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The matrix-valued numerical range over finite fields

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Abstract: In this paper we define and study the matrix-valued $k \times k$ numerical range of $n \times n$ matrices using the Hermitian product and the product with $n \times k$ unitary matrices U (on the right with U, on the left with its adjoint $U^\dagger = U^{-1}$). For all $i, j = 1, \dots, k$ we study the possible (i,j)-entries of these $k \times k$ matrices. Our results are for the case in which the base field is finite, but the same definition works over $\mathbb{C}$. Instead of the degree 2 extension $\mathbb{R} \hookrightarrow \mathbb{C}$ we use the degree 2 extension $\mathbb{F}_q \hookrightarrow \mathbb{F}_{q^2}$, q a prime power, with the Frobenius map $t \mapsto t^q$ as the nonzero element of its Galois group. The diagonal entries of the matrix numerical ranges are the scalar numerical ranges, while often the nondiagonal entries are the entire $\mathbb{F}_{q^2}$. We also define the matrix-valued numerical range map.

Key words: numerical range, finite field, unitary matrix

1. Introduction

In this paper we define and study the matrix-valued numerical range of $n \times n$-matrices with respect to a Hermitian product. Our results are for the case in which the base field is finite, but the same definition works over $\mathbb{C}$. Instead of the degree 2 extension $\mathbb{R} \hookrightarrow \mathbb{C}$ we use the degree 2 Galois extension $\mathbb{F}_q \hookrightarrow \mathbb{F}_{q^2}$, q a prime power, with the Frobenius map $t \mapsto t^q$ as the nonzero element of its Galois group. The main results of this paper only study the single entries of these numerical range matrices.

Recall that for each prime power q there is a unique, up to isomorphisms, field $\mathbb{F}_q$ with q elements and that $\mathbb{F}_{q^2}$ is a degree 2 Galois extension of $\mathbb{F}_q$ (see [10], [11, Theorem 2.5], and [12]). The nonzero element of the Galois group of the extension map $\mathbb{F}_q \subset \mathbb{F}_{q^2}$ is the Frobenius map $t \mapsto t^q$. The Frobenius map allows the definition of the following Hermitian form on $\mathbb{F}_{q^2}^n$ (exactly as the complex conjugation allows the definition of the Hermitian form $\mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$).

Let q be a prime power. For any $n > 0$ and any $u = (u_1, \dots, u_n) \in \overline{\mathbb{F}}_q^n$, $v = (v_1, \dots, v_n)$ $\overline{\mathbb{F}}_q^n$ set $\langle u, v\rangle := \sum_{i=1}^n u_i^q v_i$. For any integer $n > 0$ and any $a \in \mathbb{F}_q$ set $\mathcal{H}_{n,a} := \{u \in \mathbb{F}_{q^2}^n \mid \langle u, u\rangle = a\}$. Set $\mathrm{Num}_a(M) := \{\langle u, Mu\rangle \mid u \in \mathcal{H}_{n,a}\}$. The set $\mathrm{Num}_a(M)$ is called the a*-numerical range* of M [1–7]. The set $\mathrm{Num}(M) := \mathrm{Num}_1(M)$ is called the *numerical range* of M.

For any $M = (m_{ij}) \in M_{n,n}(\mathbb{F}_{q^2})$ set $M^\dagger = (m_{ji}^q)$. The square matrix M is said to be *Hermitian* if $M = M^\dagger$.

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Fix $E \in M_{k,k}(\mathbb{F}_{q^2})$ such that $E^\dagger = E$. For each positive integer n set

$$C_{n,k}(E) := \{A \in M_{n,k}(\mathbb{F}_{q^2}) \mid A^\dagger A = E\}.$$

If $E = a\mathbb{I}_{k,k}$, where $\mathbb{I}_{k,k}$ is the identity matrix and $a \in \mathbb{F}_q$, we write $C_{n,k}(a)$ instead of $C_{n,k}(E)$. For any matrix $M \in M_{n,n}(\mathbb{F}_{q^2})$ the (k,E)-numerical range $\mathrm{Num}_{k,E}(M)$ (or just (k,a)-numerical range $\mathrm{Num}_{k,a}(M)$ if $E = a\mathbb{I}_{k,k}$) of M is the set of all $B \in M_{k,k}(\mathbb{F}_{q^2})$ of the form $A^\dagger MA$ for some $A \in C_{n,k}(E)$. Let U be a $k \times k$ unitary matrix, i.e. take $U \in M_{k\times k}(\mathbb{F}_{q^2})$ such that $U^\dagger U = \mathbb{I}_{k\times k}$. The map $A \mapsto U^\dagger A$ gives a bijection between $C_{n,k}(E)$ and $C_{n,k}(U^\dagger EU)$ and hence E and $U^\dagger EU$ give essentially the same numerical range for $k \times k$ matrices. Obviously $\mathrm{Num}_{k,E}(\mathbb{I}_{n\times n}) = \{E\}$. If $M = M^\dagger$, then all elements of $\mathrm{Num}_{k,a}(M)$ are Hermitian (Remark 2.9).

We always have $0\mathbb{I}_{k\times k} \in \mathrm{Num}_{k,0}(M)$ (just use the zero matrix $0 \in M_{n,k}(\mathbb{F}_{q^2})$). As in [3] we call $\mathrm{Num}'_{k,0}(M)$ the set of all $B \in M_{k,k}(\mathbb{F}_{q^2})$ of the form $A^\dagger MA$ for some $A \in C_{n,k}(E)$ with $A \neq 0$. Call $\mathrm{Num}_{k,0}(M)''$ the subset of $\mathrm{Num}_{k,0}(M)$ obtained using only $A \in C_{n,k}(0)$ in which no column vector and no row vector is zero.

For all positive integers i,j such that $1 \le i \le k$ and $1 \le j \le n$ let $\pi_{ij}(\mathrm{Num}_{k,E}(M))$ (or $\pi_{ij}(\mathrm{Num}_{k,a}(M))$ if $E = a\mathbb{I}_{k\times k}$) denote the set of all $\alpha \in \mathbb{F}_{q^2}$, which are the (i,j)-entries of some $A \in \mathrm{Num}_{k,E}(M)$. These subsets of $\mathbb{F}_{q^2}$ are usually very easy to compute and often quite large (see Proposition 4.3 and Theorem 4.5). This may seem to be a disappointment, but we saw in [5] plenty of ways to use Num_a to distinguish matrices M, M' with $\mathrm{Num}_a(M) = \mathrm{Num}_a(M')$. For another way to distinguish between M and M' see the numerical map described at the end of the introduction.

Set $\mathrm{Num}_k(M) := \mathrm{Num}_{k,1}(M)$ and $\pi_{ij}(N_k(M)) := \pi_{ij}(\mathrm{Num}_{k,1}(M))$. If we know the set $\mathrm{Num}_{k,1}(M)$, then we know all sets $\mathrm{Num}_{k,a}(M)$, $a \in \mathbb{F}_q \setminus \{0\}$ (Remark 2.5).

Question 1.1 *Fix integers $n \ge 2$ and $k > 0$. What is the maximal cardinality $\gamma(q,n,k)$ (resp. $\gamma(q,n,k,0)$) of some $\mathrm{Num}_k(M)$ (resp. $\mathrm{Num}_{k,0}(M)$), $M \in M_{n,n}(\mathbb{F}_{q^2})$, and the minimum $\delta(q,n,k)$ (resp. $\delta(q,n,k,0)$) among all M that are not a multiple of the identity?*

Fix positive integers n,k, $a \in \mathbb{F}_q$, and $M \in M_{n,n}(\mathbb{F}_{q^2})$. The $(n \times k, a)$*-numerical map* $\nu_{M,n\times k,a}$: $C_{n,k}(a) \to M_{k,k}(\mathbb{F}_{q^2})$ of M is the map $C_{n,k}(a) \to M_{k,k}(\mathbb{F}_{q^2})$ defined by the formula $A \mapsto A^\dagger MA$. We have $\mathrm{Im}(\nu_{M,n\times k,a}) = \mathrm{Num}_{k,a}(M)$ and hence to give upper (resp. lower) bounds on the integer $\sharp(\mathrm{Num}_{k,a}(M))$ it is sufficient to give "very good" lower (resp. upper) bounds on the cardinality of the fibers of the map $\nu_{M,n\times k,a}$. By Remark 2.5, to know all $\nu_{M,n\times k,a}$, $a \in \mathbb{F}_q \setminus \{0\}$, it is sufficient to know $\nu_{M,n\times k,1}$. Thus, it is sufficient to study $\nu_{M,n\times k,1}$ and $\nu_{M,n\times k,0}$. See Remark 5.1 and Proposition 5.2 for some results concerning the numerical map.

2. Foundational remarks

For any matrix $M = (m_{ij}) \in M_{n,n}(\mathbb{F}_{q^2})$ let $M^\dagger = (a_{ij})$ be the matrix with $a_{ij} = m_{ji}^q$ for all i,j. M is said to be *Hermitian* if $M^\dagger = M$. Note that the diagonal elements of a Hermitian matrix are contained in $\mathbb{F}_q$. Let $e_1 = (1,0,\dots,0),\dots,e_n = (0,\dots,0,1)$ be the standard basis of $\mathbb{F}_{q^2}^n$. Let $\mathbb{I}_{n\times n}$ denote the identity $n \times n$ matrix. For any $a \in \mathbb{F}_q$ set $C_a(a) := \{z \in \mathbb{F}_{q^2}^n \mid \langle z,z\rangle = a\}$ ([1, 2, 4, 5]).

Notation 2.1 *Write* $M = (m_{ij})$, $i, j = 1, \dots, n$.

Remark 2.2 *For each* $a \in \mathbb{F}_q \setminus \{0\}$ *there are exactly* $q+1$ *elements of* $\overline{\mathbb{F}}_q$ *such that* $z^{q+1} = a$ *and all of them are in* $\mathbb{F}_{q^2}$ *(see [11, Theorem 2.8] and [1, Remark 3]).* 0 *is the only element of* $\overline{\mathbb{F}}_q$ *such that* $z^{q+1} = 0$.

Remark 2.3 *By Remark 2.2 the set* $C_2(0)$ *is the union of* $(0,0)$ *and the set of all* $(u,v) \in (\mathbb{F}_{q^2} \setminus \{0\})^2$ *such that* $v = tu$ *for some* t *with* $t^{q+1} = -1$ *and hence* $\sharp(C_2(0)) = 1 + (q^2-1)(q+1) = q^3 + q^2 - q$ *by Remark 2.2. Now take* $a \neq 0$. *The integer* $\sharp(C_2(a))$ *is the number of* $\mathbb{F}_{q^2}$*-solutions of the equation* $x^{q+1} + y^{q+1} = a$. *For any* y *such that* $y^{q+1} = a$ *(and there are* $q+1$ *such* y*s by Remark 2.2) there is a unique* x *satisfying the equation* $x^{q+1} = 0$. *For all* y *such that* $y^{q+1} \neq a$ *we get* $q+1$ *possible* y*s. Thus,* $\sharp(C_2(a)) = q+1+(q^2-q-1)(q+1) = (q+1)(q^2-q)$.

Remark 2.4 *If* $B \in \mathrm{Num}_{k,E}(M)$, *then* $B^\dagger \in \mathrm{Num}_{k,E}(M^\dagger)$, *because if* $A \in C_{n,k}(E)$, *then* $A^\dagger \in C_{n,k}(E)$ *and* $(A^\dagger M A)^\dagger = A^\dagger M^\dagger A$.

Remark 2.5 *Take* $a \in \mathbb{F}_q \setminus \{0\}$. *Fix* $c \in \mathbb{F}_{q^2}$ *such that* $c^{q+1} = a$ *(Remark 2.2). Fix* $A \in C_{n,k}(1)$. *Since* $(cA)^\dagger(cA) = c^{q+1}A^\dagger A$, *we have* $cA \in C_{n,k}(a)$. *For any* $M \in M_{n,n}(\mathbb{F}_{q^2})$ *we have* $(cA)^\dagger M(cA) = c^{q+1}A^\dagger MA = aA^\dagger MA$. *Thus,* $\mathrm{Num}_{k,a}(M) = a\mathrm{Num}_{k,1}(M)$ *and* $\pi_{ij}(\mathrm{Num}_{k,a}(M)) = a\pi_{ij}(\mathrm{Num}_{k,1}(M))$. *Thus, it is sufficient to study* $\mathrm{Num}_{k,1}(M)$ *and* $\mathrm{Num}_{k,0}(M)$.

The following two results show that the case $a = 0$ is simpler.

Lemma 2.6 *If* $B \in \mathrm{Num}_{k,0}(M)$, *then* $zB \in \mathrm{Num}_{k,0}(M)$ *for all* $z \in \mathbb{F}_q \setminus \{0\}$.

Proof Take $t \in \mathbb{F}_{q^2}$ such that $t^{q+1} = z$ (Remark 2.2) and $A \in C_{n,k}(0)$ such that $B = A^\dagger MA$. We have $tA \in C_{n,k}(0)$, because $(tA)^\dagger(tA) = t^{q+1}A^\dagger A = 0\mathbb{I}_{k\times k}$. We have $(tB)^\dagger M(tB) = t^{q+1}A^\dagger MA$. □

Remark 2.7 *Take* $a = 0$ *and* $k \geq 2$. *Fix* $t \in \mathbb{F}_{q^2}$, $M \in M_{n,n}(\mathbb{F}_{q^2})$, *and* $i \in \{1, \dots, k\}$. *Take* $A \in C_{n,k}(0)$. *Call* $u_1, \dots, u_k$ *the column vectors of* A. *Call* $A_{t,i}$ *the element of* $C_{n,k}(0)$, *which has as its column vectors the ones of* A, *except the* i*th one, which is* tu_i. *Let* z_{xy} *(resp.* w_{xy}*) be the* (x,y)*-entry of the matrix* $A^\dagger MA$ *(resp.* $A^\dagger_{t,i}MA_{t,i}$*). We have* $w_{xy} = z_{xy}$ *if* $i \notin \{x,y\}$, $w_{ii} = t^{q+1}z_{ii}$, $w_{iy} = t^q z_{iy}$ *if* $y \neq i$, *and* $w_{xi} = tz_{xi}$ *if* $x \neq i$. *Thus, if* $i \neq j$ *either* $\pi_{ij}(\mathrm{Num}_{k,0}(M)) = \mathbb{F}_{q^2}$ *or* $\pi_{ij}(\mathrm{Num}_{k,0}(M)) = \{0\}$ *and if* $c \in \pi_{ii}(\mathrm{Num}_{k,0}(M))$, *then* $t^{q+1}c \in \pi_{ii}(\mathrm{Num}_{k,0}(M))$ *for all* $t \in \mathbb{F}_{q^2}$. *Hence, by Remark 2.2 the set* $\pi_{ii}(\mathrm{Num}_{k,0}(M))$ *is an* $\mathbb{F}_q$*-linear subspace of* $\mathbb{F}_{q^2}$.

Remark 2.8 *Take* $a \in \mathbb{F}_q \setminus \{0\}$ *and* $k \geq 2$. *Fix* $t \in \mathbb{F}_{q^2}$ *such that* $t^{q+1} = 1$ *(there are* $q+1$ *such entries by Remark 2.2). Fix* $M \in M_{n,n}(\mathbb{F}_{q^2})$ *and* $i \in \{1, \dots, k\}$. *Take* $A \in C_{n,k}(a)$. *Call* $u_1, \dots, u_k$ *the column vectors of* A. *Let* $A_{t,i}$ *be the element of* $C_{n,k}(a)$ *with as its column vectors the same as the ones of* A, *except the* i*th one, which is* tu_i. *Let* z_{xy} *(resp.* w_{xy}*) be the* (x,y)*-entry of the matrix* $A^\dagger MA$ *(resp.* $A^\dagger_{t,i}MA_{t,i}$*). We have* $w_{xy} = z_{xy}$ *if* $i \notin \{x,y\}$, $w_{ii} = z_{ii}$, $w_{iy} = t^q z_{iy}$ *if* $y \neq i$, *and* $w_{xi} = tz_{xi}$ *if* $x \neq i$. *Thus, if* $i \neq j$ *and* $\alpha \in \pi_{ij}(\mathrm{Num}_{k,a}(M))$, *then* $t\alpha \in \pi_{ij}(\mathrm{Num}_{k,a}(M))$.

Remark 2.9 *For all* $M \in M_{n,n}(\mathbb{F}_{q^2})$ *and all* $A \in C_{n,k}(a)$ *we have* $(A^\dagger MA)^\dagger = A^\dagger M^\dagger A$. *Thus, if* $F \in \mathrm{Num}_{k,a}(M)$, *then* $F^\dagger \in \mathrm{Num}_{k,a}(M^\dagger)$. *Now assume* $M = M^\dagger$. *Since* $(A^\dagger MA)^\dagger = A^\dagger M^\dagger A = A^\dagger MA$ *for all* $A \in C_{n,k}(a)$, *every element of* $\mathrm{Num}_{k,a}(M)$ *is Hermitian.*

3. $C_{n,k}(a)$

Take $a \in \mathbb{F}_q$ and $A \in M_{n,k}(\mathbb{F}_{q^2})$. Let $u_1,\dots,u_k$ be the column vectors of A. We have $A \in C_{n,k}(a)$ if and only if $\langle u_i, u_j\rangle = a\delta_{ij}$ for all i,j. Thus, $A \in C_{n,k}(0)$ if and only if the linear span of its columns is contained in the Hermitian variety $C_n(0)$ over $\mathbb{F}_{q^2}$. The set $C_n(0)$ is the affine cone of the $(n-1)$-dimensional Hermitian variety (see [8, Ch. V] and [9, Ch. 23]). Thus, $C_{n,k}(0) \ne \emptyset$ for all $k > 0$, and $\lfloor (n-2)/2 \rfloor + 1$ is the maximal integer k such that there is $A \in C_{n,k}(0)$ whose columns are linearly independent [9, Lemma 23.3.1], while $C_{n,k}(0) \ne \emptyset$ if and only if $1 \le k \le n$. Take $B \in M_{n,n}(\mathbb{F}_{q^2})$ such that $B^\dagger B = \mathbb{I}_{n,n}$, i.e. take a unitary B. Since $A^\dagger B^\dagger BA = A^\dagger A$ for all $A \in M_{k,k}(\mathbb{F}_{q^2})$, left multiplication by B induces a bijection of $C_{n,k}(E)$. Hence, $\mathrm{Num}_{k,E}(B^\dagger MB) = \mathrm{Num}_{k,E}(M)$ for every $M \in M_{n,n}(\mathbb{F}_{q^2})$.

Remark 3.1 *Fix* $a \in \mathbb{F}_q$ *and positive integers* n,k.

1. *The zero-matrix is an element of* $C_{n,k}(0)$.
2. *We have* $A \in C_{k,k}(a)$, $a \in \mathbb{F}_q$, *if and only if* $A^\dagger \in C_{k,k}(a)$.
3. *Assume* $a \ne 0$. *Since* $\langle\ ,\ \rangle$ *is nondegenerate, the column vectors* $u_1,\dots,u_k$ *of any* $A \in C_{n,k}(a)$ *are linearly independent and in particular* $C_{n,k}(a) = \emptyset$ *for all* $a \ne 0$ *and* $n > k$.
4. *For all* $y > n$, *all* $x > k$, *and* $A \in C_{n,k}(0)$ *we may extend* A *to an element of* $C_{x,k}(0)$, *an element of* $C_{n,y}(0)$, *and an element of* $C_{x,y}(0)$, *adding zeros as the new entries. These new matrices cannot be used to test* Num'', *but they may be used to test* Num *and* Num'.

See [1, page 171], [8, Ch. V], and [9, Ch. 23] for the integer $\sharp(C_{n-k}(1))$ appearing in Lemma 3.6 and the main properties of Hermitian varieties.

Remark 3.2 *Take* $k > 0$, $1 \le i \le n$, $1 \le j \le n$, $a \in \mathbb{F}_q$, $c \in \mathbb{F}_{q^2}$, *and* $M \in M_{n,n}(\mathbb{F}_{q^2})$. *We have* $\mathrm{Num}_{k,a}(M - c\mathbb{I}_{n\times n}) = \mathrm{Num}_{k,a}(M) - ca$ *and hence* $\pi_{ij}(\mathrm{Num}_{k,a}((M - c\mathbb{I}_{n\times n}))) = \pi_{ij}(\mathrm{Num}_{k,a}(M)) - ca$.

Remark 3.3 *Fix* $(t_1,\dots,t_k) \in (\mathbb{F}_{q^2} \setminus \{0\})^k$ *and set* $\underline{t} := (t_1,\dots,t_k)$. *For any* $A \in M_{n,k}(\mathbb{F}_{q^2})$ *with* $u_1,\dots,u_k$ *as its column vectors let* $\underline{t}A$ *be the* $n \times k$ *matrix with* $t_1u_1,\dots,t_ku_k$ *as its column vector. Note that* $A \in C_{n,k}(0)$ *if and only if* $\underline{t}A \in C_{n,k}(0)$. *If* $t_i^{q+1} = t_1^{q+1}$ *for all* i, *then for any* $a \in \mathbb{F}_q$ $A \in C_{n,k}(a)$ *if and only if* $\underline{t}A \in C_{n,k}(t_1^{q+1}a)$.

Remark 3.4 *If* $M \in M_{n,n}(\mathbb{F}_{q^2})$ *and* $A \in M_{n,k}(\mathbb{F}_{q^2})$ *we have* $(A^\dagger MA)^\dagger = A^\dagger M^\dagger A$. *If* $A^\dagger A = E$ *with* $E = E^\dagger$, *then* $AA^\dagger E$. *Thus,* $\mathrm{Num}_{k,E}(M^\dagger)$ *is obtained from* $\mathrm{Num}_{k,E}(M)$ *taking* † *and if* $M = M^\dagger$ *every element of* $\mathrm{Num}_{k,E}(M)$ *is Hermitian.*

The next lemma describes $C_{2,2}(0)$.

Lemma 3.5 *Take*

$$A = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$$

. *We have* $A^\dagger A = 0\mathbb{I}_{2\times 2}$ *if and only if there are* $t_0, t_1 \in \mathbb{F}_{q^2}$ *such that* $t_0^{q+1} = t_1^{q+1} = -1$, $x_{11} = t_0 x_{12}$, $x_{21} = t_1 x_{22}$ *and* $t_0 x_{12}^{q+1} + t_1 x_{22}^{q+1} = 0$.

Proof The $(1,1)$ (resp. $(2,2)$) entry of $A^\dagger A$ shows that $(x_{11}, x_{12}) \in C_2(0)$ (resp. $(x_{21}, x_{22}) \in C_2(0)$) and hence there is $t_0 \in \mathbb{F}_{q^2}$ (resp. $t_1 \in \mathbb{F}_{q^2}$) such that $t_0^{q+1} = -1$ and $x_{11} = t_0 x_{12}$ (resp. $t_1^{q+1} = -1$ and $x_{21} = t_1 x_{22}$). The vanishing of the $(1,2)$ entry of $A^\dagger A$ is equivalent to $t_0 x_{12}^{q+1} + t_1 x_{22}^{q+1} = 0$, which is equivalent to $t_0^q x_{12}^{q+1} + t_1^q x_{22}^{q+1} = 0$, i.e. the vanishing of the $(2,1)$-entry of $A^\dagger A$, because $t^q = t$ for all $t \in \mathbb{F}_q$ and $t^{q+1} \in \mathbb{F}_q$ for all $t \in \mathbb{F}_{q^2}$. □

Lemma 3.6 *Take* $A \in C_{n,k}(a)$, $a \neq 0$, *and* $n > k$. *Then there are* $\sharp(C_{n-k}(1))$ *matrices* $B \in C_{n,k+1}(a)$ *whose first* k *column vectors are the ones of* A.

Proof Call $u_1, \dots, u_k \in \mathbb{F}_{q^2}^n$ the column vectors of A. Set $V := \{v \in \mathbb{F}_{q^2}^n \mid \langle v, u_i \rangle = 0 \text{ for all } i = 1, \dots, k\}$. V is a linear subspace of $\mathbb{F}_{q^2}^n$ and the restriction of $\langle\ ,\ \rangle$ to V is nondegenerate. The possible $(k+1)$th column vectors of B are the elements $v \in V$ such that $\langle v, v \rangle = a$. By Remark 3.1 we get the same number for all $a \in \mathbb{F}_q \setminus \{0\}$. Hence, the number in the lemma is the integer $\sharp(C_{n-k}(a))$. Recall that $\sharp(C_{n-k}(a)) = \sharp(C_{n-k}(1))$ for all $a \neq 0$. Thus, $(a-1)\sharp(C_{n-k}(1)) + \sharp(C_{n-k}(0)) = \sharp(C_{n-k+1}(0))$. □

4. The single entries of the numerical matrix range

Take positive integers n, k, $a \in \mathbb{F}_q$, and $M \in M_{n,n}(\mathbb{F}_{q^2})$. Write $M = (m_{uv})$, $1 \le u \le n$, $1 \le v \le n$.

Remark 4.1 *Fix integers* i, j *such that* $1 \le i \le n$ *and* $1 \le j \le n$. *Fix any* $A \in M_{k,k}(\mathbb{F}_{q^2})$ *such that* $A^\dagger A = a\mathbb{I}_{k\times k}$ *and call* $u_1, \dots, u_k$ *the column vectors of* A. *Note that* $u_i^\dagger M u_j$ *is the* (i,j)*-entry of* $A^\dagger M A$. *In particular* $\pi_{ii}(\mathrm{Num}_{k,a}(M)) \subseteq \mathrm{Num}_a(M)$ *for all* i, j, a *and equality holds if every* $u \in \mathbb{F}_{q^2}^n$ *with* $\langle u, u \rangle = a$ *is a column vector of some* $A \in M_{n,k}(\mathbb{F}_{q^2})$ *such that* $A^\dagger A = a\mathbb{I}_{k\times k}$. *This is always true if either* $a = 0$ *or* $k \le n$.

By Remark 4.1 to look at all $\pi_{ij}(\mathrm{Num}_{k,a}(M))$ it is sufficient to handle the case $i \neq j$. Obviously $\pi_{ij}(\mathrm{Num}_{k,a}(M)) = \{0\}$ for any i, j and a if $M = 0$.

Remark 4.2 *Take* $M \in M_{n,n}(\mathbb{F}_{q^2})$, $A \in C_{n,k}(a)$, *and* $(i,j) \in \{1, \dots, n\} \times \{1, \dots, n\}$. *Let* $u_1, \dots, u_k \in \mathbb{F}_{q^2}^n$ *be the column vectors of* A. *Note that* $\langle u_i, M u_j \rangle$ *is the* (i,j)*-entry of* $A^\dagger M A$.

Proposition 4.3 *Fix integers* $n \ge 2$, $k \ge 2$, *and* $1 \le i \le n$, $1 \le j \le \min\{n, k\}$, *such that* $i \neq j$. *Fix* $M \in M_{n,n}(\mathbb{F}_{q^2})$.

(i) Assume $n = k$. *We have* $\pi_{ij}(\mathrm{Num}_{n,0}(M)) = \{0\}$ *if and and only if* $m_{ii} = m_{jj}$, $m_{iy} = 0$ *for all* $y \neq i$ *and* $m_{xj} = 0$ *for all* $x \neq j$. *In all other cases we have* $\pi_{ij}(\mathrm{Num}_{k,0}(M)') = \mathbb{F}_{q^2}$.

(ii) If $n > k$, then $\pi_{ij}(\mathrm{Num}_{k,0}(M)) = \mathbb{F}_{q^2}$ if either $(m_{ii} - m_{jj}, m_{ij}, m_{ji}) \neq (0,0,0)$ or there is $y \in \{1,\dots,k\} \setminus \{j\}$ such that $m_{iy} \neq 0$.

(iii) If $n < k$, then $\pi_{ij}(\mathrm{Num}_{k,0}(M)) = \mathbb{F}_{q^2}$ if either $(m_{ii} - m_{jj}, m_{ij}, m_{ji}) \neq (0,0,0)$ or there is $x \in \{1,\dots,n\} \setminus \{j\}$ such that $m_{xj} \neq 0$.

Proof Taking $A = 0\mathbb{I}_{n,k}$ we get $0 \in \pi_{ij}(\mathrm{Num}_{k,0}(M))$ for all M.

(i) Assume $n = k$. Either $\pi_{ij}(\mathrm{Num}_{k,0}(M)) = \mathbb{F}_{q^2}$ or $\pi_{ij}(\mathrm{Num}_{k,0}(M)) = \{0\}$ (Remark 2.7). Note that the entries of $\pi_{ij}(\mathrm{Num}_{k,0}(M))$ are the numbers $u_i^\dagger M u_j$, for some ith and jth column vectors of some $A \in C_{n,k}(0)$. Thus, we see that $\pi_{ij}(\mathrm{Num}_{k,0}(M)) = \{0\}$ if $m_{ii} = m_{jj}$, $m_{iy} = 0$ for all $y \neq i$ and $m_{xj} = 0$ for all $x \neq j$.

Now we prove the "only if" part. Since $A^\dagger c\mathbb{I}_{n,n}A = cA^\dagger A = 0\mathbb{I}_{k\times k}$ for all $A \in C_{n,k}(0)$, we have $\mathrm{Num}_{k,0}(c\mathbb{I}_{n\times n}) = \{0\}$ for all $c \in \mathbb{F}_{q^2}$. Since $\mathrm{Num}_{k,0}(c\mathbb{I}_{n\times n}) = \{0\}$, we have $\pi_{ij}(\mathrm{Num}_{k,0}(M)') = \pi_{ij}(\mathrm{Num}_{k,0}(M - m_{jj}\mathbb{I}_{n\times n}))')$. Thus, we may assume $m_{jj} = 0$ and hence $m_{ii} = m_{jj}$ if and only if $m_{ii} = 0$. Up to a permutation of the indices (which is induced by Hermitian transformations) we may assume $i = 1$ and $j = 2$. Note that $m_{22} = 0$.

(a) In this step we assume $m_{22} = 0$ and $(m_{11}, m_{12}, m_{21}) \neq (0,0,0)$ and prove that $\pi_{ij}(\mathrm{Num}_{k,0}(M)) \neq \{0\}$.

(a1) Take $n = k = 2$. It is sufficient to find $A = (a_{ij}) \in C_{2,2}(0)$ with $a_{11} = x$, $a_{21} = y$, $a_{12} = u$, and $a_{22} = v$ such that $Z := m_{11}x^q u + m_{21}y^q u + m_{12}x^q v \neq 0$, because the right-hand side of the last equality is the $(1,2)$-entry of $A^\dagger MA$. We have $A \in C_{2,2}(0)$ if and only if $x^q u + y^q v = 0$ and there are $t, c \in \mathbb{F}_{q^2}$ such that $y = tx$, $v = cu$, $t^{q+1} = -1$, and $c^{q+1} = -1$. We have $Z = x^q u(m_{11} + t^q m_{21} + cm_{12})$. Thus, to get $Z \neq 0$ we need $xu \neq 0$. When $xu \neq 0$ the condition $x^q u + y^q v = 0$ is satisfied if and only if $t^q c = -1$. Hence, we get $Z \neq 0$ if and only if there is $c \in \mathbb{F}_{q^2}$ such that $c^{q+1} = -1$ and $c^2 m_{12} + cm_{11} - m_{12} \neq 0$. Since $(m_{11}, m_{12}, m_{21}) \neq (0,0,0)$, the equation $z^2 m_{21} + zm_{11} - m_{12}$ has at most 2 roots. Since there are $q+1$ elements $c \in \mathbb{F}_{q^2}$, with $c^{q+1} = -1$ (Remark 2.2), we get some $Z \neq 0$.

(a2) Take $(n,k) \neq (2,2)$. Take $A = (a_{ij}) \in C_{n,k}(0)$ with $a_{ij} = 0$ if either $i > 2$ or $j > 2$ and apply step (a1) to the upper-left corner 2×2 submatrix of A.

(b) From now on we assume $(n,k) \neq (2,2)$ and $m_{12} = m_{22} = m_{11} = m_{21} = 0$.

(b1) First assume $m_{1x} \neq 0$ for some x and $n \geq k$. Up to a permutation of the indices we assume $x = 3$. Adding zero entries as in step (a2) we see that it is sufficient to prove the case $n = k = 3$. Taking $\frac{1}{m_{13}}M$ instead of M we reduce to the case $m_{13} = 1$. Thus, M is the matrix in (4.1). Take $A = (x_{ij}) \in C_{3,3}(0)$. It is sufficient to find A such that (4.2) has a solution $Z \neq 0$. There are $q+1$ elements t of $\mathbb{F}_{q^2}$ such that $t^{q+1} = -1$ (Remark 2.2). Take $u_1 = (1,0,t)$, $u_2 = (t,0,-1)$, and $u_3 = (0,0,0)$. Since $(-1)^q = -1$ for every prime-power q, even in characteristic 2, the left-hand side of (4.2) is $h(t) = -(m_{33} + m_{23})t^q - 1$. Since $h(t)$ is a nonzero polynomial of degree at most q, we may find t with $h(t) \neq 0$.

(b2) Applying the argument of step (b1) we conclude if $k \geq n$ and there is $y \in \{3,\dots,k\}$ such that $m_{y2} \neq 0$.

(ii) Assume $n > k$. By assumption, $j \leq k$. Fix $\alpha \in \mathbb{F}_{q^2} \setminus \{0\}$. Call M_1 the submatrix of M formed by its first k rows. By step (i) there is $B \in C_{n,n}(0)$ such that α is the (i,j)-entry of $B^\dagger M_1 B$. Let A be the

$n \times k$ matrix with B in its left upper $k \times k$ corner and 0 as all its other entries. The matrix $A^{\dagger}MA$ has α as its (i,j)-entry.

(iii) Assume $n < k$. By assumption, $j \le n$. Fix $\alpha \in \mathbb{F}_{q^2} \setminus \{0\}$. By step (i) there is $B = (v_1, \dots, v_n) \in C_{n,n}(0)$ such that $B^{\dagger}MA^{\dagger}$ has α as its (i,j) entry (part (i)). Take $A \in C_{n,k}(0)$ with $v_1, \dots, v_n$ as its first n column vectors and zero as its other entries. The matrix $A^{\dagger}MA$ has α as its (i,j)-entry. □

As a corollary of Proposition 4.3, we get the following result.

Proposition 4.4 *Take $M \in M_{n,n}(\mathbb{F}_{q^2})$, which is not a multiple of the identity, and assume $k \ge 2$. Then $\sharp(\mathrm{Num}_{k,0}(M)) \ge q^2$.*

Proof Since M is not a multiple of the identity, we have $n \ge 2$ and there are $i, j \in \{1, \dots, n\}$ such that $(m_{ii} - m_{jj}, m_{ij}, m_{ji}) \ne (0,0,0)$. Up to a permutation of the indices we may assume $i = 1$ and $j = 2$. By assumption we have $k \ge 2$. We only use $u_1, \dots, u_k \in C_{n,1}(0)$ such that $\langle u_x, u_y \rangle = 0$ for all x, y, $u_x = 0$ for all $x \ge 3$, and u_1, u_2 have 0 as their $n-2$ entries. With this trick we reduce to the case $n = k = 2$. By Proposition 4.3 we have $\pi_{12}(\mathrm{Num}_{2,0}(M)) = \mathbb{F}_{q^2}$. Thus, $\sharp(\mathrm{Num}_{k,0}(M)) \ge q^2$. □

Theorem 4.5 *Fix $a \in \mathbb{F}_q \setminus \{0\}$, an integer $n \ge 3$, $i, j \in \{1, \dots, n\}$ such that $i \ne j$ and $M \in M_{n,n}(\mathbb{F}_{q^2})$ such that $(m_{ii} - m_{jj}, m_{ij}, m_{ji}) = (0,0,0)$. We have $\pi_{ij}(\mathrm{Num}_{n,a}(M)) = \mathbb{F}_{q^2}$ if either there is $x \in \{3, \dots, n\}$ such that $m_{ix} \ne 0$ or there is $y \in \{3, \dots, n\}$ such that $m_{y2} \ne 0$.*

Proof Taking $M - m_{ii}\mathbb{I}_{n\times n}$ and using Remark 3.2 we reduce to the case $m_{ii} = m_{jj} = m_{ij} = m_{ji} = 0$.

(a) Assume the existence of $x \in \{3, \dots, n\}$ such that $m_{ix} \ne 0$. Up to a permutation of the indices we may take as $A \in C_{n,n}(a)$ a matrix with B in its left upper corner, 0 for all other entries either in columns 1, 2, or 3 or rows 1, 2, 3, and with as its last $n-3$ column vectors mutually orthogonal vectors v_x of $\mathbb{F}_{q^2}^{n-3}$ with $\langle v_x, v_x \rangle = a$ for all $x = 4, \dots, k$. Thus, we reduce to the case $n = k = 3$ with a matrix M with $m_{13} \ne 0$. Taking $\frac{1}{m_{13}}M$ instead of M, we reduce to the matrix

$$M = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}. \tag{4.1}$$

Take $A \in C_{3,3}(a)$ and fix $Z \in \mathbb{F}_{q^2}$. Write $A = (x_{ij})$. We need to solve the equation

$$m_{31}x_{11}^q x_{12} + m_{32}x_{21}^q x_{22} + m_{33}x_{31}^q x_{32} + m_{23}x_{21}^q x_{32} + x_{11}^q x_{32} = Z \tag{4.2}$$

with the restriction that $A \in C_{3,3}(a)$. We call u_1, u_2, and u_3 the column vectors of A.

First assume $Z \ne 0$. Take $t \in \mathbb{F}_{q^2}$ such that $t^{q+1} = a$ (Remark 2.2). We take $x_{12} = x_{21} = x_{31} = 0$ and $x_{11} = t$. Thus, $u_1 \in C_3(a)$. We take $x_{32} = Z/t^q$. With these choices of x_{ij} (4.2) is satisfied. By Remark 2.2 we may find x_{22} such that $(0, u_{22}, Z/t^q) \in C_3(a)$. We take u_3 with $x_{13} = 0$, $u_3 \in C_3(a)$, and $\langle u_2, u_3 \rangle = 0$. Now assume $Z = 0$. We take $u_2 = (0,t,0)$, $u_1 = (t,0,0)$, $u_3 = (0,0,t)$ (again $x_{12} = x_{21} = x_{31} = 0$).

(b) Assume the existence of $y \in \{3, \dots, n\}$ such that $m_{y2} \ne 0$. It is sufficient to mimic the proof of part (a). □

5. The numerical range map

Remark 5.1 *Fix $M \in M_{n,n}(\mathbb{F}_{q^2})$ and $A, B \in C_{n,n}(1)$. We have $A^\dagger MA = B^\dagger MB$ if and only if M and $AB^\dagger$ commute. If all eigenvalues of M over $\overline{\mathbb{F}}_{q^2}$ are distinct, then this is the case if and only if $AB^\dagger$ is a polynomial of degree $\le n-1$ in M with coefficients in $\mathbb{F}_{q^2}$ and hence (since $AB^\dagger \ne 0\mathbb{I}_{n\times n}$) there are at most nq^2-1 such M. If $M = (m_{ij})$ is diagonal with $m_{ii} \ne m_{jj}$ for all $i \ne j$ the commutator of M is given by the diagonal matrices. A diagonal matrix $A = (a_{ij}) \in C_n(a)$ if and only if $a_{ii}^{q+1} = a$ for all i. Thus, for $a = 0$ we get $A = 0\mathbb{I}_{n\times n}$, while all fibers of $\nu_{M,n\times n,a}$, $a \ne 0$, have cardinality $n(q+1)$. Hence,*

$$\sharp(\mathrm{Num}_{n,1}(M)) = \frac{\sharp C_n(1))}{n(q^2-1)}.$$

Proposition 5.2 *Take $n = k = 2$, $b \in \mathbb{F}_{q^2} \setminus \{0\}$, and*

$$M = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}.$$

Then each fiber of $\nu_{M,2\times 2,1}$ has cardinality $q+1$.

Proof Taking $\frac{1}{b}M$ instead of M we see that it is sufficient to consider the case $b = 1$. If $DM = MD$, with $D = (d_{ij})$, we have $d_{21} = 0$, because the multiples of $(1,0)$ are the only eigenvectors of M. If $D \in C_{2,2}(1)$ we also get $d_{12} = 0$ and $d_{11}^{q+1} = d_{22}^{q+1} = 1$. Since $DM = MD$, we get $d_{11} = d_{22}$. The assertion on the numerical map follows from Remarks 2.2 and 5.1. $\square$

6. The case $q = 2$

As in the case $k = 1$ the cases $q = 2$ and $q \ne 2$ are quite different (see [1, Remark 8] and [6]). In this section we always assume $q = 2$. We write $M = (m_{ij})$ and $A = (a_{ij})$.

Remark 6.1 *Take $x \in \mathbb{F}_4$. Since $q = 2$, we have $q+1 = q^2-1$. Thus, $x^{q+1} = 1$ if $x \ne 0$ and $x^{q+1} = 0$ if $x = 0$.*

Remark 6.2 *We have $A \in C_{n,k}(0)$ if and only if each column vector of A has an even number of nonzero entries and different columns are pairwise orthogonal. By part (2) of Remark 3.1 $A \in C_{n,k}(0)$ if and only if each row vector of A has an even number of nonzero entries and different row vectors are pairwise orthogonal.*

Remark 6.3 *We have $A \in C_{n,k}(1)$ if and only if each column vector of A has an odd number of nonzero entries and different columns are pairwise orthogonal. By part (2) of Remark 3.1 $A \in C_{n,k}(1)$ if and only if each row vector of A has an odd number of nonzero entries and different row vectors are pairwise orthogonal.*

Example 6.4 *Take $n = k = 2$ and take $A = (a_{ij}) \in M_{2,2}(\mathbb{F}_4)$. By Remarks 6.1 and 6.2 we have $A \in C_{2,2}(0)$ if and only if either $A = 0\mathbb{I}_{2\times 2}$ or $a_{11}^2a_{21} + a_{12}^2a_{22} = 0$. Thus, for each $(a_{11}, a_{12}, a_{22}) \in (\mathbb{F}_4 \setminus \{0\})^3$ there is a unique $a_{21} \in \mathbb{F}_4 \setminus \{0\}$ such that $A \in C_{2,2}(0)$. Thus, $\sharp(C_{2,2}(0)) = 28$.*

Example 6.5 *Take* $n = k = 2$ *and take* $A = (a_{ij}) \in M_{2,2}(\mathbb{F}_4)$*. By Remarks 6.1 and 6.3 we have* $A \in C_{2,2}(1)$ *if and only if either* $a_{12} = a_{21} = 0$ *and* $a_{11}a_{22} \neq 0$ *or* $a_{11} = a_{22} = 0$ *and* $a_{12}a_{21} \neq 0$*. Thus,* $\sharp(C_{2,2}(0)) = 18$*. First assume* $a_{12} = a_{21} = 0$ *and* $a_{11}a_{22} \neq 0$*, say* $a_{11} = a$ *and* $a_{22} = b$*. We get*

$$MA = \begin{pmatrix} am_{11} & bm_{12} \\ am_{21} & bm_{22} \end{pmatrix}$$

and hence

$$A^{\dagger}MA = \begin{pmatrix} m_{11} & a^q bm_{12} \\ ab^q m_{21} & m_{22} \end{pmatrix}.$$

Varying $a, b \in \mathbb{F}_4 \setminus \{0\}$ *we get all matrices*

$$\begin{pmatrix} m_{11} & cbm_{12} \\ c^q m_{21} & m_{22} \end{pmatrix}, \ c \in \mathbb{F}_4 \setminus \{0\}.$$

Take $a_{11} = a_{22} = 0$ *and* $a_{12}a_{21} \neq 0$*, say* $a_{12} = a$ *and* $a_{21} = b$*. Since*

$$MA = \begin{pmatrix} bm_{12} & am_{11} \\ bm_{22} & am_{21} \end{pmatrix},$$

we get all matrices

$$\begin{pmatrix} m_{22} & cbm_{12} \\ c^q m_{21} & m_{11} \end{pmatrix}, \ c \in \mathbb{F}_4 \setminus \{0\}.$$

Thus, $\sharp(\mathrm{Num}_2(M)) = 1$ *if and only if* $M = m_{11}\mathbb{I}_{2\times 2}$*, while* $\gamma(2,2,2) = 6$ *and* $\sharp(\mathrm{Num}_2(M)) = 6$ *if and only if* $m_{11} \neq m_{22}$ *and* $(m_{12}, m_{21}) \neq (0,0)$*. There are* 144 *such matrices. We have* $\delta(2,2,2) = 2$ *and the minimum is achieved if and only if* M *is diagonal, but not a multiple of the identity. There are* 12 *such matrices.*

Example 6.6 *Take* $n = k = 3$ *and take* $A = (a_{ij}) \in M_{3,3}(\mathbb{F}_4)$*. By Remarks 6.1 and 6.2 we have* $A \in C_{3,3}(0)$ *if and only if either* $A = 0\mathbb{I}_{3\times 3}$ *or the rows are pairwise orthogonal and each row has exactly one zero entry. The same discussion works using columns instead of rows. Suppose that* $a_{11} \neq 0$ *and* $a_{12} \neq 0$ *and hence* $a_{13} = 0$*. If the second column is orthogonal to the first one and it has exactly one zero entry, then* $a_{23} = 0$*,* $a_{21}a_{22} \neq 0$*, and* $a_{11}^2 a_{21} + a_{12}^2 a_{22} = 0$*. In the same way we get* $a_{33} = 0$*,* $a_{31}a_{32} \neq 0$*,* $a_{11}^q a_{31} + a_{12}^q a_{32} = 0$*, and* $a_{21}^2 a_{31} + a_{22}^2 a_{32} = 0$*. Since* $t^2 = t^{-1}$ *for all* $t \in \mathbb{F}_4 \setminus \{0\}$*, the* 3 *degree* 3 *equations are equivalent to the following quadratic equations:* $a_{21}a_{12} + a_{11}a_{22} = 0$*,* $a_{31}a_{12} + a_{11}a_{32} = 0$*,* $a_{22}a_{31} + a_{21}a_{32} = 0$*, which (since all their entries are nonzero) are equivalent to* $a_{21}/a_{22} = a_{11}/a_{12}$*,* $a_{31}/a_{32} = a_{11}/a_{32}$*,* $a_{31}/a_{32} = a_{21}/a_{22}$*. Thus, we may fix arbitrary* $a_{11}, a_{12}, a_{21}, a_{31} \in \mathbb{F}_4 \setminus \{0\}$ *and then get uniquely the other* a_{ij} *s.*

The same argument works if either $a_{11} = 0$ *and* $a_{12}a_{13} \neq 0$ *or if* $a_{12} = 0$ *and* $a_{11}a_{13} \neq 0$*. We get* $\sharp(C_{3,3}(0)) = 243$*.*

7. Conclusions and further work

We have developed a matrix-valued numerical range. Now we describe (as remarks) three suggestions for the interested reader.

Remark 7.1 *(Over the complex numbers) Let* $\langle\ ,\ \rangle$ *be the usual Hermitian product on* $\mathbb{C}^n$ *(antilinear in the first variable) and take* $E \in M_{k,k}(\mathbb{C})$ *such that* $E = E^\dagger$*. Since* $\langle\ ,\ \rangle$ *is definite positive, we have* $C_{n,k}(0) = \{0\mathbb{I}_{k,k}\}$ *and* $C_{n,k}(E) = \emptyset$ *if neither* $E = 0\mathbb{I}_{k,k}$ *nor* E *is definite positive. Now assume that* E *is definite positive. Using a unitary transformation we see that to compute all numerical ranges it is sufficient to compute them when* E *is a diagonal matrix, say with* $a_1, \dots, a_k$ *diagonal entries with* $a_i > 0$ *for all* i*. For any* $A \in M_{n,k}(\mathbb{C})$ *with* $u_1, \dots, u_k$ *let* $\underline{a}A$ *be the matrix with* $(\sqrt{a_1}u_1, \dots, \sqrt{a_k}u_k)$ *as its column vectors. The map* $A \mapsto \underline{a}A$ *shows that to compute all* $n \times k$ *numerical ranges it is sufficient to compute the ones for* $E = \mathbb{I}_{k\times k}$*. Since* $\langle\ ,\ \rangle$ *is definite positive, we see that* $C_{n,k}(1)$ *is formed by the* $n \times k$ *matrices with as columns vectors a (partial) unitary frame. Thus,* $C_{n,k}(1) \neq \emptyset$ *if and only if* $1 \le k \le n$*. If* $1 \le k \le n$ *the* $(n \times k, 1)$*-numerical range of any* $n \times n$ *matrix is connected, compact, and circular.*

Remark 7.2 *As in [1, 3, 4] one could also consider the case in which we only use vectors and matrices defined over* $\mathbb{F}_q$*. As in the case of the usual numerical range we put a subscript* $_q$ *for this case, like* $C_{n,k,q}(E)$*,* $C_{n,k,q}(a)$*,* $\mathrm{Num}_{k,E}(M)_q$*, and* $\mathrm{Num}_{k,a}(M)_q$*. Since* $t^q = t$ *for each* $t \in \mathbb{F}_q$*, the equations defining* $C_{n,k,q}(a)$ *are quadratic equations. As in the case* $k = 1$*, we are dealing with a system of degree* 2 *equations in several variables (here* k^2 *equations in* nk *variables), which are homogeneous if* $a = 0$*. Thus, for* $a = 0$ *we always have nonzero solutions for* $n \gg k$ *by the Chevalley–Warning theorem (see [10, 6.9, 6.11] and [12, 3.1, 3.5]), while for* $a \neq 0$ *the same theorem shows that for* $n \gg k$ *if they have at least one solution then the set of all solutions is large. The description of these solutions should be related to, but easier than, the study of the fibers of the matrix numerical map.*

Remark 7.3 *Let* K *be a field equipped with a fixed degree* 2 *Galois extension* L*. As in [4] one can do the matrix-valued numerical range for matrices* $M \in M_{n,k}(L)$*.*

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