A NOTE ON ENTIRE FUNCTIONS

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Abstract. In this article we consider certain power series with real coefficients that represent an entire function and provide sufficient conditions for the unboundedness of these functions on the real axis.

1. Introduction

Let us consider a function of complex variable

$$f(z) = \sum_{n=0}^{\infty} a_n z^{2n}. \tag{1}$$

with $a_n > 0$ for each $n \in \mathbb{N}$. Trivially, such a function is upper unbounded on the real axis. Suppose we alter infinite coefficients a_n making them negative. It can well happen that the new function is bounded on the real axis. For example, if we consider

$$f(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$$

and we alternating the signs, we get

$$g(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = \cos z$$

which is bounded on $\mathbb{R}$. Under what conditions the series continues to be an entire function upper unbounded on the real axis?

2. A first result

Theorem 1. *Let $f(z)$ be an entire function as in (1) and let be $(c_n)_n$ a sequence of positive real numbers. Let*

$$b_n = \begin{cases} a_n & if\ n \equiv 1 \pmod 3, \\ a_n & if\ n \equiv 2 \pmod 3, \\ -c_{n/3} & if\ n \equiv 0 \pmod 3, \quad n > 0. \end{cases}$$

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be a real sequence such that the function

$$g(z) = \sum_{n=0}^{\infty} b_n z^{2n} \tag{2}$$

is still entire. If, for each $n \in \mathbb{N}$, it is

$$c_n \leq 2\sqrt{a_{3n-1}a_{3n+1}} \tag{3}$$

then the function g is upper unbounded on the real axis.

Proof. If $x \in \mathbb{R}$, we can write

$$g(x) = a_0 + a_1 x^2 + \sum_{n=1}^{\infty} x^{6n-2}\left(a_{3n-1} - c_n x^2 + a_{3n+1}x^4\right)$$

and we can call, for each $n \in \mathbb{N}$, $n \geq 1$

$$p_n(x) = a_{3n-1} - c_n x^2 + a_{3n+1}x^4.$$

Since, by hypothesis, it is

$$c_n^2 - 4a_{3n-1}a_{3n+1} \leq 0.$$

we have that $p_n(x) \geq 0$ for each $x \in \mathbb{R}$ and for each $n \geq 1$. Hence

$$g(x) \geq a_0 + a_1 x^2.$$

and so g is an upper unbounded entire function on the real axis. □

Corollary 1. *If there is $k \in \mathbb{N}$, $k \geq 2$ such that (3) for each $n \geq k$ then then the function g is upper unbounded on the real axis.*

Proof. We can write

$$g(x) = q_k(x) + \sum_{n=k}^{+\infty} x^{6n-2}\left(a_{3n-1} - c_n x^2 + a_{3n+1}x^4\right).$$

where

$$q_k(x) = a_0 + a_1 x^2 + \sum_{h=1}^{k-1} x^{6h-2} p_h(x).$$

By reasoning as before, we have that $g(x) \geq q_k(x)$ $\forall x \in \mathbb{R}$. Now, since the leading term of $q_k(x)$ is given by $a_{3k-2}x^{6k-4}$, we have that

$$\lim_{x\to\pm\infty} q_k(x) = +\infty.$$

thus the function g is upper unbounded on the real axis. □

Corollary 2. *With the same hypothesis as before, if*

$$\lim_{n\to+\infty} \sup \frac{c_n^2}{a_{3n-1}a_{3n+1}} \leq L < 4.$$

then the function g is upper unbounded on the real axis.

Proof. By definition of $\lim_{n\to+\infty}$ sup, we have that

$$\forall\varepsilon > 0\,\exists n\,(\varepsilon) : \forall n > n\,(\varepsilon) \Rightarrow \frac{c_n^2}{a_{3n-1}a_{3n+1}} < L + \varepsilon.$$

If we choose $0 < \varepsilon \leq 4 - L$, then we have that

$$\forall n > n\,(\varepsilon) \;\Rightarrow\; \frac{c_n^2}{a_{3n-1}a_{3n+1}} \leq L + 4 - L = 4.$$

By Corollary 1 we have that the function g is upper unbounded on the real axis. □

In particular, we have that

Corollary 3. *If*

$$\lim_{n\to+\infty} \frac{c_n^2}{a_{3n-1}a_{3n+1}} = L < 4.$$

then the function g is upper unbounded on the real axis.

Of course we have that

Corollary 4. *Let*

$$g(z) = \sum_{n=0}^{\infty} a_n z^n$$

be an entire function such that

(1) $a_n \in \mathbb{R}$,
(2) *There exists $n_1 \in \mathbb{N}$ such that $n > n_1$, n odd $\Rightarrow a_n = 0$,*
(3) *If $p_{n_1}(z) = \sum_{n=0}^{n_1} a_n z^n$ the function $f(z) = g(z) - p_{n_1}(z)$ satisfies the hypothesis of Corollary 3.*

then g is upper unbounded on the real axis.

Proof. Trivial. □

3. A second result

Theorem 2. *Let $f(z)$ be an entire function as in (1) and let be $(c_n)_n$, $(d_n)_n$ two sequences of positive real numbers such that the function*

$$g(z) = \sum_{n=0}^{+\infty} b_n z^n$$

is entire, with

$$\begin{cases} b_0 = a_0 \\ b_1 = a_1 \\ b_2 = a_2 \end{cases}$$

and

$$b_n = \begin{cases} a_n & if \quad n \equiv 2 \pmod 4, \\ c_{\frac{n+1}{4}} & if \quad n \equiv 3 \pmod 4, \\ -d_{\frac{n}{4}} & if \quad n \equiv 0 \pmod 4, \\ a_n & if \quad n \equiv 1 \pmod 4. \end{cases}$$

for $n \geq 3$. If

$$\lim_{n\to+\infty} \sup \frac{d_n^2}{c_n a_{4n+1}} < L < 3. \tag{4}$$

then g is upper unbounded on the real axis.

Proof. We write

$$g(x) = a_0 + a_1 x^2 + \sum_{n=1}^{\infty} x^{8n-4} q_n(x).$$

where

$$q_n(x) = a_{4n-2} + c_n x^2 - d_n x^4 + a_{4n+1} x^6 \quad \forall n \geq 1.$$

We observe that

$$q_n'(x) = 2x\left(c_n - 2d_n x^2 + 3a_{4n+1}x^4\right) \quad \forall n \geq 1.$$

thus, if

$$d_n^2 < 3c_n a_{4n+1} \quad \forall n \geq 1 \tag{5}$$

the $q_n(x)$ has only a point of local minimum at $x = 0$ we have that $q_n(0) = a_{4n-2} > 0$ by hypothesis. By condition (4) there exists an $n_1 \in \mathbb{N}$ such that for each $n > n_1$ the condition (5) holds. Therefore we have that

$$g(x) = a_0 + a_1 x^2 + \sum_{n=1}^{n_1} x^{8n-4} q_n(x) + \sum_{n=n_1+1}^{\infty} x^{8n-4} q_n(x).$$

It follows that

$$g(x) \geq a_0 + a_1 x^2 + \sum_{n=1}^{n_1} x^{8n-4} q_n(x) = p(x).$$

Since the leading term of $p(x)$ is $a_{4n_1+1}x^{8n_1+2}$, we have that

$$\lim_{x\to\pm\infty} p(x) = +\infty.$$

thus g is upper unbounded on the real axis. □

4. A THIRD RESULT

Theorem 3. *Let $f(z)$ be an entire function as in (1) and let be $(c_n)_n$, $(d_n)_n$ two sequences of positive real numbers such that the function*

$$g(z) = \sum_{n=0}^{+\infty} b_n z^n$$

is entire, with

$$\begin{cases} b_0 = a_0 \\ b_1 = a_1 \end{cases}$$

and

$$b_n = \begin{cases} -c_n & if \ n \equiv 3 \pmod 5 \\ -d_n & if \ n \equiv 0 \pmod 5 \\ a_n & elsewhere \end{cases}$$

If

$$(6) \qquad \begin{cases} \lim_{n\to+\infty} \sup \frac{c_{n+3}^2}{a_{n+2}a_{n+4}} < L_2 < 4 \\ \\ \lim_{n\to+\infty} \sup \frac{a_{n+2}d_{n+5}^2 + c_{n+3}^2 a_{n+6}}{a_{n+2}a_{n+4}a_{n+6}} < L_3 < 4 \end{cases}$$

then g is upper unbounded on the real axis.

Proof. We write

$$g(x) = a_0 + a_1 x^2 + \sum_{n=0}^{\infty} x^{10n} p_n(x)$$

where

$$p_n(x) = a_{5n+2}x^4 - c_{5n+3}x^6 + a_{5n+4}x^8 - d_{5n+5}x^{10} + a_{5n+6}x^{12} \quad \forall n \geq 1.$$

We consider now the ternary quadratic forms

$$\phi_n(y_1, y_2, y_3) = a_{n+2}y_1^2 - c_{n+3}y_1y_2 + a_{n+4}y_2^2 - d_{n+5}y_2y_3 + a_{n+6}y_3^2 \quad \forall n \geq 1.$$

and we observe that

$$\phi_n(x^2, x^4, x^6) = p_n(x) \ \ \forall n \in N, \forall x \in \mathbb{R}$$

For each $n \geq 1$ let

$$\mathbf{A}_n = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{12} & \alpha_{22} & \alpha_{23} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{pmatrix} = \begin{pmatrix} a_{n+2} & -\frac{c_{n+3}}{2} & 0 \\ -\frac{c_{n+3}}{2} & a_{n+4} & -\frac{d_{n+5}}{2} \\ 0 & -\frac{d_{n+5}}{2} & a_{n+6} \end{pmatrix}$$

be the matrices associated with the quadratic forms ϕ_n. It is well known that if, for each $n \geq 1$ it is

$$\begin{cases} \Delta_1(n) = \alpha_{11} > 0 \\ \\ \Delta_2(n) = \det \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{12} & \alpha_{22} \end{pmatrix} > 0 \\ \\ \Delta_3(n) = \det \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{12} & \alpha_{22} & \alpha_{23} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{pmatrix} > 0 \end{cases}$$

then the quadratic forms ϕ_n are strictly positive definite. We have that $\alpha_{11} = a_{n+2} > 0$ by hypothesis, and

$$\begin{cases} \Delta_2(n) > 0 \Leftrightarrow \frac{c_{n+3}^2}{a_{n+2}a_{n+4}} < 4 \\ \\ \Delta_3(n) > 0 \Leftrightarrow \frac{a_{n+2}d_{n+5}^2 + c_{n+3}^2 a_{n+6}}{a_{n+2}a_{n+4}a_{n+6}} < 4 \end{cases} \tag{7}$$

Since condition (6) holds, there exist a natural number n_1 such that for each $n > n_1$ condition (7) holds also. Arguing as in the proofs of the previous theorems, we have that g is upper unbounded on the real axis. □

In particular we have that

Corollary 5. *With the same hypothesis of Theorem 6,if*

$$\begin{cases} \lim_{n \to +\infty} \sup \frac{a_{n+3}^2}{a_{n+2}a_{n+4}} < L_2 < 4 \\ \\ \lim_{n \to +\infty} \sup \frac{a_{n+2}a_{n+5}^2 + a_{n+3}^2 a_{n+6}}{a_{n+2}a_{n+4}a_{n+6}} < L_3 < 4 \end{cases}$$

then g is upper unbounded on the real axis.

Example 1. *The entire function*

$$g(z) = 1 + \frac{z^2}{1!} + \frac{1}{2!}z^4 - \frac{1}{3!}z^6 + \frac{1}{4!}z^8 - \frac{1}{5!}z^6 + \frac{1}{6!}z^{12} + \frac{1}{7!}z^{14} - \frac{1}{8!}z^{16} + \cdots$$

is upper unbounded on the real axis because

$$\lim_{n \to +\infty} \frac{a_{n+3}^2}{a_{n+2}a_{n+4}} = 1.$$

and

$$\lim_{n \to +\infty} \frac{a_{n+2}a_{n+5}^2 + a_{n+3}^2 a_{n+6}}{a_{n+2}a_{n+4}a_{n+6}} = 1.$$

5. Conclusions

We can generalize somewhat the procedure in order to obtain further sufficient conditions, at least in principle. Indeed, if we take any fixed $k \in \mathbb{N}$ with $k \geq 3$ and if we take $\lambda_k = 2k - 1$ then we can consider the polynomials

$$p_{n,k}(x) = a_{\lambda_k n+2}x^2 + a_{\lambda_k n+3}x^4 + \cdots a_{\lambda_k n+\lambda_k+1}x^{4k} \quad n \geq 1.$$

and we can write

$$f(x) = a_0 + a_1 x^2 + \sum_{n=0}^{\infty} x^{4k-2} p_{n,k}(x).$$

where

$$p_{n,k}(x) = \sum_{h=2}^{2k} a_{\lambda_k n+h}x^{2h}.$$

Now, if we consider k positive real sequences $(c_{n,j})_n$, $j = 1..k$ and we assume that the function

$$g(x) = a_0 + a_1 x^2 + \sum_{n=0}^{\infty} x^{4k-2} q_{n,k}(x)$$

where

$$q_{n,k}(x) = a_{\lambda_k n+2}x^2 - c^{(1)}_{(\lambda_k n+3)}x^4 + a_{\lambda_k n+4}x^6 - c^{(2)}_{(\lambda_k n+3)}x^4 \cdots a_{\lambda_k n+\lambda_k+1}x^{4k}.$$

is still entire, then we can consider the quadratic forms in k variables

$$\phi_{n,k}(y_1, \cdots y_k) = a_{\lambda_k n+2}y_1^2 - c^{(1)}_{(\lambda_k n+3)}y_2^2 + a_{\lambda_k n+4}y_3^2 - c^{(2)}_{(\lambda_k n+3)}x^4 \cdots a_{\lambda_k n+\lambda_k+1}y_k^2$$

and we can require that this forms are strictly positive definite. Doing so, at least in principle, it is possible to obtain a set of conditions like those of (7).

References

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