Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2019) 43: 955 – 964
© TÜBİTAK
doi:10.3906/mat-1810-100

The numerical range of matrices over $\mathbb{F}_4$

Edoardo BALLICO*
Deptartment of Mathematics, University of Trento Povo, Italy

Received: 23.10.2018 • **Accepted/Published Online:** 15.02.2019 • **Final Version:** 27.03.2019

Abstract: For any prime power q and any $u = (x_1, \dots, x_n), v = (y_1, \dots, y_n) \in \mathbb{F}_{q^2}^n$ set $\langle u, v\rangle := \sum_{i=1}^n x_i^q y_i$. For any $k \in \mathbb{F}_q$ and any $n \times n$ matrix M over $\mathbb{F}_{q^2}$, the k-numerical range $\mathrm{Num}_k(M)$ of M is the set of all $\langle u, Mu\rangle$ for $u \in \mathbb{F}_{q^2}^n$ with $\langle u, u\rangle = k$ [5]. Here, we study the case $q = 2$, which is quite different from the case $q \neq 2$.

Key words: numerical range, finite field, binary field

1. Introduction and main results

Let q be a prime power. Let $\mathbb{F}_q$ denote the only field, up to field isomorphisms, with $|\mathbb{F}_q| = q$ ([8, Theorem 2.5]). Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{F}_{q^2}^n$. For all $v, w \in \mathbb{F}_{q^2}^n$, say $v = a_1e_1 + \cdots + a_ne_n$ and $w = b_1e_1 + \cdots + b_ne_n$, set $\langle v, w\rangle = \sum_{i=1}^n a_i^q b_i$. $\langle\ ,\ \rangle$ is the standard Hermitian form of $\mathbb{F}_{q^2}^n$. For any $n \geq 1$ and any $a \in \mathbb{F}_q$ set

$$C_n(a) := \{(x_1, \dots, x_n) \in \mathbb{F}_{q^2}^n \mid x_1^{q+1} + \cdots + x_n^{q+1} = a\}.$$

The set $C_n(1)$ is an affine chart of the Hermitian variety of $\mathbb{P}^n(\mathbb{F}_{q^2})$ ([6, Chapter 5], [7, Chapter 23]). Take $M \in M_{n,n}(\mathbb{F}_{q^2})$, i.e. let M be an $n \times n$ matrix with coefficients in $\mathbb{F}_{q^2}$. For any $k \in \mathbb{F}_q$ set $\mathrm{Num}_k(M) := \{\langle u, Mu\rangle \mid u \in C_n(k)\} \subseteq \mathbb{F}_{q^2}$. Set $\mathrm{Num}(M) := \mathrm{Num}_1(M)$. The set $\mathrm{Num}(M)$ is called the *numerical range* of M. These concepts were introduced in [5] when q is a prime $p \equiv 3 \pmod 4$ and in [1] in the general case. We always have $0 \in \mathrm{Num}_0(M)$. When $n \geq 2$, we defined in [4] the set $\mathrm{Num}_0'(M)$ as the set of all $\langle u, Mu\rangle$ for some $u \in \mathbb{F}_{q^2} \setminus \{0\}$ such that $\langle u, u\rangle = 0$. We have $\mathrm{Num}_0(M) \setminus \{0\} \subseteq \mathrm{Num}_0'(M) \subseteq \mathrm{Num}_0(M)$. For any $M = (m_{ij}) \in M_{n,n}(\mathbb{F}_{q^2})$, set $(M^\dagger)_{ij} = m_{ji}^q$. For any $M \in M_{n,n}(\mathbb{F}_{q^2})$ and any $u \in \mathbb{F}_{q^2}^n$, set $\nu_M(u) := \langle u, Mu\rangle$.

If $q = 2$ then in the definition of $C_n(a)$ we just take $a \in \mathbb{F}_2 = \{0, 1\}$. In particular for $q = 2$ (and for all even q by Remark 2.6), we only have to compute $\mathrm{Num}_1(M)$ and $\mathrm{Num}_0(M)$. The cases "$q = 2$" and "$q \neq 2$" (independently of the parity of q) are quite different, because if $x \in \mathbb{F}_4 \setminus \{0\}$, then $x^3 = 1$ and therefore when $q = 2$, the set $C_n(a)$ is just the set of all $(x_1, \dots, x_n) \in \mathbb{F}_4^n$ such that the number of nonzero entries entries x_i is $\equiv a \pmod 2$ (Remark 2.8).

*Correspondence: ballico@science.unitn.it
2010 *AMS Mathematics Subject Classification:* 15A33; 11T99
The author was partially supported by MIUR and GNSAGA of INdAM (Italy)

Obviously all diagonal entries m_{ii} of a matrix M belong to $\mathrm{Num}(M)$. If $q=2$ and $n=2$ and $M=(m_{ij})$, then $\mathrm{Num}(M)=\{m_{11},m_{22}\}$ (Remark 2.10).

We summarize our main results in the following way (here we take $q=2$ and $M=(m_{ij})\in M_{n,n}(F_4)$).

Proposition 1.1 *Assume $n\ge 3$. We have $\mathrm{Num}_0(M)=\{0\}$ if and only if $M=c\mathbb{I}_{n\times n}$ for some $c\in\mathbb{F}_4$.*

Theorem 1.2 *Assume $n\ge 2$. We have $\mathrm{Num}_0(M)=\{0,b\}$ with $b\ne 0$ if and only if $(\frac{1}{b}(M-m_{11}\mathbb{I}_{n\times n}))^\dagger=\frac{1}{b}(M-m_{11}\mathbb{I}_{n\times n})$ and $M\ne m_{11}\mathbb{I}_{n\times n}$ and this is the case if and only if $M\ne d\mathbb{I}_{n\times n}$ for any d and there is $c\in\mathbb{F}_4$ such that $(\frac{1}{b}(M-c\mathbb{I}_{n\times n}))^\dagger=\frac{1}{b}(M-c\mathbb{I}_{n\times n})$.*

Theorem 1.3 *Take $N\in M_{n,n}(\mathbb{F}_4)$, $n\ge 3$. We have $\mathrm{Num}_1(N)=\{a,b\}$ with $a\ne b$ if and only if $(\frac{1}{b-a}(N-a\mathbb{I}_{n,n}))^\dagger=\frac{1}{b-a}(N-a\mathbb{I}_{n,n})$ and $N\ne c\mathbb{I}_{n,n}$ for some c.*

Corollary 1.4 *Assume $n\ge 3$. We have $\mathrm{Num}(M)\subseteq\mathbb{F}_2$ if and only if $M^\dagger=M$.*

Proposition 1.5 *We have $|\mathrm{Num}(M)|\le 1$ if and only if $\mathrm{Num}(M)=\{m_{11}\}$.*

(a) If $n=2$ we have $|\mathrm{Num}(M)|=1$ if and only if $m_{11}=m_{22}$.

(b) If $n\ge 3$, then $\mathrm{Num}(M)=\{m_{11}\}$ if and only if $M=m_{11}\mathbb{I}_{n\times n}$.

We also prove that $\mathrm{Num}(M)=\mathbb{F}_4$ if $n\ge 3$, $M\ne 0\mathbb{I}_{n\times n}$ and M is strictly triangular (Remark 3.5).

2. Preliminaries

For any matrix $M=(m_{ij})\in M_{n,n}(\mathbb{F}_{q^2})$, let $M^\dagger=(a_{ij})$ be the matrix with $a_{ij}=m_{ji}^q$ for all i,j. M is said to be *Hermitian* if $M^\dagger=M$. Note that the diagonal elements of a Hermitian matrix are contained in $\mathbb{F}_q$. Let $e_1=(1,0,\dots,0),\dots,e_n=(0,\dots,0,1)$ be the standard basis of $\mathbb{F}_{q^2}^n$. Let $\mathbb{I}_{n\times n}$ denote the identity $n\times n$ matrix. For any $a\in\mathbb{F}_q$ and any $n>0$ we have $C_n(a)\ne\emptyset$ by [1, Remark 3] and hence, $\mathrm{Num}_a(M)\ne\emptyset$ for any a, any n, and any matrix M.

Notation 2.1 *Write $M=(m_{ij})$, $i,j=1,\dots,n$.*

Remark 2.2 *Take $M=(m_{ij})\in M_{n,n}(\mathbb{F}_q)$. The vector e_i gives $m_{ii}\in\mathrm{Num}(M)$. Hence, $\mathrm{Num}(M)$ contains all diagonal elements of M.*

Remark 2.3 *For any $a,b\in\mathbb{F}_{q^2}^*$, any $k\in\mathbb{F}_q$, and any $M\in M_{n,n}(\mathbb{F}_{q^2})$, we have $\mathrm{Num}_k(aM)=a\mathrm{Num}_k(M)$ and $\mathrm{Num}_k(M+b\mathbb{I}_{n,n})=\mathrm{Num}_k(M)+kb^{q+1}$ ([5, Proposition 3.1], [1, Remark 7], [4, Remark 2.4]).*

Remark 2.4 *Take $M=(m_{ij})\in M_{n,n}(\mathbb{F}_q)$ such that $M^\dagger=M$. For any $u\in\mathbb{F}_q^n$, we have $\langle u,Mu\rangle=\langle M^\dagger u,u\rangle=\langle Mu,u\rangle$. Hence, $\langle u,Mu\rangle\in\mathbb{F}_q$. Thus, $\mathrm{Num}_k(M)\subseteq\mathbb{F}_q$ for every $k\in\mathbb{F}_q$.*

We recall from [5] the following definitions. For any $O,P\in\mathbb{F}_{q^2}$ the *strict affine $\mathbb{F}_q$-hull* $((O,P))$ of O and P is the set $\{tO+(1-t)P\}_{t\in\mathbb{F}_q\setminus\{0,1\}}$. If $O=P$, then $((O,P))=\{O\}$. If $O\ne P$, then $((O,P))$ is the complement of $\{O,P\}$ in the affine $\mathbb{F}_q$-line of $\mathbb{F}_{q^2}\cong\mathbb{F}_q^2$ spanned by O and P and hence it has cardinality

$q-2$. For any two nonempty subsets $S, S' \subseteq \mathbb{F}_{q^2}$ set $((S,S')) := \cup_{O\in S, P\in S'}((O,P))$. With this notation we have the following lemma ([1, Lemma 1]).

Lemma 2.5 *Let M be unitarily equivalent to the direct sums of matrices A and B. Then* $\mathrm{Num}(M) = ((\mathrm{Num}(A), \mathrm{Num}(B))) \cup \{\mathrm{Num}_0(A) + \mathrm{Num}(B)\} \cup \{\mathrm{Num}(A) + \mathrm{Num}_0(B)\}$.

Remark 2.6 *Each element of $\mathbb{F}_q^*$, q even, is a square. Hence to compute all $\mathrm{Num}_k(M)$ when q is even it is sufficient to compute $\mathrm{Num}_0(M)$ and $\mathrm{Num}_1(M)$.*

Unless otherwise stated, from now on $m_{ij} \in \mathbb{F}_4$ and $q = 2$.

Notation 2.7 *Fix $e \in \mathbb{F}_4 \setminus \mathbb{F}_2$. We have $e^3 = 1$, $e^2 + e = 1$ and $\mathbb{F}_4 = \{0, 1, e, e^2\}$. If $a \in \mathbb{F}_4^*$, then $a^3 = 1$. If a, b, c are 3 different elements of $\mathbb{F}_4$, then $\{a, b, c, a+b+c\} = \mathbb{F}_4$. Hence for any $a \in \mathbb{F}_4$ the set $a + \mathbb{F}_4^*$ of all $a+b$, $b \in \mathbb{F}_4^*$ is the set $\mathbb{F}_4 \setminus \{a\}$. We fix some $e \in \mathbb{F}_4 \setminus \mathbb{F}_2$ and write $\mathbb{F}_4 = \{0, 1, e, e^2\}$.*

Remark 2.8 *For each $x \in \mathbb{F}_4^*$ we have $x^3 = 1$. We obviously have $0^3 = 0$. Take $u = (x_1, \dots, x_n) \in \mathbb{F}_4^n$. We have $x_1^3 + \cdots + x_n^3 = 1$ (resp. $x_1^3 + \cdots + x_n^3 = 0$) if and only if $x_i \neq 0$ for an odd (resp. an even) number of indices i.*

Remark 2.9 *Take $u \in \mathbb{F}_4^n$ and $t \in \mathbb{F}_4^*$. We have $\langle tu, M(tu)\rangle = t^{q+1}\langle u, Mu\rangle = \langle u, Mu\rangle$, because $t^3 = 1$.*

Remark 2.10 *Assume $n = 2$. By Remark 2.8, we have $\mathrm{Num}(M) = \{m_{11}, m_{22}\}$.*

3. Strictly triangular matrices

We first list some cases with $n = 3$ in which we prove that $\mathrm{Num}(M) = \mathbb{F}_4$. All these matrices are triangular matrices with equal entries, m_{11}, on the diagonal. By [5, Lemma 2.7] to compute $\mathrm{Num}(M)$, it is sufficient to compute $\mathrm{Num}(N)$, where N is the strictly triangular matrix $M - m_{11}\mathbb{I}_{3\times 3}$.

Proposition 3.1 *Fix $a, b, c \in \mathbb{F}_4$ with $ab \neq 0$. Take*

$$M = \begin{pmatrix} c & b & 0 \\ 0 & c & a \\ 0 & 0 & c \end{pmatrix}$$

Then $\mathrm{Num}(M) = \mathbb{F}_4$.

Proof Taking $\frac{1}{a}(M - c\mathbb{I}_{3\times 3})$ instead of M and applying [5, Lemma 2.7], we reduce to the case $c = 0$ and $a = 1$. Note that even after this reduction step, we have $b \in \mathbb{F}_4 \setminus \{0\}$. Take $u = (x_1, x_2, x_3) \in C_3(1)$, i.e. assume $x_1^3 + x_2^3 + x_3^3 = 1$. We have $\langle u, Mu\rangle = x_2(bx_1^2 + x_2x_3)$. Taking $x_1 = x_3 = 1$ and $x_2 = 0$, we get $0 \in \mathrm{Num}(M)$. From now on we always take $x_2 = 1$ and in particular $x_2^3 = 1$. Thus, we may use any x_1, x_2 with $x_1^3 + x_3^3 = 0$, i.e. any $(x_1, x_3) \in \mathbb{F}_4^2$ with either $x_1 = x_3 = 0$ or $x_1x_3 \neq 0$. Fix $c \in \mathbb{F}_4 \setminus \{b\}$. If $u = (1, 1, c-b)$, then $\langle u, Mu\rangle = 1(b + c - b) = c$. Note that $1 + e \neq 0$ and hence, $b(1+e) \neq 0$. We take $u = (e, 1, b(1+e)) = be + b + be = b$. □

Proposition 3.2 *Fix* $a, b, c, d \in \mathbb{F}_4$ *with* $abd \neq 0$. *Take*

$$M = \begin{pmatrix} c & b & d \\ 0 & c & a \\ 0 & 0 & c \end{pmatrix}$$

Then $\mathrm{Num}(M) = \mathbb{F}_4$.

Proof As in the proof of Proposition 3.1, we reduce to the case $c = 0$ and $d = 1$. Take $u = (x_1, x_2, x_3) \in C_3(1)$, i.e. assume $x_1^3 + x_2^3 + x_3^3 = 1$. We have $Mu = (bx_2 + x_3, ax_3, 0)$ and hence, $\langle u, Mu \rangle = bx_2x_1^2 + x_3x_1^2 + ax_3x_2^2$. Taking $u = (1, 0, 0)$, we get $0 \in \mathrm{Num}(M)$. From now on we always take $x_2 = 1$ and in particular $x_2^3 = 1$. Thus, we may use any x_1, x_2 with $x_1^3 + x_3^3 = 0$, i.e. any $(x_1, x_3) \in \mathbb{F}_4^2$ with either $x_1 = x_3 = 0$ or $x_1x_3 \neq 0$. Fix $c \in \mathbb{F}_4 \setminus \{0\}$. It is sufficient to find $x_1, x_3 \in \mathbb{F}_4 \setminus \{0\}$ with $x_1^2(b + x_3) = c + ax_3$. Since $a \neq 0$ and $|\mathbb{F}_4 \setminus \{0\}| = 3$, there is $w \in \mathbb{F}_4 \setminus \{0\}$ such that $b + w \neq 0$ and $c + aw \neq 0$. Since the Frobenius map $t \mapsto t^2$ induces a permutation $\mathbb{F}_4 \setminus \{0\} \to \mathbb{F}_4 \setminus \{0\}$, there is $z \in \mathbb{F}_4 \setminus \{0\}$ such that $z^2 = (c + aw)/(b + w)$. Take $u = (z, 1, w)$. □

Proposition 3.3 *Fix* $a, b, c \in \mathbb{F}_4$ *with* $(a, b) \neq (0, 0)$. *Take*

$$M = \begin{pmatrix} c & a & b \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix}$$

Then $\mathrm{Num}(M) = \mathbb{F}_4$.

Proof It is sufficient to do the case $c = 0$. Take $u = (x_1, x_2, x_3) \in C_3(1)$, i.e. assume that 2 or none among x_1, x_2, x_3 are zeroes. We have $Mu = (ax_2 + bx_3, 0, 0)$ and hence, $\langle u, Mu \rangle = x_1^2(ax_2 + bx_3)$. Taking $u = (1, 0, 0)$, we get $0 \in \mathrm{Num}(M)$. Fix $w \in \mathbb{F}_4 \setminus \{0\}$. Since $(a, b) \neq (0, 0)$, there is $(a_2, a_3) \in (\mathbb{F}_4 \setminus \{0\})^2$ such that $aa_2 + ba_3 \neq 0$. Since the Frobenius map $t \mapsto t^2$ induces a permutation $\mathbb{F}_4 \setminus \{0\} \to \mathbb{F}_4 \setminus \{0\}$, there is $z \in \mathbb{F}_4 \setminus \{0\}$ such that $z^2 = w/(aa_2 + ba_3)$. Take $u = (z, a_2, a_3)$. □

Proposition 3.4 *Take* $M = (m_{ij}) \in M_{n,n}(\mathbb{F}_4)$, $n \geq 3$, *such that* $m_{ij} = 0$ *for all* $i \geq j$. *We have* $\mathrm{Num}(M) = \mathbb{F}_4$ *if and only if* $M \neq 0$.

Proof Since $\mathrm{Num}(0\mathbb{I}_{n\times n}) = \{0\}$, we only need to prove the " if " part. Assume $m_{ij} \neq 0$ for some $i < j$. Take any principal minor of M associated to i, j and some $h \in \{1, \ldots, n\} \setminus \{i, j\}$ and apply one of the Propositions above. □

Remark 3.5 *Take* M *as in one of the Propositions 3.1, 3.2, 3.3, 3.4. A similar proof works for* M^t. *Thus, we computed* $\mathrm{Num}(M)$ *for all strictly triangular matrices and proved that* $\mathrm{Num}(M) = \mathbb{F}_4$, *unless either* $n = 2$ *or* $M = m_{11}\mathbb{I}_{3\times 3}$.

4. The proofs

Lemma 4.1 *Take* $n = 2$. *We have* $\mathrm{Num}_0(M) = \{0, m_{11} + m_{22} + m_{12} + m_{21}, m_{11} + m_{22} + m_{12}e + m_{21}e^2, m_{11} + m_{22} + m_{12}e^2 + m_{21}e\}$.

1. $\mathrm{Num}_0(M) = \{0\}$ *if and only if* $m_{11} = m_{22}$ *and* $m_{12} = m_{21} = 0$.

2. *If* $m_{11} = m_{22}$ *and* $m_{12} = m_{21} \neq 0$, *then* $\mathrm{Num}_0(M) = \{0, m_{12}\}$.

3. *Assume* $m_{11} = m_{22}$. *If either* $m_{12} = 0$ *and* $m_{21} \neq 0$ *or* $m_{21} = 0$ *and* $m_{12} \neq 0$, *then* $\mathrm{Num}_0(M) = \mathbb{F}_4$.

4. *If* $m_{11} = m_{22}$, $m_{12} \neq m_{21}$ *and* $m_{12}m_{21} \neq 0$, *then* $\mathrm{Num}_0(M) = \{0, m_{12}, m_{21}\}$.

5. *If* $m_{11} \neq m_{22}$ *and* $m_{21} = m_{12} = 0$, *then* $\mathrm{Num}_0(M) = \{0, m_{11} + m_{22}\}$.

6. *If* $m_{11} \neq m_{22}$ *and* $m_{21} = m_{12} \neq 0$, *then* $\mathrm{Num}_0(M) = \{0, m_{11} + m_{22}, m_{11} + m_{22} + m_{12}\}$*; this set has two elements if and only if* $m_{22} = m_{11} + m_{21}$.

7. *Assume* $m_{11} \neq m_{22}$ *and either* $m_{12} = 0$ *and* $m_{21} \neq 0$ *or* $m_{12} \neq 0$ *and* $m_{21} = 0$, *then* $|\mathrm{Num}_0(M)| = 3$ *and* $\mathrm{Num}_0(M) = \mathbb{F}_4 \setminus \{m_{11} + m_{22}\}$.

8. *If* $m_{11} \neq m_{22}$, $m_{21}m_{12} \neq 0$ *and* $m_{12} \neq m_{21}$, *then* $\mathrm{Num}_0(M) = \{0, m_{11} + m_{22} + em_{12} + e^2m_{21}, m_{11} + m_{22} + e^2m_{12} + em_{21}\}$*; we have* $|\mathrm{Num}_0(M)| = 2$ *if and only if either* $m_{11} + m_{22} + em_{12} + e^2m_{21} = 0$ *or* $m_{11} + m_{22} + e^2m_{12} + em_{21} = 0$.

Proof Take $u = (x_1, x_2) \in \mathbb{F}_4^2$ with $x_1^3 + x_2^3 = 0$. The case $u = (0,0)$ gives $0 \in \mathrm{Num}_0(M)$. By Remark 2.8 to compute the other elements of $\mathrm{Num}_0(M)$, we may assume $x_1 \neq 0$ and $x_2 \neq 0$. Using Remark 2.9 with $t = x_1^{-1}$ we reduce to the case $x_1 = 1$. Taking $x_2 = 1$ (resp. $x_2 = e$, resp. $x_2 = e^2$), we get $m_{11} + m_{12} + m_{21} + m_{22} \in \mathrm{Num}_0(M)$ (resp. $m_{11} + m_{12}e + m_{21}e^2 + m_{22} \in \mathrm{Num}_0(M)$, $m_{11} + m_{22} + m_{12}e^2 + m_{21}e \in \mathrm{Num}_0(M)$. We have $m_{12}e + m_{21}e^2 = m_{12}e^2 + m_{21}e$ if and only if $m_{12}(e + e^2) = m_{21}(e + e^2)$, i.e. if and only $m_{12} = m_{21}$. We have $m_{12} + m_{21} = m_{12}e + m_{21}e^2$ if and only if $m_{12}(1 + e) = m_{21}(1 + e^2)$, i.e. if and only if $m_{12}e = m_{21}$. We have $m_{12} + m_{21} = m_{12}e^2 + m_{21}e$ if and only if $(m_{12}(1 + e^2) = m_{21}(1 + e)$, i.e. if and only if $m_{12} = em_{21}$, i.e. $m_{12}e^2 = m_{21}$.

(a) Assume $m_{11} = m_{22}$. If $m_{12} = m_{21} = 0$, then $\mathrm{Num}_0(M) = \{0\}$. Now assume $m_{12} = m_{21} \neq 0$. Since $e^2 + e = 1$, we get that $\mathrm{Num}_0(M) = \{0, m_{12}e, m_{12}\}$. If $m_{12} = 0 \neq m_{21}$ (resp. $m_{21} = 0$ and $m_{12} \neq 0$), then $\mathrm{Num}_0(M)$ contains 0, m_{21}, em_{21}, e^2m_{21} (resp. 0, m_{12}, em_{12}, e^2m_{12}); in both cases we get $\mathrm{Num}_0(M) = \mathbb{F}_4$. Now assume $m_{12} \neq m_{21}$ and $m_{12}m_{21} \neq 0$. Set $t := m_{12}/m_{21}$. Either $t = e$ or $t = e^2$. Assume $t = e$ (the case $t = e^2$ being similar). $\mathrm{Num}_0(M)$ is the union of 0, $m_{21}(1 + e) = m_{21}e^2 = m_{12}$, $m_{21}(e + e^2) = m_{21}$ and $m_{21}(e^2 + e^2) = 0$. Hence, $\mathrm{Num}_0(M) = \{0, m_{12}, m_{21}\}$.

(b) Assume $m_{11} \neq m_{22}$. If $m_{21} = m_{12} = 0$, then $\mathrm{Num}_0(M) = \{0, m_{11} + m_{22}\}$. If $m_{21} = m_{12} \neq 0$, then $\mathrm{Num}_0(M)$ is the union of $\{0\}$, $m_{11} + m_{22}$ and $m_{11} + m_{22} + m_{12}$ (recall that $2m_{12} = 0$ and $e + e^2 = 1$). Assume $m_{12} = 0$ and $m_{21} \neq 0$. We get $\mathrm{Num}_0(M) = \{0, m_{11} + m_{22} + m_{21}, m_{11} + m_{22} + m_{21}e, m_{11} + m_{22} + m_{21}e^2\}$. Since $\{m_{21}, m_{21}e, m_{21}e^2\} = \mathbb{F}_4^*$, we get $|\mathrm{Num}_0(M)| = 3$ and $\mathrm{Num}_0(M) = \mathbb{F}_4 \setminus \{m_{11} + m_{22}\}$. The same answer comes if $m_{12} \neq 0$ and $m_{21} = 0$. Now assume $m_{12} \neq 0$, $m_{21} \neq 0$, and $m_{12} \neq m_{21}$. We first check that $A := \{m_{12} + m_{21}, em_{12} + e^2m_{21}, e^2m_{12} + em_{21}\}$ has cardinality 2. Since $e \neq e^2$ and $m_{12} \neq m_{21}$, we have $em_{12} + e^2m_{21} \neq e^2m_{12} + em_{21}$. We have $m_{12} + m_{21} = em_{12} + e^2m_{21}$ if and only if $(1 + e)m_{12} = (1 + e^2)m_{21}$, i.e. if and only if $e^2m_{12} = em_{21}$, i.e. if and only if $em_{12} = m_{21}$. In the same way, we see that $m_{12} + m_{21} = e^2m_{12} + em_{21}$ if and only if $e^2m_{12} = m_{21}$. Since $m_{21}/m_{12} \notin \{0, 1\}$, we have $m_{21}/m_{12} \in \{e, e^2\}$. Hence $A = \{em_{12} + e^2m_{21}, e^2m_{12} + em_{21}\}$. We get part (8). □

Lemma 4.2 *Take $q=2$, $n=3$ and $m_{11}=m_{22}=m_{33}$.*

1. *If $m_{ij}=0$ for all $i\ne j$, then $\mathrm{Num}_0(M)=\{0\}$.*

2. *If $m_{ij}=0$ and $m_{ji}\ne 0$ for some i,j, then $\mathrm{Num}_0(M)=\mathbb{F}_4$.*

3. *If $m_{ij}\ne m_{ji}$, then $\mathrm{Num}_0(M)\supseteq\{0,m_{ij},m_{ji}\}$.*

4. *If $m_{ij}=m_{ji}\ne 0$, then $\mathrm{Num}_0(M)\supseteq\{0,m_{ij}\}$.*

All nonzero m_{ij} are in $\mathrm{Num}_0(M)\setminus\{0\}$ and they are the only elements of $\mathrm{Num}_0(M)\setminus\{0\}$ unless there is i,j with $m_{ij}=0$ and $m_{ji}\ne 0$.

Proof Take $u=(x_1,x_2,x_3)\in\mathbb{F}_4$ with $x_1^3+x_2^3+x_3^3=0$. By Remark 2.8 either $u=0$ or there is $i,j\in\{1,2,3\}$ with $i\ne j$ and $x_h\ne 0$ if and only if $h\in\{i,j\}$. Apply Lemma 4.1 to the restriction of M to $\mathbb{F}_4e_i+\mathbb{F}_4e_j$. □

Remark 2.8, Lemma 4.1, and the proof of Lemma 4.2 give the following result.

Lemma 4.3 *Take $n=3$ and $m_{11}=m_{22}\ne m_{33}$. Let $A=(a_{ij})$ and $B=(b_{ij})$ be the 2×2 matrices with $a_{ij}=m_{hk}$, $h=i+1$, $k=i+1$, and $b_{ij}=m_{vw}$, $v=i$ if $i=1$, $v=3$ if $i=2$, $w=j$ if $j=1$, $w=3$ if $j=2$. A and B are as in one of the last 4 cases of Lemma 4.1.*

1. *If $m_{12}m_{21}=0$ and $m_{12}\ne m_{21}$, then $\mathrm{Num}_0(M)=\mathbb{F}_4$*

2. *If $m_{12}=m_{21}=0$, then $\mathrm{Num}_0(M)=\mathrm{Num}_0(A)\cup\mathrm{Num}_0(B)$.*

3. *If $m_{12}m_{21}\ne 0$ and $m_{12}\ne m_{21}$, then $\mathrm{Num}_0(M)=\mathrm{Num}_0(A)\cup\mathrm{Num}_0(B)\cup\{m_{12},m_{21}\}$; we have $|\mathrm{Num}_0(M)|\ge 3$; we have $\mathrm{Num}_0(M)\ne\mathbb{F}_4$ if and only if $\mathrm{Num}_0(A)\cup\mathrm{Num}_0(B)\supseteq\{0,m_{12},m_{21}\}$.*

Take the set-up of (3), i.e. assume $m_{12}m_{21}\ne 0$ and $m_{12}\ne m_{21}$. $\mathrm{Num}_0(A)\supseteq\{0,m_{12},m_{21}\}$ if and only if either $m_{13}=m_{31}=0$ and $m_{11}+m_{33}\in\{m_{12},m_{21}$ or $m_{31}=m_{13}\ne 0$ and $m_{11}+m_{33},m_{11}+m_{33}+m_{13}\}\subseteq\{0,m_{12},m_{21}$ or $m_{13}m_{31}=0$, $m_{13}\ne m_{31}$ and $\mathbb{F}_4\setminus\{m_{11}+m_{33}\}$ or $m_{31}m_{13}\ne 0$, $m_{13}\ne m_{31}$ and $\{m_{11}+m_{33}+em_{13}+e^2m_{31},m_{11}+m_{33}+e^2m_{13}+em_{31}\}\subseteq\{0,m_{12},m_{21}\}$. The same list works for B exchanging the indices 1 and 2.

Lemma 4.4 *Take $n=3$ and $|\{m_{11},m_{22},m_{33}\}|=3$. Let $A_h=(a^h_{ij})$, $i,j=1,2$, $h=1,2,3$, be the matrix obtained from M deleting the h-th row and the h-th column; each A_h is as in one of the last 4 cases of Lemma 4.1. We have $\mathrm{Num}_0(M)=\mathrm{Num}_0(A_1)\cup\mathrm{Num}_0(A_2)\cup\mathrm{Num}_0(A_3)$. Hence, $\mathrm{Num}_0(M)=\mathbb{F}_4$ if one of the following conditions is satisfied:*

1. *$m_{ij}=0$ for all $i\ne j$.*

2. *There is $i\in\{1,2,3\}$ such that, writing $\{1,2,3\}=\{i,j,h\}$, we have $m_{jh}m_{jh}=0$ and $m_{jh}\ne m_{hj}$.*

3. *There is $i\in\{1,2,3\}$ such that, writing $\{1,2,3\}=\{i,j,h\}$, we have $m_{ij}\ne 0$, $m_{ji}\ne 0$, $m_{hi}\ne 0$, $m_{ih}\ne 0$, and $\{m_{ij},m_{ji},m_{ih},m_{hi}\}=\mathbb{F}_4^*$.*

Proof Remark 2.8, Lemma 4.1, and the proof of Lemma 4.2 give $\mathrm{Num}_0(M) = \mathrm{Num}_0(A_1) \cup \mathrm{Num}_0(A_2) \cup \mathrm{Num}_0(A_3)$.

(a) Assume $m_{ij} = 0$ for all $i \neq j$. Since $m_{11}+m_{22}$, $m_{11}+m_{33}$, and $m_{22}+m_{33}$ are distinct elements of $\mathbb{F}_4^*$, we have $\{0, m_{11}+m_{22}, m_{11}+m_{33}, m_{22}+m_{33}\} = \mathbb{F}_4$.

(b) Assume $m_{12}m_{21} = m_{13}m_{31} = 0$ and $m_{12}+m_{21} \neq 0$, $m_{13}+m_{31} \neq 0$. We have $\mathrm{Num}_0(A_3) = \mathbb{F}_4 \setminus \{m_{11}+m_{22}\}$ and $\mathrm{Num}_0(A_2) = \mathbb{F}_4 \setminus \{m_{11}+m_{33}\}$ and thus, $\mathrm{Num}_0(M) = \mathbb{F}_4$. The same proof works if either $m_{23}m_{32} = m_{13}m_{31} = 0$ and $m_{23}+m_{32} \neq 0$, $m_{13}+m_{31} \neq 0$ or $m_{12}m_{21} = m_{23}m_{32} = 0$ and $m_{12}+m_{21} \neq 0$, $m_{23}+m_{32} \neq 0$.

(c) Assume $m_{12} \neq 0$, $m_{21} \neq 0$, $m_{13} \neq 0$, and $m_{31} \neq 0$ and $\{m_{12}, m_{21}, m_{31}, m_{13}\} = \mathbb{F}_4^*$. The last 2 cases in Lemma 4.1 give $\mathrm{Num}_0(A_3) \cup \mathrm{Num}_0(A_2) = \mathbb{F}_4$. The same proof works for $\mathrm{Num}_0(A_1) \cup \mathrm{Num}_0(A_2)$ and $\mathrm{Num}_0(A_3) \cup \mathrm{Num}_0(A_1)$. □

Lemma 4.5 *Assume* $n = 3$. $\mathrm{Num}(M)$ *is the union of* $\{m_{11}, m_{22}, m_{33}\}$, $\sum_{i,j=1}^{3} m_{ij}$, $m_{11}+m_{22}+m_{33}+e(m_{12}+m_{23}+m_{31})+e^2(m_{13}+m_{21}+m_{32})$, $m_{11}+m_{22}+m_{33}+e^2(m_{12}+m_{23}+m_{31})+e(m_{13}+m_{21}+m_{32})$ *and* B_h, $h = 1,2,3$, *where (writing* $\{i,j,h\} = \{1,2,3\}$*)* $B_h = \{m_{ii}+m_{ij}+m_{ji}+m_{jj}+m_{hh}+e(m_{ih}+m_{jh})+e^2(m_{hi}+m_{hj}), m_{ii}+m_{ij}+m_{ji}+m_{jj}+m_{hh}+e^2(m_{ih}+m_{jh})+e(m_{hi}+m_{hj})\}$.

Proof Take $u = (x_1, x_2, x_3) \in \mathbb{F}_4^3$ with $x_1^3 + x_2^3 + x_3^3 = 1$. By Remark 2.8, there is an odd number of indices i with $x_i \neq 0$. Taking u with exactly one nonzero coordinate, we get $\{m_{11}, m_{22}, m_{33}\} \subseteq \mathrm{Num}(M)$. Thus, it is sufficient to test all u with $x_i \neq 0$ for every i. Taking $u = (t,t,t)$ for some $t \neq 0$, we get $\sum_{i,j=1}^{3} m_{ij} \in \mathrm{Num}(M)$. Now assume that u has exactly two different entries, say $x_i = x_j$ with $i \neq j$ and $x_h \neq x_i$, $\{i,j,h\} = \{1,2,3\}$. By Remark 2.9, we may assume that $x_i = 1$ and hence, either $x_h = e$ or $x_h = e^2$. In the first case, $\langle u, Mu \rangle = m_{ii}+m_{ij}+m_{ji}+m_{jj}+m_{hh}+e(m_{ih}+m_{jh})+e^2(m_{hi}+m_{hj})$. In the second case, we have $\langle u, Mu \rangle = m_{ii}+m_{ij}+m_{ji}+m_{jj}+m_{hh}+e^2(m_{ih}+m_{jh})+e(m_{hi}+m_{hj})$. Now assume that all entries of u are different. By Remark 2.9, we may assume that $x_1 = 1$. Hence, either $(x_2, x_3) = (e, e^2)$ or $(x_2, x_3) = (e^2, e)$. In the first case, we have $\langle u, Mu \rangle = m_{11}+m_{22}+m_{33}+e(m_{12}+m_{23}+m_{31})+e^2(m_{13}+m_{21}+m_{32})$. In the second case, we have $\langle u, Mu \rangle = m_{11}+m_{22}+m_{33}+e^2(m_{12}+m_{23}+m_{31})+e(m_{13}+m_{21}+m_{32})$. □

Corollary 4.6 *Assume* $n = 3$ *and* $|\{m_{11}, m_{22}, m_{33}\}| = 3$. *We have* $\mathrm{Num}(M) = \mathbb{F}_4$ *if one of the following conditions is satisfied*

1. $m_{12}+m_{23}+m_{31} = t(m_{13}+m_{21}+m_{32})$ *with* $t \in \{e, e^2\}$;
2. $m_{ij} = m_{12}$ *for all* $i \neq j$.

Proof Since $|\{m_{11}, m_{22}, m_{33}\}| = 3$, we have $\mathbb{F}_4 = \{m_{11}, m_{22}, m_{33}, m_{11}+m_{22}+m_{33}\}$ (Remark 2.7). Hence, it is sufficient to check if $m_{11}+m_{22}+m_{33} \in \mathrm{Num}(M)$. To get (1), use the third and fourth sum in Lemma 4.5 and that $e^3 + e^3 = 0$. To get (2), use B_1 in Lemma 4.5. □

Corollary 4.7 *Assume* $n = 3$ *and* $|\{m_{11}, m_{22}, m_{33}\}| = 3$. *Fix* 5 *of the elements* m_{ij} *with* $i \neq j$, *say all except* m_{hk}. *There is a choice of* m_{hk} *with* $\mathrm{Num}(M) = \mathbb{F}_4$.

Proof Since $|\{m_{11}, m_{22}, m_{33}\}| = 3$, we have $\mathbb{F}_4 = \{m_{11}, m_{22}, m_{33}, m_{11}+m_{22}+m_{33}\}$ (Remark 2.7). Hence, it is sufficient to check that $m_{11}+m_{22}+m_{33} \in \mathrm{Num}(M)$ for some choice of m_{hk}. We do the case $h=1$, $k=2$, because the other cases are similar. We take $m_{12} := e(m_{13}+m_{21}+m_{32}) + m_{23}+m_{31}$ and apply part (1) of Corollary 4.6. The proof shows that if $m_{13}+m_{21}+m_{32} \neq 0$, we have at least 2 choices for m_{12}. □

Lemma 4.8 *Take $n=3$. We have $|\mathrm{Num}(M)| = 1$ if and only if $M = m_{11}\mathbb{I}_{3\times 3}$.*

Proof Since the "if " part is trivial, we assume $|\mathrm{Num}(M)| = 1$. Since we have $\{m_{11}, m_{22}, m_{33}\} \subseteq \mathrm{Num}(M)$ (Remark 2.2), we have $m_{11} = m_{22} = m_{33}$. Since $e^3 = 1$, Lemma 4.5 gives $m_{12}+m_{23}+m_{31} = e(m_{13}+m_{21}+m_{32})$ and $m_{12}+m_{23}+m_{31} = e^2(m_{13}+m_{21}+m_{32})$. Hence, $m_{12}+m_{23}+m_{31} = m_{13}+m_{21}+m_{32} = 0$, i.e. $m_{31} = m_{12}+m_{23}$ and $m_{32} = m_{13}+m_{21}$. Lemma 4.5 implies that $m_{ij}+m_{ji}+e(m_{ih}+m_{jh})+e^2(m_{hi}+m_{hj}) = 0$ and $m_{ij}+m_{ji}+e^2(m_{ih}+m_{jh})+e(m_{hi}+m_{hj}) = 0$ for all $\{i,j,h\} = \{1,2,3\}$. Since $e^2 - e = 1$, subtracting these equalities, we get $m_{ih}+m_{jh} = m_{hi}+m_{hj}$ for all i,h,j. In particular, we have $m_{13}+m_{23} = m_{31}+m_{32} = m_{12}+m_{23}+m_{13}+m_{21}$, i.e. $m_{12} = m_{21}$. In the same way, we get $m_{ij} = m_{ji}$ for all $i \neq j$. Then the set B_h in the statement of Lemma 4.6 gives $e(m_{ih}+m_{jh}) + e^2(m_{ih}+m_{ji}) = 0$. Since $e \neq e^2$, we get $m_{ih}+m_{jh} = 0$, i.e. $m_{ih} = m_{jh}$, for all $\{i,j,h\} = \{1,2,3\}$. Since M is symmetric, we get $m_{ij} = m_{12}$ for all $i \neq j$. Since $m_{12} = 3m_{12} = m_{12}+m_{23}+m_{31} = 0$, we get $m_{ij} = 0$ for all $i \neq j$. □

Proof [Proof of Proposition 1.1:] The " if " part is obvious, while the " only if " part follows from part (1) of Lemma 4.1. □

Proof [Proof of Proposition 1.5:] We always have $\{m_{11}, \dots, m_{nn}\} \subseteq \mathrm{Num}(M)$ (Remark 2.2) and this inclusion is an equality if $n=2$ (Remark 2.10), proving the case $n=2$. The case $n=3$ is true by Lemma 4.8. Now assume $n \geq 4$ and $|\mathrm{Num}(M)| = 1$. Hence $m_{ii} = m_{11}$ for all i. By Lemma 4.8 applied to all $\mathbb{F}_4e_i + \mathbb{F}_4e_j + \mathbb{F}_4e_h$ we have $m_{ij} = 0$ for all $i \neq j$. □

Proof [Proof of Theorem 1.2:] Taking $\frac{1}{b}M$ instead of M, we reduce to the case $b=1$ (Remark 2.3). Note that $\mathrm{Num}_0(M) = \mathrm{Num}_0(M - c\mathbb{I}_{n\times n})$ for any $c \in \mathbb{F}_4$ (Remark 2.3). Hence, Remark 2.4 and Proposition 4.4 give the "if " part. Note that $M - m_{11}\mathbb{I}_{n\times n}$ is Hermitian if and only if $m_{ij}^2 = m_{ji}$ for all $i \neq j$ and $m_{ii} - m_{11} \in \mathbb{F}_2$ for all i. Hence, $M - m_{11}\mathbb{I}_{n\times n}$ is Hermitian if and only if $M - m_{ii}\mathbb{I}_{n\times n}$ is Hermitian for some $i \in \{1,\dots,n\}$ and hence, that $M - m_{11}\mathbb{I}_{n\times n}$ is Hermitian if and only if there is $c \in \mathbb{F}_4$ with $M - c\mathbb{I}_{n\times n}$ Hermitian.

Now, assume that $\mathrm{Num}_0(M) \subseteq \{0,1\}$. Taking $A := M_{|\mathbb{F}_4e_i+\mathbb{F}_4e_j}$ with $i \neq j$ we reduce to the case $n=2$; we write $A = (a_{hk})$, $h,k = 1,2$. Then taking $M - a_{11}\mathbb{I}_{2\times 2}$, we reduce to the case $a_{11} = 0$. After this reduction, we need to prove that $M^\dagger = M$. If $n=2$ we assume $\mathrm{Num}_0(M) = \{0,1\}$, but if $n \geq 3$ we only assume that $\mathrm{Num}_0(A) \subseteq \{0,1\}$.

First, assume that $a_{22} = 0$. Set $\alpha := a_{12}+a_{21}$, $\beta := ea_{12}+e^2a_{21}$ and $\gamma := e^2a_{12}+a_{21}$. Since $2a = 0$ for all $a \in \mathbb{F}_4$, $e^2+1 = e$, $e+1 = e^2$ and $e^2+1 = \alpha$, we have $\alpha+\beta = \gamma$, $\beta+\gamma = \alpha$ and $\alpha+\gamma = \beta$. Lemma 4.1 gives $\alpha, \beta, \gamma \in \mathbb{F}_2$. First, assume that $\alpha = 0$, i.e. $a_{12} = a_{21}$. We get $\beta = (e^2+e)a_{12}$ and so $a_{12} \in \mathbb{F}_2$. Thus, A is Hermitian in this case.

Now, assume that $a_{22} \neq 0 = a_{11}$. If $a_{12} = a_{21} = 0$, then part (5) of Lemma 4.1 implies that $a_{22} \in \mathbb{F}_2$ and hence, A is Hermitian. Case (7) of Lemma 4.1 excludes the case where $a_{12}a_{21} = 0$ and

$(a_{12}, a_{21}) \neq (0,0)$. If $a_{12} = a_{21} \neq 0$, then case (6) of Lemma 4.1 gives $a_{22} = a_{11} + a_{21}$, i.e. $a_{22} = a_{12} = a_{21}$; since $a_{11}+a_{22}+a_{12}+a_{21} \in \mathrm{Num}_0(M) \subseteq \{0,1\}$ and $a_{11} = 0$, we get $a_{ij} \in \mathbb{F}_2$ for all $i,j = 1,2$. Since $a_{21} = a_{12}$, this 2×2 matrix is Hermitian. Now, assume that $a_{12} \neq a_{21}$ and $a_{12}a_{21} \neq 0$. Part (8) of Lemma 4.1 gives that either $\delta := a_{22} + ea_{12} + e^2a_{21} = 0$ or $\eta := a_{22} + e^2a_{12} + ea_{21} = 0$ and that (since $a_{12} \neq a_{21}$) either $\delta = 0$ and $\eta = 1$ or $\delta = 1$ and $\eta = 0$. Since $e^2 + e = 1$, we have $1 = \delta + \eta = (e^2+e)a_{12} + (e^2+e)a_{21} = a_{21} + a_{12}$. Since $a_{12} \neq 0$ and $a_{21} \neq 0$, we get $(a_{12}, a_{21}) \in \{(e,e^2),(e^2,e)\}$ and hence, $a_{21} = a_{12}^2$. Since $e^3 = 1$, if $(a_{12}, a_{21}) = (e^2, e)$ (resp. $(a_{12}, ma_{21}) = (e, e^2)$), then $\delta \in \mathbb{F}_2$ (resp. $\eta \in \mathbb{F}_2$) gives $a_{22} \in \mathbb{F}_2$. Hence, this 2×2 matrix is Hermitian. □

Proof [Proofs of Theorem 1.3 and Corollary 1.4]: Take $N \in M_{n,n}(\mathbb{F}_4)$ such that $\mathrm{Num}(N) = \{a,b\}$ with $a \neq b$ and set $M := \frac{1}{b-a}(N - a\mathbb{I}_{n,n})$. By Remark 2.3, we have $\mathrm{Num}(M) = \{0,1\}$. By Theorem 1.2, the matrices N and M are not a multiple of $\mathbb{I}_{n,n}$. Hence, to prove Theorem 1.3, it is sufficient to prove Corollary 1.4. Also note that in the last assertion of Theorem 1.3, it is sufficient to assume that $N \neq c\mathbb{I}_{n\times n}$ with $c \in \{a,b\}$.

If $M^\dagger = M$, then for any $u \in \mathbb{F}_4^n$, we have $\langle u, Mu\rangle = \langle Mu, u\rangle = (\langle u, Mu\rangle)^2$ and hence, $\mathrm{Num}(M) \subseteq \mathbb{F}_2$, proving the " if " part.

Now, assume that $\mathrm{Num}(M) \subseteq \mathbb{F}_2$. Since $\langle e_i, Me_i\rangle = m_{ii}$, we have $m_{ii} \in \mathbb{F}_2$ for all i. Taking the restriction to $\mathbb{F}_4e_i + \mathbb{F}_4e_j + \mathbb{F}_4e_h$ for all i,j,h with $1 \le i < j < h \le n$, we reduce to the case $n = 3$.

(a) First, assume that $m_{ii} = m_{11}$ for all i. By Remark 2.3, taking $\mathbb{I}_{2\times 2} + M$ instead of M if $m_{11} = 1$, we reduce to the case $m_{11} = m_{22} = m_{33} = 0$. Set $\alpha := m_{12} + m_{23} + m_{31}$ and $\beta := m_{13} + m_{21} + m_{32}$. By Lemma 4.5 $\alpha + \beta \in \mathbb{F}_2$, $e\alpha + e^2\beta \in \mathbb{F}_2$ and $e^2\alpha + e\beta \in \mathbb{F}_2$. Since $e^2 + e = 1$ we get $\alpha + \beta \in \mathbb{F}_2$. For any $\{i,j,h\} = \{1,2,3\}$ set $\delta_i := m_{jh} + m_{hj}$. By Lemma 4.5, each element of B_h, $h = 1,2,3$, is contained in $\mathbb{F}_2$. Since $e^2 + e = 1$, the sum of the two elements of B_h gives $\delta_j + \delta_i \in \mathbb{F}_2$.

Since $\sum_{ij} m_{ij} \in \mathrm{Num}(M)$ (Lemma 4.5), we have $\delta_1 + \delta_2 + \delta_3 \in \mathbb{F}_2$. Hence, $\delta_i \in \mathbb{F}_2$ for all i.

Let $B = (b_{ij})$, $i,j = 1,2,3$, be the 3×3-matrix with $b_{ii} = 0$ for all i, $b_{ij} = m_{ij}$ if $j < i$ and $b_{ji} = b_{ij}^2$ if $i < j$. Thus, B is a Hermitian matrix and therefore, $\mathrm{Num}(D) \subseteq \mathbb{F}_2$ if $D := M + B$. Note that $D = (d_{ij})$ $d_{ij} = 0$ if either $i \ge j$ or $i < j$ and $m_{ji} = m_{ij}^2$ and that $d_{ij} = 1$ if $i < j$ and $m_{ji} \neq m_{ji}^2$, because $x^2 + x = 1$ if $x \in \mathbb{F}_4 \setminus \mathbb{F}_2$ and $x^2 + x = 0$ if $x \in \mathbb{F}_2$.

(a1) Assume that D has exactly one nonzero entry. First, assume that $d_{12} = 1$; take $u = (1,e,1)$; we have $Du = (e,0,0)$ and $\langle u, Du\rangle = e \notin \mathbb{F}_2$, a contradiction. If $d_{13} = 1$ take $u = (1,1,e)$. If $d_{23} = 1$, take $u = (1,1,e)$.

(a2) Now, assume that D has 2 nonzero entries. First, assume that $d_{12} = d_{13} = 1$ and $d_{23} = 0$; take $u = (1,e,1)$; we have $Du = (e+1,0,0)$ and $\langle u, Du\rangle = e+1 \notin \mathbb{F}_2$, a contradiction. Now, assume that $d_{12} = 0$ and $d_{13} = d_{23} = 1$; take $u = (1,e,e)$; we have $Du = (e,e,0)$ and $\langle u, Du\rangle = e + e^3 = e+1 \notin \mathbb{F}_2$, a contradiction. Now, assume that $d_{12} = d_{23} = 1$ and $d_{13} = 0$; take $u = (1,e,e)$; we have $Du = (e,e,0)$ and $\langle u, Du\rangle = e + e^3 = e+1 \notin \mathbb{F}_2$, a contradiction.

(a3) Now, assume that $d_{12} = d_{13} = d_{23} = 1$; take $u = (1,e,1)$; since $e+1 = e^2$, we have $Du = (e^2,e,0)$ and $\langle u, Du\rangle = e^2 + e^3 \notin \mathbb{F}_2$, a contradiction.

(b) Now, assume that $m_{ii} \neq m_{jj}$ for some $i \neq j$. Let $E = (e_{ij}) \in M_{3,3}(\mathbb{F}_2)$ be the diagonal matrix with $e_{ii} = m_{ii}$ for all i. Since $m_{ii} \in \mathbb{F}_2$ for all i, we have $E^\dagger = E$ and hence, $G := M + E$ is Hermitian if and only if G is Hermitian. Since E is Hermitian, $\langle u, Eu\rangle \in \mathbb{F}_2$ for all $u \in \mathbb{F}_4^3$. Hence, $\langle u, Gu\rangle = \langle u, Mu\rangle + \langle u, Eu\rangle \in \mathbb{F}_2$ for all $u \in C_3(1)$. Since all diagonal elements of G are zero, step (a) gives $G^\dagger = G$ and so $M^\dagger = M$. □

Acknowledgments

The author was partially supported by Ministero Istruzione Università Ricerca (Italy) and Gruppo Nazionale Geometria Algebra (Istituto di Alta Matematica, Italy).

References

[1] Ballico E. On the numerical range of matrices over a finite field. Linear Algebra and its Applications 2017; 512: 162-171.

[2] Ballico E. Corrigendum to "On the numerical range of matrices over a finite field " Linear Algebra Appl. 512 (2017) 162-171. Linear Algebra and its Applications 2018; 556: 421–427.

[3] Ballico E. On the numerical range of square matrices with coefficients in a degree 2 Galois field extension. Turkish Journal of Mathematics 2018; 42: 1698-1710

[4] Ballico E. The Hermitian null-range of a matrix over a finite field. Electronic Journal of Linear Algebra 2018; 34: 205-216.

[5] Coons JI, Jenkins J, Knowles D, Luke RA, Rault PX. Numerical ranges over finite fields. Linear Algebra and its Applicatons 2016; 501: 37-47.

[6] Hirschfeld JWP. Projective geometries over finite fields. Oxford, UK: Clarendon Press, 1979.

[7] Hirschfeld JWP, Thas JA. General Galois geometries. Oxford UK: The Clarendon Press, 1991.

[8] Lindl R, Niederreiter H. Introduction to finite fields and their applications. Cambrige UK: Cambridge University Press, 1994.