# EXISTENCE OF NONTRIVIAL LOGARITHMIC CO-HIGGS STRUCTURE ON CURVES

EDOARDO BALLICO AND SUKMOON HUH

Abstract. We study various aspects on nontrivial logarithmic co-Higgs structure associated to unstable bundles on algebraic curves. We check several criteria for (non-)existence of nontrivial logarithmic co-Higgs structures and describe their parameter spaces. We also investigate the Segre invariants of these structures and see their non-simplicity. In the end we also study the higher dimensional case, specially when the tangent bundle is not semistable.

Contents



## 1. Introduction

A logarithmic co-Higgs sheaf on a complex manifold $X$ is a pair $(\mathcal{E}, \Phi)$ with a torsion-free coherent sheaf $\mathcal{E}$ on $X$ and a morphism $\Phi : \mathcal{E} \to \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$ satisfying the integrability condition $\Phi \wedge \Phi = 0$, where $\mathcal{T}_{\mathcal{D}}$ is the logarithmic tangent bundle $X$ associated to an arrangement $\mathcal{D}$ of hypersurfaces with simple normal crossings.

2010 *Mathematics Subject Classification.* Primary: 14H60; Secondary: 14D20, 14J60, 53D18.

*Key words and phrases.* co-Higgs sheaf, logarithmic tangent bundle, nilpotent.

The first author is partially supported by MIUR and GNSAGA of INDAM (Italy). The second author is supported by Basic Science Research Program 2015-037157 through NRF funded by MEST and the National Research Foundation of Korea(KRF) 2016R1A5A1008055 grant funded by the Korea government(MSIP).

When $\mathcal{D}$ is empty, it is a co-Higgs sheaf in the usual sense, introduced and developed by Hitchin and Gualtieri; see [18, 15]. When $\mathcal{E}$ is locally free, it is a generalized vector bundle on $X$ considered as a generalized complex manifold, whose co-Higgs field vanishes in the normal direction to the support of $\mathcal{D}$.

It is observed in [4, Theorem 1.1] that the semistability of a co-Higgs bundle $(\mathcal{E},\Phi)$ on $X$ with nonnegative Kodaira dimension implies the semistability of $\mathcal{E}$. In case of negative Kodaira dimension, there are several works on description of moduli space of semistable co-Higgs bundles, including the case when the associated bundle is not stable; see [27] and [10].

Now the additional condition for a co-Higgs field to vanish in the normal direction to $\mathcal{D}$ with higher degree, forces the associated bundle to be unstable. So we are mainly interested in the logarithmic co-Higgs sheaves associated to the arrangement with high degree and assume that the length of Harder-Narasimhan filtration is at least two. We fix numeric data for the Harder-Narasimhan filtration of the sheaf in consideration, i.e. fix the length $s$ at least two of the filtration together with rank $r_i$ and degree $d_i$ of the successive quotients in the Harder-Narasimhan filtration (2). Setting $\gamma := \deg \mathcal{T}_{\mathcal{D}}$ and $\mu_i := d_i/r_i$, we always assume that $\mu_s - \mu_1 \le \gamma < 0$ as the least requirement for the existence of the non-trivial co-Higgs field; see Corollary 3.9. Then we investigate the numeric criterion for the sheaf to admit a non-trivial co-Higgs field; see Proposition 3.4 and Theorem 3.7.

**Theorem 1.1.** *Fix the numeric data for the Harder-Narasimhan filtration and denote by $\mathbb{U}$ the set of the torsion-free sheaves on an algebraic curve $X$ with these data. Then the following hold:*

(i) *there exists an unstable sheaf in $\mathbb{U}$ with non-trivial co-Higgs field;*
(ii) *the inequality $\mu_s - \mu_1 \ge \gamma + 1 - g$ implies the existence of an unstable sheaf in $\mathbb{U}$ with no non-trivial co-Higgs field;*
(iii) *the inequality $\mu_s - \mu_1 < \gamma + 1 - g$ implies that every sheaf in $\mathbb{U}$ admits a non-trivial co-Higgs field.*

The existence part is induced by explicit usage of positive elementary transformations and the positive answer to the Lange conjecture [28]. Furthermore we extend the notion of Segre invariant to the setting of logarithmic co-Higgs sheaves and show that it is well-defined over curves under the assumption that $\gamma < 0$ and that this invariant is same as the usual Segre invariant under a certain condition; see Corollary 4.8 and Proposition 4.14.

**Theorem 1.2.** *For a logarithmic co-Higgs sheaf $(\mathcal{E},\Phi)$ on an algebraic curve $X$ with $\gamma < 0$, the $k^{th}$-Segre invariant $s_k(\mathcal{E},\Phi)$ is well-defined. It is also equal to the Segre invariant $s_k(\mathcal{E})$ in the usual sense, if $\mathcal{E}$ admits the complete Harder-Narasimhan filtration, i.e. $r_i = 1$ for all $i$.*

Then we check in Proposition 4.14 that co-Higgs sheaves associated to unstable bundle are usually not stable, not even simple.

Over algebraic curves the bundle $\mathcal{T}_{\mathcal{D}}$ is automatically semistable. So, as the counterpart to the case of algebraic curves, in §5 we deal with the case when the dimension of $X$ is at least 2 and $\mathcal{T}_{\mathcal{D}}$ is not semistable. Under the assumption that the biggest slope in the Harder-Narasimhan filtration of $\mathcal{T}_{\mathcal{D}}$ is negative, we give a recipe to construct all the pairs $(\mathcal{E},\Phi)$ with $\mathcal{E}$ reflexive of $\mathrm{rk}(\mathcal{E}) = r \in \{2,3\}$ and non-trivial co-Higgs field $\Phi : \mathcal{E} \to \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$. When $r = 2$ and in most cases with $r = 3$, the map $\Phi$ is always 2-nilpotent and so it is integrable. We also /point out exactly when we cannot guarantee the integrability.

## 2. Preliminary

Let $X$ be a smooth projective variety of dimension $n$ at least one with the tangent bundle $T_X$ over the field of complex numbers $\mathbb{C}$. We fix an ample line bundle $\mathcal{O}_X(1)$ and denote by $\mathcal{E}(t)$ the twist of $\mathcal{E}$ by $\mathcal{O}_X(t)$ for any coherent sheaf $\mathcal{E}$ on $X$ and $t \in \mathbb{Z}$. We also denote by $\mathcal{E}^\vee$ the dual of $\mathcal{E}$. The dimension of cohomology group $H^i(X,\mathcal{E})$ is denoted by $h^i(X,\mathcal{E})$ and we will skip $X$ in the notation, if there is no confusion. We define the slope $\mu(\mathcal{E})$ of a coherent sheaf $\mathcal{E}$ on $X$ with respect to $\mathcal{O}_X(1)$ to be $\deg \mathcal{E}/\mathrm{rk}(\mathcal{E})$.

Now consider an arrangement $\mathcal{D} = \{D_1, \dots, D_m\}$ of pairwise distinct, smooth and irreducible divisors $D_i$ on $X$, and if there is no confusion we also denote by $\mathcal{D}$ the divisor $D_1 + \dots + D_m$. We assume that the divisor $\mathcal{D}$ has simple normal crossings. Then the associated logarithmic tangent bundle $T_X(-\log \mathcal{D})$ is locally free and fits into the following exact sequence; see [13].

$$0 \to T_X(-\log \mathcal{D}) \to T_X \to \oplus_{i=1}^m \varepsilon_{i*}\mathcal{O}_{D_i}(D_i) \to 0, \tag{1}$$

where $\varepsilon_i : D_i \to X$ is the embedding. If there is no confusion, we will simply denote $T_X(-\log \mathcal{D})$ by $\mathcal{T}_{\mathcal{D}}$.

**Definition 2.1.** [4] A *$\mathcal{D}$-logarithmic co-Higgs* sheaf on $X$ is a pair $(\mathcal{E}, \Phi)$ where $\mathcal{E}$ is a torsion-free coherent sheaf on $X$ and $\Phi : \mathcal{E} \to \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$ with $\Phi \wedge \Phi = 0$. Here $\Phi$ is called the *logarithmic co-Higgs field* of $(\mathcal{E}, \Phi)$ and the condition $\Phi \wedge \Phi = 0$ is an integrability condition originating in the work of Simpson [29].

For a torsion-free coherent sheaf $\mathcal{E}$ on $X$, we consider its associated Harder-Narasimhan filtration:

$$\{0\} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_s = \mathcal{E} \tag{2}$$

with the graduation $gr(\mathcal{E}) := \oplus_{i=1}^s \mathcal{F}_i/\mathcal{F}_{i-1}$ such that each $\mathcal{F}_i/\mathcal{F}_{i-1}$ is semistable and $\mu(\mathcal{F}_i/\mathcal{F}_{i-1})$ is strictly decreasing for all $i < s$. The integer $s$ is called *the length* of the filtration, and if $s = r$, then the filtration is said to be *complete*. We denote by $\mu_+(\mathcal{E})$ and $\mu_-(\mathcal{E})$ the maximal and minimal slopes in the filtration, respectively:

$$\mu_+(\mathcal{E}) := \mu(\mathcal{F}_1)\ ,\ \mu_-(\mathcal{E}) := \mu(\mathcal{F}_s/\mathcal{F}_{s-1}).$$

**Remark 2.2.** For two torsion-free sheaves $\mathcal{A}$ and $\mathcal{B}$ on $X$, let $\mathcal{A}\overline{\otimes}\mathcal{B}$ be the quotient of $\mathcal{A} \otimes \mathcal{B}$ by its torsion. If $\mathcal{A}$ and $\mathcal{B}$ are semistable, then $\mathcal{A}\overline{\otimes}\mathcal{B}$ is also semistable by [22, Theorem 2.5]. Applying this observation to the Harder-Narasimhan filtrations of $\mathcal{A}$ and $\mathcal{B}$, we get that $\mu_+(\mathcal{A}\overline{\otimes}\mathcal{B}) = \mu_+(\mathcal{A}) + \mu_+(\mathcal{B})$.

**Lemma 2.3.** *If $f : \mathcal{A} \to \mathcal{B}$ is a nonzero map between two torsion-free sheaves on $X$, then we have $\mu_-(\mathcal{A}) \le \mu_+(\mathcal{B})$*

*Proof.* Let $\{0\} = \mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}_a = \mathcal{A}$ be the Harder-Narasimhan filtration of $\mathcal{A}$ and let $k \in \{1, \dots, a\}$ be the minimal integer such that $\mathcal{A}_k \not\subseteq \ker(f)$, i.e. the minimal integer such that $f_{|\mathcal{A}_k} \not\equiv 0$. Then we have $f_{|\mathcal{A}_{k-1}} \equiv 0$ and so $f_{|\mathcal{A}_k}$ induces a nonzero map $\widetilde{f} : \mathcal{A}_k/\mathcal{A}_{k-1} \to \mathcal{B}$.

Let $\{0\} = \mathcal{B}_0 \subset \mathcal{B}_1 \subset \cdots \subset \mathcal{B}_b = \mathcal{B}$ be the Harder-Narasimhan filtration of $\mathcal{B}$ and let $l$ be the minimal positive integer $l \le b$ such that $\widetilde{f}(\mathcal{A}_k/\mathcal{A}_{k-1}) \subseteq \mathcal{B}_l$. Then we have $\widetilde{f}(\mathcal{A}_k/\mathcal{A}_{k-1}) \not\subseteq \mathcal{B}_{l-1}$ and so $\widetilde{f}$ induces a nonzero map $\hat{f} : \mathcal{A}_k/\mathcal{A}_{k-1} \to \mathcal{B}_l/\mathcal{B}_{l-1}$.

Since $\mathcal{A}_k/\mathcal{A}_{k-1}$ and $\mathcal{B}_l/\mathcal{B}_{l-1}$ are semistable, we have $\mu(\mathcal{A}_k/\mathcal{A}_{k-1}) \le \mu(\mathcal{B}_l/\mathcal{B}_{l-1})$. By the definition of $\mu_+$ and $\mu_-$ in terms of the Harder-Narasimhan filtration we have $\mu(\mathcal{A}_k/\mathcal{A}_{k-1}) \ge \mu_-(\mathcal{A})$ and $\mu(\mathcal{B}_l/\mathcal{B}_{l-1}) \le \mu_+(\mathcal{B})$, concluding the assertion. □

Remark 2.2 and Lemma 2.3 give the following whose assertion will be assumed throughout this article.

**Corollary 2.4.** *Assuming the existence of a nonzero map $\Phi : \mathcal{E} \to \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$, we have*

$$\mu_-(\mathcal{E}) \le \mu_+(\mathcal{E}) + \mu_+(\mathcal{T}_{\mathcal{D}}) = \mu_+(\mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}). \tag{3}$$

**Remark 2.5.** Assume that $\mathcal{E}$ is not semistable and so $s \ge 2$. If there exists a nonzero map $f \in \mathrm{Hom}(\mathcal{E}/\mathcal{F}_{s-1}, \mathcal{F}_{s-1} \otimes \mathcal{T}_{\mathcal{D}})$, then we may composite the quotient map $\mathcal{E} \to \mathcal{E}/\mathcal{F}_{s-1}$ with it to get a nonzero 2-nilpotent logarithmic co-Higgs field $\Phi_f$. Note that the associated co-Higgs field is uniquely determined by the choice of a map, i.e. if $f$ and $g$ are two different nonzero maps in $\mathrm{Hom}(\mathcal{E}/\mathcal{F}_{s-1}, \mathcal{F}_{s-1} \otimes \mathcal{T}_{\mathcal{D}})$, then we get $\Phi_f \ne \Phi_g$.

If $n$ is at least two, we fix a polarization $\mathcal{O}_X(1)$ with respect to which we consider (semi-)stability. For most cases in this article we will mainly assume that $\mathcal{D}$ is of high degree so that $\mathcal{T}_{\mathcal{D}}$ is "sufficiently negative" and that $\mathcal{T}_{\mathcal{D}}$ is semistable with $\gamma = \deg \mathcal{T}_{\mathcal{D}} < 0$, except in §5.2;

**Assumption 2.6.** *We always assume that $\gamma = \deg \mathcal{T}_{\mathcal{D}}$ is negative, if there is no specification.*

**Remark 2.7.** There are manifolds with $\Omega_X^1$ ample as in [11, 12], in which cases we may even take $\mathcal{D} = \emptyset$: If instead of logarithmic co-Higgs field we use the field $T_X(-\mathcal{D}) \cong T_X \otimes \mathcal{O}_X(-\mathcal{D})$ vanishing on a divisor $\mathcal{D}$, then we may use the semistability of the tangent bundle of many Fano manifolds [26] and then take a very positive $\mathcal{D}$ to get $T_X(-\mathcal{D})$ negative and semistable.

We fix a triple of integers $(r, d, s) \in \mathbb{Z}^{\oplus 3}$ together with pairs $(r_i, d_i) \in \mathbb{Z}^{\oplus 2}$ for $1 \le i \le s$ such that $r \ge 2$, $s \ge 1$, $r_i \ge 1$ and

$$r = r_1 + \cdots + r_s\ ,\ d = d_1 + \cdots + d_s.$$

Assume further that $d_i/r_i > d_{i+1}/r_{i+1}$ for $i = 1, \dots, s-1$. Then we denote by

$$\mathbb{U} = \mathbb{U}_X(s; r_1, d_1; \dots; r_s, d_s)$$

the set of all torsion-free coherent sheaves $\mathcal{E}$ of rank $r$ such that the Harder-Narasimhan filtration (2) of $\mathcal{E}$ with respect to $\mathcal{O}_X(1)$ has $(r_1, d_1; \cdots; r_s, d_s)$ as its numerical data, i.e. each quotient sheaf $\mathcal{F}_i/\mathcal{F}_{i-1}$ is semistable of rank $r_i$ and degree $d_i$. By [22] the filtration (2) tensored by $\mathcal{T}_{\mathcal{D}}$

$$\{0\} = \mathcal{F}_0 \otimes \mathcal{T}_{\mathcal{D}} \subset \mathcal{F}_1 \otimes \mathcal{T}_{\mathcal{D}} \subset \cdots \subset \mathcal{F}_s \otimes \mathcal{T}_{\mathcal{D}} \tag{4}$$

is the Harder-Narasimhan filtration of $\mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$ if $\mathcal{T}_{\mathcal{D}}$ is semistable. We also assume the existence of a nonzero co-Higgs field $\Phi : \mathcal{E} \to \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$: if $n$ is at least two, we do not assume for the moment the integrability condition $\Phi \wedge \Phi = 0$, because in the most examples in this article it will follow from the other assumption, or from Lemma 2.8, where we assume that $s$ is at least two.

Denote by $\widetilde{\Phi}$ the following map:

$$\mathcal{E} \otimes \mathcal{T}_{\mathcal{D}} \to \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}^{\otimes 2}$$

induced by $\Phi$. Comsiting the natural map $\mathcal{T}_{\mathcal{D}}^{\otimes 2} \to \wedge^2 \mathcal{T}_{\mathcal{D}}$ with $\widetilde{\Phi} \circ \Phi$, we have $\Phi \wedge \Phi$ as an element in $\mathrm{Hom}(\mathcal{E}, \mathcal{E} \otimes \wedge^2 \mathcal{T}_{\mathcal{D}})$.

**Lemma 2.8.** *If $s$ is at least two, then we have $\widetilde{\Phi} \circ \Phi = 0$, i.e. $\Phi$ is 2-nilpotent. In particular, we have $\Phi \wedge \Phi = 0$.*

*Proof.* Since we assume that $\gamma$ is negative, the sheaf $\mathcal{F}_1 \otimes \mathcal{T}_{\mathcal{D}} \subset \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$ is the Harder-Narasimhan filtration of $\mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$ and $\mathcal{F}_1 \otimes \mathcal{T}_{\mathcal{D}}^{\otimes 2} \subset \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}^{\otimes 2}$ is the Harder-Narasimhan filtration of $\mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}^{\otimes 2}$. Thus we have $\Phi(\mathcal{E}) \subseteq \mathcal{F}_1 \otimes \mathcal{T}_{\mathcal{D}}$ and $\widetilde{\Phi}(\mathcal{F}_1 \otimes \mathcal{T}_{\mathcal{D}}) = 0$, implying that $\widetilde{\Phi} \circ \Phi = 0$. □

## 3. Curve case

Assume that $X$ is a smooth algebraic curve of genus $g$ and take $\mathcal{D} = \{p_1, \dots, p_m\}$ a set of $m$ distinct points. Then we have $\mathcal{T}_{\mathcal{D}} \cong T_X \otimes \mathcal{O}_X(-\mathcal{D})$ with degree $\gamma := 2-2g-m$. We assume that $\gamma$ is negative so that we are not in the set-up of [24]. The sequence (1) turns into the following

$$0 \longrightarrow \mathcal{T}_{\mathcal{D}} \longrightarrow T_X \longrightarrow \oplus_{i=1}^m \mathbb{C}_{p_i} \longrightarrow 0.$$

Another feature of the case $n = 1$ is that all logarithmic co-Higgs fields automatically satisfy the integrability condition.

Consider a vector bundle $\mathcal{E}$ of rank $r$ with the Harder-Narasimhan filtration (2) and we assume

$$\mu_-(\mathcal{E}) = \mu(\mathcal{F}_s/\mathcal{F}_{s-1}) \le \gamma + \mu(\mathcal{F}_1) = \gamma + \mu_+(\mathcal{E}), \tag{5}$$

which is a necessary condition for the existence of a nonzero map $\Phi : \mathcal{E} \to \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$; see Corollary 3.9.

**Remark 3.1.** For each $i \in \{1, \dots, s\}$ with $\mu(\mathcal{F}_i/\mathcal{F}_{i-1}) + \gamma \ge \mu_-(\mathcal{E})$, define

$$b(i) := \min_{i+1 \le k \le s} \{k \mid \mu(\mathcal{F}_i/\mathcal{F}_{i-1}) + \gamma \ge \mu(\mathcal{F}_k/\mathcal{F}_{k-1})\}$$

and then the map $\Phi$ induces a map $\Phi^i : \mathcal{F}_{b(i)}/\mathcal{F}_{b(i)-1} \to (\mathcal{F}_i/\mathcal{F}_{i-1}) \otimes \mathcal{T}_{\mathcal{D}}$. Similarly, for each $j \in \{2, \dots, s\}$ with $\mu(\mathcal{F}_j/\mathcal{F}_{j-1}) \le \gamma + \mu_+(\mathcal{E})$, define

$$c(j) := \max_{1 \le k \le j-1} \Big\{k \mid \mu(\mathcal{F}_j/\mathcal{F}_{j-1}) \le \gamma + \mu(\mathcal{F}_k/\mathcal{F}_{k-1})\Big\}.$$

The map $\Phi$ induces a map $\Phi_j : \mathcal{F}_j/\mathcal{F}_{j-1} \to \Big(\mathcal{F}_{c(j)}/\mathcal{F}_{c(j)-1}\Big) \otimes \mathcal{T}_{\mathcal{D}}$. Note that these maps $\Phi^i$ and $\Phi_j$ are not necessarily nonzero.

Now fix the following numeric data

$$(s\ ;\ r_1, \dots, r_s\ ;\ d_1, \dots, d_s) \in \mathbb{Z}^{\oplus(2s+1)}$$

with $s, r_i > 0$ for each $i$ such that $d_i/r_i > d_{i+1}/r_{i+1}$ for all $i$; if $g = 0$, we also assume $a_i/r_i \in \mathbb{Z}$ for each $i$. Recall that we denote by $\mathbb{U}_X(s; r_1, d_1; \dots; r_s, d_s)$ the set of all vector bundles $\mathcal{E}$ of rank $r := \sum_{i=1}^s r_i$ on $X$ with the Harder-Narasimhan filtration (2) such that $\mathrm{rk}(\mathcal{F}_i/\mathcal{F}_{i-1}) = r_i$ and $\deg \mathcal{F}_i/\mathcal{F}_{i-1} = d_i$ for each $i$. The conditions just given above for $s$, $r_i$ and $d_i$ are the necessary and sufficient conditions for the existence of a vector bundle $\mathcal{E}$ on $X$ with rank $r$ and degree $d := d_1 + \cdots + d_s$.

Indeed, for the existence part, in case $g \ge 2$ we may even take a stable bundle $\mathcal{F}_i/\mathcal{F}_{i-1}$, while in case $g = 1$ by Atiyah's classification of vector bundles on elliptic curves, we may take as $\mathcal{F}_i/\mathcal{F}_{i-1}$ a semistable bundle; we can choose either indecomposable one or polystable one, depending on our purpose.

To get parameters spaces we first get parameter spaces for the sheaves $\mathcal{E}$, then for a fixed sheaf $\mathcal{E}$ we study all logarithmic co-Higgs fields $\Phi : \mathcal{E} \to \mathcal{E}\otimes\mathcal{T}_{\mathcal{D}}$ and then we put together the informations. We have several problems coming from the sheaves $\mathcal{E}$ such as non-separatedness or often reducibility of moduli of sheaves, and then more problems bring the logarithmic co-Higgs field into the picture.

First of all, we fix enough numerical invariant to get a bounded family of pairs $(\mathcal{E},\Phi)$. Fixing an ample line bundle $\mathcal{O}_X(1)$, we consider sheaves $\mathcal{E}$ with a Harder-Narasimhan filtration (2) and we fix the Hilbert function of each subquotient $\mathcal{F}_i/\mathcal{F}_{i-1}$. Since each $\mathcal{F}_i/\mathcal{F}_{i-1}$ is assumed to be semistable, the family of all $\mathcal{F}_i/\mathcal{F}_{i-1}$ are bounded. We first see that the $\mathrm{Ext}^1$-groups involved in the extensions

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_2/\mathcal{F}_1 \longrightarrow 0$$

are upper bounded and that the set of all $\mathcal{F}_2$ is bounded. Then we consider the set of all $\mathcal{F}_3$ and so on, inductively. We may get relative $\mathrm{Ext}^1$-groups as parameter spaces, but these parameter spaces usually do not parametrizes one-to-one isomorphism classes of sheaves, even by taking into account that proportional extensions gives isomorphic sheaves.

For the relative $\mathrm{Ext}^1$ we need to have universal family parametrizing all $\mathcal{F}_i/\mathcal{F}_{i-1}$ and we usually need to work with parameter spaces of sheaves which do not parametrizes one-to-one isomorphic classes. Note that there is a flat family with isomorphic sheaves $\mathcal{E}$ whose flat limit is $gr(\mathcal{E}) = \oplus_{i=1}^s \mathcal{F}_i/\mathcal{F}_{i-1}$. Thus there is no hope of one-to-one parametrization of isomorphism classes of sheaves; when the numerology allows that some $\mathcal{F}_i/\mathcal{F}_{i-1}$ is strictly semistable, then this phenomenon occurs even for the graded subquotient $\mathcal{F}_i/\mathcal{F}_{i-1}$. Algebraic stacks of course do not parametrize isomorphism classes of sheaves, not even of vector bundles; see [14]. In the case $n = 1$ with $X = \mathbb{P}^1$, we have a unique bundle, $\mathcal{E}$ for any fixed parameter space $\mathbb{U}_{\mathbb{P}^1}(s;r_1,d_1;\cdots;r_s,d_s)$ and so the parameter space for $(\mathcal{E},\Phi)$ is the vector space $\mathrm{Hom}(\mathcal{E},\mathcal{E}\otimes\mathcal{T}_{\mathcal{D}})$, which parametrizes one-to-one the isomorphism classes of pairs $(\mathcal{E},\Phi)$. See Remark 3.12 for the case $n = 1$ and $X$ a curve of genus $g \geq 2$.

**Remark 3.2.** In the case $s = 2$, the datum of $(\mathcal{E},\Phi)$ with $[\mathcal{E}] \in \mathbb{U}_X(2;r_1,d_1;r_2,d_2)$ and $\Phi : \mathcal{E} \to \mathcal{E}\otimes\mathcal{T}_{\mathcal{D}}$ induces a holomorphic triple $\psi : \mathcal{E}/\mathcal{F}_1 \longrightarrow \mathcal{F}_1\otimes\mathcal{T}_{\mathcal{D}}$ in the sense of [9] and we may study the stability of the holomorphic triple. Conversely, for every holomorphic triple $f : \mathcal{G}_2 \to \mathcal{G}_1\otimes\mathcal{T}_{\mathcal{D}}$ such that $\mathcal{G}_1$ and $\mathcal{G}_2$ are semistable with $\mathrm{rk}(\mathcal{G}_i) = r_i$ and $\deg\mathcal{G}_i = d_i$, $i = 1,2$, and for any extension class

$$0 \longrightarrow \mathcal{G}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{G}_2 \longrightarrow 0, \tag{6}$$

we get $[\mathcal{E}] \in \mathbb{U}_X(2;r_1,d_1;r_2,d_2)$ with $0 \subset \mathcal{G}_1 \subset \mathcal{E}$ as its Harder-Narasimhan filtration and a 2-nilpotent map $\Phi : \mathcal{E} \to \mathcal{E}\otimes\mathcal{T}_{\mathcal{D}}$ induced by $f$. Two sheaves, say $\mathcal{E}$ and $\mathcal{E}'$, fitting as middle bundles in (6) for the same $\mathcal{G}_1$ and $\mathcal{G}_2$ are isomorphic if and only if their associated extensions are proportional, because $\mathcal{G}_1$ and $\mathcal{G}_2$ are assumed to be semistable with $d_1/r_1 > d_2/r_2$ and so (6) is the Harder-Narasimhan filtration of the bundle in the middle.

This argument fits very well in §5.1, where $\mathcal{T}_{\mathcal{D}}$ is assumed to be semistable, because $\mathcal{F}_i\otimes\mathcal{T}_{\mathcal{D}}$ would be in the Harder-Narasimhan filtration of $\mathcal{E}\otimes\mathcal{T}_{\mathcal{D}}$; in this case we only require that $\mathcal{E}$ is torsion-free and then define $\mathbb{U}(s;r_1,d_1;\ldots;r_s,d_s)$ with Mumford's (slope-)semistability.

3.1. **Projective line.** We take $X = \mathbb{P}^1$ and then we have $\mathcal{T}_{\mathcal{D}} \cong \mathcal{O}_{\mathbb{P}^1}(\gamma)$ with $\gamma < 0$. Any vector bundle $\mathcal{E} \cong \oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i)$ on $\mathbb{P}^1$ with $a_1 \geq \cdots \geq a_r$ can be rewritten as

$$\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(b_1)^{\oplus r_1} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(b_s)^{\oplus r_s}, \tag{7}$$

with $\sum_{i=1}^s r_i = r$ and $b_1 > \cdots > b_s$, i.e. in the Harder-Narasimhan filtration (2) associated to $\mathcal{E}$, we have $\mathcal{F}_i/\mathcal{F}_{i-1} \cong \mathcal{O}_{\mathbb{P}^1}(b_i)^{\oplus r_i}$. Now consider $\mathbb{U}_{\mathbb{P}^1}(s; r_1, d_1; \ldots; r_s, d_s)$ with $b_i := d_i/r_i$ and then it is a single point set, consisting only of $\mathcal{E}$. Set

$$\Delta := \sum_{1 \leq i < j \leq s} \max\{0, \gamma + 1 + b_i - b_j\}.$$

Then we have $h^0(\mathcal{H}om(\mathcal{E}, \mathcal{E}(\gamma))) = \Delta$. So the parameter space is a well-defined vector space, or its associated projective space if we consider nonzero co-Higgs fields up to scalar multiplication. We have $\Delta > 0$ if and only if $b_1 + \gamma \geq b_s$.

For any $\Phi \in \mathrm{Hom}(\mathcal{E}, \mathcal{E}(\gamma))$ and any positive integer $i$, let $\Phi^{(i)} : \mathcal{E} \to \mathcal{E}(i\gamma)$ be the map obtained by iterating $i$ times a shift of $\Phi$. If $b_1 + i\gamma < b_s$ for some $i$, then we have $\Phi^{(i)} = 0$ and so $\Phi$ is a nilpotent logarithmic co-Higgs field. In particular, if $b_1 + 2\gamma < b_s \leq b_1 + \gamma$, then all logarithmic co-Higgs fields are 2-nilpotents and so we have the following.

**Proposition 3.3.** *For the bundle $\mathcal{E}$ in (7) with $2\gamma \leq b_s - b_1 < \gamma$, the set of its co-Higgs structures is identified with a $\Delta$-dimensional vector space.*

Now the assumption in (5) is simply $b_1 + \gamma \geq b_s$ and let $e$ be the last integer $i$ such that $b_i > \gamma + b_1$. Then we may write

$$\mathcal{E} \cong \mathcal{E}_+ \oplus \mathcal{E}_- \,, \text{ with } \mathcal{E}_+ \cong \oplus_{i=1}^e \mathcal{O}_{\mathbb{P}^1}(b_i)^{\oplus r_i} \text{ and } \mathcal{E}_- \cong \oplus_{i=e+1}^r \mathcal{O}_{\mathbb{P}^1}(b_i)^{\oplus r_i}. \tag{8}$$

It is possible to have $e = 0$ and so $\mathcal{E}_+$ is trivial. Then we have $H^0(\mathcal{E}nd(\mathcal{E})(\gamma)) = H^0(\mathcal{H}om(\mathcal{E}_-, \mathcal{E})(\gamma))$. Thus in case of $\mathbb{P}^1$ we may rephrase our question in the set-up of holomorphic triples $(\mathcal{E}_1, \mathcal{E}_2, f)$ with $\mathcal{E}_1 = \mathcal{E}_-$, $\mathcal{E}_2 = \mathcal{E}(\gamma)$ and $f : \mathcal{E}_1 \to \mathcal{E}_2$. Here, $\mathcal{E}_1$ and $\mathcal{E}_2$ are related in a sense that $\mathcal{E}_1$ is a twist of a factor of $\mathcal{E}_2$. So our general problem concerning nonzero maps $\Phi : \mathcal{E} \to \mathcal{E}(\gamma)$ is equivalent to a problem about nonzero maps $\Phi : \mathcal{E}_- \to \mathcal{E}(\gamma)$.

3.2. **Elliptic curves.** Let $X$ be an elliptic curve and use the classification of vector bundles on elliptic curves due to M. Atiyah in [1]. We have $\mathcal{T}_{\mathcal{D}} \cong \mathcal{O}_X(-\mathcal{D})$.

**Proposition 3.4.** *Fix an integer $s \geq 2$ and consider $\mathbb{U} := \mathbb{U}_X(s; r_1, d_1; \ldots; r_s, d_s)$ with $d_s/r_s \leq d_1/r_1 + \gamma$.*

(i) *There exists $[\mathcal{E}] \in \mathbb{U}$ with $\mathrm{Hom}(\mathcal{E}, \mathcal{E}(-\mathcal{D})) \neq 0$.*
(ii) *If $d_s/r_s = d_1/r_1 + \gamma$, there is $[\mathcal{E}] \in \mathbb{U}$ with $\mathrm{Hom}(\mathcal{E}, \mathcal{E}(-\mathcal{D})) = 0$.*
(iii) *If $d_s/r_s < d_1/r_1 + \gamma$, then we have $\mathrm{Hom}(\mathcal{E}, \mathcal{E}(-\mathcal{D})) \neq 0$ for all $[\mathcal{E}] \in \mathbb{U}$.*
(iv) *If $e$ is the maximal integer such that $d_s/r_s \leq d_1/r_1 + e\gamma$, then we have $\Phi^{(e+1)} = 0$ for every $[\mathcal{E}] \in \mathbb{U}$ and $\Phi \in \mathrm{Hom}(\mathcal{E}, \mathcal{E}(-\mathcal{D})) = 0$.*

*Proof.* Take $[\mathcal{E}] \in \mathbb{U}$ and set $\mathcal{E}_s := \mathcal{F}_s/\mathcal{F}_{s-1}$. In the set-up of part (iii) we have $\mu(\mathcal{E}_s^\vee \otimes \mathcal{F}_1(-\mathcal{D})) > 0$ and so Riemann-Roch gives $h^0(\mathcal{E}_s^\vee \otimes \mathcal{F}_1(-\mathcal{D})) > 0$. Take as $\Phi$ the composition of the surjection $\mathcal{E} \to \mathcal{E}_s$ with a nonzero map $\mathcal{E}_s \to \mathcal{F}_1(-\mathcal{D})$ and then the inclusion $\mathcal{F}_1(-\mathcal{D}) \hookrightarrow \mathcal{E}(-\mathcal{D})$, proving (iii).

Now assume $d_s/r_s = d_1/r_1 + \gamma$. Take as $\mathcal{E}_i$ any semistable bundle with prescribed numeric data so that $\mathcal{E}_1$ and $\mathcal{E}_s$ are polystable and no factor of $\mathcal{E}_1(-\mathcal{D})$ is isomorphic to a factor of $\mathcal{E}_s$. Set $\mathcal{E} := \oplus_{i=1}^s \mathcal{E}_i$. Due to the slope, we have $\mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_j(-\mathcal{D})) = 0$

if $(i,j) \neq (s,1)$. Since every nonzero map between stable bundles with the same slope is an isomorphism, we have $\mathrm{Hom}(\mathcal{E}_s, \mathcal{E}_1(-\mathcal{D})) = 0$ and so $\mathrm{Hom}(\mathcal{E}, \mathcal{E}(-\mathcal{D})) = 0$, proving part (ii).

Under the same situation, set $t := \gcd(|d_s|, r_s)$ and write $r_s = at$ and $d_s = bt$. Then each indecomposable factor of $\mathcal{E}_s$ has rank $a$ and degree $b$, which is also stable. Pick one of these indecomposable factors, say $\mathcal{A}$. Now from $d_s/r_s = d_1/r_1 + \gamma$, we see that $a$ divides $r_1$. Then we have $r_1/a \in \mathbb{Z}$ and it also divides $d_1$, say $r_1 = ap$ and $d_1 = qp$. We also see that $\gcd(a,q) = \gcd(a,b) = 1$ and so $\mathcal{E}_1$ is a polystable bundle whose factors have rank $a$ and degree $q = b - a\gamma$. Let $\mathcal{G}$ be any polystable vector bundle of rank $r_1$ and degree $d_1$ with $\mathcal{A} \otimes \mathcal{O}_X(\mathcal{D})$ as one of its factors. Set $\mathcal{F} := \mathcal{G} \oplus \left(\oplus_{i=2}^s \mathcal{E}_i\right)$ and then we have $[\mathcal{F}] \in \mathbb{U}$. Since $\mathrm{Hom}(\mathcal{E}_s, \mathcal{G}(-\mathcal{D})) \neq 0$, we have $\mathrm{Hom}(\mathcal{F}, \mathcal{F}(-\mathcal{D})) \neq 0$, proving part (i).

Part (iv) is obvious. □

**Remark 3.5.** In parts (i) and (iii) of Proposition 3.4 the proof gives the existence of a nonzero 2-nilpotent co-Higgs field $\Phi$.

3.3. **Higher genus case.** Assume that $X$ has genus $g \geq 2$. Note that $\gamma \leq 2-2g$. For the pairs of integers $(r,d)$ with $r>0$, denote by $\mathbb{M}_X(r,d)$ the moduli space of the stable vector bundles of rank $r$ on $X$ with degree $d$. It is known to be a non-empty, smooth and irreducible quasi-projective variety of dimension $r^2(g-1)+1$.

Fix a point $p \in X$ and take any exact sequence on $X$:

$$0 \longrightarrow \mathcal{A} \stackrel{u}{\longrightarrow} \mathcal{B} \longrightarrow \mathbb{C}_p \longrightarrow 0 \tag{9}$$

with $\mathcal{A}$ and $\mathcal{B}$ locally free. Note that $\mathrm{rk}(\mathcal{A}) = \mathrm{rk}(\mathcal{B})$ and that $\deg \mathcal{B} = \deg \mathcal{A} + 1$. Then we say that $\mathcal{B}$ is obtained from $\mathcal{A}$ by applying *a positive elementary transformation at $p$* and that $\mathcal{A}$ is obtained from $\mathcal{B}$ by applying *a negative elementary transformation at $p$*. For a fixed $\mathcal{A}$ (resp. $\mathcal{B}$) the set of all extensions (9) is parametrized by a vector space of dimension $\mathrm{rk}(\mathcal{A})$ (resp. $\mathrm{rk}(\mathcal{B})$); since it is an irreducible variety, we may speak about the general positive elementary transformation of $\mathcal{A}$ (resp. a general negative elementary transformation of $\mathcal{B}$).

**Lemma 3.6.** *For $(r,d,k) \in \mathbb{Z}^{\oplus 3}$ with $r,k>0$, fix a general bundle $[\mathcal{A}] \in \mathbb{M}_X(r,d)$. If $\mathcal{B}$ is obtained from $\mathcal{A}$ by applying $k$ positive elementary transformations, then it is stable.*

*Proof.* Since the statement is trivial for $r=1$, we may assume $r \geq 2$.

(a) First assume $k=1$ with the sequence (9) and that $\mathcal{B}$ is not stable so that there exists a subbundle $\mathcal{G}_t \subset \mathcal{B}$ of rank $t \in \{1, \dots, r-1\}$ with $\deg \mathcal{G}_t/t \geq (d+1)/r$. Let $\mathcal{C} \subset \mathcal{A}$ be the saturation of $u^{-1}(\mathcal{G})$ and set $a := \deg \mathcal{C}$. Then we have $a \geq \deg u^{-1}(\mathcal{G}) \geq \deg \mathcal{G} - 1$. Since $\mathcal{A}$ is general, we get by [21, Theorem 3.10] or [7, Theorem 2] that $\mu(\mathcal{A}/\mathcal{C}) - \mu(\mathcal{C}) \geq g-1$, from which we get

$$\frac{d}{r} - \frac{a}{t} \geq \frac{(r-t)(g-1)}{r}.$$

Using this with $\deg(\mathcal{G}_t) \le a+1$, we get

$$\begin{aligned}\mu(\mathcal{B}) - \mu(\mathcal{G}_t) &\ge \frac{d+1}{r} - \frac{a+1}{t} \\ &\ge \frac{t-r+(r-t)(g-1)}{rt} \\ &= \frac{(r-t)(g-2)}{rt} \ge 0,\end{aligned}$$

The equality holds if and only if $g=2$ and $\deg \mathcal{G}_t = a+1$. Let $\widetilde{a}$ be the maximal degree of a rank $t$ subbundle of $\mathcal{A}$. For arbitrary $t$ and $g$, Mukai and Sakai proved in [23] that $td - \widetilde{a}r \le t(r-t)g$, while the quoted results also said that $td - \widetilde{a}r \ge t(r-t)(g-1)$. The precise value of $\widetilde{a}$ is known by an unpublished result of A. Hirschowitz in [17] and [21, Remark 3.14], which says that $td - \widetilde{a}r = t(r-t)(g-1) + \varepsilon$, where $\varepsilon$ is the only integer such that $0 \le \varepsilon < r$ and $\varepsilon + t(r-t)(g-1) \equiv td \pmod r$.

Now assume $g=2$. We conclude unless $\varepsilon = 0$, $a = \widetilde{a}$ and $\deg \mathcal{G}_t = a+1$. In this case we use that we take a general positive elementary transformation of $\mathcal{A}$. Since $\varepsilon = 0$ and $\mathcal{A}$ is general, $\mathcal{A}$ has only finitely many rank $t$ subbundles of maximal degree $a = \widetilde{a}$, say $\mathcal{N}_i$ for $1 \le i \le \delta$; see [25] and [30]. The fiber $\mathcal{N}_{i|\{p\}}$ of $\mathcal{N}_i$ at $p$ is a $t$-dimensional linear subspace of the fiber $\mathcal{A}_{|\{p\}}$ of $\mathcal{A}$ at $p$, which is an $r$-dimensional vector space. The union of these $t$-dimensional linear subspaces $\mathcal{N}_i$ for $1 \le i \le \delta$, is a proper subset of $\mathcal{A}_{|\{p\}}$. Thus, for a general positive elementary transformation $\mathcal{B}$ of $\mathcal{A}$ at $p$, the saturation $\mathcal{M}_i$ of $\mathcal{N}_i$ is just $\mathcal{N}_i$ for all $i$, i.e. $\deg u^{-1}(\mathcal{M}_i) = \deg \mathcal{M}_i$ for all $i$, contradicting the assumptions $a = \widetilde{a}$ and $\deg \mathcal{G}_t = a+1$.

(b) Now assume $k \ge 2$. The case $k=1$ proves that a general positive elementary transformation of a stable bundle is stable. Similarly a general negative transformation of a stable bundle is also stable, and so we may apply the step (a) $k$ times to get the assertion. □

**Theorem 3.7.** *Fix an integer $s \ge 2$ and consider $\mathbb{U} := \mathbb{U}_X(s; r_1, d_1; \dots; r_s, d_s)$ with $d_s/r_s \le d_1/r_1 + \gamma$.*

(i) *There exists $[\mathcal{E}] \in \mathbb{U}$ with $\mathrm{Hom}(\mathcal{E}, \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}) \neq 0$.*

(ii) *If $d_s/r_s \ge d_1/r_1 + \gamma + 1 - g$, there is $[\mathcal{E}] \in \mathbb{U}$ with $\mathrm{Hom}(\mathcal{E}, \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}) = 0$.*

(iii) *If $d_s/r_s < d_1/r_1 + \gamma + 1 - g$, then $\mathrm{Hom}(\mathcal{E}, \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}) \neq 0$ for all $[\mathcal{E}] \in \mathbb{U}$.*

*Proof.* Take $[\mathcal{E}] \in \mathbb{U}$ and set $\mathcal{E}_s := \mathcal{F}_s/\mathcal{F}_{s-1}$. Since $\mathcal{E}_s$ and $\mathcal{F}_1$ are semistable, the bundle $\mathcal{E}_s^\vee \otimes \mathcal{F}_1(-D)$ is also semistable. In the set-up of part (iii) we have $\mu(\mathcal{E}_s^\vee \otimes \mathcal{F}_1 \otimes \mathcal{T}_{\mathcal{D}}) > g-1$ and so Riemann-Roch gives $h^0(\mathcal{E}_s^\vee \otimes \mathcal{F}_1 \otimes \mathcal{T}_{\mathcal{D}}) > 0$. Take as $\Phi$ the composition of the surjection $\mathcal{E} \to \mathcal{E}_s$ with a nonzero map $\mathcal{E}_s \to \mathcal{F}_1 \otimes \mathcal{T}_{\mathcal{D}}$ and then the inclusion $\mathcal{F}_1 \otimes \mathcal{T}_{\mathcal{D}} \hookrightarrow \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$, proving (iii).

Now assume the set-up of (ii) and pick a general element $(\mathcal{E}_1, \dots, \mathcal{E}_s)$ in

$$\mathbb{M}_X(r_1, d_1) \times \cdots \times \mathbb{M}_X(r_s, d_s).$$

In particular, each $\mathcal{E}_i$ is a general stable bundle in $\mathbb{M}_X(r_i, d_i)$. Set $\mathcal{E} := \oplus_{i=1}^s \mathcal{E}_i$ and then it is sufficient to prove the following claim for (ii).

*Claim 1:* We have $\mathrm{Hom}(\mathcal{E}, \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}) = 0$.

*Proof of Claim 1:* Since $\mathcal{E} := \oplus_{i=1}^s \mathcal{E}_i$, it is enough to prove that $H^0(\mathcal{E}_i, \mathcal{E}_j \otimes \mathcal{T}_{\mathcal{D}}) = 0$ for all $i, j$. We have $H^0(\mathcal{E}_i, \mathcal{E}_i \otimes \mathcal{T}_{\mathcal{D}}) = 0$ for each $i$, because $\mathcal{E}_i$ is stable and $\gamma < 0$. Now assume $i \neq j$. Note that $(E_i^\vee, E_j \otimes \mathcal{T}_{\mathcal{D}})$ is a general element of

$$\mathbb{M}_X(r_i, -d_i) \times \mathbb{M}_X(r_j, d_j + r_j\gamma).$$

We have $\mu(\mathcal{E}_i^\vee \otimes \mathcal{E}_j \otimes \mathcal{T}_{\mathcal{D}}) = -\mu(\mathcal{E}_i) + \mu(\mathcal{E}_j) + \gamma \le g-1$. By a theorem of A. Hirschowitz in [28, Theorem 1.2], we have $h^0(\mathcal{E}_i^\vee \otimes \mathcal{E}_j \otimes \mathcal{T}_{\mathcal{D}}) = 0$, concluding the proof of *Claim 1*.

Now we prove part (i). Let $\mathcal{B}_i$ be a semistable bundle of rank $r_i$ on $X$ with degree $d_i$ for each $i$, and let $\mathcal{E} := \oplus_{i=1}^s \mathcal{B}_i$. Our strategy is to find appropriate $\mathcal{B}_1$ and $\mathcal{B}_s$ with the additional condition $\mathrm{Hom}(\mathcal{B}_s, \mathcal{B}_1 \otimes \mathcal{T}_{\mathcal{D}}) \neq 0$, which would imply part (i).

(a) Assume $r_s < r_1$. Setting $r' := r_1 - r_s$ and $d' := d_1 + \gamma r_1 - d_s$, it is enough to show the existence of an exact sequence of vector bundles on $X$:

$$0 \longrightarrow \mathcal{A}_1 \longrightarrow \mathcal{A}_2 \longrightarrow \mathcal{A}_3 \longrightarrow 0, \tag{10}$$

with $\mathcal{A}_1, \mathcal{A}_2$ semistable and $\mathcal{A}_1$ of rank $r_s$ and degree $d_s$, $\mathcal{A}_2$ of rank $r_1$ and degree $d_1 + \gamma r_1$. Then $\mathcal{A}_3$ would be of rank $r'$ and degree $d'$, and we may take $\mathcal{B}_s := \mathcal{A}_1$ and $\mathcal{B}_1 := \mathcal{A}_2 \otimes \mathcal{T}_{\mathcal{D}}^\vee$.

Note that for a quadruple of integers $(x_1, x_2, a_1, a_2) \in \mathbb{Z}^{\oplus 4}$ with $x_2 > x_1 > 0$ and $a_1/x_1 \le a_2/x_2$ (resp. $a_1/x_1 < a_2/x_2$), we have

$$\frac{a_2}{x_2} \le \frac{a_2 - a_1}{x_2 - x_1} \left(\text{resp. } \frac{a_2}{x_2} < \frac{a_2 - a_1}{x_2 - x_1}\right).$$

Using the above to $(x_1, x_2, a_1, a_2) = (r_s, r_1, d_s, d_1 + \gamma r_1)$, together with

$$\mu_-(\mathcal{E}) = d_s/r_s \le d_1/r_1 + \gamma = \mu_-(\mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}),$$

we have $d_1/r_1 + \gamma \le d'/r'$ with equality if and only if $d_s/r_s = d_1/r_1 + \gamma$. When the equality holds, we may take as $\mathcal{A}_1$ and $\mathcal{A}_3$ arbitrary semistable bundles with the prescribed ranks and degrees and then take $\mathcal{A}_2 := \mathcal{A}_1 \oplus \mathcal{A}_3$. Now assume $d_s/r_s < d_1/r_1 + \gamma$ and so $d_1/r_1 + \gamma < d'/r'$. In this case by the positive answer to the conjecture of Lange, there is an exact sequence (10) of vector bundles on $X$ with the prescribed ranks and degrees and with stable $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$; see [28, Introduction].

(b) Assume $r_s > r_1$. Similarly as in (a) we set $r'' := r_s - r_1$ and $d'' := d_1 - \gamma r_1 - d_s$. By taking $\mathcal{B}_s := \mathcal{A}_2$ and $\mathcal{B}_1 := \mathcal{A}_3 \otimes \mathcal{T}_{\mathcal{D}}^\vee$, it is sufficient to find an exact sequence (10) with $\mathcal{A}_2$ and $\mathcal{A}_3$ semistables, $\mathcal{A}_1$ of rank $r''$ and degree $d''$, $\mathcal{A}_2$ of rank $r_s$ and degree $d_s$ and $\mathcal{A}_3$ or rank $r_1$ and degree $d_1 + \gamma r_1$.

First assume $d_s/r_s = d_1/r_1 + \gamma$. In this case we have $d''/r'' = d_s/r_s$ and we take as $\mathcal{A}_1$ and $\mathcal{A}_3$ arbitrary semistable bundles with prescribed ranks and degrees and set $\mathcal{A}_2 := \mathcal{A}_1 \oplus \mathcal{A}_3$. Now assume $d_s/r_s < d_1/r_1 + \gamma$ and so $d_1/r_1 + \gamma > d''/r''$. Again by the conjecture of Lange proved in [28] we may take $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ with the prescribed ranks and degree and stable.

(c) Assume $r_s = r_1$. First assume $d_s/r_s = d_1/r_1 + \gamma$, i.e. $d_1 = d_s - \gamma r_1$. In this case we take as $\mathcal{B}_s$ any semistable bundle with rank $r_s$ and degree $d_s$ and set $\mathcal{B}_1 := \mathcal{B}_s \otimes \mathcal{T}_{\mathcal{D}}$. Now assume $k := d_1 + r_1\gamma - d_s > 0$. We take as $\mathcal{B}_s$ a general stable bundle of rank $r_s$ and degree $d_s$. Then $\mathcal{B}_s \otimes \mathcal{T}_{\mathcal{D}}$ is a general element of $\mathbb{M}_X(r_1, d_1 - t)$. We take as $\mathcal{B}_1$ a bundle obtained from $\mathcal{B}_s \otimes \mathcal{T}_{\mathcal{D}}$ by applying $k$ general positive elementary transformations. □

**Remark 3.8.** Consider a smooth algebraic curve $X$ of an arbitrary genus $g \ge 0$ and assume $s \ge 3$ together with

$$d_2/r_2 + \gamma < d_s/r_s \le d_1/r_1 + \gamma.$$

By Theorem 3.7 in case $g \ge 2$ and Proposition 3.4 for $g = 1$, there is $(\mathcal{E}, \Phi)$ with $[\mathcal{E}] \in \mathbb{U}_X(s; r_1, d_1; \dots; r_s, d_s)$ and a nonzero map $\Phi : \mathcal{E} \to \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$. Take any $[\mathcal{E}] \in$

$\mathbb{U}_X(s;r_1,d_1;\dots;r_s,d_s)$ with the Harder-Narasimhan filtration (2). Note that we have $\mathrm{Hom}(\mathcal{A},\mathcal{B})=0$ for any semistable bundles $\mathcal{A}$ and $\mathcal{B}$ with $\mu(\mathcal{A})>\mu(\mathcal{B})$. *Claim 1* in the proof of Theorem 3.7 applied to $\mathcal{E}/\mathcal{F}_1$ shows that any map $\Phi:\mathcal{E}\to\mathcal{E}\otimes\mathcal{T}_{\mathcal{D}}$ is uniquely determined by $f:\mathcal{E}/\mathcal{F}_{s-1}\to\mathcal{F}_1\otimes\mathcal{T}_{\mathcal{D}}$; moreover we get $\mathrm{Im}(\Phi)=\mathrm{Im}(f)$ and $\ker(\Phi)$ is the inverse image of $\ker(f)$ under the surjection $\mathcal{E}\to\mathcal{E}/\mathcal{F}_{s-1}$.

Conversely, for $1\le i\le s$, choose arbitrary semistable bundles $\mathcal{E}_i$ with $\mathrm{rk}(\mathcal{E}_i)=r_i$ and $\deg\mathcal{E}_i=d_i$, and a map $f:\mathcal{E}_s\to\mathcal{E}_1\otimes\mathcal{T}_{\mathcal{D}}$. To get a vector bundle $[\mathcal{E}]\in\mathbb{U}_X(s;r_1,d_1;\dots;r_s,d_s)$, we only need to consider $(s-1)$ extension classes

$$0\longrightarrow\mathcal{F}_i\longrightarrow\mathcal{F}_{i+1}\longrightarrow\mathcal{E}_{i+1}\longrightarrow 0$$

for $i=1,\dots,s-1$, where $\mathcal{F}_1:=\mathcal{E}_1$. Once $\mathcal{E}$ is chosen, the map $\Phi:\mathcal{E}\to\mathcal{E}\otimes\mathcal{T}_{\mathcal{D}}$ is uniquely determined by $f$.

In case of curves, we sometimes may improve Remark 2.3 to a strict inequality in the following way.

**Remark 3.9.** Take $[\mathcal{E}]\in\mathbb{U}:=\mathbb{U}_X(s;r_1,d_1;\dots;r_s,d_s)$ with the Harder-Narasimhan filtration (2). Assume that $s\ge 2$ with $r_s\ne r_1$,

$$\gcd(r_1,d_1)=\gcd(r_s,d_s)=1 \text{ and } d_1/r_1+\gamma\ge d_s/r_s.$$

Since $\gcd(r_1,d_1)=1$, the sheaf $\mathcal{F}_1$ is stable and so $\mathcal{F}_1\otimes\mathcal{T}_{\mathcal{D}}$ is stable. Since $\gcd(r_s,d_s)=1$, the sheaf $\mathcal{F}_s/\mathcal{F}_{s-1}$ is also stable. From $r_1\ne r_s$ we get that $\mathcal{F}_1\otimes\mathcal{T}_{\mathcal{D}}$ and $\mathcal{F}_s/\mathcal{F}_{s-1}$ are not isomorphic and so we have $\mathrm{Hom}(\mathcal{F}_s/\mathcal{F}_{s-1},\mathcal{F}_1\otimes\mathcal{T}_{\mathcal{D}})=0$. If $s\ge 3$, we obviously have $\mathrm{Hom}(\mathcal{F}_i/\mathcal{F}_{i-1},\mathcal{F}_j\otimes\mathcal{T}_{\mathcal{D}})=0$ for all $i,j\in\{1,\dots,s\}$ with $(i,j)\ne(s,1)$, even without the assumptions $r_s\ne r_1$ and $\gcd(r_1,d_1)=\gcd(r_s,d_s)=1$. Thus we have $\mathrm{Hom}(\mathcal{E},\mathcal{E}\otimes\mathcal{T}_{\mathcal{D}})=0$.

**Remark 3.10.** In the following exact sequence

$$0\longrightarrow\mathcal{A}\longrightarrow\mathcal{C}\longrightarrow\mathcal{B}\longrightarrow 0 \tag{11}$$

of vector bundles on $X$ with $\mathcal{A}$ and $\mathcal{B}$ semistable, if we have $\mu(\mathcal{A})+2-2g>\mu(\mathcal{B})$, then we have $h^1(\mathcal{A}\otimes\mathcal{B}^\vee)=0$ and so (11) splits. Thus if we have

$$\frac{d_i}{r_i}+2-2g>\frac{d_{i+1}}{r_{i+1}}$$

for all $i$, then we have $\mathcal{E}\cong gr(\mathcal{E})$ for all $[\mathcal{E}]\in\mathbb{U}:=\mathbb{U}_X(s;r_1,d_1;\dots;r_s,d_s)$.

Now assume $d_i/r_i+2-2g\ge d_{i+1}/r_{i+1}$ for all $i$. If the equality holds for some $i$, i.e. $d_i/r_i=d_{i+1}/r_{i+1}+2g-2$, then we have $r_i\ne r_{i+1}$ and that $r_h$ and $d_h$ are coprime, where $h\in\{i,i+1\}$ is the index with higher rank $r_h=\max\{r_i,r_{i+1}\}$. As in Remark 3.9 we get $\mathcal{E}\cong gr(\mathcal{E})$ for all $[\mathcal{E}]\in\mathbb{U}$.

For example, take $s=2$. We just proved that $\mathcal{E}\cong\mathcal{F}_1\oplus\mathcal{F}_2/\mathcal{F}_1$ for all $[\mathcal{E}]\in\mathbb{U}(2;1,d_1;1,d_2)$ with a nonzero map $\Phi:\mathcal{E}\to\mathcal{E}\otimes\mathcal{T}_{\mathcal{D}}$ and either $\mathcal{D}\ne 0$ or $r_1\ne r_2$, and $d_h,r_h\in\mathbb{Z}$, where $r_h=\max\{r_1,r_2\}$.

**Example 3.11.** Assume $d_1>d_2-\gamma$. For a fixed $\mathcal{R}\in\mathrm{Pic}^{d_2}(X)$, consider the set

$$\mathbb{E}:=\{(\mathcal{F}_1,\psi)\mid\mathcal{F}_1\in\mathrm{Pic}^{d_1}(X)\text{ and }0\ne\psi:\mathcal{R}\to\mathcal{F}_1\times\mathcal{T}_{\mathcal{D}}\}/\sim,$$

where the equivalent relation $\sim$ is given by $(\mathcal{F}_1,\psi)\sim(\mathcal{F}_1,c\psi)$ for all $c\in\mathbb{C}^*$. $\mathbb{E}$ is the set of all effective divisors of $X$ with degree $d_1+\gamma-d_2$ and so $\mathbb{E}$ is isomorphic to a symmetric product of $d_1+\gamma-d_2$ copies of $X$ and in particular it is irreducible. By Remark 3.10 we have $\mathcal{E}\cong\mathcal{F}_1\oplus\mathcal{F}_2/\mathcal{F}_1$ for all $[\mathcal{E}]\in\mathbb{U}_X(2;1,d_2+\gamma;1,d_2)$.

**Example 3.12.** Assume $g \geq 2$ and take $\mathcal{D} = \emptyset$ so that $\gamma = 2 - 2g$. Fix any $d \in \mathbb{Z}$ and consider $\mathcal{E} \in \mathbb{U}_X(2;1,d+2g-2;1,d)$ with a nonzero map $\Phi : \mathcal{E} \to \mathcal{E} \otimes T_X$. Set $\mathcal{R} := \mathcal{F}_2/\mathcal{F}_1 \in \mathrm{Pic}^d(X)$ and then $\Phi$ is induced by a nonzero map $\psi : \mathcal{R} \to \mathcal{F}_1 \otimes T_X$. Since $\mathcal{F}_1$ is in $\mathrm{Pic}^{d+2g-2}(X)$, the map $\psi$ is an isomorphism. Thus we get $\mathcal{F}_1 \cong \mathcal{R} \otimes \omega_X$ and that for a fixed $\mathcal{E}$ the set of all nonzero map $\Phi$ is parametrized by a nonzero scalar. From $h^1(\omega_X) = 1$ we see that there are, up to isomorphism, exactly two vector bundles $\mathcal{E}$ fitting into an exact sequence

$$0 \to \mathcal{R} \otimes \omega_X \to \mathcal{E} \to \mathcal{R} \to 0, \tag{12}$$

that is, $(\mathcal{R} \otimes \omega_X) \oplus \mathcal{R}$ and an indecomposable bundle. Thus the set of all $(\mathcal{E}, \Phi)$, up to isomorphisms, with nonzero $\Phi$, is parametrized one-to-one by the disjoint union of two copies of $\mathrm{Pic}^d(X) \times \mathbb{C}^*$. Thus no one-to-one parameter space is irreducible. We get another irreducible parameter space that is not one-to-one, by taking as parameter space, up to a nonzero constant, the relative $\mathrm{Ext}^1$ group of (12) over $\mathrm{Pic}^d(X)$; each indecomposable bundle $\mathcal{E}$ appears $\infty^1$-times and it has $gr(\mathcal{E}) \cong (\mathcal{R} \otimes \omega_X) \oplus \mathcal{R}$ as its limit inside the parameter space.

Now for $s$ at least two let us define the set $\mathbb{U}^{\mathrm{co}} = \mathbb{U}_X^{\mathrm{co}}(s; r_1, d_1; \dots ; r_s, d_s)$ of certain co-Higgs bundles associated to $\mathbb{U} = \mathbb{U}_X(s; r_1, d_1; \dots ; r_s, d_s)$ as follows.

$$\mathbb{U}^{\mathrm{co}} := \{(\mathcal{E}, \Phi) \mid [\mathcal{E}] \in \mathbb{U} \text{ and } \Phi : \mathcal{E} \to \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}} \text{ with } \mathrm{Im}(\Phi) \subseteq \mathcal{F}_1 \text{ and } \ker(\Phi) \subseteq \mathcal{F}_{s-1}$$
$$\text{such that } \mathrm{rk}(\mathrm{Im}(\Phi)) = \min\{r_1, r_s\} \text{ and } \mathcal{F}_1, \mathcal{F}_s/\mathcal{F}_{s-1} \text{ stable } \}$$

Denote by $\Gamma \subseteq \mathbb{M}_X(r_1, d_1) \times \mathbb{M}_X(r_s, d_s)$ the set of all pairs $(\mathcal{F}_1, \mathcal{F}_s/\mathcal{F}_{s-1})$ obtained from $\mathbb{U}^{\mathrm{co}}$ and call the projection from $\Gamma$ to each factor by $\pi_1$ and $\pi_2$, respectively.

**Proposition 3.13.** *Assume that*

- *each $d_i$ is positive such that $d_i/r_i > d_{i+1}/r_{i+1}$ for all $i$, and*
- *$d_1/r_1 + \gamma \geq d_s/r_s$.*

*Then we have the following assertions.*

(i) *If $r_1 = r_s$, then $\pi_1$ and $\pi_2$ are dominant.*
(ii) *If $r_1 < r_s$ (resp. $r_1 > r_s$) and $d_1/r_1 + \gamma > d_s/r_s$, then $\pi_1$ (resp. $\pi_2$) is dominant.*
(iii) *Assume $d_1/r_1 + \gamma > g - 1 + d_s/r_s$. Then $\Gamma$ contains a non-empty open subset of $\mathbb{M}_X(r_1, d_1) \times \mathbb{M}_X(r_s, d_s)$; if $r_1 \geq r_s$, then we have* $\ker(\Phi) = \mathcal{F}_{s-1}$.

*Proof.* Fix a point $([\mathcal{A}_1], [\mathcal{A}_s]) \in \mathbb{M}_X(r_1, d_1) \times \mathbb{M}_X(r_s, d_s)$. For arbitrary $[\mathcal{A}_i] \in \mathbb{M}_X(r_i, d_i)$, $i = 2, \cdots, s-1$, we consider $\mathcal{E} := \oplus_{i=1}^s \mathcal{A}_i$ with the Harder-Narasimhan filtration (2) such that $\mathcal{F}_1 \cong \mathcal{A}_1$ and $\mathcal{F}_s/\mathcal{F}_{s-1} \cong \mathcal{A}_s$. We take a map $\Phi : \mathcal{E} \to \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$ with $\Phi(\mathcal{F}_{s-1}) = 0$, which is induced by a map $\psi : \mathcal{A}_s \to \mathcal{A}_1 \otimes \mathcal{T}_{\mathcal{D}}$ whose existence is guaranteed by the assumptions.

First assume $r_s = r_1$. We need to prove the existence of a map $\psi$ of rank $r_1$ when $\mathcal{A}_1$ is general in $\mathbb{M}_X(r_1, d_1)$ and $\mathcal{A}_s$ is general in $\mathbb{M}_X(r_s, d_s)$; we do not claim here that $([\mathcal{A}_1], [\mathcal{A}_2])$ is general in $\mathbb{M}_X(r_1, d_1) \times \mathbb{M}_X(r_s, d_s)$. The dominance of $\pi_2$ is the content of Lemma 3.6. The dominance of $\pi_1$ can be proved by applying the dual map, or with the same proof as in the proof of Lemma 3.6, concluding part (i).

Now assume $r_s < r_1$ and $d_1/r_1 + \gamma > d_s/r_s$. We take as $\mathcal{A}_s$ a general element of $\mathbb{M}_X(r_s, d_s)$. The existence of a stable $\mathcal{A}_1 \in \mathbb{M}_X(r_1, d_1)$ with an embedding $\psi : \mathcal{A}_s \hookrightarrow \mathcal{A}_1 \otimes \mathcal{T}_{\mathcal{D}}$ with $\mathcal{A}_1/\psi(\mathcal{A}_s)$ stable and general in $\mathbb{M}_X(r_s - r_1, d_1 - \gamma r_1)$ is proved in part (i) of the proof of Theorem 3.7.

By using part (ii) of the proof of Theorem 3.7 instead of part (i), we get the case $r_s > r_1$ and $d_1/r_1 + \gamma > d_s/r_s$.

Now consider part (iii) and assume $d_1/r_1+\gamma > g-1+d_s/r_s$. Take a general $([\mathcal{A}_1],[\mathcal{A}_s]) \in \mathbb{M}_X(r_1,d_1)\times \mathbb{M}_X(r_s,d_s)$ and set $\mathcal{B}_1 := \mathcal{A}_1\otimes \mathcal{T}_{\mathcal{D}}$. Then it is sufficient to find $\psi : \mathcal{A}_s \to \mathcal{B}_1$ with $\mathrm{Im}(\psi) = \min\{r_1,r_s\}$. By the assumptions, we have $\mu(\mathcal{A}_s^\vee \otimes \mathcal{B}_1) > g-1$ and so Riemann-Roch gives $\mathrm{Hom}(\mathcal{A}_s,\mathcal{B}_1)\neq 0$. Take a general element $\psi \in \mathrm{Hom}(\mathcal{A}_s,\mathcal{B}_1)$ and then it is sufficient to prove that $\psi$ has rank $\min\{r_1,r_s\}$. Note that we have $h^1(\mathcal{A}_s^\vee\otimes\mathcal{B}_1)=0$ and so $\mathcal{B}_1$ is an element of the following set

$$\mathbb{W} := \{[\mathcal{F}]\in \mathbb{M}_X(r_1,d_1+\gamma r_1)\mid h^1(\mathcal{A}_s^\vee\otimes\mathcal{F})=0\}.$$

By Riemann-Roch, we have the following, for each $[\mathcal{F}]\in\mathbb{W}$,

$$h^0(\mathcal{A}_s^\vee\otimes\mathcal{F}) = \deg(\mathcal{A}_s^\vee\otimes\mathcal{F})+r_1r_2(1-g) = r_1r_2\left(\frac{d_1}{r_1}+\gamma-\frac{d_s}{r_s}+1-g\right)>0.$$

Now take the relative Hom with $\mathbb{W}$ as its parameter space, i.e. for each $[\mathcal{F}]\in\mathbb{W}$, the fibre is $\mathrm{Hom}(\mathcal{A}_s,\mathcal{F})$. The total space $\Lambda$ of this relative Hom is irreducible, because $h^0(\mathcal{A}_s^\vee\otimes\mathcal{F})$ is constant for all $[\mathcal{F}]\in\mathbb{W}$ by [6] and [20]. By [5, part (d) of Theorem 1.2], a general element $(\varphi:\mathcal{A}_s\to\mathcal{F})$ of $\Lambda$ has $\varphi$ with rank $\min\{r_1,r_s\}$. When $r_1\geq r_s$, this implies that $\ker(\Phi)=\mathcal{F}_{s-1}$, because the map $\psi:\mathcal{F}_s/\mathcal{F}_{s-1}\to\mathcal{F}_1\otimes\mathcal{T}_{\mathcal{D}}$ is injective if and only if it has rank $r_s$ . □

**Remark 3.14.** As in the end of proof of Proposition 3.13, to show that the set of the co-Higgs bundles $(\mathcal{E},\Phi)$ with certain properties is parametrized by an irreducible variety, it sometimes works to prove that (a) the set of all bundles $\mathcal{E}$ is parametrized by an irreducible variety $Y$, and (b) the integer $k := \dim \mathrm{Hom}(\mathcal{E}_y,\mathcal{E}_y\otimes\mathcal{T}_{\mathcal{D}})$ is constant for all $y\in Y$. In this case, the set of all $(\mathcal{E},\Phi)$ with no restriction on $\Phi$ is parametrized by a vector bundle of rank $k$ on $Y$.

**Example 3.15.** Assume $r_1=r_s$ and $d_1/r_1+\gamma = d_s/r_s$. Consider a bundle $[\mathcal{E}]\in\mathbb{U}_X(s;r_1,d_1;\dots;r_s,d_s)$ with an arbitrary map $\Phi:\mathcal{E}\to\mathcal{E}\otimes\mathcal{T}_{\mathcal{D}}$. Since $d_i/r_i > d_1/r_1+\gamma$ for all $i<s$, there is no nonzero map $\mathcal{F}_i/\mathcal{F}_{i-1}\to\mathcal{E}\otimes\mathcal{T}_{\mathcal{D}}$ and so we have $\mathcal{F}_{s-1}\subseteq\ker(\Phi)$. On the other hand, since $d_s/r_s > d_j/r_j+\gamma$ for all $j>1$, we have $\Phi(\mathcal{E})\subseteq\mathcal{F}_1$. We have $\mathrm{rk}(\Phi)=r_1$ if and only if $\mathcal{F}_s/\mathcal{F}_{s-1}\cong\mathcal{F}_1\otimes\mathcal{T}_{\mathcal{D}}$ and $\Phi$ is induced by an isomorphism $\mathcal{F}_s/\mathcal{F}_{s-1}\to\mathcal{F}_1\otimes\mathcal{T}_{\mathcal{D}}$.

**Example 3.16.** Assume $r_1=r_s$ and that

$$d_2/r_2+\gamma < d_s/r_s \text{ and } d_{s-1}/r_{s-1} > d_1/r_1+\gamma.$$

If we choose $(\mathcal{E},\Phi)$ with $[\mathcal{E}]\in\mathbb{U}_X(s;r_1,d_1;\dots;r_s,d_s)$, then as in Example 3.15 we see that $\Phi(\mathcal{E})\subseteq\mathcal{F}_1\otimes\mathcal{T}_{\mathcal{D}}$ and $\mathcal{F}_{s-1}\subseteq\ker(\Phi)$, because $d_i/r_i+\gamma<d_s/r_s$ for all $i>1$ and $d_j/r_j > d_1/r_1+\gamma$ for all $j<s$. Set $k:=d_1-d_2+\gamma r_1$. Then we have $\mathrm{rk}(\Phi)=r_1$ if and only if $\mathcal{F}_1\otimes\mathcal{T}_{\mathcal{D}}$ is obtained from $\mathcal{F}_s/\mathcal{F}_{s-1}$ by applying $k$ positive elementary transformations and $\Phi$ is induced by the associated inclusion $\mathcal{F}_s/\mathcal{F}_{s-1}\hookrightarrow\mathcal{F}_1\otimes\mathcal{T}_{\mathcal{D}}$.

## 4. Segre invariant

In this section, we do not assume that $\mathcal{T}_{\mathcal{D}}$ has some kind of negativity, so that we may have stable $(\mathcal{E},\Phi)$ with nonzero $\Phi$. Let $\mathcal{E}$ be a torsion-free sheaf of rank $r\geq 2$ and $\Phi:\mathcal{E}\to\mathcal{E}\otimes\mathcal{T}_{\mathcal{D}}$ a co-Higgs field. For a fixed integer $k\in\{1,\dots,r-1\}$, let us denote by $\mathcal{S}(k,\mathcal{E},\Phi)$ the set of all subsheaves $\mathcal{A}\subset\mathcal{E}$ of rank $k$ such that $\Phi(\mathcal{A})\subseteq\mathcal{A}\otimes\mathcal{T}_{\mathcal{D}}$. Define the $k^{th}$-*Segre invariant* to be

$$s_k(\mathcal{E},\Phi) := k\deg\mathcal{E} - \max_{\mathcal{A}\in\mathcal{S}(k,\mathcal{E},\Phi)} r\deg\mathcal{A}.$$

In case $\Phi = 0$, we simply denote it by $s_k(\mathcal{E})$. This is an extension of the Segre invariant, introduced in [19] with the notation $s_k(\mathcal{E})$, to the case $n \ge 2$. Over curves this notion was used in several literatures, including [5, 7, 8, 17, 21, 25, 28, 30]. If we take $\mathcal{T}_{\mathcal{D}}^\vee$ instead of $\mathcal{T}_{\mathcal{D}}$, we get a definition for logarithmic Higgs fields. Note that we always have $\mathcal{S}(k, \mathcal{E}, 0) \neq \emptyset$ and $s_k(\mathcal{E}) \le s_k(\mathcal{E}, \Phi)$.

**Lemma 4.1.** *Let $(\mathcal{E}, \Phi)$ be a 2-nilpotent co-Higgs bundle of rank $r$, and set $\mathcal{A} := \ker(\Phi)$ and $\mathcal{B} := \mathrm{Im}(\Phi)$ with $r' := \mathrm{rk}(\mathcal{A})$. Then we have the following:*

(i) $\mathcal{A} \in \mathcal{S}(r', \mathcal{E}, \Phi)$;
(ii) $\mathcal{S}(k, \mathcal{A}, 0) \subseteq \mathcal{S}(k, \mathcal{E}, \Phi)$ *for* $1 \le k < r'$;
(iii) $\mathcal{B}$ *is torsion-free and* $\Phi^{-1}(\mathcal{G}) \in \mathcal{S}(k, \mathcal{E}, \Phi)$ *for all* $\mathcal{G} \in \mathcal{S}(k - r', \mathcal{B}, 0)$ *and* $r' < k < r$;
(iv) $\mathcal{S}(k, \mathcal{E}, \Phi) \neq \emptyset$ *for all* $k$.

*Proof.* Parts (i) and (ii) are obvious. Part (iii) is true, because $\mathcal{B} \subseteq \mathcal{A} \otimes \mathcal{T}_{\mathcal{D}}$ by the definition of 2-nilpotent. Part (iv) follows from the other ones. □

**Example 4.2.** From [3, Theorem 1.1] we get a description of the set of nilpotent co-Higgs structures on a fixed stable bundle of rank two on $\mathbb{P}^n$. Indeed it is either trivial or an $(n+1)$-dimensional vector space, depending on the parity of the first Chern class and an invariant $x_{\mathcal{E}}$. We get a non-trivial set of nilpotent co-Higgs structures on $\mathcal{E}$ if and only if $c_1(\mathcal{E}) + 2x_{\mathcal{E}} = -3$, and in this case we get $s_1(\mathcal{E}, \Phi) = 1$.

4.1. **Curve case.** From now on we assume $n = 1$ with $g = g(X)$ and $\gamma < 0$. Take $(\mathcal{E}, \Phi)$ with $[\mathcal{E}] \in \mathbb{U}_X(s; r_1, d_1; \dots; r_s, d_s)$ and let (2) be the Harder-Narasimhan filtration of $\mathcal{E}$.

**Remark 4.3.** From the assumption $\gamma < 0$, we have $\Phi(\mathcal{F}_i) \subseteq \mathcal{F}_{i-1} \otimes \mathcal{T}_{\mathcal{D}}$.

Remark 4.3 immediately proves the following two lemmas.

**Lemma 4.4.** *For an integer $j \in \{1, \dots, s-1\}$, set $k(j) = \sum_{i=1}^{j} r_i$. Then*

$$s_{k(j)}(\mathcal{E}, \Phi) \le k(j) \deg \mathcal{E} - r \deg \mathcal{F}_j$$

**Remark 4.5.** We expect that the inequality in Lemma 4.4 is in fact equality, although we give the positive answers only to some special cases; see Lemma 4.6 and Proposition 4.10.

**Lemma 4.6.** *Assume $g = 0$ and take $\mathcal{E} \cong \oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i)$ with $a_i \ge a_j$ for all $i \le j$. Then we have*

$$s_k(\mathcal{E}, \Phi) = s_k(\mathcal{E}) = k(a_1 + \cdots + a_r) - r(a_1 + \cdots + a_k).$$

**Proposition 4.7.** *Fix $[\mathcal{E}] \in \mathbb{U}_X(s; r_1, d_1; \dots; r_s, d_s)$ with $s \ge 2$ and $\Phi \in \mathrm{Hom}(\mathcal{E}, \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}})$. Choose any $k \in \{r_1 + 1, \dots, r - r_s + 1\}$ such that there is $h \in \{1, \dots, s-1\}$ with $r_1 + \cdots + r_h < k < r_1 + \cdots + r_{h+1}$, and set*

$$\begin{aligned} r' &:= k - r_1 - \cdots - r_s, \\ e &:= s_k(\mathcal{F}_{h+1}/\mathcal{F}_h), \\ d' &:= r' \deg \mathcal{F}_{h+1}/\mathcal{F}_h - e r_{h+1}. \end{aligned}$$

*Let $\mathcal{B} \subset \mathcal{F}_{h+1}/\mathcal{F}_h$ be any subsheaf of rank $r'$ and degree $d'$, and set $\mathcal{A} := u^{-1}(\mathcal{B})$, where $u$ is the surjection in the exact sequence*

$$0 \longrightarrow \mathcal{F}_h \longrightarrow \mathcal{F}_{h+1} \stackrel{u}{\longrightarrow} \mathcal{F}_{h+1}/\mathcal{F}_h \longrightarrow 0.$$

*Then $\mathcal{B} \in \mathcal{S}(k, \mathcal{E}, \Phi)$ and $s_k(\mathcal{E}, \Phi) \le k \deg \mathcal{E} - k(\deg \mathcal{F}_h + e)$.*

*Proof.* Note that $d'$ is the degree of all rank $r'$ maximal degree subsheaves of $\mathcal{F}_{h+1}/\mathcal{F}_h$. Since $\deg \mathcal{A} = d'$, we have $\deg \mathcal{B} = \deg \mathcal{F}_h + d'$. Since $\Phi(\mathcal{F}_{h+1}) \subset \mathcal{F}_h \subset \mathcal{B}$, we have $\mathcal{B} \in \mathcal{S}(k, \mathcal{E}, \Phi)$. Since $\deg \mathcal{B} = \deg \mathcal{F}_h + \deg \mathcal{A}$, we get the assertion. □

Now Lemma 4.6 and Proposition 4.7 prove the following result.

**Corollary 4.8.** *The Segre invariant $s_k(\mathcal{E}, \Phi)$ is defined for all $(\mathcal{E}, \Phi)$, if $\gamma < 0$.*

Example 4.9 shows that in Proposition 4.7 we may have strict inequality; of course, to be in the set-up of Proposition 4.7 we need to have $r_{h+1} \geq 2$.

**Example 4.9.** Assume $g \geq 5$ and fix $h \in \{1, \dots, s-2\}$ with $s \geq 3$. Set $r_{h+1} = r_{h+2} = 2$ and fix $r_i > 0$ for $i \notin \{h+1, h+2\}$ and $d_i \in \mathbb{Z}$, $i = 1, \dots, s$ such that

- $d_i/r_i > d_{i+1}/r_{i+1}$ for all $i = 1, \dots, s-1$ and
- $d_{h+1} = 2d_{h+2} + 1$.

By a theorem of Nagata there is a stable bundle $\mathcal{E}_{h+1}$ of rank 2 with degree $d_{h+1}$ and $g - 1 \leq s_1(\mathcal{E}_{h+1}) \leq g$. Here, $s_1(\mathcal{E}_{h+1})$ is the only integer $t$ with $g - 1 \leq t \leq g$ and $d_{h+1} - t$ even. For $i \neq h+1$ we choose $\mathcal{E}_i$ to be any semistable bundle of degree $d_i$ and rank $r_i$. Set $\mathcal{E} := \oplus_{i=1}^{s} \mathcal{E}_i$ and then we have $\mathcal{F}_i = \oplus_{j=1}^{i} \mathcal{E}_j$ in the Harder-Narasimhan filtration (2) of $\mathcal{E}$.

Take any $\Phi : \mathcal{E} \to \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$ with $\ker(\Phi) \supseteq \oplus_{i=0}^{h+1} \mathcal{E}_i$, e.g. take $\Phi = 0$ or, for certain $\mathcal{E}_1$ and $\mathcal{E}_s$ so that there is a nonzero map $\mathcal{E}_s \to \mathcal{E}_1 \otimes \mathcal{T}_{\mathcal{D}}$, take a 2-nilpotent map $\Phi$ with $\ker(\Phi) \supseteq \mathcal{F}_{s-1}$. Let $\mathcal{A} \subset \mathcal{F}_{h+1}/\mathcal{F}_h$ be a line subbundle of maximal degree and then we have $\mathcal{B} := u^{-1}(\mathcal{A}) = (\sum_{i=1}^{h} \mathcal{E}_i) \oplus \mathcal{A}$. If we set $\mathcal{B}_1 := (\sum_{i=1}^{h} \mathcal{E}_i) \oplus \mathcal{E}_{h+2}$, then we have $\deg \mathcal{B}_1 > \deg \mathcal{B}$.

**Proposition 4.10.** *Assume $r_i = 1$ for all $i$. For an integer $k \in \{1, \dots, r-1\}$ and any co-Higgs bundle $(\mathcal{E}, \Phi)$ with $[\mathcal{E}] \in \mathbb{U}_X(r; 1, d_1; \dots; 1, d_r)$, we have*

$$s_k(\mathcal{E}) = s_k(\mathcal{E}, \Phi) = k \deg \mathcal{E} - r \deg \mathcal{F}_k$$

*and $\mathcal{F}_k$ is the only bundle achieving the minimum degree in $\mathcal{S}(k, \mathcal{E}, \Phi)$.*

*Proof.* By Remark 4.3 and Lemma 4.4, we have $[\mathcal{F}_k] \in \mathcal{S}(k, \mathcal{E}, \Phi)$. Thus it is sufficient to prove that $\mathcal{F}_k$ is the only one achieving the minimum degree in $\mathcal{S}(k, \mathcal{E}, 0)$. Take any $[\mathcal{G}] \in \mathcal{S}(k, \mathcal{E}, 0)$ with maximal degree. The maximality condition on $\deg \mathcal{G}$ implies that $\mathcal{E}/\mathcal{G}$ has no torsion and so it is a vector bundle of rank $r - k$ on $X$. We use double induction on $k$ and $r$. The case $k = 1$ is obvious, because $\mathcal{F}_1$ is the first step of the Harder-Narasimhan filtration of $\mathcal{E}$.

Assume that $k$ is at least two and the proposition holds for trivial co-Higgs fields with any $k' \in \{1, \dots, k-1\}$ and any bundles $\mathcal{E}'$ whose Harder-Narasimhan filtration has rank one bundles as subquotients.

Assume for the moment $\mathcal{F}_1 \subset \mathcal{G}$. Since $\mathcal{F}_1$ is saturated in $\mathcal{E}$, i.e. $\mathcal{E}/\mathcal{F}_1$ has no torsion, $\mathcal{F}_1$ is saturated in $\mathcal{G}$ and $\mathcal{G}/\mathcal{F}_1$ is a rank $k-1$ subsheaf of the vector bundle $[\mathcal{E}/\mathcal{F}_1] \in \mathbb{U}_X(r-1; 1, d_2; \dots; 1, d_r)$. The inductive assumption gives $\deg \mathcal{G}/\mathcal{F}_1 \leq \deg \mathcal{F}_k/\mathcal{F}_1$, with equality if and only if $\mathcal{G}/\mathcal{F}_1 \cong \mathcal{F}_k/\mathcal{F}_1$, i.e. $\deg \mathcal{G} \leq \deg \mathcal{F}_k$ with equality if and only if $\mathcal{G} \cong \mathcal{F}_k$.

Now assume $\mathcal{F}_1 \not\subseteq \mathcal{G}$. Since $\mathcal{G}$ is saturated in $\mathcal{E}$, this means that $\mathcal{F}_1 + \mathcal{G}$ has rank $k+1$. Let $\mathcal{N}$ be the saturation of $\mathcal{F}_1 + \mathcal{G}$ in $\mathcal{E}$, and then we have $\deg \mathcal{N} \geq \deg \mathcal{F}_1 + \deg \mathcal{G}$ and $\mathcal{N}/\mathcal{F}_1$ is a rank $k$ subsheaf of $\mathcal{E}/\mathcal{F}_1$. If $r \geq k-2$, then by the inductive assumption on $r$ we have $\deg \mathcal{N}/\mathcal{F}_1 \leq \deg \mathcal{F}_{k+1}/\mathcal{F}_1 < \deg \mathcal{F}_k - \deg \mathcal{F}_1$ and so $\deg \mathcal{G} < \deg \mathcal{F}_k$, a contradiction. Thus we may assume $k = r-1$ and so $\mathcal{N} \cong \mathcal{E}$. Since $\mathcal{F}_1 + \mathcal{G}$

has rank $k+1$, the natural map $\mathcal{G} \to \mathcal{E}/\mathcal{F}_1$ is injective. Thus we have $\deg \mathcal{G} \le \deg \mathcal{E} - \deg \mathcal{F}_1 < \deg \mathcal{F}_{r-1}$, a contradiction. □

Now for $k \in \{1, \dots, r-1\}$ set

$$\delta_k(\mathcal{E}, \Phi) := \max_{\mathcal{A} \in \mathcal{S}(k,\mathcal{E},\Phi)} \deg(\mathcal{A}),$$

$\delta_0(\mathcal{E}, \Phi) := 0$ and $\delta_r(\mathcal{E}, \Phi) := \deg(\mathcal{E})$. In case $\Phi = 0$, we simply denote it by $\delta_k(\mathcal{E})$.

**Proposition 4.11.** *Fix $h \in \{1, \dots, s\}$ with $s \ge 2$ and set $\rho := r_1 + \cdots + r_h$. For $[\mathcal{E}] \in \mathbb{U}_X(s; r_1, d_1; \dots; r_s, d_s)$, we have*

(i) *$s_\rho(\mathcal{E}) = \rho \deg \mathcal{E} - k \deg \mathcal{F}_h$ and $\mathcal{F}_h \subset \mathcal{E}$ is the only subsheaf of rank $\rho$ with degree $\deg \mathcal{F}_h$;*

(ii) *$\deg \mathcal{E} \le \mathcal{F}_{h-1} + (k-\rho) d_h / r_h$ for $k$ with $\rho - r_h < k < \rho$ and $[\mathcal{G}] \in \mathcal{S}(k, \mathcal{E}, 0)$;*

(iii) *$\delta_k(\mathcal{E}) - \deg(\mathcal{F}_{h-1})$ for $k$ with $\rho - r_h < k < \rho$, equals*

$$\max\{\sum_{j=h}^{s} \delta_{t_j}(\mathcal{F}_j/\mathcal{F}_{j-1}) \;\Big|\; t_h + \cdots + t_s = k + r_h - \rho \text{ with } 0 \le t_j \le r_j \text{ for all } j\}.$$

*Proof.* Set $\mu_i := \mu(\mathcal{F}_i/\mathcal{F}_{i-1}) = d_i/r_i$ for $i = 1, \dots, s$ and let $\mathcal{G} \subseteq \mathcal{E}$ be a rank $\rho$ subsheaf of maximal degree. Then part (i) is trivial if $s = h$, because $\mathcal{G} \cong \mathcal{E}$ in this case. Thus we may assume that $h < s$. Set $a_0 = 0$ and

$$a_i := \ker(\mathcal{F}_i \cap \mathcal{G}) \text{ with } k_i := a_i - a_{i-1},$$

for $i = 1, \dots, s$. If we denote by $\mathcal{R}_i \subseteq \mathcal{F}_i/\mathcal{F}_{i-1}$ the image of $\mathcal{F}_i \cap \mathcal{G}$ by the quotient map $\pi_i : \mathcal{F}_i \to \mathcal{F}_i/\mathcal{F}_{i-1}$, then $\mathcal{R}_i$ is trivial, i.e. $\mathcal{F}_i \cap \mathcal{G} \subseteq \mathcal{F}_{i-1}$, if and only if $k_i = 0$. Setting $S := \{i \in \{1, \dots, s\} \mid k_i > 0\}$, we have $\sum_{i=1}^s k_i = \sum_{i \in S} k_i = \rho$ and that $\mathcal{G} \cong \mathcal{F}_h$ if and only if $k_i = r_i$ for all $i \le h$, or equivalently $k_i = 0$ for all $i > h$. Since $\mathcal{F}_0$ is trivial, we have $\mathcal{R}_1 \cong \mathcal{F}_1 \cap \mathcal{G}$. Thus we have $\deg \mathcal{G} = \sum_{i \in S} \deg \mathcal{R}_i$. Since each $\mathcal{F}_i/\mathcal{F}_{i-1}$ is semistable, we have $\deg \mathcal{R}_i \le k_i \mu_i$ for all $i \in S$ and so we may use that $\mu_i > \mu_j$ for all $i < j$ to get part (i).

For part (ii) let $\mathcal{G} \subset \mathcal{E}$ be a rank $k$ subsheaf of maximal degree and define $k_i$, $S \subseteq \{1, \dots, s\}$ and the sheaves $\mathcal{R}_i \subset \mathcal{F}_i/\mathcal{F}_{i-1}$ as above. Then we have $\sum_{i \in S} k_i = k$ and $\deg \mathcal{G} \le \sum_{i \in S} k_i \mu_i$ and again we may use that $\mu_i > \mu_j$ for all $i < j$, to get the assertion. Part (iii) comes directly from the definition of $\delta_k(\mathcal{E})$. □

As immediate corollaries of Theorem 4.11 we get the following.

**Corollary 4.12.** *We have $s_k(\mathcal{E}, \Phi) = s_k(\mathcal{E}) = s_k(gr(\mathcal{E}))$ for all $k$.*

**Corollary 4.13.** *For $k$ with $r - r_s < k < r$, we have*

$$\delta_k(\mathcal{E}, \Phi) = \delta_k(\mathcal{E}) = d_1 + \cdots + d_{s-1} + \delta_{k+r_s-r}(\mathcal{E}/\mathcal{F}_{s-1}).$$

4.2. **Simplicity.** Again let $X$ be a smooth curve of genus $g$. Fix $\mathcal{R} \in \mathrm{Pic}(X)$ and set $\gamma := \deg \mathcal{R}$. For a map $\Phi : \mathcal{E} \to \mathcal{E} \otimes \mathcal{R}$, set

$$\mathrm{End}(\mathcal{E}, \Phi) := \{f \in \mathrm{End}(\mathcal{E}) \mid \hat{f} \circ \Phi = \Phi \circ f\},$$

where $\hat{f}$ is the map $f \otimes \mathrm{id}_{\mathcal{R}} : \mathcal{E} \otimes \mathcal{R} \to \mathcal{E} \otimes \mathcal{R}$.

In case $\gamma > 0$, it often happens that $\mathrm{End}(\mathcal{E}, \Phi)$ is properly contained in $\mathrm{End}(\mathcal{E})$ and $(\mathcal{E}, \Phi)$ is simple with $\mathcal{E}$ not simple, e.g. stable Higgs fields when $g \ge 2$ or stable co-Higgs fields when $g = 0$. In this short section, we consider the case $\gamma < 0$ and show why this is seldom the case for $\gamma < 0$.

We assume that $[\mathcal{E}] \in \mathbb{U}_X(s; r_1, d_1; \cdots; r_s, d_s)$ with the Harder-Narasimhan filtration (2) and that $\Phi : \mathcal{E} \to \mathcal{E} \otimes \mathcal{R}$ is nonzero and so $s \geq 2$. Note that every endomorphism of $\mathcal{E}$ preserves the Harder-Narasimhan filtration of $\mathcal{E}$. By Remark 4.3, every endomorphism of $(\mathcal{E}, \Phi)$ also preserves the Harder-Narasimhan filtration of $\mathcal{E}$. Now set $\mathcal{K} := \ker(\Phi)$ and then we have $\mathcal{K} \supseteq \mathcal{F}_1$ by the case $i = 1$ of Remark 4.3 or Lemma 5.3 below. For two maps $\varphi \in \mathrm{End}(\mathcal{E}/\mathcal{F}_{r-1})$ and $\psi \in \mathrm{Hom}(\mathcal{E}/\mathcal{F}_{r-1}, \mathcal{K})$, define a map $f : \mathcal{E} \to \mathcal{E}$ to be the following composition:

$$\mathcal{E} \longrightarrow \mathcal{E}/\mathcal{F}_{r-1} \longrightarrow \mathcal{K} \hookrightarrow \mathcal{E},$$

where the first map is the natural quotient and the second map is given by $\psi \circ \varphi$. By the definition of $\mathcal{K}$, we have $\Phi \circ f = 0$. If $\Phi$ is 2-nilpotent, i.e. $\mathrm{Im}(\Phi) \subseteq \mathcal{K} \otimes \mathcal{R}$, e.g. if $s = 2$ by Lemma 4.1, we have $\hat{f} \circ \Phi = 0$. So, if $\Phi$ is 2-nilpotent and $\mathrm{Hom}(\mathcal{E}/\mathcal{F}_{r-1}, \mathcal{K}) \neq 0$, then we have $\mathrm{End}(\mathcal{E}, \Phi) \not\cong \mathbb{C}$. We also see from $\mathcal{F}_1 \subseteq \mathcal{K}$ that if $\mathrm{Hom}(\mathcal{E}/\mathcal{F}_{r-1}, \mathcal{F}_1) \neq 0$, then we have $\mathrm{End}(\mathcal{E}, \Phi) \not\cong \mathbb{C}$. By Riemann-Roch, we get $\mathrm{Hom}(\mathcal{E}/\mathcal{F}_{r-1}, \mathcal{F}_1) \neq 0$, if $d_s/r_s < d_1/r_1 + g - 1$. Since $\Phi \neq 0$ and each $\mathcal{F}_i/\mathcal{F}_{i-1}$ is semistable, we have $d_s/r_s \leq d_1/r_1 + \gamma$. Now if $\mathcal{R} \cong \mathcal{T}_{\mathcal{D}}$, then we have $\gamma \leq 2 - 2g$ and so $d_s/r_s < g - 1 + d_1/r_1$ for all $g \geq 2$. Thus we get the following.

**Proposition 4.14.** *For $[\mathcal{E}] \in \mathbb{U}_X(s; r_1, d_1; \ldots; r_s, d_s)$ with a nonzero co-Higgs field $\Phi$ on a smooth curve $X$ of genus $g \geq 2$, the pair $(\mathcal{E}, \Phi)$ is not simple.*

**Remark 4.15.** In our set-up, adding a nonzero map $\Phi$ to an unstable bundle $\mathcal{E}$ does not help enough to get a semistable pair $(\mathcal{E}, \Phi)$; usually it is not simple, e.g. any endomorphism inducing $\mathcal{F}_s \to \mathcal{F}_1$ commutes with $\Phi$.

## 5. Higher dimensional case

In this section we consider the case when the dimension of $X$ is at least two. Note that a coherent sheaf $\mathcal{E}$ on $X$ is semistable if and only if $\mu_+(\mathcal{E}) = \mu_-(\mathcal{E})$.

5.1. **Case of $\mathcal{T}_{\mathcal{D}}$ semistable.** We fix a polarization $\mathcal{O}_X(1)$ with respect to which we consider slope, stability and semistability. We assume that $\mathcal{T}_{\mathcal{D}}$ is semistable with $\mu(\mathcal{T}_{\mathcal{D}}) < 0$; in case $\mu(\mathcal{T}_{\mathcal{D}}) \geq 0$, we would get that the framework would be the construction of stable or semistable co-Higgs or logarithmic co-Higgs bundles as in [4]. There are several manifolds $X$ with $T_X$ semistable, or equivalently with the semistable cotangent bundle; see [26].

Choose a pair $(\mathcal{E}, \Phi)$ with $\mathcal{E}$ a torsion-free sheaf of rank $r$ and $\Phi : \mathcal{E} \to \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$ with the Harder-Narasimhan filtration (2) of $\mathcal{E}$. Then the sheaf $\mathcal{F}_i/\mathcal{F}_{i-1}$ is a torsion-free semistable sheaf for all $i$ and $\mu(\mathcal{F}_i/\mathcal{F}_{i-1}) > \mu(\mathcal{F}_{i-1}/\mathcal{F}_{i-2})$ for every $i > 1$. As in §3 on curve case, for fixed integers $r_i$ and $d_i$, we consider the set $\mathbb{U}_X(s; r_1, d_1, \ldots, r_s, d_s)$ of torsion-free sheaves of rank $r$ on $X$ with the Harder-Narasimhan filtration (2) with subquotient $\mathcal{F}_i/\mathcal{F}_{i-1}$ of ranks $r_i$ and degrees $d_i$ for $i = 1, \ldots, s$.

Recall that in characteristic zero the tensor product of two semistable sheaves is still semistable by [22, Theorem 2.5], So if $\Phi$ is not trivial, then we get $s \geq 2$ and so $\mathcal{E}$ is not semistable with the Harder-Narasimhan filtration (4) for $\mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$. If $\mathcal{A}$ is a semistable torsion-free sheaf, then we have

$$\mu(\mathcal{A} \otimes \mathcal{T}_{\mathcal{D}}) = \mu(\mathcal{A}) + \gamma.$$

Thus if $[\mathcal{E}] \in \mathbb{U}_X(s; r_1, d_1; \ldots; r_s, d_s)$ and there is a nonzero map $\Phi : \mathcal{E} \to \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$, then we get $d_1/r_1 + \gamma \geq d_s/r_s$; see Corollary 3.9. Now let us use the same idea in Lemma

2.3. Define

$$\ell_2 = \ell_2(\mathcal{E}) := \max_{1\le i\le s}\{\, i \mid \mu(\mathcal{F}_i/\mathcal{F}_{i-1}) + \gamma \ge \mu(\mathcal{F}_s/\mathcal{F}_{s-1})\}$$

and then we have $\Phi(\mathcal{E}) \subset \mathcal{F}_{\ell_2}\otimes\mathcal{T}_{\mathcal{D}}$. From $\gamma < 0$, we get $\ell_2 \le s-1$. On the other hand, letting

$$\ell_1 = \ell_1(\mathcal{E}) := \min_{1\le j\le s}\{\, j \mid \mu(\mathcal{F}_{\ell_2}/\mathcal{F}_{\ell_2 -1}) + \gamma < \mu(\mathcal{F}_j/\mathcal{F}_{j-1})\},$$

the map $\Phi$ induces a nonzero map $\overline{\Phi} : \mathcal{E}/\mathcal{F}_{\ell_1} \longrightarrow \mathcal{F}_{\ell_2}\otimes\mathcal{T}_{\mathcal{D}}$. In particular, if $\ell_1 \ge \ell_2$, e.g. $s = 2$ or $d_2/r_2 + \gamma < d_s/r_s$, which imply $\ell_2 = 1$, then any such map $\Phi$ is 2-nilpotent.

In [4, Section 2] we consider the following exact sequence for $r \ge 2$

$$0 \longrightarrow \mathcal{O}_X^{\oplus(r-1)} \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_Z \otimes \mathcal{A} \longrightarrow 0, \tag{13}$$

where $\mathcal{A}$ is a line bundle of $\deg\mathcal{A} < 0$ with $h^0(\mathcal{T}_{\mathcal{D}}\otimes\mathcal{A}^\vee) \ge r-1$ and $Z \subset X$ is a locally complete intersection of codimension two. Under certain assumptions on $Z$, we may choose $\mathcal{E}$ to be reflexive or locally free. Then any $(r-1)$-dimensional linear subspace of $H^0(\mathcal{T}_{\mathcal{D}}\otimes\mathcal{A}^\vee)$ produces a nonzero 2-nilpotent co-Higgs field defined by the following composition:

$$\mathcal{E} \longrightarrow \mathcal{I}_Z \otimes \mathcal{A} \longrightarrow \mathcal{T}_{\mathcal{D}}^{\oplus(r-1)} \longrightarrow \mathcal{E}\otimes\mathcal{T}_{\mathcal{D}}.$$

Assume now the existence of an endomorphism $v : \mathcal{E} \to \mathcal{E}$ such that $v'\circ\Phi = \Phi\circ v$, where $v' : \mathcal{E}\otimes\mathcal{T}_{\mathcal{D}} \to \mathcal{E}\otimes\mathcal{T}_{\mathcal{D}}$ is the induces by $v$ and the identity map on $\mathcal{T}_{\mathcal{D}}$. Since we assume that $\deg\mathcal{A} < 0$, (13) is the Harder-Narasimhan filtration of $\mathcal{E}$. We also assume that (13) does not split and so every automorphism of $\mathcal{E}$ is induced by an element of $H^0(\mathcal{A}^\vee)^{\oplus(r-1)}$ and an $(r-1)\times(r-1)$-matrix of constants acting on $\mathcal{O}_X^{\oplus(r-1)}$. Note that, if $r = 2$, these assumptions imply $h^0(\mathcal{E}nd(\mathcal{E})) = 1 + h^0(\mathcal{A}^\vee)$. In this case, the co-Higgs field $\Phi$ is obtained by composing a map $\Phi_1 : \mathcal{I}_Z\otimes\mathcal{A} \to \mathcal{T}_{\mathcal{D}}$ with a map $\Phi_2 : \mathcal{T}_{\mathcal{D}} \to \mathcal{E}\otimes\mathcal{T}_{\mathcal{D}}$ induced by the inclusion in (13).

5.2. **Case of $\mathcal{T}_{\mathcal{D}}$ not semistable.** In this subsection we assume that $\mathcal{T}_{\mathcal{D}}$ is not semistable so that it admits the Harder-Narasimhan filtration

$$\{0\} = \mathcal{H}_0 \subset \mathcal{H}_1 \subset \cdots \subset \mathcal{H}_h = \mathcal{T}_{\mathcal{D}} \tag{14}$$

with $h \ge 2$. Assume further that $\mu_+(\mathcal{T}_{\mathcal{D}}) = \mu(\mathcal{H}_1) < 0$. Since $h \ge 2$, we have $\dim X \ge h \ge 2$.

Fix a torsion-free sheaf $\mathcal{E}$ of rank $r$ and degree $d$ with Harder-Narasimhan filtration (2). We assume the existence of a nonzero logarithmic co-Higgs field $\Phi : \mathcal{E} \to \mathcal{E}\otimes\mathcal{T}_{\mathcal{D}}$.

**Lemma 5.1.** *If $\mathcal{E}$ is reflexive, then $\mathcal{F}_i$ is also reflexive for each $i$.*

*Proof.* In case $n = 1$, the sheaf $\mathcal{F}_i$ in (2) is locally free and in particular reflexive. Now assume $n \ge 2$ and then we need to prove that $\mathcal{F}_i$ has depth at least two. This is true, because $\mathcal{E}$ has depth at least two and $\mathcal{E}/\mathcal{F}_i$ has no torsion and so it has positive depth. □

**Remark 5.2.** Lemma 5.1 works for arbitrary $\mathcal{T}_{\mathcal{D}}$, even in the case $n = 1$.

**Lemma 5.3.** *We have $\mathcal{F}_1 \subseteq \ker(\Phi)$ and $s \ge 2$.*

*Proof.* Assume $\Phi(\mathcal{F}_1) \neq 0$ and let $i_0$ be the minimal integer $i \in \{1,\dots,s\}$ such that $\Phi(\mathcal{F}_1) \subseteq \mathcal{F}_i \otimes \mathcal{T}_{\mathcal{D}}$. By the definition of $i_0$, the map $\Phi$ induces a nonzero map $\varphi : \mathcal{F}_1 \to (\mathcal{F}_{i_0}/\mathcal{F}_{i_0-1}) \otimes \mathcal{T}_{\mathcal{D}}$. Since the tensor product of two semistable sheaves, modulo its torsion, is again semistable by [22, Theorem 2.5] and $\mu(\mathcal{H}_1) < 0$, the sheaf $gr((\mathcal{F}_{i_0}/\mathcal{F}_{i_0-1}) \otimes \mathcal{T}_{\mathcal{D}})$ given by the Harder-Narasimhan filtration of $\mathcal{T}_{\mathcal{D}}$ has all its factors with slope less than $\mu(\mathcal{F}_1)$. Thus we get $\Phi = 0$, a contradiction.

Now $\Phi$ is a nonzero map with $\ker(\Phi) \supseteq \mathcal{F}_1$ and so we have $s \geq 2$. □

**Remark 5.4.** By Lemma 5.1, the pair $(\mathcal{F}_1, 0)$ is a logarithmic co-Higgs subsheaf of $(\mathcal{E}, \Phi)$ and so $(\mathcal{E}, \Phi)$ is not semistable. In particular, $\mathcal{E}$ is also not semistable.

5.2.1. *Rank 2 case.* In this subsection we consider the co-Higgs sheaves $(\mathcal{E}, \Phi)$ with $\mathcal{E}$ reflexive of rank two and $\Phi$ nonzero.

**Lemma 5.5.** *If $\mathcal{E}$ is reflexive of rank two, then $\Phi$ is 2-nilpotent.*

*Proof.* Since $\Phi$ is nonzero, the sheaf $\mathcal{F}_1$ has rank one by Lemma 5.3. Since $\mathcal{F}_1$ is reflexive on a smooth variety $X$ by Lemma 5.1, it is a line bundle by [16, Proposition 1.9]. Now we get that $\mathcal{E}/\mathcal{F}_1 \cong \mathcal{I}_Z \otimes \mathcal{A}$ for some line bundle $\mathcal{A}$ and some closed subscheme $Z \subset X$ with $\dim Z \leq n-2$. By definition of Harder-Narasimhan filtration, we have $\deg \mathcal{A} < \deg \mathcal{F}_1$. Let $\psi : \mathcal{E} \to (\mathcal{E}/\mathcal{F}_1) \otimes \mathcal{T}_{\mathcal{D}}$ be the map induced by $\Phi$. Since $\ker(\Phi) \supseteq \mathcal{F}_1$ by Lemma 5.3, it is sufficient to prove that $\Phi(\mathcal{E}) \subseteq \mathcal{F}_1 \otimes \mathcal{T}_{\mathcal{D}}$, i.e. $\psi = 0$. Note that $\psi$ induces a map $\widetilde{\psi} : (\mathcal{E}/\mathcal{F}_1) \to (\mathcal{E}/\mathcal{F}_1) \otimes \mathcal{T}_{\mathcal{D}}$ with $\mathrm{Im}(\psi) = \mathrm{Im}(\widetilde{\psi})$, due to $\mathcal{F}_1 \subseteq \ker(\Phi)$. Since $(\mathcal{E}/\mathcal{F}_1)$ has rank one and it is torsion-free, it is semistable. Again as in the proof of Lemma 5.3, since $\mu(\mathcal{H}_1) < 0$ and the tensor product of two semistable sheaves, modulo its torsion, is semistable by [22, Theorem 2.5], we have $\mu((\mathcal{E}/\mathcal{F}_1) \otimes \mathcal{H}_1) < \mu(\mathcal{E}/\mathcal{F}_1)$ and so $\widetilde{\psi} = 0$. Thus we have $\psi = 0$. □

Now we describe all pairs $(\mathcal{E}, \Phi)$ with $\mathcal{E}$ reflexive of rank two and $\Phi$ nonzero. By Lemma 5.3 and assumption that $\Phi$ is nonzero, the sheaf $\mathcal{E}$ is not semistable and $s = 2$. By Lemmas 5.1, 5.3, 5.5 and [16, Proposition 1.9], the map $\Phi$ is 2-nilpotent and it fits into an exact sequence

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_Z \otimes \det(\mathcal{E}) \otimes \mathcal{F}_1^{\vee} \longrightarrow 0,$$

with $Z$ a closed subscheme of $X$ with either $Z = \emptyset$ or $\dim Z = n-2$. Moreover, $\Phi$ is uniquely determined by a map $u : \det(\mathcal{E}) \otimes \mathcal{F}_1^{\vee} \to \mathcal{F}_1 \otimes \mathcal{T}_{\mathcal{D}}$. Thus the set of all logarithmic co-Higgs structures on $\mathcal{E}$ is parametrized by

$$V(\mathcal{E}) := H^0(\det(\mathcal{E})^{\vee} \otimes \mathcal{F}_1^{\otimes 2} \otimes \mathcal{T}_{\mathcal{D}}).$$

The trivial element $0 \in V(\mathcal{E})$ corresponds to the trivial co-Higgs field $\Phi = 0$. Note that $\Phi = 0$ also exists for stable sheaves.

Now we reverse the construction. Fix two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on $X$ with $\deg \mathcal{L}_1 > \deg \mathcal{L}_2$ and a closed subscheme $Z \subset X$ such that a general extension

$$0 \longrightarrow \mathcal{L}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_Z \otimes \mathcal{L}_2 \longrightarrow 0 \tag{15}$$

is reflexive. We just observed that any co-Higgs field $\Phi : \mathcal{E} \to \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$ is 2-nilpotent and that $\mathrm{Hom}(\mathcal{E}, \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}) \cong H^0(\mathcal{L}_1 \otimes \mathcal{L}_2^{\vee} \otimes \mathcal{T}_{\mathcal{D}})$. We may see [16, Theorem 4.1] for a description about the conditions on $\mathcal{L}_1$, $\mathcal{L}_2$, $\omega_X$ and $Z$ assuring the existence of a reflexive sheaf fitting in (15) when $n = 3$. Since (15) is the Harder-Narasimhan filtration of any $\mathcal{E}$ fitting into (15), so the family of the co-Higgs sheaves $(\mathcal{E}, \Phi)$ with $gr(\mathcal{E}) = \mathcal{L}_1 \oplus (\mathcal{I}_Z \otimes \mathcal{L}_2)$ is parametrized by a fibration over $\mathbb{P}\mathrm{Ext}^1_X(\mathcal{I}_Z \otimes \mathcal{L}_2, \mathcal{L}_1)$ whose fibre over $[\mathcal{E}]$ is $H^0(\mathcal{L}_1 \otimes \mathcal{L}_2^{\vee} \otimes \mathcal{T}_{\mathcal{D}})$.

**Remark 5.6.** Assume $s = 2$ and $\mu_+(\mathcal{T}_\mathcal{D}) < 0$. For a torsion-free coherent sheaf $\mathcal{E}$ of rank at least 2, as in the proof of Lemma 5.5 we see that every logarithmic co-Higgs field $\Phi : \mathcal{E} \to \mathcal{E} \otimes \mathcal{T}_\mathcal{D}$ is integrable and 2-nilpotent with

$$\mathrm{Hom}(\mathcal{E}, \mathcal{E} \otimes \mathcal{T}_\mathcal{D}) \cong \mathrm{Hom}(\mathcal{E}/\mathcal{F}_1, \mathcal{F}_1 \otimes \mathcal{T}_\mathcal{D}),$$

where $\mathcal{F}_1$ is semistable, and $\mathcal{E}/\mathcal{F}_1$ is torsion-free and semistable. Recall that if $\mathcal{E}$ is reflexive, then so is $\mathcal{F}_1$ by Lemma 5.1. Take any exact sequence

$$0 \to \mathcal{F}_1 \to \mathcal{G} \to \mathcal{E}/\mathcal{F}_1 \to 0. \tag{16}$$

Any such extension in (16) is torsion-free. For any fixed $\mathcal{G}$ fitting into (16), not necessarily reflexive, the proof of Lemma 5.5 shows that every logarithmic co-Higgs field $\Phi : \mathcal{G} \to \mathcal{G} \otimes \mathcal{T}_\mathcal{D}$ is integrable and 2-nilpotent with $\mathrm{Hom}(\mathcal{G}, \mathcal{G} \otimes \mathcal{T}_\mathcal{D}) \cong \mathrm{Hom}(\mathcal{E}/\mathcal{F}_1, \mathcal{F}_1 \otimes \mathcal{T}_\mathcal{D})$.

5.2.2. *Rank* 3 *case.* We assume $r = 3$ and that $\mathcal{E}$ is reflexive. Since we assume $\mu_+(\mathcal{T}_\mathcal{D}) < 0$, we get $s \geq 2$ by Lemma 5.3 and so $s \in \{2, 3\}$.

**Remark 5.7.** The case $s = 2$ is dealt in Remark 5.6. In this case, the sheaf $\mathcal{F}_1$ is either a line bundle or a semistable reflexive sheaf of rank two with $\mathcal{E}/\mathcal{F}_1 \cong \mathcal{I}_Z \otimes \mathcal{A}$ for some line bundle $\mathcal{A}$ and a closed subscheme $Z \subset X$ with $\dim Z \leq n - 2$. In both cases, we may apply Remark 2.8.

From now on we assume $s = 3$ and so the sheaf $\mathcal{F}_i$ in (2) has rank $i$ for each $i$. By Lemma 5.1, the sheaf $\mathcal{F}_1$ is a line bundle and $\mathcal{F}_2$ is reflexive so that $\mathcal{F}_2/\mathcal{F}_1 \cong \mathcal{I}_{Z_1} \otimes \mathcal{A}_1$ and $\mathcal{E}/\mathcal{F}_2 \cong \mathcal{I}_{Z_2} \otimes \mathcal{A}_2$ with $\mathcal{A}_1, \mathcal{A}_2$ line bundles and $Z_1, Z_2$ closed subschemes of $X$ with dimension at most $n - 2$. Here we have $\deg \mathcal{F}_1 > \deg \mathcal{A}_1 > \deg \mathcal{A}_2$. Set

$$\delta(\mathcal{E}) := \mu_-(\mathcal{F}_2) - \mu_+(\mathcal{E} \otimes \mathcal{T}_\mathcal{D}),$$

where $\mu_-(\mathcal{F}_2) = \deg \mathcal{A}_1$ and $\mu_+(\mathcal{E} \otimes \mathcal{T}_\mathcal{D}) = \deg \mathcal{F}_1 + \mu_+(\mathcal{T}_\mathcal{D})$

(a) Assume $\delta(\mathcal{E}) > 0$ and then we have $\Phi_{|\mathcal{F}_2} = 0$, i.e. $\Phi$ is uniquely induced by a map $u_1 : \mathcal{I}_{Z_2} \otimes \mathcal{A}_2 \to \mathcal{E} \otimes \mathcal{T}_\mathcal{D}$. Since $\mathcal{E}/\mathcal{F}_2 \cong \mathcal{I}_{Z_2} \otimes \mathcal{A}_2$ is of rank one and $\mu_+(\mathcal{T}_\mathcal{D}) < 0$, the composition of $u_1$ with the quotient map $\mathcal{E} \otimes \mathcal{T}_\mathcal{D} \to (\mathcal{E}/\mathcal{F}_2) \otimes \mathcal{T}_\mathcal{D}$ is trivial, implying $\mathrm{Im}(u_1) \subseteq \mathcal{F}_2 \otimes \mathcal{T}_\mathcal{D}$. Thus $\Phi$ is uniquely determined by a map $u : \mathcal{I}_{Z_2} \otimes \mathcal{A}_2 \to \mathcal{F}_2 \otimes \mathcal{T}_\mathcal{D}$. Conversely, any map $u : \mathcal{I}_{Z_2} \otimes \mathcal{A}_2 \to \mathcal{F}_2 \otimes \mathcal{T}_\mathcal{D}$ induces a 2-nilpotent logarithmic co-Higgs field on $\mathcal{E}$ by taking the composition $u \circ \pi$, where $\pi : \mathcal{E} \to \mathcal{E}/\mathcal{F}_2$ is the quotient map.

(b) Assume now $\delta(\mathcal{E}) \leq 0$. Set $\mathcal{B} := \mathrm{Im}(\Phi_{|\mathcal{F}_2})$ and $\mathcal{G} := \mathrm{Im}(\Phi)$. Since we have

$$\mu_+((\mathcal{E}/\mathcal{F}_2) \otimes \mathcal{T}_\mathcal{D}) = \mu(\mathcal{E}/\mathcal{F}_2) + \mu_+(\mathcal{T}_\mathcal{D}) < \mu(\mathcal{E}/\mathcal{F}_2),$$

the composition of $\Phi$ with the quotient map $\mathcal{E} \otimes \mathcal{T}_\mathcal{D} \to (\mathcal{E}/\mathcal{F}_2) \otimes \mathcal{T}_\mathcal{D}$ is trivial and so we have $\mathcal{G} \subseteq \mathcal{F}_2 \otimes \mathcal{T}_\mathcal{D}$. If $\mathcal{B}$ is trivial, then we may apply part (a), i.e. $\Phi$ is 2-nilpotent and it is uniquely induced by $u : \mathcal{I}_{Z_2} \otimes \mathcal{A}_2 \to \mathcal{F}_2 \otimes \mathcal{T}_\mathcal{D}$. Now we assume that $\mathcal{B}$ is not trivial. Since $\Phi(\mathcal{F}_1) = 0$ and $\mathcal{F}_2/\mathcal{F}_1$ is a torsion-free sheaf of rank one, we have $\mathcal{B} \cong \mathcal{F}_2/\mathcal{F}_1$ and so $\mathrm{rk}(\mathcal{G}) \in \{1, 2\}$. Note that we have $\mathcal{B} \subseteq \mathcal{F}_1 \otimes \mathcal{T}_\mathcal{D}$ from $\mu_+(\mathcal{E}/\mathcal{F}_2) + \mu_+(\mathcal{T}_\mathcal{D}) < \mu_+(\mathcal{E}/\mathcal{F}_2)$.

(b-i) First assume $\mathrm{rk}(\mathcal{G}) = 1$ and then $\mathcal{B}$ is a subsheaf of $\mathcal{G}$ with the same rank. Since $\mathcal{F}_1 \otimes \mathcal{T}_\mathcal{D}$ is a saturated subsheaf of $\mathcal{F}_2 \otimes \mathcal{T}_\mathcal{D}$, we have $\mathcal{G} \subseteq \mathcal{F}_1 \otimes \mathcal{T}_\mathcal{D}$. Thus $\Phi$ is uniquely determined by a map $\mathcal{E}/\mathcal{F}_1 \to \mathcal{F}_1 \otimes \mathcal{T}_\mathcal{D}$, i.e. by an element of $H^0(\mathcal{T}_\mathcal{D} \otimes \mathcal{F}_1 \otimes \mathcal{A}^\vee)$; the converse also holds, but we cannot guarantee the integrability of the associated logarithmic co-Higgs field.

(b-ii) Now assume $\mathrm{rk}(\mathcal{G}) = 2$. Since we have $\mathcal{G} = \psi(\mathcal{E}/\mathcal{F}_1)$ for the map $\psi : \mathcal{E}/\mathcal{F}_1 \to \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$, the map $\psi$ is injective as a map of sheaves and $\mathcal{G} \cong \mathcal{E}/\mathcal{F}_1$. In this case we also have $\mathcal{F}_1 = \ker(\Phi)$. We get that $\mathcal{E}$ is a reflexive sheaf fitting into an exact sequence

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{E} \stackrel{f}{\longrightarrow} \mathcal{G} \longrightarrow 0 \tag{17}$$

with $\mathcal{F}_1$ a line bundle and $\mathcal{G}$ a torsion-free unstable sheaf of rank two with $\deg \mathcal{F}_1 > \mu_+(\mathcal{G})$. The map $\Phi$ is determined by a unique injective map $v : \mathcal{G} \to \mathcal{F}_2 \otimes \mathcal{T}_{\mathcal{D}}$. Conversely, set $\mathcal{G}_1 \subset \mathcal{G}$ to be the Harder-Narasimhan filtration of $\mathcal{G}$ and $\mathcal{F}_2 = f^{-1}(\mathcal{G}_1)$, where $f$ is the surjection in (17). Then the composition of the quotient map $\mathcal{E} \to \mathcal{E}/\mathcal{F}_1$ with an injective map $\mathcal{G} \to \mathcal{F}_2 \otimes \mathcal{T}_{\mathcal{D}}$ induces a logarithmic co-Higgs field $\Phi$ with the given data $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{G})$, which does not necessarily satisfy the integrability condition. Note that if $\mathcal{G} \subset \mathcal{F}_1 \otimes \mathcal{T}_{\mathcal{D}}$, i.e. $\Phi$ comes from an injective map $\mathcal{G} \to \mathcal{F}_1 \otimes \mathcal{T}_{\mathcal{D}}$, then $\Phi$ is 2-nilpotent and so it is integrable.

**Example 5.8.** Assume that $\mathcal{T}_{\mathcal{D}}$ is not semistable with Harder-Narasimhan filtration (14) and set $\mu_2(\mathcal{T}_{\mathcal{D}}) := \mu(\mathcal{H}_2/\mathcal{H}_1)$. Let $\mathcal{E}$ be a torsion-free sheaf of rank $r$ with (2) as its Harder-Narasimhan filtration and assume $\mu_+(\mathcal{E}) - \mu_-(\mathcal{E}) < \mu_2(\mathcal{T}_{\mathcal{D}})$. In this case, for any map $\Phi : \mathcal{E} \to \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$, the sheaf $\mathrm{Im}(\Phi)$ is contained in the subsheaf $\overline{\mathcal{E} \otimes \mathcal{H}_1}$ of $\mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$, which is the image of the natural map $\mathcal{E} \otimes \mathcal{H}_1 \to \mathcal{E} \otimes \mathcal{T}_{\mathcal{D}}$. We have $\overline{\mathcal{E} \otimes \mathcal{H}_1} \cong \mathcal{E} \otimes \mathcal{H}_1$ if either $\mathcal{E}$ or $\mathcal{H}_1$ is locally free. Note that $\mathcal{H}_1$ is locally free, if it has rank one, because $\mathcal{H}_1$ is reflexive and $X$ is smooth; see [16, Proposition 1.9]. In particular, if $n = 2$, then $\mathcal{H}_1$ is a line bundle and $\mu_2(\mathcal{T}_{\mathcal{D}}) = \mu_-(\mathcal{T}_{\mathcal{D}})$. Thus under these assumptions we may repeat the observations given in the case $\mathcal{T}_{\mathcal{D}}$ semistable using $\mathcal{H}_1$ instead of $\mathcal{T}_{\mathcal{D}}$. Without any assumption on $\mu_2(\mathcal{T}_{\mathcal{D}})$ we may see at least a part of the logarithmic co-Higgs fields of $\mathcal{E}$ in this way.

## References


[1] M. F. Atiyah, *Vector bundles on an elliptic curve*, Proc. London Math. Soc. **7** (1959), 414–452.

[2] E. Ballico and S. Huh, *A note on co-Higgs bundles*, to appear in Taiwanese J. Math., preprint, arXiv:1606.01843 [math.AG].

[3] E. Ballico and S. Huh, *2-nilpotent co-Higgs structures*, preprint, arXiv:1606.02584 [math.AG].

[4] E. Ballico and S. Huh, *Logarithmic co-Higgs bundles*, preprint, arXiv:1609.03733 [math.AG].

[5] E. Ballico and B. Russo, *Maps between stable vector bundles on a smooth curve*, Bull. Polish Acad. Sci. Math. **50** (2002), no. 4, 433–446.

[6] C. Bănică, *Sur les Ext-globaux dans une déformation*, Compositio Math. **44** (1981), no. 1-3, 17–27.

[7] U. N. Bhosle, *Maximal subsheaves of torsion-free sheaves on nodal curves*, J. London Math. Soc. (2) **74** (2006), 59–74.

[8] L. Brambila Paz and H. Lange, *A stratification of the moduli space of vector bundles on curves* Dedicated to Martin Kneser on the occasion of his 70th birthday, J. Reine Angew. Math. **494** (1998), 173–187.

[9] S. B. Bradlow, O. García-Prada and P. Gothen, *Moduli spaces of holomorphic triples over compact Riemann surfaces*, Math. Ann. **328** (2004), 299–351.

[10] A. V. Colmenares, *Moduli spaces of semistable rank-2 co-Higgs bundles over* $\mathbb{P}^1 \times \mathbb{P}^1$, preprint, arXiv:1604.01372 [math.AG].

[11] O. Debarre, *Varieties with ample cotangent bundle*, Compos. Math. **141** (2005), no. 6, 1445–1459.

[12] O. Debarre, Corrigendum: Varieties with ample cotangent bundle (Compos. Math. **141** (2005), 1445–1459), Compos. Math. **149** (2013), no. 3, 505–506.

[13] I. Dolgachev, *Logarithmic sheaves attached to arrangements of hyperplanes*, J. Math. Kyoto Univ. **47** (2007), no. 1, 35–64.

[14] T. Gómez, *Algebraic stacks*, Proc. Indian Acad. Sci. Math. Sci. **111** (2001), no. 1, 1–31; arXiv: 9911199.

[15] M. Gualtieri, *Generalized complex geometry*, Ann. Math. **174** (2011), no. 1, 75–123.
[16] R. Hartshorne, *Stable reflexive sheaves*, Math. Ann. **254** (1980), no. 2, 121–176.
[17] A. Hirschowitz, *Problèmes de Brill-Noether en rang supèrieur*, unpublished preprint, Nice, 1986; available in pdf in his webpages.
[18] N. Hitchin, *Generalized Calabi-Yau manifolds*, Q. J. Math. **54** (2003), no. 3, 281–308.
[19] H. Lange, *Zur Klassifikation von Regelmannigfaltigkeiten*, Math. Ann. **262** (1983), no. 4, 447–459.
[20] H. Lange, *Universal families of extensions*, J. Algebra **83** (1983), no. 1, 101–112.
[21] H. Lange, *Some geometrrical aspects of vector bundles on curves*, Topics in algebraic geometry (Guanajuato, 1989), 53–74, Aportaciones Mat. Notes Investigatión 5, Soc. Mat. Mexicana, México, 1992.
[22] M. Maruyama, *The theorem of Grauert-Mülich-Spindler*, Math. Ann. **255** (1981), no. 3, 317–333.
[23] S. Mukai and F. Sakai, *Maximal subbundles of vector bundles on a curve*, Manuscripta Math. **2** (1985), 251–256.
[24] N. Nitsure, *Moduli space of semistable pairs on a curve*, Proc. London Math. Soc. (3) **62** (1991), no. 2, 275–300.
[25] W. Oxbury, *Varieties of maximal line subbundles*, Math. Proc. Camb. Phil. Soc. **129** (2000), 9–18.
[26] T. Peternell and J. A. Wiśniewski, *On stability of tangent bundles of Fano manifolds with* $b_2 = 1$, J. Alg. Geom. **4** (1995), 363–384.
[27] S. Rayan, *Constructing co-Higgs bundles on* $\mathbb{CP}^2$, Q. J. Math. **65** (2014), no. 4, 1437–1460.
[28] B. Russo and M. Teixidor i Bigas, *On a conjecture of Lange*, J. Alg. Geom. **8** (1999), no. 3, 483–496.
[29] C. Simpson, *Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization*, J. Amer. Math. Soc. **1** (1988), no. 4, 867–918.
[30] M. Teixidor i Bigas, *Subbundles of maximal degree*, Math. Proc. Camb. Phil. Soc. **136** (2004), 541–545.

Università di Trento, 38123 Povo (TN), Italy
*E-mail address*: edoardo.ballico@unitn.it

Sungkyunkwan University, Suwon 440-746, Korea
*E-mail address*: sukmoonh@skku.edu