structure is uniquely determined (when it exists) up to appropriate isomorphisms, indeed, to within unique such isomorphisms; so that to give an $\mathcal{A}$ with such a structure is just to give an $\mathcal{A}$ with a certain property, in this case, the property of admitting finite coproducts.

In an example so simple as that of finite coproducts, we know precisely in what sense the structure is *unique to within a unique isomorphism*; but it is not so obvious what such uniqueness should mean in the case of a general 2-monad T on a 2-category $\mathcal{K}$, even in the case where $\mathcal{K}$ is just **Cat**.

In [27], Kelly and Lack provide a useful definition in this general setting (comparing it with possible alternative or stronger forms) and to deduce mathematical consequences of a 2-monad's having this *uniqueness of structure* property, or variants thereof.

To capture such cases, they place themselves in the general context of a *strict 2-monad* (T, μ, η) on a 2-category $\mathcal{K}$.

Using this notion of T-morphism, they express more precisely what it might mean to say that an action of T on A is ***unique to within a unique isomorphism***: it means that, given two actions $a, a' \colon \mathrm{T}A \longrightarrow A$, there is a unique invertible 2-cell $\theta \colon a \Longrightarrow a'$ such that $(\mathrm{id}_A, \theta) \colon (A, a) \longrightarrow (A, a')$ is an isomorphism of T-algebras. For such a T, we may say for short that T-*algebra structure is essentially unique*.

We may say that T-*morphism structure is unique* if, given T-algebras (A, a), (B, b) and a morphism $f \colon A \longrightarrow B$ in $\mathcal{K}$, there exists at most one 2-cell $\overline{f} \colon b\mathrm{T}f \Longrightarrow fa$ such that $(f, \overline{f})$ is a T-morphism.

Accordingly to [27], the 2-monad T is said ***property-like*** when it has both essential uniqueness of algebra structure and uniqueness of morphism structure.

The theory of 2-monad, and more generally of pseudo-monads, allows us not only to give a precise definition of what is property and what is structure, but it provides also a useful instrument to understand how one can combine different pseudo-monads and structures: the *pseudo-distributive laws*.

A ***pseudo-distributive law*** consists of a pseudo-natural transformation

$$\delta \colon \mathrm{ST} \longrightarrow \mathrm{TS}$$

and four invertible modifications satisfying certain coherence conditions, for which we refer to [12, 46, 47, 55, 56].

The existence of a pseudo-distributive law between the pseudo-monads T and S, implies that TS is a pseudo-monad, an that the 2-category **Ps**-TS-**Alg** of pseudo-algebras is equivalent to the 2-category **Ps**-$\widetilde{\mathrm{T}}$-**Alg**, where $\widetilde{\mathrm{T}}$ is the lifting of the pseudo-monad T on the the 2-category **Ps**-S-**Alg**. Again we refer to [54, 55, 56] for a detailed analysis of these topics.

This means that for an object C of $\mathcal{K}$, $\mathrm{TS}(C)$ has both canonical pseudo-T-algebra and pseudo-S-algebra structures on it.

Generalized exact completion

In category theory one can find various notions of completing a category to an exact category initiated by Freyd's exact completion of a regular category [16], and they include also the exact completion of a category with certain weak finite limits, see [6, 10].

In recent works [41, 42, 43, 44], Maietti and Rosolini generalize these exact completions by relativizing the basic data to a doctrine equipped with just the structure sufficient to present the notion of an equivalence relation. In particular, they determined the exact completion of an *elementary existential doctrine* $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ with (weak) full comprehensions and comprehensive diagonals.

They use a weakened notion of Lawvere *hyperdoctrine* [36, 37, 38], called *elementary doctrine*.

The exact completion of an elementary, existential doctrine with full comprehensions and comprehensive diagonals, which are called ***existential m-variational doctrine***, can be obtained in several, but equivalent ways.

The first is noting that an m-variational existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$, rises to a proper, stable factorization system $\langle \mathcal{E}, \mathcal{M} \rangle$ on the base category $\mathcal{C}$, where morphisms of $\mathcal{M}$ are comprehensions, and morphisms of $\mathcal{E}$ are arrows of $\mathcal{C}$ such that $\exists_g(\top_A) = \top_B$ for $g\colon A \longrightarrow B$.

Moreover the doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is equivalent to the doctrine $\mathrm{Sub}_{\mathcal{M}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ of $\mathcal{M}$-subobjects.

This construction can be extended to an equivalence between the 2-category **Ex-mVar** of m-variational existential doctrines, and the 2-category **LFS** whose objects are categories with finite limits together with a proper, stable factorization system.

This equivalence is a translation in terms of doctrines of the work of Hughes and Jacobs [19].

To conclude this construction one apply other two free constructions: the first is the construction of a regular category starting from a category $\mathcal{D}$ with finite limits together with a proper, stable factorization system $\langle \mathcal{E}', \mathcal{M}' \rangle$, introduced by Kelly in [26], and the second is the exact completion of a regular category, see [6].

We can summarize this exact completion for m-variational existential doctrines with the following diagram

$$\mathbf{Ex\text{-}mVar} \xrightarrow{\cong} \mathbf{LFS} \xrightarrow{\mathbf{Map\ Rel}(-)} \mathbf{Reg} \xrightarrow{(-)_{\mathrm{ex/reg}}} \mathbf{Xct}\,.$$

The exact completion of an m-variational existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ computed by the previous completion, is denoted by $(\mathbf{Ef}_P)_{\mathrm{ex/reg}}$.

A second notion of exact completion for doctrines is provided by the *quotient completion* of an elementary doctrine introduced by Maietti and Rosolini in [43] together with the construction of the *category of entire functional relation* associated to an m-variational existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$.

The last instance of exact completion for an m-variational existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is provided by the tripos-to-topos construction $\mathcal{T}_P$, see [20, 51].

Finally, in [42, 43, 44] Maietti and Rosolini show that an arbitrary elementary existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ can be completed to an m-variational existential doctrine $(P)_{cd}\colon \mathcal{X}_{P_c}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$, so what we obtain combining these constructions with the exact completion for m-variational existential doctrines, is that the 2-functor

$$\mathbf{Xct} \longrightarrow \mathbf{EED}$$

which sends an exact category $\mathcal{X}$ to the subobjects doctrine $\mathrm{Sub}_{\mathcal{X}}\colon \mathcal{X}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ (which is elementary and existential since the base category is exact) has a left bi-adjoint.

The aim of this thesis

In the first part of this work we give a complete description in all the details of the previous exact completions for an elementary existential doctrine, and we compare the different instruments which are involved in these constructions: regular and exact categories, factorization systems, fibrations and doctrines.

The main purpose of this thesis is to combine the categorical approach to logic given by the study of doctrines, with the universal algebraic techniques given by the theory of the pseudo-monads and pseudo-distributive laws.

Every completions of doctrines is then formalized by a pseudo-monad, and then combinations of these are studied by the analysis of the pseudo-distributive laws.

The starting point are the works of Maietti and Rosolini [42, 43], in which they describe three completions for elementary doctrines: the first which adds full comprehensions, the second comprehensive diagonals, and the third quotients.

We give an explicit description of the pseudo-functors and the pseudo-adjunctions obtained from these completions, and we start our analysis of the pseudo-monads

$$\mathrm{T}_c, \mathrm{T}_d, \mathrm{T}_q\colon\ \mathbf{ElD} \longrightarrow \mathbf{ElD}$$

where $\mathbf{ElD}$ denotes the 2-category of elementary doctrines.

We prove that all these pseudo-monads are property-like (as pseudo-monads), and the following equivalences of 2-categories hold

$$\mathbf{CE} \equiv \mathbf{Ps}\text{-}\mathrm{T}_{\mathrm{c}}\text{-}\mathbf{Alg}$$

$$\mathbf{CED} \equiv \mathbf{Ps}\text{-}\mathrm{T}_{\mathrm{d}}\text{-}\mathbf{Alg}$$

$$\mathbf{QED} \equiv \mathbf{Ps}\text{-}\mathrm{T}_{\mathrm{q}}\text{-}\mathbf{Alg}$$

where **CE** is the 2-category of elementary doctrines with full comprehensions, **CED** is the 2-category of elementary doctrines with comprehensive diagonals, and **QED** is the 2-category of elementary doctrines with stable quotients.

The inclusion of each of these categories into **ElD** is obviously not full, as morphisms are those that preserves the relevant structures.

Our analysis of pseudo-distributive laws starts from the pseudo-monad T_d. It is proved [42, 43] this free construction preserves comprehensions and quotients, and we use this result to define a lifting $\widetilde{\mathrm{T}_d}$ of T_d on the 2-categories $\mathbf{Ps}\text{-}\mathrm{T_c}\text{-}\mathbf{Alg}$ and $\mathbf{Ps}\text{-}\mathrm{T_q}\text{-}\mathbf{Alg}$.

The existence of these lifting is equivalent to prove that there exist two pseudo-distributive laws $\delta_1\colon \mathrm{T}_c\mathrm{T}_d \longrightarrow \mathrm{T}_d\mathrm{T}_c$ and $\delta_2\colon \mathrm{T}_q\mathrm{T}_d \longrightarrow \mathrm{T}_d\mathrm{T}_q$, and then $\mathrm{T}_d\mathrm{T}_c$ and $\mathrm{T}_d\mathrm{T}_q$ are pseudo-monads.

The third pseudo-distributive law we describe is $\delta_3\colon \mathrm{T}_c\mathrm{T}_q \longrightarrow \mathrm{T}_q\mathrm{T}_c$, which again exists because the quotients completion preserves full comprehensions, and then we can define a lifting $\widetilde{\mathrm{T}_q}$ of T_q on the 2-category $\mathbf{Ps}\text{-}\mathrm{T_c}\text{-}\mathbf{Alg}$.

Finally we prove using the same arguments as before, the existence of two pseudo-distributive laws $\delta_4\colon \mathrm{T}_c\mathrm{T}_d\mathrm{T}_q \longrightarrow \mathrm{T}_d\mathrm{T}_q\mathrm{T}_c$ and $\delta_5\colon \mathrm{T}_q\mathrm{T}_c\mathrm{T}_d \longrightarrow \mathrm{T}_d\mathrm{T}_q\mathrm{T}_c$, and then we conclude that the 2-endofunctor $\mathrm{T}_d\mathrm{T}_q\mathrm{T}_c$ is a pseudo-monad.

In the second work we present a free construction that given a primary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ and a class $\mathcal{A}$ of morphisms of $\mathcal{C}$ closed under pullbacks, compositions and which contains the identities, provides a doctrine $P^{\mathrm{ex}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ which has left adjoint along the morphisms of $\mathcal{A}$, and these satisfy Beck-Chevalley conditions and Frobenius reciprocity.

In particular, if the class $\mathcal{A}$ is the class of the projections of $\mathcal{C}$, then the doctrine $P^{\mathrm{ex}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is existential in the sense of [42, 43, 44].

This construction extends to a 2-functor

$$\mathrm{E}\colon \mathbf{PD} \longrightarrow \mathbf{ExD}$$

from the 2-category **PD** of primary doctrines to the 2-category **ExD** of existential doctrines, and this 2-functor is 2-left-adjoint to the forgetful functor $\mathrm{U}\colon \mathbf{ExD} \longrightarrow \mathbf{PD}$.

Then we consider the 2-monad $\mathrm{T}_e\colon \mathbf{PD} \longrightarrow \mathbf{PD}$, and we prove that it is lax-idempotent, and in particular property-like.

Moreover we have the equivalence of 2-categories

$$\mathbf{ExD} \equiv \mathrm{T_e}\text{-}\mathbf{Alg}\,.$$

The existential completion preserves the elementary structure in the sense that if $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is an elementary doctrine, then $P^{\mathrm{ex}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is an elementary and existential doctrine.

Therefore we can generalize the exact completion for elementary existential doctrines to an arbitrary elementary doctrine, as the following composition

$$\mathbf{ElD} \xrightarrow{(-)^{\mathrm{ex}}} \mathbf{ExD} \xrightarrow{(-)^{cd}} \mathbf{Ex\text{-}mVar} \xrightarrow{\cong} \mathbf{LFS} \xrightarrow{\mathbf{Map\ Rel}(-)} \mathbf{Reg} \xrightarrow{(-)_{\mathrm{ex/reg}}} \mathbf{Xct}\,.$$

We give an explicit description of the exact category $\mathcal{T}_{P^{\mathrm{ex}}}$ constructed by an elementary doctrine $P\colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ combining the existential completion with the tripos-to-topos construction.

The last completion we consider is the elementary completion of a primary doctrine.

In this case we can not apply the general construction defined before because the class of morphisms on which we need to add left adjoints is not closed under pullbacks and compositions.

We can use it, for example, if the base category C of a primary doctrine $P\colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ has finite limits, applying the existential completion with the class $\mathcal{A}$ generated by morphisms of the form $\mathrm{id}_A \times \Delta_B$ for A and B objects of C.

There is another interesting class of categories that we show can be easily completed to a category with finite limits, and these are the categories which are the syntactic category of some first order theory.

Given a first order theory $\mathbb{T}$, the syntactic category $C_{\mathbb{T}}$ has the property that if two morphisms $f,g\colon A \longrightarrow B$ have an arrow

$$\begin{tikzcd} H \arrow[r, "h"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

such that $fh = gh$, then f and g has an equalizer.

This observation follows when we formalize the *unification problem*, [48, 52] in the syntactic category.

Recall that the unification problem in the first order logic can be expressed as follows: given two terms containing the same variables, find, if it exists, the simplest substitution which makes the two terms equal. The resulting substitution is called ***most general unifier***, and it is unique up to variable renaming.

The key point is that if two terms admit an unifier, then there exists a most general one.

We observe that this can be translated in the syntactic category in a direct way, and the result is that if two arrows admit a morphism which equalizes them, then there exists an equalizer.

Therefore given a syntactic category $C_{\mathbb{T}}$, we can complete it to a category with finite limits $C_{\mathbb{T}}^{0}$ just adding an initial objects 0.

So a primary doctrine $P\colon C_{\mathbb{T}}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ can be easily completed to a primary doctrine $P^0\colon (C_{\mathbb{T}}^{0})^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ whose base category has finite limits, and then we can apply the existential completion on the class $\mathcal{A}$ of arrows generated by morphisms of the form $\mathrm{id}_A \times \Delta_B$.

Finally we give a complete description of the elementary completion of a primary doctrine $P\colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ whose base category is discrete with free products in all the details. From a logical point of view we are looking for a first order theory in a language in which no function symbols are considered.

In this case the description of the new elementary doctrine constructed from P can be simplified, and it is more natural from a logical point of view.

We conclude with the analysis of the 2-monad T_{el} coming from this construction, proving that it is fully property-like.

Moreover we have the following equivalence of 2-categories

$$\mathrm{T}_{\mathrm{el}}\text{-}\mathbf{Alg} \equiv \mathbf{PdD}$$

where **PdD** is the 2-category of primary doctrines whose base category is discrete with free products.

Since the existential completion does not change the base category and preserves the elementary structure, there exists a pseudo-distributive law $\delta\colon \mathrm{T}_{\mathrm{el}}\mathrm{T}_{\mathrm{ex}} \longrightarrow \mathrm{T}_{\mathrm{ex}}\mathrm{T}_{\mathrm{el}}$ by the same argument used before.

Contents of chapters

In chapter 1 we introduce the notions of monad and distributive law, and their generalization as pseudo-monads on a 2-category and pseudo-distributive law.

We introduce also the notion of property-like 2-monads, explaining how these kind of 2-monads are able to capture the differences between what is *structure* and what is *property*.

This will be also useful to pass from the ordinary case of a monad to the pseudo-setting.

I chapter 2 we present a classical categorical approach to logic, using *regular* and *exact* categories. We recall some known facts about the categorical semantic which will be useful in the following chapters to understand the meaning from a logical point of view of what is a doctrine and what the 1-cells and 2-cells of the 2-category **PD** mean.

Moreover the definition of *stable, proper factorization system* is recalled and we present two free constructions which are used later: the exact completion of a regular category, and the regular completion of a category with a stable, proper factorization system.

Chapter 3 is devoted to the introduction of other two categorical instruments, which are fibrations and doctrines.

In the first part of this chapter we compare them, showing to what kind of fibration an existential m-variational doctrine is equivalent. We will see also that the 2-category **Ex-mVar** is equivalent to the 2-category of **LFS**, whose objects are categories together with a proper, stable factorization system. This result is suggested by the work [19], in which Hughes and Jacobs prove a similar result for what they call factorization fibrations.

In the last part we introduce the quotients completion of an elementary doctrine, and we present the exact completion for elementary existential doctrine in all the details, comparing some different ways to define an exact category starting from an elementary existential doctrine.

The lasts three chapters of this thesis are composed by three works developed during my doctorate, which are under submission.

In the first we construct and characterize three pseudo-monads obtained by the completions with quotients, comprehensive diagonals and full comprehensions.

Moreover a complete study of their pseudo-distributive laws is given.

In the second work we develop the existential completion for primary doctrines, showing that this is a free construction and that it can be applied in a more general context.

Again we study the 2-monad which comes from this completion, showing that it is lax-idempotent and proving the equivalence $\mathbf{ExD} \equiv \mathrm{T}_e\text{-}\mathbf{Alg}$.

Moreover we show that the existential completion preserves the elementary structure of a doctrine, and this allows us to generalize the exact completion to an arbitrary elementary doctrine.

In the last work we give a categorical interpretation of the problem of unification in the context of a syntactic category.

In particular we show that using a known result which state that if there exists an unifier, then there exists a most general one, we can easily complete a syntactic category to a category with finite limits.

Then, for these categories, we can apply the general existential completion to obtain an elementary doctrine, and we conclude this work with a detailed description of the elementary completion for a primary doctrine whose base category is discrete with free products. Again we obtain a 2-monad which is lax-idempotent.

All the references are given at the beginning of every sections.

Contents

Chapter 1
Preliminaries

This chapter contains definitions of category theoretic terms used in this thesis.

In particular we fix the notation and we introduce the notions monads and their algebras, 2-categories, pseudo-functors, pseudo-monads.

We refer to [5, 40, 45] for a general introduction to the theory of monads, and to [33, 54] for more details.

We conclude with the definition of pseudo-distributive laws for pseudo-monads introduced by Beck in [1] and with some useful results which will be fundamental later. We follow the notation used by Tanaka and Power in [54, 55, 56].

For a more detailed discussion and analysis of the coherence axioms for pseudo-distributive laws we refer to the works of Marmolejo [46, 47], while for a more standard introduction to the theory of 2-monads and pseudo-monads there are several works as [12, 53, 28, 3].

1.1 Monads and their algebras

Recall that a ***closure operation*** on a preoredered set $\mathcal{A} = (|\mathcal{A}|, \leq)$ is a mapping $\mathrm{T}\colon |\mathcal{A}| \longrightarrow |\mathcal{A}|$ with the following properties

1. if $A \leq B$ then $\mathrm{T}(A) \leq \mathrm{T}(B)$;
2. $A \leq \mathrm{T}(A)$;
3. $\mathrm{T}^2(A) \leq \mathrm{T}(A)$;

for all the elements A and B of $|\mathcal{A}|$.

This notion has been generalized from preordered sets to arbitrary categories and is then called a ***monad***.

Definition 1.1.1. A ***monad*** on a category $\mathcal{A}$ is a triple (T, μ, η) consisting of a functor $\mathrm{T}\colon \mathcal{A} \longrightarrow \mathcal{A}$, and two natural transformations, the ***multiplication*** $\mu\colon T^2 \longrightarrow \mathrm{T}$ and the ***unit*** $\eta\colon \mathrm{id}_{\mathcal{A}} \longrightarrow \mathrm{T}$ such that the following diagrams, one for the associativity of μ and another for the left and right unity of η, commute:

$$
\begin{tikzcd}
\mathrm{T}^3 \arrow[r, "\mathrm{T}\mu"] \arrow[d, "\mu\mathrm{T}"'] & \mathrm{T}^2 \arrow[d, "\mu"] \\
\mathrm{T}^2 \arrow[r, "\mu"'] & \mathrm{T}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{T} \arrow[r, "\eta\mathrm{T}"] \arrow[dr, "\mathrm{id}_T"'] & \mathrm{T}^2 \arrow[d, "\mu"] & \mathrm{T} \arrow[l, "\mathrm{T}\eta"'] \arrow[dl, "\mathrm{id}_T"] \\
& \mathrm{T} &
\end{tikzcd}
$$

Definition 1.1.2. Given a monad T on a category $\mathcal{A}$, a T-***algebra*** is a pair (A,α), where A is an object of $\mathcal{A}$ and α is a morphism $\alpha\colon \mathrm{T}A \longrightarrow A$ called the ***structure map***, such that the following diagrams commute:

$$
\begin{tikzcd}
\mathrm{T}^2A \arrow[r, "\mathrm{T}a"] \arrow[d, "\mu_A"'] & \mathrm{T}A \arrow[d, "a"] \\
\mathrm{T}A \arrow[r, "a"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[dr, "1_A"'] & \mathrm{T}A \arrow[d, "a"] \\
& A
\end{tikzcd}
$$

A T-***morphism*** $f\colon (A,\alpha) \longrightarrow (B,\beta)$ of T-algebras is a morphism $f\colon A \longrightarrow B$ of $\mathcal{A}$ such that the following diagram commutes:

$$
\begin{tikzcd}
TA \arrow[r, "Tf"] \arrow[d, "\alpha"'] & TB \arrow[d, "\beta"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
$$

Definition 1.1.3. The category whose objects are T-algebras and whose morphisms are T-morphisms is denoted by T-**Alg** or $\mathcal{A}^{\mathrm{T}}$, and it is called ***Eilenberg-Moore category***.

Proposition 1.1.4. *Any adjunction*

$$
(\mathrm{F},\mathrm{G},\eta,\varepsilon)\colon\ \mathcal{A} \underset{\mathrm{G}}{\overset{\mathrm{F}}{\rightleftarrows}} \mathcal{B}
$$

induces a monad $(\mathrm{GF},\mathrm{G}\varepsilon\mathrm{F},\eta)$.

Proof. It is a direct verification. See [5, Proposition 4.2.1]. □

There arises the question to which extent every monad is induced by an adjunction.

The answer will be positive but in most cases there is not a unique such adjunction even up to isomorphism. We will show that there is a minimal and a maximal solution to this problem.

Definition 1.1.5. Let (T,μ,η) be a monad on $\mathcal{A}$. A ***resolution*** $(\mathcal{B},\mathrm{F},\mathrm{G},\varepsilon)$ of this monad consists of a category $\mathcal{B}$ a pair of adjoint functors $\mathrm{F}\colon \mathcal{A} \longrightarrow \mathcal{B}$ and $\mathrm{G}\colon \mathcal{B} \longrightarrow \mathcal{A}$ such that $\mathrm{GF} = \mathrm{T}$, the unit of the adjunction is η and $\mu = \mathrm{G}\varepsilon\mathrm{F}$.

The resolutions of a given monad form a category whose morphisms

$$\Phi\colon (\mathcal{B},\mathrm{F},\mathrm{G},\varepsilon) \longrightarrow (\mathcal{B}',\mathrm{F}',\mathrm{G}',\varepsilon)$$

are functor $\Phi\colon \mathcal{B} \longrightarrow \mathcal{B}'$ such that the diagram

$$\begin{tikzcd}
 & \mathcal{B} \arrow[dr, "\mathrm{G}"] \arrow[dd, "\Phi"] & \\
\mathcal{A} \arrow[ur, "\mathrm{F}"] \arrow[dr, "\mathrm{F}'"'] & & \mathcal{A} \\
 & \mathcal{B}' \arrow[ur, "\mathrm{G}'"'] &
\end{tikzcd}$$

commutes and $\Phi\varepsilon = \varepsilon'\Phi$.

Proposition 1.1.6. *The Eilenberg-Moore category* $\mathcal{A}^{\mathrm{T}}$ *of a monad* (T,μ,η) *on* $\mathcal{A}$ *gives rise to a resolution* $(\mathcal{A}^{\mathrm{T}},\mathrm{F}^{\mathrm{T}},\mathrm{G}^{\mathrm{T}},\varepsilon^{\mathrm{T}})$ *which is a terminal object in the category of all resolutions. Thus, given a resolution* $(\mathcal{B},\mathrm{F},\mathrm{G},\varepsilon)$ *there is a unique functor* $\mathrm{K}^{\mathrm{T}}\colon \mathcal{B} \longrightarrow \mathcal{A}^{\mathrm{T}}$ *, called the* **comparison functor***, such that* $\mathrm{K}^{\mathrm{T}}\mathrm{F} = \mathrm{F}^{\mathrm{T}}$, $\mathrm{G}^{\mathrm{T}}\mathrm{K}^{\mathrm{T}} = \mathrm{G}$ *and* $\mathrm{K}^{\mathrm{T}}\varepsilon = \varepsilon^{\mathrm{T}}\mathrm{K}^{\mathrm{T}}$.

Proof. We define the resolution $(\mathcal{A}^{\mathrm{T}},\mathrm{F}^{\mathrm{T}},\mathrm{G}^{\mathrm{T}},\varepsilon^{\mathrm{T}})$ as follows:

1. we define the functor $\mathrm{G}^{\mathrm{T}}\colon \mathcal{A}^{\mathrm{T}} \longrightarrow \mathcal{A}$ by

$$\mathrm{G}^{\mathrm{T}}(A,\alpha) = A, \quad \mathrm{G}^{\mathrm{T}}g = g$$

for every T-algebra (A,α) and for every T-morphism $g\colon (A,\alpha) \longrightarrow (B,\beta)$;

2. we define the functor $\mathrm{F}^{\mathrm{T}}\colon \mathcal{A} \longrightarrow \mathcal{A}^{\mathrm{T}}$ by

$$\mathrm{F}^{\mathrm{T}}A = (\mathrm{T}A,\mu_A), \quad \mathrm{F}^{\mathrm{T}}g = \mathrm{T}g$$

for every object A of $\mathcal{A}$ and every morphism $g\colon A \longrightarrow B$. It is easy to check that $\mathrm{G}^{\mathrm{T}}\mathrm{F}^{\mathrm{T}} = \mathrm{T}$;

3. the natural transformation $\varepsilon^{\mathrm{T}}\colon \mathrm{F}^{\mathrm{T}}\mathrm{G}^{\mathrm{T}} \longrightarrow \mathrm{id}_{\mathcal{A}^{\mathrm{T}}}$ is defined by components as follows: let (A,α) be an object of $\mathcal{A}^{\mathrm{T}}$, we define $\varepsilon^{\mathrm{T}}_{(A,\alpha)} = \alpha$. Since (A,α) is a T-algebra the following diagram commutes:

$$\begin{tikzcd}
\mathrm{T}^2A \arrow[r, "\mathrm{T}\alpha"] \arrow[d, "\mu_A"'] & \mathrm{T}A \arrow[d, "\alpha"] \\
\mathrm{T}A \arrow[r, "\alpha"'] & A.
\end{tikzcd}$$

Therefore $\varepsilon^{\mathrm{T}}_{(A,\alpha)}$ is a T-morphism, and it is easy to check that it extends to a natural transformation. Moreover for every object A of $\mathcal{A}$ we have

$$\mathrm{G}^{\mathrm{T}}\varepsilon^{\mathrm{T}}_{(\mathrm{T}A,\mu_A)} = \mathrm{G}^{\mathrm{T}}\mu_A = \mu_A$$

thus $\mathrm{G}^{\mathrm{T}}\varepsilon^{\mathrm{T}}\mathrm{F} = \mu$.

4. It is direct to prove the functor F^{T} is left adjoint to G^{T} with unit η and counit ε^{T}, because
$$\varepsilon^{\mathrm{T}}_{(\mathrm{T}A,\mu_A)}\mathrm{T}\eta_A = \mu_A\mathrm{T}\eta_A = \mathrm{id}_{\mathrm{T}A}$$
and similarly for the other triangle equality.

We have proved that $(\mathcal{A}^{\mathrm{T}}, \mathrm{F}^{\mathrm{T}}, \mathrm{G}^{\mathrm{T}}, \varepsilon^{\mathrm{T}})$ is a resolution. Now we prove that it is a terminal object. Consider another resolution $(\mathcal{B}, \mathrm{F}, \mathrm{G}, \varepsilon)$ of the monad (T, μ, η). For every object A of $\mathcal{A}^{\mathrm{T}}$ and every morphism $g\colon A \longrightarrow B$ of $\mathcal{A}^{\mathrm{T}}$ we define

$$\mathrm{K}^{\mathrm{T}}A = (\mathrm{G}A, \mathrm{G}\varepsilon_A), \quad \mathrm{K}^{\mathrm{T}}(g) = \mathrm{G}(g).$$

Then we have
$$\mathrm{G}^{\mathrm{T}}\mathrm{K}^{\mathrm{T}}A = \mathrm{G}A$$
and
$$\mathrm{K}^{\mathrm{T}}\mathrm{F}A = (\mathrm{GF}A, G\varepsilon_{\mathrm{F}A}) = (\mathrm{T}A, \mu_A) = \mathrm{F}^{\mathrm{T}}A$$
Moreover for every object A of $\mathcal{B}$ we have

$$\varepsilon^{\mathrm{T}}_{\mathrm{K}^{\mathrm{T}}A} = \varepsilon^{\mathrm{T}}_{(\mathrm{G}A,\mathrm{G}\varepsilon_A)} = \mathrm{G}\varepsilon_A$$

and
$$\mathrm{K}^{\mathrm{T}}\varepsilon_A = \mathrm{G}\varepsilon_A.$$
Therefore we can conclude that $\mathrm{K}^{\mathrm{T}}\varepsilon = \varepsilon^{\mathrm{T}}\mathrm{K}^{\mathrm{T}}$. □

Definition 1.1.7. The ***Kleisli category*** $\mathcal{A}_{\mathrm{T}}$ of a monad (T, μ, η) on a category $\mathcal{A}$ is defined as follows: the object of $\mathcal{A}_{\mathrm{T}}$ are the same as those of $\mathcal{A}$, and for every A and B in $\mathcal{A}_{\mathrm{T}}$ we define $\mathcal{A}_{\mathrm{T}}(A, B) = \mathcal{A}(A, \mathrm{T}B)$. To define the composition we consider $f \in \mathcal{A}_{\mathrm{T}}(A, B)$ and $g \in \mathcal{A}_{\mathrm{T}}(B, C)$. Then the composition $g * f$ is defined as

$$g * f = \mu_C\mathrm{T}gf.$$

In particular we can observe that

$$\eta_B * f = \mu_B \mathrm{T}\eta_B f = f$$

and

$$f * \eta_A = \mu_B \mathrm{T} f \eta_A = \mu_B \mathrm{T}\eta_B f = f.$$

Therefore $\eta_A : A \longrightarrow \mathrm{T}A$ is the identity of $\mathcal{A}_\mathrm{T}(A, A)$.

Proposition 1.1.8. *The Kleisli category $\mathcal{A}_\mathrm{T}$ of a monad (T, μ, η) on $\mathcal{A}$ gives rise to a resolution $(\mathcal{A}_\mathrm{T}, \mathrm{F}_\mathrm{T}, \mathrm{G}_\mathrm{T}, \varepsilon_\mathrm{T})$ which is an initial object in the category of all resolutions. Thus, given a resolution $(\mathcal{B}, \mathrm{F}, \mathrm{G}, \varepsilon)$ there is a unique functor $\mathrm{K}_\mathrm{T} : \mathcal{A}_\mathrm{T} \longrightarrow \mathcal{B}$, such that $\mathrm{K}_\mathrm{T}\mathrm{F}_\mathrm{T} = \mathrm{F}$, $\mathrm{G}\mathrm{K}_\mathrm{T} = \mathrm{G}_\mathrm{T}$ and $\mathrm{K}_\mathrm{T}\varepsilon_\mathrm{T} = \varepsilon\mathrm{K}_\mathrm{T}$.*

Proof. We define the resolution $(\mathcal{A}_\mathrm{T}, \mathrm{F}_\mathrm{T}, \mathrm{G}_\mathrm{T}, \varepsilon_\mathrm{T})$ as follows:

1. we define the functor $\mathrm{G}_\mathrm{T} : \mathcal{A}_\mathrm{T} \longrightarrow \mathcal{A}$ by:

   $$\mathrm{G}_\mathrm{T} A = \mathrm{T}A, \quad \mathrm{G}_\mathrm{T} g = \mu_B \mathrm{T} g$$

   for every object A of $\mathcal{A}_\mathrm{T}$ and every morphism $g : A \longrightarrow B$ of $\mathcal{A}_\mathrm{T}$;
2. we define the functor $\mathrm{F}_\mathrm{T} : \mathcal{A} \longrightarrow \mathcal{A}_\mathrm{T}$ by:

   $$\mathrm{F}_\mathrm{T} A = A, \quad \mathrm{F}_\mathrm{T} g = \eta_B \mathrm{T} g$$

   for every object A of $\mathcal{A}$ and every morphism $g : A \longrightarrow B$ of $\mathcal{A}$. It is easy to check that $\mathrm{G}_\mathrm{T}\mathrm{F}_\mathrm{T} = \mathrm{T}$.
3. We define the natural transformation $\varepsilon_\mathrm{T} : \mathrm{G}_\mathrm{T}\mathrm{F}_\mathrm{T} \longrightarrow \mathrm{id}_{\mathcal{A}_\mathrm{T}}$ by putting $\varepsilon_{\mathrm{T}A} = \mathrm{id}_{\mathrm{T}A}$ in $\mathcal{A}$. Moreover we have

   $$(\mathrm{G}_\mathrm{T}\varepsilon_\mathrm{T}\mathrm{F}_\mathrm{T})_A = \mathrm{G}_\mathrm{T}(\varepsilon_{\mathrm{T}A}) = \mathrm{G}_\mathrm{T}(\mathrm{id}_{\mathrm{T}A}) = \mu_A$$

   thus $\mathrm{G}_\mathrm{T}\varepsilon_\mathrm{T}\mathrm{F}_\mathrm{T} = \mu$.
4. As in the case of Proposition 1.1.6 it is direct to prove the functor F_T is left adjoint to G_T with unit η and counit ε_T

We have proved that $(\mathcal{A}_\mathrm{T}, \mathrm{F}_\mathrm{T}, \mathrm{G}_\mathrm{T}, \varepsilon_\mathrm{T})$ is a resolution. Now we prove that it is an initial object. Consider another resolution $(\mathcal{B}, \mathrm{F}, \mathrm{G}, \varepsilon)$ of the monad (T, μ, η). For every object A of $\mathcal{A}_\mathrm{T}$ and every morphism $g : A \longrightarrow B$ of $\mathcal{A}_\mathrm{T}$ we define

$$\mathrm{K}_\mathrm{T} A = \mathrm{F}A, \quad \mathrm{K}_\mathrm{T}(g) = \varepsilon_{\mathrm{F}B}\mathrm{F}g.$$

Then we have

$$\mathrm{G}\mathrm{K}_\mathrm{T} A = \mathrm{G}\mathrm{F}A = \mathrm{T}A = \mathrm{G}_\mathrm{T} A$$

and

$$\mathrm{G}\mathrm{K}_\mathrm{T} g = \mathrm{G}(\varepsilon_{\mathrm{F}B}\mathrm{F}g) = (\mathrm{G}\varepsilon\mathrm{F})_B \mathrm{G}\mathrm{F}g = \mu_B \mathrm{T}(g) = \mathrm{G}_\mathrm{T} g$$

thus $\mathrm{G}\mathrm{K}_\mathrm{T} = \mathrm{G}_\mathrm{T}$. Moreover for every A of $\mathcal{A}$ and $g : A \longrightarrow B$ of $\mathcal{A}$ we have

$$\mathrm{K}_\mathrm{T}\mathrm{F}_\mathrm{T} A = \mathrm{F}A$$

and

$$\mathrm{K_T F_T} g = \varepsilon_{\mathrm{F}B} \mathrm{F} \eta_B \mathrm{F} f = \mathrm{F} f$$

thus $\mathrm{F_T K_T} = \mathrm{F}$. Moreover we have for every A of $\mathcal{A}_\mathrm{T}$

$$(\mathrm{K_T}\varepsilon_\mathrm{T})_A = \mathrm{K_T}\,\mathrm{id}_{\mathrm{T}A} = \varepsilon_{\mathrm{F}A} = (\varepsilon \mathrm{K_T})_A.$$

This completes the proof. □

Corollary 1.1.9. *The comparison functor* $\mathrm{K}^\mathrm{T} : \mathcal{A}_\mathrm{T} \longrightarrow \mathcal{A}^\mathrm{T}$ *is full and faithful.*

Proof. It is easy to see that the comparison functor in this case is full. We prove that it is faithful. Let $g : A \longrightarrow \mathrm{T}B$ be a morphism of $\mathcal{A}$. By definition of K^T we have

$$\mathrm{K}^\mathrm{T} g = \mathrm{G_T} g = \mu_B \mathrm{T} g$$

hence

$$g = \mu_B \eta_{\mathrm{T}B} g = \mu_B \mathrm{T} g \eta_A = \mathrm{K}^\mathrm{T}(g)\eta_A.$$

Therefore it is faithful. □

Remark 1.1.10. The comparison functor $\mathrm{K}^\mathrm{T} : \mathcal{A}_\mathrm{T} \longrightarrow \mathcal{A}^\mathrm{T}$ sends an object A of $\mathcal{A}_\mathrm{T}$ in the free algebra $(\mathrm{G_T}A, (\mathrm{G_T}\varepsilon_\mathrm{T})_A) = (\mathrm{T}A, \mu_A)$.

Corollary 1.1.11. *The Kleisli category of a monad* (T, μ, η) *is equivalent to the full subcategory of the Eilenberg-Moore category consisting of all the free algebras.*

Proof. It is a direct consequence of Corollary 1.1.9 and Remark 1.1.10. □

Definition 1.1.12. Let $(\mathrm{S}, \mu^S, \eta^S)$ and $(\mathrm{T}, \mu^T, \eta^T)$ be two monads on a category $\mathcal{A}$. A ***lifting*** of T on S-**Alg** we mean a monad $\widetilde{\mathrm{T}}$ on the category S-**Alg** such that

$$\mathrm{T}\mathrm{G}^S = \mathrm{G}^S \widetilde{\mathrm{T}}$$

Definition 1.1.13. A ***distributive law*** from a monad S over a monad T is a natural transformation

$$\delta : \mathrm{ST} \longrightarrow \mathrm{TS}$$

such that the following diagrams commute

$$
\begin{tikzcd}
\mathrm{S}^2\mathrm{T} \arrow[r, "\mathrm{S}\delta"] \arrow[d, "\mu^S\mathrm{T}"'] & \mathrm{STS} \arrow[r, "\delta\mathrm{S}"] & \mathrm{TS}^2 \arrow[d, "\mathrm{T}\mu^S"] \\
\mathrm{ST} \arrow[rr, "\delta"'] & & \mathrm{TS}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{T} \arrow[d, "\eta^S\mathrm{T}"'] \arrow[dr, "\mathrm{T}\eta^S"] & \\
\mathrm{ST} \arrow[r, "\delta"'] & \mathrm{TS}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathrm{ST}^2 \arrow[r, "\delta\mathrm{T}"] \arrow[d, "\mathrm{S}\mu^T"'] & \mathrm{TST} \arrow[r, "\mathrm{T}\delta"] & \mathrm{T}^2\mathrm{S} \arrow[d, "\mu^T\mathrm{S}"] \\
\mathrm{ST} \arrow[rr, "\delta"'] & & \mathrm{TS}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{S} \arrow[d, "\mathrm{S}\eta^T"'] \arrow[dr, "\eta^T\mathrm{S}"] & \\
\mathrm{ST} \arrow[r, "\delta"'] & \mathrm{TS}.
\end{tikzcd}
$$

Theorem 1.1.14. *To give a distributive law* $\delta\colon \mathrm{ST} \longrightarrow \mathrm{TS}$ *is equivalent to give a lifting* $\widetilde{\mathrm{T}}$ *of* T *on* S-**Alg**.

Proof. Given $\delta\colon \mathrm{ST} \longrightarrow \mathrm{TS}$ and a S-algebra (A, a), we define

$$\widetilde{\mathrm{T}}(A, a) = (\mathrm{T}A, \mathrm{T}a \circ \delta_A).$$

We show that $(\mathrm{T}A, \mathrm{T}a \circ \delta_A)$ is a S-algebra, which means that the diagram

$$
\begin{tikzcd}
\mathrm{S}^2\mathrm{T}A \arrow[r, "\mathrm{S}(\mathrm{T}a\circ\delta_A)"] \arrow[d, "\mu^S{}_{\mathrm{T}A}"'] & \mathrm{ST}A \arrow[d, "\mathrm{T}a\circ\delta_A"] \\
\mathrm{ST}A \arrow[r, "\mathrm{T}a\circ\delta_A"'] & \mathrm{T}A
\end{tikzcd}
$$

must commute. By naturality of δ we have

$$\delta_A \circ \mathrm{ST}a = \mathrm{TS}a \circ \delta_{\mathrm{S}A}.$$

Then

$$(\mathrm{T}a \circ \delta_A) \circ (\mathrm{ST}a \circ \mathrm{S}\delta_A) = \mathrm{T}a \circ \mathrm{TS}a \circ \delta_{\mathrm{S}A} \circ \mathrm{S}\delta_A = \mathrm{T}(a \circ \mathrm{S}a) \circ \delta_{\mathrm{S}A} \circ \mathrm{S}\delta_A.$$

By hypothesis $a \circ \mathrm{S}a = a \circ \mu^S$, then

$$(\mathrm{T}a \circ \delta_A) \circ (\mathrm{ST}a \circ \mathrm{S}\delta_A) = \mathrm{T}a \circ \mathrm{T}\mu^S \circ \delta_{\mathrm{S}A} \circ \mathrm{S}\delta_A$$

and since δ is a distributive law then

$$(\mathrm{T}a \circ \delta_A) \circ (\mathrm{ST}a \circ \mathrm{S}\delta_A) = \mathrm{T}a \circ \delta_A \circ \mu^S_{\mathrm{T}A}.$$

Let $f\colon (A,a) \longrightarrow (B,b)$ be a morphism of S-algebras. We define $\widetilde{\mathrm{T}}f = \mathrm{T}f$, and it is direct to show that it is a morphism of S-algebras

$$\mathrm{T}f \circ (\mathrm{T}a \circ \delta_A) = \mathrm{T}b \circ \mathrm{TS}f \circ \delta_A = \mathrm{T}b \circ \delta_B \circ \mathrm{ST}f.$$

it is routine to extend the multiplication and unit of T to its lifting $\widetilde{\mathrm{T}}$.

For the converse construction we apply $\widetilde{\mathrm{T}}$ to the S-algebra $(\mathrm{S}A, \mu^S_A)$, and this yields a morphism $\mathrm{T}\mu^S_A\colon \mathrm{STS}A \longrightarrow \mathrm{TS}A$. We define δ_A as the composition

$$\mathrm{ST}A \xrightarrow{\mathrm{ST}\eta^S_A} \mathrm{STS}A \xrightarrow{\mathrm{T}\mu^S_A} \mathrm{TS}A.$$

It is further routine to verify that it satisfies the axioms and that these construction are mutually inverse. For all the detail we refer to [54, 55]. □

Given a monad S, all the lifting of a monad T on $\mathscr{A}$ to S-**Alg**, form a category denoted by $\mathbf{Lift}_{\text{S-}\mathbf{Alg}}$, and all the distributive laws over S form a category denoted by $\mathbf{Dist}_\mathrm{S}$. In particular Theorem 1.1.16 can be extended to an isomorphism between the category $\mathbf{Dist}_\mathrm{S}$ and $\mathbf{Lift}_{\text{S-}\mathbf{Alg}}$. We refer to [54] and [55] for all the detail.

Theorem 1.1.15. *The category* $\mathbf{Dist}_\mathrm{S}$ *and* $\mathbf{Lift}_{\text{S-}\mathbf{Alg}}$ *are isomorphic.*

Proof. See [54, Theorem 3.19 and Corollary 3.29]. □

This theorem can be extended in the context of pseudo-monad, but there we do not have an isomorphism of 2-categories, but only and equivalence.

We conclude this section with a central result about the theory of monads. It is known that in general, given two monad T and S on the same category $\mathscr{A}$, the composition TS is not a monad.

However if there exists a distributive law $\delta\colon \mathrm{ST} \longrightarrow \mathrm{TS}$ then one can prove that the composition TS is again a monad and its category of algebras is isomorphic to the category $\widetilde{\mathrm{T}}$-**Alg**.

Theorem 1.1.16. *Let* $\delta\colon \mathrm{ST} \longrightarrow \mathrm{TS}$ *be a distributive law. Then*

1. *the functor* TS *acquires the structure of monad, with multiplication given by*

$$\mathrm{TSTS} \xrightarrow{\mathrm{T}\delta\mathrm{S}} \mathrm{T}^2\mathrm{S}^2 \xrightarrow{\mu^T\mathrm{S}^2} \mathrm{TS}^2 \xrightarrow{\mathrm{T}\mu^S} \mathrm{TS}$$

2. TS-**Alg** *is canonically isomorphic to* $\widetilde{\mathrm{T}}$-**Alg**.

Proof. The proof is a direct verification. We refer to [54, 55] for all the detail. □

1.2 2-Categories

In this section we recall some definitions about 2-category theory and we fix the notation for the rest of this work.

There are several other equivalent way to introduce the notion 2-category, and the more natural and elegant is given using enrichment, and we refer to [25, 28, 32, 39].

Definition 1.2.1. A ***2-category*** $\mathcal{A}$ consists of the following data:

- a class $\mathcal{A}_0$ of objects, called ***0-cells***;
- for each pair of 0-cells A and B, a category $\mathcal{A}(A,B)$, whose objects are called ***1-cells*** of $\mathcal{A}$ and whose morphisms are called ***2-cells*** of $\mathcal{A}$;
- for each triple of 0-cells A, B and C a functor:

$$\mathrm{c}_{A,B,C}\colon \mathcal{A}(B,C)\times\mathcal{A}(A,B) \longrightarrow \mathcal{A}(A,C)$$

  called ***composition functor***;
- for each 0-cell A of $\mathcal{A}$ a functor:

$$\mathrm{u}_A\colon \mathcal{I} \longrightarrow \mathcal{A}(A,A)$$

  called ***unit functor***. The category $\mathcal{I}$ denotes the category with one object and one morphism.

These data are required to satisfy the following axioms:

$$\begin{tikzcd}
\mathcal{A}(C,D)\times\mathcal{A}(B,C)\times\mathcal{A}(A,B) \arrow[r, "\mathrm{c}_{B,C,D}\times\mathrm{id}_{\mathcal{A}(A,B)}"] \arrow[d, "\mathrm{id}_{\mathcal{A}(C,D)}\times\mathrm{c}_{A,B,C}"'] & \mathcal{A}(B,D)\times\mathcal{A}(A,B) \arrow[d, "\mathrm{c}_{A,B,D}"] \\
\mathcal{A}(C,D)\times\mathcal{A}(A,C) \arrow[r, "\mathrm{c}_{A,C,D}"'] & \mathcal{A}(A,D)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{A}(A,B)\times\mathcal{I} \arrow[r] \arrow[d, "\mathrm{u}_A\times\mathrm{id}_{\mathcal{A}(A,B)}"'] & \mathcal{A}(A,B) \arrow[d, "\mathrm{id}_{\mathcal{A}(A,B)}"] & \mathcal{I}\times\mathcal{A}(A,B) \arrow[l] \arrow[d, "\mathrm{u}_B\times\mathrm{id}_{\mathcal{A}(A,B)}"] \\
\mathcal{A}(A,B)\times\mathcal{A}(A,A) \arrow[r, "\mathrm{c}_{A,A,B}"'] & \mathcal{A}(A,B) & \mathcal{A}(B,B)\times\mathcal{A}(A,B). \arrow[l, "\mathrm{c}_{A,B,B}"]
\end{tikzcd}$$

The fact that 1-cells are defined as objects of a category and 2-cells as arrows implies the associativity and the unit law for the vertical composition of 2-cells, and the two previous diagrams imply the associativity and the unit law for both the horizontal composition of 2-cells and the composition of 1-cells.

Definition 1.2.2. Let $\mathcal{A}$ and $\mathcal{B}$ be 2-categories. A ***2-functor*** F from $\mathcal{A}$ to $\mathcal{B}$ consists of:

- for every 0-cell A of $\mathcal{A}$ a 0-cell $\mathrm{F}A$ of $\mathcal{B}$;
- for each pair A and B of 0-cells of $\mathcal{A}$ a functor $\mathrm{F}_{A,B} \colon \mathcal{A}(A,B) \longrightarrow \mathcal{B}(\mathrm{F}A,\mathrm{F}B)$

subject to the commutativity of the following diagrams:

$$\begin{tikzcd}
\mathcal{A}(B,C) \times \mathcal{A}(A,B) \arrow[r, "\mathrm{F}_{B,C} \times \mathrm{F}_{A,B}"] \arrow[d, "\mathrm{c}_{A,B,C}"'] & \mathcal{B}(\mathrm{F}B,\mathrm{F}C) \times \mathcal{B}(\mathrm{F}B,\mathrm{F}A) \arrow[d, "\mathrm{c}_{\mathrm{F}A,\mathrm{F}B,\mathrm{F}C}"] \\
\mathcal{A}(A,C) \arrow[r, "\mathrm{F}_{A,C}"'] & \mathcal{B}(\mathrm{F}A,\mathrm{F},C)
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & \mathcal{A}(A,A) \arrow[d, "\mathrm{F}_{A,A}"] \\
\mathcal{I} \arrow[ur, "\mathrm{u}_A"] \arrow[r, "\mathrm{u}_{\mathrm{F}A}"'] & \mathcal{B}(\mathrm{F}A,\mathrm{F}A)
\end{tikzcd}$$

Definition 1.2.3. Let F and G be 2-functors between 2-categories $\mathcal{A}$ and $\mathcal{B}$. A ***2-natural transformation*** α from F to G consists of a collection of 1-cells of $\mathcal{B}$ indexed by 0-cells of $\mathcal{A}$, such that for every component $\alpha_A \colon \mathrm{F}A \longrightarrow \mathrm{G}A$ at the 0-cell A the following diagram commutes:

$$\begin{tikzcd}
\mathcal{A}(A,B) \arrow[r, "\mathrm{F}_{A,B}"] \arrow[d, "\mathrm{G}_{A,B}"'] & \mathcal{B}(\mathrm{F}A,\mathrm{F}B) \arrow[d, "\alpha_B \circ -"] \\
\mathcal{B}(\mathrm{G}A,\mathrm{G}B) \arrow[r, "- \circ \alpha_A"'] & \mathcal{B}(\mathrm{F}A,\mathrm{G}B).
\end{tikzcd}$$

Example 1.2.4. The 2-category **Cat**: the 0-cells are given by all small categories, 1-cells are given by functors between them, and 2-cells are given by natural transformations.

Definition 1.2.5. Let $\mathcal{A}$ and $\mathcal{B}$ be 2-categories. A ***pseudo-functor*** $(\mathrm{F}, h, \overline{h})$ from $\mathcal{A}$ to $\mathcal{B}$ consists of the data for a 2-functor plus:

- for each triple A, B and C of 0-cells of $\mathcal{A}$, an invertible natural transformation

$$\begin{tikzcd}
\mathcal{A}(C,B)\times\mathcal{A}(A,B) \arrow[rr, "\mathrm{F}_{B,C}\times\mathrm{F}_{A,B}"] \arrow[d, "\mathrm{c}_{A,B,C}"'] & & \mathcal{B}(\mathrm{F}C,\mathrm{F}B)\times\mathcal{B}(\mathrm{F}A,\mathrm{F}B) \arrow[d, "\mathrm{c}_{\mathrm{F}A,\mathrm{F}B,\mathrm{F}C}"] \\
\mathcal{A}(A,C) \arrow[rr, "\mathrm{F}_{A,C}"'] \arrow[urr, phantom, "\Downarrow h_{A,B,C}"] & & \mathcal{B}(\mathrm{F}A,\mathrm{F}B)
\end{tikzcd}$$

- for each 0-cell A an invertible 2-cell

$$\begin{tikzcd}
\mathcal{I} \arrow[d, "\mathrm{u}_A"'] \arrow[dr, "\mathrm{u}_{\mathrm{F}A}"] & \\
\mathcal{A}(A,A) \arrow[r, "\mathrm{F}_{A,A}"'] \arrow[ur, phantom, "\swarrow \overline{h}_A"] & \mathcal{B}(\mathrm{F}A,\mathrm{F}A)
\end{tikzcd}$$

subject to the following coherence axioms:

- ***composition axiom***: for every triple of 1-cells

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{l} D$$

in $\mathcal{A}$, the following equality of 2-cells holds

$$h_{g\circ f,l} \circ (i_{\mathrm{F}l}.h_{f,g}) = h_{f,l\circ g} \circ (h_{g,h}.i_{\mathrm{F}f}).$$

This means that the diagram

$$\begin{tikzcd}
\mathrm{F}l\circ\mathrm{F}g\circ\mathrm{F}f \arrow[r, "i_{\mathrm{F}l}.h_{f,g}"] \arrow[d, "h_{g,l}.i_{\mathrm{F}f}"'] & \mathrm{F}l\circ\mathrm{F}(g\circ f) \arrow[d, "h_{g\circ f,l}"] \\
\mathrm{F}(g\circ l)\circ\mathrm{F}f \arrow[r, "h_{f,l\circ g}"'] & \mathrm{F}(l\circ g\circ f)
\end{tikzcd}$$

commutes;

- ***unit axioms***: for every 1-cell

$$f\colon A \longrightarrow B$$

in $\mathcal{A}$ the following equality of 2-cells holds

$$h_{1_A,f} \circ (i_{\mathrm{F}f}.\overline{h}_A) = i_{\mathrm{F}f}, \quad h_{f,1_B} \circ (\overline{h}_B.i_{\mathrm{F}f}).$$

This means that the diagrams

$$\begin{tikzcd} \mathrm{F}f \circ 1_{\mathrm{F}A} \arrow[r, "i_{\mathrm{F}f}.\overline{h}_A"] \arrow[dr, "i_{\mathrm{F}f}"'] & \mathrm{F}f \circ \mathrm{F}1_A \arrow[d, "h_{1_A,f}"] \\ & \mathrm{F}f \end{tikzcd} \qquad \begin{tikzcd} 1_{\mathrm{F}B} \circ \mathrm{F}f \arrow[r, "\overline{h}_B.i_{\mathrm{F}f}"] \arrow[dr, "i_{\mathrm{F}f}"'] & \mathrm{F}1_B \circ \mathrm{F}f \arrow[d, "h_{f,1_B}"] \\ & \mathrm{F}f \end{tikzcd}$$

commute.

Definition 1.2.6. Let $(\mathrm{F}, h, \overline{h})$ and $(\mathrm{G}, k, \overline{k})$ be pseudo-functors from $\mathcal{A}$ to $\mathcal{B}$. A ***pseudo-natural transformation*** (α, τ) from F to G consists of the following data:

- for each 0-cell A in $\mathcal{A}$ a 1-cell $\alpha_A \colon \mathrm{F}A \longrightarrow \mathrm{G}A$;
- for each pair of 0-cells A and B in $\mathcal{A}$, an invertible natural transformation $\tau^{A,B}$, called ***pseudo-naturality*** of α:

$$\tau^{A,B} \colon \mathrm{G} \circ \alpha_A \longrightarrow \alpha_B \circ \mathrm{F}.$$

These data are required to satisfy the following coherence axioms

- for each pair of 1-cells

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in $\mathcal{A}$ the following equality of 2-cells must holds

$$\tau^{A,C}_{g\circ f} \circ (k_{f,g}.i_{\alpha_A}) = (i_{\alpha_C}.h_{f,g}) \circ (\tau^{B,C}_g.i_{\mathrm{F}f}) \circ (i_{\mathrm{G}f}.\tau^{A,B}_f).$$

This means that the diagram

$$\begin{tikzcd} \mathrm{G}g \circ \mathrm{G}f \circ \alpha_A \arrow[r, "i_{\mathrm{G}g}.\tau^{A,B}_f"] \arrow[d, "k_{f,g}.i_{\alpha_A}"'] & \mathrm{G}g \circ \alpha_B \circ \mathrm{F}f \arrow[r, "\tau^{B,C}_g.i_{\mathrm{F}f}"] & \alpha_C \circ \mathrm{F}g \circ \mathrm{F}f \arrow[d, "i_{\alpha_C}.h_{f,g}"] \\ \mathrm{G}(g \circ f) \circ \alpha_A \arrow[rr, "\tau^{A,C}_{g\circ f}"'] & & \alpha_C \circ \mathrm{F}(g \circ f) \end{tikzcd}$$

commutes;
- for each 0-cell A in $\mathcal{A}$ the following equality of 2-cells must holds

$$(i_{\alpha_A}.\overline{h}_A) \circ i_{\alpha_A} = \tau^{A,A}_{1_A} \circ (\overline{k}_A.i_{\alpha_A}) \circ i_{\alpha_A}.$$

This means that the diagram

$$\begin{tikzcd}
\alpha_A \arrow[r, "i_{\alpha_A}"] \arrow[d, "i_{\alpha_A}"'] & 1_{GA} \circ \alpha_A \arrow[r, "i_{\alpha_A}.\bar{k}_A"] & G1_A \circ \alpha_A \arrow[d, "\tau^{A,A}_{1_A}"] \\
\alpha_A \circ 1_{FA} \arrow[rr, "i_{\alpha_A}.\bar{h}_A"'] & & \alpha_A \circ F1_A
\end{tikzcd}$$

commutes.

Notation: we usually suppress the superscripts ^{A,B} whenever they are clear from the context.

Definition 1.2.7. Let (α,τ) and (β,γ) be pseudo-natural transformations. A ***modification*** χ from (α,τ) to (β,γ) consists of a collection of 2-cells $\{\ \chi_A\colon \alpha_A \longrightarrow \beta_A\ \}$ indexed by 0-cell of $\mathcal{A}$, such that for every 1-cell $f\colon A \longrightarrow B$ in $\mathcal{A}$ the following equality holds

$$\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[d, bend right=40, "\alpha_A"'] \arrow[d, bend left=40, "\beta_A"] \arrow[d, phantom, "\overset{\chi_A}{\Rightarrow}"] & FB \arrow[d, "\beta_B"] \\
GA \arrow[r, "Gf"'] \arrow[ur, phantom, "\Downarrow\gamma_f^{A,B}"] & GB
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[d, "\alpha_A"'] & FB \arrow[d, bend right=40, "\alpha_B"'] \arrow[d, bend left=40, "\beta_B"] \arrow[d, phantom, "\overset{\chi_B}{\Rightarrow}"] \\
GA \arrow[r, "Gf"'] \arrow[ur, phantom, "\tau_f^{A,B}\Downarrow"] & GB
\end{tikzcd}$$

1.2.1 Property-like 2-monads

A ***2-monad*** (T,μ,η) on a 2-category $\mathcal{A}$ is a 2-functor $\mathrm{T}\colon \mathcal{A} \longrightarrow \mathcal{A}$ together 2-natural transformations $\mu\colon \mathrm{T}^2 \longrightarrow \mathrm{T}$ and $\eta\colon 1 \longrightarrow \mathrm{T}$ such that the following diagrams commute

$$\begin{tikzcd}
\mathrm{T}^3 \arrow[r, "\mathrm{T}\mu"] \arrow[d, "\mu\mathrm{T}"'] & \mathrm{T}^2 \arrow[d, "\mu"] \\
\mathrm{T}^2 \arrow[r, "\mu"'] & \mathrm{T}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{T} \arrow[r, "\eta\mathrm{T}"] \arrow[dr, "\mathrm{id}"'] & \mathrm{T}^2 \arrow[d, "\mu"] & \mathrm{T} \arrow[l, "\mathrm{T}\eta"'] \arrow[dl, "\mathrm{id}"] \\
& \mathrm{T} &
\end{tikzcd}$$

A T-***algebra*** is a pair (A,a) where, A is an object of $\mathcal{A}$ and $a\colon \mathrm{T}A \longrightarrow A$ is a 1-cell such that the diagrams

$$\begin{tikzcd} \mathrm{T}^2A \arrow[r, "\mathrm{T}a"] \arrow[d, "\mu_A"'] & \mathrm{T}A \arrow[d, "a"] \\ \mathrm{T}A \arrow[r, "a"'] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "1_A"'] & \mathrm{T}A \arrow[d, "a"] \\ & A \end{tikzcd}$$

commute. A ***strict*** T-***morphism*** from a T-algebra (A,a) to a T-algebra (B,b) is a 1-cell $f\colon A \longrightarrow B$ such that the following diagram commutes:

$$\begin{tikzcd} \mathrm{T}A \arrow[r, "\mathrm{T}f"] \arrow[d, "a"'] & \mathrm{T}B \arrow[d, "b"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

while a ***lax*** T-***morphism*** from a T-algebra (A,a) to a T-algebra (B,b) is a pair $(f,\overline{f})$ where f is a 1-cell $f\colon A \longrightarrow B$ and $\overline{f}$ is a 2-cell

$$\begin{tikzcd} \mathrm{T}A \arrow[r, "\mathrm{T}f"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\overline{f}"] & \mathrm{T}B \arrow[d, "b"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

which satisfies the following ***coherence*** conditions:

$$\begin{tikzcd} \mathrm{T}^2A \arrow[r, "\mathrm{T}^2f"] \arrow[d, "\mu_A"'] & TB \arrow[d, "\mu_B"] \\ \mathrm{T}A \arrow[r, "\mathrm{T}f"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\overline{f}"] & \mathrm{T}B \arrow[d, "b"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \quad = \quad \begin{tikzcd} \mathrm{T}^2A \arrow[r, "\mathrm{T}^2f"] \arrow[d, "\mathrm{T}a"'] \arrow[dr, phantom, "\Downarrow\mathrm{T}\overline{f}"] & TB \arrow[d, "\mathrm{T}b"] \\ \mathrm{T}A \arrow[r, "\mathrm{T}f"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\overline{f}"] & \mathrm{T}B \arrow[d, "b"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

and

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\eta_A"'] & B \arrow[d, "\eta_B"] \\
\mathrm{T}A \arrow[r, "\mathrm{T}f"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\overline{f}"] & \mathrm{T}B \arrow[d, "b"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "1_A"'] & B \arrow[d, "1_B"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
$$

Observe that regions in which no 2-cell is written commute, so they are deemed to contain the identity 2-cell.

A lax morphism $(f, \overline{f})$ in which $\overline{f}$ is invertible is said T-***morphism***. So a strict T-morphism is a T-morphism where $\overline{f}$ is the identity 2-cell.

The category of T-algebras and lax T-morphisms becomes a 2-category T-$\mathbf{Alg}_l$ introducing the T-transformations as 2-cells: a ***T-transformation*** from the 1-cell $(f, \overline{f})\colon (A, a) \longrightarrow (B, b)$ to $(g, \overline{g})\colon (A, a) \longrightarrow (B, b)$ is a 2-cell $\alpha\colon f \Longrightarrow g$ in $\mathcal{A}$ which satisfies the following coherence condition

$$
\begin{tikzcd}
\mathrm{T}A \arrow[r, bend left, "\mathrm{T}f"] \arrow[r, bend right, "\mathrm{T}g"'] \arrow[r, phantom, "\Downarrow\mathrm{T}\alpha"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\overline{g}"] & \mathrm{T}B \arrow[d, "b"] \\
A \arrow[r, bend right, "g"'] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathrm{T}A \arrow[r, bend left, "\mathrm{T}f"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\overline{f}"] & \mathrm{T}B \arrow[d, "b"] \\
A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B
\end{tikzcd}
$$

expressing compatibility of α with $\overline{f}$ and $\overline{g}$.

It is observed in [27] that, using the notion of T-morphism, one can express in precise mathematical terms what it means that an action of a 2-monad T on an object A is *unique up to a unique isomorphism*.

In [27] an T-algebra structure is essentially unique if, given two actions $a, a'\colon \mathrm{T}A \longrightarrow A$, there is a unique invertible 2-cell $\alpha\colon a \Longrightarrow a'$ such that $(1_A, \alpha)\colon (A, a) \longrightarrow (A, a')$ is a morphism of T-algebras. This is fixed by the following definition of property-like 2-monad.

A 2-monad (T, μ, η) is said ***property-like***, if it satisfies the following conditions:

- for every T-algebras (A, a) and (B, b), and for every invertible 1-cell $f\colon A \longrightarrow B$ there exists a unique invertible 2-cell $\overline{f}$

$$\begin{tikzcd} \mathrm{T}A \arrow[r, "\mathrm{T}f"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\overline{f}"] & \mathrm{T}B \arrow[d, "b"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

such that $(f, \overline{f})\colon (A, a) \longrightarrow (B, b)$ is a morphism of T-algebras;

- for every T-algebras (A, a) and (B, b), and for every 1-cell $f\colon A \longrightarrow B$ if there exists a 2-cell $\overline{f}$

$$\begin{tikzcd} \mathrm{T}A \arrow[r, "\mathrm{T}f"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\overline{f}"] & \mathrm{T}B \arrow[d, "b"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

  such that $(f, \overline{f})\colon (A, a) \longrightarrow (B, b)$ is a lax morphism of T-algebras, then it is the unique 2-cell with such property.

We say that a 2-monad (T, μ, η) is ***lax-idempotent*** when, for every T-algebras (A, a) and (B, b), and for every 1-cell $f\colon A \longrightarrow B$, there exists a unique 2-cell $\overline{f}$

$$\begin{tikzcd} \mathrm{T}A \arrow[r, "\mathrm{T}f"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\overline{f}"] & \mathrm{T}B \arrow[d, "b"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

such that $(f, \overline{f})\colon (A, a) \longrightarrow (B, b)$ is a lax morphism of T-algebras, then a useful result in [27] is Proposition 6.1.

Proposition 1.2.8. *Every lax-idempotent 2-monad is property-like.*

Proof. See [27, Proposition 6.1]. □

1.3 Pseudo-monads and pseudo-distributive laws

This section is devoted to the formal definition of pseudo-monad and pseudo-distributive law.

The technicalities involved with the definitions are quite complex, but the idea is straightforward.

As in the case for the definition of ordinary monad and distributive law, we follow the notation of Tanaka and Power [56, 55, 54].

Definition 1.3.1. A ***pseudo-monad*** $(\mathrm{T}, \mu, \eta, \tau, \rho, \lambda)$ on a 2-category $\mathcal{A}$ consists of

- a pseudo-functor $\mathrm{T}\colon \mathcal{A} \longrightarrow \mathcal{A}$;
- a pseudo-natural transformation $\mu\colon \mathrm{T}^2 \longrightarrow \mathrm{T}$;
- a pseudo-natural transformation $\eta\colon \mathrm{id}_{\mathcal{A}} \longrightarrow \mathrm{T}$;
- an invertible modification

$$\begin{tikzcd}
\mathrm{T}^3 \arrow[r, "\mathrm{T}\mu"] \arrow[d, "\mu\mathrm{T}"'] \arrow[dr, phantom, "\Downarrow\tau"] & \mathrm{T}^2 \arrow[d, "\mu"] \\
\mathrm{T}^2 \arrow[r, "\mu"'] & T
\end{tikzcd}$$

- invertible modifications

$$\begin{tikzcd}
\mathrm{T} \arrow[r, "\mathrm{T}\eta"] \arrow[dr, "\mathrm{id}"'] & \mathrm{T}^2 \arrow[d, "\mu"] \arrow[dl, phantom, "\Leftarrow\lambda"] \\
& \mathrm{T}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathrm{T} \arrow[r, "\eta\mathrm{T}"] \arrow[dr, "\mathrm{id}"'] & \mathrm{T}^2 \arrow[d, "\mu"] \arrow[dl, phantom, "\Leftarrow\rho"] \\
& \mathrm{T}
\end{tikzcd}$$

subject to two coherence axioms

$$\begin{tikzcd}
\mathrm{T}^4 \arrow[r, "\mathrm{T}^2\mu"] \arrow[d, "\mu\mathrm{T}^2"'] \arrow[dr, "\mathrm{T}\mu\mathrm{T}"] & \mathrm{T}^3 \arrow[dr, "\mathrm{T}\mu"] \arrow[d, phantom, "\Downarrow\mathrm{T}\tau"] & \\
\mathrm{T}^3 \arrow[dr, "\mu\mathrm{T}"'] \arrow[r, phantom, "\Downarrow\tau\mathrm{T}"] & \mathrm{T}^3 \arrow[r, "\mathrm{T}\mu"'] \arrow[d, "\mu\mathrm{T}"] \arrow[dr, phantom, "\Downarrow\tau"] & T^2 \arrow[d, "\mu"] \\
& \mathrm{T}^2 \arrow[r, "\mu"'] & \mathrm{T}
\end{tikzcd}
=
\begin{tikzcd}
\mathrm{T}^4 \arrow[r, "\mathrm{T}^2\mu"] \arrow[d, "\mu\mathrm{T}^2"'] \arrow[dr, phantom, "\cong"] & \mathrm{T}^3 \arrow[d, "\mu\mathrm{T}"] \arrow[dr, "\mathrm{T}\mu"] & \\
\mathrm{T}^3 \arrow[r, "\mathrm{T}\mu"] \arrow[dr, "\mu\mathrm{T}"'] & \mathrm{T}^2 \arrow[dr, "\mu"] \arrow[d, phantom, "\Downarrow\tau"] \arrow[r, phantom, "\Downarrow\tau"] & \mathrm{T}^2 \arrow[d, "\mu"] \\
& \mathrm{T}^2 \arrow[r, "\mu"'] & \mathrm{T}
\end{tikzcd}$$

$$
\begin{tikzcd}
\mathrm{T}^3 \arrow[r, "\mathrm{id}"] \arrow[dr, phantom, "\Downarrow\lambda\mathrm{T}"] & \mathrm{T}^3 \arrow[r, "\mathrm{T}\mu"] \arrow[d, "\mu\mathrm{T}"] \arrow[dr, phantom, "\Downarrow\tau"] & \mathrm{T}^2 \arrow[d, "\mu"] \\
\mathrm{T}^2 \arrow[u, "\mathrm{T}\eta\mathrm{T}"] \arrow[r, "\mathrm{id}"'] & \mathrm{T}^2 \arrow[r, "\mu"'] & \mathrm{T}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathrm{T}^3 \arrow[r, "\mathrm{id}"] \arrow[dr, phantom, "\Downarrow\mathrm{T}\rho"] & \mathrm{T}^3 \arrow[r, "\mathrm{T}\mu"] & \mathrm{T}^2 \arrow[d, "\mu"] \\
\mathrm{T}^2 \arrow[u, "\mathrm{T}\eta\mathrm{T}"] \arrow[r, "\mathrm{id}"'] & \mathrm{T}^2 \arrow[ur, "\mathrm{id}"] \arrow[r, "\mu"'] & \mathrm{T}.
\end{tikzcd}
$$

Definition 1.3.2. A ***pseudo-algebra*** (A, a, a_μ, a_η) for a pseudo-monad $(\mathrm{T}, \mu, \eta, \tau, \rho, \lambda)$ consists of

- a 0-cell A in $\mathcal{A}$;
- a 1-cell $a\colon \mathrm{T}A \longrightarrow A$;
- invertible 2-cells

$$
\begin{tikzcd}
\mathrm{T}^2A \arrow[r, "\mathrm{T}a"] \arrow[d, "\mu_A"'] \arrow[dr, phantom, "\Downarrow a_\mu"] & \mathrm{T}A \arrow[d, "a"] \\
\mathrm{T}A \arrow[r, "a"'] & A
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[dr, "1_A"'] \arrow[dr, phantom, shift left=3, "\Swarrow a_\eta"] & \mathrm{T}A \arrow[d, "a"] \\
 & A
\end{tikzcd}
$$

subject to two coherence axioms:

$$
\begin{tikzcd}
\mathrm{T}^3A \arrow[r, "\mathrm{T}^2a"] \arrow[d, "\mu_{\mathrm{T}A}"'] \arrow[dr, "\mathrm{T}\mu_A"] & \mathrm{T}^2A \arrow[dr, "\mathrm{T}a"] \arrow[d, phantom, "\Downarrow\mathrm{T}a_\mu"] & \\
\mathrm{T}^2A \arrow[dr, "\mu_A"'] \arrow[r, phantom, "\Downarrow\tau_A"] & \mathrm{T}^2A \arrow[r, "\mathrm{T}a"'] \arrow[d, "\mu_A"] \arrow[dr, phantom, "\Downarrow a_\mu"] & \mathrm{T}A \arrow[d, "a"] \\
 & \mathrm{T}A \arrow[r, "a"'] & A
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathrm{T}^3A \arrow[r, "\mathrm{T}^2a"] \arrow[d, "\mu_{\mathrm{T}A}"'] \arrow[dr, phantom, "\cong"] & \mathrm{T}^2A \arrow[d, "\mu_A"] \arrow[dr, "\mathrm{T}a"] & \\
\mathrm{T}^2A \arrow[r, "\mathrm{T}a"] \arrow[dr, "\mu_A"'] \arrow[dr, phantom, shift left=3, "\Downarrow a_\mu"] & \mathrm{T}A \arrow[dr, "a"] \arrow[r, phantom, "\Downarrow a_\mu"] & \mathrm{T}A \arrow[d, "a"] \\
 & \mathrm{T}A \arrow[r, "a"'] & A
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathrm{T}^2A \arrow[r, "\mathrm{id}"] \arrow[dr, phantom, "\Downarrow\lambda_A"] & \mathrm{T}^2A \arrow[r, "\mathrm{T}a"] \arrow[d, "\mu_A"] \arrow[dr, phantom, "\Downarrow a_\mu"] & \mathrm{T}A \arrow[d, "a"] \\
\mathrm{T}A \arrow[u, "\mathrm{T}\eta_A"] \arrow[r, "\mathrm{id}"'] & \mathrm{T}A \arrow[r, "a"'] & A
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathrm{T}^2A \arrow[r, "\mathrm{id}"] \arrow[dr, phantom, "\Downarrow\mathrm{T}a_\eta"] & \mathrm{T}^2A \arrow[r, "Ta"] & \mathrm{T}A \arrow[d, "a"] \\
\mathrm{T}A \arrow[u, "\mathrm{T}\eta_A"] \arrow[r, "\mathrm{id}"'] & \mathrm{T}A \arrow[ur, "\mathrm{id}"] \arrow[r, "a"'] & A
\end{tikzcd}
$$

A second identity axiom, one for the composite of a_μ with $\eta_{\mathrm{T}A}$, follows from these two axioms.

Definition 1.3.3. A ***pseudo-morphism*** of pseudo-T-algebras from (A, a, a_μ, a_η) to (B, b, b_μ, b_η) consists of a pair $(f, \overline{f})$ where $f\colon A \longrightarrow B$ is a 1-cell and $\overline{f}$ is an invertible 2-cell

$$\begin{tikzcd}
\mathrm{T}A \arrow[r, "\mathrm{T}f"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\overline{f}"] & \mathrm{T}B \arrow[d, "b"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

subject to two coherence axioms:

$$\begin{tikzcd}
\mathrm{T}^2A \arrow[r, "\mathrm{T}^2f"] \arrow[d, "\mu_A"'] \arrow[dr, "\mathrm{T}a"] & \mathrm{T}^2B \arrow[dr, "\mathrm{T}b"] \arrow[d, phantom, "\Downarrow\mathrm{T}\overline{f}"] & \\
\mathrm{T}A \arrow[dr, "a"'] \arrow[r, phantom, "\Downarrow a_\mu"] & \mathrm{T}A \arrow[r, "\mathrm{T}f"'] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\overline{f}"] & \mathrm{T}B \arrow[d, "b"] \\
& A \arrow[r, "f"'] & B
\end{tikzcd}
=
\begin{tikzcd}
\mathrm{T}^2A \arrow[r, "\mathrm{T}^2f"] \arrow[d, "\mu_A"'] \arrow[dr, phantom, "\cong"] & \mathrm{T}^2B \arrow[dr, "\mathrm{T}b"] \arrow[d, "\mu_B"'] & \\
\mathrm{T}A \arrow[dr, "a"'] \arrow[r, "\mathrm{T}f"] & \mathrm{T}B \arrow[dr, "b"] \arrow[d, phantom, "\Downarrow\overline{f}"] \arrow[r, phantom, "\Downarrow b_\mu"] & \mathrm{T}B \arrow[d, "b"] \\
& A \arrow[r, "f"'] & B
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "1_A"'] \arrow[dr, "\eta_A"] & B \arrow[dr, "\eta_B"] \arrow[d, phantom, "\cong"] & \\
A \arrow[dr, "1_A"'] \arrow[r, phantom, "\Downarrow a_\eta"] & \mathrm{T}A \arrow[r, "\mathrm{T}f"'] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\overline{f}"] & \mathrm{T}B \arrow[d, "b"] \\
& A \arrow[r, "f"'] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "1_A"'] & B \arrow[dr, "\eta_B"] \arrow[d, "1_B"'] & \\
A \arrow[dr, "1_A"'] \arrow[r, "f"] & B \arrow[dr, "1_B"] \arrow[r, phantom, "\Downarrow b_\eta"] & \mathrm{T}B \arrow[d, "b"] \\
& A \arrow[r, "f"'] & B.
\end{tikzcd}$$

Definition 1.3.4. A ***pseudo*-T-*transformations*** from $(f,\overline{f})\colon (A,a) \longrightarrow (B,b)$ to $(g,\overline{g})\colon (A,a) \longrightarrow (B,b)$ is an invertible 2-cell $\alpha\colon f \Longrightarrow g$ in $\mathcal{A}$ satisfies the following coherence axiom

$$\begin{tikzcd}
\mathrm{T}A \arrow[r, "\mathrm{T}f"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\overline{f}"] & \mathrm{T}B \arrow[d, "b"] \\
A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B
\end{tikzcd}
=
\begin{tikzcd}
\mathrm{T}A \arrow[r, bend left, "\mathrm{T}f"] \arrow[r, bend right, "\mathrm{T}g"'] \arrow[r, phantom, "\Downarrow\mathrm{T}\alpha"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\overline{g}"] & \mathrm{T}B \arrow[d, "b"] \\
A \arrow[r, "g"'] & B.
\end{tikzcd}$$

The above definitions together form a 2-category.

Definition 1.3.5. Let $(\mathrm{T}, \mu, \eta, \tau, \rho, \lambda)$ be a pseudo-monad. We define the 2-category $\mathbf{Ps}$-T-$\mathbf{Alg}$ where 0-cells are pseudo-T-algebras, 1-cells are pseudo-morphisms, and 2-cells are pseudo-T-transformations. The composition functor is defined as

$$\mathrm{c}_{A,B,C}\colon\ \mathbf{Ps}\text{-}\mathrm{T}\text{-}\mathbf{Alg}\,((B,b),(C,c)) \times \mathbf{Ps}\text{-}\mathrm{T}\text{-}\mathbf{Alg}\,((A,a),(B,b)) \longrightarrow \mathbf{Ps}\text{-}\mathrm{T}\text{-}\mathbf{Alg}\,((A,a),(C,c))$$

which send a pair 1-cells

$$(f,\overline{f})\colon (A,a,a_\mu,a_\eta) \longrightarrow (B,b,b_\mu,b_\eta) \quad (g,\overline{g})\colon (B,b,b_\mu,b_\eta) \longrightarrow (C,c,c_\mu,c_\eta)$$

to

$$(gf,\overline{gf})\colon (A,a,a_\mu,a_\eta) \longrightarrow (C,c,c_\mu,c_\eta)$$

where gf is the composite of 1-cells in $\mathcal{A}$ and $\overline{gf} = (\overline{g}.i_{\mathrm{T}f}) \circ (i_g.\overline{f})$ as shown below

$$\begin{tikzcd}
\mathrm{T}A \arrow[r, "\mathrm{T}f"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\overline{f}"] & \mathrm{T}B \arrow[r, "\mathrm{T}g"] \arrow[d, "b"] \arrow[dr, phantom, "\Downarrow\overline{g}"] & \mathrm{T}C \arrow[d, "c"] \\
A \arrow[r, "f"'] & B \arrow[r, "g"'] & C
\end{tikzcd}$$

It is straightforward to prove that $(gf,\overline{gf})$ satisfies the axioms of Definition 1.3.3. The composition functor defines the composition of 2-cells as the horizontal composition.

Definition 1.3.6. Let $(\mathrm{T}, \mu^T, \eta^T, \tau^T, \rho^T, \lambda^T)$ and $(\mathrm{S}, \mu^S, \eta^S, \tau^S, \rho^S, \lambda^S)$ be pseudo-monads on a 2-category $\mathcal{A}$. A ***pseudo-distributive law*** $(\delta, \overline{\mu}^S, \overline{\mu}^T, \overline{\eta}^S, \overline{\eta}^T)$ of S over T consists of

- a pseudo-natural transformation $\delta\colon \mathrm{ST} \longrightarrow \mathrm{TS}$;
- invertible modifications

$$\begin{tikzcd}
\mathrm{S}^2\mathrm{T} \arrow[r, "\mathrm{S}\delta"] \arrow[d, "\mu^S\mathrm{T}"'] & \mathrm{STS} \arrow[r, "\delta_S"] \arrow[d, phantom, "\Downarrow\overline{\mu}^S"] & \mathrm{TS}^2 \arrow[d, "\mathrm{T}\mu^S"] \\
\mathrm{ST} \arrow[rr, "\delta"'] & & \mathrm{TS}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{ST}^2 \arrow[r, "\delta\mathrm{T}"] \arrow[d, "\mathrm{S}\mu^T"'] & \mathrm{TST} \arrow[r, "\mathrm{T}\delta"] \arrow[d, phantom, "\Downarrow\overline{\mu}^T"] & \mathrm{T}^2\mathrm{S} \arrow[d, "\mu^T\mathrm{S}"] \\
\mathrm{ST} \arrow[rr, "\delta"'] & & \mathrm{TS}
\end{tikzcd}$$

- invertible modifications

$$
\begin{tikzcd}
\mathrm{T} \arrow[d, "\eta^S\mathrm{T}"'] \arrow[dr, "\mathrm{T}\eta^S"] \arrow[dr, phantom, shift right=3, "\overline{\eta}^S \Nearrow"] & \\
\mathrm{ST} \arrow[r, "\delta"'] & \mathrm{TS}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{S} \arrow[d, "\mathrm{S}\eta^T"'] \arrow[dr, "\eta^T\mathrm{S}"] \arrow[dr, phantom, shift right=3, "\overline{\eta}^T \Nearrow"] & \\
\mathrm{ST} \arrow[r, "\delta"'] & \mathrm{TS}
\end{tikzcd}
$$

subject to the ten coherence axioms we list below.

1. The first axioms involves $\overline{\eta}^T$ and $\overline{\eta}^S$ and is self-dual

$$
\begin{tikzcd}
\mathrm{id}_{\mathcal{A}} \arrow[r, "\eta^S"] \arrow[dr, "\eta^T"'] & \mathrm{S} \arrow[r, "\mathrm{S}\eta^T"] \arrow[dr, "\eta^T\mathrm{S}"'] \arrow[d, phantom, "\cong"] & \mathrm{ST} \arrow[d, "\delta"] \arrow[dl, phantom, "\overline{\eta}^T \Swarrow"] \\
& \mathrm{T} \arrow[r, "\mathrm{T}\eta^S"'] & \mathrm{TS}
\end{tikzcd}
=
\begin{tikzcd}
\mathrm{id}_{\mathcal{A}} \arrow[r, "\eta^S"] \arrow[dr, "\eta^T"'] & \mathrm{S} \arrow[r, "\mathrm{S}\eta^T"] \arrow[d, phantom, "\cong"] & \mathrm{ST} \arrow[d, "\delta"] \arrow[dl, phantom, "\Downarrow\overline{\eta}^S"] \\
& \mathrm{T} \arrow[ur, "\eta^S\mathrm{T}"] \arrow[r, "\mathrm{T}\eta^S"'] & \mathrm{TS}
\end{tikzcd}
$$

2. the second is a coherence axiom involving μ^S, η^S and λ^S

$$
\begin{tikzcd}
& \mathrm{S}^2\mathrm{T} \arrow[r, "\mathrm{S}\delta"] \arrow[d, "\mu^S\mathrm{T}"] \arrow[dl, phantom, "\Downarrow\lambda^S\mathrm{T}"] & \mathrm{STS} \arrow[r, "\delta\mathrm{S}"] \arrow[d, phantom, "\Downarrow\overline{\mu}^S"] & \mathrm{TS}^2 \arrow[d, "\mathrm{T}\mu^S"] \\
\mathrm{ST} \arrow[ur, "\mathrm{S}\eta^S\mathrm{T}"] \arrow[r, "\mathrm{id}_{\mathrm{ST}}"'] & \mathrm{ST} \arrow[rr, "\delta"'] & & \mathrm{TS}
\end{tikzcd}
=
\begin{tikzcd}
\mathrm{S}^2\mathrm{T} \arrow[r, "\mathrm{S}\delta"] & \mathrm{STS} \arrow[r, "\delta\mathrm{S}"] \arrow[dl, phantom, "\Searrow\mathrm{S}\overline{\eta}^S"] & \mathrm{TS}^2 \arrow[dr, "\mathrm{T}\mu^S"] \arrow[d, phantom, shift left=4, "\mathrm{T}\lambda^S \Downarrow"] & \\
\mathrm{ST} \arrow[u, "\mathrm{S}\eta^S\mathrm{T}"] \arrow[ur, "\mathrm{ST}\eta^S \cong"'] \arrow[rr, "\delta"'] & & \mathrm{TS} \arrow[u, "\mathrm{TS}\eta^S"] \arrow[r, "\mathrm{id}_{\mathrm{TS}}"'] & \mathrm{TS}
\end{tikzcd}
$$

3. the third axiom is a coherence axiom involving μ^S, η^S and ρ^S

$$
\begin{tikzcd}
& \mathrm{S}^2\mathrm{T} \arrow[r, "\mathrm{S}\delta"] \arrow[d, "\mu^S\mathrm{T}"] \arrow[dl, phantom, "\Downarrow\rho^S\mathrm{T}"] & \mathrm{STS} \arrow[r, "\delta\mathrm{S}"] \arrow[d, phantom, "\Downarrow\overline{\mu}^S"] & \mathrm{TS}^2 \arrow[d, "\mathrm{T}\mu^S"] \\
\mathrm{ST} \arrow[ur, "\eta^S\mathrm{ST}"] \arrow[r, "\mathrm{id}_{\mathrm{ST}}"'] & \mathrm{ST} \arrow[rr, "\delta"'] & & \mathrm{TS}
\end{tikzcd}
=
\begin{tikzcd}
\mathrm{S}^2\mathrm{T} \arrow[r, "\mathrm{S}\delta"] \arrow[dr, phantom, "\cong"] & \mathrm{STS} \arrow[r, "\delta\mathrm{S}"] \arrow[dr, phantom, "\overline{\eta}^S\mathrm{S} \Nearrow"] & \mathrm{TS}^2 \arrow[dr, "\mathrm{T}\mu^S"] \arrow[d, phantom, shift left=4, "\mathrm{T}\rho^S \Downarrow"] & \\
\mathrm{ST} \arrow[u, "\eta^S\mathrm{ST}"] \arrow[rr, "\delta"'] & & \mathrm{TS} \arrow[ul, "\eta^S\mathrm{TS}"] \arrow[u, "\mathrm{T}\eta^S\mathrm{S}"'] \arrow[r, "\mathrm{id}_{\mathrm{TS}}"'] & \mathrm{TS}
\end{tikzcd}
$$

4. axiom 4 is, in a sense, dual to axiom 2, and it involves η^T, μ^T and λ^T

$$
\begin{tikzcd}
 & \mathrm{ST}^2 \arrow[r, "\delta\mathrm{T}"] \arrow[d, "\mathrm{S}\mu^T"] \arrow[drr, phantom, "\Downarrow\overline{\mu}^T"] & \mathrm{TST} \arrow[r, "\mathrm{T}\delta"] & \mathrm{T}^2\mathrm{S} \arrow[d, "\mu^T\mathrm{S}"] \\
\mathrm{ST} \arrow[ur, "\mathrm{ST}\eta^T"] \arrow[r, "\mathrm{id}_{\mathrm{ST}}"'] \arrow[r, phantom, shift left=3, "\Downarrow\mathrm{S}\lambda^T"] & \mathrm{ST} \arrow[rr, "\delta"'] & & \mathrm{TS}
\end{tikzcd}
=
\begin{tikzcd}
\mathrm{ST}^2 \arrow[r, "\delta\mathrm{T}"] \arrow[dr, phantom, "\cong"] & \mathrm{TST} \arrow[r, "\mathrm{T}\delta"] \arrow[dr, phantom, "\mathrm{T}\overline{\eta}^T \Nearrow"] & \mathrm{T}^2\mathrm{S} \arrow[dr, "\mu^T\mathrm{S}"] \arrow[d, phantom, shift left=4, "\lambda^T\mathrm{S} \Downarrow"] & \\
\mathrm{ST} \arrow[u, "\mathrm{ST}\eta^T"] \arrow[rr, "\delta"'] & & \mathrm{TS} \arrow[ul, "\mathrm{TS}\eta^T"] \arrow[u, "\mathrm{T}\eta^T\mathrm{S}"'] \arrow[r, "\mathrm{id}_{\mathrm{TS}}"'] & \mathrm{TS}
\end{tikzcd}
$$

5. axiom 5 is, in a sense, dual to axiom 3, and it involves η^T, μ^T and ρ^T

$$
\begin{tikzcd}
 & \mathrm{ST}^2 \arrow[r, "\delta\mathrm{T}"] \arrow[d, "\mathrm{S}\mu^T"] \arrow[drr, phantom, "\Downarrow\overline{\mu}^T"] & \mathrm{TST} \arrow[r, "\mathrm{T}\delta"] & \mathrm{T}^2\mathrm{S} \arrow[d, "\mu^T\mathrm{S}"] \\
\mathrm{ST} \arrow[ur, "\mathrm{S}\eta^T\mathrm{T}"] \arrow[r, "\mathrm{id}_{\mathrm{ST}}"'] \arrow[r, phantom, shift left=3, "\Downarrow\mathrm{S}\rho^T"] & \mathrm{ST} \arrow[rr, "\delta"'] & & \mathrm{TS}
\end{tikzcd}
=
\begin{tikzcd}
\mathrm{ST}^2 \arrow[r, "\delta\mathrm{T}"] \arrow[dr, phantom, "\Searrow\overline{\eta}^T\mathrm{T}"] & \mathrm{TST} \arrow[r, "\mathrm{T}\delta"] \arrow[dr, phantom, "\cong"] & \mathrm{T}^2\mathrm{S} \arrow[dr, "\mu^T\mathrm{S}"] \arrow[d, phantom, shift left=4, "\rho^T\mathrm{S} \Downarrow"] & \\
\mathrm{ST} \arrow[u, "\mathrm{S}\eta^T\mathrm{T}"] \arrow[ur, "\eta^T\mathrm{ST}"'] \arrow[rr, "\delta"'] & & \mathrm{TS} \arrow[u, "\eta^T\mathrm{TS}"] \arrow[r, "\mathrm{id}_{\mathrm{TS}}"'] & \mathrm{TS}
\end{tikzcd}
$$

6. this axiom involves $\overline{\mu}^S$ and $\overline{\eta}^T$

$$
\begin{tikzcd}
 & \mathrm{S}^2\mathrm{T} \arrow[dr, "\mathrm{S}\delta"] \arrow[dd, "\mu^S\mathrm{T}"] & & \\
 & & \mathrm{STS} \arrow[dr, "\delta\mathrm{S}"] \arrow[d, phantom, "\Downarrow\overline{\mu}^S"] & \\
\mathrm{S}^2 \arrow[uur, "\mathrm{S}^2\eta^T"] \arrow[d, "\mu^S"'] \arrow[r, phantom, "\cong"] & \mathrm{ST} \arrow[drr, "\delta"] \arrow[dr, phantom, "\Downarrow\overline{\eta}^T"] & & \mathrm{TS}^2 \arrow[d, "\mathrm{T}\mu^S"] \\
\mathrm{S} \arrow[ur, "\mathrm{S}\eta^T"] \arrow[rrr, "\eta^T\mathrm{S}"'] & & & \mathrm{TS}
\end{tikzcd}
=
\begin{tikzcd}
 & \mathrm{S}^2\mathrm{T} \arrow[dr, "\mathrm{S}\delta"] \arrow[d, phantom, "\Downarrow\mathrm{S}\overline{\eta}^T"] & & \\
 & & \mathrm{STS} \arrow[dr, "\delta\mathrm{S}"] \arrow[d, phantom, "\Downarrow\overline{\eta}^T\mathrm{S}"] & \\
\mathrm{S}^2 \arrow[uur, "\mathrm{S}^2\eta^T"] \arrow[urr, "\mathrm{S}\eta^T\mathrm{S}"] \arrow[rrr, "\eta^T\mathrm{S}^2"'] \arrow[d, "\mu^S"'] \arrow[drrr, phantom, "\cong"] & & & \mathrm{TS}^2 \arrow[d, "\mathrm{T}\mu^S"] \\
\mathrm{S} \arrow[rrr, "\eta^T\mathrm{S}"'] & & & \mathrm{TS}
\end{tikzcd}
$$

7. this axiom involves $\overline{\mu}^T$ and $\overline{\eta}^S$

$$
\begin{tikzcd}
& \mathrm{ST}^2 \arrow[dr, "\delta\mathrm{T}"] \arrow[dd, "\mathrm{S}\mu^T"] & & \\
& & \mathrm{TST} \arrow[dr, "\mathrm{T}\delta"] \arrow[d, phantom, "\Downarrow\overline{\mu}^T"] & \\
\mathrm{T}^2 \arrow[uur, "\eta^S\mathrm{T}^2"] \arrow[r, phantom, "\cong"] \arrow[d, "\mu^T"'] & \mathrm{ST} \arrow[drr, "\delta"] & {} \arrow[d, phantom, "\Downarrow\overline{\eta}^S"] & \mathrm{T}^2\mathrm{S} \arrow[d, "\mu^T\mathrm{S}"] \\
\mathrm{T} \arrow[ur, "\eta^S\mathrm{T}"'] \arrow[rrr, "\mathrm{T}\eta^T S"'] & & {} & \mathrm{TS}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& \mathrm{ST}^2 \arrow[dr, "\delta\mathrm{T}"] \arrow[d, phantom, "\Downarrow\overline{\eta}^S\mathrm{T}"] & & \\
& {} & \mathrm{TST} \arrow[dr, "\mathrm{T}\delta"] \arrow[d, phantom, "\Downarrow\mathrm{T}\overline{\eta}^S"] & \\
\mathrm{T}^2 \arrow[uur, "\eta^S\mathrm{T}^2"] \arrow[urr, "\mathrm{T}\eta^S\mathrm{T}"] \arrow[rrr, "\mathrm{T}^2\eta^S"'] \arrow[d, "\mu^T"'] & & {} \arrow[d, phantom, "\cong"] & \mathrm{T}^2\mathrm{S} \arrow[d, "\mu^T\mathrm{S}"] \\
\mathrm{T} \arrow[rrr, "\mathrm{T}\eta^S"'] & & {} & \mathrm{TS}
\end{tikzcd}
$$

8. this axioms involves $\overline{\mu}^S$ and τ^S

$$
\begin{tikzcd}
& \mathrm{S}^3\mathrm{T} \arrow[r, "\mathrm{S}^2\delta"] \arrow[dl, "\mu^S\mathrm{ST}"'] \arrow[d, "\mathrm{S}\mu^S\mathrm{T}"] & \mathrm{S}^2\mathrm{TS} \arrow[r, "\mathrm{S}\delta\mathrm{S}"] \arrow[d, phantom, "\Downarrow\mathrm{S}\overline{\mu}^S"] & \mathrm{STS}^2 \arrow[r, "\delta\mathrm{S}^2"] \arrow[d, "\mathrm{ST}\mu^S"'] \arrow[r, phantom, bend right, "\cong"] & \mathrm{TS}^3 \arrow[d, "\mathrm{TS}\mu^S"] \\
\mathrm{S}^2\mathrm{T} \arrow[r, phantom, "\Downarrow\tau^S\mathrm{T}"] \arrow[dr, "\mu^S\mathrm{T}"'] & \mathrm{S}^2\mathrm{T} \arrow[rr, "\mathrm{S}\delta"] \arrow[d, "\mu^S\mathrm{T}"] & {} \arrow[d, phantom, "\Downarrow\overline{\mu}^S"] & \mathrm{STS} \arrow[r, "\delta\mathrm{S}"] & \mathrm{TS}^2 \arrow[d, "\mathrm{T}\mu^S"] \\
& \mathrm{ST} \arrow[rrr, "\delta"'] & {} & & \mathrm{TS}
\end{tikzcd}
$$

$$\|$$

$$
\begin{tikzcd}
\mathrm{S}^3\mathrm{T} \arrow[r, "\mathrm{S}^2\delta"] \arrow[d, "\mu^S\mathrm{ST}"'] \arrow[dr, phantom, "\cong"] & \mathrm{S}^2\mathrm{TS} \arrow[r, "\mathrm{S}\delta\mathrm{S}"] \arrow[d, "\mu^S\mathrm{TS}"] & \mathrm{STS}^2 \arrow[r, "\delta\mathrm{S}^2"] \arrow[d, phantom, "\Downarrow\overline{\mu}^S\mathrm{S}"] & \mathrm{TS}^3 \arrow[d, "\mathrm{T}\mu^S\mathrm{S}"'] \arrow[dr, "\mathrm{TS}\mu^S"] & \\
\mathrm{S}^2\mathrm{T} \arrow[r, "\mathrm{S}\delta"'] \arrow[d, "\mu^S\mathrm{T}"'] & \mathrm{STS} \arrow[rr, "\delta\mathrm{S}"'] \arrow[d, phantom, "\Downarrow\overline{\mu}^S"] & {} & \mathrm{TS}^2 \arrow[d, "\mathrm{T}\mu^S"'] \arrow[r, phantom, "\Downarrow\mathrm{T}\tau^S"] & \mathrm{TS}^2 \arrow[dl, "\mathrm{T}\mu^S"] \\
\mathrm{ST} \arrow[rrr, "\delta"'] & {} & & \mathrm{TS} &
\end{tikzcd}
$$

9. this axioms involves $\overline{\mu}^T$ and τ^T

$$\begin{tikzcd}
\mathrm{ST}^3 \arrow[r, "\delta\mathrm{T}^2"] \arrow[d, "\mathrm{S}\mu^T\mathrm{T}"'] & \mathrm{TST}^2 \arrow[r, "\mathrm{T}\delta\mathrm{T}"] \arrow[d, phantom, "\Downarrow\overline{\mu}^T\mathrm{T}"] & \mathrm{T}^2\mathrm{ST} \arrow[r, "\mathrm{T}^2\delta"] \arrow[d, "\mu^T\mathrm{ST}"'] \arrow[dr, phantom, "\cong"] & \mathrm{T}^3\mathrm{S} \arrow[d, "\mu^T\mathrm{TS}"] \arrow[dr, "\mathrm{T}\mu^T\mathrm{S}"] & \\
\mathrm{ST}^2 \arrow[rr, "\delta\mathrm{T}"'] \arrow[d, "\mathrm{S}\mu^T"'] \arrow[drrr, phantom, "\Downarrow\overline{\mu}^T"] & & \mathrm{TST} \arrow[r, "\mathrm{T}\delta"'] & \mathrm{T}^2\mathrm{S} \arrow[d, "\mu^T\mathrm{S}"] \arrow[r, phantom, "\Downarrow\tau^T\mathrm{S}"] & \mathrm{TS}^2 \arrow[dl, "\mu^T\mathrm{S}"] \\
\mathrm{ST} \arrow[rrr, "\delta"'] & & & \mathrm{TS} &
\end{tikzcd}$$

$$\|$$

$$\begin{tikzcd}
& \mathrm{ST}^3 \arrow[r, "\delta\mathrm{T}^2"] \arrow[dl, "\mathrm{S}\mu^T\mathrm{T}"'] \arrow[d, "\mathrm{ST}\mu^T"] \arrow[dr, phantom, "\cong"] & \mathrm{TST}^2 \arrow[r, "\mathrm{T}\delta\mathrm{T}"] \arrow[d, "\mathrm{TS}\mu^T"] & \mathrm{T}^2\mathrm{ST} \arrow[r, "\mathrm{T}^2\delta"] \arrow[d, phantom, "\Downarrow\mathrm{T}\overline{\mu}^T"] & \mathrm{T}^3\mathrm{S} \arrow[d, "\mathrm{T}\mu^T\mathrm{S}"] \\
\mathrm{ST}^2 \arrow[dr, "\mathrm{S}\mu^T"'] \arrow[r, phantom, "\Downarrow\mathrm{S}\tau^T"] & \mathrm{ST}^2 \arrow[r, "\delta\mathrm{T}"] \arrow[d, "\mathrm{S}\mu^T"] & \mathrm{TST} \arrow[rr, "\mathrm{T}\delta"] \arrow[dr, phantom, "\Downarrow\overline{\mu}^T"] & & \mathrm{T}^2\mathrm{S} \arrow[d, "\mu^T\mathrm{S}"] \\
& \mathrm{ST} \arrow[rrr, "\delta"'] & & & \mathrm{TS}
\end{tikzcd}$$

10. the last axiom is self dual and it involves $\overline{\mu}^T$ and $\overline{\mu}^S$

$$\begin{tikzcd}
S^2T^2 \arrow[r, "S\delta T"] \arrow[d, "S^2\mu^T"'] & STST \arrow[r, "ST\delta"] \arrow[d, phantom, "\Downarrow S\bar{\mu}^T"] & ST^2S \arrow[r, "\delta TS"] \arrow[d, "S\mu^T S"] & TSTS \arrow[r, "T\delta S"] \arrow[d, phantom, "\Downarrow \bar{\mu}^T S"] & T^2S^2 \arrow[d, "\mu^T S"] \\
S^2T \arrow[rr, "S\delta"'] \arrow[d, "\mu^S T"'] & & STS \arrow[rr, "\delta S"'] \arrow[d, phantom, "\Downarrow \bar{\mu}^S"] & & TS^2 \arrow[d, "T\mu^S"] \\
ST \arrow[rrrr, "\delta"'] & & & & TS
\end{tikzcd}$$

$$\|$$

$$\begin{tikzcd}
& & & ST^2S \arrow[dr, "\delta TS"] \arrow[d, phantom, "\cong"] & & & \\
& S^2T^2 \arrow[r, "S\delta T"] \arrow[dl, "S^2\mu^T"'] \arrow[d, "\mu^S T^2"] & STST \arrow[r, "\delta ST"] \arrow[ur, "ST\delta"] \arrow[d, phantom, "\Downarrow \mu^S T"] & TS^2T \arrow[r, "TS\delta"] \arrow[d, "T\mu^S T"] & TSTS \arrow[r, "T\delta S"] \arrow[d, phantom, "\Downarrow T\bar{\mu}^S"] & T^2S^2 \arrow[d, "T^2\mu^S"'] \arrow[dr, "\mu^T S^2"] & \\
S^2T \arrow[r, phantom, "\cong"] \arrow[dr, "\mu^S T"'] & ST^2 \arrow[rr, "\delta T"'] \arrow[d, "S\mu^T"] & & TST \arrow[rr, "T\delta"'] \arrow[d, phantom, "\Downarrow \bar{\mu}^T"] & & T^2S \arrow[r, phantom, "\cong"] \arrow[d, "\mu^T S"'] & TS^2 \arrow[dl, "T\mu^S"] \\
& ST \arrow[rrrr, "\delta"'] & & & & TS &
\end{tikzcd}$$

The definition of lifting for pseudo-monads is a natural generalization of Definition 1.1.12.

Definition 1.3.7. Let $(\mathrm{S}, \mu^S, \eta^S, \tau^S, \lambda^S, \rho^S)$ and $(\mathrm{T}, \mu^T, \eta^T, \tau^T, \lambda^T, \rho^T)$ be two pseudo-monads on a 2-category $\mathcal{A}$. A ***lifting*** of the pseudo-monad T on **Ps**-S-**Alg** is a pseudo-monad $\widetilde{\mathrm{T}}$ on the category **Ps**-S-**Alg** such that

$$\mathrm{T}\mathrm{G}^S = \mathrm{G}^S\widetilde{\mathrm{T}}$$

where G^S is the forgetful 2-functor for the pseudo-monad S.

As in the ordinary case, given a pseudo-monad S we can define the 2-category $\mathbf{Ps}\text{-}\mathbf{Dist}_{\mathrm{S}}$ of pseudo-distributive laws and the 2-category $\mathbf{Lift}_{\mathbf{Ps}\text{-}\mathrm{S}\text{-}\mathbf{Alg}}$ of liftings over **Ps**-S-**Alg**.

The following theorems are the extensions in the pseudo setting of Theorems 1.1.14 and 1.1.16.

Theorem 1.3.8. *The 2-category* $\mathbf{Ps\text{-}Dist}_{\mathrm{S}}$ *and* $\mathbf{Lift}_{\mathbf{Ps}\text{-}\mathrm{S}\text{-}\mathbf{Alg}}$ *are equivalent.*

Proof. The constructions are essentially the same as those for ordinary distributive laws and ordinary lifting as in Theorem 1.1.14. However it is tedious and straightforward to complete the proof because we need to take care about to all the pseudo-maps. Therefore we refer to [54] for the complete proof of these result. □

Theorem 1.3.9. *Let* $\delta\colon \mathrm{ST} \longrightarrow \mathrm{TS}$ *be a pseudo-distributive law of pseudo-monads on a 2-category* $\mathcal{A}$*. Then*

1. *the pseudo-functor* TS *acquires the structure of pseudo-monad, with multiplication given by*

$$\mathrm{TSTS} \xrightarrow{\mathrm{T}\delta\mathrm{S}} \mathrm{T}^2\mathrm{S}^2 \xrightarrow{\mu^T \mathrm{S}^2} \mathrm{TS}^2 \xrightarrow{\mathrm{T}\mu^S} \mathrm{TS}$$

2. $\mathbf{Ps}$-TS-$\mathbf{Alg}$ *is canonically isomorphic to* $\mathbf{Ps}$-$\widetilde{\mathrm{T}}$-$\mathbf{Alg}$.

Proof. See [54, Proposition 7.8 and Theorem 7.9]. □

Chapter 2
Regular Categories and Factorization Systems

In this chapter we introduce the notions of regular and exact categories, and we examine the relationship between these kind of categories and first-order predicate logic. We refer to [4, 5] for an introduction to the study of this kind of categories, and to [24, 45, 33] for the applications in logic.

In the first section some general results on the theory of regular category are recalled, and we present the so called ***exact completion*** of a regular category, which will play a central rule in the rest of this work. See [6, 8, 10].

This completion provides a left biadjoint to the inclusion

$$\mathbf{Xct} \longrightarrow \mathbf{Reg}$$

of the 2-category $\mathbf{Xct}$ whose objects are exact categories into the 2-category $\mathbf{Reg}$ whose objects are regular categories.

In the second section we explain the categorical semantic in a regular category, and this will provide the starting point for the more general approach to logic using doctrines and fibrations.

In the last section we analyse the works of Kelly [26] on the calculus of relations in a finitely complete category together with a factorization system.

In particular we emphasise two points which emerge from this work: the first is that we do not need necessary a regular category in order to have a calculus of relations with associative composition; it suffices that a finitely-complete category has a proper, stable, factorization system.

The second point is that the inclusion

$$\mathbf{Reg} \longrightarrow \mathbf{LFS}$$

has a left biadjoint, where $\mathbf{LFS}$ is the 2-category whose objects are finitely complete categories with a proper, stable, factorization system.

Therefore combining the exact completion with this results, we get that the inclusion

$$\mathbf{Xct} \longrightarrow \mathbf{LFS}$$

has a left biadjoint.

2.1 Regular categories and exact completion

The notions of a regular and of an exact category are among the most interesting notions studied in category theory. In fact, several important mathematical situations can be axiomatized in categorical terms as regular or exact categories satisfying some typical axioms. For instance small regular categories are the basis for an invariant definition of first-order (intuitionistic) theories, see [45, 24, 16].

All monadic categories over a power of **Set**, and in particular algebraic categories, are exact. Abelian categories and Grothendieck toposes are other examples of exact categories.

As it is always the case in mathematics, when a new relevant structure emerges and begins to be studied as such, an immediate question is the study of the *free* such structures. Of course, *free* refers to a given forgetful functor, and in the case of regular and exact categories there are several such forgetful functors whose corresponding free functor (left adjoint to the forgetful) should be investigated.

One of the most relevant free construction is the "exact completion" of a regular category, which is based on the theory of relations. We refer to [6, 11, 10] for a detailed description of this topic and for the presentation of the exact completion of a finitely complete category.

Let C be a category, and let A be an object of C. We write $\mathrm{Sub}(A)$ for the full subcategory of the slice category C/A whose objects are subobjects of A. This category is of course a preorder and we follow the usual custom of denoting its unique protomorphism by $\leq$. If C/A is finitely complete, so is $\mathrm{Sub}(A)$, and following the notation of [24] , products in $\mathrm{Sub}(A)$ are called ***intersections*** and are denoted by $\cap$ rather then $\times$.

If the category C has finite limits, then every morphism $f\colon A \longrightarrow B$ induces two functors: the first is the post-composition functor

$$\textstyle\sum_f\colon C/A \longrightarrow C/B$$

and the other is

$$f^*\colon C/B \longrightarrow C/A$$

which sends an object g of C/B to the object f^*g of C/A which is the left-vertical map in the pullback square

$$\begin{tikzcd} P \arrow[r] \arrow[d, "f^*g"'] & C \arrow[d, "g"] \\ A \arrow[r, "f"'] & B. \end{tikzcd}$$

This functor can be restricted to a functor

$$f^*\colon \operatorname{Sub}(B) \longrightarrow \operatorname{Sub}(A)$$

because the pullback of a monomorphism is a monomorphism, and we shall again denote it by f^*.

We say that $\mathcal{C}$ has ***images*** if we can assign to every morphism f a subobject $\operatorname{im} f$ of its codomain, which is the least (in the sense of the preorder $\leq$) subobject of $\operatorname{cod} f$ through which f factors.

Lemma 2.1.1. *Let $\mathcal{C}$ be a category with pullbacks, then the following are equivalent:*

1. *$\mathcal{C}$ has images;*
2. *for every object A the inclusion $\operatorname{Sub}(A) \to \mathcal{C}/A$ has a left adjoint;*
3. *for every morphism $f\colon A \longrightarrow B$ the pullback functor $f^*\colon \operatorname{Sub}(B) \longrightarrow \operatorname{Sub}(A)$ has a left adjoint $\exists_f\colon \operatorname{Sub}(A) \longrightarrow \operatorname{Sub}(B)$.*

Proof.[Sketch] The equivalence of the first two points follows from directly the definitions. If $\mathcal{C}$ has images, then we define $\exists_f$ to be composite

$$\operatorname{Sub}(A) \longrightarrow \mathcal{C}/A \xrightarrow{\Sigma_f} \mathcal{C}/B \xrightarrow{\operatorname{im}} \operatorname{Sub}(B) \qquad (2.1)$$

where the first functor is the inclusion the functor $\sum_f$ acts as post-composition, and im sends an arrow $h\colon C \longrightarrow B$ to the image $\operatorname{im}(h)$. One can verify that composition (2.1) gives a left adjoint to f^*. See [24, Lemma 1.3.1] for all the details. □

The canonical morphism $g\colon \operatorname{dom} f \longrightarrow \operatorname{dom}(\operatorname{im} f)$ which is the unit of the last adjunction of Lemma 2.1.1 is said ***cover***. We use the convention $g\colon \operatorname{dom} f \twoheadrightarrow \operatorname{dom}(\operatorname{im} f)$ to indicate that g is a cover.

Remark 2.1.2. A morphism $f\colon A \longrightarrow B$ is a cover if and only if there exists a monomorphism $B \overset{m}{\rightarrowtail} C$ such that for every commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "l"'] & B \arrow[d, "m"] \arrow[dl, dotted, "k"] \\ D \arrow[r, tail, "g"'] & C \end{tikzcd}$$

where $D \overset{g}{\rightarrowtail} C$ is a monomorphism, there exists a morphism $k\colon B \longrightarrow D$ such that $kf = l$ and $kg = m$. Moreover this morphism is unique since g is monic.

Remark 2.1.3. Observe that if $f\colon A \twoheadrightarrow B$ is a cover then it cannot factor through any proper subobject of its codomain: suppose that f is the unit of an arrow $g\colon A \longrightarrow C$ and that $f = mp$, where m is a monomorphism.

Then $(\operatorname{im} gm)p$ is a factorization of g, hence there exists an arrow r such that $(\operatorname{im} gm)r = \operatorname{im} g$, and since $\operatorname{im} g$ is a monomorphism, we have $mr = \mathrm{id}$. Now we have that m is a monomorphism and $mr = \mathrm{id}$, then we can conclude that m is an isomorphism.

Remark 2.1.4. A regular epimorphism $f\colon A \longrightarrow B$ is a cover, because for any factorization of it through a subobject

$$\begin{tikzcd} C \arrow[r, shift left, "h"] \arrow[r, shift right, "g"'] & A \arrow[r, "f"] \arrow[d, "i"] & B \\ & E \arrow[ur, tail, "m"'] & \end{tikzcd}$$

the morphism m is an isomorphism, since it is a monomorphism and it is the coequalizer of the pair ih and ig.

Remark 2.1.5. We can also observe that if the category C has equalizers, then every cover is an epimorphism, since it cannot factor through the equalizer of any distinct pair of morphisms.

Lemma 2.1.6. *Let C be a category with pullbacks, and let $f\colon A \longrightarrow B$ be a morphism of C. The the following are equivalent:*

1. f is a cover;
2. for every commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \arrow[dl, dashed, "k"] \\ C \arrow[r, tail, "m"'] & D \end{tikzcd}$$

there exists a unique $k\colon B \longrightarrow C$ such that $kf = g$ and $mk = h$.

Proof. By Remark 2.1.2 the first point is a special case of the second. Conversely, the existence of a commutative square and the fact that C has pullback say that f factors through the subobject h^*m of B, and since f is a cover then this must be an isomorphism. This means that h factors through m. Writing $k\colon B \longrightarrow C$ for this factorization, we have $mkf = hf = mg$, whence $kf = g$ since m is monic. □

Morphisms with the property described in Lemma 2.1.6 are sometimes called ***strong epimorphism*** or ***extremal epimorphism***, see for example [4] and [5].

Now we can give the definition of regular category, following the definition of [24].

Definition 2.1.7. A category C is said ***regular*** if it has finite limits, has images, and every cover is stable under pullbacks. A functor between regular categories is called ***regular*** if it preserves finite limits and covers.

Remark 2.1.8. In a regular category C, for every pair of morphisms f and g with common codomain we

$$g^* \operatorname{im} f \cong \operatorname{im}(g^* f)$$

because monomorphisms and covers are stable under pullback. See the following diagram

$$\begin{tikzcd}
\bullet \arrow[r, ->>] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \bullet \arrow[r, >->, "g^* \operatorname{im} f"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \bullet \arrow[d, "g"] \\
\bullet \arrow[r, ->>] & \bullet \arrow[r, >->, "\operatorname{im} f"'] & \bullet .
\end{tikzcd}$$

Example 2.1.9. The category **Set** is regular: covers are surjective functions and images are the usual set-theoretic ones. Similarly one can show that the category **Gp** of groups is regular and more generally, any category monadic over **Set** is regular. Observe that in **Set** and **Gp** covers coincide with the epimorphisms, but for example, in the category **Mon** of monoids, this not holds. An important example of category which has images but it is not regular is the category **Top** of topological spaces: the images are injective continuous functions, and covers are surjection $X \longrightarrow Y$ such that Y is topologized as a quotient space of X. However these covers are not stable under pullbacks. See [13].

The above examples and Remark 2.1.5 suggest that covers and epimorphims not always coincide. The following proposition gives a characterisation of covers in regular categories.

Proposition 2.1.10. *In a regular category, the covers are exactly the regular epimorphisms.*

Proof.[Sketch] By Remark 2.1.4 we already know that regular epimorphism are covers. Conversely let $f\colon A \longrightarrow B$ be a cover in a regular category, and let

$$\begin{tikzcd}
R \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & A \arrow[r, "f"] & B
\end{tikzcd}$$

be its kernel pair. We shall prove that f is the coequalizer of a and b. Let $c\colon A \longrightarrow C$ be a morphism such that $ca = cb$, and let

$$A \xrightarrow{\ d\ } D \xrightarrow{\langle g,h\rangle} B \times C$$

the image of the factorization of $\langle f,c\rangle\colon A \longrightarrow B \times C$. If we prove that g is an isomorphism then $d = g^{-1} f$ and then $hg^{-1} f = c$. Moreover this is the unique factorization of c through f because covers are epimorphisms by Remark 2.1.5. To this end it is sufficient to prove that g is monic, since the cover f factors through it. See [24, Lemma 1.3.4] for all the details. $\square$

Corollary 2.1.11. *In a regular category strong epimorphisms and regular epimorphisms coincide.*

Proof. It follows from Proposition 2.1.10 and Lemma 2.1.6. □

Lemma 2.1.12 (Frobenius reciprocity). *Let* $f\colon A \longrightarrow B$ *be a morphism of a regular category* $\mathcal{C}$*. Then for every subobjects* $A' \rightarrowtail A$ *and* $B' \rightarrowtail B$ *we have*

$$\exists_f(A' \cap f^*(B')) \cong \exists_f(A') \cap B'$$

in $\mathrm{Sub}(B)$.

Proof. Consider the following diagram

$$\begin{tikzcd}
A' \cap f^*(B') \arrow[rr, two heads] \arrow[dr, tail] \arrow[dd, tail] & & \exists_f(A') \cap B' \arrow[dr, tail] \arrow[dd, tail] & \\
& f^*(B') \arrow[rr] \arrow[dd, tail] & & B' \arrow[dd, tail] \\
A' \arrow[rr, two heads] \arrow[dr, tail] & & \exists_f(A') \arrow[dr, tail] & \\
& A \arrow[rr, "f"'] & & B
\end{tikzcd}$$

in which the front, left and right faces are pullbacks. The base of the cube commutes by definition of $\exists_f$. Then the back face is also a pullback, and since the category $\mathcal{C}$ is regular then its top edge is a cover. Therefore, since monomorphisms are stable under pullback, we have that $\exists_f(A') \cap B'$ is the image of the composite morphisms

$$\begin{tikzcd}
A' \cap f^*(B') \arrow[rr] \arrow[dr, two heads] & & B \\
& \exists_f(A') \cap B \arrow[ur, tail] &
\end{tikzcd}$$

and then $\exists_f(A' \cap f^*(B')) \cong \exists_f(A') \cap B'$. □

A regular category needs not have coequalizer for arbitrary pairs of morphisms; we can only prove that it has coequalizers for those pairs which occur as kernel-pairs of a morphism f, since we can factor this morphism and prove that the cover given from this factorization is the coequalizer of this kernel-pair. See [5, 24].

A pair of morphisms which occurs as kernel-pair has some interesting properties, in particular it is an equivalence relation in the sense of the following definition.

Definition 2.1.13. Let $\langle a, b\rangle : R \rightrightarrows A$ be a pair of parallel morphisms in a finitely complete category.

1. We say that $\langle a, b\rangle$ is a ***relation*** if $\langle a, b\rangle : R \longrightarrow A \times A$ is monic;
2. we say $\langle a, b\rangle$ is ***reflexive*** if there exists $r : A \longrightarrow R$ such that $ar = br = \mathrm{id}_A$;
3. we say $\langle a, b\rangle$ is ***symmetric*** if there exists $s : R \longrightarrow R$ such that $as = b$ and $bs = a$;
4. we say $\langle a, b\rangle$ is ***transitive*** if there exists $t : P \longrightarrow R$, where P is the pullback

$$\begin{tikzcd} P \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, "a"] \\ R \arrow[r, "b"'] & A \end{tikzcd}$$

   such that $at = ap$ and $bt = bq$;
5. we say that $\langle a, b\rangle$ is an ***equivalence relation*** if it has all four the above properties.

Remark 2.1.14. Note that if $\langle a, b\rangle$ is a relation, then the morphism r, s and t which verify the other three properties are unique if they exist.

Remark 2.1.15. The kernel pair of any morphism $f : A \longrightarrow B$ in a regular category is an equivalence relation.

We say that an equivalence relation is ***effective*** if it occurs as a kernel-pair. There are some regular categories in which some equivalence relation are not effective, such as the category of torsion free abelian groups: it is regular, but not every equivalence relation is effective. See [24].

Definition 2.1.16. A regular category C is said ***exact*** if every equivalence relation is effective.

Example 2.1.17. The categories **Gp** and **Set** are exacts, and more generally, any category which is monadic over a power of **Set** is exact.

The notion of regular category is precisely the one that allows to develop the calculus of relations as an equational calculus over graphs.

We define a ***relation*** R ***from*** A ***to*** B as a subobject $R \rightarrowtail A \times B$, and the existence of images in a regular category allows us to define the ***composite*** of two relations as follows: if $S \rightarrowtail B \times C$ is another relation from B to C, then the composite $SR \rightarrowtail A \times C$ is

$$SR := \mathrm{im}\left[\sum_{\mathrm{pr}_{A\times C}} \left(\mathrm{pr}^*_{A\times B}(R) \cap \mathrm{pr}^*_{B\times C}(S)\right)\right]$$

where pr's denote projections from $A \times B \times C$. The stability of covers under pullback means that the above composition is ***associative***, see [29], determining in this way the ***category*** $\mathbf{Rel}(C)$ ***of relations of*** C, whose identity morphisms are given by diagonal subobjects.

Notice that $\mathbf{Rel}(C)$ has extra structure:

1. a ***local order*** preserved by composition, which has finite intersections;
2. an ***involution*** $(-)^\circ \colon \mathbf{Rel}(C) \longrightarrow \mathbf{Rel}(C)$ which is the identity on objects and which preserves the local order;
3. an ***embedding*** $C \longrightarrow \mathbf{Rel}(C)$ given by the construction of the graph; it is the identity on the objects and it sends a morphism $f \colon X \longrightarrow Y$ to the relation $\langle \mathrm{id}_A, f\rangle \colon X \longrightarrow A \times B$. The graph of an arrow in $\mathbf{Rel}(C)$ is called ***function***, and sometime we use the notation $f \colon A \longrightarrow B$ to indicate such a relation.

This structure allows to give purely algebraic proofs about facts in C, by using the following lemma.

Lemma 2.1.18. *Let C be a regular category, then*

1. the relation $R \colon A \longrightarrow B$ tabulated by $\langle a, b\rangle$ is a function if and only if

$$RR^\circ \leq \mathrm{id}_A \textit{ and } \mathrm{id}_B \leq R^\circ R;$$

2. a morphism $f \colon A \longrightarrow B$ in C is a monomorphism if and only if $f^\circ f = \mathrm{id}_A$ and it is a regular epimorphism if and only if $ff^\circ = \mathrm{id}_B$;

Proof.[Sketch] Let $f \colon A \longrightarrow B$ be a function. The relation $f^\circ f$ is tabulated by the kernel pair of f, whence

$$\mathrm{id}_B \leq f^\circ f.$$

In particular the equality holds if and only if $f \colon A \longrightarrow B$ is a mononomorphism. Moreover the relation ff° is tabulated by $\mathrm{im}\langle f, f\rangle = \Delta_B \,\mathrm{im}\, f$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, two heads] & B \arrow[r, "\Delta_B"] & B \times B \\
 & I \arrow[u, "\mathrm{im}\, f"] \arrow[ur, "\mathrm{im}\langle f,f\rangle"'] &
\end{tikzcd}$$

and then

$$ff^\circ \leq \mathrm{id}_A \,.$$

In particular the equality holds if and only if $\mathrm{im}\, f$ is an isomorphism, which means that f is a regular epimorphism. For the other implication we refer to [26, 7, 10]. □

One of the uses of the theory of relations is to describe the left-biadjoint to the forgetful 2-functor from the 2-category of exact categories to the one of regular categories. See [11, 10, 6] for all the details.

Definition 2.1.19. Let C be a regular category. The ***exact completion*** $(C)_{\mathrm{ex/reg}}$ is defined as follow:

- an ***object*** is a pair (A, E) where A is an object of C and $E \longrightarrow A \times A$ is an equivalence relations in C;
- a ***morphisms*** $R\colon (A, E) \longrightarrow (B, F)$ is a relation $R\colon A \longrightarrow B$ in C such that

$$RE = FR = R$$

and

$$E \leq R^{\circ}R,\, RR^{\circ} \leq F.$$

The composition is the relations composition, and the identity on (A, E) is E itself.

For the proof that the category $(C)_{\mathrm{ex/reg}}$ is an exact category we refer to [11, 16].

Theorem 2.1.20 (Exact Completion). *Let C be a regular category, and let $\mathcal{A}$ be an exact category. The category $(C)_{\mathrm{ex/reg}}$ is exact and the embedding*

$$C \longrightarrow (C)_{\mathrm{ex/reg}}$$

induces an equivalence between the category $\mathbf{Reg}(C, \mathcal{A})$ of regular functors from C to $\mathcal{A}$ and the category $\mathbf{Xct}((C)_{\mathrm{ex/reg}}, \mathcal{A})$ of regular functors from $(C)_{\mathrm{ex/reg}}$ to $\mathcal{A}$.

Remark 2.1.21. The embedding of 2-categories $\mathbf{Xct} \longrightarrow \mathbf{Reg}$ is full, and then the exact completion is an ***idempotent*** process. Moreover a regular category C is exact if and only if the unit $\eta\colon C \longrightarrow (C)_{\mathrm{ex/reg}}$ is an equivalence.

Remark 2.1.22. A new description of the exact completion $(C)_{\mathrm{ex/reg}}$ of a regular category C is given in [31] using a certain topos $\mathbf{Sh}(C)$ of sheaves on C. In this case the exact completion is then constructed as the closure of C in $\mathbf{Sh}(C)$ under finite limits and coequalizers of equivalence relations. A disadvantage of this approach is that this completion can be applied only to small regular categories.

2.2 First-order categorical logic

Regular categories have exactly what we need for the interpretation of a fragment of a first order language in a category.

We describe the interpretation of the so called regular formulas, and for a general description of this topic we refer to [24, 45].

In particular this part will make clear the relationship between category theory and predicate logic, and it is a direct generalization of the traditional definition due to A. Tasrki of satisfaction of first-order formulae in ordinary set-valued structures.

Definition 2.2.1. A ***first-order signature*** Σ consists of the following data:

1. a set Σ-Sort of ***sorts***;
2. a set Σ-Fun of ***function symbols***, together with a map assigning to each function symbol its ***type***, that is a finite non-empty list of sorts, where the last one is separated from the others by an arrow:

$$f\colon A_1, A_2, \dots, A_n \longrightarrow B$$

   and if $n = 0$ we will say that f is a ***constant***;
3. a set Σ-Rel of ***relation symbols***, together with a map assigning to each relation symbol its ***type***, that is a finite list of sorts:

$$R \rightarrowtail A_1, \dots, A_n$$

   and if $n = 0$, we will say that R is an ***atomic proposition***.

For each sort A of a signature Σ we assume given a supply of ***variables*** of sort A.

Definition 2.2.2. The collection of ***terms*** is defined recursively by the following rules:

- $x\colon A$ is a term for every variable x of sort A;
- $f(t_1, \dots, t_n)\colon B$ for every function symbol $f\colon A_1, \dots, A_n \longrightarrow B$ and $t_1\colon A_1, \dots, t_n\colon A_n$;

We have use denoted $t\colon A$ to say that t is a term of sort A.

Definition 2.2.3. The set of ***regular formulae*** is defined recursively by the following clauses, together with, for every formula ϕ, the finite set $\mathrm{FV}(\phi)$ of the ***free variables*** of ϕ:

1. *Relations*: $R(t_1, \dots, t_n)$ is a formula, if $t_1\colon A_1, \dots, t_n\colon A_n$ are terms and $R \rightarrowtail A_1, \dots, A_n$ is a relation symbol. The free variable of this formula are all the variables occurring in t_i.
2. *Equality*: $s =_A t$ is a formula, if s and t are terms with the same sort A. $\mathrm{FV}(s = t) = \mathrm{FV}(s) \cup \mathrm{FV}(t)$;
3. *Truth*: $\top$ is a formula. $\mathrm{FV}(\top) = \emptyset$;
4. *Binary Conjunction*: $\phi \wedge \psi$ is a formula, if ϕ and ψ are. $\mathrm{FV}(\phi \wedge \psi) = \mathrm{FV}(\phi) \cup \mathrm{FV}(\psi)$;
5. *Existential Quantification*: $(\exists x\colon A)\phi$ is a formula for every formula ϕ. $\mathrm{FV}((\exists x\colon A)\phi) = \mathrm{FV}(\phi) \setminus \{x\}$.

Definition 2.2.4. A ***context*** is a finite list of distinct variables $\vec{x} := x_1, ..., x_n$. The ***type of a context*** is the list of sorts of the variables appearing in it. We say that a context $\vec{x}$ is ***suitable*** for a formula ϕ if all the free variables of ϕ occur in $\vec{x}$; a ***regular formula-in-context*** is an expression of the form $\vec{x}.\phi$, where ϕ is a regular formula and $\vec{x}$ is a suitable contest for ϕ. Similarly, a ***term-in-context*** is an expression of the form $\vec{x}.t$ where t is a term and $\vec{x}$ is a context containing all the variables appearing in t.

Now we introduce the formal expressions which will serve as axioms for the logical theories we wish to consider.

Definition 2.2.5. By a ***regular sequent*** over a signature Σ we mean a formal expression

$$\phi \vdash_{\vec{x}} \psi$$

where ϕ and ψ are regular formulae over Σ and $\vec{x}$ is a context suitable for both of them.

Definition 2.2.6. By a ***regular theory*** over a signature Σ we mean as set $\mathbb{T}$ of regular sequents over Σ, whose elements are called ***axioms*** of $\mathbb{T}$.

Since in the rest of this section we work always with regular sequents and regular theories, we will called it simply sequents and theories.

We conclude this part with two examples of regular theories.

Example 2.2.7 (Elementary theory of abstract categories). A fundamental example of regular theory is the ***elementary theory of abstract categories***. It can be express over a signature of two sort, see [24], but we present it following the notation of Lawvere, see [35]. The signature is given by the following data:

- one sort M, which represents the morphisms;
- two unary function symbols $\mathrm{dom}\colon M \longrightarrow M$ and $\mathrm{cod}\colon M \longrightarrow M$;
- one relation symbol $\Gamma \rightarrowtail M, M, M$.

The axioms of the theory are:

1. $\top \vdash_x \mathrm{cod}(\mathrm{dom}(x)) = \mathrm{dom}(x)$ and $\top \vdash_x \mathrm{dom}(\mathrm{cod}(x)) = \mathrm{cod}(x)$;
2. $\Gamma(x, y, u) \vdash_{x,y,u} \mathrm{dom}(x) = \mathrm{dom}(u) \wedge \mathrm{cod}(y) = \mathrm{cod}(u)$;
3. $\Gamma(x, y, u) \wedge \Gamma(x, y, u') \vdash_{x,y,u,u'} u = u'$;
4. $\mathrm{dom}(y) = \mathrm{cod}(x) \vdash_{x,y} (\exists u)\Gamma(x, y, u)$;
5. $(\exists u)\Gamma(x, y, u) \vdash_{x,y} \mathrm{dom}(y) = \mathrm{cod}(x)$;
6. identity axiom: $\top \vdash_x \Gamma(\mathrm{dom}(x), x, x) \wedge \Gamma(x, \mathrm{cod}(x), x)$;
7. associativity axiom: $\Gamma(x, y, u) \wedge \Gamma(y, z, w) \wedge \Gamma(x, w, f) \wedge \Gamma(u, z, g) \vdash_{x,y,z,u,w} f = g$.

The meaning of the formula $\mathrm{dom}(x) = y$ is "the domain of x is y" (and similarly for cod), and $\Gamma(x, y, u)$ means that "u is the composition x followed by y". Besides the usual means of abbreviating formulas, the following (as well as. others) are special to the elementary theory of abstract categories

$$f\colon x \longrightarrow y \ \text{ means } \ \mathrm{dom}(f) = x \wedge \mathrm{cod}(f) = y$$

and

$$fg = h \text{ means } \Gamma(g, f, h).$$

In this presentation with a signature of only one sort, the objects are identified with the identity morphisms.

Example 2.2.8 (Theory of divisible abelian groups). Another example of regular theory is the theory of divisible, abelian groups; the signature is defined by one sort A, two function symbol $+ \colon A, A \longrightarrow A$, $(-)^{-1} \colon A \longrightarrow A$ and a constant symbol e. This theory is obtained from the theory of abelian groups, which has the following axioms

- $\top \vdash_{x,y,z} (x + y) + z = x + (y + z)$;
- $\top \vdash_x (x)^{-1} + x = e$;
- $\top \vdash_x x + e = x$;
- $\top \vdash_{x,y} x + y = y + x$;

and for every $n > 1$ we add the axiom

$$\top \vdash_x (\exists y)(ny = x).$$

2.2.1 Categorical semantic

Definition 2.2.9. Let C a category with finite products, and let Σ be a signature. A Σ-***structure*** M in C is given by the following data:

1. for every sort A of Σ-sort, is given a object MA in C, and for every finite string of sorts $A_1, \ldots, A_n$ we define

$$M(A_1, \ldots, A_n) := MA_1 \times \ldots \times MA_n.$$

   If the string is the empty one, we define $M([])$ as the terminal object of C;
2. for every function symbol $f \colon A_1, \ldots, A_n \longrightarrow B$, is defined a morphism

$$Mf \colon M(A_1, \ldots, MA_n) \longrightarrow MB$$

   in C;
3. for every relation symbol $R \rightarrowtail A_1, \ldots A_n$, is define a subobject

$$MR \rightarrowtail M(A_1, \ldots A_n)$$

   in C.

Definition 2.2.10. The Σ-structures in C form a category Σ-$\mathbf{Str}(C)$ whose morphisms are called Σ-***structure homomorphisms***: an homomorphisms $h \colon M \longrightarrow N$ consists in a collection of morphisms $h_A \colon MA \longrightarrow NA$ in C, indexed by the sorts of Σ, satisfying the following properties:

1. for every function symbol $f\colon A_1,\dots,A_n \longrightarrow B$ the diagram

$$\begin{tikzcd} M(A_1,\dots,A_n) \arrow[r, "Mf"] \arrow[d, "h_{A_1}\times\dots\times h_{A_n}"'] & MB \arrow[d, "h_B"] \\ N(A_1,\dots,A_n) \arrow[r, "Nf"'] & NB \end{tikzcd}$$

   commutes:
2. for every relation symbol $R \rightarrowtail A_1,\dots A_n$ of Σ there is a commutative diagram in $\mathcal{C}$ of the form:

$$\begin{tikzcd} MR \arrow[r, tail] \arrow[d] & M(A_1,\dots,A_n) \arrow[d, "h_{A_1}\times\dots\times h_{A_n}"] \\ NR \arrow[r, tail] & N(A_1,\dots,A_n). \end{tikzcd}$$

Identities and compositions in Σ-$\mathbf{Str}(\mathcal{C})$ are defined component-wise from those in $\mathcal{C}$.

Remark 2.2.11. Observe that every functor $T\colon \mathcal{C} \longrightarrow \mathcal{D}$ which preserves finite products and monomorphisms induces a functor $\Sigma\text{-}\mathbf{Str}(T)\colon \Sigma\text{-}\mathbf{Str}(\mathcal{C}) \longrightarrow \Sigma\text{-}\mathbf{Str}(\mathcal{D})$ in the natural way; any natural transformation $\alpha\colon T_1 \longrightarrow T_2$ between such functors induces a natural transformation $\Sigma\text{-}\mathbf{Str}(\alpha)\colon \Sigma\text{-}\mathbf{Str}(T_1) \longrightarrow \Sigma\text{-}\mathbf{Str}(T_2)$. Thus the construction Σ-$\mathbf{Str}(-)$ is 2-functorial.

Definition 2.2.12. Let $\mathcal{C}$ be a category with finite products and let M be an object of Σ-$\mathbf{Str}(\mathcal{C})$. Consider a term-in-contest $\vec{x}.t$, where the type of $\vec{x}$ is $A_1,\dots,A_n$, and $t\colon B$. We define the morphism

$$[\![\vec{x}.t]\!]_M\colon M(A_1,\dots,A_n) \longrightarrow MB$$

in $\mathcal{C}$ recursively by the following clauses:

1. if t is a variable, then it must be of the for $x_i\colon A_i$ for some $i \leq n$, and then we define $[\![\vec{x}.t]\!]_M = \mathrm{pr}_i$, where $\mathrm{pr}_i\colon M(A_1,\dots,A_n) \longrightarrow MA_i$ is the projection;
2. if t is $f(t_1,\dots,t_m)$, where $t_i\colon C_i$ and $f\colon C_1,\dots C_m \longrightarrow B$, then $[\![\vec{x}.t]\!]_M$ is defined as the composition of

$$M(A_1,\dots,A_n) \xrightarrow{\langle [\![\vec{x}.t_1]\!]_M,\dots,[\![\vec{x}.t_m]\!]_M\rangle} M(C_1,\dots C_m) \xrightarrow{Mf} MB.$$

Lemma 2.2.13 (Substitution Property). *Let $\vec{y}$ be a suitable contest for $t\colon C$ with $y_i\colon B_i$. Let $\vec{s}$ be a string of terms of the same length and type as $\vec{y}$, and let $\vec{x}$ be a suitable contest for $\vec{s}$ with $x_i\colon A_i$. Then $[\![\vec{x}.t[\vec{s}/\vec{y}]]\!]_M$ is the composite*

$$M(A_1,\ldots,A_n) \xrightarrow{\langle [\![\vec{x}.s_1]\!]_M,\ldots,[\![\vec{x}.s_m]\!]_M\rangle} M(B_1,\ldots B_m) \xrightarrow{[\![\vec{x}.t]\!]_M} MC$$

Proof. Straightforward induction on the structure of the term t. See [24, Lemma 1.2.4]. □

Remark 2.2.14 (Weakening Property). Observe that if $\vec{y}$ is a suitable contest for a term t, we can apply the previous lemma to the string $\vec{s} = \vec{y}$, and take as suitable contest for $\vec{s}$ a contest $\vec{x}$ containing $\vec{y}$. Then we obtain

$$[\![\vec{x}.t]\!]_M = [\![\vec{y}.t]\!]_M \circ \mathrm{pr}$$

where pr is an opportune projection.

Lemma 2.2.15. *Let* $h\colon M \longrightarrow N$ *be an homomorphism of* Σ*-structures in a category* C *with finite products, and let* $\vec{x}.t$ *be a term-in-contest, with* $x_i\colon A_i$ *and* $t\colon B$. *Then the diagram*

$$\begin{tikzcd} M(A_1,\ldots,A_n) \arrow[r, "{[\![\vec{x}.t]\!]_M}"] \arrow[d, "h_{A_1}\times\ldots\times h_{A_n}"'] & MB \arrow[d, "h_B"] \\ N(A_1,\ldots,A_n) \arrow[r, "{[\![\vec{x}.t]\!]_N}"'] & NB \end{tikzcd}$$

commutes.

Proof. The proof is again an induction on the structure of the term t. See [24, Lemma 1.2.5]. □

We turn next to the interpretation of regular formulae in a Σ-structure in a regular category, and it will be clear why one need this structure in order to interpret this kind of formulae.

Definition 2.2.16. Let M be a Σ-structure in a regular category C. A formula in contest $\vec{x}.\phi$, where $x_i\colon A_i$, is interpreted as a subobject

$$[\![\vec{x}.\phi]\!]_M \rightarrowtail M(A_1,\ldots,A_n)$$

according to the following recursive clauses:

1. if ϕ is of the form $R(t_1,\ldots,t_m)$, where R is a relation symbol of type $B_1,\ldots,B_m$, then $[\![\vec{x}.\phi]\!]_M$ is the pullback

$$\begin{tikzcd}
\llbracket \vec{x}.\phi \rrbracket_M \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & MR \arrow[d, tail] \\
M(A_1, \ldots, A_n) \arrow[r, "\langle \llbracket \vec{x}.t_1 \rrbracket_M, \ldots, \llbracket \vec{x}.t_n \rrbracket_M \rangle"'] & M(B_1, \ldots, B_m)
\end{tikzcd}$$

2. if ϕ is of the form $s =_B t$, then $\llbracket \vec{x}.\phi \rrbracket_M$ is the equalizer

$$\begin{tikzcd}
\llbracket \vec{x}.\phi \rrbracket_M \arrow[r, tail] & M(A_1, \ldots, A_n) \arrow[r, shift left, "\llbracket \vec{x}.s \rrbracket"] \arrow[r, shift right, "\llbracket \vec{x}.t \rrbracket"'] & MB
\end{tikzcd}$$

3. if ϕ is $\top$ then $\llbracket \vec{x}.\phi \rrbracket_M$ is the top element of $\mathrm{Sub}(A_1, \ldots, A_n)$;
4. if ϕ is $\gamma \wedge \psi$ then $\llbracket \vec{x}.\phi \rrbracket_M$ is the pullback

$$\begin{tikzcd}
\llbracket \vec{x}.\gamma \wedge \psi \rrbracket_M \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \llbracket \vec{x}.\gamma \rrbracket_M \arrow[d, tail] \\
\llbracket \vec{x}.\psi \rrbracket_M \arrow[r, tail] & M(A_1, \ldots, A_n)
\end{tikzcd}$$

5. if ϕ is $(\exists y \colon B)\psi$ then $\llbracket \vec{x}.\phi \rrbracket_M$ is the image of the following composition

$$\begin{tikzcd}
\llbracket \vec{x}, y.\psi \rrbracket_M \arrow[r, tail] & M(A_1, \ldots, A_n, B) \arrow[r, "\pi"] & M(A_1, \ldots, A_n). \\
& \llbracket \vec{x}.\phi \rrbracket_M \arrow[ur, tail] &
\end{tikzcd}$$

Lemma 2.2.17 (Substitution Property). *Let $\vec{y}.\phi$ be a regular formula, with $y_i \colon B_i$, and let M be a Σ-structure on a regular category $\mathcal{C}$. Let $\vec{s}$ be a string a terms of the same length and type of $\vec{y}$, and let $\vec{x}$ be a suitable conntest for all the terms of $\vec{s}$, with $x_j \colon A_j$. Then $\llbracket \vec{x}.\phi[\vec{s}/\vec{y}] \rrbracket_M$ is the pullback of the following diagram:*

$$\begin{tikzcd}
\llbracket \vec{x}.\phi[\vec{s}/\vec{y}] \rrbracket_M \arrow[r] \arrow[d, tail] & \llbracket \vec{y}.\phi \rrbracket_M \arrow[d, tail] \\
M(A_1, \ldots, A_n) \arrow[r, "\langle \llbracket \vec{x}.s_1 \rrbracket_M, \ldots, \llbracket \vec{x}.s_m \rrbracket_M \rangle"'] & M(B_1, \ldots, B_m)
\end{tikzcd}$$

Proof. One can prove the lemma using induction on the structure of the formula ϕ. See [24, Lemma 1.2.7]. $\square$

Lemma 2.2.18. *Let $\mathcal{C}$ be a regular category, and let $h\colon M \longrightarrow N$ be an homomorphism of Σ-structure. Then for every regular formula-in-context the $\vec{x}.\phi$ the diagram*

$$\begin{tikzcd} \llbracket \vec{x}.\phi \rrbracket_M \arrow[r, tail] \arrow[d] & M(A_1,\dots,A_n) \arrow[d, "h_{A_1}\times\dots\times h_{A_n}"] \\ \llbracket \vec{x}.\phi \rrbracket_N \arrow[r, tail] & N(A_1,\dots,A_n) \end{tikzcd}$$

commutes.

Proof. One can prove this lemma using induction on the structure of the formula ϕ. See [24, Lemma 1.2.9]. □

2.2.2 Structural rules

The definitions of the last two subsections provide a useful tool for constructing objects and morphisms with prescribed properties in a given category $\mathcal{C}$, but first-order logic is more than a convenient shorthand for describing particular objects and morphisms of a category; it is also a tool for proving things about them via suitable deduction-system.

We develop such deduction system for the fragment of first order logic we have considered and we prove that it is sound for the categorical semantic. This means that anything is formally derivable in the deduction system is valid in any structure for a given signature in a regular category.

Our deduction-system will be formulated as ***sequent calculi***, following the notation of [24]. It provide rules for inferring the validity of certain sequents.

Given the axioms and inference rules below, the notion of ***proof*** (or derivation) is the usual one: a chain of inference rules whose premises are the axioms in the system and whose conclusion is the given sequent.

Allowing the axioms of theory $\mathbb{T}$ to be taken as premises yields the notion of proof relative to a theory $\mathbb{T}$.

Definition 2.2.19. 1. The ***structural rules*** consist of:

a. ***identity axiom***

$$\frac{}{\phi \vdash_{\vec{x}} \phi}$$

b. ***substitution rule***

$$\frac{\phi \vdash_{\vec{x}} \psi}{\phi[\vec{s}/\vec{x}] \vdash_{\vec{y}} \psi[\vec{s}/\vec{x}]}$$

where $\vec{y}$ is a suitable contest for every term of the string $\vec{s}$ and $\vec{s}$ has the same length and type of $\vec{x}$;

c. ***cut rule***

$$\frac{\phi \vdash_{\vec{x}} \psi \qquad \psi \vdash_{\vec{x}} \chi}{\phi \vdash_{\vec{x}} \chi}$$

2. the ***equality rules*** are

$$\frac{}{\top \vdash_{x} (x = x)}$$

and

$$\frac{}{(\vec{x} = \vec{y}) \wedge \phi \vdash_{\vec{z}} \phi[\vec{y}/\vec{x}]}$$

where $\vec{z}$ is a suitable contest for ϕ, and it contains $\vec{x}$ and $\vec{y}$;

3. the ***rules for finite conjunction*** are

$$\frac{}{\phi \vdash_{\vec{x}} \top}$$

$$\frac{}{\phi \wedge \psi \vdash_{\vec{x}} \phi}$$

$$\frac{}{\phi \wedge \psi \vdash_{\vec{x}} \psi}$$

and the rule

$$\frac{\phi \vdash_{\vec{x}} \psi \qquad \phi \vdash_{\vec{x}} \chi}{\phi \vdash_{\vec{x}} \psi \wedge \chi}$$

4. the ***rule for existential quantification*** consists of the double rule

$$\frac{\phi \vdash_{\vec{x},y} \psi}{\overline{(\exists y \colon B)\phi \vdash_{\vec{x}} \psi}}$$

5. the ***Frobenius axiom*** consist of the following

$$\frac{}{\phi \wedge (\exists y \colon B)\psi \vdash_{\vec{x}} (\exists y \colon B)(\phi \wedge \psi)}$$

Remark 2.2.20 (Weakening Rule). Observe that the substitution rule allows us to derive form a sequent $\phi \vdash_{\vec{x}} \psi$, a sequent $\phi \vdash_{\vec{y}} \psi$, where the context $\vec{y}$ contains the context $\vec{x}$.

Remark 2.2.21. Observe that the Frobenius axiom is provable in a full first order logic, using the rules for implication.

Definition 2.2.22. We say that a regular sequent σ is ***provable*** in a regular theory $\mathbb{T}$, if there exists a derivation relative to $\mathbb{T}$ (using the rules described previously), which has σ at the bottom line.

Definition 2.2.23. Let M be a Σ-structure over a regular category $\mathcal{C}$.

1. If $\sigma = (\phi \vdash_{\vec{x}} \psi)$ is a sequent with $x_i : A_i$, we say σ is ***satisfied in*** M if
$$[\![\vec{x}.\phi]\!] \leq [\![\vec{x}.\psi]\!]$$
in $\mathrm{Sub}(M(A_1, ..., A_n))$, and we will write $M \vDash \sigma$.
2. If $\mathbb{T}$ is a regular theory over Σ, we say M is a ***model*** of $\mathbb{T}$ if all the axioms of $\mathbb{T}$ are satisfied in M, and we will write $M \vDash \mathbb{T}$.
3. We define $\mathbb{T}$-$\mathbf{Mod}(C)$ the full subcategory of Σ-$\mathbf{Str}(C)$, whose objects are models of $\mathbb{T}$.

Example 2.2.24. 1. A ***topological group*** can be seen as a model of the theory of groups in the category of topological spaces.
2. Similarly, an ***algebraic*** (resp. Lie) group is a model of the algebraic theory of groups in the category of algebraic varieties (resp. the category of smooth manifolds).

Lemma 2.2.25. *Let* $T : C \longrightarrow \mathcal{D}$ *be a regular functor between regular categories, let M be a Σ-structure in C and let σ be a sequent over Σ. If $M \vDash \sigma$, then* Σ-$\mathbf{Str}(T)(M) \vDash \sigma$ *in* $\mathcal{D}$.

Proof. It is again an induction on the structure. See [24, Lemma 1.2.13]. □

Remark 2.2.26. Observe that by Lemma 2.2.25 we can restrict the functor defined in Remark 2.2.11 to $\mathbb{T}\text{-}\mathbf{Mod}(T) : \mathbb{T}\text{-}\mathbf{Mod}(C) \longrightarrow \mathbb{T}\text{-}\mathbf{Mod}(\mathcal{D})$.

Theorem 2.2.27 (Soundness). *Let $\mathbb{T}$ be a regular theory over a signature Σ, and let M be a model of $\mathbb{T}$ in a cartesian category C. If σ is a regular sequent which is provable in $\mathbb{T}$, then $M \vDash \sigma$.*

Proof. See [24, Proposition 1.3.2]. □

2.2.3 Internal language

Let C be a regular category, we can define a canonical signature Σ_C called ***internal language*** as follow: the sorts of Σ_C are the objects of C, and for every non-empty list of object of C $A_1, ..., A_n, B$ and every morphism $f : A_1 \times ... \times A_n \longrightarrow B$, we define a function symbol $f : A_1, ..., A_n \longrightarrow B$ in Σ_C.

Observe that every morphism of the form $f : A_1 \times ... \times A_n \longrightarrow B$ induces more function symbols: the first one is the n-ary function symbol $f : A_1, ..., A_n \longrightarrow B$, then there is the $(n-1)$-ary function symbol $f : A_1, ..., A_{n-2}, (A_{n-1} \times A_n) \longrightarrow B$

and so on until the unary one $f\colon A_1 \times \ldots \times A_n \longrightarrow B$. In the same way for every subobjects $R \rightarrowtail A_1, \ldots, A_n$ we define the relation symbols of the signature.

Moreover there is a canonical structure for Σ_C in C, called ***tautological structure***, which assigns to every sort A the corresponding object A in C and to every function symbol the corresponding morphism in C. The usefulness of this notion lies in the fact that properties of C or constructions in it can often be formulated in terms of satisfaction of certain formulae over Σ_C in the canonical structure. The internal language can thus be used for proving things about C. See [24, 50] for all details.

2.2.4 Syntactic category

In Subsection 2.2.2 we have seen a Soundness Theorem, asserting that "anything is provable is true". Now we look at the converse result, asserting that "anything is true is provable"; this result is known to logicians as a Completeness Theorem.

Starting from a regular theory $\mathbb{T}$ over a signature Σ we want to construct a category $C_{\mathbb{T}}$ of the appropriate kind and a particular model $M_{\mathbb{T}}$ for this theory.

We call this category the ***syntactic category*** $C_{\mathbb{T}}$, and the model $M_{\mathbb{T}}$ ***generic model***. As for the previous section we follow the notation of [24], and we suggest for further reading [45].

Definition 2.2.28. Let $\mathbb{T}$ be a regular theory over a signature Σ. We define the ***syntactic category*** $C_{\mathbb{T}}$ as follow:

- ***objects***: the objects of $C_{\mathbb{T}}$ are α-equivalence classes $\{\vec{x}.\phi\}$ of regular-formula-in-contest, where $\vec{x}.\phi$ and $\vec{y}.\psi$ are said to be α-equivalent if $\vec{x}$ and $\vec{y}$ have the same length and type, and if $\phi[\vec{y}/\vec{x}]$ is exactly ψ. Observe that by Lemma 2.2.17, if $\vec{x}.\phi$ and $\vec{y}.\psi$ are α-equivalent, then $\|\vec{x}.\phi\|_M$ is equal to $\|\vec{y}.\psi\|_M$.
- ***morphisms***: let $\{\vec{x}.\phi\}$ and $\{\vec{y}.\psi\}$ be objects of $C_{\mathbb{T}}$. A $\mathbb{T}$-***provably functional*** proposition θ from $\vec{x}.\phi$ to $\vec{y}.\psi$ is a regular formula whose free variables are in $\vec{x}, \vec{y}$ and such that the following sequents are provable in $\mathbb{T}$:

  1. $\theta \vdash_{\vec{x},\vec{y}} \phi \wedge \psi$;
  2. $\theta \wedge \theta\{\vec{z}/\vec{y}\} \vdash_{\vec{x},\vec{y},\vec{z}} \vec{z} = \vec{y}$;
  3. $\phi \vdash_{\vec{x}} (\exists \vec{y})\theta$.

We take the morphisms of $C_{\mathbb{T}}$ to be $\mathbb{T}$-provable-equivalence classes of formulae-in-contest which are $\mathbb{T}$-provably functional, and we denote a class of this type by $[\theta]$.

Now consider the following diagram

$$\{\vec{x}.\phi\} \xrightarrow{[\theta]} \{\vec{y}.\psi\} \xrightarrow{[\gamma]} \{\vec{z}.\chi\}.$$

The composition $[\gamma] \circ [\theta]$ is defined as $[\exists \vec{y}(\theta \wedge \psi)]$. It is direct to check that this formula is $\mathbb{T}$-provably functional, for example the first sequent is provable as follow

$$\dfrac{\dfrac{\theta\wedge\gamma\vdash_{\vec{x},\vec{y},\vec{z}}\theta \qquad \dfrac{\theta\vdash_{\vec{x},\vec{y},\vec{z}}\phi\wedge\psi}{\theta\vdash_{\vec{x},\vec{y},\vec{z}}\phi}}{\theta\wedge\gamma\vdash_{\vec{x},\vec{y},\vec{z}}\phi} \qquad \dfrac{\theta\wedge\gamma\vdash_{\vec{x},\vec{y},\vec{z}}\gamma \qquad \dfrac{\gamma\vdash_{\vec{x},\vec{y},\vec{z}}\psi\wedge\chi}{\gamma\vdash_{\vec{x},\vec{y},\vec{z}}\chi}}{\theta\wedge\gamma\vdash_{\vec{x},\vec{y},\vec{z}}\chi}}{\dfrac{\theta\wedge\gamma\vdash_{\vec{x},\vec{y},\vec{z}}\phi\wedge\chi}{\exists\vec{y}(\theta\wedge\gamma)\vdash_{\vec{x},\vec{z}}\phi\wedge\chi}}$$

Similarly we can verify the others sequents, and the associativity of the composition. Moreover the identity morphism on $\{\vec{x}.\phi\}$ is the equivalence class

$$\{\vec{x}.\phi\} \xrightarrow{[\phi\wedge(x=z)]} \{\vec{z}.\phi[\vec{z}/\vec{x}]\}.$$

Theorem 2.2.29. *$\mathcal{C}_{\mathbb{T}}$ is a category, and it is regular.*

Proof. See [24, Lemma 1.4.2 and Lemma 1.4.10]. □

Lemma 2.2.30. *Any subobject of $\{\vec{x}.\phi\}$ in $\mathcal{C}_{\mathbb{T}}$ is isomorphic to one of the form*

$$\{\vec{x'}.\psi[\vec{x'}/\vec{x}]\} \xrightarrow{[\psi\wedge(\vec{x}=\vec{x'})]} \{\vec{x}.\phi\}$$

where ψ is a formula such that the sequent $\psi\vdash_{\vec{x}}\phi$ is provable in $\mathbb{T}$. Moreover for two subobjects ψ and χ we have $\{\vec{x}.\psi\}\leq\{\vec{x}.\phi\}$ in $\mathrm{Sub}_{\mathcal{C}}(\{\vec{x}.\phi\})$ if and only if the sequent $\psi\vdash_{\vec{x}}\chi$ is provable in $\mathbb{T}$.

Proof. See [24, Lemma 1.4.4 (iv)]. □

Observe that we have a canonical Σ-structure $M_{\mathbb{T}}$ in $\mathcal{C}_{\mathbb{T}}$, which assigns to a sort A the object $\{x.\top\}$, where $x\colon A$, to every function symbol $f\colon A_1,\ldots,A_n \longrightarrow B$ the morphism

$$\{x_1,\ldots,x_n.\top\} \xrightarrow{[f(x_1,\ldots,x_n)=y]} \{y.\top\}$$

and to a relation symbol $R\rightarrowtail A_1,\ldots,A_n$ the suboject of $\{x_1,\ldots,x_n.\top\}$ whose domain is $\{x_1,\ldots,x_n.R(x_1,\ldots,x_n)\}$.

Lemma 2.2.31. *Let $\mathbb{T}$ be a regular theory.*

- *For any term-in-contest $\vec{x}.t$ over Σ, the interpretation in $M_{\mathbb{T}}$ is the morphism*

$$[t(x)=y]\colon\{\vec{x}.\top\}\longrightarrow\{y.\top\}$$

- *For every formula in contest $\vec{x}.\phi$ the interpretation in $M_{\mathbb{T}}$ is the subobject*

$$\{\vec{x}.\phi\}\rightarrowtail\{\vec{x}.\top\}$$

- *A sequent $\phi \vdash_{\vec{x}} \psi$ is satisfied in $M_{\mathbb{T}}$ is and only if it is provable in $\mathbb{T}$.*

Proof. See [24, Lemma 1.4.5]. □

Theorem 2.2.32 (Completeness). *Let $\mathbb{T}$ be a regular theory. If a sequent in $\mathbb{T}$ is satisfied in all the models of $\mathbb{T}$, then it is provable in $\mathbb{T}$.*

Proposition 2.2.33. *Let $\mathbb{T}$ be a regular theory. Then for any regular category $\mathcal{D}$ the functor*

$$\mathbf{Reg}(\mathcal{C}_{\mathbb{T}}, \mathcal{D}) \to \mathbb{T}\text{-}\mathbf{Mod}(\mathcal{D})$$

which sends a regular functor $F\colon \mathcal{C}_{\mathbb{T}} \longrightarrow \mathcal{D}$ to $F(M_{\mathbb{T}})$ is an equivalence of categories.

Remark 2.2.34. Observe that the previous theorem tell us that the functor $\mathbb{T}$-$\mathbf{Mod}(-)$ is in some sense representable. In other words, it states that studying models of a regular theory is equivalent to study regular functors from the syntactic category to the category on which we want to give an interpretation of the theory.

Definition 2.2.35. Let $\mathbb{T}$ and $\mathbb{T}'$ be regular theories. We said that $\mathbb{T}$ and $\mathbb{T}'$ are ***Morita-equivalent*** if $\mathcal{C}_{\mathbb{T}}$ and $\mathcal{C}_{\mathbb{T}'}$ are equivalent.

2.3 Factorization systems

A number of author have observed that the regularity of category $\mathcal{C}$ is not necessary for the existence of a "calculus of relations" in $\mathcal{C}$ with an associative composition of relations.

In this section we will see that it is sufficient that the finitely complete category $\mathcal{C}$ has a proper factorization system $\langle \mathcal{E}, \mathcal{M} \rangle$ whose class $\mathcal{E}$ is stable under pullbacks.

We begin with a review of factorization systems. For more details on the former we refer to [15] and [26].

Definition 2.3.1. Let $\mathcal{E}$ and $\mathcal{M}$ be subclasses of the category $\mathcal{C}^{\to}$ of arrows in an arbitrary category $\mathcal{C}$. We say that $\langle \mathcal{E}, \mathcal{M} \rangle$ is a ***factorization system*** for $\mathcal{C}$ if the following hold:

1. Iso $\subset \mathcal{E} \cap \mathcal{M}$, where Iso denotes the class of isomorphisms of $\mathcal{C}$;
2. $\mathcal{E}$ and $\mathcal{M}$ are closed under composition;
3. $\mathcal{E}$ and $\mathcal{M}$ satisfy the ***diagonal fill-in property***, namely, for every commutative square

$$\begin{tikzcd} \bullet \arrow[r, "e"] \arrow[d] & \bullet \arrow[d] \arrow[dl, dashed, "f"] \\ \bullet \arrow[r, "m"'] & \bullet \end{tikzcd}$$

where $e \in \mathcal{E}$ and $m \in \mathcal{M}$ there is a unique f making the previous diagram commutative;
4. every arrow f in C factors as $f = me$, where $e \in \mathcal{E}$ and $m \in \mathcal{M}$, and we shall call m the ***image*** of f;

Remark 2.3.2. Observe that the condition 3 in Definition 2.3.1 is equivalent to the following: if $fme = m'e'f'$, where $e, e' \in \mathcal{E}$ and $m, m' \in \mathcal{M}$, there exists a unique w such that the diagram

$$\begin{tikzcd} \bullet \arrow[r, "e"] \arrow[d, "f'"'] & \bullet \arrow[r, "m"] \arrow[d, dashed, "w"] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "e'"'] & \bullet \arrow[r, "m'"'] & \bullet \end{tikzcd}$$

commutes.

Definition 2.3.3. A factorization system $\langle \mathcal{E}, \mathcal{M} \rangle$ is said to be ***proper*** if every morphism in $\mathcal{E}$ is an epimorphism, and every morphism in $\mathcal{M}$ is a monomorphism.

Remark 2.3.4. Suppose that $\langle \mathcal{E}, \mathcal{M} \rangle$ is a stable factorization system on a finitely complete category C. When $\mathcal{M}$ is the class of all monomorphisms, $\mathcal{E}$ consists of the strong epimorphisms, so that C is a regular category.

Definition 2.3.5. Assume that C has finite limits. A proper factorization system $\langle \mathcal{E}, \mathcal{M} \rangle$ for C is said to be ***stable*** if the class $\mathcal{E}$ is stable under pulling back.

Remark 2.3.6. If C has finite limits and $\langle \mathcal{E}, \mathcal{M} \rangle$ is a factorization system, then by the diagonal fill-in property, the class $\mathcal{M}$ is stable under pullbacks.

Example 2.3.7. The category **Top** of topological spaces is not regular, as it is observed in Example 2.1.9, but it has a stable factorization system $\langle \mathcal{E}, \mathcal{M} \rangle$ where $\mathcal{E}$ is the class of epimorphisms, and $\mathcal{M}$ is the class of strong monomorphisms.

Following the same idea used in Section 2.1, the notion of relation can be generalized in the context of factorization systems.

Definition 2.3.8. Let C be finitely complete category and let $\langle \mathcal{E}, \mathcal{M} \rangle$ be a proper factorization system. A ***relation*** R ***from*** A ***to*** B is a subobject

$$\langle r_1, r_2 \rangle \colon R \longrightarrow A \times B$$

such that the inclusion $\langle r_1, r_2 \rangle$ lies in $\mathcal{M}$.

For the rest of this section we fix a category with finite limits C and a stable factorization system $\langle \mathcal{E}, \mathcal{M} \rangle$. As for the case of regular categories, we want to define a category whose morphisms are relations. Therefore we shall define how one can compose relations, and we prove that the composition is associative.

We can compose two relations $R \colon A \longrightarrow B$ and $Q \colon B \longrightarrow C$ by forming the diagram

$$\begin{tikzcd}
& & P \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & \\
& R \arrow[dl, "r_2"'] \arrow[dr, "r_1"] & & Q \arrow[dl, "q_2"'] \arrow[dr, "q_1"] & \\
A & & B & & C
\end{tikzcd}$$

Fig. 2.1: Composition of Relations

where the diamond is a pullback and taking for QR the image of

$$\langle r_1 p_1, q_2 p_2 \rangle \colon P \longrightarrow A \times C \ .$$

To prove that the composition of relations is associative we need the following lemma, and here we can see that it is fundamental that $\mathcal{E}$ is stable under pullbacks.

Lemma 2.3.9. *A morphism* $g\colon A \longrightarrow B$ *of* $\mathcal{C}$ *factorizes through the image of* $f\colon C \longrightarrow B$ *if and only of if we have* $gh = ft$ *for some* $h \in \mathcal{E}$ *and some* t*.*

Proof. Let $f = me$ be the $\langle \mathcal{E}, \mathcal{M} \rangle$-factorization. If h and t as above exist, by the diagonal fill-in property, see Definition 2.3.1, we have an s such that

$$\begin{tikzcd}
\bullet \arrow[r, "h"] \arrow[d, "et"'] & \bullet \arrow[d, "g"] \arrow[dl, "s"] \\
\bullet \arrow[r, "m"'] & \bullet
\end{tikzcd}$$

commutes, since $h \in \mathcal{E}$ and $m \in \mathcal{M}$. Conversely if $g = ml$, then we consider the pullback

$$\begin{tikzcd}
\bullet \arrow[r, "h"] \arrow[d, "t"'] \arrow[dr, phantom, "\lrcorner", very near start] & \bullet \arrow[d, "l"] \\
\bullet \arrow[r, "e"'] & \bullet
\end{tikzcd}$$

and we have $met = mlh$. Thus $ft = gh$ with $h \in \mathcal{E}$ because the factorization system is stable. □

Given a relation $R\colon A \longrightarrow B$, we say that a span $\langle a, b \rangle \colon X \longrightarrow A \times B$ ***belongs to*** R, written $b\,(R)\,a$, if it factors through the inclusion $\langle r_1, r_2 \rangle \colon R \longrightarrow A \times B$.

Note that the graph $\langle \mathrm{id}_A, f\rangle \colon A \longrightarrow A\times B$ of a morphism f is a relation from A to B since it is a coretraction and hence certainly lies in $\mathcal{M}$, because $\mathcal{M}$ contains isomorphisms. Following the notation of [26], we identify this relation with f, and we call it a ***function***. Note that $b\,(f)\,a$ means $b = fa$. Apply Lemma 2.3.9 we can prove the following result.

Proposition 2.3.10. *Let $R\colon A \longrightarrow B$ and $Q\colon B \longrightarrow C$ be two relations, and let $QR\colon A \longrightarrow C$ be the composition. For a span $\langle a,c\rangle \colon X \longrightarrow A\times C$ we have $c\,(QR)\,a$ if and only if, for some $e \in \mathcal{E}$ and some b, we have $ce\,(Q)\,b$ and $b\,(R)\,ae$.*

Proof. [Sketch] If $c\,(QR)\,a$ then $\langle a,c\rangle$ factors through QR, hence it factor through the image of $\langle r_1p_1, q_2p_2\rangle$, where p_1 and p_2 are the arrows of the pullback

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Q \arrow[d, "q_2"] \\ R \arrow[r, "r_2"'] & B. \end{tikzcd}$$

By Lemma 2.3.9 there exist $e \in \mathcal{E}$ and t such that

$$\langle a,c\rangle e = \langle r_1p_1, q_2p_2\rangle t$$

and then $ae = r_1p_1t$ and $ce = q_2p_2t$. We define $b := r_2p_1t$, and by definition of p_1 and p_2 we have $b = q_1p_2t$. Therefore $\langle ae, b\rangle = \langle r_1, r_2\rangle p_1t$, which means that $b\,(R)\,ae$, and $\langle b, ce\rangle = \langle q_1, q_2\rangle p_2t$, hence $ce\,(Q)\,b$. The converse is similar, and we refer to [26] and [9] for the proof. □

Corollary 2.3.11. *The composition of relations is associative.*

Proof. Consider a span $\langle a,d\rangle \colon X \longrightarrow A\times D$ and three relations $P\colon A \longrightarrow B$, $Q\colon B \longrightarrow C$ and $R\colon C \longrightarrow D$. We want to prove that $d\,((RQ)P)\,a$ if and only if $d\,((RQ)P)\,a$.

By Proposition 2.3.10 $d\,((RQ)P)\,a$ holds if and only if there exist $e_1, e_2 \in \mathcal{E}$ and some morphisms b_1, b_2 such that

1. $b_1\,(P)\,ae_1$;
2. $b_2\,(Q)\,b_1e_2$;
3. $de_1e_2\,(R)\,b_2$.

Similarly $d\,(R(QP))\,a$ holds if and only if there exist $\bar{e}_1, \bar{e}_2 \in \mathcal{E}$ and some morphisms $\bar{b}_1, \bar{b}_2$ such that

1. $\bar{b}_2\,(P)\,a\bar{e}_1\bar{e}_2$;
2. $\bar{b}_1\bar{e}_2\,(Q)\,\bar{b}_2$;

3. $d\bar{e}_1\,(R)\,\bar{b}_1$.

If we have $d\,((RQ)P)\,a$ then we have $b_1\,(P)\,ae_1$. By definition this means that $\langle ae_1, b_1\rangle$ factors through P, and then $\langle ae_1, b_1\rangle e_2$ factors through P. Therefore we have $b_1e_2\,(P)\,ae_1e_2$. Defining $\bar{e}_1 := e_1e_2$, $\bar{e}_2 := \mathrm{id}$, $\bar{b}_1 := b_2$ and $\bar{b}_2 := b_1e_2$ we obtain that

$$b_1\,(P)\,ae_1,\ \bar{b}_1\bar{e}_2\,(Q)\,\bar{b}_2,\ de_1e_2\,(R)\,b_2.$$

Thus we have that $d\,((RQ)P)\,a$ implies $d\,(R(QP))\,a$. Similarly we can prove the converse, and we can conclude that the composition of relations is associative. □

Remark 2.3.12. We have proved that if the factorization system is stable then the composition in associative, but there is a strong result, see [29] and [26], which is that the composition is associative if and only if the factorization system $\langle \mathcal{E}, \mathcal{M}\rangle$ is stable.

So the objects of C and the relations with respect a fixed stable, proper factorization system $\langle \mathcal{E}, \mathcal{M}\rangle$, form the category $\mathbf{Rel}(C; \mathcal{E}, \mathcal{M})$, or $\mathbf{Rel}(C)$ if the factorization system is clear from the context. In particular this is a 2-category when we order the relations from A to B in the usual way as subobjects.

By Remark 2.3.6 every pullback along a morphism of $\mathcal{M}$ is in $\mathcal{M}$, hence this 2-category has local finite infima, $R \wedge R'$ being the usual intersection. Moreover the top element of $\mathbf{Rel}(C)(A, B)$ is the relation $\mathrm{id}_{A\times B}\colon A \times B \longrightarrow A \times B$.

As in the case of regular categories, the 2-category $\mathbf{Rel}(C)$ has an anti-involution sending $R\colon A \longrightarrow B$ to $R^\circ\colon B \longrightarrow A$ given by $\langle r_2, r_1\rangle\colon R \longrightarrow B \times A$, and there is an embedding $C \longrightarrow \mathbf{Rel}(C)$ sending morphisms of C to functions of $\mathbf{Rel}(C)$.

Observe that when the relation R is a function $f\colon A \longrightarrow B$ we do not need to pass to an image when we consider the composition Qf since $\langle p_1, q_2p_2\rangle$ is already in $\mathcal{M}$ because it is the pullback of $\langle q_1, q_2\rangle$ along $f \times \mathrm{id}_C$ and $\mathcal{M}$ is stable under pullbacks. Thus for functions f and g the composition $g^\circ f$ is the relation R tabulated by $\langle r_1, r_2\rangle$ in the pullback

$$\begin{tikzcd} & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \\ A \arrow[dr, "f"'] & & C \arrow[dl, "g"] \\ & B & \end{tikzcd}$$

In particular $f^\circ f$ is tabulated by the kernel pair of f, hence

$$\mathrm{id}_A \leq f^\circ f \tag{2.2}$$

and the equality holds if and only if f is a monomorphism.

When, however, R is an arbitrary relation and Q is a function $k\colon B \longrightarrow C$ the pullback in the composition 2.1 is trivial, but we are obligated to take the image of $\langle r_1, kr_2\rangle$. Taking R to be h° with $h\colon B \longrightarrow A$, the relation kh° is given by the image of $\langle h, k\rangle\colon B \longrightarrow A \times C$. Thus we have that for a given relation $S\colon A \longrightarrow C$ we have $k\,(S)\,h$ if and only if $kh^\circ \leq S$. This means that if S is tabulated by s_1 and s_2 then

$$S = s_2 s_1^\circ \tag{2.3}$$

Moreover taking $k = h\colon B \longrightarrow A$ we have that hh° is tabulated by the image of $\langle h, h\rangle\colon B \longrightarrow A \times A$, which is $\Delta_A i$ where $i\colon I \longrightarrow A$ is the image of h, because $\Delta_A \in \mathcal{M}$ and $\mathcal{M}$ is closed under composition. Thus we have

$$hh^\circ \leq id_A \tag{2.4}$$

with equality if and only if $h \in \mathcal{E}$.

Recall that an arrow in a 2-category is often called a ***map*** if it has a right adjoint. The origin of this name being the observation that the maps in $\mathbf{Rel}(C)$ for a regular C are precisely the functions.

Remark 2.3.13. From 2.4 and 2.2 follows that every function f is a ***map*** in the 2-category $\mathbf{Rel}(C)$, because it has f° as right adjoint. Observe that if C is a regular category these are the only maps, as it is observed 2.1.18.

We denote by Σ the class of monomorphisms which are also morphisms of $\mathcal{E}$.

Proposition 2.3.14. *A relation $R\colon A \longrightarrow B$ tabulated by r_1 and r_2 is a map if and only if $r_1 \in \Sigma$. In this case we have $R \dashv R^\circ$.*

Proof. If $r_1 \in \Sigma$ we have $R \dashv R^\circ$ since (2.3) and (2.4) give

$$RR^\circ = r_2 r_1^\circ r_1 r_2^\circ = r_2 r_2^\circ \leq \mathrm{id}_A$$

and (2.3) and (2.2) give

$$R^\circ R = r_1 r_2^\circ r_2 r_1^\circ \geq r_1 r_1^\circ = \mathrm{id}_B\,.$$

Suppose conversely that R has a right adjoint $Q\colon B \longrightarrow A$. Since $\mathrm{id}_A \leq QR$ by Proposition 2.3.10 there exists some $e \in \mathcal{E}$ and b such that $b\,(R)\,e$, hence $p_1 t = e$ for some t. So r_1 lies in $\mathcal{E}$. It remains to show that r_1 is a monomorphism. Let $x, y\colon K \longrightarrow R$ be two morphisms such that $r_1 x = r_2 y$. If we prove that also $r_2 x = r_2 y$ then we can conclude that $x = y$ because $\langle r_1, r_2\rangle$ is in $\mathcal{M}$. So consider $\langle r_1 x, r_1 x\rangle$, and since it factorizes through the identity relation id_A and $\mathrm{id}_A \leq QR$, then we have $r_1 x\,(QR)\,r_1 x$. Using again Proposition 2.3.10, we get some $e \in \mathcal{E}$ and some b with $r_1 xe\,(Q)\,b$. Since trivially $r_2 xe\,(R)\,r_1 xe$ we have that

$$r_2 xe\,(RQ)\,b.$$

Thus $\langle b, r_2 xe\rangle$ factorizes through id_B, since $RQ \leq \mathrm{id}_B$, and then $r_2 xe = b$. Moreover we also obtain that $r_2 ye = b$ because $r_1 x = r_1 y$. Since $e \in \mathcal{E}$ is an epimorphism we have $r_2x = r_2y$, and then we have the equality $\langle r_1, r_2\rangle x = \langle r_1, r_2\rangle y$, which implies $x = y$. □

Since an invertible arrow in a 2-category is in particular a map, then applying Proposition 2.3.14 we have the following Corollary.

Corollary 2.3.15. *A relation* $R\colon A \longrightarrow B$ *tabulated by* r_1 *and* r_2 *is invertible in* $\mathbf{Rel}(C)$ *if and only if* $r_1, r_2 \in \Sigma$. *In particular a function* $f\colon A \longrightarrow B$ *is invertible in* $\mathbf{Rel}(C)$ *if and only if* $f \in \Sigma$.

Let us now write $\mathcal{B}$ for the category $\mathbf{Map}\,\mathbf{Rel}(C)$ of maps of $\mathbf{Rel}(C)$, with $J\colon C \longrightarrow \mathcal{B}$ for the inclusion. Recall that the objects of $\mathbf{Map}\,\mathbf{Rel}(C)$ are the objects of C, 1-cells are the maps of $\mathbf{Rel}(C)$ and 2-cells are defined as in $\mathbf{Rel}(C)$.

A morphism $R\colon A \longrightarrow B$ in $\mathcal{B}$ is tabulated by $\langle r_1, r_2\rangle$ where $r_1 \in \Sigma$ by Proposition 2.3.14, and if $Q \leq R$ with $Q\colon A \longrightarrow B$ and Q is tabulated by $\langle q_1, q_2\rangle$, then there exists a morphism h such that $r_1h = q_1$ and $r_2h = q_2$. Then h lies in Σ because $r_1, q_1 \in \Sigma$, and h lies in $\mathcal{M}$. In particular h is invertible and we can conclude that $Q = R$. In other word $\mathcal{B}$ is a mere category; the 2-categorical structure it inherits from $\mathbf{Rel}(C)$ is locally discrete.

We see that Corollary 2.3 means that the class Σ consists precisely in the arrows inverted by the functor $J\colon C \longrightarrow \mathcal{B}$, and in general this inclusion turn to be the universal $J\colon C \longrightarrow C[\Sigma^{-1}]$ inverting the class Σ. See [26].

Theorem 2.3.16. *The inclusion* $J\colon C \longrightarrow \mathcal{B}$ *is the projection of* C *to its category of fractions* $C[\Sigma^{-1}]$. *Moreover the category* $\mathcal{B}$ *is regular and the inclusion preserves finite limits.*

Proof. See [26]. □

We define **LFS** the 2-category whose objects are finitely complete categories with stable factorization system, a 1-cell $F\colon C \longrightarrow C'$ is a left-exact functor such that $F\mathcal{E} \subset \mathcal{E}'$ and $F\mathcal{M} \subset \mathcal{M}'$, and 2-cells are natural transformations. It has a full sub-2-category **Reg** given by the regular categories with $\mathcal{M}$ consisting of the monomorphisms. A 1-cell in **LFS** between regular categories is just a left-exact functor that preserves strong epimorphisms.

Theorem 2.3.17. *The inclusion* $\mathbf{Reg} \longrightarrow \mathbf{LFS}$ *has a left biadjoint functor. In particular* $J\colon C \longrightarrow \mathcal{B}$ *is the reflection of the 2-category* **LFS** *into* **Reg**.

Proof.[Sketch] Let $T\colon C \longrightarrow \mathcal{D}$ be a 1-cell in **LFS**, and let $\mathcal{D}$ be a regular category. Since T is left-exact it preserves monomorphisms, and since $T\mathcal{E}$ are strong epimorphisms, then it inverts every element of Σ. By Theorem 2.3.16, $T = SJ$ for an unique $S\colon \mathcal{B} \longrightarrow \mathcal{D}$, and S is a 1-cell of **Reg**. Then we have the universal property of J also for 2-cells is classical, for any category of fractions. See [26] and [17] for all the details. □

A functor $T\colon \mathcal{C} \longrightarrow \mathcal{C}'$ in **LFS** induces a 2-functor $\mathbf{Rel}(T)\colon \mathbf{Rel}(\mathcal{C}) \longrightarrow \mathbf{Rel}(\mathcal{C}')$ which sends an object A to TA, and sends a relation $R\colon A \longrightarrow B$ to $TR\colon TA \longrightarrow TB$ tabulated by $\langle Tr_1, Tr_2\rangle\colon TR \longrightarrow TA \times TB$. Moreover it preserves inequalities. We refer to [26] for more details and for the proof of the following theorem.

Theorem 2.3.18. *For* $\mathcal{B} = \mathbf{Map}\ \mathbf{Rel}(\mathcal{C})$*, the 2-functor* $\mathbf{Rel}(J)\colon \mathbf{Rel}(\mathcal{C}) \longrightarrow \mathbf{Rel}(\mathcal{B})$ *induced by* $J\colon \mathcal{C} \longrightarrow \mathcal{B}$ *is an isomorphism of 2-categories.*

Chapter 3
Elementary Doctrines and Exact Completion

In this chapter we introduce the notion of primary, elementary and existential doctrine, and we presents some free completions which allow us to generalize both the regular completion of a category with finite limits and the exact completion of a regular category introduced in [6, 8, 10] in the context of elementary existential doctrines. We refer to the works of Maietti and Rosolini [41, 42, 43, 44] for all the details.

The construction of an exact category starting from an elementary existential doctrine is not trivial, and we divide this construction in several intermediate steps.

The first result that we want to prove is that the 2-category of existential m-variational doctrine **Ex-mVar** is 2-equivalent to the 2-category of stable factorizations systems **LFS**.

The second is to prove that every elementary existential doctrine can be completed to an existential m-variational doctrine.

In order to show the first equivalence we introduce the notion of fibrations, see [5, 21], and we use the result proved by Hughes and Jacobs in [19], where they show that factorizations systems are equivalent to bifibrations with full subset types, strong coproducts and coproducts.

Then we show that m-variational existential doctrines are equivalent to this kind of fibrations, and we give a complete description of the factorization systems constructed from this doctrines.

After that we analyse the regular category $\mathbf{Ef}_P$ constructed from an existential m-variational doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ which is the result of the composition of the following functors

$$\mathbf{Ex\text{-}mVar} \xrightarrow{\equiv} \mathbf{LFS} \xrightarrow{\mathbf{Map}\ \mathbf{Rel}(-)} \mathbf{Reg}\,.$$

It is shown in [42, 43] that an elementary existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ can be completed to an existential m-variational one $(P)_{cd}$, applying two free constructions: the first which produces an elementary existential doctrine with full comprehensions, and the second which enforces the comprehensive diagonals.

Composing the previous free constructions together with the exact completion of a regular category, we obtain a first instance of exact completion of an elementary existential doctrine

$$\mathbf{EED} \xrightarrow{(-)_{cd}} \mathbf{Ex\text{-}mVar} \xrightarrow{\equiv} \mathbf{LFS} \xrightarrow{\mathbf{Map\ Rel}(-)} \mathbf{Reg} \xrightarrow{(-)_{\mathrm{ex/reg}}} \mathbf{Xct}\,.$$

Moreover Maietti and Rosolini observed that if the base category of an existential m-variational doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ has quotients, stable and of effective descents, then the category $\mathbf{Ef}_P$ is exact.

In particular we have the following equivalence of exact categories

$$\mathbf{Ef}_{(P)_{cqd}} \equiv (\mathbf{Ef}_{(P)_{cd}})_{\mathrm{ex/reg}}.$$

So the quotients completion provides a second way to complete an existential m-variational doctrine to an exact category.

We conclude this chapter with a comparison between the previous exact completions and the tripos-to-topos construction. See [20, 51].

We introduce a generalized tripos to topos construction for elementary existential doctrine, which provides an exact category $\mathcal{T}_P$ starting from an elementary existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$. See [41, 44] for all the details.

A direct calculation will show that the category $\mathcal{T}_P$ is equivalent to the category $\mathbf{Ef}_{(P)_{cqd}}$.

3.1 Fibrations and factorization systems

It is a known fact that a factorization system on a category with sufficient pullbacks give rise to a fibration.

In [19] the fibrations that arise in such a way are characterized, by making precise the logical structure that is given by the factorization system. The original motivation for this investigation comes from the Birkhoff's result about definability and deductibility for universal algebras [2].

In this section we describe how factorization systems give rise to bifibration with certain logical properties, and then we describe how one can go in the reverse direction: from bifibrations with this structure to factorization systems.

We refer to [19] for all the details about these constructions, and to [21] for a complete description of fibrations and their relation with structural aspects of logic and type theory.

Definition 3.1.1. Let $p\colon \mathcal{G} \longrightarrow \mathcal{C}$ be a functor, and $f\colon X \longrightarrow Y$ an arrow in $\mathcal{G}$, with $pf = u\colon A \longrightarrow B$. We say that f is ***Cartesian over*** u if for every morphism $g\colon Z \longrightarrow Y$ in $\mathcal{G}$ such that pg factors through u, $pg = u \circ w$, there exists a unique $h\colon Z \longrightarrow X$ such that $g = f \circ h$ and $ph = u$.

Definition 3.1.2. A ***fibration*** is a functor $p\colon \mathcal{G} \longrightarrow \mathcal{C}$ such that, for every Y in $\mathcal{G}$ and every $u\colon I \longrightarrow pY$, there exists a Cartesian $f\colon X \longrightarrow Y$ over u.

For a given fibration $p\colon \mathcal{G} \longrightarrow \mathcal{C}$, and any A in $\mathcal{C}$, let $\mathcal{G}_A$ be the ***fibre category*** over A: the objects of $\mathcal{G}_A$ are the objects X of $\mathcal{G}$ such that $pX = A$, and the morphisms of $\mathcal{G}_A$ are the morphisms $f\colon X \longrightarrow Y$ of $\mathcal{G}$ such that $pf = \mathrm{id}_A$, and they are called ***vertical morphisms***.

Let $p\colon \mathcal{G} \longrightarrow \mathcal{C}$ be a fibration, and let X be an object of $\mathcal{G}$ such that $pX = A$. For every morphism $u\colon B \longrightarrow A$ we fix a Cartesian morphism $\overline{u}Y$ above u and we denote $\mathrm{dom}(\overline{u}Y) = u^*(Y)$ the domain of the morphism $\overline{u}Y$. Then we can define the ***substitution functor***

$$u^*\colon \mathcal{G}_A \longrightarrow \mathcal{G}_B$$

sending X to $u^*(X)$, and a morphism $f\colon X \longrightarrow Y$ of $\mathcal{G}_A$ to u^*f, which is defined as the unique morphism such that the square

$$\begin{tikzcd}
u^*(X) \arrow[r, "\overline{u}X"] \arrow[d, dashed, "u^*f"'] & X \arrow[d, "f"] \\
u^*(Y) \arrow[r, "\overline{u}Y"'] & Y
\end{tikzcd}$$

commutes.

Observe that this morphism exists because $p(\overline{u}X \circ f) = u = p(\overline{u}Y)$, and then there is a unique vertical arrow making the previous diagram commutative.

Example 3.1.3 (Codomain fibration). For every category $\mathcal{C}$ with finite limits we define the ***codomain fibration*** $\mathrm{cod}\colon \mathcal{C}^{\rightarrow} \longrightarrow \mathcal{C}$, sending an object $f\colon A \longrightarrow B$ of the arrows category $\mathcal{C}^{\rightarrow}$ to B. The Cartesian morphisms in $\mathcal{C}^{\rightarrow}$ coincide with pullback squares in $\mathcal{C}$.

Example 3.1.4 (Subobjects fibration). We consider the category of subobjects $\mathbf{Sub}(\mathcal{C})$ of $\mathcal{C}$ (with finite limits), whose objects are equivalence classes of monomorphisms, where the relation we are considering is the usual which identify two monomorphisms m and n if $m \leq n$ and $n \leq m$. Then the restriction of the codomain functor to $\mathrm{cod}\colon \mathbf{Sub}(\mathcal{C}) \longrightarrow \mathcal{C}$ is a fibration, and it is called ***subobjects fibration***. This fibration is used to describe the so called internal logic of $\mathcal{C}$. See [21].

Example 3.1.5 (Equivalence Relations). Recall from Definition 2.1.13 that a relation on an object A of a category $\mathcal{C}$ with finite limits is just a monomorphism $R \rightarrowtail A \times A$. We can define a subcategory $\mathbf{Rel}(\mathcal{C})$ of $\mathbf{Sub}(\mathcal{C})$ whose objects are relations, and then we define the fibration $p\colon \mathbf{Rel}(\mathcal{C}) \longrightarrow \mathcal{C}$ which sends an object $R \rightarrowtail A \times A$ to A. Moreover we can consider the subcategory $\mathbf{ERel}(\mathcal{C})$ of $\mathbf{Rel}(\mathcal{C})$ whose objects are equivalence relations, and we can restrict the previous functor to the fibration $p\colon \mathbf{ERel}(\mathcal{C}) \longrightarrow \mathcal{C}$.

Example 3.1.6. Let $\mathcal{C}$ be a category with finite limits, and let $\langle \mathcal{E}, \mathcal{M} \rangle$ be a factorization system for $\mathcal{C}$. Since $\mathcal{M}$ is stable under pullbacks, the functor

$$\mathrm{cod}\colon \mathcal{M} \longrightarrow \mathcal{C}$$

is a sub-fibration of the codomain fibration. Given $A \in \mathcal{C}$ the fibre category $\mathcal{M}_A$ over A consists of $\mathcal{M}$-morphisms with codomain A. Given a morphism $f\colon A \longrightarrow B$ in $\mathcal{C}$ the substitution functor

$$f^*\colon \mathcal{M}_B \longrightarrow \mathcal{M}_A$$

is defined by pullback along f. Moreover the fibration is a ***fibred pre-order*** if and only if $\langle \mathcal{E}, \mathcal{M} \rangle$ is a proper factorization.

Throughout what follows, we assume that $\mathcal{C}$ has finite limits.

Definition 3.1.7. Let $p\colon \mathcal{G} \longrightarrow \mathcal{C}$ be a fibration. We say that p is a ***op-fibration*** if

$$p^{\mathrm{op}}\colon \mathcal{G}^{\mathrm{op}} \longrightarrow \mathcal{C}^{\mathrm{op}}$$

is a fibration. If p is both a fibration and a op-fibration, we say that p is a ***bifibration.***

Let $p\colon \mathcal{G} \longrightarrow \mathcal{C}$ be a bifibration, let X be an object of $\mathcal{G}$ such that $pX = A$.

Consider a morphism $u\colon A \longrightarrow B$ of $\mathcal{C}$. We denote by $\underline{u}X$ the op-morphism above u and by $\coprod_u X$ the codomain of this morphism. We recall an equivalent characterization of bifibrations. See [21] for the details.

Lemma 3.1.8. *Let $p\colon \mathcal{G} \longrightarrow \mathcal{C}$ be a fibration. It is a bifibration if and only if for every morphism $u\colon A \longrightarrow B$ we have $\coprod_u \dashv u^*$.*

A bifibration $p\colon \mathcal{G} \longrightarrow \mathcal{C}$ is said to satisfy ***Beck-Chevalley*** just in case, for every pullback square in $\mathcal{C}$

$$\begin{tikzcd} A \arrow[r, "v"] \arrow[d, "r"'] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, "s"] \\ C \arrow[r, "u"'] & D \end{tikzcd}$$

the canonical natural transformation $\coprod_v r^* \longrightarrow s^* \coprod_u$ is an isomorphism.

In this case we say that the fibration p ***has coproducts***. As it is observed in [21, 19], not all bifibrations satisfy Beck-Chevalley.

Example 3.1.9. Given a factorization system $\langle \mathcal{E}, \mathcal{M} \rangle$ on $\mathcal{C}$ the codomain fibration $\mathrm{cod}\colon \mathcal{M} \longrightarrow \mathcal{C}$ defined in Example 3.1.3 is a bifibration. Indeed for every $f\colon A \longrightarrow B$ in $\mathcal{C}$ let

$$\mathrm{im}(f \circ -)\colon \mathcal{M}_A \longrightarrow \mathcal{M}_B$$

be the functor taking $m \colon M \longrightarrow A$ to the image of $\mathrm{im}(f \circ m)$. It is easy to check that $\mathrm{im}(f \circ -) \dashv f^*$. Moreover the induced bifibration satisfies Beck-Chevalley just in the case the factorization system is stable. See [19].

Lemma 3.1.10. *The bifibration* $\mathrm{cod} \colon \mathcal{M} \longrightarrow \mathcal{C}$ *induced by a factorization system* $\langle \mathcal{E}, \mathcal{M} \rangle$ *has coproducts if and only if the class* $\mathcal{E}$ *is stable under pullbacks.*

Proof.[Sketch] Suppose that $\mathrm{cod} \colon \mathcal{M} \longrightarrow \mathcal{C}$ has coproducts and the arrow $u \colon I \longrightarrow J$ is an $\mathcal{E}$-morphism, and consider the following pullback

$$\begin{tikzcd}
A \arrow[r, "v"] \arrow[d, "r"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "s"] \\
I \arrow[r, "u"'] & J.
\end{tikzcd}$$

We want to prove that $\mathrm{im}(v) \cong \mathrm{id}_B$. Thus

$$\mathrm{im}(v) \cong \mathrm{im}(v \circ -)\,\mathrm{id}_A \cong \mathrm{im}(v \circ -) r^*\,\mathrm{id}_I \cong s^*\,\mathrm{im}(u \circ -)\,\mathrm{id}_I$$

where the last isomorphism holds by Beck-Chevalley. Since u is an $\mathcal{E}$-morphism, we have $\mathrm{im}(u) \cong \mathrm{id}_J$ and then

$$\mathrm{im}(v) \cong s^*\,\mathrm{im}(u \circ -)\,\mathrm{id}_I \cong s^*\,\mathrm{id}_j \cong \mathrm{id}_B\,.$$

Therefore we can conclude that $\mathcal{E}$ is stable under pullbacks. For the other implication we refer to [19]. $\square$

Definition 3.1.11. Let $p \colon \mathcal{G} \longrightarrow \mathcal{C}$ be a fibration. We say that p has ***subset type***, if p has a right adjoint $\top \colon \mathcal{C} \longrightarrow \mathcal{G}$, where $p \circ \top = \mathrm{id}_{\mathcal{C}}$, and $\top$ has a further right adjoint $\{-\} \colon \mathcal{G} \longrightarrow \mathcal{C}$.

The logical interpretation of the Definition 3.1.11 is the following: given a fibration $p \colon \mathcal{G} \longrightarrow \mathcal{C}$ we view the category $\mathcal{G}$ as providing predicates over the types in $\mathcal{C}$, and the functor p takes a predicate to the type of its free variable. If p has a right adjoint $\top \colon \mathcal{C} \longrightarrow \mathcal{G}$ such that $p\top = \mathrm{id}_{\mathcal{C}}$, then this adjoint picks out the maximal or "true" predicate for each type.

A right adjoint $\{-\} \colon \mathcal{G} \longrightarrow \mathcal{C}$ to $\top$ is interpreted as mapping a predicate to its extension in $\mathcal{C}$.

Definition 3.1.12. For X in $\mathcal{G}$, define the ***projection*** $\pi_X \colon \{X\} \longrightarrow pX$, to be $p\varepsilon_X$, where

$$\varepsilon \colon \top\{-\} \Longrightarrow \mathrm{id}_{\mathcal{G}}$$

is the counit of the adjunction between $\top$ and $\{-\}$. If the functor $X \mapsto \pi_X$ from $\mathcal{G}$ and $\mathcal{C}^{\to}$ is full and faithful, we say that p has ***full subset types***.

Example 3.1.13. The subobject fibration defined in Example 3.1.4 has full subset type. The associate functor $\{-\}\colon \mathbf{Sub}(C) \longrightarrow C$ takes a representation $(X \rightarrowtail A)$ of a subobject to its codomain $A \in C$.

Example 3.1.14. Given a factorization system $\langle \mathcal{E}, \mathcal{M} \rangle$, the codomain fibration $\mathrm{cod}\colon \mathcal{M} \longrightarrow C$ defined in Example 3.1.6 has full subset types. The functor $\top\colon C \longrightarrow \mathcal{M}$ is given by $\top(A) = \mathrm{id}_A\colon A \longrightarrow A$, and the right adjoint is the domain functor $\mathrm{dom}\colon \mathcal{M} \longrightarrow C$ which sends a morphism to its domain.

The following definition basically says that the subset projections are closed under composition. We use the same terminology of [21, 19], but the original name "strong coproducts" comes from dependent type theory, see [49].

Definition 3.1.15. Let $p\colon \mathcal{G} \longrightarrow C$ be a bifibration with full subset type. We say that p admits ***strong coproducts*** along subset projections just in the case, for every X in $\mathcal{G}$ and Y in $\mathcal{G}_{\{X\}}$, the canonical arrow $\{\underline{\pi}_X Y\}$ is an isomorphism.

$$\begin{tikzcd}
\{Y\} \arrow[r, "\{\underline{\pi}_X Y\}"] \arrow[d, "\pi"'] & \{\coprod_{\pi_X} Y\} \arrow[d, "\pi"] \\
\{X\} \arrow[r, "\pi_X"'] & pX
\end{tikzcd}$$

Example 3.1.16. For any factorization system $\langle \mathcal{E}, \mathcal{M} \rangle$ the bifibration $\mathrm{cod}\colon \mathcal{M} \longrightarrow C$ admits strong coproducts with respect to projections. Indeed for every $\mathcal{M}$-morphisms $m\colon M \longrightarrow B$ and $n\colon B \longrightarrow C$, we have the following diagram

$$\begin{tikzcd}
M \arrow[r, "\cong"] \arrow[d, "m=\pi_m"'] & \mathrm{im}(n \circ m) \arrow[d, "\pi"] \\
B \arrow[r, "n=\pi_n"'] & C
\end{tikzcd}$$

where the top arrow is an isomorphism because $\mathcal{M}$ is closed under compositions.

Thus we have proved the following result.

Theorem 3.1.17. *Let C have a factorization system $\langle \mathcal{E}, \mathcal{M} \rangle$. Then the bifibration $\mathrm{cod}\colon \mathcal{M} \longrightarrow C$ has full subset types and admits strong coproducts along subset projections. Moreover $\mathcal{E}$ is stable under pullbacks if and only if $\mathrm{cod}\colon \mathcal{M} \longrightarrow C$ has coproducts.*

We have shown that factorization systems induce bifibration with full subset types and strong coproducts along subset projections. Now we see how to construct a factorization system from such bifibration.

Definition 3.1.18. Consider a fibration $p\colon \mathcal{G} \longrightarrow \mathcal{C}$ which satisfies the following conditions:

1. p is a bifibration;
2. p has full subset type;
3. p has strong coproducts along subset projections.

We call such p a ***factorization fibration***.

Lemma 3.1.19. *Let $p\colon \mathcal{G} \longrightarrow \mathcal{C}$ be a factorization fibration. Any morphism $f\colon A \longrightarrow B$ in $\mathcal{C}$ can be factored as*

$$A \xrightarrow{\ u\ } \{X\} \xrightarrow{\ \pi_X\ } B$$

where u is of the form $u = \{\underline{f}\top A\} \circ \eta_A$ and η is the unit of the adjunction $\top \dashv \{-\}$.

Proof. We take the factorization

$$A \xrightarrow{\ \eta_A\ } \{\top A\} \xrightarrow{\ \{\underline{f}\top A\}\ } \{\textstyle\coprod_f \top A\} \xrightarrow{\ \pi_{\coprod_f \top A}\ } B$$

and we see that this works since

$$\pi_{\coprod_f \top A}\{\underline{f}\top A\}\eta_A = p(\varepsilon_{\coprod_f \top A})p\top(\{\underline{f}\top A\}\eta_A) = p(\varepsilon_{\coprod_f \top A}\top(\{\underline{f}\top A\}\eta_A))$$

and since ε is a natural transformation, we have $\varepsilon_{\coprod_f \top A}\top\{\underline{f}\top A\} = \underline{f}\top A\varepsilon_{\top A}$. Thus

$$\pi_{\coprod_f \top A}\{\underline{f}\top A\}\eta_A = p(\underline{f}\top A(\varepsilon_{\top A}\top\eta_A)) = p(\underline{f}\top A) = f.$$

□

Observe that Lemma 3.1.19 suggests to define the factorization system $\langle \mathcal{E}, \mathcal{M}\rangle$ associated to a factorization fibration in the following way: the abstract epis $\mathcal{E}$ will consist of composites

$$A \xrightarrow{\ u\ } \{X\} \xrightarrow{\ \cong\ } B \tag{3.1}$$

where u is of the form defined in Lemma 3.1.19. The abstract monos $\mathcal{M}$ will consist of composites

$$B \xrightarrow{\ \cong\ } \{X\} \xrightarrow{\ \pi_X\ } pX. \tag{3.2}$$

Theorem 3.1.20. *Let $p\colon \mathcal{G} \longrightarrow \mathcal{C}$ be a factorization fibration. The fibration p induces a factorization system $\langle \mathcal{E}, \mathcal{M}\rangle$ on $\mathcal{C}$ where the arrows of $\mathcal{E}$ are of the form* (3.1) *and the arrows of $\mathcal{M}$ are of the form* (3.2).

Proof. We refer to [19, Theorem 3.6] for the complete proof of this result. □

The next two theorems show that this construction is coherent, in the following sense: if we consider a factorization system $\langle \mathcal{E}, \mathcal{M} \rangle$, and we construct the system associated with the codomain fibration $\mathrm{cod}\colon \mathcal{M} \longrightarrow \mathcal{C}$, we get $\langle \mathcal{E}, \mathcal{M} \rangle$ again.

On the other hand, if we consider a factorization fibration $p\colon \mathcal{G} \longrightarrow \mathcal{C}$, and we construct the associated factorization system $\langle \mathcal{E}, \mathcal{M} \rangle$ and the codomain fibration $\mathrm{cod}\colon \mathcal{M} \longrightarrow \mathcal{C}$, we do not get $p\colon \mathcal{G} \longrightarrow \mathcal{C}$ again, but an equivalent fibration. As corollary we see that this construction is idempotent.

For the proof of the following results see [19, Theorem 3.7] and [19, Theorem 3.8].

Theorem 3.1.21. *Let $\langle \mathcal{E}, \mathcal{M} \rangle$ be a factorization system on $\mathcal{C}$. Let $\langle \mathcal{E}', \mathcal{M}' \rangle$ be the factorization system constructed via the codomain fibration $\mathrm{cod}\colon \mathcal{M} \longrightarrow \mathcal{C}$, as in Theorem 3.1.20. Then $\mathcal{E}' = \mathcal{E}$ and $\mathcal{M}' = \mathcal{M}$.*

Theorem 3.1.22. *Let $p\colon \mathcal{G} \longrightarrow \mathcal{C}$ be a factorization fibration and let $\langle \mathcal{E}, \mathcal{M} \rangle$ be the corresponding factorization system, constructed via Theorem 3.1.20. Then we have the following equivalence*

$$\begin{tikzcd} \mathcal{G} \arrow[rr, bend left, "\sim"] \arrow[dr, "p"'] & & \mathcal{M} \arrow[dl, "\mathrm{cod}"] \\ & \mathcal{C} & \end{tikzcd}$$

Corollary 3.1.23. *Let $p\colon \mathcal{G} \longrightarrow \mathcal{M}$ be a factorization fibration. The the class $\mathcal{E}$ of the induced factorization system $\langle \mathcal{E}, \mathcal{M} \rangle$ is stable under pullbacks if and only if the factorization system has coproducts. Moreover the factorization system $\langle \mathcal{E}, \mathcal{M} \rangle$ is stable in the sense of Definition 2.3.5 if and only if p is a fibred pre-order and it has coproducts.*

3.2 Doctrines

In Section 2.1 we have seen one of the common development of the categorical approach to predicate logic, in which formulas in context are interpreted as subobjects in categories. See for example the classic text by Makkai and Reyes [45].

In this section we review the notion of primary, elementary and existential doctrine from [43, 42, 44], which is appropriate to analyse the notion of quotient of an equivalence relation and comprehensions. For more details we refer to the previous articles and [41].

The notion of primary doctrine is an obvious generalization of that of a ***hyper-doctrine***. Hyperdoctrines were introduced, in a series of seminal papers, by F.W. Lawvere to synthesize the structural properties of logical systems, see [36, 37, 38]. His intuition was to consider logical languages and theories as indexed categories and to study their 2-categorical properties.

Recall from [36] that a hyperdoctrine is a functor $F\colon C^{\mathrm{op}} \longrightarrow \mathbf{Heyt}$ from a cartesian closed category C to the category of Heyting algebras satisfying some further conditions: for every morphism $f\colon A \longrightarrow B$ in C, the morphism $F_f\colon FB \longrightarrow FA$ of Heyting algebras, where F_f denotes the action of the functor F on the morphism f, has a left adjoint $\exists_f$ and a right adjoint $\forall_f$ satisfying the Beck–Chevalley condition.

The intuition is that a hyperdoctrine determines an appropriate categorical structure to abstract both notions of first order theory and of interpretation.

Finally there are also some hyperdoctrines, called ***triposes***, which provide a notion of model for higher order logic, see [51].

These were introduced under the name ***formal topos*** by J. Bénabou already beginning of the 1970ies and later reinvented by Hyland, Johnstone and Pitts around 1980.

Definition 3.2.1. Let C be a category with finite products. A ***primary doctrine*** is a functor $P\colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ from the opposite of the category C to the category of inf-semilattices.

The structure of a primary doctrine is just what is needed to handle a many-sorted logic with binary conjunctions and a true constant, as seen in the following example.

Example 3.2.2. Let $\mathbb{T}$ be a theory in a first order language $\mathbf{Sg}$. We define the ***Lindenbaum-Tarski*** primary doctrine

$$LT\colon C_{\mathbb{T}}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$$

where $C_{\mathbb{T}}$ is the category of lists of variables and term substitutions:

- ***objects*** of $C_{\mathbb{T}}$ are finite lists of variables $\vec{x} := (x_1, \ldots, x_n)$, and we include the empty list ();
- a ***morphisms*** from $(x_1, \ldots, x_n)$ into $(y_1, \ldots, y_m)$ is a substitution $[t_1/y_1, \ldots, t_m/y_m]$ where the terms t_i are built in $\mathbf{Sg}$ on the variable $x_1, \ldots, x_n$;
- the ***composition*** of two morphisms $[\vec{t}/\vec{y}]\colon \vec{x} \longrightarrow \vec{y}$ and $[\vec{s}/\vec{z}]\colon \vec{y} \longrightarrow \vec{z}$ is given by the substitution

$$[s_1[\vec{t}/\vec{y}]/z_k, \ldots, s_k[\vec{t}/\vec{y}]/z_k]\colon \vec{x} \longrightarrow \vec{z}\ .$$

The functor $LT\colon C_{\mathbb{T}}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ sends a list $(x_1, \ldots, x_n)$ in the class $LT(x_1, \ldots, x_n)$ of all well formed formulas in the context $(x_1, \ldots, x_n)$. We say that $\psi \leq \phi$ where $\phi, \psi \in LT(x_1, \ldots, x_n)$ if $\psi \vdash_{\mathbb{T}} \phi$, and then we quotient in the usual way to obtain a partial order on $LT(x_1, \ldots, x_n)$. Given a morphism of $C_{\mathbb{T}}$

$$[t_1/y_1, \ldots, t_m/y_m]\colon (x_1, \ldots, x_n) \longrightarrow (y_1, \ldots, y_m)$$

then the functor $LT_{[\vec{t}/\vec{y}]}$ acts as the substitution $LT_{[\vec{t}/\vec{y}]}(\psi(y_1,\dots,y_m)) = \psi[\vec{t}/\vec{y}]$. For all the detail we refer to [43], and for the case of a many sorted first order theory we refer to [50].

Example 3.2.3. Let C be a category with finite limits. The functor

$$\mathrm{Sub}_C \colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$$

assigns to an object A in C the poset $\mathrm{Sub}_C(A)$ of subobjects of A in C and, for an arrow $f\colon B \longrightarrow A$, the functor $\mathrm{Sub}_C(f)\colon \mathrm{Sub}_C(A) \longrightarrow \mathrm{Sub}_C(B)$ is given by pulling a subobject back along f. We denote the objects of $\mathrm{Sub}_C(A)$ by $[\, B \xrightarrow{f} A \,]$.

Example 3.2.4. Consider a category $\mathcal{D}$ with finite products and weak pullbacks: the doctrine is given by the functor of weak subobjects

$$\Psi_{\mathcal{D}} \colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$$

where $\Psi_{\mathcal{D}}(A)$ is the poset reflection of the slice category $\mathcal{D}/A$, and for an arrow $f\colon B \longrightarrow A$, the functor $\Psi_{\mathcal{D}}(f)\colon \Psi_{\mathcal{D}}(A) \longrightarrow \Psi_{\mathcal{D}}(B)$ is given by a weak pullback of an arrow $g\colon X \longrightarrow A$ with f.

Example 3.2.5. The following example of primary doctrine $S\colon \mathbf{Set}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is the set-theoretic hyperdoctrine and it can be considered in any axiomatic set theory such as ZF. We briefly recall its definition:

- the category **Set** is the category of sets and functions;
- $S(A)$ is is the poset category of subsets of the set A whose morphisms are inclusions;
- a functor $S_f\colon S(B) \longrightarrow S(A)$ acts as the inverse image $f^{-1}(U)$ for every subset U of B.

For the rest of the section let C be a category with binary products. An ***elementary doctrine*** on C is a primary doctrine $P\colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ such that for every A in C there is an object δ_A in $P(A \times A)$ such that

1. the assignment
$$\exists_{\langle \mathrm{id}_A, \mathrm{id}_A\rangle}(\alpha) := P_{\mathrm{pr}_1}(\alpha) \wedge \delta_A$$
for α in PA determines a left adjoint to $P_{\langle \mathrm{id}_A, \mathrm{id}_A\rangle}\colon P(A\times A) \longrightarrow PA$;
2. for every morphism e of the form $\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_2\rangle\colon X \times A \longrightarrow X \times A \times A$ in C, the assignment
$$\exists_e(\alpha) := P_{\langle \mathrm{pr}_1, \mathrm{pr}_2\rangle}(\alpha) \wedge P_{\langle \mathrm{pr}_2, \mathrm{pr}_3\rangle}(\delta_A)$$
for α in $P(X\times A)$ determines a left adjoint to $P_e\colon P(X\times A\times A) \longrightarrow P(X \times A)$.

Remark 3.2.6. We make a few comments about this definition, recalling [42, Remark 2.2]:

1. the first condition of the previous definition implies the uniqueness of δ_A;
2. since $\langle \mathrm{pr}_2, \mathrm{pr}_1 \rangle \circ \langle \mathrm{id}_A, \mathrm{id}_A \rangle = \langle \mathrm{id}_A, \mathrm{id}_A \rangle$, the first condition of the definition of elementary doctrine implies

$$\exists_{\langle \mathrm{id}_A, \mathrm{id}_A \rangle}(\alpha) := P_{\mathrm{pr}_2}(\alpha) \wedge \delta_A$$

3. if C has a terminal object, the second condition implies the first one.

Example 3.2.7. Let $\mathbb{T}$ be a first order theory. The primary doctrine $LT\colon C_{\mathbb{T}}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$, as defined in Example 3.2.2, is elementary when $\mathbb{T}$ has an equality predicate.

Example 3.2.8. The subobject doctrine and the weak subobject doctrine defined in Example 3.2.3 and 3.2.4 are elementary, and the structure is given by the post-composition with an equalizer, see [43].

Definition 3.2.9. A primary doctrine $P\colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is ***existential*** if, for every A_1 and A_2 in C, for any projection $\mathrm{pr}_i\colon A_1 \times A_2 \longrightarrow A_i$, $i = 1, 2$, the functor

$$P_{\mathrm{pr}_i}\colon P(A_i) \longrightarrow P(A_1 \times A_2)$$

has a left adjoint $\exists_{\mathrm{pr}_i}$, and these satisfy:

1. ***Beck-Chevalley condition:*** for any pullback diagram

$$\begin{tikzcd} X' \arrow[r, "\mathrm{pr}'"] \arrow[d, "f'"'] & A' \arrow[d, "f"] \\ X \arrow[r, "\mathrm{pr}"'] & A \end{tikzcd}$$

with pr and pr' projections, for any β in $P(X)$ the canonical arrow

$$\exists_{\mathrm{pr}'} P_{f'}(\beta) \leq P_f \exists_{\mathrm{pr}}(\beta)$$

is an isomorphism;
2. ***Frobenius reciprocity:*** for any projection $\mathrm{pr}\colon X \longrightarrow A$, α in $P(A)$ and β in $P(X)$, the canonical arrow

$$\exists_{\mathrm{pr}}(P_{\mathrm{pr}}(\alpha) \wedge \beta) \leq \alpha \wedge \exists_{\mathrm{pr}}(\beta)$$

in $P(A)$ is an isomorphism.

Remark 3.2.10. In the definition of elementary doctrine Back-Chevalley condition and Frobenius reciprocity are not required because they follow from the explicit form of the left adjoints. See [43].

Remark 3.2.11. Given an existential elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$, for every map $f\colon A \longrightarrow B$ in $\mathcal{C}$ the functor P_f has a left adjoint $\exists_f$ that can be computes as:

$$\exists_{\mathrm{pr}_2}(P_{f\times \mathrm{id}_B}(\delta_B) \wedge P_{\mathrm{pr}_1}(\alpha))$$

for α in $P(A)$, where pr_1 and pr_2 are the projections from $A \times B$.

Example 3.2.12. The primary doctrine $LT\colon \mathcal{C}_{\mathbb{T}}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$, as defined in Example 3.2.2 for a first order theory $\mathbb{T}$, is existential. An existential left adjoint to P_{pr} is computed by quantifying existentially the variables that are not involved in the substitution given by the projection: if we consider a projection $\mathrm{pr} = [x/z]\colon (x, y) \longrightarrow (z)$ and a formula $\phi \in LT(x, y)$, then $\exists_{\mathrm{pr}}(\phi) = \exists_y(\phi[z/x])$. In this case the meaning of the Beck-Chevalley condition is clear: consider the following pullback

$$\begin{tikzcd}
(w_1, \ldots, w_n, y) \arrow[r, "{[w_1/w_1, \ldots, w_n/w_n]}"] \arrow[d, "{[t/x, y/y]}"'] & (w_1, \ldots, w_n) \arrow[d, "{[t/z]}"] \\
(x, y) \arrow[r, "{[x/z]}"'] & (z)
\end{tikzcd}$$

Then Beck–Chevalley condition rewrites the fact that substitution commutes with quantification as

$$\exists_y(\phi[t/x]) = (\exists_y(\phi[z/x]))[t/z]$$

since the declaration $(w_1, \ldots, w_n)$ ensures that y does not appear in t.

Example 3.2.13. For a cartesian category $\mathcal{D}$ with weak pullbacks, the doctrine of weak subobjects $\Psi_{\mathcal{D}}\colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ defined in Example 3.2.4 is existential. Existential left adjoints are given by post-composition.

Example 3.2.14. The doctrine $S\colon \mathbf{Set}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ defined in Example 3.2.5 is existential: on a subset P of a set A, the left adjoint $\exists_{\mathrm{pr}}$, for any projection $\mathrm{pr}\colon A \longrightarrow B$, must be evaluated as $\exists_{\mathrm{pr}}(P) = \{b \in B | \exists a \in A[a \in \mathrm{pr}^{-1}\{b\} \cap P]\}$.

Example 3.2.15. The doctrine $\mathrm{Sub}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ defined in Example 3.2.3 is elementary, but it is not existential in general. We will see in Section 3.3 that this doctrine is existential if and only if $\mathcal{C}$ is regular.

The category of elementary doctrines **ElD** is a 2-category, where:

- a ***1-cell*** is a pair (F, b)

$$\begin{tikzcd} \mathcal{C}^{\mathrm{op}} \arrow[dd, "F^{\mathrm{op}}"'] \arrow[dr, "P"] & \\ {} \arrow[r, phantom, "b \Downarrow"] & \mathbf{InfSL} \\ \mathcal{D}^{\mathrm{op}} \arrow[ur, "R"'] & \end{tikzcd}$$

such that $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is a functor preserving products, and $b\colon P \longrightarrow R \circ F^{\mathrm{op}}$ is a natural transformation preserving the structures. More explicitly, for every object A in $\mathcal{C}$, the function b_A preserves finite infima and

$$b_{A\times A}(\delta_A) = R_{\langle F\,\mathrm{pr}_1, F\,\mathrm{pr}_2\rangle}(\delta_{FA})$$

- a ***2-cell*** is a natural transformation $\theta\colon F \longrightarrow G$ such that for every A in $\mathcal{C}$ and every α in PA, we have
$$b_A(\alpha) \leq R_{\theta_A}(c_A(\alpha))$$

Consider the 2-subcategory **ExD** of **ElD** whose objects are elementary existential doctrines. The 1-cells of this category are those pair (F, b) in **ElD** such that b preserves the left adjoints along projections.

The notion of structure for a given signature seen in Subsection 2.2.1 can be generalized in the context of doctrines, see for example [50] or [45]. In particular the requirement that the functor F in a 1-cells (F, b) preserves products, and the conditions on the natural transformation b, guarantee that 1-cells preserve the structures.

Let us recall briefly how is defined the semantic for first order logic on a primary doctrine. Given a first order signature Σ of sorts A, function symbols $f\colon A_1, \dots, A_n \longrightarrow B$, and relation symbols $R \rightarrowtail A_1, \dots, A_n$, a ***structure*** $\llbracket - \rrbracket$ for the signature in a primary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ assigns an object $\llbracket A \rrbracket$ of $\mathcal{C}$ to each sort A, a morphism $\llbracket f \rrbracket\colon \llbracket A_1 \rrbracket \times \cdots \times \llbracket A_n \rrbracket \longrightarrow \llbracket B \rrbracket$ to each function symbol, and an object $\llbracket R \rrbracket$ of $P(\llbracket A_1 \rrbracket \times \cdots \times \llbracket A_n \rrbracket)$ to each relation symbol.

Then each term in context $t : B \quad [\Gamma]$ can be interpreted as a morphism $\llbracket t : B \ \ [\Gamma] \rrbracket\colon \llbracket \Gamma \rrbracket \longrightarrow \llbracket B \rrbracket$ in $\mathcal{C}$. Each formula $\psi \ \ [\Gamma]$ can be interpreted as an object $\llbracket \psi \ \ [\Gamma] \rrbracket$ of $P(\llbracket \Gamma \rrbracket)$.

The definitions of $\llbracket t : B \ \ [\Gamma] \rrbracket$ and $\llbracket \psi \ \ [\Gamma] \rrbracket$ proceed by induction on the structure of those expressions. As in the case of Section 2.1, we consider only regular formulas. For example, the formula $t_1 = t_2 \ [\Gamma]$, where t_1 and t_2 are terms of sort A, is mapped in $P_{\langle \llbracket t_1 \ [\Gamma] \rrbracket, \llbracket t_2 \ [\Gamma] \rrbracket \rangle}(\delta_A)$, and we see that to interpret formulas of this kind we need the elementary structure. A formula of the form $\exists x : A.\psi \ \ [\Gamma]$ is interpreted as $\exists_{\llbracket x:A \ \ [\Gamma] \rrbracket}(\psi \ [\Gamma])$, and in this case we need the existential structure.

We say that a structure ***satisfies*** a sequent $\psi \vdash \phi \ [\Gamma]$ if

$$\llbracket \psi \ [\Gamma] \rrbracket \leq \llbracket \phi \ [\Gamma] \rrbracket.$$

This notion of satisfaction is sound for an opportune fragment of first order intuitionistic logic, in the sense that all provable sentences are satisfied. It is also complete, in the sense that a sequent is provable if it is satisfied by all structures in first order doctrine. This completeness result is not very informative because the collection of such structures includes one (in a Lindenbaum-Tarski doctrine constructed from syntax) in which satisfaction coincides with provability. See Example 3.2.2.

A more useful consequence of this connection between first order logic and doctrines is the ability to use the familiar language of first order logic to give constructions in a doctrine that would otherwise involve complicated, order-enriched commutative diagrams. To do this one uses the following language, which is called the ***internal language*** of a doctrine. The idea is to generalize the construction seen in Subsection 2.2.3.

In particular one can associate to a doctrine $P\colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ a signature having a sort for every object of C, an n-ary function symbol $f\colon A_1,\dots,A_n \longrightarrow B$ for each finite list of objects $A_1,\dots,A_n,B$ and every morphism $f\colon A_1\times\cdots\times A_n \longrightarrow B$ of C, and an n-ary relation symbol $R \rightarrowtail A_1,\dots,A_n$ for every list $A_1,\dots,A_n$ of objects of C and every object of $P(A_1\times\cdots\times A_n)$. The terms and first order formulas over this signature form the ***internal language*** of the doctrine $P\colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$. We refer to [50], [51] for a detailed description of the internal language of a doctrine and an hyperdoctrine.

3.2.1 Elementary quotients completion

The structure of elementary doctrine is suitable to describe the notion of an equivalence relation and that of a quotient for such a relation.

Given an elementary doctrine $P\colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$, an object A in C, and an object ρ in $P(A\times A)$, we say that ρ is a ***P-equivalence relation on*** A if it satisfies:

- ***reflexivity:*** $\delta_A \le \rho$;
- ***symmetry:*** $\rho \le P_{\langle \mathrm{pr}_2,\mathrm{pr}_1\rangle}(\rho)$, for $\mathrm{pr}_1,\mathrm{pr}_2\colon A\times A \longrightarrow A$ the first and the second projection, respectively;
- ***transitivity:*** $P_{\langle \mathrm{pr}_1,\mathrm{pr}_2\rangle}(\rho)\wedge P_{\langle \mathrm{pr}_2,\mathrm{pr}_3\rangle}(\rho) \le P_{\langle \mathrm{pr}_1,\mathrm{pr}_3\rangle}(\rho)$ for

$$\mathrm{pr}_1,\mathrm{pr}_2,\mathrm{pr}_3\colon A\times A\times A \longrightarrow A$$

  the first, the second, and the third projection, respectively.

Remark 3.2.16. The P-equivalence relations are exactly the equivalence relations in the internal language of P. So an object $\rho \in P(A\times A)$ is an P-equivalence relation if the following sequents are provable in the internal language:

- $a_1 =_A a_2 \vdash \rho(a_1,a_2)\ [a_1 : A, a_2 : A]$;
- $\rho(a_1,a_2) \vdash \rho(a_2,a_1)\ [a_1 : A, a_2 : A]$;
- $\rho(a_1,a_2)\wedge\rho(a_2,a_3) \vdash \rho(a_1,a_3)\ [a_1 : A, a_2 : A, a_3 : A]$.

Remark 3.2.17. For an elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$, the object δ_A is a P-equivalence relation, and for every morphism $f\colon A \longrightarrow B$, the functor

$$P_{f\times f}\colon P(B\times B) \longrightarrow P(A\times A)$$

takes a P-equivalence relation σ on B to a P-equivalence relation on A.

The ***P-kernel*** of a morphism $f\colon A \longrightarrow B$, is the object $P_{f\times f}(\delta_B)$, and by Remark 3.2.17, it is a P-equivalence relation on A. An equivalence relation is said ***effective*** if it is the P-kernel of a morphism..

Remark 3.2.18. The P-kernel of $f\colon A \longrightarrow B$ in the internal language is the formula $f(a_1) =_B f(a_2)\ [a_1 : A, a_2 : A]$.

Definition 3.2.19. Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be an elementary doctrine, and let ρ be an P-equivalence relation on A. A ***P-quotient*** of ρ is a morphism $q\colon A \longrightarrow A/\rho$ in $\mathcal{C}$ such that $P_{q\times q}(\delta_{A/\rho}) \geq \rho$ and for every morphism $f\colon A \longrightarrow Z$ such that $P_{f\times f}(\delta_Z) \geq \rho$, there exists a unique morphism $g\colon A/\rho \longrightarrow Z$ such that $g\circ q = f$.

Remark 3.2.20. In the internal language a quotient of $\rho \in P(A\times A)$ is a term $q(a) : A/\rho\ \ [a : A]$ such that

$$\rho(a_1,a_2) \vdash q(a_1) =_{A/\rho} q(a_2)\ [a_1 : A, a_2 : A]$$

and for every term $f(a) : Z\ \ [a : A]$ such that

$$\rho(a_1,a_2) \vdash f(a_1) =_Z f(a_2)\ [a_1 : A, a_2 : A]$$

there exists a unique term $g(a') : Z\ \ [a' : A/\rho]$ such that $f(a) = g(q(a))$.

We say that such a P-quotient is ***stable*** if in every pullback

$$\begin{tikzcd} A' \arrow[r, "q'"] \arrow[d, "f'"'] & C' \arrow[d, "f"] \\ A \arrow[r, "q"'] & A/\rho \end{tikzcd}$$

in $\mathcal{C}$, the morphism $q'\colon A' \longrightarrow C'$ is a P-quotient.

In the following example we see that the notion of P-equivalence relation, quotients and effective morphism coincide with usual notion seen in Section 2.1.

Example 3.2.21. Consider the subobjects doctrine $\mathrm{Sub}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ obtained form a category with finite limits as defined in Example 3.2.3. A quotient of a $\mathrm{Sub}_{\mathcal{C}}$-equivalence relation $[\ R \xrightarrow{\langle r_1, r_2\rangle} A\times A\ ]$ is the coequalizer

$$R \underset{r_2}{\overset{r_1}{\rightrightarrows}} A \xrightarrow{q} A/R$$

since $\mathrm{Sub}_{q\times q}(\delta_{A/R}) = [\ P \xrightarrow{\langle p_1,p_2\rangle} A\ ]$ is the kernel pair of $q\colon A \longrightarrow A/R$

$$\begin{tikzcd} P \arrow[r, tail, "{\langle p_1,p_2\rangle}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A\times A \arrow[d, "q\times q"] \\ A/R \arrow[r, tail, "\Delta_{A/R}"'] & A/R \times A/R. \end{tikzcd}$$

Thus we have that $[\ P \xrightarrow{\langle p_1,p_2\rangle} A\times A\ ]$ is an effective equivalence relation. In particular, all the Sub_{C}-equivalence relations have stable, effective quotients if and only if the C category is exact. See [43] for more details.

The abstract theory that captures the essential action of a quotient is that of descent. We recall some basic concepts from that in our particular case of interest of an elementary doctrine. See [22, 23] for a survey on descent theory.

Definition 3.2.22. Given an elementary doctrine $P\colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ and a P-equivalence relation ρ on an object A in C, the partial order of ***descent data*** Des_{ρ} is the sub-order of $P(A)$ on those α such that

$$P_{\mathrm{pr}_1}(\alpha) \wedge \rho \le P_{\mathrm{pr}_2}(\alpha)$$

where $\mathrm{pr}_1, \mathrm{pr}_2\colon A\times A \longrightarrow A$ are the projections.

Remark 3.2.23. Again we translate in the internal language the previous definition: $\psi(a)\ [a : A]$ is a descent data for a relation $\rho(a_1, a_2)\ [a_1 : A, a_2 : A]$ if

$$\psi(a_1) \wedge \rho(a_1, a_2) \vdash \psi(a_2)\ [a_1 : A, a_2 : A]$$

Remark 3.2.24. Given an elementary doctrine $P\colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$, consider a morphism $f\colon A \longrightarrow B$ in C and let ρ be the P-kernel $P_{f\times f}(\delta_A)$. The functor $P_f\colon P(B) \longrightarrow P(A)$ takes values in $Des_{\rho} \subseteq P(A)$.

Definition 3.2.25. Given an elementary doctrine $P\colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$, consider a morphism $f\colon A \longrightarrow B$ in C, let ρ be its P-kernel. The arrow is of ***effective descent*** if the functor $P_f\colon P(B) \longrightarrow Des_{\rho}$ is an isomorphism.

Example 3.2.26. In the Example 3.2.5 of the doctrine $S\colon \mathbf{Set}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$, every canonical surjection $f\colon A \longrightarrow A/\!\sim$ in the quotient of an equivalence relation $\sim$

on A, is of effective descent. The condition in Definition 3.2.25 recognizes the fact that the subsets of the $A/\sim$ are in bijection with those subsets U of A which are closed with respect to the equivalence relation, in the sense that for $a_1, a_2 \in A$ such $a_1 \sim a_2$ and $a_1 \in U$ one has also that $a_2 \in U$.

Consider the 2-full 2-subcategory **QED** of **ElD** whose object are elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ with stable effective quotients of P-equivalence relations and of effective descent.

The 1-cells of the category **QED** are those 1-cells of **ElD**

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \arrow[dd, "F^{\mathrm{op}}"'] \arrow[dr, "P"{name=U}] & \\
& \mathbf{InfSL} \\
\mathcal{D}^{\mathrm{op}} \arrow[ur, "R"'{name=D}] &
\arrow[Rightarrow, from=U, to=D, "b"']
\end{tikzcd}$$

such that F preserves quotients.

In [43, 42, 44] Maietti and Rosolini present a construction that produces an elementary doctrine with quotients. We shall present it in the following, and we see that this is a generalization of the exact completion seen in Section 2.1 in the contest of elementary doctrines.

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be an elementary doctrine. We define the ***elementary quotient completion*** of P the doctrine $P_q\colon \mathcal{Q}_P^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ where:

- ***an object of*** $\mathcal{Q}_P$ is a pair (A,ρ) such that ρ is a P-equivalence relation on A;
- ***an arrow of*** $\mathcal{Q}_P$ $f\colon (A,\rho) \longrightarrow (B,\sigma)$ is a morphism $f\colon A \longrightarrow B$ of $\mathcal{C}$ such that $\rho \le P_{f\times f}(\sigma)$.

Compositions and identities are given by $\mathcal{C}$.

The indexed partial inf-semilattice $P_q\colon \mathcal{Q}_P^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ on $\mathcal{Q}_P$ is given by the categories of descent data:

$$P_q(A,\rho) := Des_\rho$$

and the following lemma is instrumental to give the assignment on morphisms using the action of P on morphisms. See [42, Lemma 4.1] for the proof.

Lemma 3.2.27. *Let (A,ρ) and (B,σ) be objects in $\mathcal{Q}_P$, and let β be in Des_σ. Then if $f\colon (A,\rho) \longrightarrow (B,\sigma)$ is an arrow of $\mathcal{Q}_P$ then $P_f(\beta)$ is in Des_ρ.*

The previous construction gives a well defined elementary doctrine as it is proved in [42, Lemma 4.2], and this doctrine has descent quotients of P_q-equivalence relations. See [42, Lemma 4.4].

Lemma 3.2.28. *With the notation used above, the functor $P_q\colon \mathcal{Q}_P^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is an elementary doctrine. Moreover it has descent quotients of P_q-equivalence* *relations and quotients are stable and effective descent, and P_q-equivalence relations are effective.*

There is an obvious 1-morphism $(J, j)\colon P \longrightarrow P_q$ in $\mathbf{ElD}$, where the functor $J\colon \mathcal{C} \longrightarrow \mathcal{Q}_P$ sends an object A in $\mathcal{C}$ to (A, δ_A) and a morphism $f\colon A \longrightarrow B$ to $f\colon (A, \delta_A) \longrightarrow (B, \delta_B)$ since $\delta_A \le P_{f\times f}(\delta_B)$, and $j_A\colon P(A) \longrightarrow P_q(A, \delta_A)$ is the identity because

$$P_q(A, \delta_A) = Des_{\delta_A} = P(A).$$

It is immediate to see that J is full and faithful and that (J, j) is just a change of base.

In [42, 43] the authors show that the quotient completion is a free completion in the sense that there is a left biadjoint to the forgetful 2-functor

$$\mathrm{U}\colon\ \mathbf{QED} \longrightarrow \mathbf{ElD}\ .$$

We refer to [42, Theorem 4.5] for the proof of the following theorem.

Theorem 3.2.29. *For every elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ the pre-composition with the 1-morphism*

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \arrow[dr, "P"] \arrow[dd, "J^{\mathrm{op}}"'] & \\
{} \arrow[r, phantom, "\Downarrow j"] & \mathbf{InfSL} \\
\mathcal{Q}_P{}^{\mathrm{op}} \arrow[ur, "P_q"'] &
\end{tikzcd}$$

in $\mathbf{ElD}$ *gives an essential equivalence of categories*

$$-\circ (J, j)\colon\ \mathbf{QED}(P_q, Z) \longrightarrow \mathbf{ElD}(P, Z)$$

for every Z in $\mathbf{QED}$.

Proposition 3.2.30. *Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be an elementary existential doctrine, and let $\mathcal{C}$ be a finitely complete category. Then $P_q\colon \mathcal{Q}_P^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is elementary and existential and the category $\mathcal{Q}_P$ is regular.*

Proof. [Sketch] Let $\mathcal{E}$ be class of quotients, and let $\mathcal{M}$ be the class of monomorphisms. These two class are a proper, stable factorization system for $\mathcal{Q}_P$ since quotients are stable. $\square$

3.2.2 Set-like doctrines

In [43, 42, 44] Maietti and Rosolini intend to develop doctrines that may interpret constructive theories for mathematics. They observe that there are two crucial properties that an elementary doctrine should verify in order to sustain such interpretations. One relates to the axiom of comprehension and to equality.

Let $P\colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be an elementary doctrine and let α an object of $P(A)$. A ***comprehension*** of α is an arrow $\{\!|\alpha|\!\}\colon X \longrightarrow A$ such that $P_{\{\!|\alpha|\!\}} = \top_X$ and, for every $f\colon Z \longrightarrow A$ such that $P_f(\alpha) = \top_Z$, there exists a unique map $g\colon Z \longrightarrow X$ such that $f = \{\!|\alpha|\!\} \circ g$.

One says that P ***has comprehensions*** if every α has a comprehension, and that P ***has full comprehensions*** if, moreover, $\alpha \leq \beta$ in $P(A)$ whenever $\{\!|\alpha|\!\}$ factors through $\{\!|\beta|\!\}$.

Intuitively, the comprehension morphism represents the subsets of elements in the object A obtained by comprehension with the predicate α.

Remark 3.2.31. In the internal language of an elementary doctrine $P\colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$, a comprehension of a formula $\phi(a)$ $[a : A]$ is a term $\{\!|a : A \mid \phi(a)|\!\}(x) : A \ \ [x : X]$ such that

$$\top \vdash \phi(\{\!|a : A|\phi(a)|\!\}(x))\ [x : X]$$

and any other term which this property can be obtained from $\{\!|a : A \mid \phi(a)|\!\}(x)$ by an unique substitution.

Example 3.2.32. The doctrine $S\colon \mathbf{Set}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ defined in Example 3.2.5 has comprehensions given by the trivial remark that a subset determines an actual function by inclusion.

Example 3.2.33. The doctrine $\mathrm{Sub}_C\colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ defined in Example 3.2.3. In this case for every object A and every $\alpha = [\ B \overset{\alpha}{\rightarrowtail} A\ ]$ in $\mathrm{Sub}_C(A)$, the comprehension $\{\!|\alpha|\!\}$ is the arrow in C $B \overset{\alpha}{\rightarrowtail} A$. Moreover the comprehensions are full.

Remark 3.2.34. For every $f\colon A' \longrightarrow A$ in C then the mediating arrow between the comprehensions $\{\!|\alpha|\!\}\colon X \longrightarrow A$ and $\{\!|P_f(\alpha)|\!\}\colon X' \longrightarrow A'$ produces a pullback

$$\begin{tikzcd}
X' \arrow[r, "\{P_f(\alpha)\}"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A' \arrow[d, "f"] \\
X \arrow[r, "\{\alpha\}"'] & A.
\end{tikzcd}$$

Thus comprehensions are stable under pullbacks.

Remark 3.2.35. If $\{\!|\alpha|\!\}\colon B \longrightarrow A$ is a comprehension of α, then $\{\!|\alpha|\!\}$ is monic.

Given an elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$, and an object α in $P(A)$, a ***weak comprehension*** of α is an arrow $\{\!|\alpha|\!\}\colon X \longrightarrow A$ in $\mathcal{C}$ such that $\top_X \leq P_{\{\!|\alpha|\!\}}(\alpha)$ and for every $g\colon Z \longrightarrow A$ such that $\top_Z \leq P_g(\alpha)$, there is an arrow $g\colon Z \longrightarrow A$ such that $f = \{\!|\alpha|\!\} \circ g$.

We say that an elementary doctrine ***has weak comprehensions*** if every α has a weak comprehension, and that the doctrine ***has full weak comprehensions*** if, moreover, $\alpha \leq \beta$ in $P(A)$ if $\{\!|\alpha|\!\}$ factors through $\{\!|\beta|\!\}$.

Example 3.2.36. Following the Example 3.2.33 one can see that the doctrine $\Psi_{\mathcal{D}}\colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ of weak subobjects defined in 3.2.4 has full weak comprehensions.

Recall from [21] that the fibration of vertical maps on the category of points freely adds comprehensions to a given fibration producing an indexed poset in case the given fibration is such. For a doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ the indexed poset consists of the base category $\mathcal{G}_P$ of points where

- ***an object*** is a pair (A,α) where A is in $\mathcal{C}$ and α is in $P(A)$;
- ***an arrow*** $f\colon (A,\alpha) \longrightarrow (B,\beta)$ is an arrow $f\colon A \longrightarrow B$ of $\mathcal{C}$ such that $\alpha \leq P_f(\beta)$.

The indexed functor extends to $P_c\colon \mathcal{G}_P^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ by setting

- $P_c(A,\alpha) := \{\gamma \in P(A) \mid \gamma \leq \alpha\}$;
- $P_c(f)\colon (B,\beta) \longrightarrow (A,\alpha)$ sends $\gamma \leq \beta$ to $P_f(\gamma) \wedge \alpha$.

Moreover the comprehensions of $P_c\colon \mathcal{G}_P^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ are full, as is observed in [43, 44, 42].

As for the case of quotient completion, there is a natural embedding $(I,i)\colon P \longrightarrow P_c$ in **ElD** which maps and object A in $\mathcal{C}$ to $(A,\top_A)$.

Let **CE** be the 2-category of elementary doctrines with full comprehension.

Then the previous construction give the following result. For the proof we refer to [44, Theorem 3.1], [43], and [42].

Theorem 3.2.37. *The association to an elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ of the doctrine $P_c\colon \mathcal{G}_P^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ determines a left bi-adjoint to the inclusion of* **CE** *into* **ElD**. *If the doctrine P is existential, then P_c is also existential.*

Proposition 3.2.38. *If $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ has comprehensions then its quotient completion $P_q\colon \mathcal{Q}_P^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ also has comprehensions.*

A special case of comprehensions are the diagonal morphisms and the following definition considers just that possibility.

An elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ ***has comprehensive diagonals*** if every diagonal arrow $\langle \mathrm{id}_A, \mathrm{id}_A\rangle\colon A \longrightarrow A \times A$ is the comprehension of δ_A.

Example 3.2.39. An elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ has comprehensive diagonals if and only if for every pair of morphisms $A \overset{f}{\underset{g}{\rightrightarrows}} B$ in $\mathcal{C}$ we have

$$f = g \text{ in } \mathcal{C} \text{ if and only if } \top \vdash f(a) =_B g(a)\ [a : A]$$

For elementary doctrine we have the following useful characterization. See [41, Proposition 2.12].

Proposition 3.2.40. *Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be an elementary doctrine. The following are equivalent:*

1. *P has comprehensive diagonals;*
2. *for any two arrows $f, g\colon A \longrightarrow B$ in $\mathcal{C}$ it is*

$$f = g \textit{ if and only if } \top_A \leq P_{\langle f,g\rangle}(\delta_B).$$

Thanks to Proposition 3.2.40, there is a 2- reflection of elementary doctrines from **ElD** to its full 2-subcategory **CED** of elementary doctrines with comprehensive diagonals once one notices that the condition

$$\top_A \leq P_{\langle f,g\rangle}(\delta_B)$$

ensures that $P_f = P_g$. So the reflection takes an elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ to the elementary doctrine

$$P_d\colon \mathcal{X}_P^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$$

induced by P on the quotient category $\mathcal{X}_P$ of $\mathcal{C}$ with respect to the equivalence relation where $f \sim g$ when

$$\top_A \leq P_{\langle f,g\rangle}(\delta_B).$$

Following the notation of [43, 42, 44] we refer to the doctrine P_d as the ***extensional reflection of*** P.

Remark 3.2.41. If an elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ has comprehensions then $P_d\colon \mathcal{X}_P^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ has also comprehensions. Moreover if P has quotients then $P_d\colon \mathcal{X}_P^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ has also quotients. See [42, 41] for all the details.

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be an elementary doctrine. We say that P is a ***variational doctrine*** if it has weak full comprehensions and comprehensive diagonals. We say that P is an ***m-variational doctrine*** if it has full comprehensions and comprehensive diagonals. The category of m-variational doctrines is denoted by **mVar**.

As for the case of the quotient completion, the construction of an m-variational doctrine can be extended to a bi-adjunction as it is proved in [42] and [41].

Theorem 3.2.42. *The association to an elementary doctrine* $P\colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *of the doctrine* $(P)_{cd}$ *determines a left bi-adjoint to the inclusion of* $\mathbf{mVar}$ *into* $\mathbf{ElD}$. *If P is existential, then* $(P)_{cd}$ *is also existential.*

Remark 3.2.43. Let $P\colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be an existential m-variational doctrine. For every element α in $P(A)$ we have

$$\alpha = \exists_{\{\alpha\}}(\top_A) \tag{3.3}$$

because the comprehension $\{\!\{\alpha\}\!\}$ factorizes on $\{\!\{\exists_{\{\alpha\}}(\top_A)\}\!\}$, and since the comprehensions in P are full, then $\alpha \leq \exists_{\alpha}(\top_A)$. Moreover we have that $\exists_{\{\alpha\}}(\top_A) \leq \alpha$ if and only if $\top_A \leq P_{\{\alpha\}}(\alpha)$, and then the equality (3.3) holds.

Remark 3.2.44. If an elementary doctrine $P\colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is m-variational the base category C has equalizers. In particular for every pair of arrows $A \underset{g}{\overset{f}{\rightrightarrows}} B$ in C, the equalizer is

$$E \xrightarrow{\{P_{\langle f,g\rangle}(\delta_B)\}} A \underset{g}{\overset{f}{\rightrightarrows}} B$$

because comprehensions are stable under pullbacks and $\Delta_B\colon B \longrightarrow B \times B$ is $\Delta_B = \{\!\{\delta_B\}\!\}$. Hence the square

$$\begin{tikzcd} E \arrow[r, "\{P_{\langle f,g\rangle}(\delta_B)\}"] \arrow[d, "a"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "\langle f,g\rangle"] \\ B \arrow[r, "\Delta_B"'] & B \times B \end{tikzcd}$$

is a pullback and then $\{\!\{P_{\langle f,g\rangle}(\delta_B)\}\!\}\colon E \longrightarrow A$ is an equalizer for $A \underset{g}{\overset{f}{\rightrightarrows}} B$.

Thus the category C has finite limits, and pullbacks can be computed as follows

$$\begin{tikzcd} X \arrow[rr] \arrow[dd] \arrow[dr, "\{P_{g\times f}(\delta_B)\}"] & & A \arrow[dd, "f"] \\ & Y \times X \arrow[ur, "\mathrm{pr}_2"'] \arrow[dl, "\mathrm{pr}_1"] & \\ Y \arrow[rr, "g"'] & & B. \end{tikzcd}$$

Proposition 3.2.45. *Let* $P\colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *be an existential m-variational doctrine. Then the left adjoint functors* $\exists_f$ *satisfy the Beck-Chevalley condition with respect to pullbacks.*

Proof. See [41, Proposition 2.19]. $\square$

The assignment of comprehensions extends to a 1-arrow

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \arrow[dd, "\mathrm{id}_{\mathcal{C}}^{\mathrm{op}}"'] \arrow[dr, "P"] & \\
 & \mathbf{InfSL} \arrow[l, phantom, "\Downarrow \{\!|-|\!\}"] \\
\mathcal{C}^{\mathrm{op}} \arrow[ur, "\mathrm{Sub}_{\mathcal{C}}"'] &
\end{tikzcd}$$

from P to the doctrine of the subobjects in **ElD**. Moreover the functor

$$\{\!|-|\!\}\colon P(A) \longrightarrow \mathrm{Sub}_{\mathcal{C}}(A)$$

is fully faithful.

By Remark 3.2.44 one can think that comprehensions and comprehensive diagonals force an elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ to "look like" a poset of subobjects of $\mathcal{C}$.

The previous observation can be extended in the case of an elementary doctrine with weak comprehensions, and the result is that if an elementary doctrine is variational then it can be seen as a "subdoctrine" of the weak subjobject doctrine. See [41].

Remark 3.2.46. Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be an existential variational doctrine. Consider α and β in $P(A)$. We can observe that $\{\!|\alpha \wedge \beta|\!\} = \{\!|\alpha|\!\} \wedge \{\!|\beta|\!\}$ in $\Psi_{\mathcal{C}}(A)$

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[dr, "\{\!|\alpha\wedge\beta|\!\}"] & X \arrow[d, "\{\!|\beta|\!\}"] \\
Y \arrow[r, "\{\!|\alpha|\!\}"'] & A
\end{tikzcd}$$

This means that $\{\!|-|\!\}\colon P(A) \longrightarrow \Psi_{\mathcal{C}}(A)$ is a natural homomorphism. In particular, since the doctrine has weak comprehensive diagonals, it preserves fibered equalities, and then it is a 1-arrow in **ExD**

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \arrow[dd, "\mathrm{id}_{\mathcal{C}^{\mathrm{op}}}"'] \arrow[dr, "P"] & \\
 & \mathbf{InfSL} \arrow[l, phantom, "\Downarrow \{\!|-|\!\}"] \\
\mathcal{C}^{\mathrm{op}} \arrow[ur, "\Psi_{\mathcal{C}}"'] &
\end{tikzcd}$$

Now we define another functor

$$\Psi_C(A) \xrightarrow{\exists_{-}\top_A} P(A).$$

$$[B \xrightarrow{f} A] \longmapsto \exists_f(\top_B)$$

Observe that it extends to a morphism in the category **InfSL**, and this is a left adjoint to $\{\!|-|\!\}\colon P(A) \longrightarrow \Psi_C(A)$. Moreover we have that

$$\exists_{-}\top_A([\ \Delta_A\colon A \longrightarrow A\times A\ ]) = \delta_A.$$

Hence it provides a 1-arrow in **ExD**

$$\begin{tikzcd} C^{\mathrm{op}} \arrow[dd, "\mathrm{id}_{C^{\mathrm{op}}}"'] \arrow[dr, "\Psi_C"] & \\ & \mathbf{InfSL} \arrow[l, phantom, "\Downarrow \exists_{-}\top"] \\ C^{\mathrm{op}} \arrow[ur, "P"'] & \end{tikzcd}$$

In [41] Maietti, Rosolini and Pasquali show that for an existential variational doctrine $P\colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$, the adjunction of Remark 3.2.46 is an equivalence if and only if the doctrine satisfies the ***Rule of Choice***, which means that for every $\phi \in P(A\times B)$ such that

$$\top_A \leq \exists_{\mathrm{pr}_1}(\phi)$$

there is an arrow $f\colon A \longrightarrow B$ such in C that

$$\top_A \leq P_{\langle \mathrm{id}_A, f\rangle}(\phi).$$

A similar characterization can be given for an existential m-variational doctrine $P\colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$, in particular C is a regular category and P is the doctrine of subobjects if and only if P satisfies the ***Rule of Unique Choice***, which means that for every pair of objects A and B and every entire functional relation ϕ from A to B there is an arrow $f\colon A \longrightarrow B$ in C such that

$$\top_A \leq P_{\langle \mathrm{id}_A, f\rangle}(\phi).$$

Finally, given elementary existential doctrine $P\colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$, the completion P_{cd} satisfies the Rule of Choice, if and only if the doctrine P is equipped with ε-operators by [41, Theorem 51.5].

We refer to [41, 21] for all the details about this final results.

3.3 Existential m-variation doctrines, factorization systems and exact completion

In Section 3.1 we have seen the connection between fibrations and factorization system, and recall that starting from a factorization fibration, the resulting factorization system is not necessary proper or stable. Again we refer to [19] for all the details.

In this section we show what kind of fibration we can construct starting from an existential m-variational doctrine, and we see that the resulting fibration is a factorization fibration with coproducts and it is a fibred pre-order.

Therefore we can use Theorem 3.1.20 and 3.1.23 to construct a stable, proper factorization system $\langle \mathcal{M}, \mathcal{E} \rangle$ from an existential m-variational doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$.

Moreover we see that every existential m-variational doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is equivalent to the doctrine of $\mathcal{M}$-subobject

$$\mathrm{Sub}_{\mathcal{M}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$$

where $\langle \mathcal{E}, \mathcal{M} \rangle$ is the stable, proper factorization system induced by the doctrine.

It is a known fact that primary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ determines a faithful fibration

$$p_P\colon \mathcal{G}_P \longrightarrow \mathcal{C}$$

by Grothendieck construction, see [21, 43]. We recall very briefly that construction in the present situation.

The data for the ***total category*** $\mathcal{G}_P$ are:

- **an object** is a pair (A, α), where A is in $\mathcal{C}$ and α is in $P(A)$
- **an arrow** $f\colon (A,\alpha) \longrightarrow (B,\beta)$ is an arrow $f\colon A \longrightarrow B$ of $\mathcal{C}$ such that $\alpha \leq P_f(\beta)$.

The projection on the first component extends to a functor $p_P\colon \mathcal{G}_P \longrightarrow \mathcal{C}$ which is faithful, with a right inverse right adjoint.

Remark 3.3.1. Let A be an object of $\mathcal{C}$. Observe that in our case the objects of the fibre category $(\mathcal{G}_P)_A$ are of the form (A,α), and for every pair (A,α) and (A,β) there is at most one morphism in $(\mathcal{G}_P)_A$, that is $\mathrm{id}_A\colon (A,\alpha) \longrightarrow (A,\beta)$. Therefore the category $(\mathcal{G}_P)_A$ is an inf-semilattice, since $P(A)$ is.

Let (A,α) be an object of $\mathcal{G}_P$. For every morphism $u\colon B \longrightarrow A$ in $\mathcal{C}$, we can fix a Cartesian morphism $u\colon (B, P_u(\alpha)) \longrightarrow (A,\alpha)$ above u. This morphism induces a functor

$$u^*\colon (\mathcal{G}_P)_A \longrightarrow (\mathcal{G}_P)_B$$

where $u^*(A,\alpha) := (B, P_u(\alpha))$. It is direct to check that it preserves the order since $(A,\alpha) \leq (A,\gamma)$ implies $(B,P_u(\alpha)) \leq (B,P_u(\gamma))$.

Using Remark 3.2.11 we can prove that every elementary existential doctrine induces a bifibration. In particular we can see that we need both the existential and the elementary structure, because we need left adjoint to every functor of the form P_f with f morphism in $\mathcal{C}$.

Proposition 3.3.2. *Let* $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *be an existential elementary doctrine, then it induces a bifibration* $p_P\colon \mathcal{G}_P \longrightarrow \mathcal{C}$.

Proof. Consider an object (A,α) of $\mathcal{G}_P$, and let $f\colon A \longrightarrow B$ be an arrow in $\mathcal{C}$. By Remark 3.2.11 the functor P_f has a left adjoint functor $\exists_f$, and then $f\colon (A,\alpha) \longrightarrow (B,\exists_f(\alpha))$ is a morphism in $\mathcal{G}_P$ because

$$\alpha \leq P_f\exists_f(\alpha).$$

Let $g\colon (Z,\gamma) \longrightarrow (B,\exists_f(\alpha))$ be a morphism in $\mathcal{G}_P$, and consider the following diagram

$$\begin{tikzcd} Z & & \\ & A \arrow[ul, "g"] \arrow[r, "f"'] & B. \arrow[ull, bend right, "h"'] \end{tikzcd}$$

Since $\alpha \leq P_g(\gamma)$, we have $\alpha \leq P_{hf}(\gamma)$, and then $\alpha \leq P_f(P_h(\gamma))$. Applying the functor $\exists_f$ to both the element, we have

$$\exists_f(\alpha) \leq P_h(\gamma)$$

because $\exists_f P_f \leq \mathrm{id}_{P(B)}$. Thus the diagram

$$\begin{tikzcd} (Z,\gamma) & & \\ & (A,\alpha) \arrow[ul, "g"] \arrow[r, "f"'] & (B,\exists_f(\alpha)) \arrow[ull, bend right, "h"'] \end{tikzcd}$$

commutes in $\mathcal{G}_P$. Therefore we can conclude that p is an op-fibration, and then it is a bifibration. $\square$

Let $p_P\colon \mathcal{G}_P \longrightarrow \mathcal{C}$ be a fibration coming from an existential elementary doctrine. Since it is a bifibration, for every morphism $u\colon A \longrightarrow B$ in $\mathcal{C}$, the functor $u^*\colon \mathcal{G}_B \longrightarrow \mathcal{G}_A$ has a left adjoint $\coprod_u \dashv u^*$ by Lemma 3.1.8. In this case the left adjoint $\coprod_u\colon \mathcal{G}_A \longrightarrow \mathcal{G}_B$ sends (A,α) in $(B,\exists_u(\alpha))$.

Remark 3.3.3. If the doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is existential and m-variational, then by Proposition 3.2.45, every functor $\exists_f$ satisfies Back-Chevalley condition. Therefore the bifibration $p_P\colon \mathcal{G}_P \longrightarrow \mathcal{C}$ has coproducts, since for every pullback in $\mathcal{C}$

$$\begin{tikzcd} K \arrow[r, "v"] \arrow[d, "r"'] \arrow[dr, phantom, "\lrcorner", very near start] & L \arrow[d, "s"] \\ I \arrow[r, "u"'] & J \end{tikzcd}$$

we have

$$\coprod_u r^*(I,\iota) = \coprod_u (K, P_r(\iota)) = (L, \exists_v P_r(\iota))$$

and this is equal to

$$s^* \coprod_u (I,\iota) = s^*(J, \exists_u(\iota)) = (L, P_s \exists_u(\iota))$$

because the doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ satisfies Beck-Chevalley for any pullaback.

Proposition 3.3.4. *Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be an existential elementary doctrine with comprehensions, then it induces a fibration $p_P\colon \mathcal{G}_P \longrightarrow \mathcal{C}$ with subset type.*

Proof. We define $\top\colon \mathcal{C} \longrightarrow \mathcal{G}_P$ the functor which sends an object A to $(A, \top_A)$ and a morphism $f\colon A \longrightarrow B$ to the arrow $f\colon (A,\top_A) \longrightarrow (B,\top_B)$. It is direct to prove that it is a right adjoint to p, and clearly $p \circ \top = \mathrm{id}_{\mathcal{C}}$. Now we construct a right adjoint to $\top$. For every (A,α) we choose a comprehension of α:

$$\{\!|\alpha|\!\}\colon A_\alpha \longrightarrow A.$$

We define $\{(A,\alpha)\} := A_\alpha$, and observe that $\mathcal{G}_P(\top(A),(B,\beta)) \cong \mathcal{C}(A, B_\beta)$ because every morphism

$$f\colon (A,\top_A) \longrightarrow (B,\beta)$$

is such that $\top_A = P_f(\beta)$, and then f factors in a unique way through $\{\!|\beta|\!\}$.

Therefore $\mathcal{G}_P(\top(-),(B,\beta))$ is representable for every (B,β) in $\mathcal{G}_P$, and then we can conclude that there exists a right adjoint $\top \dashv \{-\}$. Moreover for every morphism $f\colon (B,\beta) \longrightarrow (A,\alpha)$ the arrow $\{f\}\colon B_\beta \longrightarrow A_\alpha$ is defined as the unique morphism such that the diagram

$$\begin{tikzcd} B_\beta \arrow[r, "\{\!|\beta|\!\}"] \arrow[dr, dashed, "\{f\}"'] & B \arrow[r, "f"] & A \\ & A_\alpha \arrow[ur, "\{\!|\alpha|\!\}"'] & \end{tikzcd}$$

commutes. $\square$

In the previous proposition we have proved that $\top \dashv \{-\}$, and we can observe that the counit $\varepsilon\colon \top \circ \{-\} \Longrightarrow \mathrm{id}_{\mathcal{G}_P}$ of this adjunction is defined as:

$$\varepsilon_{(B,\beta)}\colon (B_\beta, \top_{B_\beta}) \longrightarrow (B,\beta)$$

where $\varepsilon_{(B,\beta)} := \{\!|\beta|\!\}$. Using the same notation of Definition 3.1.12, we have that for every (A,α) in $\mathcal{G}_P$, the arrow $\pi_{(A,\alpha)}\colon \{(A,\alpha)\} \longrightarrow A$ is a comprehension of α.

Remark 3.3.5. We define a functor from $\mathcal{G}_P$ to $\mathcal{C}^{\rightarrow}$, sending (A,α) into $\pi_{(A,\alpha)}$. In particular if the existential elementary doctrine has full comprehension, this functor is full and faithful, since for every commutative diagram

$$\begin{tikzcd}
A_\alpha \arrow[r, "\{\!|\alpha|\!\}"] \arrow[d, "f"'] & A \arrow[d, "g"] \\
B_\beta \arrow[r, "\{\!|\beta|\!\}"'] & B
\end{tikzcd}$$

we have that

$$P_{\{\!|\alpha|\!\}}(P_g(\beta)) = P_f(P_{\{\!|\beta|\!\}}(\beta)) = \top_{A_\alpha}$$

and then $\alpha \leq P_g(\beta)$ because the doctrine has full comprehensions.

The following observation will allows us to conclude that a fibration induced by an existential elementary doctrine with full comprehensions has strong coproduct.

Proposition 3.3.6. *Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow$ **InfSL** be an existential m-variational doctrine. Then every composition of comprehensions is again a comprehension.*

Proof. Let $\{\!|\beta|\!\}\colon C \longrightarrow B$ and $\{\!|\alpha|\!\}\colon B \longrightarrow A$ be comprehensions and consider the comprehension

$$\{\!|\exists_{\{\!|\alpha|\!\}}(\beta)|\!\}\colon D \longrightarrow A\ .$$

It is direct to verify that

$$P_{\{\!|\beta|\!\}}P_{\{\!|\alpha|\!\}}\exists_{\{\!|\alpha|\!\}}(\beta) = \top_C.$$

Therefore there exists a unique $g\colon C \longrightarrow D$ such that the following commutes

$$\begin{tikzcd}
C \arrow[r, "\{\!|\beta|\!\}"] \arrow[dr, "g"'] & B \arrow[r, "\{\!|\alpha|\!\}"] & A \\
& D. \arrow[ur, "\{\!|\exists_{\{\!|\alpha|\!\}}(\beta)|\!\}"'] &
\end{tikzcd}$$

Observe that for every γ in $P(B)$ we have that $\gamma \leq \top_B = P_{\{\!|\alpha|\!\}}(\alpha)$ implies

$$\exists_{\{\!|\alpha|\!\}}(\gamma) \leq \exists_{\{\!|\alpha|\!\}}P_{\{\!|\alpha|\!\}}(\alpha) \leq \alpha.$$

In particular we have $\exists_{\{\alpha\}}(\beta) \leq \alpha$, and then

$$P_{\{\exists_{\{\alpha\}}(\beta)\}}(\alpha) = \top_D.$$

Hence there exists a unique $h\colon D \longrightarrow B$ such that the following diagram commutes

$$\begin{tikzcd}
B \arrow[rr, "\{\alpha\}"] & & A \\
& D. \arrow[ul, "h"] \arrow[ur, "\{\exists_{\{\alpha\}}(\beta)\}"'] &
\end{tikzcd}$$

Now we can observe that

$$P_h(P_{\{\alpha\}}(\exists_{\{\alpha\}}(\beta)) = \top_D$$

implies $P_h(\beta) = \top_D$ because we have $P_{\{\alpha\}}\exists_{\{\alpha\}}(\beta) = \beta$ by Proposition 3.2.45. Therefore there is a unique $l\colon D \longrightarrow C$ such that the diagram

$$\begin{tikzcd}
C \arrow[r, "\{\beta\}"] & B \arrow[r, "\{\alpha\}"] & A \\
& D \arrow[ul, "l"] \arrow[u, "h"] \arrow[ur, "\{\exists_{\{\alpha\}}(\beta)\}"'] &
\end{tikzcd}$$

commutes. Then we can conclude that $g \circ l = \mathrm{id}_D$, and since g is a monomorphism, it is an isomorphism. $\square$

The previous proposition has the following consequence.

Proposition 3.3.7. *Let* $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *be an existential m-variational doctrine. Then the fibration* $p_P\colon \mathcal{G}_P \longrightarrow \mathcal{C}$ *has strong coproducts.*

By Proposition 3.3.2, 3.3.4, 3.3.7 and Remark 3.3.5 we have the following corollary.

Corollary 3.3.8. *Every existential m-variational doctrine* $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *induces a factorization fibration with coproducts*

$$p_P\colon \mathcal{G}_P \longrightarrow \mathcal{C}.$$

Moreover this fibration is a fibred pre-order.

Combining Corollary 3.1.23 and Corollary 3.3.8 we obtain the following result.

Theorem 3.3.9. *Every existential m-variational doctrine induces a stable factorization system* $\langle \mathcal{E}, \mathcal{M} \rangle$ *where* $\mathcal{M}$ *is the class of comprehensions, and the morphisms of* $\mathcal{E}$ *are those arrows* $u\colon A \longrightarrow B$ *such that* $\exists_u(\top_A) = \top_B$.

Remark 3.3.10. If $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is an existential m-variational doctrine every arrow $f\colon A \longrightarrow B$ admits the following factorization

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "g"'] & & B \\ & I. \arrow[ur, "\{\exists_f(\top_A)\}"'] & \end{tikzcd}$$

Moreover we have that g satisfies $\exists_g(\top_A) = \top_I$ since

$$\exists_g(\top_A) = P_{\{\exists_f(\top_A)\}}\exists_{\{\exists_f(\top_A)\}}\exists_g(\top_A) = P_{\{\exists_f(\top_A)\}}(\exists_f(\top_A)) = \top_I.$$

Observe that $P_{\{\exists_f(\top_A)\}}\exists_{\{\exists_f(\top_A)\}} = \mathrm{id}_{P(I)}$ because comprehensions are monomorphisms and in an existential m-variational doctrine the Beck-Chevalley condition holds for every morphism. In particular it holds for the following pullback

$$\begin{tikzcd} I \arrow[r, "\mathrm{id}_I"] \arrow[d, "\mathrm{id}_I"'] \arrow[dr, phantom, "\lrcorner", very near start] & I \arrow[d, "\{\exists_f(\top_A)\}"] \\ I \arrow[r, "\{\exists_f(\top_A)\}"'] & A. \end{tikzcd}$$

Remark 3.3.11. Consider a stable, proper factorization system $\langle \mathcal{E}, \mathcal{M}\rangle$ for a category $\mathcal{C}$ with finite limits. The codomain fibration induces an existential m-variational doctrine

$$\mathrm{Sub}_{\mathcal{M}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$$

which sends an object A into the category of $\mathcal{M}$-subobjects of A.

Proposition 3.3.12. *Let* $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *be an existential m-variational doctrine, and let* $\langle \mathcal{E}, \mathcal{M}\rangle$ *be the factorization system induced by Corollary 3.3.8. Then the doctrine* $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *is equivalent to* $\mathrm{Sub}_{\mathcal{M}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$.

Recall that **LFS** is the 2-category whose objects are $(\mathcal{C}, \langle \mathcal{E}, \mathcal{M}\rangle)$, where $\langle \mathcal{E}, \mathcal{M}\rangle$ is a stable, proper factorization system for a category $\mathcal{C}$ with finite limits, and whose morphisms are functors preserving the factorizations.

Theorem 3.3.13. *The 2-category* **LFS** *is 2-equivalent to the 2-category* **Ex-mVar** *of existential m-variational doctrines.*

We can combine now the three free completions we have studied in the previous section, and we obtain the exact completion for existential m-variational doctrines:

$$\mathbf{Ex\text{-}mVar} \xrightarrow{\ \cong\ } \mathbf{LFS} \xrightarrow{\mathbf{Map}\ \mathbf{Rel}(-)} \mathbf{Reg} \xrightarrow{(-)_{\mathrm{ex/reg}}} \mathbf{Xct}\,.$$

One can give a more concrete description of the regular category given by the composition of the firsts two previous functors.

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be an m-variational existential doctrine, and let $\langle \mathcal{E}, \mathcal{M}\rangle$ be the stable, proper factorization system on $\mathcal{C}$ defined in Theorem 3.3.9.

We shall understand how we can characterize the relations and the maps in this particular factorization system.

Recall that a relation $R = \langle r_1, r_2\rangle\colon A \longrightarrow B$ in $\mathbf{Rel}(\mathcal{C}, \langle \mathcal{E}, \mathcal{M}\rangle)$ from A to B is a map if and only if $r_1 \in \Sigma = \mathcal{E} \cap \mathrm{mono}$ by Proposition 2.3.14.

In our case we have that R is a relation if and only if $R = \{\!|\alpha|\!\}$ for some $\alpha \in P(A \times B)$.

In particular R is a map if and only if $\mathrm{pr}_1\{\!|\alpha|\!\} \in \Sigma$. Observe that $\mathrm{pr}_1\{\!|\alpha|\!\} \in \mathcal{E}$ implies that $\exists_{\mathrm{pr}_1\{\alpha\}}(\top_A) = \top_A$, and by Remark 3.2.43 we have

$$\exists_{\mathrm{pr}_1\{\alpha\}}(\top_A) = \top_A \text{ if and only if } \exists_{\mathrm{pr}_1}(\alpha) = \top_A$$

An α in $P(A \times B)$ such that $\exists_{\mathrm{pr}_1}(\alpha) = \top_A$ is said ***entire from** A **to** B*.

The condition $r_1 \in \mathrm{mono}$ means that α is ***functional from** A **to** B*, which implies that

$$P_{\langle \mathrm{pr}_1, \mathrm{pr}_2\rangle}(\alpha) \wedge P_{\langle \mathrm{pr}_1, \mathrm{pr}_3\rangle}(\alpha) \leq P_{\langle \mathrm{pr}_2, \mathrm{pr}_3\rangle}(\delta_B)$$

in $P(A \times B \times B)$.

Therefore we can give a direct description of the category $\mathbf{Map}\ \mathbf{Rel}(\mathcal{C})$ of maps of $\mathbf{Rel}(\mathcal{E}, \mathcal{M}, \mathcal{C})$, and we denote this category $\mathbf{Ef}_P$. Objects of $\mathbf{Ef}_P$ are the objects of $\mathcal{C}$, and morphisms are entire functional relations.

As results we have that the category $\mathbf{Ef}_P$ is regular, it is called ***regular completion*** of the m-variational existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$.

Example 3.3.14. The regular completion $(\mathcal{D})_{\mathrm{reg/lex}}$ of a category $\mathcal{D}$ with finite limit in [6] is equivalent to the regular completion $\mathbf{Ef}_{(\mathrm{Sub}_{\mathcal{D}})_{cd}}$ of the doctrine

$$\mathrm{Sub}_{\mathcal{D}}\colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$$

of subobjects of $\mathcal{D}$.

The ***exact completion*** of a m-variational doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is given by $(\mathbf{Ef}_P)_{\mathrm{ex/reg}}$.

Moreover we can generalize the regular and the exact completion to an arbitrary elementary existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$, obtaining the regular category $\mathbf{Ef}_{(P)_{cd}}$ and the exact category $(\mathbf{Ef}_{(P)_{cd}})_{\mathrm{ex/reg}}$.

We can summarize the exact completion of an elementary existential doctrine as the composition of the followings

$$\mathbf{EED} \xrightarrow{(-)_{cd}} \mathbf{Ex\text{-}mVar} \xrightarrow{\cong} \mathbf{LFS} \xrightarrow{\mathbf{Map}\ \mathbf{Rel}(-)} \mathbf{Reg} \xrightarrow{(-)_{\mathrm{ex/reg}}} \mathbf{Xct}\,.$$

Now we look at the quotient completion and we denote by $\mathbf{QD}$ the 2-category of existential m-variational doctrines with stable, effective quotients.

In this case, a doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ of $\mathbf{QD}$ provides two stable proper factorization systems for the base category $\mathcal{C}$: the first one comes from the m-variational structure as above, and we denote it by $\langle \mathcal{E}_1, \mathcal{M}_1 \rangle$, and the second one $\langle \mathcal{E}_2, \mathcal{M}_2 \rangle$ is given by the quotients.

The class $\mathcal{E}_2$ consists of all the morphisms which are quotients, and the class $\mathcal{M}_2$ consists of arrows $f\colon A \longrightarrow B$ of $\mathcal{C}$ such that $P_{f\times f}(\delta_B) = \delta_A$.

In particular $\mathcal{M}_2$ is the class of monomorphisms of $\mathcal{C}$, because if a morphism f of $\mathcal{C}$ is mono then $P_{f\times f}(\delta_B) = \delta_A$ by [43, Corollary 4.8], while if a morphism $f\colon A \longrightarrow B$ satisfies $P_{f\times f}(\delta_B) = \delta_A$, then we can construct the kernel pair as follows

$$\begin{tikzcd}
X \arrow[rr] \arrow[dd] \arrow[dr, "\{\!\!\{P_{f\times f}(\delta_B)\}\!\!\}"] & & A \arrow[dd, "f"] \\
& A\times A \arrow[ur, "\mathrm{pr}_2"'] \arrow[dl, "\mathrm{pr}_1"] & \\
A \arrow[rr, "f"'] & & B.
\end{tikzcd}$$

By Remark 3.2.44 we can conclude that f is mono because the doctrine P is m-variational and then $\{\!\!\{P_{f\times f}(\delta_B)\}\!\!\} = \{\!\!\{\delta_A\}\!\!\} = \Delta_A$.

Moreover if we consider a commutative square

$$\begin{tikzcd}
A \arrow[r, "q"] \arrow[d, "u"'] & B \arrow[d, "v"] \\
C \arrow[r, "m"'] & D
\end{tikzcd}$$

where $q \in \mathcal{E}_2$ and $m \in \mathcal{M}_2$, then we have

$$\delta_B \leq P_{v\times v}(\delta_D)$$

and then

$$P_{q\times q}(\delta_B) \leq P_{q\times q}(P_{v\times v}(\delta_D)) = P_{u\times u}(P_{m\times m}(\delta_D)) = P_{u\times u}(\delta_C).$$

Thus there exists a unique $s\colon B \longrightarrow C$ such that $u = sq$, since q is a quotient of $P_{q\times q}(\delta_B)$. Hence we have

$$msq = mu = vq$$

and then $ms = v$ because q is an epimorphism.

Therefore $\langle \mathcal{E}_2, \mathcal{M}_2 \rangle$ is a factorization system, proper and it is stable because quotients are stable in every doctrine of $\mathbf{QD}$.

Thus the factorizations system $\langle \mathcal{E}_2, \mathcal{M}_2 \rangle$ has as class $\mathcal{M}_2$ all the monomorphisms, and then the category $\mathcal{C}$ is regular. See [9].

Note that $\langle \mathcal{E}_1, \mathcal{M}_1 \rangle$ and $\langle \mathcal{E}_2, \mathcal{M}_2 \rangle$ are not equal in general: they are the same factorization system if and only if the doctrine P satisfies the rules of unique choice, see [41].

As it is observed in [41], the construction of the category $\mathbf{Ef}_P$ for a doctrine P of **QD** forces the rule of unique choice, in the sense that the category $\mathbf{Ef}_P$ is an exact category.

3.3.1 Tripos to topos

We conclude this chapter comparing the three different exact completions of an elementary existential doctrine.

Recall from [51] the construction of a topos from a tripos. In [41] it is shown that this construction can be stated in the case of an elementary existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$. We refer to [41, 44] for a complete analysis of that.

Given an elementary existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ the category $\mathcal{T}_P$ consists of

- ***objects:*** pair (A, ρ) such that ρ is in $P(A\times A)$and satisfies symmetry and transitivity properties as in Subsection 3.2.1;
- ***arrows***: an arrow $\phi\colon (A, \rho) \longrightarrow (A, \sigma)$ is an object ϕ in $P(A \times B)$ such that
  1. $\phi \leq P_{\langle \mathrm{pr}_1, \mathrm{pr}_1 \rangle}(\rho) \wedge P_{\langle \mathrm{pr}_2, \mathrm{pr}_2 \rangle}(\sigma)$;
  2. $P_{\langle \mathrm{pr}_1, \mathrm{pr}_2 \rangle}(\rho) \wedge P_{\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle}(\phi) \leq P_{\langle \mathrm{pr}_1, \mathrm{pr}_3 \rangle}(\phi)$ in $P(A \times A \times B)$ where the pr_i's are the projections from $A \times A \times B$;
  3. $P_{\langle \mathrm{pr}_1, \mathrm{pr}_2 \rangle}(\phi) \wedge P_{\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle}(\sigma) \leq P_{\langle \mathrm{pr}_1, \mathrm{pr}_3 \rangle}(\phi)$ in $P(A \times B \times B)$ where the pr_i's are the projections from $A \times B \times B$;
  4. $P_{\langle \mathrm{pr}_1, \mathrm{pr}_2 \rangle}(\phi) \wedge P_{\langle \mathrm{pr}_1, \mathrm{pr}_3 \rangle}(\phi) \leq P_{\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle}(\sigma)$ in $P(A \times B \times B)$ where the pr_i's are the projections from $A \times B \times B$;
  5. $P_{\Delta_A}(\rho) \leq \exists_{\mathrm{pr}_1}(\phi)$ in $P(A)$ where the pr_i's are the projections from $A \times B$.

The composition of $\phi\colon (A, \rho) \longrightarrow (B, \sigma)$ and $\psi\colon (B, \sigma) \longrightarrow (C, \tau)$ is defined as

$$\exists_{\langle \mathrm{pr}_1, \mathrm{pr}_3 \rangle}(P_{\langle \mathrm{pr}_1, \mathrm{pr}_2 \rangle}(\phi) \wedge P_{\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle}(\psi))$$

and the identity on (A, ρ) is the arrow $\rho\colon (A, \rho) \longrightarrow (A, \rho)$.

This construction in called in [41, 44] the ***exact completion*** of an elementary existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$.

Example 3.3.15. The main examples of this construction are localic toposes and realizability toposes obtained from a tripos, see [20, 51].

In [44, 41] it is proved that the category $\mathcal{T}_P$ obtained from the tripos to topos construction for an elementary existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is exact and it is equivalent to the category $(\mathbf{Ef}_{(P)_{cd}})_{\mathrm{ex/reg}}$.

Moreover this construction can be extended to a 2-functor $\mathbf{EED} \longrightarrow \mathbf{Xct}$ which sends and elementary existential doctrine to the category $\mathcal{T}_P$, and this 2-functor is biadjoint to the 2-functor $\mathbf{Xct} \longrightarrow \mathbf{EED}$ which sends an exact category $\mathcal{X}$ to the doctrine $\mathrm{Sub}_{\mathcal{X}} \colon \mathcal{X}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$. See [41, Theorem 4.9] for all the details.

Theorem 3.3.16. *Let* $P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *be an elementary existential doctrine. Then the category* $\mathcal{T}_P$ *is exact and the 2-functor* $\mathbf{Xct} \longrightarrow \mathbf{EED}$ *that takes an exact category to the elementary existential doctrine of its subobjects has a left biadjoint which associates the exact category* $\mathcal{T}_P$ *to an elementary existential doctrine* P.

We conclude this section comparing the tripos-to-topos construction

Theorem 3.3.17. *Let* $P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *be an elementary existential doctrine. Then the category* $\mathcal{T}_P$ *is equivalent to* $(\mathbf{Ef}_{(P)_{cd}})_{\mathrm{ex/reg}}$.

Theorem 3.3.18. *Let* $P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *be an elementary existential doctrine. Then the category* $\mathcal{T}_P$ *is equivalent to* $\mathbf{Ef}_{(P)_{cqd}}$.

Chapter 4
Completions of Elementary Doctrines and Pseudo-Distributive Laws

Abstract In this paper we construct three pseudo-monads related to the completion with quotients, the completion with comprehension and the completion with comprehensive diagonals, and prove that they all are pseudo-property-like. This produces an algebraic description of the the 2-categories of elementary doctrines with each of the previous structures. In particular, we prove that each such 2-category is equivalent to the 2-category of pseudo-algebras of the pseudo-monad related to the appropriate completion. Finally we show that there are pseudo-distributive laws between certain pairs among the three pseudo-monads, hence we obtain that the composition of such a pair is again a pseudo-monad.

4.1 Introduction

Category theory provides a language to study at the same time the syntax and the semantics of formal systems and to compare different theories even if they are in different logical languages.

F.W. Lawvere introduced this approach to logic in [36, 37, 38]. He had the intuition that it is possible to study the properties of logical theories using indexed categories, introducing what he called hyperdoctrines.

It is emphasized in several works, see for instance [24, 42, 51, 50], how every first order theory corresponds (up to isomorphism) to a unique syntactic hyperdoctrine, which contains all the information about the syntax and the semantics of the theory. In the same way one can study higher order theories, see [51].

In recent work [43, 42, 44, 41], Maietti and Rosolini studied a more general notion than hyperdoctrines, namely primary and elementary doctrines, and they generalized the exact completion of Carboni, see [8, 6], by relativizing the basic data to a doctrine equipped with just enough structure to talk about the notion of an equivalence relation.

In category theory, in order to give a precise meaning to the notion of "completion", one can take the notion of a left adjoint functor to the forgetful functor between

2-categories. A possible counterpart of this in logic can be seen in the extension of a first order theory with new constructors and new axioms.

It is known that starting from an adjunction one can construct a monad, and more generally, starting from a pseudo-adjunction one can construct a pseudo-monad. This allows to give an algebraic interpretation of the completion one considers, and to understand if the structure added by completing is just a new property.

In order to understand the previous distinction, as explained in [27], one may look at the 2-monad coming from the completion, and study the 2-category of its algebras.

In the present paper we study the following pseudo-monads together with the categories of pseudo-algebras coming from three completions of elementary doctrines: the completion with comprehensions, the completion with comprehensive diagonals, and the completion with quotients. We prove that all these pseudo-monads are property-like in the sense of [27]. Moreover we present how these pseudo-monads can be composed, in other words we find pseudo-distributive laws between certain pairs of them.

In sections 4.2, 4.3 and 4.4 we construct the pseudo-functors and the pseudo-monads coming from the three completions mentioned before, and we prove that all three pseudo-monads are pseudo-property like. The first completion we present is the completion with comprehensive diagonals, because it is the easiest and the other two are done following similar arguments.

In section 4.5 we present the pseudo-distributive laws, and explain what one obtains composing the pseudo-monads.

4.2 Elementary doctrines with comprehensive diagonals

In this section we consider the biadjunction determined by the completion to force diagonals to be comprehensive for elementary doctrines. We show that in this case, the biadjunction is a 2-adjunction, and we shall explain how every elementary doctrine with comprehensive diagonals can be seen as an algebra for the 2-monad. In order to compute such 2-monad we first compute explicitly the 2-functor left adjoint to the forgetful 2-functor.

Consider the full 2-subcategory **CED** of **ElD**, whose objects are elementary doctrines with comprehensive diagonals. With the same notation used in [42], we want to verify the existence of the left adjoint to the forgetful 2-functor:

$$\mathrm{D}\colon \mathbf{ElD} \longrightarrow \mathbf{CED}$$

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be an elementary doctrine, we define $\mathcal{X}_P$ the ***extensional collapse*** of P:

- **the objects** of $\mathcal{X}_P$ are the objects of $\mathcal{C}$;

- **a morphism** $[f]\colon A \longrightarrow B$ is an equivalence class of morphisms $f\colon A \longrightarrow B$ such that $\delta_A \leq_{A\times A} P_{f\times f}(\delta_B)$ with respect to the equivalence $f \sim f'$ when $\delta_A \leq_{A\times A} P_{f\times f'}(\delta_B)$.

The indexed inf-semilattice $P_x\colon \mathcal{X}_P^{op} \longrightarrow \mathbf{InfSL}$ will be given by P itself: indeed for every A in $\mathcal{C}$, $P_x(A) = P(A)$ and for every $[f]\colon A \longrightarrow B$, $P_x([f]) = P(f)$ as one shows that $P(f) = P(f')$ when $f \sim f'$. See [42, Lemma 5.5].

The idea is that the assignment $\mathrm{D}(P) = P_x$ can be extend to a 2-functor. We need to describe how it acts one the 1-cells and 2-cells. Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ and $R\colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be elementary doctrines, and consider a 1-cell (F, b):

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \arrow[dd, "F^{\mathrm{op}}"'] \arrow[dr, "P"] & \\
 & \mathbf{InfSL} \\
\mathcal{D}^{\mathrm{op}} \arrow[ur, "R"'] &
\end{tikzcd} \qquad b\colon P \Rightarrow R F^{\mathrm{op}}$$

Let $(\widetilde{F}, b)$ be the pair where

- $\widetilde{F}(A)$ is $F(A)$ for every $A \in \mathcal{X}_P$;
- $\widetilde{F}([f])$ is $[F(f)]$ for every $[f]\colon A \longrightarrow B$.

Proposition 4.2.1. *$(\widetilde{F}, b)$ is a 1-morphism in* **CED**.

Proof. First we prove that $\tilde{F}\colon \mathcal{X}_P \longrightarrow \mathcal{X}_R$ is a well-defined functor. If $f\colon A \longrightarrow B$ and $g\colon A \longrightarrow B$ are a morphism in $\mathcal{C}$, such that $\delta_A \leq P_{g\times f}(\delta_B)$, then we have

$$b_{A\times A}(\delta_A) \leq b_{A\times A}(P_{g\times f}(\delta_B))$$

Since b is a natural transformation, the following diagram commutes

$$\begin{tikzcd}
P(B\times B) \arrow[r, "P_{g\times f}"] \arrow[d, "b_{B\times B}"'] & P(A\times A) \arrow[d, "b_{A\times A}"] \\
RF(B\times B) \arrow[r, "R_{F(g\times f)}"'] & RF(A\times A)
\end{tikzcd}$$

Hence we have

$$b_{A\times A}(\delta_A) \leq R_{F(g\times f)}(b_{B\times B}(\delta_B))$$

By definition, $b_{A\times A}(\delta_A) = R_{\langle F(\mathrm{pr}_1), F(\mathrm{pr}_2)\rangle}(\delta_{F(B)})$; thus

$$R_{\langle F(\mathrm{pr}_1), F(\mathrm{pr}_2)\rangle}(\delta_{F(A)}) \leq R_{\langle F(\mathrm{pr}'_1), F(\mathrm{pr}'_2)\rangle \circ F(g\times f)}(\delta_{F(B)})$$

where $\mathrm{pr}_i\colon A \times A \longrightarrow A$ and $\mathrm{pr}'_i\colon B \times B \longrightarrow B$ are the projections. Finally

$$F(g \times f) \circ \langle F(\mathrm{pr}_1), F(\mathrm{pr}_2)\rangle^{-1} = \langle F(\mathrm{pr}'_1), F(\mathrm{pr}'_2)\rangle \circ F(g) \times F(f),$$

so

$$\delta_A \leq R_{F(g)\times F(f)}(\delta_B).$$

It is now easy to check that $\tilde{F}$ is a functor from $\mathcal{X}_P$ to $\mathcal{X}_R$. Next we have that $(\tilde{F}, b)$ is a 1-cell observing that

$$b_{A\times A}(\delta_A) = (R_x)_{\langle \widetilde{F}([\mathrm{pr}_1]), \widetilde{F}([\mathrm{pr}_2])\rangle}(\delta_{\widetilde{F}(B)})$$

because $\widetilde{F}([\mathrm{pr}_i]) = [F(\mathrm{pr}_i)]$, $\widetilde{F}(B) = F(B)$ by definition of $\widetilde{F}$, and

$$\langle \widetilde{F}([\mathrm{pr}_1]), \widetilde{F}([\mathrm{pr}_2])\rangle = [\langle F(\mathrm{pr}_1), F(\mathrm{pr}_2)\rangle]$$

by [42, Lemma 5.4], and

$$(R_x)_{\langle \widetilde{F}([\mathrm{pr}_1]), \widetilde{F}([pr2])\rangle} = (R_x)_{[\langle F(\mathrm{pr}_1), F(\mathrm{pr}_2)\rangle]} = R_{\langle F(\mathrm{pr}_1), F(\mathrm{pr}_2)\rangle}$$

□

As for a 2-cell $\theta\colon (F, b) \Longrightarrow (G, c)$, where (F, b) and (G, c) are 1-cells in $\mathbf{ElD}(P, R)$, define $\widetilde{\theta}\colon \widetilde{F} \longrightarrow \widetilde{G}$ as the natural transformation with $\widetilde{\theta}_A = [\theta_A]$. Since it is a 2-cell in $\mathbf{ElD}$,

$$b_A(\alpha) \leq_{F(A)} R_{\theta_A}(c_A(\alpha)).$$

By definition of R_x and $\widetilde{F}$,

$$R_{\theta_A}(c_A(\alpha)) = (R_x)_{[\theta_A]}(c_A(\alpha)) = (R_x)_{\widetilde{\theta}_A}(c_A(\alpha)),$$

so

$$b_A(\alpha) \leq_{\widetilde{F}(A)} (R_x)_{\widetilde{\theta}_A}(c_A(\alpha)).$$

Proposition 4.2.2. *Let* $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *and* $R\colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *be elementary doctrines. The map*

$$\mathrm{D}_{P,R}\colon \mathbf{ElD}(P, R) \longrightarrow \mathbf{CED}(P_x, R_x)$$

such that $\mathrm{D}_{P,R}(F, b) = (\widetilde{F}, b)$ *and* $\mathrm{D}_{P,R}(\theta) = \widetilde{\theta}$ *is a functor and*

$$\mathrm{D}\colon \mathbf{ElD} \longrightarrow \mathbf{CED}$$

is a 2-functor with the assignment $\mathrm{D}(P) = P_x$.

We prove that the 2-functor $\mathrm{D}\colon \mathbf{ElD} \longrightarrow \mathbf{CED}$ is left adjoint to the forgetful 2-functor. Recall from [42] the equivalence

$$- \circ (K, k)\colon\ \mathbf{CED}(P_x, Z) \equiv \mathbf{ElD}(P, Z)$$

where $K\colon \mathcal{C} \longrightarrow \mathcal{X}_P$ is the quotient functor and k_A is the identity. For more details see [42, Theorem 5.5].

For an elementary doctrine $P \in \mathbf{ElD}$, let

$$\eta_P\colon P \longrightarrow \mathrm{U} \circ \mathrm{D}(P)$$

be the image of the identity on $\mathrm{D}(P)$, under the equivalence

$$- \circ (K_P, k_P)\colon\ \mathbf{CED}(\mathrm{D}(P), \mathrm{D}(P)) \equiv \mathbf{ElD}(P, \mathrm{U} \circ \mathrm{D}(P))$$

which means that η_P is the 1-morphism (K_P, k_P). It is direct to check that the assignment

$$\eta\colon\ \mathrm{id}_{\mathbf{ElD}} \longrightarrow \mathrm{U} \circ \mathrm{D}$$

is a 2-natural transformation.

Remark 4.2.3. In the case P is of the form P_x we have that

$$\mathbf{CED}(\mathrm{D}(P_x), P_x) \cong \mathbf{ElD}(P_x, P_x)$$

because **CED** is a full 2-subcategory of **ElD**. Then η_{P_x} is isomorphic to the identity on P_x.

Remark 4.2.4. Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be an elementary doctrine with comprehensive diagonals, and let $f\colon A \longrightarrow B$ and $g\colon A \longrightarrow B$ be morphisms such that $\delta_A \leq P_{f\times g}(\delta_B)$. We have that $\top_A \leq P_{f\times g\circ\Delta_A}(\delta_B) = P_{\langle f,g\rangle}(\delta_B)$. Thus there exists a unique morphism $h\colon A \longrightarrow B$ such that the following diagram commutes:

$$\begin{tikzcd} B \arrow[rr, "\Delta_B"] & & B \times B \\ & A \arrow[ul, dashed, "h"] \arrow[ur, "\langle f,g\rangle"'] & \end{tikzcd}$$

By Remark 4.2.4, if $P \in \mathbf{CED}$ then $f \sim g$ if and only if $f = g$. For this reason we can define a 1-cell $(T_P, t_P)\colon P_x \longrightarrow P$ such that

- T_P sends A in A and $[f]$ in f;
- t_P is the identity.

Moreover it is easy to see that $(T_P, t_P) \circ (K_P, k_P) = 1_P$ and $(K_P, k_P) \circ (T_P, t_P) = 1_{P_x}$. Thus we denote $\varepsilon_P := (T_P, t_P)$ and the a 2-natural transformation

$$\varepsilon\colon \mathrm{D} \circ \mathrm{U} \longrightarrow \mathrm{id}_{\mathbf{CED}}$$

Remark 4.2.5. If $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is an elementary doctrine of **CED** we have

$$\mathrm{D}(T_P, t_P) = (T_{P_x}, t_{P_x})$$

Proposition 4.2.6. *For every elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ with comprehensive diagonals, the following equalities hold:*

$$\varepsilon_P \circ \eta_P = 1_P$$

and

$$\eta_P \circ \varepsilon_P = 1_{P_x}.$$

Proof. The first is a consequence of the definition of η_P end ε_P, and the second by Remark 4.2.4. □

Proposition 4.2.7. *For every elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ we have*

$$\varepsilon_{\mathrm{D}(P)} \circ \mathrm{D}(\eta_P) = 1_{\mathrm{D}(P)}.$$

Proof. It follows from 4.2.5. □

We are now in the position to compute the 2-monad:

- let $\mathrm{T}_d\colon \mathbf{ElD} \longrightarrow \mathbf{ElD}$ be the 2-functor $\mathrm{T} = \mathrm{U} \circ \mathrm{D}$;
- let $\eta\colon \mathrm{id}_{\mathbf{ElD}} \longrightarrow \mathrm{T}$ be the unit of the 2-adjunction;
- let $\mu\colon \mathrm{T}_d^2 \longrightarrow \mathrm{T}_d$ be the 2-natural transformation $\mu := \mathrm{U}\varepsilon\mathrm{D}$;

Remark 4.2.8. Observe $\mu_P\colon \mathrm{T}_d^2 P \longrightarrow \mathrm{T}_d P$ is an isomorphism.

Proposition 4.2.9. *The triple $(\mathrm{T}_d, \mu, \eta)$ is a 2-monad.*

Proof. The following diagram commutes by Remark 4.2.5

$$\begin{tikzcd}
\mathrm{T}_d^3 \arrow[r, "\mu \mathrm{T}_d"] \arrow[d, "\mathrm{T}_d \mu"'] & \mathrm{T}_d^2 \arrow[d, "\mu"] \\
\mathrm{T}_d^2 \arrow[r, "\mu"'] & \mathrm{T}_d
\end{tikzcd}$$

Moreover, we have $\eta_{P_x} = \mathrm{T}(\eta_P)$, and then the following diagram commutes

$$\begin{tikzcd}
\mathrm{id}_{\mathbf{ElD}} \circ \mathrm{T}_d \arrow[r, "\eta \mathrm{T}_d"] \arrow[dr, "\mathrm{id}"'] & \mathrm{T}_d^2 \arrow[d, "\mu"] & \mathrm{T}_d \circ \mathrm{id}_{\mathbf{ElD}} \arrow[l, "\mathrm{T}_d \eta"'] \arrow[dl, "\mathrm{id}"] \\
& \mathrm{T}_d &
\end{tikzcd}$$

Therefore T_d is a 2-monad. □

Proposition 4.2.10. *Let* $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *be an elementary doctrine. If it admits an action* $a\colon \mathrm{T}_d P \longrightarrow P$ *such that* (P, a) *is a pseudo-*T_d*-algebra, then* $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *has comprehensive diagonals, and the action preserves them.*

Proof. Let (P, a) be a pseudo-T_d-algebra, so in particular the identity axiom holds

$$\begin{tikzcd}
P \arrow[r, "\eta_P"] \arrow[dr, "1_P"'] \arrow[dr, phantom, shift left=3, "\Downarrow a_\eta"] & \mathrm{T}_d P \arrow[d, "a"] \\
& P.
\end{tikzcd}$$

Let $f\colon C \longrightarrow A \times A$ be a morphism of $\mathcal{C}$ such that $P_f(\delta_A) \geq \top_C$. Since P_x has comprehensive diagonals, there exists a unique $[g]$ such that the following diagram commutes

$$\begin{tikzcd}
A \arrow[r, "{[\Delta_A]}"] & A \times A \\
& C. \arrow[ul, "{[g]}"] \arrow[u, "{[f]}"']
\end{tikzcd}$$

So

$$\begin{tikzcd}
aA \arrow[r, "{a[\Delta_A]}"] & a(A \times A) \\
& aC \arrow[ul, "{a[g]}"] \arrow[u, "{a[f]}"']
\end{tikzcd}$$

also commutes. Now we use the fact that $a_\eta\colon a\eta_P \Longrightarrow \mathrm{id}_P$ is a natural transformation, where all the components are isomorphisms. So the upper triangle and all the squares of the following diagram commute

$$\begin{tikzcd}
aA \arrow[rr, "{a[\Delta_A]}"] \arrow[dd, "a_{\eta_A}"'] & & a(A\times A) \arrow[dd, "a_{\eta_{(A\times A)}}"] \\
& aC \arrow[ul, "{a[g]}"'] \arrow[ur, "{a[f]}"] \arrow[dd, "a_{\eta_C}"'] & \\
A \arrow[rr, "\Delta_A"'] & & A\times A \\
& C. \arrow[ul, "g"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Thus the bottom triangle commutes. Moreover g is certainly unique. $\square$

Remark 4.2.11. Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be an elementary doctrine with comprehensive diagonals. The diagram

$$\begin{tikzcd}
\mathrm{T}_d^2 P \arrow[r, "{\mathrm{T}_d(T_P,t_P)}"] \arrow[d, "\mu_P"'] & \mathrm{T}_d P \arrow[d, "{(T_P,t_P)}"] \\
\mathrm{T}_d P \arrow[r, "{(T_P,t_P)}"'] & P
\end{tikzcd}$$

commutes by Remark 4.2.5, since $\mu_P = \varepsilon_{D(P)} = (T_{P_x}, t_{P_x}) = D(T_P, t_P) = \mathrm{T}(T_P, t_P)$ in **ElD**. Thus every elementary doctrine of **CED** can be regarded with an action $a\colon \mathrm{T}_d P \longrightarrow P$ which makes the previous diagram commutes. This means that an elementary doctrine with comprehensive diagonals can be seen as a T_d-algebra, endowed with the action $a = (T_P, t_P)$.

Remark 4.2.12. Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ and $R\colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be elementary doctrines with comprehensive diagonals, and let $(F, f)\colon P \longrightarrow R$ be a 1-cell in **CED**. By Remark 4.2.4 and definition of T_d, we conclude that the following diagram commutes

$$\begin{tikzcd}
T_d P \arrow[r, "{T_d(F,f)}"] \arrow[d, "{(T_P,t_P)}"'] & T_d R \arrow[d, "{(T_R,t_R)}"] \\
T_d P \arrow[r, "{(F,f)}"'] & T_d R
\end{tikzcd}$$

commutes. By the same argument as in Remark 4.2.11 we conclude that every 2-cell in **CED** induces a 2-cell in $\mathrm{T_d}$-**Alg**, and we have the following inclusion of 2-categories

$$\mathbf{CED} \hookrightarrow \mathrm{T_d}\text{-}\mathbf{Alg} \hookrightarrow \mathrm{T_d}\text{-}\mathbf{Alg}_l \hookrightarrow \mathbf{ElD}$$

Theorem 4.2.13. *The 2-monad* $\mathrm{T}_d\colon \mathbf{ElD} \longrightarrow \mathbf{ElD}$ *is pseudo-idempotent. In particular it is fully property-like.*

Proof. The proof of the previous theorem is a direct consequence of [27, Proposition 9.6]. In fact we can see that the condition (ii) here is satisfied by Propositions 4.2.6. □

Combining Proposition 4.2.10 and Theorem 4.2.13 we obtain the following corollary.

Corollary 4.2.14. *We have the following equivalence of categories*

$$\mathrm{T_d}\text{-}\mathbf{Alg} \cong \mathbf{CED}$$

4.3 Elementary doctrines with comprehensions

In this section we consider the completion with comprehensions of an elementary doctrines. We prove that in this case, the biadjunction is a pseudo-adjunction, and explain how every elementary doctrine with comprehensions can be seen as an algebra for the pseudo-monad constructed from the pseudo-adjunction.

Let **CE** be the 2-category of elementary doctrines with full comprehension. We recall the construction used in [42]: given an elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ we define a new category $\mathcal{G}_P$.

- **an object** of $\mathcal{G}_P$ is a pair (A,α), where A is in $\mathcal{C}$ and α is in $P(A)$;
- **a morphism** $f\colon (A,\alpha) \longrightarrow (B,\beta)$ is a morphism $f\colon A \longrightarrow B$ in $\mathcal{C}$ such that $\alpha \le P_f(\beta)$;

The indexed functor extends to $P_c\colon \mathcal{G}_P^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ by setting

- $P_c(A,\alpha) = \{\gamma \in P(A) \,|\, \gamma \le \alpha\}$;
- $P_c(f)\colon P_c(B,\beta) \longrightarrow P_c(A,\alpha)$ sends $\gamma \le \beta$ into $P(f)(\gamma) \wedge \alpha$.

Remark 4.3.1. We can observe that for every object (A,α) of $\mathcal{G}_P$ we have

$$\delta_{(A,\alpha)} = \delta_A \wedge \alpha \boxtimes \alpha$$

where $\alpha \boxtimes \alpha := P_{\mathrm{pr}_1}(\alpha) \wedge P_{\mathrm{pr}_2}(\alpha)$.

Following the structure of Section 4.2 we prove that the assignment $\mathrm{C}(P) = P_c$ can be extended to 2-functor

$$\mathrm{C}\colon\ \mathbf{ElD} \longrightarrow \mathbf{CE}$$

and we start defining how it acts on the 1-cells and 2-cells in **ElD**.

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ and $R\colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be elementary doctrines, and consider a 1-cell (F, b) in **ElD**:

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \arrow[dd, "F^{\mathrm{op}}"'] \arrow[dr, "P"] \arrow[dd, phantom, shift left=8ex, "\Downarrow b"] & \\
& \mathbf{InfSL} \\
\mathcal{D}^{\mathrm{op}} \arrow[ur, "R"'] &
\end{tikzcd}$$

We want to prove that the pair $(\widehat{F}, \widehat{b})$ where:

- $\widehat{F}(A,\alpha)$ is $(FA, b_A(\alpha))$ for every $(A,\alpha) \in \mathcal{G}_P$;
- $\widehat{F}(f)$ is $F(f)$ for every $f\colon (A,\alpha) \longrightarrow (B,\beta)$;
- $\widehat{b}$ is the restriction of b on P_c;

is a 1-cell in **CE**:

$$\begin{tikzcd}
\mathcal{G}_P^{op} \arrow[dd, "\widehat{F}^{op}"'] \arrow[dr, "P_c"] \arrow[dd, phantom, shift left=8ex, "\Downarrow \widehat{b}"] & \\
& \mathbf{InfSL} \\
\mathcal{G}_R^{op} \arrow[ur, "R_c"'] &
\end{tikzcd}$$

Proposition 4.3.2. $(\widehat{F}, \widehat{b})$ *is a 1-cell in* **CE**.

Proof. First we prove that $\widehat{F}\colon \mathcal{G}_P \longrightarrow \mathcal{G}_R$ is a functor.

If $f\colon (A,\alpha) \longrightarrow (B,\beta)$ is a morphism in $\mathcal{G}_P$ then

$$\alpha \le P_f(\beta).$$

Therefore

$$b_A(\alpha) \le b_A(P_f(\beta)) = R_{F(f)}(b_B(\beta)).$$

Now observe that

$$(R_c)_{\langle F(\mathrm{pr}_1), F(\mathrm{pr}_2)\rangle}(\delta_{(FA, b_A(\alpha))}) = R_{\langle F(\mathrm{pr}_1), F(\mathrm{pr}_2)\rangle}(b_A(\alpha) \boxtimes b_A(\alpha) \wedge \delta_{FA}) \wedge b_{A\times A}(\alpha \boxtimes \alpha)$$

which is equal to

$$R_{\langle F(\mathrm{pr}_1), F(\mathrm{pr}_2)\rangle}(R_{\mathrm{pr}'_1}(b_A(\alpha)) \wedge R_{\mathrm{pr}'_2}(b_A(\alpha))) \wedge b_{A\times A}(\delta_A) \wedge b_{A\times A}(\alpha \boxtimes \alpha)$$

where $\mathrm{pr}'_i \colon FA \times FA \longrightarrow FA$. Moreover we know that b_A is a natural transformation, hence the diagram

$$\begin{tikzcd} PA \arrow[r, "P_{\mathrm{pr}_i}"] \arrow[d, "b_A"'] & P(A \times A) \arrow[d, "b_{A\times A}"] \\ RFA \arrow[r, "R_{F(\mathrm{pr}_i)}"'] & RF(A \times A). \end{tikzcd}$$

commutes. This implies that

$$(R_c)_{\langle F(\mathrm{pr}_1),F(\mathrm{pr}_2)\rangle}(\delta_{(FA,b_A(\alpha))}) = b_{A\times A}(P_{\mathrm{pr}_1}(\alpha)\wedge P_{\mathrm{pr}_2}(\alpha))\wedge b_{A\times A}(\delta_A)\wedge b_{A\times A}(\alpha\boxtimes\alpha)$$

and

$$b_{A\times A}(P_{\mathrm{pr}_1}(\alpha) \wedge P_{\mathrm{pr}_2}(\alpha)) = b_{A\times A}(\alpha \boxtimes \alpha).$$

Hence we conclude that $(\widehat{F},\widehat{b})$ is a 1-cell since

$$\widehat{b}_{(A,\alpha)\times(A,\alpha)}(\delta_{(A,\alpha)}) = b_{A\times A}(\delta_A \wedge \alpha \boxtimes \alpha) = (R_c)_{\langle F(\mathrm{pr}_1),F(\mathrm{pr}_2)\rangle}(\delta_{\widehat{F}(A,\alpha)}).$$

Finally we must prove that $(\widehat{F},\widehat{b})$ preserves comprehensions. We start observing that every comprehension in $\mathcal{G}_P$ is of the form

$$\{\!|\gamma|\!\} \colon (A,\gamma) \longrightarrow (A,\alpha)$$

where $\gamma \in P_c(A,\alpha)$, and $\{\!|\gamma|\!\}$ is the identity on A. Then

$$F(\{\!|\gamma|\!\}) \colon (FA, b_A(\gamma)) \longrightarrow (FA, b_A(\alpha))$$

and $F(\{\!|\gamma|\!\})$ is id_{FA} by definition of $\widehat{F}$, so it is a comprehension of $b_A(\gamma)$. □

Proposition 4.3.3. *Let* (F,b) *and* (G,c) *be two objects in* $\mathbf{ElD}(P,R)$ *and let* $\theta \colon (F,b) \longrightarrow (G,c)$ *be a 2-cell in* $\mathbf{ElD}$*. We define*

$$\widehat{\theta} \colon (\widehat{F},\widehat{b}) \longrightarrow (\widehat{G},\widehat{c})$$

where

$$\widehat{\theta}_{(A,\alpha)} \colon (FA, b_A(\alpha)) \longrightarrow (GA, c_A(\alpha))$$

is θ_A*. Then it is a 2-cell in* $\mathbf{CE}$*.*

Proof. Let (A,α) be an object of $\mathcal{G}_P$. We have that

$$b_A(\alpha) \le R_{\theta_A}(c_A(\alpha))$$

because θ is a 2-morphism. Therefore

$$\theta_A \colon (FA, b_A(\alpha)) \longrightarrow (GA, c_A(\alpha))$$

is a morphism in $\mathcal{G}_R$. Let γ be an object in $P_c(A,\alpha)$. Then

$$(R_c)_{\theta_A}(\widehat{c}_A(\gamma)) = R_{\theta_A}(c_A(\gamma)) \wedge b_A(\alpha)$$

by definition of R_c. Finally observe that $b_A(\gamma) \leq b_A(\alpha)$ since $\gamma \in P_c(A,\alpha)$, and $b_A(\gamma) \leq R_{\theta_A}(c_A(\gamma))$, and then we can conclude that

$$\widehat{b}_A(\gamma) = b_A(\gamma) \leq R_{\theta_A}(c_A(\gamma)) \wedge b_A(\alpha) = (R_c)_{\theta_A}(\widehat{c}_A(\gamma)).$$

$\square$

Proposition 4.3.4. *The assignment*

$$\mathrm{C}_{P,R} \colon \mathbf{ElD}(P,R) \longrightarrow \mathbf{CE}(P_c,R_c)$$

which maps (F,b) *into* $(\widehat{F},\widehat{b})$ *and a 2-cell* $\theta \colon (F,b) \longrightarrow (G,c)$ *into* $\widehat{\theta} \colon (\widehat{F},\widehat{b}) \longrightarrow (\widehat{G},\widehat{c})$ *is a functor and*

$$\mathrm{C} \colon \mathbf{ElD} \longrightarrow \mathbf{CE}$$

is a 2-functor with the assignment $\mathrm{C}(P) = P_c$.

We prove that the 2-functor $\mathrm{C} \colon \mathbf{ElD} \longrightarrow \mathbf{CE}$ is left adjoint to the forgetful 2-functor. Recall from [42] the equivalence

$$- \circ (I,i) \colon \mathbf{CE}(P_c,Z) \equiv \mathbf{ElD}(P,Z)$$

where $I \colon \mathcal{C} \longrightarrow \mathcal{R}_P$ sends an object A into $(A,\top_A)$, a morphism $f \colon A \longrightarrow B$ to $f \colon (A,\top_A) \longrightarrow (B,\top_B)$ and i_A is the identity. For more details see [42, Theorem 4.8]. For an elementary doctrine $P \in \mathbf{ElD}$, let

$$\eta_P \colon P \longrightarrow \mathrm{U} \circ \mathrm{C}(P)$$

be the image of the identity on $\mathrm{C}(P)$, under the equivalence

$$- \circ (I_P,i_P) \colon \mathbf{CE}(\mathrm{C}(P),\mathrm{C}(P)) \equiv \mathbf{ElD}(P,\mathrm{U} \circ \mathrm{C}(P))$$

which means that η_P is the 1-cell (I_P,i_P). It is direct to check that the assignment

$$\eta \colon \mathrm{id}_{\mathbf{ElD}} \longrightarrow \mathrm{U} \circ \mathrm{C}$$

is a 2-natural transformation.

Remark 4.3.5. For every $P \in \mathbf{CE}$ the equivalence

$$- \circ (I_P, i_P) \colon \ \mathbf{CE}(\mathrm{C} \circ \mathrm{U}(P), P) \equiv \mathbf{ElD}(\mathrm{U}(P), \mathrm{U}(P))$$

is essentially surjective by definition, and then there exists a 1-cell (T_P, t_P) such that

$$(T_P, t_P) \circ (I_P, i_P) \cong 1_P.$$

Let $\theta \colon (T_P, t_P) \circ (I_P, i_P) \Longrightarrow 1_P$ be the invertible 2-cell and let $\varepsilon_P := (T_P, t_P)$ be the previous 1-cell.

Remark 4.3.6. For every morphism $f \colon A \longrightarrow B$ in $\mathcal{C}$, the following diagram commutes

$$\begin{tikzcd}
T_P(A, \top_A) \arrow[r, "T_P(f)"] \arrow[d, "\theta^P_A"'] & T_P(B, \top_B) \arrow[d, "\theta^P_B"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

where $\theta^P \colon T_P \circ J_P \Longrightarrow 1_P$ is the isomorphism defined in Remark 4.3.5.

Remark 4.3.7. Let $P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be an elementary doctrine with comprehensions, and consider the 1-cells $(F, b), (G, c) \colon P_c \longrightarrow P$

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \arrow[dd, "I^{\mathrm{op}}"'] \arrow[ddr, "P"] & \\
 & \arrow[d, phantom, "\Downarrow i_P"] \\
\mathcal{G}_P^{\mathrm{op}} \arrow[r, "P_c"] \arrow[dd, bend right, "G^{\mathrm{op}}"'] \arrow[dd, bend left, "F^{\mathrm{op}}"] & \mathbf{InfSL} \arrow[d, phantom, "\overset{b}{\Downarrow}\;\overset{c}{\Downarrow}"] \\
 & {} \\
\mathcal{C}^{\mathrm{op}}. \arrow[uur, "P"'] &
\end{tikzcd}$$

Consider an invertible 2-cell $\theta \colon (F, b) \circ (I_P, i_P) \Longrightarrow (G, c) \circ (I_P, i_P)$. Then for every $f \colon A \longrightarrow B$ the diagram

$$\begin{tikzcd}
F(A,\top_A) \arrow[r, "Ff"] \arrow[d, "\theta_A"'] & F(B,\top_B) \arrow[d, "\theta_B"] \\
G(A,\top_A) \arrow[r, "Gf"'] & G(B,\top_B)
\end{tikzcd}$$

commutes. We want to prove that this isomorphism can be extended to every object of $\mathcal{G}_P$. Observe that every (A,α) can be seen as a comprehension of α in $\mathcal{G}_P$

$$(A,\alpha) \xrightarrow{\{\alpha\}} (A,\top_A)$$

and

$$F(A,\alpha) \xrightarrow{F\{\alpha\}} F(A,\top_A) \qquad\qquad G(A,\alpha) \xrightarrow{G\{\alpha\}} G(A,\top_A)$$

are comprehensions of $b_A(\alpha)$ and $c_A(\alpha)$. Moreover we have $b_A(\alpha) = P_{\theta_A}(c_A(\alpha))$ for every α in PA because θ is invertible. Using [43, Remark 4.2] we have the following pullback square

$$\begin{tikzcd}
F(A,\alpha) \arrow[r, "F\{\alpha\}"] \arrow[d, "\theta_{(A,\alpha)}"'] & F(A,\top_A) \arrow[d, "\theta_A"] \\
G(A,\alpha) \arrow[r, "G\{\alpha\}"'] & G(A,\top_A).
\end{tikzcd}$$

In order to prove the naturality we can consider a morphism $f\colon (A,\alpha) \longrightarrow (B,\beta)$ in $\mathcal{G}_P$, and we observe that the following diagram

$$\begin{tikzcd}
F(A,\alpha) \arrow[r, "F\{\alpha\}"] \arrow[d, "Ff"'] \arrow[ddr, dashed] & F(A,\top_A) \arrow[d, "Ff"] \arrow[ddr, dashed] & \\
F(B,\beta) \arrow[r, "F\{\beta\}"'] \arrow[ddr, dashed] & F(B,\top_B) \arrow[ddr, dashed] & \\
& G(A,\alpha) \arrow[r, "G\{\alpha\}"] \arrow[d, "Gf"'] & G(A,\top_A) \arrow[d, "Gf"] \\
& G(B,\beta) \arrow[r, "G\{\beta\}"'] & G(B,\top_B)
\end{tikzcd}$$

commutes, where the diagonals arrows are components of θ. Then we have proved that $(F,b)\circ(I_P,i_P)\cong(G,c)\circ(I_P,i_P)$ implies $(F,b)\cong(G,c)$.

Proposition 4.3.8. *The assignment*

$$\varepsilon\colon \mathrm{C}\circ\mathrm{U}\longrightarrow \mathrm{id}_{\mathbf{CE}}$$

where ε_P is defined as in 4.3.5, is a pseudo-natural transformation.

Proof. Let $P\colon \mathcal{C}^{\mathrm{op}}\longrightarrow \mathbf{InfSL}$ and $R\colon \mathcal{D}^{\mathrm{op}}\longrightarrow \mathbf{InfSL}$ be two elementary doctrine with comprehensions, we define

$$\tau_{PR}\colon\ \mathbf{CE}(\varepsilon_P,1_R)\longrightarrow \mathbf{CE}(1_{P_c},\varepsilon_R)\circ\mathrm{C}\circ\mathrm{U}$$

where the 2-morphisms

$$\tau_{PR_{(F,b)}}\colon\ \mathbf{CE}(\varepsilon_P,1_{,R}\,)(F,b)\Longrightarrow \mathbf{CE}(1_{P_c},\varepsilon_R)\circ\mathrm{C}(F,b)$$

are defined as

$$(\tau_{PR_{(F,b)}})_{(A,\top_A)}:=(\theta^R_{FA})^{-1}\circ F(\theta^P_A).$$

We can define $\tau_{PR_{(F,b)}}$ just on the elements of the form $(A,\top_A)$ because this definition can be extended to every object (A,γ) by Remark 4.3.7 since both P and P_c have comprehensions, and the 1-cells in $\mathbf{CE}$ preserve them. Now we must prove the naturality of τ_{PR}. Consider a 2-morphism $\phi\colon (F,b)\Longrightarrow(G,c)$ and observe that

$$\mathbf{CE}(1_{P_c},\varepsilon_R)\circ\mathrm{C}\circ\mathrm{U}(\phi)_{(A,\alpha)}=T_R(\widehat{\phi}_{(A,\alpha)})$$

and

$$\mathbf{CE}(\varepsilon_P,1_R)(\phi)_{(A,\alpha)}=\phi_{T_P(A,\alpha)}.$$

The diagram

$$\begin{tikzcd}
FT_P(A,\top_A) \arrow[r, "\phi_{T_P(A,\alpha)}"] \arrow[d, "F(\theta^P_A)"'] & GT_P(A,\top_A) \arrow[d, "G(\theta^P_A)"] \\
F(A) \arrow[r, "\phi_A"] \arrow[d, "(\theta^R_{FA})^{-1}"'] & G(A) \arrow[d, "(\theta^R_{GA})^{-1}"] \\
T_R\widehat{F}(A,\top_A) \arrow[r, "T_R(\widehat{\phi}_{(A,\top_A)})"'] & T_R\widehat{G}(A,\top_A)
\end{tikzcd}$$

commutes since the following commutes by Remark 4.3.6

$$\begin{tikzcd}
T_R\widehat{F}(A,\top_A) \arrow[r, "T_R(\widehat{\phi}_{(A,\top_A)})"] \arrow[d, "(\theta^R_{FA})"'] & T_R\widehat{G}(A,\top_A) \arrow[d, "(\theta^R_{GA})"] \\
F(A) \arrow[r, "\phi_A"'] & G(A).
\end{tikzcd}$$

So

$$\begin{tikzcd}
FT_P(A,\top_A) \arrow[r, "F(\theta^P_A)"] \arrow[d, "\phi_{T_P(A,\top_A)}"'] & F(A) \arrow[d, "\phi_A"] \\
GT_P(A,\top_A) \arrow[r, "G(\theta^P_A)"'] & G(A)
\end{tikzcd}$$

commutes since $\phi\colon (F,b) \Longrightarrow (G,c)$ is a natural transformation. It is straightforward to prove that the coherence axioms of the definition of lax-natural transformation are satisfied. □

Remark 4.3.9. Let $P\colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be an elementary doctrine, and consider the following 1-cell

$$\begin{tikzcd}
C^{\mathrm{op}} \arrow[dd, "I_P{}^{\mathrm{op}}"'] \arrow[dr, "P"] \arrow[dr, phantom, shift right=4, "\Downarrow i_P"] & \\
& \mathbf{InfSL} \\
\mathcal{G}_P^{\mathrm{op}} \arrow[ur, "P_c"'] &
\end{tikzcd}$$

Applying the functor C to it we obtain the 1-cell

$$\begin{tikzcd}
\mathcal{G}_P^{\mathrm{op}} \arrow[dd, "\widehat{I_P}{}^{\mathrm{op}}"'] \arrow[dr, "P_c"] \arrow[dr, phantom, shift right=4, "\Downarrow \widehat{i}_P"] & \\
& \mathbf{InfSL} \\
\mathcal{G}_{P_c}^{\mathrm{op}} \arrow[ur, "(P_c)_c"'] &
\end{tikzcd}$$

We can observe that $(\widehat{I_P}, \widehat{i}_P) = (I_{P_c}, i_{P_c})$, because

$$\widehat{I_P}(A,\alpha) = ((A,\alpha),\alpha) = ((A,\alpha),\top_{(A,\alpha)}) = I_{P_c}(A,\alpha).$$

Moreover we have that, for every morphism $f\colon (A,\alpha) \longrightarrow (B,\beta)$ in $\mathcal{G}_P$,

$$\widehat{I_P}(f) = I_P(f) = f = I_{P_c}(f)$$

Thus $i_{P_c} = \widehat{i_P}$ since they are both the identity.

Remark 4.3.10. Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be an elementary doctrine in **CE**. By definition of ε_P, we have

$$(T_P, j_P) \circ (I_P, i_P) \cong \mathrm{id}_P\,.$$

Hence we have

$$\mathrm{C}(T_P, j_P) \circ \mathrm{C}(J_P, j_P) \cong \mathrm{id}_{\mathrm{C}(P)}$$

and by Remark 4.3.9 we have

$$\mathrm{C}(T_P, j_P) \circ (I_{P_c}, i_{P_c}) \cong \mathrm{id}_{\mathrm{C}(P)}\,.$$

So we can assume that $\varepsilon_{P_c} = (T_{P_c}, t_{P_c}) = \mathrm{C}(T_P, t_P)$.

Remark 4.3.11. Combining Remark 4.3.6 and Remark 4.3.10, we can assume that $\mathrm{C}(\theta^P) = \theta^{P_c}$ for every elementary doctrine in **CE**. This choice is going to simplify many calculations in the following. In particular this implies that $C\varepsilon = \varepsilon C$ as pseudo-natural transformation.

Proposition 4.3.12. *For every elementary doctrine* $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *we have*

$$\varepsilon_{\mathrm{C}(P)} \circ \mathrm{C}(\eta_P) \cong 1_{\mathrm{C}(P)}$$

Proof. By Remark 4.3.9 and Remark 4.3.11 we have

$$\varepsilon_{\mathrm{C}(P)} \circ \mathrm{C}(\eta_P) = \varepsilon_{\mathrm{C}(P)} \circ (I_{P_c}, i_{P_c})$$

and the conclusion follows by definition of $\varepsilon_{\mathrm{C}(P)}$. □

Proposition 4.3.13. *For every elementary doctrine* $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *in* **CE**, *the following isomorphism holds:*

$$\varepsilon_P \circ \eta_P \cong 1_P$$

Proof. It follows directly from the definitions of ε_P and η_P. □

Remark 4.3.14. The isomorphism in Proposition 4.3.13 can be extended to an invertible modification between the pseudo-natural transformation $(\varepsilon \circ \eta, \tau')$ and $1_{\mathbf{CE}}$, where τ'_{PR} is given by $i_{\mathbf{CE}(\eta_P, 1_R)}.\tau_{PR}$. We define $\lambda\colon (\varepsilon, \tau) \circ (\eta, \mathrm{id}) \Longrightarrow 1_{\mathbf{CE}}$ where $\lambda_P := \theta_P$. Next we prove that it satisfies the following equation

$$
\begin{tikzcd}
P \arrow[d, "{(F,b)}"'] \arrow[dr, "\varepsilon_P\eta_P"] & & & P \arrow[d, "\mathrm{id}_P"'] \arrow[dr, "\varepsilon_P\eta_P"] \arrow[dr, phantom, shift right=3, "\Swarrow\lambda_P"'] & \\
R \arrow[d, "\mathrm{id}_R"'] \arrow[dr, "\varepsilon_R\eta_R"] \arrow[r, phantom, "\Downarrow\tau'_{(F,b)}"] \arrow[dr, phantom, shift right=3, "\Swarrow\lambda_R"'] & P \arrow[d, "{(F,b)}"] & = & P \arrow[r, "\mathrm{id}_P"'] \arrow[d, "{(F,b)}"'] & P \arrow[d, "{(F,b)}"] \\
R \arrow[r, "\mathrm{id}_R"'] & R & & R \arrow[r, "\mathrm{id}_R"'] & R
\end{tikzcd}
$$

and this means that the following equality must holds

$$(\lambda_R.i_{(F,b)}) \circ \tau'_{(F,b)} = i_{(F,b)}.\lambda_P.$$

It is straightforward to verify the following identities

- $(\lambda_R.i_{(F,b)})_A = \theta^R_{FA}$;
- $(i_{\mathbf{CE}(\eta_P,1_R)}.\tau_{PR})_A = (\tau_{PR(F,b)})_{(A,\top_A)} = (\theta^R_{FA})^{-1} \circ F(\theta^P_A)$;
- $(i_{(F,b)}.\lambda_P)_A = F(\lambda_{RA}) = F(\theta^P_A)$;

Therefore we can conclude that $\lambda\colon (\varepsilon,\tau)\circ(\eta,\mathrm{id}) \Longrightarrow 1_{\mathbf{CE}}$ is an invertible modification.

Remark 4.3.15. Using the same argument of 4.3.14 we can prove that the isomorphism

$$\varepsilon_{\mathrm{C}(P)} \circ \mathrm{C}(\eta_P) \cong 1_{\mathrm{C}(P)}$$

can be extended to an invertible modification $\rho\colon (\varepsilon,\tau)C \circ C(\eta,\mathrm{id}) \Longrightarrow 1_{\mathbf{CE}}$.

We are now in the position to compute the pseudo-monad:

- let $\mathrm{T}_c\colon \mathbf{ElD} \longrightarrow \mathbf{ElD}$ be the 2-functor $\mathrm{T}_c = \mathrm{U}\circ\mathrm{C}$;
- let $\eta\colon \mathrm{id}_{\mathbf{ElD}} \longrightarrow \mathrm{T}$ be the unit of the pseudo-adjunction;
- let $\mu\colon \mathrm{T}^2_c \longrightarrow \mathrm{T}_c$ be the pseudo-natural transformation $\mu = \mathrm{U}\varepsilon\mathrm{C}$.

Proposition 4.3.16. *The triple* (T_c,μ,η) *is a pseudo-monad, the following diagram*

$$
\begin{tikzcd}
\mathrm{T}^3_c \arrow[r, "\mu\mathrm{T}_c"] \arrow[d, "\mathrm{T}_c\mu"'] & \mathrm{T}^2_c \arrow[d, "\mu"] \\
\mathrm{T}^2_c \arrow[r, "\mu"'] & \mathrm{T}_c
\end{tikzcd}
$$

commutes and the modifications

$$
\begin{tikzcd}
\mathrm{T}_c \arrow[r, "\mathrm{T}_c\eta"] \arrow[dr, "\mathrm{id}"'] & \mathrm{T}_c^2 \arrow[d, "\mu"] \arrow[dl, phantom, "\Swarrow\rho"] \\
 & \mathrm{T}_c
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{T}_c \arrow[r, "\eta\mathrm{T}_c"] \arrow[dr, "\mathrm{id}"'] & \mathrm{T}_c^2 \arrow[d, "\mu"] \arrow[dl, phantom, "\Swarrow\lambda"] \\
 & \mathrm{T}_c
\end{tikzcd}
$$

satisfy the coherence axioms for pseudo-monads.

Proof. By Remark 4.3.10

$$\mu_{\mathrm{T}_c(P)} = \varepsilon_{\mathrm{C}(P_c)} = C(T_{P_c}, t_{P_c}).$$

So we have

$$\mathrm{T}_c(\mu)_P = \mathrm{T}_c(\varepsilon_{\mathrm{C}(P)}) = \mathrm{C}(T_{P_c}, t_{P_c})$$

Moreover the pseudo-natural transformations $T\mu$ and μT have the same isomorphisms τ by Remark 4.3.11 and by definition of τ in Proposition 4.3.8.

The axiom is satisfied since we have the following equality

$$\eta_{\mathrm{C}(P)} = (I_{P_c}, i_{P_c}) = (\widehat{I_P}, \widehat{j_P}) = \mathrm{T}_c(\eta_P)$$

by Remark 4.3.9, and then we have that λ and ρ are the same modification.

□

Remark 4.3.17. Consider an elementary doctrine $P\colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ in **CE**. By Remark 4.3.10, the following diagram commutes

$$
\begin{tikzcd}
T_c^2 P \arrow[r, "T_c(T_P, t_P)"] \arrow[d, "\mu"'] & T_c P \arrow[d, "(T_P, t_P)"] \\
T_c P \arrow[r, "(T_P, t_P)"'] & P.
\end{tikzcd}
$$

In other words we can regard every elementary doctrine in **CE** with an action such that $(P, (T_P, t_P))$ is a pseudo-T_c-algebra. Moreover since ε is a pseudo-natural transformation, every 1-cell in **CE** induces a pseudo-morphism in **Ps**-T_c-**Alg**, and the same holds for every 2-cell. So we have the following inclusions of 2-categories

$$\mathbf{CE} \hookrightarrow \mathbf{Ps}\text{-}\mathrm{T}_c\text{-}\mathbf{Alg} \hookrightarrow \mathbf{ElD}$$

Using the same argument of Proposition 4.2.10, we can prove the following proposition.

Proposition 4.3.18. *Let (P, a) be a pseudo-T-algebra. Then the elementary doctrine $P\colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ has comprehensions.*

Theorem 4.3.19. *Let (P, a) and (R, b) be two pseudo-*T_c*-algebras, and let* $f\colon P \longrightarrow R$ *be a 1-cell in* **CE**. *Then there exists a unique invertible 2-cell such that*

$$\begin{tikzcd} \mathrm{T}_c P \arrow[r, "\mathrm{T}_c f"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow"] & \mathrm{T}_c R \arrow[d, "b"] \\ P \arrow[r, "f"'] & R \end{tikzcd}$$

is a pseudo-morphism of pseudo-T_c*-algebras.*

Proof. The pseudo-T_c-algebra (P, a) has comprehensions by Proposition 4.3.18, and we have the following isomorphism

$$\varepsilon_P \eta_P \cong 1_P \cong a\eta_P.$$

So for every object A of C we have $a(A, \top_A) \cong \varepsilon_P(A, \top_A)$, and by Remark 4.3.7 we can conclude that $\varepsilon_P \cong a$. The isomorphism gives a pseudo-morphism of pseudo-algebras

$$\begin{tikzcd} \mathrm{T}_c P \arrow[r, "\mathrm{T}_c\, \mathrm{id}_P"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow"] & \mathrm{T}_c P \arrow[d, "\varepsilon_P"] \\ P \arrow[r, "\mathrm{id}_P"'] & P \end{tikzcd}$$

By the second coherence condition of pseudo-morphisms such isomorphism is unique. Since ε is a pseudo-natural transformation, we have the following commutative diagram

$$\begin{tikzcd} \mathrm{T}_c P \arrow[r, "\mathrm{T}_c\, \mathrm{id}_P"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow"] & \mathrm{T}_c P \arrow[r, "\mathrm{T}_c f"] \arrow[d, "\varepsilon_P"] \arrow[dr, phantom, "\Downarrow \tau_f"] & \mathrm{T}_c R \arrow[r, "T_c\, \mathrm{id}_R"] \arrow[d, "\varepsilon_R"] \arrow[dr, phantom, "\Downarrow"] & \mathrm{T}_c R \arrow[d, "b"] \\ P \arrow[r, "\mathrm{id}_P"'] & P \arrow[r, "f"'] & R \arrow[r, "\mathrm{id}_R"'] & R \end{tikzcd}$$

for the pseudo-T_c-algebras (P, a) and (R, b), and for every 1-cell in **CE** $f\colon P \longrightarrow R$. Therefore for every 1-cell in **CE** there exists an invertible 2-cell

$$\begin{tikzcd} \mathrm{T}_c P \arrow[r, "\mathrm{T}_c f"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow"] & \mathrm{T}_c R \arrow[d, "b"] \\ P \arrow[r, "f"'] & R \end{tikzcd}$$

such that the previous diagram is a 1-cell in $\mathbf{Ps}$-$\mathrm{T_c}$-$\mathbf{Alg}$. The uniqueness follows from the second coherence condition of pseudo-morphism and the fact that the doctrines P and R have comprehension by Proposition 4.3.18. $\square$

Remark 4.3.20. Observe that if (P,a) and (R,b) are pseudo-T_c-algebras, and the following square is a pseudo-morphism of pseudo-algebras

$$\begin{tikzcd} \mathrm{T}_c P \arrow[r, "\mathrm{T}_c f"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow"] & \mathrm{T}_c R \arrow[d, "b"] \\ P \arrow[r, "f"'] & R \end{tikzcd}$$

then $f\colon P \longrightarrow R$ preserves comprehensions.

Corollary 4.3.21. *There is an equivalence of 2-categories*

$$\mathbf{CE} \equiv \mathbf{Ps}\text{-}\mathrm{T_c}\text{-}\mathbf{Alg}$$

Proof. By Remark 4.3.17, Proposition 4.3.18 and Theorem 4.3.19, we need only to prove that every 2-cell $\theta\colon (F,b) \Longrightarrow (G,c)$ in $\mathbf{CE}$ is a 2-cell in $\mathbf{Ps}$-$\mathrm{T_c}$-$\mathbf{Alg}$, which means that θ must satisfy the coherence conditions. This follows directly from the pseudo-naturality of ε. $\square$

4.4 Elementary doctrines with quotients

In this section we consider the completion with quotients of an elementary doctrines.

Consider the 2-full 2-subcategory $\mathbf{QED}$ of $\mathbf{ElD}$ whose objects are the elementary doctrines $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ in which every P-equivalence relation has a P-quotient that is a stable effective descent morphism.

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be an elementary doctrine, and consider the category $\mathcal{R}_P$ of ***P-equivalence relation***:

- **an object** of $\mathcal{R}_P$ is a pair (A,ρ) such that ρ is a P-equivalence relation on A;
- **a morphism** $f\colon (A,\rho) \longrightarrow (B,\sigma)$ is a morphism $f\colon A \longrightarrow B$ such that $\rho \leq P_{f\times f}(\sigma)$.

The indexed poset $P_q\colon \mathcal{R}_P^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ will be given by the categories of descent data:

$$P_q(A,\rho) = Des_\rho$$

and for every morphism $f\colon (A,\rho) \longrightarrow (B,\sigma)$ we define

$$P_q(f) = P(f)$$

This is a well defined elementary doctrine, see [42, Lemma 4.2], and it has descent quotients of P-equivalence relations, see [42, Lemma 4.4].

Following the structure of sections 4.2 and 4.3 we prove that the assignment $\mathrm{Q}(P) = P_q$ can be extended to 2-functor

$$\mathrm{Q}\colon \mathbf{ElD} \longrightarrow \mathbf{QED}$$

and we start defining how it acts on the 1-cells and 2-cells in **ElD**.

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ and $R\colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be elementary doctrines, and consider a 1-cell (F,b):

$$\begin{tikzcd} \mathcal{C}^{\mathrm{op}} \arrow[dd, "F^{\mathrm{op}}"'] \arrow[dr, "P"{name=P}] & \\ & \mathbf{InfSL} \\ \mathcal{D}^{\mathrm{op}} \arrow[ur, "R"'{name=R}] & \arrow[from=P, to=R, "b"', shorten=2mm] \end{tikzcd}$$

We want to prove that the pair $(\overline{F},\overline{b})$ where:

- $\overline{F}(A,\rho)$ is $(FA, R_{\langle F(\mathrm{pr}_1),F(\mathrm{pr}_2)\rangle^{-1}}(b_{A\times A}(\rho)))$ for every $A \in \mathcal{R}_P$;
- $\overline{F}(f)$ is $F(f)$ for every $f\colon (A,\rho) \longrightarrow (B,\sigma)$;
- $\overline{b}$ is b restricted to the categories of descent data;

is a 2-morphism in **QED**:

$$\begin{tikzcd} \mathcal{R}_P^{op} \arrow[dd, "\overline{F}^{op}"'] \arrow[dr, "P_q"{name=P}] & \\ & \mathbf{InfSL} \\ \mathcal{R}_R^{op} \arrow[ur, "R_q"'{name=R}] & \arrow[from=P, to=R, "\overline{b}"', shorten=2mm] \end{tikzcd}$$

Lemma 4.4.1. *Let (A,ρ) be an object in $\mathcal{R}_P$ and let $\mathrm{pr}_1,\mathrm{pr}_2\colon A\times A \longrightarrow A$ be the two projections. Then $R_{\langle F(\mathrm{pr}_1),F(\mathrm{pr}_2)\rangle^{-1}}(b_{A\times A}(\rho))$ is a P-equivalence relation on FA.*

Proof. Reflexivity: ρ is an equivalence relation on A implies $b_{A\times A}(\delta_A) \le b_{A\times A}(\rho)$ and by definition of $b_{A\times A}$ we have $R_{\langle F(\mathrm{pr}_1),F(\mathrm{pr}_2)\rangle}(\delta_{FA}) \le b_{A\times A}(\rho)$. Since F preserves products $\langle F(\mathrm{pr}_1), F(\mathrm{pr}_2)\rangle$ is an isomorphism. So

$$\delta_{FA} \le R_{\langle F(\mathrm{pr}_1),F(\mathrm{pr}_2)\rangle^{-1}}(b_{A\times A}(\rho)).$$

Symmetry and transitivity are proved similarly. □

Lemma 4.4.2. *Let* $f\colon (A,\rho) \longrightarrow (B,\sigma)$ *be a morphism in* $\mathcal{R}_P$*, and let* $\mathrm{pr}_i\colon A\times A \longrightarrow A$ *and* $\mathrm{pr}'_i\colon B\times B \longrightarrow B$, $i = 1,2$ *be the projections. Then*

$$F(f)\colon (FA, R_{\langle F(\mathrm{pr}_1),F(\mathrm{pr}_2)\rangle^{-1}}(b_{A\times A}(\rho))) \longrightarrow (FB, R_{\langle F(\mathrm{pr}'_1),F(\mathrm{pr}'_2)\rangle^{-1}}(b_{B\times B}(\sigma)))$$

is a morphism in $\mathcal{R}_R$.

Proof. Since $f\colon (A,\rho) \longrightarrow (B,\sigma)$ is a 1-cell, $\rho \le P_{f\times f}(\sigma)$. Thus

$$b_{A\times A}(\rho) \le b_{A\times A}(P_{f\times f}(\sigma)) = R_{F(f\times f)}(b_{B\times B}(\sigma)).$$

Hence

$$R_{\langle F(\mathrm{pr}_1),F(\mathrm{pr}_2)\rangle^{-1}}(b_{A\times A}(\rho)) \le R_{\langle F(\mathrm{pr}_1),F(\mathrm{pr}_2)\rangle^{-1}}(R_{F(f\times f)}(b_{B\times B}(\sigma))).$$

Since

$$F(f\times f)\circ \langle F(\mathrm{pr}_1), F(\mathrm{pr}_2)\rangle^{-1} = \langle F(\mathrm{pr}'_1), F(\mathrm{pr}'_2)\rangle^{-1} \circ F(f)\times F(f)$$

it is

$$R_{\langle F(\mathrm{pr}_1),F(\mathrm{pr}_2)\rangle^{-1}}(b_{A\times A}(\rho)) \le R_{F(f)\times F(f)}(R_{\langle F(\mathrm{pr}'_1),F(\mathrm{pr}'_2)\rangle^{-1}}(b_{B\times B}(\sigma))).$$

□

Remark 4.4.3. Consider $(A,\rho) \in \mathcal{R}_P$, if $\alpha \in Des_\rho$ then

$$b_A(\alpha) \in Des_{R_{\langle F(\mathrm{pr}_1),F(\mathrm{pr}_2)\rangle^{-1}}(b_{A\times A}(\rho))}.$$

Corollary 4.4.4. *Given* $(F,b) \in \mathbf{ElD}(P,R)$ *then* $(\overline{F},\overline{b}) \in \mathbf{QED}(P_q,R_q)$.

Proof. By Remark 4.4.3 and [42, Lemma 4.2]

$$b_{A\times A}(\rho) = R_{\langle F(\mathrm{pr}_1),F(\mathrm{pr}_2)\rangle}(R_{\langle F(\mathrm{pr}_1),F(\mathrm{pr}_2)\rangle^{-1}}(b_{A\times A}(\rho)))$$

So

$$\overline{b}_{(A,\rho)\times(A,\rho)}(\delta_{(A,\rho)}) = (R_q)_{\langle\overline{F}(\mathrm{pr}_1),\overline{F}(\mathrm{pr}_2)\rangle}(\delta_{\overline{F}(A,\rho)}).$$

By Lemma 4.4.2 and Lemma 4.4.1 we can conclude that $(\overline{F},\overline{b}) \in \mathbf{ElD}(P_q, R_q)$. It remains to verify that $\overline{F}$ preserves all the quotients.

Consider a P_q-equivalence relation τ on (A,ρ). A P_q-quotient of τ is

$$\mathrm{id}_A\colon (A,\rho) \longrightarrow (A,\tau)$$

and

$$\mathrm{id}_{FA}\colon (FA, R_{\langle F(\mathrm{pr}_1),F(\mathrm{pr}_2)\rangle^{-1}}(b_{A\times A}(\rho)) \longrightarrow (FA, R_{\langle F(\mathrm{pr}_1),F(\mathrm{pr}_2)\rangle^{-1}}(b_{A\times A}(\tau))$$

is a R_q-quotient of $R_{\langle F(\mathrm{pr}_1),F(\mathrm{pr}_2)\rangle^{-1}}(b_{A\times A}(\tau))$. So $\overline{F}$ preserves quotients, and $(\overline{F},b)$ is a 1-cell in **QED**. □

Proposition 4.4.5. *Let* θ *be a morphism in* $\mathbf{ElD}(P,R)$

$$\theta\colon (F,b) \longrightarrow (G,c)\ .$$

Then θ *is also a morphism in* $\mathbf{QED}(P_q,R_q)$

$$\theta\colon (\overline{F},\overline{b}) \longrightarrow (\overline{G},\overline{c})\ .$$

Proof. We must prove that for every $(A,\rho) \in \mathcal{R}_P$

$$\theta_A\colon (FA, R_{\langle F(\mathrm{pr}_1),F(\mathrm{pr}_2)\rangle^{-1}}(b_{A\times A}(\rho))) \longrightarrow (GA, R_{\langle G(\mathrm{pr}_1),G(\mathrm{pr}_2)\rangle^{-1}}(c_{A\times A}(\rho)))$$

is a morphism in $\mathcal{R}_R$. Indeed, by definition of 2-morphism we have $b_{A\times A}(\rho) \leq R_{\theta_{A\times A}}(c_{A\times A}(\rho))$ then

$$R_{\langle F(\mathrm{pr}_1),F(\mathrm{pr}_2)\rangle^{-1}}(b_{A\times A}(\rho)) \leq R_{\langle F(\mathrm{pr}_1),F(\mathrm{pr}_2)\rangle^{-1}}(R_{\theta_{A\times A}}(c_{A\times A}(\rho)))$$

and, since θ is a natural transformation,

$$R_{\langle F(\mathrm{pr}_1),F(\mathrm{pr}_2)\rangle^{-1}}(b_{A\times A}(\rho)) \leq R_{\theta_A\times\theta_A}(R_{\langle G(\mathrm{pr}_1),G(\mathrm{pr}_2)\rangle^{-1}}((c_{A\times A}(\rho)))).$$

Finally for every $\alpha \in Des_{R_{\langle F(\mathrm{pr}_1),F(\mathrm{pr}_2)\rangle^{-1}}(b_{A\times A}(\rho))}$ we have

$$\overline{b}_A(\alpha) \leq (R_q)_{\theta_A}(\overline{c}_A(\alpha))$$

because $\overline{b}_A(\alpha) = b_A(\alpha)$, $\overline{c}_A(\alpha) = c_A(\alpha)$ and $R_q(\theta_A) = R(\theta_A)$. □

Proposition 4.4.6. *The assignment*

$$\mathrm{Q}_{P,R}\colon\ \mathbf{ElD}(P,R) \longrightarrow \mathbf{QED}(P_q,R_q)$$

which maps (F,b) *into* $(\overline{F},\overline{b})$ *and a 2-cell* $\theta\colon (F,b) \longrightarrow (G,c)$ *into* $\theta\colon (\overline{F},\overline{b}) \longrightarrow (\overline{G},\overline{c})$ *is a functor and*

$$\mathrm{Q}\colon\ \mathbf{ElD} \longrightarrow \mathbf{QED}$$

is a 2-functor with the assignment $\mathrm{Q}(P) = P_q$.

We prove that the 2-functor $\mathrm{Q}\colon\ \mathbf{ElD} \longrightarrow \mathbf{QED}$ is left adjoint to the forgetful 2-functor. Recall from [42] the crucial equivalence

$$-\circ(J,j)\colon\ \mathbf{QED}(P_q,Z) \equiv \mathbf{ElD}(P,Z)$$

where $J\colon \mathcal{C} \longrightarrow \mathcal{R}_P$ sends an object A to (A,δ_A) and a morphism $f\colon A \longrightarrow B$ to $f\colon (A,\delta_A) \longrightarrow (B,\delta_B)$ and j_A is the identity. For more details, see [42, Theorem 4.5].

Let P be an elementary doctrine in $\mathbf{QED}$. We define

$$\eta_P\colon P \longrightarrow \mathrm{U}\circ\mathrm{Q}(P)$$

the image of the identity on $\mathrm{Q}(P)$, under the equivalence

$$-\circ(J_P,j_P)\colon\ \mathbf{QED}(\mathrm{Q}(P),\mathrm{Q}(P)) \equiv \mathbf{ElD}(P,\mathrm{U}\circ\mathrm{Q}(P)).$$

It means that η_P is the 1-morphism (J_P,j_P). It is direct to check that the assignment

$$\eta\colon\ \mathrm{id}_{\mathbf{ElD}} \longrightarrow \mathrm{U}\circ\mathrm{Q}$$

is a 2-natural transformation.

Remark 4.4.7. For every $P\in\mathbf{QED}$ the equivalence

$$-\circ(J_P,j_P)\colon\ \mathbf{QED}(\mathrm{Q}\circ\mathrm{U}(P),P) \equiv \mathbf{ElD}(\mathrm{U}(P),\mathrm{U}(P))$$

is essentially surjective by definition. Then there exists a 1-morphism (T_P,t_P) such that

$$(T_P,t_P)\circ(J_P,j_P) \cong 1_P.$$

Let $\theta\colon (T_P,t_P)\circ(I_P,i_P) \Longrightarrow 1_P$ be the invertible 2-cell and let $\varepsilon_P := (T_P,t_P)$ be the previous 1-cell.

Remark 4.4.8. For every morphism $f\colon A \longrightarrow B$ in $\mathcal{C}$, the following diagram commutes

$$\begin{tikzcd} T_P(A,\delta_A) \arrow[r, "T_P(f)"] \arrow[d, "\theta^P_A"'] & T_P(B,\delta_B) \arrow[d, "\theta^P_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

where $\theta^P : T_P \circ J_P \Longrightarrow 1_P$ is the isomorphism in Remark 4.4.7.

Proposition 4.4.9. *The assignment*

$$\varepsilon \colon \mathrm{Q} \circ \mathrm{U} \longrightarrow \mathrm{id}_{\mathbf{QED}}$$

where ε_P is defined as in 4.4.7, is a pseudo-natural transformation.

Proof. We can use the same argument of Proposition 4.3.8, observing that we can restrict our attention to the elements of the form (A,δ_A). $\square$

Remark 4.4.10. Let $P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be an elementary doctrine, and consider the following 1-cell

$$\begin{tikzcd} \mathcal{C}^{\mathrm{op}} \arrow[dr, "P"] \arrow[dd, "{J_P}^{\mathrm{op}}"'] & \\ & \mathbf{InfSL} \\ \mathcal{R}_P^{\mathrm{op}} \arrow[ur, "P_q"'] \arrow[ur, phantom, shift left=6, "\Downarrow j_P"] & \end{tikzcd}$$

Applying the functor Q we obtain the 1-cell

$$\begin{tikzcd} \mathcal{R}_P^{\mathrm{op}} \arrow[dr, "P_q"] \arrow[dd, "{\overline{J}_P}^{\mathrm{op}}"'] & \\ & \mathbf{InfSL} \\ \mathcal{R}_{P_q}^{\mathrm{op}} \arrow[ur, "(P_q)_q"'] \arrow[ur, phantom, shift left=6, "\Downarrow \overline{j}_P"] & \end{tikzcd}$$

It is $(\overline{J}_P, \overline{j}_P) = (J_{P_q}, j_{P_q})$ because

$$\overline{J}_P(A,\rho) = ((A,\rho),(P_q)_{\langle J_P(\mathrm{pr}_1), J_P(\mathrm{pr}_2)\rangle^{-1}}((j_P)_{A\times A}(\rho))) = ((A,\rho),\delta_{(A,\rho)}) = J_{P_q}(A,\rho)$$

and, for $f \colon (A,\rho) \longrightarrow (B,\sigma)$ in $\mathcal{R}_P$,

$$\overline{J_P}(f) = J_P(f) = f = J_{P_q}(f).$$

Also $j_{P_q} = \overline{j}_P$ since they are both the identity.

Remark 4.4.11. Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be an elementary doctrine in **QED**. Then by definition of ε_P, we have

$$(T_P, j_P) \circ (J_P, j_P) \cong \mathrm{id}_P\,.$$

Hence

$$\mathrm{Q}(T_P, j_P) \circ \mathrm{Q}(J_P, j_P) \cong \mathrm{id}_{\mathrm{Q}(P)}$$

and by Remark 4.4.10

$$\mathrm{Q}(T_P, j_P) \circ (J_{P_q}, j_{P_q}) \cong \mathrm{id}_{\mathrm{Q}(P)}\,.$$

So we can assume that $\varepsilon_{P_q} = (T_{P_q}, t_{P_q}) = \mathrm{Q}(T_P, t_P)$.

Proposition 4.4.12. *For every elementary doctrine* $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *we have*

$$\varepsilon_{\mathrm{Q}(P)} \circ \mathrm{Q}(\eta_P) \cong 1_{\mathrm{Q}(P)}.$$

Proof. By Remark 4.4.10 we have

$$\varepsilon_{\mathrm{Q}(P)} \circ \mathrm{Q}(\eta_P) = \varepsilon_{\mathrm{Q}(P)} \circ (J_{P_q}, j_{P_q})$$

and the conclusion follows by definition of $\varepsilon_{\mathrm{Q}(P)}$. ☐

Proposition 4.4.13. *For every elementary doctrine* $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *in* **QED**, *it is*

$$\varepsilon_P \circ \eta_P \cong 1_P.$$

Proof. Immediate by definition of ε_P and η_P. ☐

Remark 4.4.14. Using the same argument as in 4.3.14 and 4.3.15 we can conclude that there are two invertible modifications $\rho\colon \varepsilon C \circ C\eta \Longrightarrow 1_{\mathbf{ElD}}$ and $\lambda\colon \varepsilon \circ \eta \Longrightarrow 1_{\mathbf{QED}}$.

We use the same argument of Sections 4.3 and 4.2 to introduce the following pseudo-monad:

- let $\mathrm{T}_q\colon \mathbf{ElD} \longrightarrow \mathbf{ElD}$ be the 2-functor $\mathrm{T}_q = \mathrm{U} \circ \mathrm{Q}$;
- let $\eta\colon \mathrm{id}_{\mathbf{ElD}} \longrightarrow \mathrm{T}_q$ be the unit of the pseudo-adjunction;
- let $\mu\colon \mathrm{T}_q^2 \longrightarrow \mathrm{T}_q$ is the pseudo-natural transformation $\mu := \mathrm{U}\varepsilon\mathrm{Q}$.

Proposition 4.4.15. *The triple* $(\mathrm{T}_q, \mu, \eta)$ *is a pseudo-monad, the following diagram commutes*

$$\begin{tikzcd} \mathrm{T}_q^3 \arrow[r, "\mu \mathrm{T}_q"] \arrow[d, "\mathrm{T}_q \mu"'] & \mathrm{T}_q^2 \arrow[d, "\mu"] \\ \mathrm{T}_q^2 \arrow[r, "\mu"'] & \mathrm{T}_q \end{tikzcd}$$

and the modifications

$$\begin{tikzcd} \mathrm{T}_q \arrow[r, "\mathrm{T}_q \eta"] \arrow[dr, "\mathrm{id}"'] & \mathrm{T}_q^2 \arrow[d, "\mu"] \arrow[dl, phantom, "\Swarrow\rho"] \\ & \mathrm{T}_q \end{tikzcd} \qquad \begin{tikzcd} \mathrm{T}_q \arrow[r, "\eta \mathrm{T}_q"] \arrow[dr, "\mathrm{id}"'] & \mathrm{T}_q^2 \arrow[d, "\mu"] \arrow[dl, phantom, "\Swarrow\lambda"] \\ & \mathrm{T}_q \end{tikzcd}$$

satisfy the coherence axiom for pseudo-monad.

Proof. By Remark 4.4.11 we have

$$\mu_{\mathrm{T}_q(P)} = \varepsilon_{\mathrm{Q}(P_q)} = Q(T_{P_q}, t_{P_q})$$

and

$$\mathrm{T}_q(\mu)_P = \mathrm{T}_q(\varepsilon_{\mathrm{Q}(P)}) = \mathrm{Q}(T_{P_q}, t_{P_q}).$$

Moreover the pseudo-natural transformations $\mu\mathrm{T}_q$ and $\mathrm{T}_q\mu$ have the same isomorphism τ, since the action of the 2-functor T_q on a 2-cell gives essentially the same 2-cell by Proposition 4.4.5.

The axiom is satisfied since we have the following equality

$$\mu_{\mathrm{Q}(P)} = (J_{P_q}, j_{P_q}) = (\overline{J}_P, \overline{j}_P) = \mathrm{T}_q(\mu_P)$$

by Remark 4.4.10, which means that ρ and λ are the same modifications. □

Remark 4.4.16. Consider an elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ in $\mathbf{QED}$. By Remark 4.4.11, the diagram

$$\begin{tikzcd} \mathrm{T}_q^2 P \arrow[r, "\mathrm{T}_q(T_P, t_P)"] \arrow[d, "\mu"'] & \mathrm{T}_q P \arrow[d, "(T_P, t_P)"] \\ \mathrm{T}_q P \arrow[r, "(T_P, t_P)"'] & P \end{tikzcd}$$

commutes. Then we can regard every elementary doctrine in **QED** with an action (T_P, t_P) such that $(P, (T_P, t_P))$ is an object in $\mathbf{Ps}$-$\mathrm{T_q}$-$\mathbf{Alg}$. Since ε is a pseudo-natural transformation, every 1-cell in **QED** induces a 1-cell in $\mathbf{Ps}$-$\mathrm{T_q}$-$\mathbf{Alg}$, and the same for every 2-cells. So we have the following inclusions of 2-categories

$$\mathbf{QED} \hookrightarrow \mathbf{Ps}\text{-}\mathrm{T_q}\text{-}\mathbf{Alg} \hookrightarrow \mathbf{ElD}$$

Remark 4.4.17. We can use the same argument of Proposition 4.3.18 to prove that every pseudo-T_q-algebra (P, a) consists of an elementary doctrine with quotients and an action which preserves them, and every morphism $f\colon P \longrightarrow R$ in **QED** can be regarded as a pseudo-morphism of pseudo-T_q-algebras.

The same arguments used in the proof of Theorem 4.3.19 can be adapted to the case of elementary doctrine with quotients. Thus we have the following result.

Theorem 4.4.18. *Let (P, a) and (R, b) be two pseudo-T_q-algebras, and let $f\colon P \longrightarrow R$ be a 1-cell in* **QED**. *Then there exists a unique invertible 2-cell such that*

$$\begin{tikzcd} \mathrm{T}_c P \arrow[r, "\mathrm{T}_c f"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow"] & \mathrm{T}_c R \arrow[d, "b"] \\ P \arrow[r, "f"'] & R \end{tikzcd}$$

is a pseudo-morphism of pseudo-T_q-algebras.

Corollary 4.4.19. *We have the following equivalence of 2-categories*

$$\mathbf{QED} \equiv \mathbf{Ps}\text{-}\mathrm{T_q}\text{-}\mathbf{Alg}\,.$$

4.5 Pseudo-distributive laws

In this section we study the pseudo-distributive laws between the pseudo-monads T_c, T_d and T_q.

First we consider the pseudo-monads T_q and T_c, and in order to prove that there exists a pseudo-distributive law $\delta\colon \mathrm{T}_c\mathrm{T}_q \longrightarrow \mathrm{T}_q\mathrm{T}_c$, we shall construct a lifting of T_q in the sense of [55, 56].

Proposition 4.5.1. *The assignment*

$$\widetilde{\mathrm{T}_q}_{(P,a)(R,c)}\colon\ \mathbf{Ps}\text{-}\mathrm{T_c}\text{-}\mathbf{Alg}((P,a),(R,c)) \longrightarrow \mathbf{Ps}\text{-}\mathrm{T_c}\text{-}\mathbf{Alg}((P_q, \varepsilon_{P_q}), (R_q, \varepsilon_{R_q}))$$

mapping a 1-cell $(f, \overline{f})$ to

$$(\mathrm{T}_q f, \tau_{\mathrm{T}_q f})$$

and a 2-cell $\theta\colon (f,\overline{f}) \Longrightarrow (g,\overline{g})$ *to*

$$\mathrm{T}_q\theta\colon (\mathrm{T}_q f, \tau_{\mathrm{T}_q f}) \Longrightarrow (\mathrm{T}_q g, \tau_{\mathrm{T}_q g})$$

is a functor.

Proof. We recall that since (P,a) is a pseudo-T_c-algebra, by Remark 4.3.17 P has comprehensions, and we know that P_q has comprehensions by [43, Lemma 5.3]. Moreover we can observe that

$$\mathrm{T}_q\theta\colon (\mathrm{T}_q f, \tau_{\mathrm{T}_q f}) \Longrightarrow (\mathrm{T}_q g, \tau_{\mathrm{T}_q g})$$

is a morphism of pseudo-T_c-algebras because ε is a pseudo natural transformation, and since T_q is a 2-functor we can conclude that the composition and the identity axioms holds. Therefore we conclude that $\widetilde{\mathrm{T}}_{q(P,a)(R,c)}$ is a functor. □

Proposition 4.5.2. *The functor defined in 4.5.1 can be extended to a 2-functor*

$$\widetilde{\mathrm{T}_q}\colon \ \mathbf{Ps}\text{-}\mathrm{T}_c\text{-}\mathbf{Alg} \longrightarrow \mathbf{Ps}\text{-}\mathrm{T}_c\text{-}\mathbf{Alg}$$

where $\widetilde{\mathrm{T}_q}(P,a) := (P_q, \varepsilon_{P_q})$.

Proof. We prove the compatibility with composition. Consider the following 1-cells

$$\begin{tikzcd}
\mathrm{T}_c P \arrow[r, "\mathrm{T}_c f"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\overline{f}"] & \mathrm{T}_c R \arrow[r, "\mathrm{T}_c g"] \arrow[d, "b"] \arrow[dr, phantom, "\Downarrow\overline{g}"] & \mathrm{T}_c S \arrow[d, "c"] \\
P \arrow[r, "f"'] & R \arrow[r, "g"'] & S
\end{tikzcd}$$

then $(g,\overline{g}) \circ (f,\overline{f}) = (g \circ f, (i_{\mathrm{T}_c f}.\overline{g}) \circ (i_g.\overline{f}))$. Next consider the following diagram

$$\begin{tikzcd}
\mathrm{T}_c P_q \arrow[r, "\mathrm{T}_c\mathrm{T}_q f"] \arrow[d, "\varepsilon_{P_q}"'] \arrow[dr, phantom, "\Downarrow\tau_f"] & \mathrm{T}_c R_q \arrow[r, "\mathrm{T}_c\mathrm{T}_q g"] \arrow[d, "\varepsilon_{R_q}"] \arrow[dr, phantom, "\Downarrow\tau_g"] & \mathrm{T}_c S_q \arrow[d, "\varepsilon_{S_q}"] \\
P_q \arrow[r, "\mathrm{T}_q f"'] & R_q \arrow[r, "\mathrm{T}_q g"'] & S_q .
\end{tikzcd}$$

Since ε is a pseudo-natural transformation, we have that $(i_{\mathrm{T}_q g}.\tau_f) \circ (\tau_g.i_{\mathrm{T}_c\mathrm{T}_q f}) = \tau_{g\circ f}$. Moreover we have the compatibility with the composition of 2-cells since T_q

is a 2-functor. Finally one can check that also the unit axion is satisfied. Then we can conclude that $\widetilde{T_q}$ is a 2-functor. □

Remark 4.5.3. The multiplication and the identity of the pseudo-monad T_q can be extended to a multiplication and identity on the functor $\widetilde{\mathrm{T}_q}$. Therefore $\widetilde{\mathrm{T}_q}$ is a pseudo-monad. Moreover we can observe that, if we consider the forgetful 2-functor $U_{\mathrm{T}_c}\colon \mathbf{Ps}\text{-}\mathrm{T_c}\text{-}\mathbf{Alg} \longrightarrow \mathbf{ElD}$, we have the equality $\mathrm{T}_q U_{\mathrm{T}_c} = U_{\mathrm{T}_c}\widetilde{\mathrm{T}_q}$.

Theorem 4.5.4. *There exists a distributive law* $\delta\colon \mathrm{T}_c\mathrm{T}_q \longrightarrow \mathrm{T}_q\mathrm{T}_c$.

Proof. Remark 4.5.3 tells us that $\widetilde{\mathrm{T}_q}$ is a lifting of T_q. Apply Theorem [55, Theorem 1] to conclude the proof. □

Corollary 4.5.5. *The 2-functor* $\mathrm{T}_q\mathrm{T}_c$ *is a pseudo-monad.*

Proof. It follows by [54, Proposition 7.8 and Theorem 7.9]. □

We can use the same arguments of Proposition 4.5.2 and 4.5.1 to prove that the 2-monad T_d can be lifted to a pseudo-monad on $\mathbf{Ps}\text{-}\mathrm{T_q}\text{-}\mathbf{Alg}$, since T_d preserves quotients by [42, Lemma 5.8]. Therefore we have the following results.

Theorem 4.5.6. *The 2-functor* $\mathrm{T}_d\mathrm{T}_q$ *is a pseudo-monad, and since* T_d *preserves comprehensions, also 2-functor* $\mathrm{T}_d\mathrm{T}_q\mathrm{T}_c$ *is a pseudo-monad.*

It is easy to observe that every pseudo-monad that we have described admits a trivial pseudo-distributive law, which is the identity since they have the property that $\mathrm{T}\mu = \mu\mathrm{T}$. Then we can conclude with the following propositions.

Proposition 4.5.7. *For every natural number n,* T_c^n*,* T_d^n *and* T_q^n *are pseudo-monads.*

Applying [54, Proposition 7.8 and Theorem 7.9] we obtain the following result.

Theorem 4.5.8. *We have the following isomorphisms*

- $\mathbf{Ps}\text{-}\mathrm{T_q}\mathrm{T_c}\text{-}\mathbf{Alg} \cong \mathbf{Ps}\text{-}\widetilde{\mathrm{T_q}}\text{-}\mathbf{Alg}$, *where* $\widetilde{\mathrm{T}_q}$ *is the lifting of* T_q *on* $\mathbf{Ps}\text{-}\mathrm{T_c}\text{-}\mathbf{Alg}$*;*
- $\mathbf{Ps}\text{-}\mathrm{T_d}\mathrm{T_c}\text{-}\mathbf{Alg} \cong \mathbf{Ps}\text{-}\widetilde{\mathrm{T_d}}\text{-}\mathbf{Alg}$, *where* $\widetilde{\mathrm{T}_d}$ *is the lifting of* T_d *on* $\mathbf{Ps}\text{-}\mathrm{T_c}\text{-}\mathbf{Alg}$*;*
- $\mathbf{Ps}\text{-}\mathrm{T_d}\mathrm{T_q}\text{-}\mathbf{Alg} \cong \mathbf{Ps}\text{-}\widetilde{\mathrm{T_d}}\text{-}\mathbf{Alg}$, *where* $\widetilde{\mathrm{T}_d}$ *is the lifting of* T_d *on* $\mathbf{Ps}\text{-}\mathrm{T_q}\text{-}\mathbf{Alg}$*;*
- $\mathbf{Ps}\text{-}\mathrm{T_d}\mathrm{T_q}\mathrm{T_c}\text{-}\mathbf{Alg} \cong \mathbf{Ps}\text{-}\widetilde{\mathrm{T_d}\mathrm{T_q}}\text{-}\mathbf{Alg}$, *where* $\widetilde{\mathrm{T}_d\mathrm{T}_q}$ *is the lifting of* $\mathrm{T}_d\mathrm{T}_q$ *on* $\mathbf{Ps}\text{-}\mathrm{T_c}\text{-}\mathbf{Alg}$*;*
- $\mathbf{Ps}\text{-}\mathrm{T_d}\mathrm{T_q}\mathrm{T_c}\text{-}\mathbf{Alg} \cong \mathbf{Ps}\text{-}\widetilde{\mathrm{T_d}}\text{-}\mathbf{Alg}$, *where* $\widetilde{\mathrm{T}_d}$ *is the lifting of* T_d *on t*$\mathbf{Ps}\text{-}\mathrm{T_q}\mathrm{T_c}\text{-}\mathbf{Alg}$*;*
- $\mathbf{Ps}\text{-}\mathrm{T_c^n}\text{-}\mathbf{Alg} \cong \mathbf{Ps}\text{-}\widetilde{\mathrm{T_c^{n\text{-}1}}}\text{-}\mathbf{Alg}$, *where* $\widetilde{\mathrm{T}_c^{n-1}}$ *is the lifting of* T_c^{n-1} *on* $\mathbf{Ps}\text{-}\mathrm{T_c}\text{-}\mathbf{Alg}$*;*
- $\mathbf{Ps}\text{-}\mathrm{T_q^n}\text{-}\mathbf{Alg} \cong \mathbf{Ps}\text{-}\widetilde{\mathrm{T_q^{n\text{-}1}}}\text{-}\mathbf{Alg}$, *where* $\widetilde{\mathrm{T}_q^{n-1}}$ *is the lifting of* T_q^{n-1} *on* $\mathbf{Ps}\text{-}\mathrm{T_q}\text{-}\mathbf{Alg}$*;*
- $\mathbf{Ps}\text{-}\mathrm{T_d^n}\text{-}\mathbf{Alg} \cong \mathbf{Ps}\text{-}\widetilde{\mathrm{T_d^{n\text{-}1}}}\text{-}\mathbf{Alg}$, *where* $\widetilde{\mathrm{T}_d^{n-1}}$ *is the lifting of* T_d^{n-1} *on* $\mathbf{Ps}\text{-}\mathrm{T_d}\text{-}\mathbf{Alg}$.

Chapter 5
The Existential Completion

Abstract We determine the existential completion of a primary doctrine, and we prove that the 2-monad obtained from it is lax-idempotent, and that the 2-category of existential doctrines is isomorphic to the 2-category of algebras for this 2-monad. We also show that the existential completion of an elementary doctrine is again elementary. Finally we extend the notion of exact completion of an elementary existential doctrine to an arbitrary elementary doctrine.

5.1 Introduction

In recent years, many relevant logical completions have been extensively studied in category theory. The main instance is the exact completion, see [6, 8, 10], which is the universal extension of a category with finite limits to an exact category. In [42, 43, 44], Maietti and Rosolini introduce a categorical version of quotient for an equivalence relation, and they study that in a doctrine equipped with a sufficient logical structure to describe the notion of an equivalence relation. In [44] they show that both the exact completion of a regular category and the exact completion of a category with binary products, a weak terminal object and weak pullbacks can be seen as instances of a more general completion with respect to an elementary existential doctrine.

In this paper we present the existential completion of a primary doctrine, and we give an explicit description of the 2-monad $\mathrm{T}_e \colon \mathbf{PD} \longrightarrow \mathbf{PD}$ constructed from the 2-adjunction, where **PD** is the 2-category of primary doctrines.

It is well known that pseudo-monads can express uniformly and elegantly many algebraic structure; we refer the reader to [56, 55, 27] for a detailed description of these topics. We show that every existential doctrine $P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ admits an action $a \colon \mathrm{T}_e P \longrightarrow P$ such that (P, a) is a T_e-algebra, and that if (R, b) is T_e-algebra then the doctrine is existential, and this gives an equivalence between the 2-category T_{e}-**Alg** and the 2-category **ExD** whose objects are existential doctrines.

Here the action encodes the existential structure for a doctrine, and we prove that this structure is uniquely determined up to an appropriate isomorphism and that the 2-monad T_e is property-like and lax-idempotent in the sense of Kelly and Lack [27].

We also prove that the existential completion preserves elementary doctrines, and then we generalize the bi-adjunction $\mathbf{EED} \to \mathbf{Xct}$ presented in [44, 41] to a bi-adjunction from the 2-category $\mathbf{ElD}$ of elementary doctrines to the 2-category of exact categories $\mathbf{Xct}$.

In the first two sections we recall definitions and results on pseudo-monads, and on primary and existential doctrines as needed for the rest of the paper.

In section 3 we describe the existential completion. We introduce a functor $\mathrm{E}\colon \mathbf{PD} \longrightarrow \mathbf{ExD}$ from the 2-category of primary doctrines to the 2-category of existential doctrines, and we prove that it is a left 2-adjoint to the forgetful functor $\mathrm{U}\colon \mathbf{ExD} \longrightarrow \mathbf{PD}$.

In sections 4 we prove that the 2-monad T_e constructed from the 2-adjunction is lax-idempotent and, in section 5, that the category T_{e}-$\mathbf{Alg}$ is 2-equivalent to the 2-category of existential doctrine.

In section 6 we show that the existential completion of an elementary doctrine is elementary, and we use this fact to extend the notion of exact completion to elementary doctrines.

5.2 A brief recap of two-dimensional monad theory

This section is devoted to the formal definition of 2-monad on a 2-category and a characterization of the definitions. We use 2-categorical pasting notation freely, following the usual convention of the topic as used extensively in [3], [55] and [56].

You can find all the details of the main results of this section in the works of Kelly and Lack [27]. For a more general and complete description of these topics, and a generalization for the case of pseudo-monad, you can see the Ph.D thesis of Tanaka [54], the articles of Marmolejo [47], [46] and the work of Kelly [28]. Moreover we refer to [4] and [39] for all the standard results and notions about 2-category theory.

A ***2-monad*** (T,μ,η) on a 2-category $\mathcal{A}$ is a 2-functor $\mathrm{T}\colon \mathcal{A} \longrightarrow \mathcal{A}$ together 2-natural transformations $\mu\colon \mathrm{T}^2 \longrightarrow \mathrm{T}$ and $\eta\colon 1_{\mathcal{A}} \longrightarrow \mathrm{T}$ such that the following diagrams

$$\begin{tikzcd}
\mathrm{T}^3 \arrow[r, "\mathrm{T}\mu"] \arrow[d, "\mu\mathrm{T}"'] & \mathrm{T}^2 \arrow[d, "\mu"] \\
\mathrm{T}^2 \arrow[r, "\mu"'] & \mathrm{T}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{T} \arrow[r, "\eta\mathrm{T}"] \arrow[dr, "\mathrm{id}"'] & \mathrm{T}^2 \arrow[d, "\mu"'] & \mathrm{T} \arrow[l, "\mathrm{T}\eta"'] \arrow[dl, "\mathrm{id}"] \\
& \mathrm{T} &
\end{tikzcd}$$

commute. Let (T, μ, η) be a 2-monad on a 2-category $\mathcal{A}$. A T-***algebra*** is a pair (A, a) where, A is an object of $\mathcal{A}$ and $a\colon \mathrm{T}A \longrightarrow A$ is a 1-cell such that the following diagrams commute

$$\begin{tikzcd}
\mathrm{T}^2A \arrow[r, "\mathrm{T}a"] \arrow[d, "\mu_A"'] & \mathrm{T}A \arrow[d, "a"] \\
\mathrm{T}A \arrow[r, "a"'] & A
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[dr, "1_A"'] & \mathrm{T}A \arrow[d, "a"] \\
& A.
\end{tikzcd}$$

A ***lax*** T-***morphism*** from a T-algebra (A, a) to a T-algebra (B, b) is a pair $(f, \overline{f})$ where f is a 1-cell $f\colon A \longrightarrow B$ and $\overline{f}$ is a 2-cell

$$\begin{tikzcd}
\mathrm{T}A \arrow[r, "\mathrm{T}f"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\overline{f}"] & \mathrm{T}B \arrow[d, "b"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

which satisfies the following ***coherence*** conditions

$$\begin{tikzcd}
\mathrm{T}^2A \arrow[r, "\mathrm{T}^2f"] \arrow[d, "\mu_A"'] & TB \arrow[d, "\mu_B"] \\
\mathrm{T}A \arrow[r, "\mathrm{T}f"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\overline{f}"] & \mathrm{T}B \arrow[d, "b"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathrm{T}^2A \arrow[r, "\mathrm{T}^2f"] \arrow[d, "\mathrm{T}a"'] \arrow[dr, phantom, "\Downarrow\mathrm{T}\overline{f}"] & \mathrm{T}B \arrow[d, "\mathrm{T}b"] \\
\mathrm{T}A \arrow[r, "\mathrm{T}f"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\overline{f}"] & \mathrm{T}B \arrow[d, "b"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

and

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\eta_A"'] & B \arrow[d, "\eta_B"] \\
\mathrm{T}A \arrow[r, "\mathrm{T}f"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\overline{f}"] & \mathrm{T}B \arrow[d, "b"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "1_A"'] & B \arrow[d, "1_B"] \\
A \arrow[r, "f"'] & B.
\end{tikzcd}
$$

The regions in which no 2-cell is written always commute by the naturality of η and μ, and are deemed to contain the identity 2-cell.

A lax morphism $(f, \overline{f})$ in which $\overline{f}$ is invertible is said T-***morphism***. And it is ***strict*** when $\overline{f}$ is the identity.

The category of T-algebras and lax T-morphisms becomes a 2-category T-$\mathbf{Alg}_{\mathrm{l}}$, when provided with 2-cells the T-***transformations***. Recall from [27] that a T-***transformation*** from $(f, \overline{f})\colon (A, a) \longrightarrow (B, b)$ to $(g, \overline{g})\colon (A, a) \longrightarrow (B, b)$ is a 2-cell $\alpha\colon f \Longrightarrow g$ in $\mathcal{A}$ which satisfies the following coherence condition

$$
\begin{tikzcd}
\mathrm{T}A \arrow[r, bend left, "\mathrm{T}f"] \arrow[r, bend right, "\mathrm{T}g"'] \arrow[r, phantom, "\Downarrow\mathrm{T}\alpha"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\overline{g}"] & \mathrm{T}B \arrow[d, "b"] \\
A \arrow[r, bend right, "g"'] & B
\end{tikzcd}
=
\begin{tikzcd}
\mathrm{T}A \arrow[r, bend left, "\mathrm{T}f"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\overline{f}"] & \mathrm{T}B \arrow[d, "b"] \\
A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B
\end{tikzcd}
$$

expressing compatibility of α with $\overline{f}$ and $\overline{g}$.

It is observed in [27] that using this notion of T-morphism, one can express more precisely what it may mean that an action of a monad T on an object A is ***unique to within a unique isomorphism***. In our case it means that, given two action $a, a'\colon \mathrm{T}A \longrightarrow A$ there is a unique invertible 2-cell $\alpha\colon a \Longrightarrow a'$ such that $(1_A, \alpha)\colon (A, a) \longrightarrow (A, a')$ is a morphism of T-algebras (in particular it is an isomorphism of T-algebras). In this case we will say that the T-***algebra structure is essentially unique***. More precisely a 2-monad (T, μ, η) is said ***property-like***, if it satisfies the following conditions:

- for every T-algebra (A, a) and (B, b), and for every invertible 1-cell $f\colon A \longrightarrow B$ there exists a unique invertible 2-cell $\overline{f}$

$$\begin{tikzcd}
\mathrm{T}A \arrow[r, "\mathrm{T}f"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\overline{f}"] & \mathrm{T}B \arrow[d, "b"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

such that $(f,\overline{f})\colon (A,a) \longrightarrow (B,b)$ is a morphism of T-algebras;

- for every T-algebra (A,a) and (B,b), and for every 1-cell $f\colon A \longrightarrow B$ if there exists a 2-cell $\overline{f}$

$$\begin{tikzcd}
\mathrm{T}A \arrow[r, "\mathrm{T}f"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\overline{f}"] & \mathrm{T}B \arrow[d, "b"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

  such that $(f,\overline{f})\colon (A,a) \longrightarrow (B,b)$ is a lax morphism of T-algebras, then it is the unique 2-cell with such property.

We conclude this section recalling a stronger property on a 2-monads (T,μ,η) on $\mathcal{A}$ which implies that T is property-like: a 2-monad (T,μ,η) is said ***lax-idempotent***, if for every T-algebras (A,a) and (B,b), and for every 1-cell $f\colon A \longrightarrow B$ there exists a unique 2-cell $\overline{f}$

$$\begin{tikzcd}
\mathrm{T}A \arrow[r, "\mathrm{T}f"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\overline{f}"] & \mathrm{T}B \arrow[d, "b"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

such that $(f,\overline{f})\colon (A,a) \longrightarrow (B,b)$ is a lax morphism of T-algebras. In particular every lax-idempotent monad is property like. See [27, Proposition 6.1].

5.3 Primary and existential doctrines

The notion of hyperdoctrine was introduced by F.W. Lawvere in a series of seminal papers [36, 38]. We recall from *loc. cit.* some definitions which will be useful in the following. The reader can find all the details about the theory of elementary and existential doctrine also in [43, 42, 44].

Definition 5.3.1. Let C be a category with finite products. A ***primary doctrine*** is a functor $P\colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ from the opposite of the category C to the category of inf-semilattices.

Definition 5.3.2. A primary doctrine $P\colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is ***elementary*** if for every A in C there exists an object δ_A in $P(A \times A)$ such that

1. the assignment
$$\exists_{\langle \mathrm{id}_A, \mathrm{id}_A \rangle}(\alpha) := P_{\mathrm{pr}_1}(\alpha) \wedge \delta_A$$
for α in PA determines a left adjoint to $P_{\langle \mathrm{id}_A, \mathrm{id}_A \rangle}\colon P(A \times A) \longrightarrow PA$;
2. for every morphism e of the form $\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_2 \rangle\colon X \times A \longrightarrow X \times A \times A$ in C, the assignment
$$\exists_e(\alpha) := P_{\langle \mathrm{pr}_1, \mathrm{pr}_2 \rangle}(\alpha) \wedge P_{\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle}(\delta_A)$$
for α in $P(X \times A)$ determines a left adjoint to $P_e\colon P(X \times A \times A) \longrightarrow P(X \times A)$.

Definition 5.3.3. A primary doctrine $P\colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is ***existential*** if, for every A_1 and A_2 in C, for any projection $\mathrm{pr}_i\colon A_1 \times A_2 \longrightarrow A_i$, $i = 1, 2$, the functor
$$P_{\mathrm{pr}_i}\colon P(A_i) \longrightarrow P(A_1 \times A_2)$$
has a left adjoint $\exists_{\mathrm{pr}_i}$, and these satisfy:

1. ***Beck-Chevalley condition:*** for any pullback diagram
$$\begin{tikzcd} X' \arrow[r, "\mathrm{pr}'"] \arrow[d, "f'"'] & A' \arrow[d, "f"] \\ X \arrow[r, "\mathrm{pr}"'] & A \end{tikzcd}$$
with pr and pr' projections, for any β in $P(X)$ the canonical arrow
$$\exists_{\mathrm{pr}'} P_{f'}(\beta) \leq P_f \exists_{\mathrm{pr}}(\beta)$$
is an isomorphism;
2. ***Frobenius reciprocity:*** for any projection $\mathrm{pr}\colon X \longrightarrow A$, α in $P(A)$ and β in $P(X)$, the canonical arrow
$$\exists_{\mathrm{pr}}(P_{\mathrm{pr}}(\alpha) \wedge \beta) \leq \alpha \wedge \exists_{\mathrm{pr}}(\beta)$$
in $P(A)$ is an isomorphism.

Remark 5.3.4. In an existential elementary doctrine, for every map $f\colon A \longrightarrow B$ in C the functor P_f has a left adjoint $\exists_f$ that can be computed as

$$\exists_{\mathrm{pr}_2}(P_{f\times \mathrm{id}_B}(\delta_B)\wedge P_{\mathrm{pr}_1}(\alpha))$$

for α in $P(A)$, where pr_1 and pr_2 are the projections from $A \times B$.

Example 5.3.5. The following examples are discussed in [36].

1. Let C be a category with finite limits. The functor

$$\mathrm{Sub}_C \colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$$

   assigns to an object A in C the poset $\mathrm{Sub}_C(A)$ of subobjects of A in C and, for an arrow $B \xrightarrow{f} A$ the morphism $\mathrm{Sub}_C(f)\colon \mathrm{Sub}_C(A) \longrightarrow \mathrm{Sub}_C(B)$ is given by pulling a subobject back along f. The fiber equalities are the diagonal arrows. This is an existential elementary doctrine if and only if the category C has a stable, proper factorization system $\langle \mathcal{E}, \mathcal{M}\rangle$. See [19].
2. Consider a category $\mathcal{D}$ with finite products and weak pullbacks: the doctrine is given by the functor of weak subobjects

$$\Psi_{\mathcal{D}} \colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$$

   where $\Psi_{\mathcal{D}}(A)$ is the poset reflection of the slice category $\mathcal{D}/A$, and for an arrow $B \xrightarrow{f} A$, the homomorphism $\Psi_{\mathcal{D}}(f)\colon \Psi_{\mathcal{D}}(A) \longrightarrow \Psi_{\mathcal{D}}(B)$ is given by a weak pullback of an arrow $X \xrightarrow{g} A$ with f. This doctrine is elementary and existential, and the existential left adjoint are given by the post-composition.
3. Let $\mathbb{T}$ be a theory in a first order language $\mathbf{Sg}$. We define a primary doctrine

$$LT \colon C_{\mathbb{T}}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$$

   where $C_{\mathbb{T}}$ is the category of lists of variables and term substitutions:

   - ***objects*** of $C_{\mathbb{T}}$ are finite lists of variables $\vec{x} := (x_1, \dots, x_n)$, and we include the empty list ();
   - a ***morphisms*** from $(x_1, \dots, x_n)$ into $(y_1, \dots, y_m)$ is a substitution $[t_1/y_1, \dots, t_m/y_m]$ where the terms t_i are built in $\mathbf{Sg}$ on the variable $x_1, \dots, x_n$;
   - the ***composition*** of two morphisms $[\vec{t}/\vec{y}]\colon \vec{x} \longrightarrow \vec{y}$ and $[\vec{s}/\vec{z}]\colon \vec{y} \longrightarrow \vec{z}$ is given by the substitution

$$[s_1[\vec{t}/\vec{y}]/z_k, \dots, s_k[\vec{t}/\vec{y}]/z_k]\colon \vec{x} \longrightarrow \vec{z}\ .$$

   The functor $LT \colon C_{\mathbb{T}}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ sends a list $(x_1, \dots, x_n)$ in the class $LT(x_1, \dots, x_n)$ of all well formed formulas in the context $(x_1, \dots, x_n)$. We say that $\psi \leq \phi$ where $\phi, \psi \in LT(x_1, \dots, x_n)$ if $\psi \vdash_{\mathbb{T}} \phi$, and then we quotient in the usual way to obtain a partial order on $LT(x_1, \dots, x_n)$. Given a morphism of $C_{\mathbb{T}}$

$$[t_1/y_1, \dots, t_m/y_m]\colon (x_1, \dots, x_n) \longrightarrow (y_1, \dots, y_m)$$

the functor $LT_{[\vec{t}/\vec{y}]}$ acts as the substitution $LT_{[\vec{t}/\vec{y}]}(\psi(y_1,\dots,y_m)) = \psi[\vec{t}/\vec{y}]$.

The doctrine $LT\colon \mathcal{C}_{\mathbb{T}}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is elementary exactly when $\mathbb{T}$ has an equality predicate and it is existential. For all the detail we refer to [43], and for the case of a many sorted first order theory we refer to [50].

5.4 Existential completion

In this section we construct an existential doctrine $P^{\mathrm{ex}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$, starting from a primary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$.

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be a fixed primary doctrine for the rest of the section, and let $a \subset \mathcal{C}_1$ be a subclass of morphisms closed under pullbacks, compositions and such that it contains the identity morphisms. In our case *closed under pullbacks* means that for every $f \in a$ and for every morphism g in $\mathcal{C}$ the pullback

$$\begin{tikzcd} A \arrow[r, "g^*f"] \arrow[d, "f^*g"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

exists and $f^*g \in a$.

For every object A of $\mathcal{C}$ consider the following preorder:

- the objects are pairs $(B \xrightarrow{g\in a} A,\ \alpha \in PB)$;
- $(B \xrightarrow{h\in a} A,\ \alpha \in PB) \leq (D \xrightarrow{f\in a} A,\ \gamma \in PD)$ if there exists $w\colon B \longrightarrow D$ such that

$$\begin{tikzcd} & B \arrow[dl, "w"'] \arrow[d, "h"] \\ D \arrow[r, "f"'] & A \end{tikzcd}$$

  commutes and $\alpha \leq P_w(\gamma)$.

It is easy to see that the previous data give a preorder. Let $P^{\mathrm{ex}}(A)$ be the partial order obtained by identifying two objects when

$$(B \xrightarrow{h\in a} A,\ \alpha \in PB) \gtrless (D \xrightarrow{f\in a} A,\ \gamma \in PD)$$

in the usual way. With abuse of notation we denote the equivalence class of an element in the same way.

Given a morphism $f\colon A \longrightarrow B$ in $\mathcal{C}$, let $P^{\mathrm{ex}}_f(C \xrightarrow{g\in a} B\ ,\ \beta \in PC)$ be the object

$$(D \xrightarrow{g^*f} A\ ,\ P_{f^*g}(\beta) \in PD)$$

where

$$\begin{tikzcd} D \arrow[r, "g^*f"] \arrow[d, "f^*g"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ C \arrow[r, "g"'] & B \end{tikzcd}$$

is a pullback because $g \in a$. Note that P^{ex}_f is well defined, because isomorphisms are stable under pullbacks.

Proposition 5.4.1. *Let* $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *be a primary doctrine. Then* $P^{\mathrm{ex}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *is a primary doctrine, in particular:*

(i) for every object A *in* $\mathcal{C}$*,* $P^{\mathrm{ex}}(A)$ *is a inf-semilattice;*
(ii) for every morphism $f\colon A \longrightarrow B$ *in* $\mathcal{C}$*,* P^{ex}_f *is an homomorphism of inf-semilattices.*

Proof. (i) For every A we have the top element $(A \xrightarrow{\mathrm{id}_A} A\ ,\ \top_A)$. Consider $(A_1 \xrightarrow{h_1} A\ ,\ \alpha_1 \in PA_1)$ and $(A_2 \xrightarrow{h_2} A\ ,\ \alpha_2 \in PA_2)$. In order to define the greatest lower bound of the two objects consider a pullback

$$\begin{tikzcd} A_1 \wedge A_2 \arrow[r, "h_1^*h_2"] \arrow[d, "h_2^*h_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A_2 \arrow[d, "h_2"] \\ A_1 \arrow[r, "h_1"'] & A \end{tikzcd}$$

which exists because $h_1 \in a$ (and $h_2 \in a$). We claim that

$$(A_1 \wedge A_2 \xrightarrow{h_1(h_2^*h_1)} A\ , P_{h_2^*h_1}(\alpha_1) \wedge P_{h_1^*h_2}(\alpha_2))$$

is such an infimum. It is easy to check that

$$(A_1 \wedge A_2 \xrightarrow{h_1(h_2^*h_1)} A\ , P_{h_2^*h_1}(\alpha_1) \wedge P_{h_1^*h_2}(\alpha_2)) \le (A_i \xrightarrow{h_i} A\ ,\ \alpha_i \in PA_i)$$

for $i = 1, 2$. Next consider $(B \xrightarrow{g} A\ ,\ \beta \in PB) \le (A_i \xrightarrow{h_i} A\ ,\ \alpha_i \in PA_i)$ for $i = 1, 2$ and $g = h_i w_i$. Then there is a morphism $w\colon C \longrightarrow A_1 \wedge A_2$ such that

$$\begin{tikzcd}
B \arrow[dr, dashed, "w"] \arrow[drr, bend left, "w_2"] \arrow[ddr, bend right, "w_1"'] & & \\
& A_1 \wedge A_2 \arrow[r, "h_1^* h_2"] \arrow[d, "h_2^* h_1"'] & A_2 \arrow[d, "h_2"] \\
& A_1 \arrow[r, "h_1"'] & A
\end{tikzcd}$$

commutes and $P_w(P_{h_2^* h_1}(\alpha_1) \wedge P_{h_1^* h_2}(\alpha_2)) = P_{w_1}(\alpha_1) \wedge P_{w_2}(\alpha_2) \geq \beta$.

(ii) We first prove that for every morphism $f \colon A \longrightarrow B$ the P_f^{ex} preserves the order. Consider $(C_1 \xrightarrow{g_1 \in a} B \ , \ \alpha_1 \in PC_1) \leq (C_2 \xrightarrow{g_2 \in a} B \ , \ \alpha_2 \in PC_2)$ with $g_2 w = g_1$ and $P_w(\alpha_2) \geq \alpha_1$. We want to prove that

$$(D_1 \xrightarrow{g_1^* f} A \ , \ P_{f^* g_1}(\alpha_1) \in PD_1) \leq (D_2 \xrightarrow{g_2^* f} A \ , \ P_{f^* g_2}(\alpha_2) \in PD_1)$$

We can observe that $g_2 w(f^* g_1) = g_1(f^* g_1) = f(g_1^* f)$. Then there exists a unique $\overline{w} \colon D_1 \longrightarrow D_2$ such that the following diagram commutes

$$\begin{tikzcd}
D_1 \arrow[dr, dashed, "\overline{w}"] \arrow[drr, bend left, "g_1^* f"] \arrow[ddr, bend right, "w f^* g_1"'] & & \\
& D_2 \arrow[r, "g_2^* f"] \arrow[d, "f^* g_2"'] & A \arrow[d, "f"] \\
& C_2 \arrow[r, "g_2"'] & B.
\end{tikzcd}$$

Moreover $P_{\overline{w}}(P_{f^* g_2}(\alpha_2)) = P_{f^* g_1}(P_w(\alpha_2)) \geq P_{f^* g_1}(\alpha_1)$, and it is easy to see that P_f^{ex} preserves top elements. Finally it is straightforward to prove that $P_f^{\mathrm{ex}}(\alpha \wedge \beta) = P_f^{\mathrm{ex}}(\alpha) \wedge P_f^{\mathrm{ex}}(\beta)$. It is straightforward to prove that $P_f^{\mathrm{ex}}(\alpha \wedge \beta) = P_f^{\mathrm{ex}}(\alpha) \wedge P_f^{\mathrm{ex}}(\beta)$. $\square$

Proposition 5.4.2. *Given a morphism $f \colon A \longrightarrow B$ of a, let*

$$\exists_f^{\mathrm{ex}}(C \xrightarrow{h} A \ , \alpha \in PC) := (C \xrightarrow{fh} B \ , \alpha \in PC)$$

when $(C \xrightarrow{h} A \ , \alpha \in PC)$ is in $P^{\mathrm{ex}}(A)$. Then $\exists_f^{\mathrm{ex}}$ is left adjoint to P_f^{ex}.

Proof. Let $\alpha := (\ C_1 \xrightarrow{g_1} B\ ,\ \alpha_1 \in PC_1)$ and $\beta := (\ D_2 \xrightarrow{f_2} A\ ,\ \beta_2 \in PD_2)$. Now we assume that $\beta \leq P^{\mathrm{ex}}_f(\alpha)$. This means that

$$\begin{tikzcd}
 & D_2 \arrow[dl, "w"'] \arrow[d, "f_2"] \\
D_1 \arrow[r, "g_1^* f"] \arrow[d, "f^* g_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
C_1 \arrow[r, "g_1"'] & B
\end{tikzcd}$$

and $P_w(P_{f^*g_1}(\alpha_1)) \geq \beta_2$. Then we have

$$\begin{tikzcd}
 & D_2 \arrow[dl, "f^* g_1 w"'] \arrow[d, "\Delta_A f_2"] \\
C_1 \arrow[r, "g"'] & B
\end{tikzcd}$$

and $P_{wf^*g_1}(\alpha_1) \geq \beta$. Then $\exists^{\mathrm{ex}}_f(\beta) \leq \alpha$.

Now assume $\exists^{\mathrm{ex}}_f(\beta) \leq \alpha$

$$\begin{tikzcd}
 & D_2 \arrow[dl, "\overline{w}"'] \arrow[d, "f f_2"] \\
C_1 \arrow[r, "g_1"'] & B
\end{tikzcd}$$

with $P_{\overline{w}}(\alpha_1) \geq \beta_2$. Then there exists $w\colon D_2 \longrightarrow D_1$ such that the following diagram commutes

$$\begin{tikzcd}
D_2 \arrow[dr, dashed, "w"] \arrow[drr, bend left, "f_2"] \arrow[ddr, bend right, "\overline{w}"'] & & \\
 & D_1 \arrow[r, "g_1^* f"] \arrow[d, "f^* g_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
 & C_1 \arrow[r, "g_1"'] & B
\end{tikzcd}$$

and $P_w(P_{f^*g_1}(\alpha_1) = P_{\overline{w}}(\alpha_1) \geq \beta_1$. Then we can conclude that $\beta \leq P^{\mathrm{ex}}_f(\alpha)$. □

Theorem 5.4.3. *For every primary doctrine* $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$, $P^{\mathrm{ex}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *satisfies:*

(i) **Beck-Chevalley Condition***: for any pullback*

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A' \arrow[d, "f"] \\
X \arrow[r, "g"'] & A
\end{tikzcd}$$

with $g \in a$ (hence also $g' \in a$), for any $\beta \in P^{\mathrm{ex}}(X)$ the following equality holds

$$\exists^{\mathrm{ex}}_{g'} P^{\mathrm{ex}}_{f'}(\beta) = P^{\mathrm{ex}}_f \exists^{\mathrm{ex}}_g(\beta).$$

*(ii)***Frobenius Reciprocity***: for every morphism $f\colon X \longrightarrow A$ of a, for every $\alpha \in P^{\mathrm{ex}}(A)$ and $\beta \in P^{\mathrm{ex}}(X)$, the following equality holds*

$$\exists^{\mathrm{ex}}_f(P^{\mathrm{ex}}_f(\alpha) \wedge \beta) = \alpha \wedge \exists^{\mathrm{ex}}_f(\beta).$$

Proof. (i) Consider the following pullback square

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A' \arrow[d, "f"] \\
X \arrow[r, "g"'] & A
\end{tikzcd}$$

where $g, g' \in a$, and let $\beta := (\ C_1 \xrightarrow{h_1} X\ ,\ \beta_1 \in PC_1) \in P^{\mathrm{ex}}(X)$. Consider the following diagram

$$\begin{tikzcd}
D_1 \arrow[r, "h_1^* f'"] \arrow[d, "f'^* h_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A' \arrow[d, "f"] \\
C_1 \arrow[r, "h_1"'] & X \arrow[r, "g"'] & A.
\end{tikzcd}$$

Since the two square are pullbacks, then the big square is a pullback, and then

$$(\ D_1 \xrightarrow{g'(h_1^* f')} A\ , P_{f'^* h_1}(\beta_1)) = (\ D_1 \xrightarrow{(gh_1)^* f} A\ , P_{f^*(gh_1)}(\beta_1))$$

and these are by definition

$$\exists^{\mathrm{ex}}_{g'} P^{\mathrm{ex}}_{f'}(\beta) = P^{\mathrm{ex}}_{f} \exists^{\mathrm{ex}}_{g}(\beta).$$

Therefore the Beck-Chevalley Condition is satisfied.

(*ii*) Consider a morphism $f\colon X \longrightarrow A$ of a, an element $\alpha := (C_1 \xrightarrow{h_1} A\ ,\ \alpha_1 \in PC_1)$ in $P^{\mathrm{ex}}(A)$, and an element $\beta = (D_2 \xrightarrow{h_2} X\ ,\ \beta_2 \in PD_2)$ in $P^{\mathrm{ex}}(X)$. Observe that the following diagram is a pullback

$$\begin{tikzcd}
D_2 \wedge D_1 \arrow[r, "h_2^*(h_1^* f)"] \arrow[d, "(h_1^* f)^* h_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & D_1 \arrow[r, "f^* h_1"] \arrow[d, "h_1^* f"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_1 \arrow[d, "h_1"] \\
D_2 \arrow[r, "h_2"'] & X \arrow[r, "f"'] & A
\end{tikzcd}$$

and this means that

$$\exists^{\mathrm{ex}}_{f}(P^{\mathrm{ex}}_{f}(\alpha) \wedge \beta) = \alpha \wedge \exists^{\mathrm{ex}}_{f}(\beta).$$

Therefore the Frobenius Reciprocity is satisfied. □

Remark 5.4.4. In the case that a is the class of the product projections, then from primary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ it can be constructed an existential doctrine $P^{\mathrm{ex}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ in the sense of Definition 5.3.3. Therefore the notion of existential doctrine can be generalized in the sense that an existential doctrine can be defined as a pair

$$(P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}\ , a)$$

where $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is a primary doctrine and a is a class of morphisms of $\mathcal{C}$ closed by pullbacks, composition and identities, such that P_f has a left adjoint for every f in a, and these satisfy Beck-Chevalley condition and Frobenius Reciprocity as in Theorem 5.4.3.

Remark 5.4.5. Let $P\colon \mathcal{C} \longrightarrow \mathbf{Pos}_{\top}$ be a functor where $\mathbf{Pos}_{\top}$ is the category of posets with top element. We can apply the existential completion since we have not used the hypothesis that PA has infimum during the proofs; we have proved that if it has a infimum it is preserved by the completion. Moreover we can express the Frobenius condition without using infima, and also this condition is preserved by completion.

Since a poset of the category $\mathbf{Pos}_{\top}$ has a top element, one has an injection from the doctrine $P\colon \mathcal{C} \longrightarrow \mathbf{Pos}_{\top}$ into $P^{\mathrm{ex}}\colon \mathcal{C} \longrightarrow \mathbf{Pos}_{\top}$. From a logical point of view, one can think of extending a theory without existential quantification to one with that quantifier, requiring that the theorems of the previous theory are preserved.

In the rest of the section we assume that the morphisms of a are all the projections. We define a 2-functor $\mathrm{E}\colon \mathbf{PD} \longrightarrow \mathbf{ExD}$ sending a primary doctrine $P\colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ into the existential doctrine $P^{\mathrm{ex}}\colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$. For all the standard notions about 2-category theory we refer to [4, 39].

Proposition 5.4.6. *Consider the category* $\mathbf{PD}(P,R)$*. We define*

$$\mathrm{E}_{P,R}\colon\ \mathbf{PD}(P,R) \longrightarrow \mathbf{ExD}(P^{\mathrm{ex}},R^{\mathrm{ex}})$$

as follow:

- *for every 1-cell* (F,b)*,* $\mathrm{E}_{P,R}(F,b) := (F,b^{\mathrm{ex}})$*, where* $b^{\mathrm{ex}}_A\colon P^{\mathrm{ex}}A \longrightarrow R^{\mathrm{ex}}FA$ *sends an object* $(\ C \xrightarrow{g} A\ ,\ \alpha)$ *in the object* $(\ FC \xrightarrow{Fg} FA\ ,\ b_C(\alpha))$*;*
- *for every 2-cell* $\theta\colon (F,b) \Longrightarrow (G,c)$ *,* $\mathrm{E}_{P,R}\theta$ *is essentially the same.*

With the previous assignment E *is a 2-functor.*

Proof. We prove that $(F,b^{\mathrm{ex}})\colon P^{\mathrm{ex}} \longrightarrow R^{\mathrm{ex}}$ is a 1-cell of $\mathbf{ExD}(P^{\mathrm{ex}},R^{\mathrm{ex}})$. We first prove that for every $A \in C$, b^{ex}_A preserves the order.

If $(\ C_1 \xrightarrow{g_1} A\ ,\ \alpha_1) \le (\ C_2 \xrightarrow{g_2} A\ ,\ \alpha_2)$, we have a morphism $w\colon C_1 \longrightarrow C_2$ such that the following diagram commutes

$$\begin{tikzcd} & C_1 \arrow[dl, "w"'] \arrow[d, "g_1"] \\ C_2 \arrow[r, "g_2"'] & A \end{tikzcd}$$

and $\alpha_1 \le P_w(\alpha_2)$. Since b is a natural transformation, we have that $b_{C_1}P_w = R_{Fw}b_{C_2}$. Then we can conclude that $(\ FC_1 \xrightarrow{Fg_1} FA\ ,\ b_{C_1}(\alpha_1)) \le (\ FC_2 \xrightarrow{Fg_2} FA\ ,\ b_{C_2}(\alpha_2))$ because $Fg_2Fw = Fg_1$ and $R_{Fw}(b_{C_2}\alpha_2) = b_{C_1}P_w(\alpha_2) \ge b_{C_1}(\alpha_1)$. Moreover, since F preserves products, we can conclude that b^{ex}_A preserves inf.

One can prove that $b^{\mathrm{ex}}\colon P^{\mathrm{ex}} \longrightarrow R^{\mathrm{ex}}F^{\mathrm{op}}$ is a natural transformation using the facts that F preserves products. Moreover we can easily see that b^{ex} preserves the left adjoints along projections. Then (F,b^{ex}) is a 1-cell of $\mathbf{ExD}$.

Now consider a 2-cell $\theta\colon (F,b) \Longrightarrow (G,c)$, and let $\alpha := (\ C_1 \xrightarrow{g_1} A\ ,\ \alpha_1)$ be an object of $P^{\mathrm{ex}}(A)$. Then

$$b^{\mathrm{ex}}_A(\alpha) = (\ FC_1 \xrightarrow{Fg_1} FA\ ,\ b_{C_1}(\alpha_1))$$

and

$$R^{\mathrm{ex}}_{\theta_A}c^{\mathrm{ex}}_A(\alpha) = (\ D_1 \xrightarrow{Gg_1^*\theta_A} FA\ ,\ R_{\theta_A^*Gg_1}c_{C_1}(\alpha_1))$$

where

$$\begin{tikzcd} D_1 \arrow[r, "Gg_1^*\theta_A"] \arrow[d, "\theta_A^* Gg_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & FA \arrow[d, "\theta_A"] \\ GC_1 \arrow[r, "Gg_1"'] & GA. \end{tikzcd}$$

Now observe that since $\theta\colon F \longrightarrow G$ is a natural transformation, there exists a unique $w\colon FC_1 \longrightarrow D_1$ such that the diagram

$$\begin{tikzcd} FC_1 \arrow[dr, dashed, "w"] \arrow[drr, bend left, "Fg_1"] \arrow[ddr, bend right, "\theta_{C_1}"'] & & \\ & D_1 \arrow[r, "Gg_1^*\theta_A"] \arrow[d, "\theta_A^* Gg_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & FA \arrow[d, "\theta_A"] \\ & GC_1 \arrow[r, "Gg_1"'] & GA \end{tikzcd}$$

commutes and then $R_w R_{\theta_A^* Gg_1} c_{C_1}(\alpha_1) = R_{\theta_{C_1}} c_{C_1}(\alpha_1) \geq b_{C_1}(\alpha_1)$. Therefore we can conclude that $b_A^{\mathrm{ex}}(\alpha) \leq R_{\theta_A}^{\mathrm{ex}} c_A^{\mathrm{ex}}(\alpha)$, and then $\theta\colon F \longrightarrow G$ can is a 2-cell $\theta\colon (F, b^{\mathrm{ex}}) \Longrightarrow (G, c^{\mathrm{ex}})$, and $\mathrm{E}_{P,R}(\theta\gamma) = \mathrm{E}_{P,R}(\theta)\mathrm{E}_{P,R}(\gamma)$.

Finally one can prove that the following diagram commutes observing that for every $(F, b) \in \mathbf{PD}(P, R)$ and $(G, c) \in \mathbf{PD}(R, D)$, $(GF, c^{\mathrm{ex}} b^{\mathrm{ex}}) = (GF, (cb)^{\mathrm{ex}})$.

$$\begin{tikzcd} \mathbf{PD}(P,R) \times \mathbf{PD}(R,D) \arrow[r, "c_{PRD}"] \arrow[d, "\mathrm{E}_{PR} \times \mathrm{E}_{RD}"'] & \mathbf{PD}(P,D) \arrow[d, "\mathrm{E}_{PD}"] \\ \mathbf{ExD}(P^{\mathrm{ex}}, R^{\mathrm{ex}}) \times \mathbf{ExD}(R^{\mathrm{ex}}, D^{\mathrm{ex}}) \arrow[r, "c_{P^{\mathrm{ex}} R^{\mathrm{ex}} D^{\mathrm{ex}}}"'] & \mathbf{ExD}(P^{\mathrm{ex}}, D^{\mathrm{ex}}) \end{tikzcd}$$

and the same for the unit diagram. Therefore we can conclude that E is a 2-functor. □

Now we prove the 2-functor $\mathrm{E}\colon \mathbf{PD} \longrightarrow \mathbf{ExD}$ is left adjoint to the forgetful functor $\mathrm{U}\colon \mathbf{ExD} \longrightarrow \mathbf{PD}$.

Proposition 5.4.7. *Let* $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *be an elementary doctrine. Then*

$$(\mathrm{id}_{\mathcal{C}}, \iota_P)\colon P \longrightarrow P^{\mathrm{ex}}$$

where $\iota_{PA}\colon PA \longrightarrow P^{\mathrm{ex}}A$ *sends* α *into (* $A \xrightarrow{\mathrm{id}_A} A$ *,* α*) is a 1-cell. Moreover the assignment*

$$\eta\colon \mathrm{id}_{\mathbf{ExD}} \longrightarrow \mathrm{UE}$$

where $\eta_P := (\mathrm{id}_{\mathcal{C}}, \iota_P)$*, is a 2-natural transformation.*

Proof. It is easy to prove that $\iota_{PA}\colon PA \longrightarrow P^{\mathrm{ex}}A$ preserves all the structures. For every morphism $f\colon A \longrightarrow B$ of $\mathcal{C}$, it one can see that the following diagram commutes

$$\begin{tikzcd} PB \arrow[r, "P_f"] \arrow[d, "\iota_{PB}"'] & PA \arrow[d, "\iota_{PA}"] \\ P^{\mathrm{ex}}B \arrow[r, "P^{\mathrm{ex}}_f"'] & P^{\mathrm{ex}}A. \end{tikzcd}$$

Then we can conclude that $(\mathrm{id}_{\mathcal{C}}, \iota_P)\colon P \longrightarrow P^{\mathrm{ex}}$ is a 1-cell of **ExD** and it is a direct verification the proof the η is a 2-natural transformation. □

Proposition 5.4.8. *Let* $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *be an existential doctrine. Then*

$$(\mathrm{id}_{\mathcal{C}}, \zeta_P)\colon P^{\mathrm{ex}} \longrightarrow P$$

where $\zeta_{PA}\colon P^{\mathrm{ex}}A \longrightarrow PA$ *sends* $(\, C \xrightarrow{f} A\,,\ \alpha)$ *in* $\exists_f(\alpha)$ *is a 1-cell. Moreover the assignment*

$$\varepsilon\colon \mathrm{EU} \longrightarrow \mathrm{id}_{\mathbf{ExD}}$$

where $\varepsilon_P = (\mathrm{id}_{\mathcal{C}}, \zeta_P)$*, is a 2-natural transformation.*

Proof. Suppose $(\, C_1 \xrightarrow{g_1} A\,,\ \alpha_1) \leq (\, C_2 \xrightarrow{g_2} A\,,\ \alpha_2)$, with $w\colon C_1 \longrightarrow C_2$, $g_2 w = g_1$ and $P_w(\alpha_2) \geq \alpha_1$. Then by Beck-Chevalley we have the equality

$$\exists_{g_2^* g_1} P_{g_1^* g_2}(\alpha_2) = P_{g_1} \exists_{g_2}(\alpha_2)$$

and

$$P_{g_1} \exists_{g_2}(\alpha_2) = P_w P_{g_2} \exists_{g_2}(\alpha_2) \geq P_w(\alpha_2) \geq \alpha_1.$$

Then

$$\exists_{g_1}(\alpha_1) \leq \exists_{g_1} \exists_{g_2^* g_1} P_{g_1^* g_2}(\alpha_2) = \exists_{g_2} \exists_{g_1^* g_2} P_{g_1^* g_2}(\alpha_2) \leq \exists_{g_2}(\alpha_2)$$

and $\delta_A = \zeta_A(\, A \xrightarrow{\mathrm{id}_A} A\,,\ \top_A)$. Now we prove the naturality of ζ_P. Let $f\colon A \longrightarrow B$ be a morphism of $\mathcal{C}$. Then the following diagram commutes

$$\begin{tikzcd}
P^{\mathrm{ex}}B \arrow[r, "P^{\mathrm{ex}}_f"] \arrow[d, "\zeta_B"'] & P^{\mathrm{ex}}A \arrow[d, "\zeta_A"] \\
PB \arrow[r, "P_f"'] & PA
\end{tikzcd}$$

because for every ($C \xrightarrow{g} B$, $\beta \in PC$) we have $\exists_{g^*f} P_{f^*g}(\beta) = P_f \exists_g(\beta)$ by Beck-Chevalley. Moreover it is easy to see that ζ_P preserves left-adjoints. Then we an conclude that for every existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$, ζ_P is a 1-cell of $\mathbf{ExD}$.

The proof of the naturality of ε is a routine verification. One must use the fact that we are working in $\mathbf{ExD}$, and then for every 1-cell (F, b), b preserves left-adjoints along the projections. □

Proposition 5.4.9. *For every primary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ we have*

$$\varepsilon_{P^{\mathrm{ex}}} \circ \eta_P{}^{\mathrm{ex}} = \mathrm{id}_P .$$

Proof. Consider the following diagram

$$\begin{tikzcd}
\mathcal{C}^{op} \arrow[d, "\mathrm{id}_{\mathcal{C}}^{op}"'] \arrow[drr, "P^{\mathrm{ex}}"] & & \\
\mathcal{C}^{op} \arrow[d, "\mathrm{id}_{\mathcal{C}}^{op}"'] \arrow[rr, "(P^{\mathrm{ex}})^{\mathrm{ex}}"'] & \arrow[u, phantom, "\Downarrow \iota^{\mathrm{ex}}"] \arrow[d, phantom, "\Downarrow \zeta_{P^{\mathrm{ex}}}"] & \mathbf{InfSL} \\
\mathcal{C} \arrow[urr, "P^{\mathrm{ex}}"'] & &
\end{tikzcd}$$

and let ($C \xrightarrow{g} A$, $\alpha \in PA$) be an element of $P^{\mathrm{ex}}A$. Then

$$\iota_{P}{}^{\mathrm{ex}}_{A}(C \xrightarrow{g} A \,,\ \alpha \in PC) = (A \xrightarrow{\mathrm{id}_A} A \,,\ (C \xrightarrow{g} A \,,\ \alpha \in PC) \in P^{\mathrm{ex}}A)$$

and

$$\zeta_{P^{\mathrm{ex}}A}(A \xrightarrow{\mathrm{id}_A} A \,,\ (C \xrightarrow{g} A \,,\ \alpha \in PC) \in P^{\mathrm{ex}}A) = \exists^{\mathrm{ex}}_{\mathrm{id}_A}(C \xrightarrow{g} A \,,\ \alpha \in PC).$$

By definition of $\exists^{\mathrm{ex}}$ we have

$$\exists^{\mathrm{ex}}_{\mathrm{id}_A}(C \xrightarrow{g} A \,,\ \alpha \in PC) = (C \xrightarrow{g} A \,,\ \alpha \in PC).$$

Then we can conclude that for every $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$, we have $\varepsilon_{P^{\mathrm{ex}}} \circ \eta_P{}^{\mathrm{ex}} = \mathrm{id}_{P^{\mathrm{ex}}}$. □

Corollary 5.4.10. $\varepsilon\mathrm{E} \circ \mathrm{E}\eta = \mathrm{id}_{\mathrm{E}}$.

Proposition 5.4.11. *For every existential doctrine* $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *we have*

$$\varepsilon_P \circ \eta_P = \mathrm{id}_P .$$

Proof. It is a direct verification. □

Corollary 5.4.12. $\mathrm{U}\varepsilon \circ \eta\mathrm{U} = \mathrm{id}_{\mathrm{U}}$.

By Corollary 5.4.10 and Corollary 5.4.12, we can conclude this section with the following theorem.

Theorem 5.4.13. *The 2-functor* E *is 2-adjoint to the 2-functor* U.

5.5 The 2-monad $\mathbf{T}_e$

In this section we construct a 2-monad $\mathrm{T}_e\colon \mathbf{PD} \longrightarrow \mathbf{PD}$, and we prove that every existential doctrine can be seen as an algebra for this 2-monad. Finally we show that the 2-monad T_e is lax-idempotent.

We define:

- $\mathrm{T}_e\colon \mathbf{ExD} \longrightarrow \mathbf{ExD}$ the 2-functor $\mathrm{T} = \mathrm{U} \circ \mathrm{E}$;
- $\eta\colon \mathrm{id}_{\mathbf{ExD}} \longrightarrow \mathrm{T}_e$ is the 2-natural transformation defined in Proposition 5.4.7;
- $\mu\colon \mathrm{T}_e^2 \longrightarrow \mathrm{T}_e$ is the 2-natural transformation $\mu = \mathrm{U}\varepsilon\mathrm{E}$.

Proposition 5.5.1. T_e *is a 2-monad.*

Proof. One can easily check that the following diagrams commute

$$\begin{tikzcd}
\mathrm{T}_e^3 \arrow[r, "\mu\mathrm{T}_e"] \arrow[d, "T_e\mu"'] & \mathrm{T}_e^2 \arrow[d, "\mu"] \\
\mathrm{T}_e^2 \arrow[r, "\mu"'] & \mathrm{T}_e
\end{tikzcd}$$

$$
\begin{tikzcd}
\mathrm{id}_{\mathbf{ExD}} \circ \mathrm{T}_e \arrow[r, "\eta \mathrm{T}_e"] \arrow[dr, "\mathrm{id}"'] & \mathrm{T}_e^2 \arrow[d, "\mu"] & \mathrm{T}_e \circ \mathrm{id}_{\mathbf{ExD}} \arrow[l, "\mathrm{T}_e \eta"'] \arrow[dl, "\mathrm{id}"] \\
& \mathrm{T}_e . &
\end{tikzcd}
$$

□

Remark 5.5.2. Observe that $\mu_P : \mathrm{T}_e^2 \cong \mathrm{T}_e$ is an isomorphism.

Proposition 5.5.3. *Let* $P \colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *be an existential doctrine. Then* (P, ε_P) *is a* T_e*-algebra.*

Proof. It is a direct verification. □

Proposition 5.5.4. *Let* $P \colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *be an primary doctrine, and let* $(P, (F, a))$ *be a* T_e*-algebra. Then* $P \colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *is existential,* $F = \mathrm{id}_C$ *and* $a = \varepsilon_P$*.*

Proof. By the unit axiom for T_e-algebras, we know that F must be the identity functor, and $a_A \iota_A = \mathrm{id}_{PA}$.

$$
\begin{tikzcd}
P \arrow[r, "\eta_P"] \arrow[dr, "\mathrm{id}_P"'] & P^{\mathrm{ex}} \arrow[d, "(F,a)"] \\
& P .
\end{tikzcd}
$$

For every morphism $f \colon A \longrightarrow B$ of C, where f is a projection, we claim that

$$\exists_f(\alpha) := a_B \exists_f^{\mathrm{ex}} \iota_A(\alpha)$$

is left adjoint to P_f. Let $\alpha \in PA$ and $\beta \in PB$, and suppose that $\alpha \le P_f(\beta)$. Then we have that

$$(A \xrightarrow{f} B , \alpha) \le (B \xrightarrow{\mathrm{id}_B} B , \beta)$$

in $P^{\mathrm{ex}}B$ and $(A \xrightarrow{f} B , \alpha) = \exists_f^{\mathrm{ex}}(A \xrightarrow{\mathrm{id}_A} A , \alpha)$. Therefore, by definition of ι, we have

$$\exists_f^{\mathrm{ex}} \iota_A(\alpha) \le \iota_B(\beta).$$

Hence

$$a_B \exists_f^{\mathrm{ex}} \iota_A(\alpha) \le a_B \iota_B(\beta) = \beta.$$

Now suppose that $\exists_f(\alpha) \le \beta$. Then

$$a_B(\ A \xrightarrow{f} B\ ,\ \alpha) \leq \beta$$

so

$$P_f a_B(\ A \xrightarrow{f} B\ ,\ \alpha) \leq P_f(\beta).$$

By the naturality of a, we have

$$P_f a_B(\ A \xrightarrow{f} B\ ,\ \alpha) = a_A P_f^{\mathrm{ex}}(\ A \xrightarrow{f} B\ ,\ \alpha).$$

Now observe that $P_f^{\mathrm{ex}}(\ A \xrightarrow{f} B\ ,\ \alpha) \geq (\ A \xrightarrow{\mathrm{id}_A} A\ ,\ \alpha) = \iota_A(\alpha)$. Therefore we can conclude that

$$\alpha = a_A \iota_A(\alpha) \leq P_f a_B(\ A \xrightarrow{f} B\ ,\ \alpha) \leq P_f(\beta).$$

Now we prove that Bech-Chevalley holds. Consider the following pullback

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A' \arrow[d, "f"] \\ X \arrow[r, "g"'] & A \end{tikzcd}$$

and $\alpha \in PX$. Then we have

$$\exists_{g'} P_{f'}(\alpha) = a_{A'} \exists_{g'}^{\mathrm{ex}} \iota_{X'}(P_{f'}\alpha) = a_{A'}(\ X' \xrightarrow{g'} A'\ , P_{f'}(\alpha)).$$

Observe that

$$(\ X' \xrightarrow{g'} A'\ , P_{f'}(\alpha)) = P_f^{\mathrm{ex}}(\ X \xrightarrow{g} A\ ,\ \alpha)$$

and since a is a natural transformation, we have

$$a_{A'} P_f^{\mathrm{ex}}(\ X \xrightarrow{g} A\ ,\ \alpha) = P_f a_A(\ X \xrightarrow{g} A\ ,\ \alpha).$$

Finally we can conclude that Bech-Chevalley holds because

$$P_f \exists_g(\alpha) = P_f a_A \exists_g^{\mathrm{ex}} \iota_X(\alpha) = P_f a_A(\ X \xrightarrow{g} A\ ,\ \alpha).$$

Hence

$$\exists_{g'} P_{f'}(\alpha) = P_f \exists_g(\alpha).$$

Now consider a projection $f\colon A \longrightarrow B$, and two elements $\beta \in PB$ and $\alpha \in PA$. We want to prove that the Frobenius reciprocity holds.

$$\exists_f(P_f(\beta)\wedge\alpha)=a_B\exists_f^{\mathrm{ex}}\iota_A(P_f(\beta)\wedge\alpha)=a_B(\ A\xrightarrow{f}B\ ,\ P_f(\beta)\wedge\alpha)$$

and

$$\beta\wedge\exists_f(\alpha)=a_B\iota_B(\beta)\wedge a_B(\ A\xrightarrow{f}B\ ,\ \alpha)$$

and

$$a_B\iota_B(\beta)\wedge a_B(\ A\xrightarrow{f}B\ ,\ \alpha)=a_B((\ B\xrightarrow{\mathrm{id}_B}B\ ,\ \beta)\wedge(\ A\xrightarrow{f}B\ ,\ \alpha)).$$

We can observe that

$$a_B((\ B\xrightarrow{\mathrm{id}_B}B\ ,\ \beta)\wedge(\ A\xrightarrow{f}B\ ,\ \alpha))=a_B(\ A\xrightarrow{f}B\ ,\ P_f(\beta)\wedge\alpha)$$

and conclude that

$$\exists_f(P_f(\beta)\wedge\alpha)=\beta\wedge\exists_f(\alpha).$$

Therefore the primary doctrine $P\colon C^{\mathrm{op}}\longrightarrow\mathbf{InfSL}$ is existential. Finally we can observe that

$$a_A(\ C\xrightarrow{g}A\ ,\alpha)=a_A\exists_g^{\mathrm{ex}}(\ C\xrightarrow{\mathrm{id}_C}C\ ,\ \alpha)=a_A\exists_g^{\mathrm{ex}}\iota_C(\alpha)=\exists_g(\alpha).$$

□

Proposition 5.5.5. *Let (P,ε_P) and (R,ε_R) be two T_e-algebras. If $(F,b)\colon(P,\varepsilon_P)\longrightarrow(R,\varepsilon_R)$ is a morphism of T_e-algebras, then (F,b) is a 1-cell of* **ExD**. *Moreover every 1-cell of* **ExD** *induces a morphism of T_e algebras.*

Proof. By definition of morphism of T_e-algebras, the following diagram commutes

$$\begin{tikzcd}
P^{\mathrm{ex}} \arrow[r, "(F{,}b^{\mathrm{ex}})"] \arrow[d, "\varepsilon_P"'] & R^{\mathrm{ex}} \arrow[d, "\varepsilon_R"] \\
P \arrow[r, "(F{,}b)"'] & R
\end{tikzcd}$$

then for every object $(\ C\xrightarrow{g}A\ ,\ \alpha\in PC)$ of $P^{\mathrm{ex}}A$ we have

$$\exists^R_{Fg}b_C(\alpha)=b_A\exists^P_g(\alpha)$$

and this means that for every projection $g\colon C\longrightarrow A$ the following diagram commutes

$$\begin{tikzcd} PC \arrow[r, "\exists^P_g"] \arrow[d, "b_C"'] & PA \arrow[d, "b_A"] \\ RFC \arrow[r, "\exists^R_{Fg}"'] & RFA. \end{tikzcd}$$

We can prove the converse using the same arguments. □

Corollary 5.5.6. *We have the following isomorphism of 2-categories*

$$\mathrm{T_e}\text{-}\mathbf{Alg} \cong \mathbf{ExD}\,.$$

Proof. It follows from Proposition 5.5.5 and Proposition 5.5.4. □

Proposition 5.5.7. *Let* $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *be a primary doctrine, and let* $(P,(F,a))$ *be a pseudo-*T_e*-algebra. Then* $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *is existential.*

Proof. Let $(P,(F,a))$ be a pseudo-algebra, then there exists an invertible 2-cell

$$\begin{tikzcd} P \arrow[r, "\eta_A"] \arrow[dr, "\mathrm{id}_P"'] \arrow[dr, phantom, bend left, "\Downarrow a_\eta"] & P^{\mathrm{ex}} \arrow[d, "(F,a)"] \\ & P \end{tikzcd}$$

and by definition, it is a natural transformation $a_\eta\colon F \longrightarrow \mathrm{id}_{\mathcal{C}}$, and for every $A \in \mathcal{C}$ and $\alpha \in PA$ we have $a_A\iota_A(\alpha) = P_{a_{\eta_A}}(\alpha)$.

Now consider a morphism $f\colon A \longrightarrow B$ in $\mathcal{C}$ and $\alpha \in PA$. We define

$$\exists_f(\alpha) := P_{{a_{\eta_A}}^{-1}} a_B \exists^{\mathrm{ex}}_f \iota_A(\alpha).$$

Using the same argument of Proposition 5.5.4 we can conclude that the elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is existential. □

Proposition 5.5.8. *The family* $\lambda_P\colon \mathrm{id}_{P^{\mathrm{ex}\,\mathrm{ex}}} \Longrightarrow \eta_{P^{\mathrm{ex}}}\mu_P$ *defined as* $\lambda_P := \mathrm{id}_{\mathcal{C}}$ *is a 2-cell in* $\mathbf{ExD}$.

Proof. It is clearly a natural transformation. We must check that for every $\alpha \in (P^{\mathrm{ex}})^{\mathrm{ex}}A$

$$\alpha \le \iota_{P^{\mathrm{ex}}A}\zeta_{P^{\mathrm{ex}}A}(\alpha).$$

Let $\alpha := (\ C \xrightarrow{g} A\ ,\ (\ D \xrightarrow{f} C\ ,\ \beta \in PD))$. Then we have

$$\iota_{P^{\mathrm{ex}}A}\zeta_{P^{\mathrm{ex}}A}(\alpha) = \iota_{P^{\mathrm{ex}}A}(\ D \xrightarrow{gf} A\ ,\ \beta \in PD) = (\ A \xrightarrow{\mathrm{id}_A} A\ ,(\ D \xrightarrow{gf} A\ ,\ \beta \in PD)).$$

Now we want to prove that

$$P_g^{\mathrm{ex}}(\ D \xrightarrow{gf} A\ ,\ \beta \in PD) \geq (\ D \xrightarrow{f} C\ ,\ \beta \in PD).$$

To see this inequality we can observe that the following diagram commutes

$$\begin{tikzcd}
D_2 \arrow[dr, dashed, "w"] \arrow[drrr, bend left, "f"] \arrow[ddr, bend right, "\mathrm{id}_D"'] & & & \\
& L \arrow[r, "m_2"] \arrow[d, "m_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & H \arrow[r, "h_2"] \arrow[d, "h_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "g"] \\
& D \arrow[r, "f"'] & C \arrow[r, "g"'] & A
\end{tikzcd}$$

since every square is a pullback, hence $P_w(P_{m_1}(\beta)) = \beta$. □

Corollary 5.5.9. *The 2-cell* $\lambda\colon \mathrm{id}_{T_e^2} \longrightarrow \eta T_e \mu$ *is a modification.*

Theorem 5.5.10. *The 2-cell μ is left adjoint to $\eta\mathrm{T}_e$, where the unit of the adjunction is λ and the counit is the identity.*

Proof. It follows from the fact that for every $P\colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$, the first component of the 1-cells μ_P, $\eta\mathrm{T}_e$ are the identity functor, and since λ_P is the identity natural transformation, when we look at the conditions of adjoint 1-cell in the 2-category **Cat**, we can observe that all the components are identities. □

Corollary 5.5.11. *The 2-monad* $\mathrm{T}_e\colon \mathbf{PD} \longrightarrow \mathbf{PD}$ *is lax-idempotent.*

Proof. It is a direct consequence of [27, Theorem 6.2] and Theorem 5.5.10 □

Observe that we can prove that the 2-monad T_e is lax-idempotent directly.

Proposition 5.5.12. *Let (P, ε_P) and (R, ε_R) be T_e algebras, and let* $(F, b)\colon P \longrightarrow R$ *be a 1-cell of* **PD**. *Then $((F, b), \mathrm{id}_F)$ is lax-morphism of algebras, and* $\mathrm{id}_F : \varepsilon_R(F, b^{\mathrm{ex}}) \Longrightarrow (F, b)\varepsilon_P$ *is the unique 2-cell making $(\mathrm{id}_F, (F, b))$ a lax-morphism.*

Proof. Consider the following diagram

$$\begin{tikzcd}
P^{\mathrm{ex}} \arrow[r, "{(F,b^{\mathrm{ex}})}"] \arrow[d, "\varepsilon_P"'] \arrow[dr, phantom, "\Downarrow \mathrm{id}_F"] & R^{\mathrm{ex}} \arrow[d, "\varepsilon_R"] \\
P \arrow[r, "{(F,b)}"'] & R.
\end{tikzcd}$$

We must prove that for every object A of $\mathcal{C}$ and every ($C \xrightarrow{f} A$, α) in $P^{\mathrm{ex}}A$

$$\exists^R_{Ff} b_C(\alpha) \leq b_A \exists^P_f(\alpha)$$

but the previous property holds if and only if

$$b_C(\alpha) \leq R_{Ff} b_A \exists^P_f(\alpha) = b_C P_f \exists^P_f(\alpha)$$

and this holds since $\alpha \leq P_f \exists^P_f(\alpha)$.

Finally it is easy to see that $\mathrm{id}_F : \varepsilon_R(F, b^{\mathrm{ex}}) \Longrightarrow (F, b)\varepsilon_P$ satisfies the coherence conditions for lax-T_e-morphisms.

Now suppose there exists another 2-cell $\theta\colon \varepsilon_R(F, b^{\mathrm{ex}}) \Longrightarrow (F, b)\varepsilon_P$ such that $((F, b), \theta)$ is a lax-morphism

$$\begin{tikzcd}
P^{\mathrm{ex}} \arrow[r, "{(F,b^{\mathrm{ex}})}"] \arrow[d, "\varepsilon_P"'] \arrow[dr, phantom, "\Downarrow \theta"] & R^{\mathrm{ex}} \arrow[d, "\varepsilon_R"] \\
P \arrow[r, "{(F,b)}"'] & R.
\end{tikzcd}$$

Then it must satisfy the following condition

$$\begin{tikzcd}
P \arrow[r, "{(F,b)}"] \arrow[d, "\eta_A"'] & R \arrow[d, "\eta_B"] \\
P^{\mathrm{ex}} \arrow[r, "{(F,b^{\mathrm{ex}})}"] \arrow[d, "\varepsilon_P"'] \arrow[dr, phantom, "\Downarrow \theta"] & R^{\mathrm{ex}} \arrow[d, "\varepsilon_R"] \\
P \arrow[r, "{(F,b)}"'] & R
\end{tikzcd}
=
\begin{tikzcd}
P \arrow[r, "{(F,b)}"] \arrow[dd, "1_P"'] & R \arrow[dd, "1_B"] \\
& \\
P \arrow[r, "{(F,b)}"'] & R
\end{tikzcd}$$

and this means that $\theta = \mathrm{id}_F$. $\square$

5.6 Exact completion for elementary doctrine

It is proved in [44] that there is a biadjunction between the categories **EED** → **Xct** given by the composition of the following 2-functors: the first is the left biadjoint to the inclusion of **Ex-mVar** into **EED**, see [44, Theorem 3.1]. The second is the biequivalence between **Ex-mVar** and the 2-category **LFS** of categories with finite limits and a proper stable factorization system, see [19]. The third is provided in [26], where it is proved that the inclusion of the 2-category **Reg** of regular categories (with exact functors) into **LFS** has a left biadjoint. The last functor is the biadjoint to the forgetful functor from the 2-category **Xct** into **Reg**, see [10].

In this section we combine these results with the existential completion for elementary doctrine, proving that the completion presented in Section 5.4 preserves the elementary structure, in the sense that if $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is an elementary doctrine, then $P^{\mathrm{ex}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is an elementary existential doctrine.

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be an elementary doctrine, and consider its existential completion $P^{\mathrm{ex}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$. Given two objects A and C of $\mathcal{C}$ we define

$$\exists^{\mathrm{ex}}_{\Delta_A\times \mathrm{id}_C}\colon P^{\mathrm{ex}}(A\times C) \longrightarrow P^{\mathrm{ex}}(A\times A\times C)$$

on $\alpha := (\ A\times C\times D \xrightarrow{\mathrm{pr}} A\times C\ ,\ \alpha\in P(A\times C\times D))$ as

$$\exists^{\mathrm{ex}}_{\Delta_A\times \mathrm{id}_C}(\alpha) := (\ A\times A\times C\times D \xrightarrow{\mathrm{pr}} A\times A\times C\ ,\ \exists_{\Delta_A\times \mathrm{id}_{C\times D}}(\alpha)\in P(A\times A\times C\times D)).$$

Remark 5.6.1. We can prove that $\exists^{\mathrm{ex}}_{\Delta_A\times \mathrm{id}_C}$ is a well defined functor for every A and C: consider two elements of $P^{\mathrm{ex}}(A\times C)$

$$\overline{\alpha} := (\ A\times C\times D \xrightarrow{\mathrm{pr}} A\times C\ ,\ \alpha\in P(A\times C\times D))$$

and

$$\overline{\beta} = (\ A\times C\times B \xrightarrow{\mathrm{pr}'} A\times C\ ,\ \beta\in P(A\times C\times B))$$

and suppose that $\overline{\alpha}\le\overline{\beta}$. By definition there exists a morphism $f\colon A\times C\times D \longrightarrow B$ such that the following diagram commutes

$$\begin{tikzcd} & A \times C \times D \arrow[dl, "\langle \mathrm{pr}_{A\times C}, f\rangle"'] \arrow[d, "\mathrm{pr}_{A\times C}"] \\ A \times C \times B \arrow[r, "\mathrm{pr}'_{A\times C}"'] & A \times C \end{tikzcd}$$

and $P_{\langle \mathrm{pr}_{A\times C}, f\rangle}(\beta) \geq \alpha$. Since the doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is elementary we have

$$\beta \leq P_{\Delta_A \times \mathrm{id}_{C\times B}} \exists_{\Delta_A \times \mathrm{id}_{C\times B}}(\beta)$$

and then

$$\alpha \leq P_{\langle \mathrm{pr}_{A\times C}, f\rangle}(P_{\Delta_A \times \mathrm{id}_{C\times B}} \exists_{\Delta_A \times \mathrm{id}_{C\times B}}(\beta)).$$

Now observe that $(\Delta_A \times \mathrm{id}_{C\times B})(\langle \mathrm{pr}_{A\times C}, f\rangle) = (\langle \mathrm{pr}_{A\times A\times C}, f\, \mathrm{pr}_{A\times C\times D}\rangle)(\Delta_A \times \mathrm{id}_{C\times D})$, and this implies

$$\alpha \leq P_{\Delta_A \times \mathrm{id}_{C\times D}}(P_{\langle \mathrm{pr}_{A\times A\times C}, f\, \mathrm{pr}_{A\times C\times D}\rangle} \exists_{\Delta_A \times \mathrm{id}_{C\times B}}(\beta)).$$

Therefore we conclude

$$\exists_{\Delta_A \times \mathrm{id}_{C\times D}}(\alpha) \leq P_{\langle \mathrm{pr}_{A\times A\times C}, f\, \mathrm{pr}_{A\times C\times D}\rangle} \exists_{\Delta_A \times \mathrm{id}_{C\times B}}(\beta).$$

It is easy to observe that the last inequality implies

$$\exists^{\mathrm{ex}}_{\Delta_A \times \mathrm{id}_C}(\overline{\alpha}) \leq \exists^{\mathrm{ex}}_{\Delta_A \times \mathrm{id}_C}(\overline{\beta}).$$

Proposition 5.6.2. *With the notation used before the functor*

$$\exists^{\mathrm{ex}}_{\Delta_A \times \mathrm{id}_C}\colon P^{\mathrm{ex}}(A \times C) \longrightarrow P^{\mathrm{ex}}(A \times A \times C)$$

is left adjoint to the functor

$$P^{\mathrm{ex}}_{\Delta_A \times \mathrm{id}_C}\colon P^{\mathrm{ex}}(A \times A \times C) \longrightarrow P^{\mathrm{ex}}(A \times C).$$

Proof. Consider an element $\overline{\alpha} \in P^{\mathrm{ex}}(A \times C)$,

$$\overline{\alpha} := (\ A \times C \times B \xrightarrow{\mathrm{pr}} A \times C\ ,\ \alpha \in P(A \times C \times B))$$

and an element $\overline{\beta} \in P^{\mathrm{ex}}(A \times A \times C)$,

$$\overline{\beta} := (\ A \times A \times C \times D \xrightarrow{\mathrm{pr}'} A \times A \times C\ ,\ \beta \in P(A \times A \times C \times D))$$

and suppose that

$$\exists^{\mathrm{ex}}_{\Delta_A \times \mathrm{id}_C}(\overline{\alpha}) \leq \overline{\beta}$$

which means that there exists $f\colon A \times A \times C \times B \longrightarrow D$

$$\begin{tikzcd}
 & A\times A\times C\times B \arrow[dl, "\langle \mathrm{pr}_{A\times A\times C}, f\rangle"'] \arrow[d, "\mathrm{pr}_{A\times A\times C}"] \\
A\times A\times C\times D \arrow[r, "\mathrm{pr}_{A\times A\times C}"'] & A\times A\times C
\end{tikzcd}$$

such that $\exists_{\Delta_A\times \mathrm{id}_{C\times B}}(\alpha) \leq P_{\langle \mathrm{pr}_{A\times A\times C}, f\rangle}(\beta)$. Therefore we have

$$\alpha \leq P_{\Delta_A\times \mathrm{id}_{C\times B}} P_{\langle \mathrm{pr}_{A\times A\times C}, f\rangle}(\beta)$$

and since

$$(\langle \mathrm{pr}_{A\times A\times C}, f\rangle)(\Delta_A\times \mathrm{id}_{C\times B}) = (\Delta_A\times \mathrm{id}_{C\times D})\, \mathrm{pr}_{A\times C\times D}(\langle \mathrm{pr}_{A\times A\times C}, f\rangle)(\Delta_A\times \mathrm{id}_{C\times B})$$

we can conclude that

$$\alpha \leq P_{\mathrm{pr}_{A\times C\times D}(\langle \mathrm{pr}_{A\times A\times C}, f\rangle)(\Delta_A\times \mathrm{id}_{C\times B})}(P_{\Delta_A\times \mathrm{id}_{C\times D}}(\beta)).$$

Then

$$\overline{\alpha} \leq P^{\mathrm{ex}}_{\Delta_A\times \mathrm{id}_C}(\overline{\beta})$$

because

$$P^{\mathrm{ex}}_{\Delta_A\times \mathrm{id}_C}(\overline{\beta}) = (\ A\times C\times D \xrightarrow{\mathrm{pr}_{A\times C}} A\times C\ ,\ P_{\Delta_A\times \mathrm{id}_{C\times D}}(\beta))$$

In the same way we can prove that $\overline{\alpha} \leq P^{\mathrm{ex}}_{\Delta_A\times \mathrm{id}_C}(\overline{\beta})$ implies $\exists^{\mathrm{ex}}_{\Delta_A\times \mathrm{id}_C}(\overline{\alpha}) \leq \overline{\beta}$. $\square$

Proposition 5.6.3. *For every A and C in $\mathcal{C}$, $\exists^{\mathrm{ex}}_{\Delta_A\times \mathrm{id}_C}$ satisfies the Frobenius condition.*

Proof. Consider $\overline{\alpha} \in P^{\mathrm{ex}}(A\times A\times C)$,

$$\overline{\alpha} := (\ A\times A\times C\times D \xrightarrow{\mathrm{pr}_{A\times A\times C}} A\times A\times C\ ,\ \alpha \in P(A\times A\times C\times D))$$

and $\overline{\beta} \in P^{\mathrm{ex}}(A\times C)$,

$$\overline{\beta} := (\ A\times C\times B \xrightarrow{\mathrm{pr}_{A\times C}} A\times C\ ,\ \beta \in P(A\times C\times B)).$$

We can observe that

$$P^{\mathrm{ex}}_{\Delta_A\times \mathrm{id}_C}(\overline{\alpha}) = (\ A\times C\times D \xrightarrow{\mathrm{pr}_{A\times C}} A\times C\ ,\ P_{\Delta_A\times \mathrm{id}_{C\times D}}(\alpha))$$

and

$$P^{\mathrm{ex}}_{\Delta_A\times \mathrm{id}_C}(\overline{\alpha})\wedge\beta = (\ A\times C\times D\times B \xrightarrow{\mathrm{pr}_{A\times C}} A\times C\ ,\ P_{\langle \mathrm{pr}_A, \mathrm{pr}_C, \mathrm{pr}_D\rangle} P_{\Delta_A\times \mathrm{id}_{C\times D}}(\alpha)\wedge P_{\langle \mathrm{pr}_A, \mathrm{pr}_C, \mathrm{pr}_B\rangle}(\beta)).$$

Moreover we can observe that $(\Delta_A \times \mathrm{id}_{C\times D})\langle \mathrm{pr}_A, \mathrm{pr}_C, \mathrm{pr}_D\rangle = \mathrm{pr}_{A\times A\times C\times D}(\Delta_A \times \mathrm{id}_{C\times D\times B})$. Therefore we have

$$\exists^{\mathrm{ex}}_{\Delta_A\times \mathrm{id}_C}(P^{\mathrm{ex}}_{\Delta_A\times \mathrm{id}_C}(\overline{\alpha}) \wedge \overline{\beta})$$

is equal to

$$(\ A\times A\times C\times D\times B \xrightarrow{\mathrm{pr}} A\times A\times C\ ,\ \exists_{\Delta_A\times \mathrm{id}_{C\times D\times B}}(P_{(\Delta_A\times \mathrm{id}_{C\times D})\langle \mathrm{pr}_A,\mathrm{pr}_C,\mathrm{pr}_D\rangle}(\alpha)\wedge P_{\langle \mathrm{pr}_A,\mathrm{pr}_C,\mathrm{pr}_B\rangle}(\beta))).$$

Now we can observe that

$$\exists_{\Delta_A\times \mathrm{id}_{C\times D\times B}}(P_{(\Delta_A\times \mathrm{id}_{C\times D})\langle \mathrm{pr}_A,\mathrm{pr}_C,\mathrm{pr}_D\rangle}(\alpha) \wedge P_{\langle \mathrm{pr}_A,\mathrm{pr}_C,\mathrm{pr}_B\rangle}(\beta))$$

is by definition

$$\exists_{\Delta_A\times \mathrm{id}_{C\times D\times B}}(P_{\Delta_A\times \mathrm{id}_{C\times D\times B}}P_{\mathrm{pr}_{A\times A\times C\times D}}(\alpha) \wedge P_{\langle \mathrm{pr}_A,\mathrm{pr}_C,\mathrm{pr}_B\rangle}(\beta)).$$

which, in turn, is equal to

$$P_{\mathrm{pr}_{A\times A\times C\times D}}(\alpha) \wedge \exists_{\Delta_A\times \mathrm{id}_{C\times D\times B}}P_{\langle \mathrm{pr}_A,\mathrm{pr}_C,\mathrm{pr}_B\rangle}(\beta)$$

since the Frobenius Reciprocity holds for $\exists$ in the elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is elementary. Thus

$$\exists^{\mathrm{ex}}_{\Delta_A\times \mathrm{id}_C}(P^{\mathrm{ex}}_{\Delta_A\times \mathrm{id}_C}(\overline{\alpha}) \wedge \overline{\beta})$$

is equal to

$$(\ A\times A\times C\times D\times B \xrightarrow{\mathrm{pr}_{A\times A\times C}} A\times A\times C\ ,\ P_{\mathrm{pr}_{A\times A\times C\times D}}(\alpha)\wedge\exists_{\Delta_A\times \mathrm{id}_{C\times D\times B}}P_{\langle \mathrm{pr}_A,\mathrm{pr}_C,\mathrm{pr}_B\rangle}(\beta)).$$

Now we look for $\overline{\alpha} \wedge \exists^{\mathrm{ex}}_{\Delta_A\times \mathrm{id}_C}(\overline{\beta})$. It is straightforward to prove that the previous is equal to

$$(\ A\times A\times C\times D\times B \xrightarrow{\mathrm{pr}_{A\times A\times C}} A\times A\times C\ ,\ P_{\mathrm{pr}_{A\times A\times C\times D}}(\alpha)\wedge P_{\langle \mathrm{pr}_A,\mathrm{pr}_A,\mathrm{pr}_C,\mathrm{pr}_B\rangle}\exists_{\Delta_A\times \mathrm{id}_{C\times B}}(\beta)).$$

Since $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is elementary we know that

$$\exists_{\Delta_A\times \mathrm{id}_{C\times B}}(\beta) = P_{\langle \mathrm{pr}'_A,\mathrm{pr}_C,\mathrm{pr}_B\rangle}(\beta) \wedge P_{\langle \mathrm{pr}_A,\mathrm{pr}'_A\rangle}(\delta_A)$$

where $\mathrm{pr}'_A\colon A\times A\times C\times B \longrightarrow A$ is the projection on the second component. By a direct computation we have

$$P_{\langle \mathrm{pr}_A,\mathrm{pr}_A,\mathrm{pr}_C,\mathrm{pr}_B\rangle}(P_{\langle \mathrm{pr}'_A,\mathrm{pr}_C,\mathrm{pr}_B\rangle}(\beta)\wedge P_{\langle \mathrm{pr}_A,\mathrm{pr}'_A\rangle}(\delta_A)) = P_{\langle \mathrm{pr}'_A,\mathrm{pr}_C,\mathrm{pr}_B\rangle}(\beta)\wedge P_{\langle \mathrm{pr}_A,\mathrm{pr}'_A\rangle}(\delta_A)$$

and

$$\exists_{\Delta_A\times \mathrm{id}_{C\times D\times B}}(P_{\langle \mathrm{pr}_A,\mathrm{pr}_C,\mathrm{pr}_B\rangle}(\beta)) = P_{\langle \mathrm{pr}'_A,\mathrm{pr}_{C\times D\times B}\rangle}(P_{\langle \mathrm{pr}_A,\mathrm{pr}_C,\mathrm{pr}_B\rangle}(\beta))\wedge P_{\langle \mathrm{pr}_A,\mathrm{pr}'_A\rangle}(\delta_A)).$$

It is easy to see that

$$P_{\langle \mathrm{pr}'_A, \mathrm{pr}_{C\times D\times B}\rangle}(P_{\langle \mathrm{pr}_A, \mathrm{pr}_C, \mathrm{pr}_B\rangle}(\beta)) \wedge P_{\langle \mathrm{pr}_A, \mathrm{pr}'_A\rangle}(\delta_A)) = P_{\langle \mathrm{pr}'_A, \mathrm{pr}_C, \mathrm{pr}_B\rangle}(\beta) \wedge P_{\langle \mathrm{pr}_A, \mathrm{pr}'_A\rangle}(\delta_A).$$

Therefore the Frobenius condition is satisfied. □

Corollary 5.6.4. *For every elementary doctrine* $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *, the doctrine* $P^{\mathrm{ex}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *is elementary and existential.*

We combine the existential completion for elementary doctrines with the completions stated at the begin of this section, obtaining a general version of the exact completion described in [41, 44]. We can summarise this operation with the following diagram

$$\mathbf{ElD} \longrightarrow \mathbf{EED} \longrightarrow \mathbf{Ex\text{-}mVar} \longrightarrow \mathbf{LFS} \longrightarrow \mathbf{Reg} \longrightarrow \mathbf{Xct}\,.$$

It is proved in [41, 42, 43] that given an elementary existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ the completion $\mathbf{EED} \to \mathbf{Xct}$ produces an exact category denoted by $\mathbb{T}_P$ and this category is defined following the same idea used to define a topos from a tripos. See [20, 51].

We conclude giving a complete description of the exact category $\mathbb{T}_{P^{\mathrm{ex}}}$ obtained from an elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$.

Given an elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$, consider the category $\mathbb{T}_{P^{\mathrm{ex}}}$, called ***exact completion of the elementary doctrine*** P, whose

objects are pair (A,ρ) such that ρ is in $P(A \times A \times C)$ for some C and satisfies:

1. there exists a morphism $f\colon A \times A \times C \longrightarrow C$ such that

$$\rho \leq P_{\langle \mathrm{pr}_2, \mathrm{pr}_1, f\rangle}(\rho)$$

   in $P(A \times A \times C)$ where $\mathrm{pr}_1, \mathrm{pr}_2\colon A \times A \times C \longrightarrow A$;
2. there exists a morphism $g\colon A \times A \times A \times C \longrightarrow C$ such that

$$P_{\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_4\rangle}(\rho) \wedge P_{\langle \mathrm{pr}_2, \mathrm{pr}_3, \mathrm{pr}_4\rangle}(\rho) \leq P_{\langle \mathrm{pr}_1, \mathrm{pr}_3, g\rangle}(\rho)$$

   where $\mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_3\colon A \times A \times A \times C \longrightarrow A$;

a morphism $\phi\colon (A,\rho) \longrightarrow (B,\sigma)$, where $\rho \in P(A \times A \times C)$ and $\sigma \in P(B \times B \times D)$, is an object ϕ of $P(A \times B \times E)$ for some E such that

1. there exists a morphism $\langle f_1, f_2\rangle\colon A \times B \times E \longrightarrow C \times D$ such that

$$\phi \leq P_{\langle \mathrm{pr}_1, \mathrm{pr}_1, f_1\rangle}(\rho) \wedge P_{\langle \mathrm{pr}_2, \mathrm{pr}_2, f_2\rangle}(\sigma)$$

   where the pr_i's are the projections from $A \times B \times E$;

2. there exists a morphism $h\colon A\times A\times B\times C\times E \longrightarrow E$ such that

$$P_{\langle \mathrm{pr}_1,\mathrm{pr}_2,\mathrm{pr}_4\rangle}(\rho)\wedge P_{\langle \mathrm{pr}_2,\mathrm{pr}_3,\mathrm{pr}_5\rangle}(\phi)\leq P_{\langle \mathrm{pr}_1,\mathrm{pr}_3,h\rangle}(\phi)$$

   where the pr_i's are the projections from $A\times A\times B\times C\times E$;
3. there exists a morphism $k\colon A\times B\times B\times D\times E \longrightarrow E$ such that

$$P_{\langle \mathrm{pr}_2,\mathrm{pr}_3,\mathrm{pr}_4\rangle}(\sigma)\wedge P_{\langle \mathrm{pr}_1,\mathrm{pr}_2,\mathrm{pr}_5\rangle}(\phi)\leq P_{\langle \mathrm{pr}_1,\mathrm{pr}_3,k\rangle}(\phi)$$

   where the pr_i's are the projections from $A\times B\times B\times D\times E$;
4. there exists a morphism $l\colon A\times B\times B\times E \longrightarrow D$ such that

$$P_{\langle \mathrm{pr}_1,\mathrm{pr}_2,\mathrm{pr}_4\rangle}(\phi)\wedge P_{\langle \mathrm{pr}_1,\mathrm{pr}_3,\mathrm{pr}_4\rangle}(\phi)\leq P_{\langle \mathrm{pr}_2,\mathrm{pr}_3,l\rangle}(\sigma)$$

   where the pr_i's are the projections from $A\times B\times B\times E$;
5. there exists a morphism $\langle g_1,g_2\rangle\colon A\times C \longrightarrow B\times E$ such that

$$P_{\langle \mathrm{pr}_1,\mathrm{pr}_1,\mathrm{pr}_2\rangle}(\rho)\leq P_{\langle \mathrm{pr}_1,g_1,g_2\rangle}(\phi)$$

   where the pr_i's are the projections from $A\times C$.

The composition of two morphisms is defined following the same structure of the tripos to topos.

Therefore we conclude with the following theorem which generalized the exact completion for an elementary existential doctrine to an arbitrary elementary doctrine.

Theorem 5.6.5. *The 2-functor* $\mathbf{Xct}\to\mathbf{ExD}$ *that takes an exact category to the elementary doctrine of its subobjects has a left biadjoint which associates the exact category* $\mathbb{T}_{P^{\mathrm{ex}}}$ *to an elementary doctrine* $P\colon\mathcal{C}^{\mathrm{op}}\longrightarrow\mathbf{InfSL}$.

Chapter 6
Unification in the Syntactic Category and Elementary Completion

Abstract We present the elementary completion for a primary doctrine whose base category has finite limits. In particular we prove that, using a general results about unification for first order languages, we can easily add finite limits to a syntactic category, and then apply the elementary completion for syntactic doctrines. We conclude with a complete description of elementary completion for primary doctrine whose base category is the free product completion of a discrete category, and we show that the 2-monad constructed from the 2-adjunction is lax-idempotent.

6.1 Introduction

The topic of completing a given structure with quotient to get a richer one and in particular the exact completion has been widely employed in category theory and logic, see [21, 6, 8].

In particular one of the main relevant free construction discussed by Carboni in [6] is the exact completion of a left exact category, and in the recent works [44, 42], Maietti and Rosolini generalized this notion by relativizing the basic data to a doctrine equipped with just the structure sufficient to present the notion of equivalence relation. The exact completion of a regular category $\mathcal{R}$ is the exact completion of the doctrine of subobjects on $\mathcal{R}$. The exact completion of a category with finite limits C is the exact completion of the doctrine of week subobjects on C.

The exact completion of an elementary existential doctrine can be seen a generalization to the tripos-to-topos construction of Hyland, Johnstone and Pitts, see [20, 51]. In [57] we present the existential completion of a primary doctrine, and we show that this construction preserves the elementary structure of a doctrine. This allows to generalize the exact completion for an arbitrary elementary doctrine, and a general version of tripos-to-topos is presented.

In this work we analyse the elementary completion, and we show that the construction presented in [57] can be generalized and applying to obtain the elementary completion for every primary doctrine $P\colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ whose base catand applaying toegory C has finite limits. The key point of the existential completion is that we add left adjoint to the class of the projections, but what is really necessary is the fact that this class is closed for pullbacks, compositions, and it contains units. Therefore given a doctrine $P\colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ and a class $\mathfrak{a}$ of morphisms with these properties, we can generalized the existential completion adding left adjoint obtaining a doctrine $P^{\mathfrak{a}}\colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ such that all the functor of the form $P^{\mathfrak{a}}_f$ for $f \in \mathfrak{a}$.

An interesting example of primary doctrine on which this construction can be applied is the syntactic primary doctrine, in the sense that we are considering the doctrine $\mathcal{L}\colon C_{\mathbb{T}}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ associated to a first order theory $\mathbb{T}$. Syntactic doctrine and syntactic categories appears in many works in categorical logic, see [42, 43] for the case of syntactic doctrine, [21, 24] for a general description of syntactic categories, and for the case of syntactic hyperdoctrine see [36, 37, 38, 50].

A syntactic category $C_{\mathbb{T}}$ has an interesting property coming from the underlying logic which allows the elementary completion. It is known that in a first order language if two formulas admit a unifier then there exists a most general one, and it is essentially unique, see [48, 52]. This fact implies that in the syntactic category associated to a first order language, if two morphisms have a morphism which equalizes them, then there exists an equalizer for such pair of arrows. Therefore we show that syntactic category $C_{\mathbb{T}}$ can be easily completed to a category $C_{\mathbb{T}}^0$ with finite limits, simply adding an initial object.

Using this property we can complete a primary doctrine $\mathcal{L}\colon C_{\mathbb{T}}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ to a primary doctrine $\mathcal{L}^0\colon (C_{\mathbb{T}}^0)^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ where the base category $C_{\mathbb{T}}^0$ is obtaining from $C_{\mathbb{T}}$ adding an initial object, and $\mathcal{L}^0$ is the natural extension of the functor $\mathcal{L}$ on $C_{\mathbb{T}}^0$. Then in the new doctrine $\mathcal{L}^0\colon (C_{\mathbb{T}}^0)^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ we can consider the closure for pullbacks, compositions, and identities of the class of morphisms of the form $\mathrm{id}_A \times \Delta_X$, and we denote it by $\mathfrak{a}_{\mathrm{el}}$. Now we are in the condition to apply the general existential completion on the class $\mathfrak{a}_{\mathrm{el}}$, obtaining an elementary doctrine.

We combine this results with the exact completion for primary doctrine proved in [57], and we show that every primary doctrine $\mathcal{L}\colon C_{\mathbb{T}}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ can be completed to an exact category. See also [45] for the construction of an exact category starting from a first order theory.

We conclude this work with a complete description of the elementary completion for a primary doctrine whose base category is discrete with free product, and we analyse the 2-monad obtained from the completion, proving that it is property-like in the sense of [27]. We conclude with some considerations on the pseudo-distributive law which can be constructed between the pseudo-monads obtained from various completions.

In the first section we recall the unification problem, and we explain how it can be translated in categorical terms, in particular we shows that most general unifier means equalizer in the syntactic category of a first order language. We conclude the section proving that adding an initial object to a syntactic category $C_{\mathbb{T}}$ we obtain a category with finite limits.

In the second section we introduce the notion of primary, elementary and existential doctrine following the notation of Maietti and Rosolini in [42, 43, 44].

In section 3 we present the general version of the existential completion defined in [57], and we prove that every syntactic primary doctrine can be completed to an exact category.

The last section is dedicated to an explicit description of the elementary completion for a primary doctrine whose base category is discrete with free products and to the study of the 2-monads constructed from the completion.

6.2 Unification in the syntactic category

The unification problem was introduced by J. A. Robinson, see [52], and in the first order logic can be expressed as follows: given two terms find, if it exists, the simplest substitution which makes the two terms equal. Such a substitution is called ***most general unifier***, and it is unique up to variable renaming.

In this section we introduce the problem of unification following Martelli and Montanari, see [48], and we explain how it can be stated in categorical terms using syntactic categories. In particular we see that the notion of most general unifier corresponds to a particular equalizer. For all the details about syntactic category we refer to [20, 43, 50, 51].

The problem of unification can be considered in the general context of equational theories, and in this case it is not required that the two terms coincide syntactically, but they are provably equal in the given equational theory. In this context the problem is called ***E-unification***, and the unifiers are called ***E-unifiers***.

An important difference between unification and E-unification is that in the first case is proved that if an unifier exists then there exists a most general unifier, see [48], while in the second case this would not hold.

There are some known example of equational theory which admits E-unifiers, but not a most general one. For a complete description of E-unification problem we refer to [18].

Let **Sg** be a one-sorted signature, consisting of a countably finite set of **Var** of ***variables*** and a ***ranked alphabet***

$$A = \bigcup_{i=0\dots} A_i$$

where A_i contains the i-adic function symbols and the elements of A_0 are called ***constant symbols***. The ***terms*** are defined recursively as usual:

- constant symbols and variables are terms;
- if $t_1, \dots, t_n$ are terms, $n \geq 1$, and $f \in A_n$ then $f(t_1, \dots, t_n)$ is a term.

We denote the set of terms as **Terms**.

A ***substitution*** is a function $\sigma\colon \mathbf{Var} \longrightarrow \mathbf{Terms}$ between the set of the variables into the set of terms, with at most a finite number of variables which are not fixed by σ. We represent a substitution as a list

$$\sigma := [t_1/x_1, \ldots, t_n/x_n]$$

where the variables x_i are distinct for $i = 1, \ldots, n$ and the variables which do not appear in the previous list are assumed to be fixed by the substitution. Sometimes we use the notation $\sigma = [\vec{t}/\vec{x}]$ when the length of the list is clear from the context.

The standard unification problem can be written as an equation

$$t' = t''$$

and a solution of this equation, if it exists, is a substitution σ making the the two terms identical. Such a substitution is called ***unifier*** of t' and t''. Moreover we can generalize the previous problem and consider a finite set of equations

$$t'_j = t''_j, \text{ for } j = 1, \ldots, m.$$

In this case a unifier is a substitution σ making all the terms identical simultaneously.

Recall from [48] two transformations which given a set S of equations, produce an equivalent set of equation S', where equivalent means that they have the same unifiers:

- ***Term Reduction***. Let
$$f(t_1, \ldots, t_m) = f(t'_1, \ldots, t'_n)$$
be an equation where both therms are not variables and where the two function symbols are identical. Then the new set of equations is obtained by replacing that equation with the following:
$$t_1 = t'_1, \ldots, t_n = t'_n$$
So in case $n = 0$ the equation is erased.
- ***Variable Elimination***. Let $x = t$ be an equation and x is a variable and t is any term. The new set of equations is obtained by applying the substitution $[t/x]$ to both terms of all other equations in the set (without erasing $x = t$).

Theorem 6.2.1. *Let S be a set of equations, and let $f(t_1, \ldots, t_n) = f'(t'_1, \ldots, t'_n)$ be an equation of S. If $f \neq f'$ then S has no unifier, otherwise the system of equations S' obtained applying Term Reduction is equivalent to S.*

Proof. See [48, Theorem 2.1]. □

Theorem 6.2.2. *Let S be a set of equations, and let $x = t$ be an equation of S. If the variable x occurs in t and t is not x then S has not unifier. Otherwise applying Variable Elimination we obtain a set of equations S' which is equivalent to the set S.*

Proof. See [48, Theorem 2.2]. □

A set of equations S is ***in solved form*** if it satisfies the following conditions:

- the equations are of the form $x_i = t_i$ for $i = 1, \ldots, n$;
- a variable which is the left member of some equation occurs only there.

Lemma 6.2.3. *Let S be a set of equations in solved form. Then it has a canonical solution:*

$$\sigma = [t_1/x_1, \ldots, t_n/x_n].$$

Every other unifier can be obtained as

$$[t_1[\vec{t'}/\vec{x'}]/x_1, \ldots, t_n[\vec{t'}/\vec{x'}]/x_n, \vec{t'}/\vec{x'}]$$

where the variables x'_j are all different from the variables of the form x_i.

Proof. See [18, Lemma 3.4]. □

The substitution σ in Lemma 6.2.3 is called ***most general unifier***.

Example 6.2.4. If we consider a set $S = \{x_1 = f_1(x_3, x_4), x_2 = f_2(x_4, x_5)\}$ then

$$\sigma = [f_1/x_1, f_2/x_2]$$

and

$$\sigma' = [f_1(x_3, f_3(x_4))/x_1, f_2(f_3(x_4), f_4(x_6))/x_2, f_3(x_4)/x_4, f_4(x_6)/x_5]$$

are solutions for S. We denote $\alpha = [f_3(x_4)/x_4, f_4(x_6)/x_5]$ and we observe that

$$\sigma' = [f_1\alpha/x_1, f_2\alpha/x_2, \alpha]$$

since σ is the most general unifier.

Now recall from [48] a non-deterministic algorithm which shows that every set of equations S can be transformed into an equivalent system of equations in solved form.

Given a set of equations S repeatedly perform the following transformation. If no transformation applies you can stop with success:

- select any equation of the form
$$t = x$$
where t is not a variable and rewrite it as
$$x = t$$
- select any equation of the form
$$x = x$$
and erase it;

- select any equation of the form
$$t = t'$$
where t and t' are not variables. If the root function symbols of the two terms are different then stop with failure, otherwise apply Term Reduction;
- select any equation of the form
$$x = t$$
and if x is a variable occurring in t then stop with failure, otherwise apply Variable Elimination.

Theorem 6.2.5. *Given a set S of equations the previous algorithm always terminates. If the algorithm terminates with failure, then S has no unifier. Otherwise the set S is transformed into an equivalent set in solved form.*

Proof. See [48, Theorem 2.3]. □

Consider now the ***syntactic category*** $\mathcal{C}_{\mathbf{Sg}}$ associated to a first order signature $\mathbf{Sg}$:

- ***objects***: the objects are finite lists of distinct variables $\vec{x} := (x_1, \ldots, x_n)$, and we include the empty list ();
- ***morphisms***: a morphism from $(x_1, \ldots, x_n)$ into $(y_1, \ldots, y_m)$ is a substitution
$$[t_1/y_1, \ldots, t_m/y_m]$$
where the terms t_i are built in $\mathbf{Sg}$ on the variable $x_1, \ldots, x_n$;
- ***composition***: consider two morphisms $[\vec{t}/\vec{y}]\colon \vec{x} \longrightarrow \vec{y}$ and $[\vec{s}/\vec{z}]\colon \vec{y} \longrightarrow \vec{z}$, then their composition is given by
$$[s_1[\vec{t}/\vec{y}]/z_k, \ldots, s_k[\vec{t}/\vec{y}]/z_k]\colon \vec{x} \longrightarrow \vec{z}\ .$$

The category $\mathcal{C}_{\mathbf{Sg}}$ has finite products, where the product of $\vec{x} \times \vec{y}$ is the list
$$(x_1, \ldots, x_n, y_1, \ldots, y_m)$$
as long as the variables are all distinct, see [43, 50] for more details.

Therefore given a set of equation $S = \{t_1 = s_1, \ldots, t_n = s_n\}$, it has a most general unifier if and only if the morphisms $[t_1/y_1, \ldots, t_n/y_n]$ and $[s_1/y_1, \ldots, s_n/y_n]$ have equalizer in the syntactic category corresponding to the signature.

This means that if the syntactic category of a signature is finitely complete then every finite set of equations has a most general unifier.

Proposition 6.2.6. *Let* $\mathbf{Sg}$ *be a first order signature. In the syntactic category* $\mathcal{C}_{\mathbf{Sg}}$ *given two morphisms* $f, g\colon B \longrightarrow C$ *if there is a morphism* h
$$\begin{tikzcd} A \arrow[r, "h"] & B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & C \end{tikzcd}$$

such that $fh = gh$, then f and g have an equalizer.

Proof. It is a direct consequence from the fact that if a finite set of terms equations have a unifier, then there exists a most general unifier. See Lemma 6.2.3. □

Theorem 6.2.7. *Let* **Sg** *be a first order signature, and let $\mathcal{C}_{\mathbf{Sg}}$ be its syntactic category. If $\mathcal{C}^0_{\mathbf{Sg}}$ is $\mathcal{C}_{\mathbf{Sg}}$ with the addition of an initial object, then it is finitely complete.*

Proof. Consider the diagram $B \overset{f}{\underset{g}{\rightrightarrows}} C$. If B is the initial object 0 then $f = g$ and $\mathrm{id}_B \colon B \longrightarrow B$ is the obvious equalizer. If B is not the initial object, there are two cases: if there exists a morphism $h \colon A \longrightarrow B$ of $\mathcal{C}_{\mathbf{Sg}}$ such that $fh = gh$, then by Proposition 6.2.6 there exists an equalizer in $\mathcal{C}_{\mathbf{Sg}}$, which is an equalizer in $\mathcal{C}^0_{\mathbf{Sg}}$. Otherwise there is no morphism of $\mathcal{C}_{\mathbf{Sg}}$ which equalizes the diagram, hence

$$0 \xrightarrow{\ !\ } B \overset{f}{\underset{g}{\rightrightarrows}} C$$

is an equalizer. □

Recall the general definition of E-unification, see [18] for further detail. Let

$$E \subseteq \mathbf{Terms} \times \mathbf{Terms}$$

be a set of pairs of terms, and let $=_E$ the smallest reflexive, symmetric and transitive binary relation containing E. A substitution σ is an ***E-unifier*** of the pair $(s,t) \in E$ if $(\sigma(t), \sigma(s)) \in =_E$. We will denote $(\sigma(t), \sigma(s)) \in =_E$ as $\sigma(t) =_E \sigma(s)$.

Observe that the problem of unification is a particular case of E-unification where $E = \emptyset$. If we want to translate the problem of E-unification in a syntactic category, we must require that $=_E$ is closed for substitutions and it is monotonic, which means that if $t =_E s$ then $\sigma(t) =_E \sigma(s)$ and $f(\dots, t, \dots) =_E f(\dots, s, \dots)$ for every function symbols.

We can construct a syntactic category denoted by $\mathcal{C}_E$ as done before, but in this case we identify two morphisms if all their components are E-provably equal: we say that two morphisms

$$\sigma = [t_1/y_1, \dots, t_m/y_m] \colon (x_1, \dots, x_n) \longrightarrow (y_1, \dots, y_m)$$

and

$$\sigma' = [s_1/y_1, \dots, s_m/y_m] \colon (x_1, \dots, x_n) \longrightarrow (y_1, \dots, y_m)$$

are ***E-provably equal***, and denoted by $\sigma =_E \sigma'$, if $t_i =_E s_i$ for $i = 1, \dots, m$.

Morphisms in $\mathcal{C}_E$ are equivalence classes of morphisms of $\mathcal{C}_{\mathbf{Sg}}$, and the reason why we require that $=_E$ is monotonic and closed for substitutions, is that we want composition to be well defined. Indeed monotonicity and closure for substitution imply that

$$\frac{\sigma =_E \sigma' : \vec{x} \longrightarrow \vec{y} \qquad \gamma =_E \gamma' : \vec{y} \longrightarrow \vec{z}}{\gamma \circ \sigma =_E \gamma' \circ \sigma' : \vec{x} \longrightarrow \vec{z}}$$

Moreover, starting from a category C with finite products, one can construct a signature $\mathbf{Sg}_C$ taking the internal language of C, and a class E_C consisting of the equation which are satisfied by the canonical structure in C. The reader can find all the details in [50, Section 4.3]. The main result is that every category with finite products is equivalent of a syntactic category of this kind. See [50, Section 4.3]. In particular if C is finitely complete, then corresponding signature $\mathbf{Sg}_C$ and class E_C have the property that every finite set S of terms equations admits a most general E-unifier.

By Theorem 6.2.7, given a syntactic category $C_{\mathbf{Sg}}$ corresponding to a first order signature, we can make it finitely complete simply adding an initial object. This means that, given such a signature, we can construct a signature $\mathbf{Sg}'$ and a set E of equations such that every finite set of terms equations in the new signature admits a most general E-unifier.

6.3 Doctrines

The notion of hyperdoctrine was introduced by F.W. Lawvere in a series of seminal papers, see [36, 37, 38], together with the more general notion of existential elementary doctrine.

This section is devoted to introduce the definitions of primary, elementary and existential doctrines following the recent works on the topics of M. E. Maietti and G. Rosolini [41, 42, 43, 44].

Definition 6.3.1. Let C be a category with finite products. A ***primary doctrine*** is a functor $P\colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ from the opposite of the category C to the category of inf-semilattices.

Definition 6.3.2. A primary doctrine $P\colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is ***elementary*** if for every A in C there exists an object δ_A in $P(A \times A)$ such that

1. the assignment
$$\exists_{\langle \mathrm{id}_A, \mathrm{id}_A \rangle}(\alpha) := P_{\mathrm{pr}_1}(\alpha) \wedge \delta_A$$
for α in PA determines a left adjoint to $P_{\langle \mathrm{id}_A, \mathrm{id}_A \rangle}\colon P(A \times A) \longrightarrow PA$;
2. for every morphism e of the form $\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_2 \rangle\colon X \times A \longrightarrow X \times A \times A$ in C, the assignment
$$\exists_e(\alpha) := P_{\langle \mathrm{pr}_1, \mathrm{pr}_2 \rangle}(\alpha) \wedge P_{\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle}(\delta_A)$$

for α in $P(X\times A)$ determines a left adjoint to $P_e\colon P(X\times A\times A) \longrightarrow P(X\times A)$.

Definition 6.3.3. A primary doctrine $P\colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is ***existential*** if, for every A_1 and A_2 in C, for any projection $\mathrm{pr}_i\colon A_1\times A_2 \longrightarrow A_i$, $i = 1,2$, the functor

$$P_{\mathrm{pr}_i}\colon P(A_i) \longrightarrow P(A_1\times A_2)$$

has a left adjoint $\exists_{\mathrm{pr}_i}$, and these satisfy:

1. ***Beck-Chevalley condition:*** for any pullback diagram

$$\begin{tikzcd} X' \arrow[r, "\mathrm{pr}'"] \arrow[d, "f'"'] & A' \arrow[d, "f"] \\ X \arrow[r, "\mathrm{pr}"'] & A \end{tikzcd}$$

   with pr and pr$'$ projections, for any β in $P(X)$ the canonical arrow

$$\exists_{\mathrm{pr}'}P_{f'}(\beta) \leq P_f\exists_{\mathrm{pr}}(\beta)$$

   is an isomorphism;
2. ***Frobenius reciprocity:*** for any projection $\mathrm{pr}\colon X \longrightarrow A$, α in $P(A)$ and β in $P(X)$, the canonical arrow

$$\exists_{\mathrm{pr}}(P_{\mathrm{pr}}(\alpha)\wedge\beta) \leq \alpha\wedge\exists_{\mathrm{pr}}(\beta)$$

   in $P(A)$ is an isomorphism.

As observed in [43, Remark 2.4] there is a well known connection between doctrine and fibrations, and all the previous definition can be given in that contest. We refer to [19, 21] for all the details.

We refer to [21, 38, 41] for a complete characterization of existential elementary doctrines, and we recall the following result which will be useful later.

Proposition 6.3.4. *Let $P\colon C^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be an existential elementary doctrine, then for every map $f\colon A \longrightarrow B$ in C the functor P_f has a left adjoint $\exists_f$ that can be computes as:*

$$\exists_{\mathrm{pr}_2}(P_{f\times\mathrm{id}_B}(\delta_B)\wedge P_{\mathrm{pr}_1}(\alpha))$$

for α in $P(A)$, where pr_1 and pr_2 are the projection from $A\times B$.

Observe that primary doctrines, elementary doctrines, and existential doctrines have a 2-categorical structure given in following way.

Definition 6.3.5. The class of primary doctrines **PD** is a 2-category, where:

- ***0-cells*** are primary doctrines;

- ***1-cells*** are pairs of the form (F, b)

$$\begin{tikzcd} \mathcal{C}^{\mathrm{op}} \arrow[dd, "F^{\mathrm{op}}"'] \arrow[drr, "P"] & & \\ & \arrow[r, phantom, "\Downarrow b"] & \mathbf{InfSL} \\ \mathcal{D}^{\mathrm{op}} \arrow[urr, "R"'] & & \end{tikzcd}$$

  such that $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is a functor preserving products, and $b\colon P \longrightarrow R \circ F^{\mathrm{op}}$ is a natural transformation preserving the structures;
- ***2-cells*** are natural transformations $\theta\colon F \longrightarrow G$ such that for every A in $\mathcal{C}$ and every α in PA, we have

$$b_A(\alpha) \leq R_{\theta_A}(c_A(\alpha))$$

  and [41].

Similarly we can define two subcategories of **PD**: the 2-category of elementary doctrine **ElD**, and the 2-category of existential doctrine **ExD**.

In this case one should require that the 1-cells preserve the appropriate structure. We refer to [41, 42, 43] for all the details.

Example 6.3.6. The following examples are discussed in [36, 41, 42, 43, 44].

Let $\mathbb{T}$ be a first order theory over a signature **Sg**. We define a primary doctrine

$$\mathcal{L}\colon \mathcal{C}_{\mathbb{T}}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$$

where the base category is the syntactic category of signature **Sg** and $\mathcal{L}(x_1, \dots, x_n)$ is the class of all well formed formulas in the context $(x_1, \dots, x_n)$. We say that $\psi \leq \phi$ where $\phi, \psi \in \mathcal{L}(x_1, \dots, x_n)$ if $\psi \vdash_{\mathbb{T}} \phi$, and then we quotient in the usual way to obtain a partial order on $\mathcal{L}(x_1, \dots, x_n)$. Now consider a morphism of $\mathcal{C}_{\mathbb{T}}$

$$[t_1/y_1, \dots, t_m/y_m]\colon (x_1, \dots, x_n) \longrightarrow (y_1, \dots, y_m)$$

then $\mathcal{L}_{[\vec{t}/\vec{y}]}(\psi(y_1, \dots, y_m)) = \psi[\vec{t}/\vec{y}]$.

2. Let $\mathcal{C}$ be a category with finite limits. The functor

$$\mathrm{Sub}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$$

assigns to an object A in $\mathcal{C}$ the poset $\mathrm{Sub}_{\mathcal{C}}(A)$ of subobjects of A in $\mathcal{C}$. Given an arrow $B \xrightarrow{f} A$ of $\mathcal{C}$, the functor $\mathrm{Sub}_{\mathcal{C}}(f)\colon \mathrm{Sub}_{\mathcal{C}}(A) \longrightarrow \mathrm{Sub}_{\mathcal{C}}(B)$ is given by pulling a subobject back along f. The fibre equalities are the diagonal arrows. This is an elementary doctrine, and it is existential if the category $\mathcal{C}$ is regular, see [19].

3. Consider a category $\mathcal{D}$ with finite products and weak pullbacks: the doctrine is given by the functor of weak subobjects

$$\Psi_{\mathcal{D}} \colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$$

where $\Psi_{\mathcal{D}}(A)$ is the poset reflection of the slice category $\mathcal{D}/A$, and for an arrow $B \xrightarrow{f} A$, the functor $\Psi_{\mathcal{D}}(f) \colon \Psi_{\mathcal{D}}(A) \longrightarrow \Psi_{\mathcal{D}}(B)$ is given by a weak pullback of an arrow $X \xrightarrow{g} A$ with f. This doctrine is elementary and existential, and the existential left adjoints are given by the post-composition.

6.4 Existential and elementary completions

In [57] we have seen that starting, from a primary doctrine $P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ and the class of projections $\mathfrak{a} \subset \mathcal{C}_1$, we can construct a doctrine $P^{\mathrm{ex}} \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ in which every arrow of the form P^{ex}_f for $f \in \mathfrak{a}$ has a left adjoint.

This construction can be generalized to an arbitrary class of morphisms closed under pullbacks, compositions, and which contains units morphisms. In particular we want to use it to construct the elementary completion of a primary doctrine $P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$. In general the class of arrows of the form $\mathrm{id}_A \times \Delta_X$ is not closed under pullbacks and compositions, therefore we consider the case in which $\mathcal{C}$ is finitely complete, and then we can close the class of morphisms of that form for compositions and pullbacks in order to apply the completion.

In this section we present the existential completion from [57] for an arbitrary class of morphisms $\mathfrak{a}$ closed for pullbacks, compositions, and with units, which adds the left adjoints to all the images of morphisms of $\mathfrak{a}$ and we explain how it can be applied to get the elementary and existential completions.

Consider a primary doctrine $P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ and for every object A of $\mathcal{C}$ consider the following preorder:

- the objects are pairs ($B \xrightarrow{g \in \mathfrak{a}} A$, $\alpha \in PB$);
- ($B \xrightarrow{h \in \mathfrak{a}} A$, $\alpha \in PB$) $\leq$ ($D \xrightarrow{f \in \mathfrak{a}} A$, $\gamma \in PD$) if there exists $w \colon B \longrightarrow D$ such that

$$\begin{tikzcd}
 & B \arrow[dl, "w"'] \arrow[d, "h"] \\
D \arrow[r, "f"'] & A
\end{tikzcd}$$

commutes and $\alpha \leq P_w(\gamma)$.

It is easy to see that the previous data give a preorder.

Let $P^{\mathfrak{a}}(A)$ be the partial order obtained by identifying two objects as usual when $(B \xrightarrow{h \in \mathfrak{a}} A \, , \ \alpha \in PB) \gtrless (D \xrightarrow{f \in \mathfrak{a}} A \, , \ \gamma \in PD)$. With abuse of notation we will denote the equivalence class of an element in the same way.

Given a morphism $f \colon A \longrightarrow B$ in $\mathcal{C}$, let $P^{\mathfrak{a}}{}_{f}(C \xrightarrow{g \in \mathfrak{a}} B \, , \ \beta \in PC)$ be the object

$$(D \xrightarrow{g^* f} A \, , \ P_{f^* g}(\beta) \in PD)$$

where

$$\begin{tikzcd} D \arrow[r, "g^* f"] \arrow[d, "f^* g"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ C \arrow[r, "g"'] & B \end{tikzcd}$$

is a pullback because $g \in \mathfrak{a}$. Note that $P^{\mathfrak{a}}_{f}$ is well defined, because isomorphisms are stable under pullback.

Proposition 6.4.1. *Let* $P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *be a primary doctrine, and let* $\mathfrak{a}$ *be a class of morphisms of* $\mathcal{C}$ *closed for pullback, compositions, and which contains the identity morphisms. Then* $P^{\mathfrak{a}} \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *is a primary doctrine, which means that:*

1. *for every object* A *of* $\mathcal{C}$*,* $P^{\mathfrak{a}}(A)$ *is a inf-semilattice;*
2. *for every* $f \colon A \longrightarrow B$ *,* $P^{\mathfrak{a}}_{f}$ *is an homomorphism of inf-semilattices.*

Proof. It is easy to see that the proof in [57] can be generalized for an arbitrary class of morphisms of $\mathcal{C}$ with the previous properties. □

Proposition 6.4.2. *Given a morphism* $f \colon A \longrightarrow B$ *of* $\mathfrak{a}$*, we define*

$$\exists^{\mathfrak{a}}_{f}(C \xrightarrow{h} A \, , \alpha \in PC) := (C \xrightarrow{fh} B \, , \alpha \in PC)$$

where $(C \xrightarrow{h} A \, , \alpha \in PC)$ *is in* $P^{\mathfrak{a}}(A)$*. Then* $\exists^{\mathfrak{a}}_{f}$ *is left adjoint to* $P^{\mathfrak{a}}_{f}$*.*

Proof. Let $\alpha := (C_1 \xrightarrow{g_1} B \, , \ \alpha_1 \in PC_1)$ and $\beta := (D_2 \xrightarrow{f_2} A \, , \ \beta_2 \in PD_2)$. Now we assume that $\beta \leq P^{\mathfrak{a}}{}_{f}(\alpha)$. This means that

$$
\begin{tikzcd}
& D_2 \arrow[dl, "w"'] \arrow[d, "f_2"] \\
D_1 \arrow[r, "g_1^* f"] \arrow[d, "f^* g_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
C_1 \arrow[r, "g_1"'] & B
\end{tikzcd}
$$

and $P_w(P_{f^*g_1}(\alpha_1)) \geq \beta_2$. Then we have

$$
\begin{tikzcd}
& D_2 \arrow[dl, "f^* g_1 w"'] \arrow[d, "\Delta_A f_2"] \\
C_1 \arrow[r, "g"'] & B
\end{tikzcd}
$$

and $P_{wf^*g_1}(\alpha_1) \geq \beta$. Then $\exists^{\mathfrak{a}}_f(\beta) \leq \alpha$. Now assume $\exists^{\mathfrak{a}}_f(\beta) \leq \alpha$

$$
\begin{tikzcd}
& D_2 \arrow[dl, "\overline{w}"'] \arrow[d, "f f_2"] \\
C_1 \arrow[r, "g_1"'] & B
\end{tikzcd}
$$

with $P_{\overline{w}}(\alpha_1) \geq \beta_2$ Then there exists $w \colon D_2 \longrightarrow D_1$ such that the following diagram commutes

$$
\begin{tikzcd}
D_2 \arrow[dr, dashed, "w"] \arrow[drr, bend left, "f_2"] \arrow[ddr, bend right, "\overline{w}"'] & & \\
& D_1 \arrow[r, "g_1^* f"] \arrow[d, "f^* g_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
& C_1 \arrow[r, "g_1"'] & B
\end{tikzcd}
$$

and $P_w(P_{f^*g_1}(\alpha_1) = P_{\overline{w}}(\alpha_1) \geq \beta_1$. Then we can conclude that $\beta \leq P^{\mathfrak{a}}_f(\alpha)$. $\square$

Theorem 6.4.3. *For every primary doctrine* $P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$, $P^{\mathfrak{a}} \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *satisfies:*

(i) **Beck-Chevalley Condition***: for every pullback*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A' \arrow[d, "f"] \\ X \arrow[r, "g"'] & A \end{tikzcd}$$

with $g \in \mathfrak{a}$ (hence also $g' \in \mathfrak{a}$), for any $\beta \in P^{\mathfrak{a}}(X)$ the following equality holds

$$\exists^{\mathfrak{a}}_{g'} P^{\mathfrak{a}}_{f'}(\beta) = P^{\mathfrak{a}}_{f} \exists^{\mathfrak{a}}_{g}(\beta)$$

*(ii)***Frobenius Reciprocity***: for every morphism $f : X \longrightarrow A$ of $\mathfrak{a}$, for every $\alpha \in P^{\mathfrak{a}}(A)$ and $\beta \in P^{\mathfrak{a}}(X)$, the following equality holds:*

$$\exists^{\mathfrak{a}}_{f}(P^{\mathfrak{a}}_{f}(\alpha) \wedge \beta) = \alpha \wedge \exists^{\mathfrak{a}}_{f}(\beta)$$

Proof. (i) Consider the following pullback square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A' \arrow[d, "f"] \\ X \arrow[r, "g"'] & A \end{tikzcd}$$

where $g, g' \in \mathfrak{a}$, and let $\beta := (\ C_1 \xrightarrow{h_1} X\ ,\ \beta_1 \in PC_1) \in P^{\mathfrak{a}}(X)$. Now consider the following diagram

$$\begin{tikzcd} D_1 \arrow[r, "h_1^* f'"] \arrow[d, "f'^* h_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A' \arrow[d, "f"] \\ C_1 \arrow[r, "h_1"'] & X \arrow[r, "g"'] & A \end{tikzcd}$$

Since the two square are pullbacks, then the big square is a pullback, and then

$$(\ D_1 \xrightarrow{g' h_1^* f'} A\ , P_{f'^* h_1}(\beta_1)) = (\ D_1 \xrightarrow{g h_1^* f} A\ , P_{f^* g h_1}(\beta_1))$$

and these are exactly

$$\exists^{\mathfrak{a}}_{g'} P^{\mathfrak{a}}_{f'}(\beta) = P^{\mathfrak{a}}_{f} \exists^{\mathfrak{a}}_{g}(\beta).$$

(*ii*) Consider a morphism $f\colon X \longrightarrow A$ of $\mathfrak{a}$, an element $\alpha := (\ C_1 \xrightarrow{h_1} A\ ,\ \alpha_1 \in PC_1)$ in $P^{\mathfrak{a}}(A)$, and an element $\beta = (\ D_2 \xrightarrow{h_2} X\ ,\ \beta_2 \in PD_2)$ in $P^{\mathfrak{a}}(X)$. Observe that the following diagram is a pullback

$$\begin{tikzcd}
D_2 \wedge D_1 \arrow[r, "h_2^*(h_1^*f)"] \arrow[d, "(h_1^*f)^*h_2"'] \arrow[dr, phantom, very near start, "\lrcorner"] & D_1 \arrow[r, "f^*h_1"] \arrow[d, "h_1^*f"'] \arrow[dr, phantom, very near start, "\lrcorner"] & C_1 \arrow[d, "h_1"] \\
D_2 \arrow[r, "h_2"'] & X \arrow[r, "f"'] & A
\end{tikzcd}$$

and this means that

$$\exists^{\mathfrak{a}}_f(P^{\mathfrak{a}}_f(\alpha) \wedge \beta) = \alpha \wedge \exists^{\mathfrak{a}}_f(\beta).$$

□

The first example is the special case of ***existential completion***, presented in [57]. In this case $\mathfrak{a}$ is the class of product projections and we can apply directly the previous construction, and one has that given a primary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$, the doctrine $P^{\mathrm{ex}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is existential and this construction extends to a 2-functor $\mathrm{E}\colon \mathbf{PD} \longrightarrow \mathbf{ExD}$ from the 2-category of primary doctrines into the category of existential doctrines, and it is left 2-adjoint to the forgetful functor. See [57]. Moreover, if $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is elementary, then the doctrine $P^{\mathrm{ex}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is elementary and existential, see [57] for all the details.

When the base category of a primary doctrine $P\colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ has finite limits, we can apply the previous completion to obtain an elementary doctrine. In this case we speak of an ***elementary completion***.

Theorem 6.4.4. *Let $\mathcal{D}$ be a category finitely complete, and let $\mathfrak{a}_{\mathrm{el}}$ be the closure for pullback and compositions of the class of morphisms of the form $\mathrm{id}_A \times \Delta_X$. Then a primary doctrine $P\colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ can be completed to an elementary doctrine $P^{\mathrm{el}}\colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$. Moreover this construction extends to a 2-functor from the 2-category of primary doctrines with base category finitely complete into the category of elementary doctrines, and it is left 2-adjoint to the forgetful functor.*

Proof. The proof that $P\colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is elementary is a direct consequence of the Proposition 6.4.2. Moreover this construction can be extended to a 2-functor since the morphisms between primary doctrines are pairs (F, b) where F is a functor preserving products. Therefore all the results about the 2-adjunction and about the characterization of the 2-monads proved in [57] can be extended for the elementary completion. □

Corollary 6.4.5. *Let $\mathcal{D}$ be a category finitely complete, and let* $P\colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *be a primary doctrine. Then* $(P^{\mathrm{el}})^{\mathrm{ex}}\colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *is a primary existential doctrine.*

Proof. It follows from Theorem 6.4.4 and from the fact that the existential completions preserves the elementary structure, see [57]. □

Next consider a first order theory $\mathbb{T}$ and the primary doctrine $\mathcal{L}\colon \mathcal{C}_{\mathbb{T}}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ defined in Example 6.3.6.

Theorem 6.4.6. *The primary doctrine* $\mathcal{L}\colon \mathcal{C}_{\mathbb{T}}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *can be completed to a primary doctrine* $\mathcal{L}^0\colon (\mathcal{C}_{\mathbb{T}}^0)^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *where* $\mathcal{C}_{\mathbb{T}}^0$ *has finite limits. Moreover the doctrine*

$$(\mathcal{L}^0)^{\mathrm{el}}\colon (\mathcal{C}_{\mathbb{T}}^0)^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$$

is elementary, and the doctrine

$$((\mathcal{L}^0)^{\mathrm{el}})^{\mathrm{ex}}\colon (\mathcal{C}_{\mathbb{T}}^0)^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$$

is elementary existential.

Proof. It is a direct consequence of Theorem 6.2.7 and Theorem 6.4.4 and Corollary 6.4.5. □

We conclude this section with a comparison between the exact completion presented by Carboni in [6, 10] and a review on the general version presented in [57].

In [41, 44] it is proved that various notions of completing a category to an exact category can be seen as an instance of the exact completion for elementary existential doctrine. In [57] we generalize this result proving that every elementary doctrine can be complete to an exact category.

By Theorem 6.4.4 and Corollary 6.4.5 we can extend the exact completion presented in [57] for a primary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ such that there exists a class of morphisms $\mathfrak{a}$ containing all the morphisms of the form $\mathrm{id}_A \times \Delta_X$ and closed for pullbacks, compositions, and containing units arrows. A primary doctrine of this kind can be completed to an exact category $\mathbb{T}_{(P^{\mathrm{el}})^{\mathrm{ex}}}$.

In particular by Theorem 6.4.6, given first order theory $\mathbb{T}$ in which formulas are only atomic formulas or finite conjunction of atomic formulas, and the symbols $\top$, the primary doctrine $\mathcal{L}\colon \mathcal{C}_{\mathbb{T}}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ can be completed to a exact category $\mathbb{T}_{((\mathcal{L}^0)^{\mathrm{el}})^{\mathrm{ex}}}$.

6.5 Applications

In this section we present a detailed description of the elementary completion for a primary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ whose base category $\mathcal{C}$ is the free finite product completion of a discrete category $\mathcal{A}$.

From a logical point of view we are looking at a first order theory in a language in which no function symbols are considered.

We give a compact description of the doctrine $P^{\mathrm{el}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ with respect to the one used in Section 6.4. Recall that in this case the class $\mathfrak{a}_{\mathrm{el}}$ is the closure for pullbacks and compositions of the class of morphisms of the form $\mathrm{id}_A \times \Delta_X$.

Let A be an object of $\mathcal{C}$, then we have that by definition the elements of the poset $P^{\mathrm{el}}(A)$ are of the form

$$(C \xrightarrow{h} A ,\alpha \in PC)$$

where $h \in \mathfrak{a}_{\mathrm{el}}$.

If the object A is not of the form $B \times X \times X$, where X is a non-terminal object of $\mathcal{C}$, then the only morphisms of $\mathfrak{a}_{\mathrm{el}}$ with codomain A are the identities, so we can define for this kind of objects

$$P^{\mathrm{el}}(A) = P(A).$$

Otherwise, for the case of objects of the form $A \times X \times X$, we can give an equivalent and more synthetic description of the poset $P^{\mathrm{el}}(A \times X \times X)$: it is a class where the objects are pairs of the form $(\alpha, \bot)$ or $(\alpha, \top)$ where $\alpha \in P(A \times X \times X)$.

Now we define the partial order on $P^{\mathrm{el}}(A \times X \times X)$. We say that $(\alpha, k_1) \leq (\beta, k_2)$ if one of the following possibilities holds

- $k_1 = \bot$ and $P_{\mathrm{id}_A \times \Delta_X}(\alpha) \leq P_{\mathrm{id}_A \times \Delta_X}(\beta)$;
- $k_1 = k_2 = \top$ and $\alpha \leq \beta$.

It is direct to check that this is a preorder, and we identify as usual two objects if $(\alpha, k_1) \leq (\beta, k_2)$ and $(\alpha, k_1) \geq (\beta, k_2)$ to obtain a partial order.

Observe that the meet of two elements in $P^{\mathrm{el}}(A \times X \times X)$ is

$$(\alpha, k_1) \wedge (\beta, k_2) = (\alpha \wedge \beta, k_1 \wedge k_2)$$

and the top element of $P^{\mathrm{el}}(A \times X \times X)$ is $(\top_{A \times X \times X}, \top)$. Therefore the poset $P^{\mathrm{el}}(A \times X \times X)$ is an inf-semilattice.

Consider a projection $\mathrm{pr}_i\colon A \times X \times X \longrightarrow A$. We define

$$P^{\mathrm{el}}_{\mathrm{pr}_1}\colon P^{\mathrm{el}}(A) \longrightarrow P^{\mathrm{el}}(A \times X \times X)$$

as

$$P^{\mathrm{el}}_{\mathrm{pr}_1}(\alpha) := (P_{\mathrm{pr}_1}(\alpha), \top)$$

and the same for the other projections.

Now consider $\Delta_A \colon A \longrightarrow A \times A$. We define

$$P^{\mathrm{el}}_{\Delta_A} \colon P^{\mathrm{el}}(A \times A) \longrightarrow P^{\mathrm{el}}(A)$$

as

$$P^{\mathrm{el}}_{\Delta_A}(\alpha, k) = P_{\Delta_A}(\alpha)$$

for $k = \top, \bot$.

Theorem 6.5.1. *Let* $P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *be an elementary doctrine. Then, with the previous assignments,* $P^{\mathrm{el}} \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *is an elementary doctrine.*

Proof. It is easy to check that $P^{\mathrm{el}} \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is a primary doctrine. Let

$$\exists^{\mathrm{el}}_{\mathrm{id}_A \times \Delta_X} \colon P^{\mathrm{el}}(A \times X) \longrightarrow P^{\mathrm{el}}(A \times X \times X)$$

be defined as

$$\exists^{\mathrm{el}}_{\mathrm{id}_A \times \Delta_X}(\alpha) := (P_{\langle \mathrm{pr}'_1, \mathrm{pr}'_2 \rangle}(\alpha), \bot)$$

where $\langle \mathrm{pr}'_1, \mathrm{pr}'_2 \rangle \colon A \times X \times X \longrightarrow A \times X$. To check that

$$\exists^{\mathrm{el}}_{\mathrm{id}_A \times \Delta_X} \dashv P^{\mathrm{el}}_{\mathrm{id}_A \times \Delta_X}$$

let $\alpha \in P^{\mathrm{el}}(A \times X)$. So

$$P^{\mathrm{el}}_{\mathrm{id}_A \times \Delta_X} \exists^{\mathrm{el}}_{\mathrm{id}_A \times \Delta_X}(\alpha) = P^{\mathrm{el}}_{\mathrm{id}_A \times \Delta_X}(P_{\langle \mathrm{pr}'_1, \mathrm{pr}'_2 \rangle}(\alpha), \bot) = P_{\mathrm{id}_A \times \Delta_X} P_{\langle \mathrm{pr}'_1, \mathrm{pr}'_2 \rangle}(\alpha) = \alpha.$$

Thus

$$\mathrm{id}_{A \times C} \leq P^{\mathrm{el}}_{\mathrm{id}_A \times \Delta_X} \exists^{\mathrm{el}}_{\mathrm{id}_A \times \Delta_X}.$$

Now consider $(\alpha, k) \in P^{\mathrm{el}}(A \times X \times X)$. By definition we have

$$\exists^{\mathrm{el}}_{\mathrm{id}_A \times \Delta_X} P^{\mathrm{el}}_{\mathrm{id}_A \times \Delta_X}(\alpha, k) = \exists^{\mathrm{el}}_{\mathrm{id}_A \times \Delta_X}(P_{\mathrm{id}_A \times \Delta_X}(\alpha)) = (P_{\langle \mathrm{pr}'_1, \mathrm{pr}'_2 \rangle} P_{\mathrm{id}_A \times \Delta_X}(\alpha), \bot)$$

and

$$(P_{\langle \mathrm{pr}'_1, \mathrm{pr}'_2 \rangle} P_{\mathrm{id}_A \times \Delta_X}(\alpha), \bot) = (P_{\langle \mathrm{pr}'_1, \mathrm{pr}'_2, \mathrm{pr}'_2 \rangle}(\alpha), \bot).$$

By definition again we have that

$$(P_{\langle \mathrm{pr}'_1, \mathrm{pr}'_2, \mathrm{pr}'_2 \rangle}(\alpha), \bot) \leq (\alpha, k)$$

if and only if

$$P_{\mathrm{id}_A \times \Delta_X}(P_{\langle \mathrm{pr}'_1, \mathrm{pr}'_2, \mathrm{pr}'_2 \rangle}(\alpha)) \leq P_{\mathrm{id}_A \times \Delta_X}(\alpha)$$

but these are equal. This prove that the doctrine is elementary. □

Remark 6.5.2. Following the notation of [42, 43], we can define $\delta_X = (\top_{X \times X}, \bot)$.

The previous construction induces a 2-functor which is left-adjoint to the forgetful functor.

Consider the 2-category **PdD** of primary doctrines whose base category is the free products completion of a discrete category, and its 2-subcategory **EdD** of elementary doctrines. We define

$$\mathrm{El}\colon \mathbf{PdD} \longrightarrow \mathbf{EdD}$$

on the objects as

$$\mathrm{El}(P) := P^{\mathrm{el}}$$

for a given primary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$.

Consider two primary doctrines $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ and $R\colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ of **PdD**. We define

$$\mathrm{El}_{P,R}\colon \mathbf{PdD}(P,R) \longrightarrow \mathbf{EdD}(P^{\mathrm{el}}, R^{\mathrm{el}})$$

as

$$\mathrm{El}(F,b) = (F, b^{\mathrm{el}})$$

where $b^{\mathrm{el}}\colon P^{\mathrm{el}} \Longrightarrow R^{\mathrm{el}}F^{\mathrm{op}}$ is the natural transformation defined as follow:

- for every $A \in \mathcal{A}$, the 1-cell $b^{\mathrm{el}}_A\colon P^{\mathrm{el}}A \longrightarrow R^{\mathrm{el}}FA$ is exactly $b_A\colon PA \longrightarrow RFA$;
- for every $A, X \in \mathcal{C}$, the 1-cell

$$b^{\mathrm{el}}_{A\times X\times X}\colon P^{\mathrm{el}}(A\times X\times X) \longrightarrow R^{\mathrm{el}}(FA\times FX\times FX)$$

  sends an element (α, k) into $(b_{A\times X\times X}(\alpha), k)$.

It is direct to verify that this is a 1-cell of elementary doctrines. Moreover observe that the functor El does not change the first component of a 1-cell. Then for every 2-cell $\theta\colon (F,b) \Longrightarrow (G,c)$ we can define $\mathrm{El}(\theta) := \theta$. Therefore we can summarize the previous results into the following proposition.

Proposition 6.5.3. $\mathrm{El}\colon \mathbf{PdD} \longrightarrow \mathbf{EdD}$ *is a 2-functor.*

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be a doctrine of **PdD** we define the 1-cell

$$(\mathrm{id}_{\mathcal{C}}, \eta)\colon P \longrightarrow P^{\mathrm{el}}$$

where $\eta_A\colon PA \longrightarrow P^{\mathrm{el}}A$ is the identity for every $A \in \mathcal{A}$, and

$$\eta_{A\times X\times X}\colon P(A\times X\times X) \longrightarrow P^{\mathrm{el}}(A\times X\times X)$$

is defined as $\eta_{A\times X\times X}(\alpha) = (\alpha, \top)$. It is direct to check that $\eta\colon P \longrightarrow P^{\mathrm{el}}$ is a natural transformation.

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ be a doctrine of **EdD** we define the 1-cell

$$(\mathrm{id}_C, \varepsilon)\colon P^{\mathrm{el}} \longrightarrow P$$

where $\varepsilon_A\colon P^{\mathrm{el}}A \longrightarrow PA$ is the identity for every A and

$$\varepsilon_{A\times X\times X}\colon P^{\mathrm{el}}(A \times X \times X) \longrightarrow P(A \times X \times X)$$

is defined as $\varepsilon_{A\times X\times X}(\alpha, \top) = \alpha$ and $\varepsilon_{A\times X\times X}(\alpha, \bot) = \alpha \wedge P_{\langle \mathrm{pr}_2, \mathrm{pr}_3\rangle}(\delta_X)$. Again one can check directly that $\varepsilon\colon P^{\mathrm{el}} \longrightarrow P$ is a natural transformation and that $(\mathrm{id}_C, \varepsilon)$ is a 1-cell of **EdD**.

Proposition 6.5.4. *The previous families of 1-cells define two 2-natural transformations*

$$\eta\colon \mathrm{id}_{\mathbf{PdD}} \longrightarrow \mathrm{UEl}$$

and

$$\varepsilon\colon \mathrm{ElU} \longrightarrow \mathrm{id}_{\mathbf{EdD}}$$

Moreover $\mathrm{El} \dashv \mathrm{U}$ *and the unit and counit of this 2-adjunction are* η *and* ε*.*

Proof. It is a straightforward verification. □

We construct a 2-monad $\mathrm{T}_{\mathrm{el}}\colon \mathbf{PdD} \longrightarrow \mathbf{PdD}$ from the 2-adjunction of Proposition 6.5.4 , and we prove that every elementary doctrine can be seen as an algebra for this 2-monad.

Finally we will show that the 2-monad T_{el} is lax-idempotent. For all the details about the theory of 2-monads we refer to [27, 28, 54, 55, 56].

Definition 6.5.5. We define:

- $\mathrm{T}_{\mathrm{el}}\colon \mathbf{PdD} \longrightarrow \mathbf{PdD}$ the 2-functor $\mathrm{T}_{\mathrm{el}} = \mathrm{U} \circ \mathrm{El}$;
- $\eta\colon \mathrm{id}_{\mathbf{PdD}} \longrightarrow \mathrm{T}_{\mathrm{el}}$ is the 2-natural transformation defined in Proposition 6.5.4;
- $\mu\colon \mathrm{T}_{\mathrm{el}}^2 \longrightarrow \mathrm{T}_{\mathrm{el}}$ is the 2-natural transformation $\mu = \mathrm{U}\varepsilon\mathrm{El}$.

Proposition 6.5.6. T_{el} *is a 2-monad.*

Proof. One can easily check that the following diagrams commute

$$\begin{tikzcd}
\mathrm{T}_{\mathrm{el}}^3 \arrow[r, "\mu \mathrm{T}_{\mathrm{el}}"] \arrow[d, "T_{\mathrm{el}}\mu"'] & \mathrm{T}_{\mathrm{el}}^2 \arrow[d, "\mu"] \\
\mathrm{T}_{\mathrm{el}}^2 \arrow[r, "\mu"'] & \mathrm{T}_{\mathrm{el}}
\end{tikzcd}$$

$$
\begin{tikzcd}
\mathrm{id}_{\mathbf{PdD}} \circ \mathrm{T}_{\mathrm{el}} \arrow[r, "\eta \mathrm{T}_{\mathrm{el}}"] \arrow[dr, "\mathrm{id}"'] & \mathrm{T}_{\mathrm{el}}^2 \arrow[d, "\mu"] & \mathrm{T}_{\mathrm{el}} \circ \mathrm{id}_{\mathbf{PdD}} \arrow[l, "\mathrm{T}_{\mathrm{el}}\eta"'] \arrow[dl, "\mathrm{id}"] \\
& \mathrm{T}_{\mathrm{el}} &
\end{tikzcd}
$$

□

Proposition 6.5.7. *Let* $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *be a doctrine of* **EdD**. *Then* $(P, (\mathrm{id}_{\mathcal{C}}, \varepsilon_P))$ *is an object of the category* T_{el}-**Alg** *of* T_{el}*-algebras.*

Proof. We prove that the following diagram commutes

$$
\begin{tikzcd}
\mathrm{T}_{\mathrm{el}}^2 P \arrow[r, "\varepsilon_{P^{\mathrm{el}}}"] \arrow[d, "{\varepsilon_P}^{\mathrm{el}}"'] & \mathrm{T}_{\mathrm{el}} P \arrow[d, "\varepsilon_P"] \\
\mathrm{T}_{\mathrm{el}} P \arrow[r, "\varepsilon_P"'] & P
\end{tikzcd}
$$

By definition of ε_P, we need only to check the element of the form $(\alpha, k_2) \in (P^{\mathrm{el}})^{\mathrm{el}}(A \times X \times X)$, since on the other elements, ε_P acts as the identity. Consider an element $\alpha = (\alpha_1, k_1)$, then

$$\varepsilon_P \varepsilon_{P^{\mathrm{el}}}(\alpha, k_2)$$

is:

- α_1 if $k_2 = \top$ and $k_1 = \top$;
- $\alpha_1 \wedge P_{\langle \mathrm{pr}_2, \mathrm{pr}_1 \rangle}(\delta_X)$ otherwise.

On the other side we have

$$\varepsilon_P {\varepsilon_P}^{\mathrm{el}}(\alpha, k_2)$$

and this is:

- α_1 if $k_2 = \top$ and $k_1 = \top$;
- $\alpha_1 \wedge P_{\langle \mathrm{pr}_2, \mathrm{pr}_1 \rangle}(\delta_X)$ otherwise.

Therefore the diagram commutes. Now we consider the condition on the unit. It is easy to observe that

$$\varepsilon_P \eta_P = \mathrm{id}_P$$

since

$$\varepsilon_P \eta_P(\alpha) = \varepsilon_P(\alpha, \top) = \alpha$$

for every $\alpha \in P(A \times X \times X)$, and both ε_P and η_P are the identity on the other objects. Therefore we have that (P, ε_P) is a mT_{el}-algebra. □

Proposition 6.5.8. *Let* $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *be a doctrine of* **PdD**, *and consider a 1-cell* $(F,a)\colon P^{\mathrm{el}} \longrightarrow P$ *such that* $(P,(F,a))$ *is a* T_{el}*-algebra. Then the doctrine* $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ *is elementary. Moreover* $F = \mathrm{id}_{\mathcal{C}}$ *and* $(\mathrm{id}_{\mathcal{C}},a)\colon P^{\mathrm{el}} \longrightarrow P$ *is exactly* ε_P.

Proof. By definition of algebra for a monad we have that the following diagram commutes:

$$\begin{tikzcd} P \arrow[r, "\eta_P"] \arrow[dr, "\mathrm{id}_P"'] & P^{\mathrm{el}} \arrow[d, "(F,a)"] \\ & P. \end{tikzcd}$$

Thus $F\colon \mathcal{C} \longrightarrow \mathcal{C}$ must be the identity functor. Now consider two objects A, X of $\mathcal{C}$ and the arrow $\mathrm{id}_A \times \Delta_X\colon A\times X \longrightarrow A\times X\times X$. We define

$$\exists_{\mathrm{id}_A\times\Delta_X}(\alpha) := a_{A\times X\times X}\exists^{\mathrm{el}}_{\mathrm{id}_A\times\Delta_X}\eta_{A\times X}(\alpha).$$

Suppose $\alpha \in P(A\times X)$, $\beta \in P(A\times X\times X)$ and $\alpha \le P_{\mathrm{id}_A\times\Delta_X}(\beta)$. Then since $\eta_{A\times X}$ preserves the order we have

$$\eta_{A\times X}(\alpha) \le \eta_{A\times X}P_{\mathrm{id}_A\times\Delta_X}(\beta)$$

and by the naturality of $\eta_{A\times X}$ we have

$$\eta_{A\times X}(\alpha) \le P^{\mathrm{el}}_{\mathrm{id}_A\times\Delta_X}\eta_{A\times X\times X}(\beta).$$

Now we use the fact that P^{el} is primary, and then

$$\exists^{\mathrm{el}}_{\mathrm{id}_A\times X\times X}\eta_{A\times X}(\alpha) \le \eta_{A\times X\times X}(\beta)$$

and then

$$a_{A\times X\times X}\exists^{\mathrm{el}}_{\mathrm{id}_A\times X\times X}\eta_{A\times X}(\alpha) \le a_{A\times X\times X}\eta_{A\times X\times X}(\beta).$$

Then we can conclude that

$$\exists^{\mathrm{el}}_{\mathrm{id}_A\times\Delta_X}(\alpha) \le \beta$$

because $a_{A\times X\times X}\eta_{A\times X\times X}$ is the identity by hypothesis.

Now we prove the convers. Suppose that $\alpha \le P_{\mathrm{id}_A\times\Delta_X}(\beta)$. Then by definition of $\exists_{\mathrm{id}_A\times\Delta_X}$ we have

$$a_{A\times X\times X}\exists^{\mathrm{el}}_{\mathrm{id}_A\times X\times X}\eta_{A\times X}(\alpha) \le \beta$$

then we have

$$P_{\mathrm{id}_A\times\Delta_X}a_{A\times X\times X}\exists^{\mathrm{el}}_{\mathrm{id}_A\times X\times X}\eta_{A\times X}(\alpha) \le P_{\mathrm{id}_A\times\Delta_X}(\beta).$$

Using the naturality of a we have

$$a_{A\times X}P^{\mathrm{el}}_{\mathrm{id}_A\times\Delta_X}\exists^{\mathrm{el}}_{\mathrm{id}_A\times X\times X}\eta_{A\times X}(\alpha)\leq P_{\mathrm{id}_A\times\Delta_X}(\beta)$$

and since $\mathrm{id}\leq P^{\mathrm{el}}_{\mathrm{id}_A\times\Delta_X}\exists^{\mathrm{el}}_{\mathrm{id}_A\times X\times X}$, we have

$$\alpha=a_{A\times X}\eta_{A\times X}(\alpha)\leq P_{\mathrm{id}_A\times\Delta_X}(\beta)$$

and we can conclude that $\exists_{\mathrm{id}_A\times\Delta_X}\dashv P_{\mathrm{id}_A\times\Delta_X}$. Finally observe that if we consider $(\top_{A\times A},\bot)\in P^{\mathrm{el}}(A\times A)$ we have

$$a_{A\times A}(\top_{A\times A},\bot)=a_{A\times A}\exists^{\mathrm{el}}_{\Delta_A}\eta_A(\top_A)=\exists_{\Delta_A}(\top_A)=\delta_A.$$

Now we can observe that for every $(\alpha,\bot)\in P^{\mathrm{el}}(A\times X\times X)$ we have

$$(\alpha,\bot)=(\alpha,\top)\wedge(\top_{A\times X\times X},\bot)=(\alpha,\top)\wedge\exists^{\mathrm{el}}_{\mathrm{id}_A\times\Delta_X}(\top_{A\times X}).$$

Therefore

$$a(\alpha,\bot)=(\alpha,\top)\wedge(\top_{A\times X\times X},\bot)=a(\alpha,\top)\wedge a\exists^{\mathrm{el}}_{\mathrm{id}_A\times\Delta_X}(\top_{A\times X})=\alpha\wedge\exists_{\mathrm{id}_A\times\Delta_X}(\top_{A\times X})$$

and, since P is elementary, we have $\exists_{\mathrm{id}_A\times\Delta_X}(\top_{A\times X})=P_{\langle\mathrm{pr}_2,\mathrm{pr}_3\rangle}(\delta_X)$. Hence we have

$$a(\alpha,\bot)=\alpha\wedge P_{\langle\mathrm{pr}_2,\mathrm{pr}_3\rangle}(\delta_X)$$

and we can conclude that $a=\varepsilon_P$. □

Proposition 6.5.9. *The 2-category* $\mathrm{T_{el}}$-**Alg** *is isomorphic as 2-category to the category* **EdD**.

Proof. It follows from Proposition 6.5.8 and Proposition 6.5.7 and from the fact that if we consider a 1-cell $(F,a)\colon(P,\varepsilon_P)\longrightarrow(R,\varepsilon_R)$ of $\mathrm{T_{el}}$-algebras then it is a 1-cell of **EdD**. □

Following the notation of [27] we prove that the 2-monad $\mathrm{T_{el}}\colon\mathbf{PdD}\longrightarrow\mathbf{PdD}$ pseudo-idempotent.

Theorem 6.5.10. *Let* (P,ε_P) *and* (R,ε_R) *be* $\mathrm{T_{el}}$ *algebras, and let* $(F,b)\colon P\longrightarrow R$ *be a 1-cell of* **PD**. *Then* $((F,b),\mathrm{id}_F)$ *is lax-morphism of algebras, and the 2-cell* $\mathrm{id}_F\colon\varepsilon_R(F,b^{\mathrm{el}})\Longrightarrow(F,b)\varepsilon_P$ *is the unique 2-cell making* $(\mathrm{id}_F,(F,b))$ *a lax-morphism. Moreover, we have that* id_F *is invertible as 2-cell, and then the 2-monad* $\mathrm{T_{el}}$ *is pseudo-idempotent.*

Proof. Consider the following diagram

$$\begin{tikzcd}
P^{\mathrm{el}} \arrow[r, "{(F,b^{\mathrm{el}})}"] \arrow[d, "\varepsilon_P"'] \arrow[dr, phantom, "\Downarrow \mathrm{id}_F"] & R^{\mathrm{el}} \arrow[d, "\varepsilon_R"] \\
P \arrow[r, "{(F,b)}"'] & R.
\end{tikzcd}$$

Let $(\alpha, k) \in P^{\mathrm{el}}(A \times X \times X)$. We have that $(\varepsilon_R(F, b^{\mathrm{el}}))_{A\times X\times X}(\alpha, k)$ is equal to

- $b_{A\times X\times X}(\alpha)$ if $k = \top$;
- $b_{A\times X\times X}(\alpha) \wedge R_{\langle \mathrm{pr}_2,\mathrm{pr}_3\rangle}(\delta_{FX})$ if $k = \bot$, with the usual notation for the functor $R_{\langle \mathrm{pr}_2,\mathrm{pr}_3\rangle} \colon R(FA \times FX) \longrightarrow R(FA \times FX \times FX)$;

One can check that we obtain the same results if we consider $((F, b)\varepsilon_P)_{A\times X\times X}(\alpha, k)$. Finally it is easy to see that $\mathrm{id}_F \colon \varepsilon_R(F, b^{\mathrm{el}}) \Longrightarrow (F, b)\varepsilon_P$ trivially satisfies the coherence conditions for lax-T_{el}-morphisms, because they are equal.

Now suppose there exists another 2-cell $\theta \colon \varepsilon_R(F, b^{\mathrm{el}}) \Longrightarrow (F, b)\varepsilon_P$ such that $((F, b), \theta)$ is a lax-morphism

$$\begin{tikzcd}
P^{\mathrm{el}} \arrow[r, "{(F,b^{\mathrm{el}})}"] \arrow[d, "\varepsilon_P"'] \arrow[dr, phantom, "\Downarrow \theta"] & R^{\mathrm{el}} \arrow[d, "\varepsilon_R"] \\
P \arrow[r, "{(F,b)}"'] & R.
\end{tikzcd}$$

Then it must satisfy the following condition

$$\begin{tikzcd}
P \arrow[r, "{(F,b)}"] \arrow[d, "\eta_P"'] & R \arrow[d, "\eta_R"] \\
P^{\mathrm{el}} \arrow[r, "{(F,b^{\mathrm{el}})}"] \arrow[d, "\varepsilon_P"'] \arrow[dr, phantom, "\Downarrow \theta"] & R^{\mathrm{el}} \arrow[d, "\varepsilon_R"] \\
P \arrow[r, "{(F,b)}"'] & R
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
P \arrow[r, "{(F,b)}"] \arrow[d, "1_P"'] & R \arrow[d, "1_R"] \\
P \arrow[r, "{(F,b)}"'] & R
\end{tikzcd}$$

and this means that $\theta = \mathrm{id}_F$. □

Corollary 6.5.11. *The 2-monad* $\mathrm{T}_{\mathrm{el}} \colon$ **PdD** $\longrightarrow$ **PdD** *is lax-idempotent and co-lax idempotent.*

Proof. It follows from [27, Proposition 6.9]. □

Corollary 6.5.12. *The 2-monad* $\mathrm{T_{el}}\colon \mathbf{PdD} \longrightarrow \mathbf{PdD}$ *is fully property-like.*

Proof. It follows from Corollary 6.5.11 and [27, Proposition 6.7]. □

Remark 6.5.13. The considerations on the 2-monad $\mathrm{T_{el}}$ on **PdD** can be extended for the general case of elementary completion, and the fact that the existential completion preserves the elementary structure suggests that there exists a distributive law $\delta\colon \mathrm{T_{el}}\mathrm{T_{ex}} \longrightarrow \mathrm{T_{ex}}\mathrm{T_{el}}$. Moreover we can compose these 2-monads with the pseudo-monads adding comprehensive diagonals, comprehensions and quotients. The key point is that every completions preserves the previous structure, and therefore we can define at every step a pseudo-distributive laws between the compositions of the pseudo-monads.

References

[1] J. Beck. Distributive laws. In *Lecture Notes in Mathematics*, volume 80. Springer, 1985.

[2] G. Birkhoff. On the structure of abstract algebras. *Proceedings of the Cambridge Philosophical Society*, 31(4):433–454, 1935.

[3] R. Blackwell, G.M. Kelly, and J. Power. Two-dimensional monad theory. *J. Pure Appl. Algebra*, 59:1–41, 1989.

[4] F. Borceux. *Handbook of Categorical Algebra 1: Basic Category Theory*, volume 51 of *Encyclopedia of Mathematics and its Applications*. Cambridge Univ. Press, 1994.

[5] F. Borceux. *Handbook of Categorical Algebra 2: Categories and Structures*, volume 51 of *Encyclopedia of Mathematics and its Applications*. Cambridge Univ. Press, 1994.

[6] A. Carboni. Some free constructions in realizability and proof theory. *J. Pure Appl. Algebra*, 103:117–148, 1995.

[7] A. Carboni, S. Kasangian, and R. Street. Bicategories of spans and relations. *J. Pure Appl. Algebra*, 33:259–267, 1984.

[8] A. Carboni and R. Cecilia Magno. The free exact category on a left exact one. *J. Aust. Math. Soc.*, 33:295–301, 1982.

[9] A. Carboni and R.Street. Order ideals in categories. *Pacific J. Math.*, 124:275–288, 1986.

[10] A. Carboni and E. Vitale. Regular and exact completions. *J. Pure Appl. Algebra*, 125:79–117, 1998.

[11] A. Carboni and R.F.C. Walters. Cartesian bicategories i. *J. Pure Appl. Algebra*, 49:11–32, 1987.

[12] E. Cheng, M. Hyland, and J. Power. Pseudo-distributive laws. *Electron. Notes Theor. Comput. Sci.*, 83:227–245, 2003.

[13] B.J. Day and G.M. Kelly. On topological quotient maps preserved by pullbacks or products. *Math. Proc. Cambridge Philos. Soc.*, 67:553–558, 1970.

[14] S. Eilenberg and J. C. Moore. Adjoint functors and triples. *Illinois J. Math.*, 9:381–398, 1965.

[15] P.J. Freyd and G.M. Kelly. Categories of continuous functors 1. *J. Pure Appl. Algebra*, 2:169–191, 1972.

[16] P.J. Freyd and A. Scedrov. *Categories, Allegories*, volume 39 of *Mathematical Library*. North-Holland, 1990.

[17] P. Gabriel and M. Zisman. *Calculus of Fractions and Homotopy Theory*, volume 35 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, 1967.

[18] J.H. Gallier and W. Snyder. Complete sets of transformations for general e-unification. *Theoret. Comput. Sci.*, 67:203–260, 1989.

[19] J. Hughes and B. Jacobs. Factorization systems and fibrations: toward a fibered birkhoff variety theorem. *Electron. Notes Theor. Comput. Sci.*, 69:156–182, 2003.

[20] J.M.E. Hyland, P.T. Johnstone, and A.M. Pitts. Tripos theory. *Math. Proc. Camb. Phil. Soc.*, 88:205–232, 1980.

[21] B. Jacobs. *Categorical Logic and Type Theory*, volume 141 of *Studies in Logic and the foundations of mathematics*. North Holland Publishing Company, 1999.

[22] G. Janelidze and W. Tholen. Facets of descent i. *Appl. Categ. Structures*, 2(3):245–281, 1994.

[23] G. Janelidze and W. Tholen. Facets of descent ii. *Appl. Categ. Structures*, 5(3):229–248, 1997.

[24] P.T. Johnstone. *Sketches of an elephant: a topos theory compendium*, volume 2 of *Studies in Logic and the foundations of mathematics*. Oxford Univ. Press, 2002.

[25] G. M. Kelly. *Basic Concepts of Enriched Category Theory*, volume 64 of *Lecture Notes in Mathematics*. Cambridge University Press, 1982.

[26] G.M Kelly. A note on relations relative to a factorization system. In I. C. Pedicchio A. Carboni and G. Rosolini, editors, *Category Theory '90*, volume 1488, pages 249–261. Springer-Verlag, 1992.

[27] G.M. Kelly and S. Lack. On property-like structure. *Theory Appl. Categ.*, 3(9):213–250, 1997.

[28] G.M. Kelly and R. Street. Review of the elements of 2-categories. In *Category Seminar. Lecture Notes in Mathematics*, volume 420, pages 75–103. Springer, 1974.

[29] A. Klein. Relations in categories. *Illinois J. Math.*, 14:536–550, 1970.

[30] H. Kleisli. Every standard construction is induced by a pair of adjoint functors. *Proc. Amer. Math. Soc.*, 16:544–546, 1965.

[31] S. Lack. A note on the exact completion of a regular category, and its infinitary generalization. *Theory Appl. Categ.*, 5(3):70–80, 1999.

[32] S. Lack. *A 2-Categories Companion*, pages 105–191. Springer New York, New York, NY, 2010.

[33] J. Lambek and P. J. Scott. *Introduction to Higher Order Categorical Logic*. Cambridge Univ. Press, 1986.

[34] F. W. Lawvere. *Functorial Semantics of Algebraic Theories*. PhD thesis, Columbia University, 1963.

[35] F.W. Lawvere. The category of categories as a foundation for mathematics. In Eilenberg, Harrison, MacLane, and Röhrl, editors, *Proceedings of the Conference on Categorical Algebra*, pages 1–20. Springer, 1966.

[36] F.W. Lawvere. Adjointness in foundations. *Dialectica*, 23(3/4):281–296, 1969.

[37] F.W. Lawvere. Diagonal arguments and cartesian closed categories. In *Category Theory, Homology Theory and their Applications*, volume 2, pages 134–145. Springer, 1969.

[38] F.W. Lawvere. Equality in hyperdoctrines and comprehension schema as an adjoint functor. In A. Heller, editor, *New York Symposium on Application of Categorical Algebra*, volume 2, pages 1–14. American Mathematical Society, 1970.

[39] T. Leinster. *Higher Operads, Higher Categories*. London Mathematical Society Lecture Notes Series. Cambridge Univ. Press, 2003.

[40] S. MacLane and I. Moerdijk. *Sheaves in Geometry and Logic. A First Introduction to Topos Theory*. Spinger, 1992.

[41] M. Maietti, F. Pasquali, and G. Rosolini. Triposes, exact completions, and hilbert's ε-operator. *Tbil. Math. J.*, 10(3):141–166, 2017.

[42] M.E. Maietti and G. Rosolini. Elementary quotient completion. *Theory Appl. Categ.*, 27(17):445–463, 2013.

[43] M.E. Maietti and G. Rosolini. Quotient completion for the foundation of constructive mathematics. *Log. Univers.*, 7(3):371–402, 2013.

[44] M.E. Maietti and G. Rosolini. Unifying exact completions. *Appl. Categ. Structures*, 23:43–52, 2013.

[45] M. Makkai and G. Reyes. *First Order Categorical Logic*, volume 611 of *Lecture Notes in Math*. Springer-Verlag, 1977.

[46] F. Marmolejo. Distributive laws for pseudo monads. *Theory Appl. Categ.*, 5(5):91–147, 1999.

[47] F. Marmolejo and R.J. Wood. Coherence for pseudodistributive laws revisited. *Theory Appl. Categ.*, 20(6):74–84, 2008.

[48] A. Martelli and U. Montanari. An efficient unification algorithm. *ACM Trans. Program. Lang. Syst.*, 4(2):258–282, 1982.

[49] P. Martin-Löf. *Intuitionistic Type Theory*. Studies in Proof Theory. Bibliopolis, 1984.

[50] A. M. Pitts. Categorical logic. In S. Abramsky, D. M. Gabbay, and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science*, volume 6, pages 39–129. Oxford Univ. Press, 1995.

[51] A. M. Pitts. Tripos theory in retrospect. *Math. Struct. in Comp. Science*, 12:265–279, 2002.

[52] J.A. Robinson. A machine-oriented logicbased on the resolution principle. *J. ACM*, 12(1):23–41, 1965.

[53] R. Street and S. Lack. The formal theory of monads ii. *J. Pure Appl. Algebra*, 175:243–265, 2002.

[54] M. Tanaka. *Pseudo-distributive laws and a unified framework for variable binding*. PhD thesis, The University of Edinburgh, 2004.

[55] M. Tanaka and J. Power. Pseudo-distributive laws and axiomatics for variable binding. *Higher-Order Symb. Comput.*, 19:305–337, 2006.

[56] M. Tanaka and J. Power. A unified category-theoretic semantics for binding signatures in substructural logics. *J. Logic Comput.*, 16(1):5–25, 2006.

[57] D. Trotta. Existential completion. submitted, 2019.