INDUCED NILPOTENT ORBITS OF THE SIMPLE LIE ALGEBRAS OF EXCEPTIONAL TYPE

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Dedicated to Nodar Berikashvili
on the occasion of his 80th birthday

Abstract. We describe algorithms for computing the induced nilpotent orbits in semisimple Lie algebras. We use them to obtain the induction tables for the Lie algebras of exceptional type. This also yields the classification of the rigid nilpotent orbits in those Lie algebras.

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1. Introduction

Let $\mathfrak{g}$ be a simple complex Lie algebra, and let G be a connected algebraic group with Lie algebra $\mathfrak{g}$. Then G acts on $\mathfrak{g}$, and a natural question is what the G-orbits in $\mathfrak{g}$ are. The nilpotent G-orbits in $\mathfrak{g}$ have been studied in great detail (see for example [8]). They have been classified in terms of so-called weighted Dynkin diagrams. In [18] the notion of *induced nilpotent orbit* was introduced. Let $\mathfrak{p} \subset \mathfrak{g}$ be a parabolic subalgebra with Levi decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$, where $\mathfrak{n}$ is the nilradical. Let $\mathcal{O}_{\mathfrak{l}}$ be a nilpotent orbit in $\mathfrak{l}$. Then in [18] it is shown that there is a unique nilpotent orbit $Ge \subset \mathfrak{g}$ such that $Ge \cap (\mathcal{O}_{\mathfrak{l}} \oplus \mathfrak{n})$ is dense in $\mathcal{O}_{\mathfrak{l}} \oplus \mathfrak{n}$. The orbit Ge is said to be induced from $\mathcal{O}_{\mathfrak{l}}$.

Naturally this led to the question which nilpotent orbits are induced, and which are not. For the simple Lie algebras of classical type this question was treated by Spaltenstein ([19]) and later by Kempken ([16]). The same problem for the exceptional types was first solved by Elashvili. Elashvili (exceptional case) and Spaltenstein (classical case) announced these results in a joint talk at the 1979 Oberwolfach conference on Transformation Groups and Invariant Theory. Later ([10], see also [19]) Elashvili has published tables which, for the Lie algebras of exceptional type, list for each induced nilpotent orbit exactly from which data it is induced (a Levi subalgebra, and a nilpotent orbit in it). We call these lists *induction tables*.

It is the objective of this paper to give algorithms that compute the induction table for a given semisimple Lie algebra (Sections 2, 3). We have implemented these algorithms in the computer algebra system GAP4 ([11]). Using this we recomputed Elashvili's tables (and fortunately this confirmed their correctness).

They are given in Section 4. This serves two purposes. Firstly, these computations constitute an independent check of the correctness of the tables. Secondly, it is our objective to make the tables more easily available. A new feature of our tables is that they contain a representative for each induced orbit. That is a nilpotent element with two properties: it lies in a particular subalgebra (denoted $\mathfrak{u}(\widetilde{D})$, see Section 3) of the parabolic subalgebra associated with the induction, and it is a representative of the induced nilpotent orbit. Also the validity of these representatives has been checked by computer.

2. Preliminaries

In this section we recall some notions from the literature. Secondly, we describe some basic algorithms that we use, and that we believe to be of independent interest. Our computational set up is as in [12]. In particular, $\mathfrak{g}$ will be a simple complex Lie algebra given by a multiplication table relative to a Chevalley basis. This means that all structure constants are integers. Therefore, all computations will take place over the base field $\mathbb{Q}$.

2.1. **Finding $\mathfrak{sl}_2$-triples.** Let $e \in \mathfrak{g}$ be a nilpotent element. Then by the Jacobson-Morozov lemma (cf. [15]) there are $f, h \in \mathfrak{g}$ with $[h, e] = 2e$, $[h, y] = -2y$, $[e, f] = h$. The triple (h, e, f) is said to be an $\mathfrak{sl}_2$-triple. The proof of the Jacobson-Morozov lemma in [15] translates to a straightforward algorithm to find such a triple containing a given nilpotent element e, which takes the following steps:

(1) By solving a system of non-homogeneous linear equations we can find $z, h \in \mathfrak{g}$ with $[e, z] = h$ and $[h, e] = 2e$.
(2) Set $R = C_{\mathfrak{g}}(e)$, the centralizer of e in $\mathfrak{g}$. Then the map $\mathrm{ad}\, h + 2 : R \to R$ is non-singular; hence there exists $u_1 \in R$ with $(\mathrm{ad}\, h+2)(u_1) = u_0$, where $u_0 = [h, z] + 2z$. We find u_1 by solving a non-homogeneous system of linear equations.
(3) Set $f = z - u_1$; then (h, e, f) is a $\mathfrak{sl}_2$-triple.

For a proof of the following theorem we refer to [8], Chapter 3.

Theorem 2.1. *Let $e_1, e_2 \in \mathfrak{g}$ be two nilpotent elements lying in $\mathfrak{sl}_2$-triples (h_1, e_1, f_1) and (h_2, e_2, f_2). Then e_1 and e_2 are G-conjugate if and only if the two $\mathfrak{sl}_2$-triples are G-conjugate, if and only if h_1 and h_2 are G-conjugate.*

Remark 2.2. Of course the elements z, h found in Step (1) of the algorithm are not necessarily unique. Indeed, let $u \in C_{\mathfrak{g}}(e) \cap [e, \mathfrak{g}]$, and let $v \in \mathfrak{g}$ be such that $[e, v] = u$. Then $z' = z + v$, $h' = h + u$ is also a solution. However, because of Theorem 2.1, this non-uniqueness does not lead to problems.

2.2. **The weighted Dynkin diagram.** Let $e \in \mathfrak{g}$ be a nilpotent element lying in an $\mathfrak{sl}_2$-triple (h, e, f). Then by the representation theory of $\mathfrak{sl}_2$ we get a direct sum decomposition $\mathfrak{g} = \oplus_{k\in\mathbb{Z}}\, \mathfrak{g}(h, k)$, where $\mathfrak{g}(h, k) = \{x \in \mathfrak{g} \mid [h, x] = kx\}$. Fix a Cartan subalgebra H of $\mathfrak{g}$ with $h \in H$. Let Φ be the corresponding root system of $\mathfrak{g}$. For $\alpha \in \Phi$ we let x_α be a corresponding root vector. For each

α there is a $k \in \mathbb{Z}$ with $x_\alpha \in \mathfrak{g}(h,k)$. We write $\eta(\alpha) = k$. It can be shown that there exists a basis of simple roots $\Delta \subset \Phi$ such that $\eta(\alpha) \geq 0$ for all $\alpha \in \Delta$. Furthermore, for such a Δ we have $\eta(\alpha) \in \{0,1,2\}$ for all $\alpha \in \Delta$ (see [6]). Write $\Delta = \{\alpha_1, \dots, \alpha_l\}$. Then the Dynkin diagram of Φ has l nodes, the i-th node corresponding to α_i. Now to each node we add the label $\eta(\alpha_i)$; the result is called the weighted Dynkin diagram. It depends only on e, and not on the choice of $\mathfrak{sl}_2$-triple containing e. Furthermore, it completely identifies the nilpotent orbit Ge. In other words, e and e' ly in the same G-orbit if and only if they have the same weighted Dynkin diagrams.

It is possible to formulate an algorithm for computing the weighted Dynkin diagram of a given nilpotent $e \in \mathfrak{g}$. However, computing the set of roots relative to a Cartan subalgebra H will in many cases be rather time consuming, and in the worst cases even prove to be infeasible (for example if H is not split over the rationals, and the construction of field extensions is required). Therefore we consider a different approach. Let $e \in \mathfrak{g}$ be nilpotent, and let (h,e,f) be an $\mathfrak{sl}_2$-triple. We consider the direct sum decomposition $\mathfrak{g} \oplus_{k\in\mathbb{Z}} \mathfrak{g}(h,k)$; and form the sequence $s(e) = (\dim \mathfrak{g}(h,0), \dim \mathfrak{g}(h,1), \dots, \dim \mathfrak{g}(h,m))$, where m is maximal with $\dim \mathfrak{g}(h,m) \neq 0$. We note that it does not depend on the choice of h (indeed, if (h',e,f') is an $\mathfrak{sl}_2$-triple, then h and h' are G-conjugate by Theorem 2.1). Secondly, for all $e' \in Ge$ we have $s(e') = s(e)$. In other words, $s(e)$ only depends on the G-orbit of e. We call $s(e)$ the signature of the orbit Ge.

Proposition 2.3. *Let $\mathfrak{g}$ be of exceptional type. Then all nilpotent orbits in $\mathfrak{g}$ have a different signature.*

Proof. We have verified this statement by a straightforward computer calculation. A more conceptual argument goes as follows. We note that knowing the signature of the orbit Ge amounts to knowing the character of the $\mathfrak{sl}_2$-triple (h,e,f) on the module $\mathfrak{g}$. Hence the signature determines the sizes of the Jordan blocks of the adjoint map $\mathrm{ad}_{\mathfrak{g}}(e)$. Now, it is known that each nilpotent orbit gives rise to a different set of sizes of Jordan blocks (cf. [17]). $\square$

Using Proposition 2.3 we compute weighted Dynkin diagrams in the following way. We first compute the list of all weighted Dynkin diagrams, with their corresponding signatures. Then, in order to compute the weighted Dynkin diagram of a nilpotent element, we compute an $\mathfrak{sl}_2$-triple containing it, compute the corresponding signature, and look it up in the list. Of course, the table of signatures is computed only once, and then stored.

We remark that it is straightforward to compute the signature of a nilpotent orbit, given its weighted Dynkin diagram. Indeed, we form the vector $\bar{w} = (w_1, \dots, w_l)$, where w_i is the label corresponding to α_i in the weighted Dynkin diagram. We write a root α of Φ as a linear combination of simple roots $\alpha = \sum_i a_i \alpha_i$. Let $\bar{a} = (a_1, \dots, a_l)$. Then the root space $\mathfrak{g}_\alpha$ is contained in $\mathfrak{g}(h,k)$ if and only if the inner product $\bar{w} \cdot \bar{a}$ equals k.

Remark 2.4. We can prove the statement of Proposition 2.3 also for simple Lie algebras of type A_n and C_n. For type C_n the argument goes as follows. Here $\mathfrak{g}$ is isomorphic to $\mathfrak{sp}_{2n}(\mathbb{C})$. Let V be its natural module. Then the adjoint module is isomorphic to the symmetric square $S^2(V)$. Let $\mathfrak{s}$ denote the subalgebra of $\mathfrak{g}$ spanned by the $\mathfrak{sl}_2$-triple (h, e, f). Let $\chi_V(x)$ denote the character of $\mathfrak{s}$ on V. Then

$$\chi_V(x) = \sum_{k\in\mathbb{Z}} n_k x^k,$$

where n_k is the dimension of the weight space with weight k. We have

$$2\chi_{S^2(V)}(x) = \chi_V(x)^2 + \chi_V(x^2).$$

From this it follows that from the knowledge of the character of $\mathfrak{s}$ on $S^2(V)$ we can recover the character of $\mathfrak{s}$ on V. But that determines the G-conjugacy class of h, and hence the nilpotent orbit Ge. The proof for A_n is simpler, as here $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}_{n+1}(\mathbb{C})$. In this case, from the character of $\mathfrak{s}$ on $\mathfrak{g}$ we directly get the character of $\mathfrak{s}$ on $V \otimes V^*$. From the latter character we recover the character of $\mathfrak{s}$ on V.

However, the statement of the proposition does not hold for $\mathfrak{g}$ of type D_n. First of all, if the weighted Dynkin diagrams of two nilpotent orbits can be transformed into each other by a diagram automorphism, then the two orbits will have the same signature. Secondly, also in other cases two different orbits can have the same signature. Indeed, consider the Lie algebra of type D_{64}, and the nilpotent orbits corresponding to the partitions $(5, 3^{27}, 1^{42})$ and $(4^8, 2^{48})$ (we refer to [8] for an account of the parametrization of nilpotent orbits by partitions). These two orbits have "essentially" different weighted Dynkin diagrams (that is, they cannot be transformed into each other by a diagram automorphism), and also have the same signature. We are grateful to Oksana Yakimova for indicating this example to us.

2.3. **Levi and parabolic subalgebras.** Let Φ denote the root system of $\mathfrak{g}$, with set of simple roots Δ. For $\alpha \in \Phi$ let $\mathfrak{g}_\alpha$ denote the corresponding root space, spanned by the root vector x_α.

A subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is said to be parabolic if it contains a Borel subalgebra (i.e., a maximal solvable subalgebra). Let $\Pi \subset \Delta$, and let $\mathfrak{p}_\Pi$ be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{h}$, $\mathfrak{g}_\alpha$ for $\alpha > 0$ and $\mathfrak{g}_{-\alpha}$, for $\alpha \in \Pi$. Then $\mathfrak{p}_\Pi$ is a parabolic subalgebra. Furthermore, any parabolic subalgebra of $\mathfrak{g}$ is G-conjugate to a subalgebra of the form $\mathfrak{p}_\Pi$. These $\mathfrak{p}_\Pi$ are called standard parabolic subslgabras.

Let $\Psi \subset \Phi$ be the root subsystem generated by Π, and let $\mathfrak{p} = \mathfrak{p}_\Pi$. Then $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$, where $\mathfrak{l}$ is spanned by $\mathfrak{h}$ along with $\mathfrak{g}_\alpha$ for $\alpha \in \Psi$, and $\mathfrak{n}$ is spanned by $\mathfrak{g}_\alpha$ for $\alpha \in \Phi$ such that $\alpha > 0$ and $\alpha \notin \Psi$. The subalgebra $\mathfrak{l}$ is reductive, and it is called a (standard) Levi subalgebra.

Let $\mathfrak{l}_1$, $\mathfrak{l}_2$ be two Levi subalgebras, of standard parabolic subalgebra corresponding to $\Pi_1, \Pi_2 \subset \Delta$. They may or may not be G-conjugate (even if they are isomorphic as abstract Lie algebras, they may not be G-conjugate). We use the following algorithm to decide whether they are conjugate:

(1) For $i = 1, 2$ set
$$u_i = \sum_{\alpha \in \Pi_i} x_\alpha.$$
(2) Compute the weighted Dynkin diagrams D_i of u_i.
(3) If $D_1 = D_2$ then $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are G-conjugate, otherwise they are not.

Lemma 2.5. *The previous algorithm is correct.*

Proof. The u_i are representatives of the principal nilpotent orbit of $\mathfrak{l}_i$ ([8], proof of Theorem 4.1.6). This orbit is distinguished (this follows from the cited proof, along with [8], Lemma 8.2.1). In other words, $\mathfrak{l}_i$ is the minimal Levi subalgebra of $\mathfrak{g}$ containing u_i. Now suppose that $D_1 = D_2$. Then there is a $g \in G$ with $gu_1 = u_2$. So $g\,\mathfrak{l}_1$ is a minimal Levi subalgebra containing u_2. Hence, by [8], Theorem 8.1.1, there is a $g' \in G$ with $g'g\,\mathfrak{l}_1 = \mathfrak{l}_2$. The reverse direction is trivial. □

3. Induced Nilpotent Orbits

Let $\mathfrak{p} \subset \mathfrak{g}$ be a parabolic subalgebra, and write $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$. Let $L \subset G$ be the connected subgroup with Lie algebra $\mathfrak{l}$. Let $Le_0 \subset \mathfrak{l}$ be a nilpotent orbit in $\mathfrak{l}$. The following is part of the content of [8], Theorem 7.1.1.

Theorem 3.1. (1) *There is a unique nilpotent orbit $Ge \subset \mathfrak{g}$ such that $Ge \cap (Le_0 \oplus \mathfrak{n})$ is dense in $Le_0 \oplus \mathfrak{n}$.*
(2) $\dim Ge = \dim Le_0 + 2 \dim \mathfrak{n}$.
(3) *Ge is the unique nilpotent orbit in $\mathfrak{g}$ of that dimension meeting $Le_0 \oplus \mathfrak{n}$.*

The orbit Ge of the theorem is said to be induced from the orbit Le_0. It only depends on the Levi subalgebra $\mathfrak{l}$ and not on the parabolic subalgebra $\mathfrak{p}$ ([8], Theorem 7.1.3). Therefore we write $Ge = \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(Le_0)$. Also, induction is transitive: if $\mathfrak{l}_1 \subset \mathfrak{l}_2$ are two Levi subalgebras, and L_1e_1 is a nilpotent orbit in $\mathfrak{l}_1$, then
$$\mathrm{Ind}_{\mathfrak{l}_2}^{\mathfrak{g}}(\mathrm{Ind}_{\mathfrak{l}_1}^{\mathfrak{l}_2}(L_1e_1)) = \mathrm{Ind}_{\mathfrak{l}_1}^{\mathfrak{g}}(L_1e_1)$$
([8], Proposition 7.1.4). Furthermore, a nilpotent orbit in $\mathfrak{g}$ that is not induced from a nilpotent orbit of a Levi subalgebra is said to be rigid.

The problem considered in this paper is to determine $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(Le_0)$ for all G-conjugacy classes of Levi subalgebras $\mathfrak{l} \subset \mathfrak{g}$, and nilpotent orbits $Le_0 \subset \mathfrak{l}$. Of course, because of transitivity we may restrict to the rigid nilpotent orbits of $\mathfrak{l}$.

Consider the union of all G-orbits in $\mathfrak{g}$ of the same dimension d. The irreducible components of these varieties are called the sheets of $\mathfrak{g}$ (cf. [3], [4]). The sheets of $\mathfrak{g}$ are parametrised by the G-conjugacy classes of pairs $(\mathfrak{l}, Le)$, where $\mathfrak{l}$ is a Levi subalgebra, and Le a rigid nilpotent orbit in $\mathfrak{l}$. Furthermore, in the sheet corresponding to $(\mathfrak{l}, Le)$ there is a unique nilpotent orbit, namely $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(Le)$.

Let $\Pi \subset \Delta$ and let $\mathfrak{p} = \mathfrak{p}_\Pi$ be the corresponding standard parabolic subalgebra. Write $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$, and let Le_0 be a rigid nilpotent orbit in $\mathfrak{l}$. Let $\widetilde{D}$ be the Dynkin diagram of $\mathfrak{g}$, which we label in the following way. If $\alpha \notin \Pi$ then we give the node corresponding to α the label 2. Let D_0 be the weighted Dynkin diagram of Le_0. Since Π is a set of simple roots of $\mathfrak{l}$, the nodes of D_0 correspond to the elements of Π. We give the node in $\widetilde{D}$ corresponding to $\alpha \in \Pi$ its label in D_0, under this correspondence. We call the diagram $\widetilde{D}$ together with its labeling, the sheet diagram of the sheet corresponding to $(\mathfrak{l}, Le_0)$.

We note that from the sheet diagram we can recover the sheet. Indeed, from [8], Theorem 7.1.6 we get that the weighted Dynkin diagram of a rigid nilpotent element only has labels 0 and 1. Hence, from the nodes with label 0 or 1 we recover Π. Then from the labels in those nodes we get the weighted Dynkin diagram of the orbit Le_0. Hence we recover the orbit Le_0 as well.

Let $\widetilde{D}$ be the sheet diagram corresponding to $(\mathfrak{l}, Le_0)$. Let $\omega : \Phi^+ \to \mathbb{Z}$ be the additive function such that for $\alpha \in \Delta$, $\omega(\alpha)$ is the label of the node corresponding to α in $\widetilde{D}$. Let $\mathfrak{u}(\widetilde{D})$ be the subspace of $\mathfrak{g}$ spanned by all $\mathfrak{g}_\alpha$ with $\omega(\alpha) \geq 2$. Let $Ge = \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(Le_0)$.

Lemma 3.2. *$Ge \cap \mathfrak{u}(\widetilde{D})$ is dense in $\mathfrak{u}(\widetilde{D})$.*

Proof. Recall that Π is a set of simple roots of the root system Ψ of $\mathfrak{l}$. If $e_0 = 0$ then $\mathfrak{u}(\widetilde{D}) = \mathfrak{n}$, and the lemma follows by Theorem 3.1. So now we assume that $e_0 \neq 0$. Note that $\mathfrak{l}' = [\mathfrak{l}, \mathfrak{l}]$ is the semisimple part of $\mathfrak{l}$. Let $\mathfrak{h}' = \mathfrak{h} \cap [\mathfrak{l}, \mathfrak{l}]$; then $\mathfrak{h}'$ is a Cartan subalgebra of $\mathfrak{l}'$. Let $h_0 \in \mathfrak{h}'$ be such that $\alpha(h_0)$ is the label in D corresponding to α, for $\alpha \in \Pi$. After possibly replacing e_0 by a L-conjugate we may assume that e_0 lies in an $\mathfrak{sl}_2$-triple (h_0, e_0, f_0). Let $\mathfrak{l} = \oplus_{k\in\mathbb{Z}} \mathfrak{l}(h_0, k)$ be the corresponding grading of $\mathfrak{l}$. Set $\mathfrak{l}_{\geq 2} = \oplus_{k\geq 2} \mathfrak{l}(h_0, k)$. Then by [8], Lemma 4.1.4, $Le_0 \cap \mathfrak{l}_{\geq 2}$ is dense in $\mathfrak{l}_{\geq 2}$.

Let $\beta \in \Phi^+$, but not in Ψ. Then written as a linear combination of simple roots, β has at least one $\alpha \in \Delta \setminus \Pi$ with positive coefficient. Hence $\omega(\beta) \geq 2$. Furthemore, a $\beta \in \Psi$ has $\omega(\beta) = \beta(h_0)$; hence $\omega(\beta) \geq 2$ if and only if $\mathfrak{g}_\beta \subset \mathfrak{l}_{\geq 2}$. It follows that $\mathfrak{u}(\widetilde{D}) = \mathfrak{l}_{\geq 2} \oplus \mathfrak{n}$.

We conclude that $Le_0 \oplus \mathfrak{n} \cap \mathfrak{u}(\widetilde{D})$ is dense in $\mathfrak{u}(\widetilde{D})$. Since $Ge \cap Le_0 \oplus \mathfrak{n}$ is dense in $Le_0 \oplus \mathfrak{n}$, we get that $Ge \cap \mathfrak{u}(\widetilde{D})$ is dense in $\mathfrak{u}(\widetilde{D})$. $\square$

Lemma 3.3. *Write $s = \dim Ge$, and let $e' \in \mathfrak{u}(\widetilde{D})$. Then $e' \in Ge$ if and only if $\dim C_{\mathfrak{g}}(e') = \dim \mathfrak{g} - s$.*

Proof. First of all, if $e' \in Ge$ then $\dim C_{\mathfrak{g}}(e') = \dim C_{\mathfrak{g}}(e) = \dim \mathfrak{g} - s$. For the converse, let t be the minimum dimension of a centralizer $C_{\mathfrak{g}}(u)$, for $u \in \mathfrak{u}(\widetilde{D})$. Then the set of elements of $\mathfrak{u}(\widetilde{D})$ with centralizer of dimension t is dense in $\mathfrak{u}(\widetilde{D})$. But two dense sets always meet. Hence from lemma 3.2 we get $t = \dim C_{\mathfrak{g}}(e) = \dim \mathfrak{g} - s$. So if $e' \in \mathfrak{u}(\widetilde{D})$, and $\dim C_{\mathfrak{g}}(e') = \dim \mathfrak{g} - s$, then the dimension of $C_{\mathfrak{g}}(e')$ is minimal among all elements of $\mathfrak{u}(\widetilde{D})$. Hence the dimension of the orbit Ge' is maximal. Now from [4], Satz 5.4 it follows that e' lies in the sheet corresponding to $\widetilde{D}$. Therefore, $e' \in Ge$. $\square$

Now we describe an algorithm for listing the induced nilpotent orbits in $\mathfrak{g}$. The output is a list of pairs $(D, \widetilde{D})$, where $\widetilde{D}$ runs through the sheet diagrams of

$\mathfrak{g}$, and D is the weighted Dynkin diagram of the corresponding nilpotent orbit. We use a set $\Omega = \{0, 1, \dots, N\}$ of integers; where N is a previously chosen parameter. The algorithm takes the following steps:

(1) Using the method of Section 2.3 get representatives of the G-conjugacy classes of Levi subalgebras, each lying inside a standard parabolic subalgebra.
(2) Set $\mathcal{I} = \varnothing$. For each Levi subalgebra $\mathfrak{l}$ from the list, and each rigid nilpotent orbit $Le_0 \subset \mathfrak{l}$ do the following:
   (a) Construct the sheet diagram $\widetilde{D}$ of the pair $(\mathfrak{l}, Le_0)$.
   (b) Compute a basis $\{u_1, \dots, u_m\}$ of the space $\mathfrak{u}(\widetilde{D})$.
   (c) Set $s = \dim Le_0 + 2\dim \mathfrak{n}$ (where $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$ is the parabolic subalgebra containing $\mathfrak{l}$).
   (d) Let $e' = \sum_{i=1}^{m} \alpha_i u_i \in \mathfrak{u}(\widetilde{D})$, where the $\alpha_i \in \Omega$ are chosen randomly, uniformly, and independently.
   (e) If $\dim C_{\mathfrak{g}}(e') = \dim \mathfrak{g} - s$ then compute the weighted Dynkin diagram (cf. Section 2.2) D of the orbit Ge' and add $(D, \widetilde{D})$ to $\mathcal{I}$. Otherwise return to Step 2(d).
(3) Return $\mathcal{I}$.

Proposition 3.4. *The previous algorithm is correct, and terminates for large enough N.*

Proof. Let $\widetilde{D}$ be a sheet diagram, and let Ge be the corresponding induced nilpotent orbit. Then by Lemma 3.3, steps 2(d) and 2(e) are executed until an element e' of Ge is found. So, if the algorithm terminates, then it returns the correct output. On the other hand, by Lemma 3.2, the set $Ge \cap \mathfrak{u}(\widetilde{D})$ is dense in $\mathfrak{u}(\widetilde{D})$. Hence for large enough N, the random element e' lies in Ge with high probability. Therefore, the algorithm will terminate. □

Remark 3.5. In practice, it turns out that selecting a rather small N (e.g., $N = 10$) suffices in order that the algorithm terminates. Furthermore, if the algorithm needs to many rounds for a given N, then one can try again with a higher value for N.

4. The Tables

In this section we give the tables of the induced nilpotent orbits in the Lie algebras of exceptional type, computed with the algorithm of the previous section.[1] In order to use this algorithm, we need to know the rigid nilpotent orbits of each Levi subalgebra. For Levi subalgebras of classical type, this is described in [8], §7.3. For the Lie algebras of exceptional type it follows from our calculations what the rigid nilpoten orbits are. We summarise this in the following theorem.

[1]The program needed 17, 282, 9055, 3 and 0.1 seconds respectively for E_6, E_7, E_8, F_4 and G_2; the computations were done on a 2GHz processor with 1GB of memory for GAP.

Theorem 4.1. *The weighted Dynkin diagrams of the rigid nilpotent orbits* (*except the zero orbit*) *in E_6, E_7, E_8, F_4 and G_2 are given in Tables* 1, 2, 3, 4, 5.

Table 1: Rigid nilpotent orbits in E_6.

label	characteristic
A_1	$\begin{array}{ccccc} & & 1 & & \\ 0&0&0&0&0 \end{array}$
$3A_1$	$\begin{array}{ccccc} & & 0 & & \\ 0&0&1&0&0 \end{array}$
$2A_2+A_1$	$\begin{array}{ccccc} & & 0 & & \\ 1&0&1&0&1 \end{array}$

Table 2: Rigid nilpotent orbits in E_7.

label	characteristic
A_1	$\begin{array}{cccccc} & & 0 & & & \\ 1&0&0&0&0&0 \end{array}$
$2A_1$	$\begin{array}{cccccc} & & 0 & & & \\ 0&0&0&0&1&0 \end{array}$
$(3A_1)'$	$\begin{array}{cccccc} & & 0 & & & \\ 0&1&0&0&0&0 \end{array}$
$4A_1$	$\begin{array}{cccccc} & & 1 & & & \\ 0&0&0&0&0&1 \end{array}$
A_2+2A_1	$\begin{array}{cccccc} & & 0 & & & \\ 0&0&1&0&0&0 \end{array}$
A_1+2A_2	$\begin{array}{cccccc} & & 0 & & & \\ 0&1&0&0&1&0 \end{array}$
$(A_1+A_3)'$	$\begin{array}{cccccc} & & 0 & & & \\ 1&0&1&0&0&0 \end{array}$

Table 3: Rigid nilpotent orbits in E_8.

label	characteristic
A_1	$\begin{array}{ccccccc} & & 0 & & & & \\ 0&0&0&0&0&0&1 \end{array}$
$2A_1$	$\begin{array}{ccccccc} & & 0 & & & & \\ 1&0&0&0&0&0&0 \end{array}$
$3A_1$	$\begin{array}{ccccccc} & & 0 & & & & \\ 0&0&0&0&0&1&0 \end{array}$
$4A_1$	$\begin{array}{ccccccc} & & 1 & & & & \\ 0&0&0&0&0&0&0 \end{array}$
A_2+A_1	$\begin{array}{ccccccc} & & 0 & & & & \\ 1&0&0&0&0&0&1 \end{array}$
A_2+2A_1	$\begin{array}{ccccccc} & & 0 & & & & \\ 0&0&0&0&1&0&0 \end{array}$
A_2+3A_1	$\begin{array}{ccccccc} & & 0 & & & & \\ 0&1&0&0&0&0&0 \end{array}$

Rigid nilpotent orbits in E_8.	
$2A_2 + A_1$	$\begin{array}{ccccccc} &&0&&&& \\ 1&0&0&0&0&1&0 \end{array}$
$A_3 + A_1$	$\begin{array}{ccccccc} &&0&&&& \\ 0&0&0&0&1&0&1 \end{array}$
$2A_2 + 2A_1$	$\begin{array}{ccccccc} &&0&&&& \\ 0&0&0&1&0&0&0 \end{array}$
$A_3 + 2A_1$	$\begin{array}{ccccccc} &&0&&&& \\ 0&1&0&0&0&0&1 \end{array}$
$D_4(a_1) + A_1$	$\begin{array}{ccccccc} &&1&&&& \\ 0&0&0&0&0&1&0 \end{array}$
$A_3 + A_2 + A_1$	$\begin{array}{ccccccc} &&0&&&& \\ 0&0&1&0&0&0&0 \end{array}$
$2A_3$	$\begin{array}{ccccccc} &&0&&&& \\ 1&0&0&1&0&0&0 \end{array}$
$D_5(a_1) + A_2$	$\begin{array}{ccccccc} &&0&&&& \\ 0&1&0&0&1&0&1 \end{array}$
$A_5 + A_1$	$\begin{array}{ccccccc} &&0&&&& \\ 1&0&1&0&0&0&1 \end{array}$
$A_4 + A_3$	$\begin{array}{ccccccc} &&0&&&& \\ 0&0&1&0&0&1&0 \end{array}$

Table 4: Rigid nilpotent orbits in F_4.

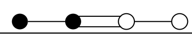
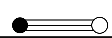
label	characteristic
	
A_1	1 0 0 0
$\widetilde{A}_1$	0 0 0 1
$A_1 + \widetilde{A}_1$	0 1 0 0
$A_2 + \widetilde{A}_1$	0 0 1 0
$\widetilde{A}_2 + A_1$	0 1 0 1

Table 5: Rigid nilpotent orbits in G_2.

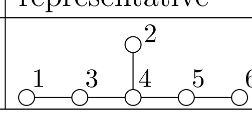
label	characteristic 
A_1	1 0
$\widetilde{A}_1$	0 1

The tables with the induced orbits have one row for each sheet. There are six columns. In the first column we give the label of the induced orbit corresponding to the sheet (where we use the same labels as in [8]). The second and third columns contain, respectively, the dimension and the weighted Dynkin diagram D (here called *characteristic*) of the induced nilpotent orbit. The dimensions of the nilpotent orbits are well-known; they were calculated in [9], and are also contained in the tables of [8]. The fourth column has the sheet diagram $\widetilde{D}$. The fifth column contains the rank of the sheet. This notion is defined as follows. Let the sheet correspond to $(\mathfrak{l}, Le)$, where $\mathfrak{l}$ is a Levi subalgebra, and Le a rigid nilpotent orbit in it. Then the rank of the sheet is the dimension of the center of $\mathfrak{l}$. It is straightforward to see that this equals the number of labels 2 in the sheet diagram $\widetilde{D}$. Finally, in the last column we give a representative of the induced orbit, i.e., an element of $\mathfrak{u}(\widetilde{D})$ with weighted Dynkin diagram D. Such a representative e is given as a sum of positive root vectors, $e = x_{\beta_1} + \cdots + x_{\beta_r}$. Then to e there corresponds a Dynkin diagram, which is simply the Dynkin diagram of the roots β_i. This diagram has r nodes, and node i is connected to node j by $\langle \beta_i, \beta_j^\vee \rangle \langle \beta_j, \beta_i^\vee \rangle = 0, 1, 2, 3$ lines. Furthermore, if these scalar products are positive, then the lines are dotted. Finally, if the root β_i is long, then node i is black. For each representative we give the corresponding Dynkin diagram, where each node has a numerical label, which denotes the position of the corresponding positive root as used by GAP4 ([13] contains lists of those roots). In other words, if the Dynkin diagram of a representative has labels $i_1, \ldots, i_k$, then the corresponding representative is the sum of the root vectors corresponding to the i_j-th positive root (in the order in which they appear in GAP4) for $1 \leq j \leq k$.

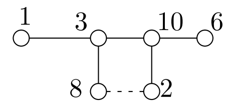
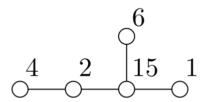
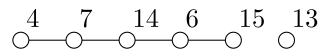
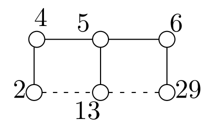
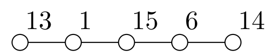
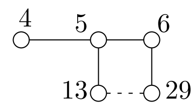
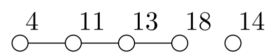
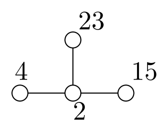
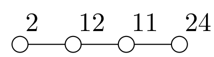
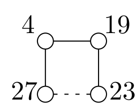
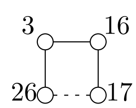
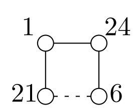
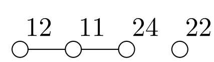
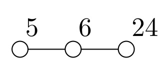
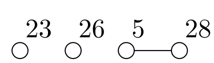
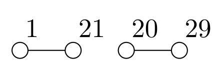
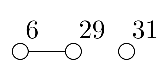
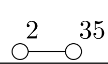
These representatives have been found as folllows. First of all, for each nilpotent orbit one or more Dynkin diagrams of representatives are known (some are described in [13], many have been found by Elashvili). For an induced nilpotent orbit with weighted Dynkin diagram D and sheet diagram $\widetilde{D}$ we construct the set S of roots $\alpha > 0$ such that $\mathfrak{g}_\alpha$ is contained in $\mathfrak{u}(\widetilde{D})$. We have written a simple-minded program in GAP4 that, for a given Dynkin diagram of a representative, tries to find a subset of S such that its Dynkin diagram is the given one (basically by trying all possibilities). We executed this program for all known Dynkin diagrams of representatives of the nilpotent orbit. In all cases we managed to find a representative this way. Furthermore, the element found was shown to be a representative of the nilpotent orbit by checking that its weighted Dynkin diagram was equal to D.

We note that in our tables the first three columns contain information relative to the induced orbit, wheras the last three columns contain information about the sheet. On some occasions it happens that a nilpotent orbit is induced in more than one way (i.e., it occurs in more than one sheet). In these cases we have not repeated the information in the first three columns; instead we have grouped the rows in the last three columns together by using a curly brace.

Table 6: Induced nilpotent orbits in the Lie algebra of type E_6.

label	dim	characteristic	sheet diagram	rk	representative
E_6	72	$\begin{smallmatrix} & & 2 & & \\ 2 & 2 & 2 & 2 & 2 \end{smallmatrix}$	$\begin{smallmatrix} & & 2 & & \\ 2 & 2 & 2 & 2 & 2 \end{smallmatrix}$	6	

Induced orbits in type E_6

$E_6(a_1)$	70	$2\,2\,\overset{2}{0}\,2\,2$	$2\,2\,\overset{2}{0}\,2\,2$	5	
D_5	68	$2\,0\,\overset{2}{2}\,0\,2$	$2\,0\,\overset{2}{2}\,0\,2$	4	
$E_6(a_3)$	66	$2\,0\,\overset{0}{2}\,0\,2$	$2\,0\,\overset{0}{2}\,0\,2$	3	
			$0\,0\,\overset{2}{2}\,2\,2$	4	
A_5	64	$2\,1\,\overset{1}{0}\,1\,2$	$2\,1\,\overset{1}{0}\,1\,2$	2	
$D_5(a_1)$	64	$1\,1\,\overset{2}{0}\,1\,1$	$0\,0\,\overset{0}{2}\,2\,2$	3	
A_4+A_1	62	$1\,1\,\overset{1}{0}\,1\,1$	$0\,0\,\overset{0}{2}\,0\,2$	2	
D_4	60	$0\,0\,\overset{2}{2}\,0\,0$	$0\,0\,\overset{2}{2}\,0\,0$	2	
A_4	60	$2\,0\,\overset{2}{0}\,0\,2$	$2\,0\,\overset{2}{0}\,0\,2$	3	
$D_4(a_1)$	58	$0\,0\,\overset{0}{2}\,0\,0$	$0\,0\,\overset{0}{2}\,0\,0$	1	
			$0\,2\,\overset{0}{0}\,0\,2$	2	
			$2\,0\,\overset{0}{1}\,0\,2$	2	
A_3+A_1	56	$0\,1\,\overset{1}{0}\,1\,0$	$1\,0\,\overset{0}{1}\,0\,2$	1	
A_3	52	$1\,0\,\overset{2}{0}\,0\,1$	$0\,0\,\overset{0}{0}\,2\,2$	2	
A_2+2A_1	50	$0\,1\,\overset{0}{0}\,1\,0$	$0\,0\,\overset{0}{0}\,2\,0$	1	
$2A_2$	48	$2\,0\,\overset{0}{0}\,0\,2$	$2\,0\,\overset{0}{0}\,0\,2$	2	
A_2+A_1	46	$1\,0\,\overset{1}{0}\,0\,1$	$0\,1\,\overset{0}{0}\,0\,2$	1	
A_2	42	$0\,0\,\overset{2}{0}\,0\,0$	$0\,0\,\overset{2}{0}\,0\,0$	1	

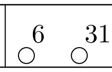
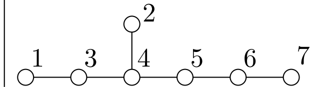
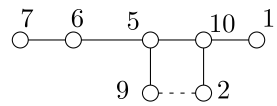
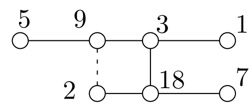
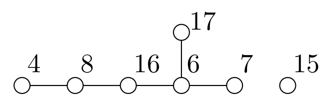
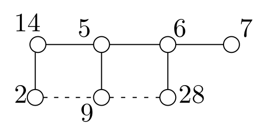
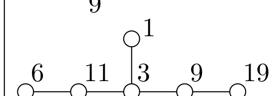
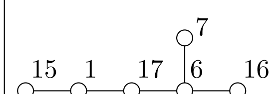
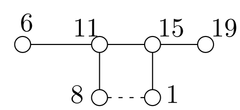
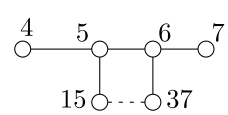
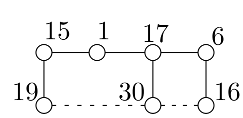
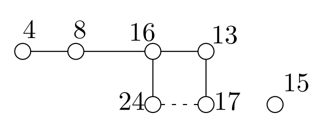
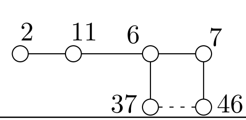
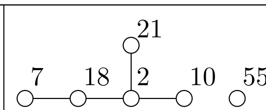
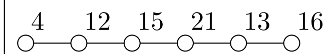
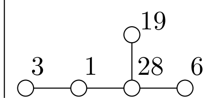
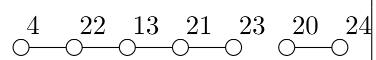
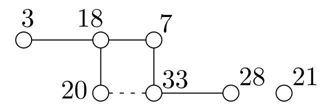
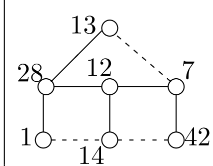
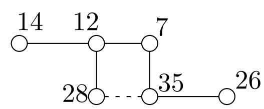
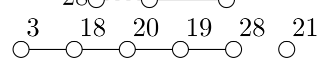
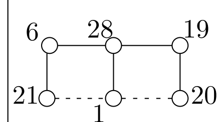
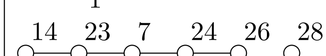
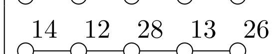
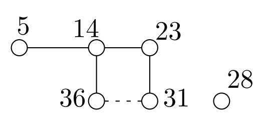
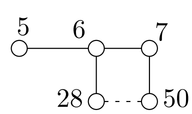
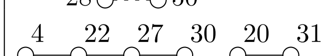
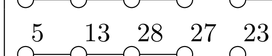
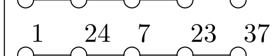
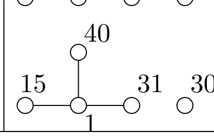
Induced orbits in type E_6					
$2A_1$	32	$1\ 0\ \overset{0}{0}\ 0\ 1$	$0\ 0\ \overset{0}{0}\ 0\ 2$	1	

Table 7: Induced nilpotent orbits in the Lie algebra of type E_7.

label	dim	characteristic	sheet diagram	rk	representative
E_7	126	$2\ 2\ \overset{2}{2}\ 2\ 2\ 2$	$2\ 2\ \overset{2}{2}\ 2\ 2\ 2$	7	
$E_7(a_1)$	124	$2\ 2\ \overset{2}{0}\ 2\ 2\ 2$	$2\ 2\ \overset{2}{0}\ 2\ 2\ 2$	6	
$E_7(a_2)$	122	$2\ 2\ \overset{2}{0}\ 2\ 0\ 2$	$2\ 2\ \overset{2}{0}\ 2\ 0\ 2$	5	
$E_7(a_3)$	120	$2\ 0\ \overset{0}{2}\ 0\ 2\ 2$	$\left\{\begin{array}{l} 2\ 0\ \overset{0}{2}\ 0\ 2\ 2 \\ 0\ 0\ \overset{2}{2}\ 2\ 2\ 2 \end{array}\right.$	4 5	 
E_6	120	$2\ 2\ \overset{0}{2}\ 0\ 2\ 0$	$2\ 2\ \overset{0}{2}\ 0\ 2\ 0$	4	
D_6	118	$2\ 1\ \overset{1}{0}\ 1\ 2\ 2$	$2\ 1\ \overset{1}{0}\ 1\ 2\ 2$	3	
$E_6(a_1)$	118	$2\ 0\ \overset{0}{2}\ 0\ 2\ 0$	$\left\{\begin{array}{l} 2\ 0\ \overset{0}{2}\ 0\ 2\ 0 \\ 0\ 0\ \overset{0}{2}\ 2\ 2\ 2 \end{array}\right.$	3 4	 
$E_7(a_4)$	116	$2\ 0\ \overset{0}{2}\ 0\ 0\ 2$	$\left\{\begin{array}{l} 2\ 1\ \overset{1}{0}\ 1\ 2\ 0 \\ 2\ 0\ \overset{0}{2}\ 0\ 0\ 2 \end{array}\right.$	2 3	 
$D_6(a_1)$	114	$2\ 1\ \overset{1}{0}\ 1\ 0\ 2$	$0\ 0\ \overset{2}{0}\ 2\ 2\ 2$	4	

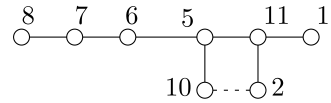
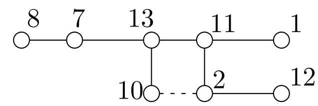
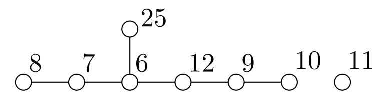
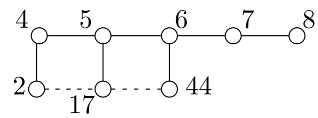
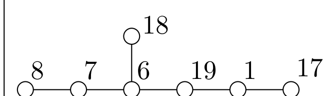
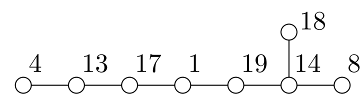
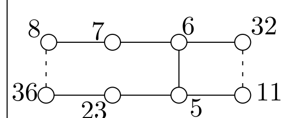
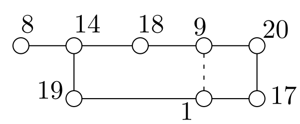
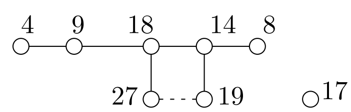
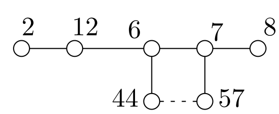
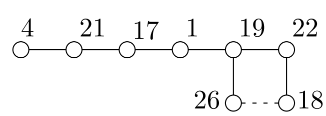
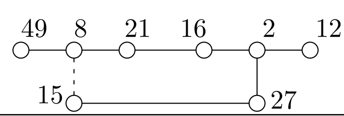
Induced orbits in type E_7

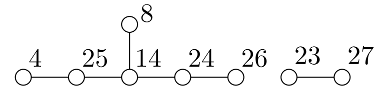
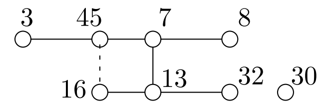
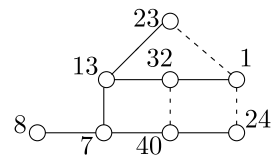
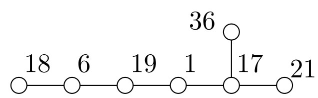
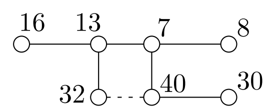
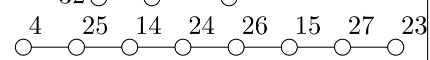
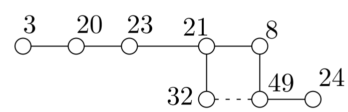
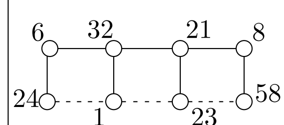
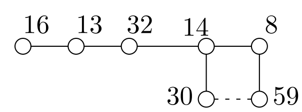
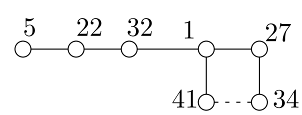
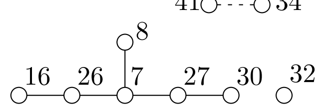
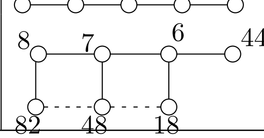
$D_5 + A_1$	114	$2\,1\,\overset{1}{0}\,1\,1\,0$	$0\,0\,\overset{2}{2}\,0\,0\,2$	3	
A_6	114	$0\,0\,\overset{0}{2}\,0\,2\,0$	$0\,0\,\overset{0}{2}\,0\,2\,0$	2	
D_5	112	$2\,2\,\overset{0}{0}\,0\,2\,0$	$2\,2\,\overset{0}{0}\,0\,2\,0$	3	
$E_7(a_5)$	112	$0\,0\,\overset{0}{2}\,0\,0\,2$	$0\,0\,\overset{0}{2}\,0\,0\,2$	2	
			$0\,2\,\overset{0}{0}\,0\,2\,2$	3	
			$2\,0\,\overset{0}{1}\,0\,2\,2$	3	
$D_6(a_2)$	110	$0\,1\,\overset{1}{0}\,1\,0\,2$	$1\,0\,\overset{0}{1}\,0\,2\,2$	2	
$E_6(a_3)$	110	$0\,2\,\overset{0}{0}\,0\,2\,0$	$0\,2\,\overset{0}{0}\,0\,2\,0$	2	
			$2\,0\,\overset{0}{1}\,0\,2\,0$	2	
$A_5 + A_1$	108	$1\,0\,\overset{0}{1}\,0\,1\,2$	$1\,0\,\overset{0}{1}\,0\,1\,2$	1	
$(A_5)'$	108	$1\,0\,\overset{0}{1}\,0\,2\,0$	$1\,0\,\overset{0}{1}\,0\,2\,0$	1	
$D_5(a_1) + A_1$	108	$2\,0\,\overset{0}{0}\,2\,0\,0$	$2\,0\,\overset{0}{0}\,2\,0\,0$	2	
$D_5(a_1)$	106	$2\,0\,\overset{0}{1}\,0\,1\,0$	$0\,0\,\overset{0}{0}\,2\,2\,2$	3	
$A_4 + A_2$	106	$0\,0\,\overset{0}{2}\,0\,0\,0$	$0\,0\,\overset{0}{2}\,0\,0\,0$	1	
$A_4 + A_1$	104	$1\,0\,\overset{0}{1}\,0\,1\,0$	$0\,0\,\overset{0}{0}\,2\,0\,2$	2	
$(A_5)''$	102	$2\,0\,\overset{0}{0}\,0\,2\,2$	$2\,0\,\overset{0}{0}\,0\,2\,2$	3	
$D_4 + A_1$	102	$2\,1\,\overset{1}{0}\,0\,0\,1$	$2\,1\,\overset{1}{0}\,0\,0\,1$	1	

Induced orbits in type E_7					
A_4	100	$2\,0\,\overset{0}{0}\,0\,2\,0$	$2\,0\,\overset{0}{0}\,0\,2\,0$	2	1 — 24 — 30 — 37
			$0\,1\,\overset{0}{0}\,0\,2\,2$	2	24 — 7 — 23 — 37
$A_3+A_2+A_1$	100	$0\,0\,\overset{0}{0}\,2\,0\,0$	$0\,0\,\overset{0}{0}\,2\,0\,0$	1	5 — 35 — 34 30 — 37 33
A_3+A_2	98	$0\,0\,\overset{0}{1}\,0\,1\,0$	$0\,1\,\overset{0}{0}\,0\,2\,0$	1	6 — 37 — 19 40 — 44
			$2\,0\,\overset{0}{0}\,1\,0\,1$	1	19 — 14 — 40 26 — 41
$D_4(a_1)+A_1$	96	$0\,1\,\overset{1}{0}\,0\,0\,1$	$0\,0\,\overset{2}{0}\,0\,0\,2$	2	2 — 25, 2 — 50, 25 — 37, 50 - - 37 55
D_4	96	$2\,2\,\overset{0}{0}\,0\,0\,0$	$2\,2\,\overset{0}{0}\,0\,0\,0$	2	3 — 1 — 28, 1 — 49
A_3+2A_1	94	$1\,0\,\overset{0}{0}\,1\,0\,1$	$0\,0\,\overset{0}{1}\,0\,0\,2$	1	28 — 13 — 43 42 51
$D_4(a_1)$	94	$0\,2\,\overset{0}{0}\,0\,0\,0$	$0\,2\,\overset{0}{0}\,0\,0\,0$	1	3 — 32, 3 — 52, 32 — 49, 52 - - 49
			$2\,0\,\overset{0}{1}\,0\,0\,0$	1	1 — 28, 1 — 49, 28 — 55, 49 - - 55
$(A_3+A_1)''$	86	$2\,0\,\overset{0}{0}\,0\,0\,2$	$2\,0\,\overset{0}{0}\,0\,0\,2$	2	1 — 31 — 53 30
A_3	84	$2\,0\,\overset{0}{0}\,0\,1\,0$	$2\,0\,\overset{0}{0}\,0\,1\,0$	1	1 — 49 — 37
$2A_2$	84	$0\,0\,\overset{0}{0}\,0\,2\,0$	$0\,0\,\overset{0}{0}\,0\,2\,0$	1	6 — 45 34 — 53
A_2+3A_1	84	$0\,0\,\overset{2}{0}\,0\,0\,0$	$0\,0\,\overset{2}{0}\,0\,0\,0$	1	2 — 50 41 48 55
A_2+A_1	76	$1\,0\,\overset{0}{0}\,0\,1\,0$	$0\,0\,\overset{1}{0}\,0\,0\,2$	1	7 — 53 49
A_2	66	$2\,0\,\overset{0}{0}\,0\,0\,0$	$2\,0\,\overset{0}{0}\,0\,0\,0$	1	1 — 62
$(3A_1)''$	54	$0\,0\,\overset{0}{0}\,0\,0\,2$	$0\,0\,\overset{0}{2}\,0\,0\,2$	2	47 48 49

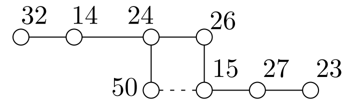
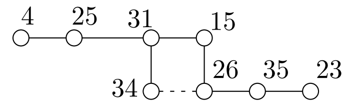
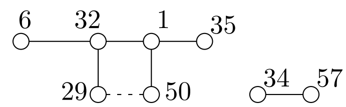
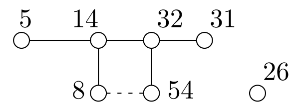
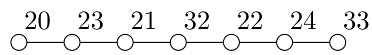
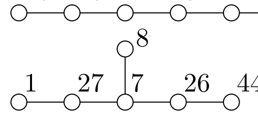
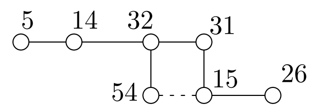
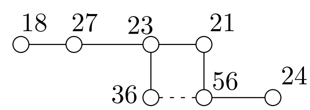
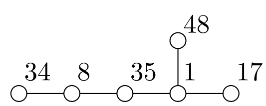
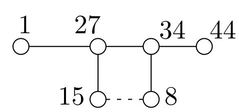
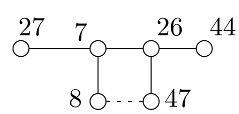
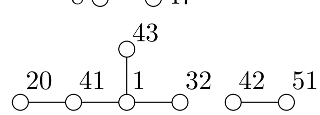
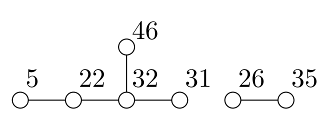
Table 8: Induced nilpotent orbits in the Lie algebra of type E_8.

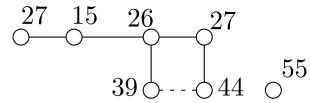
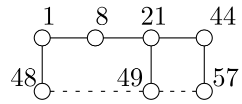
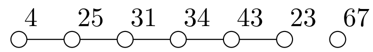
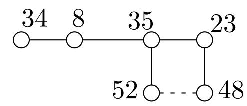
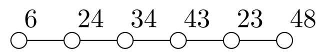
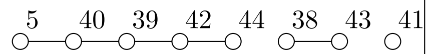
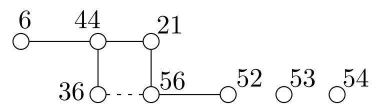
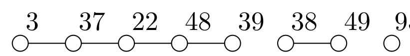
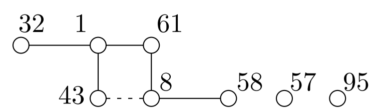
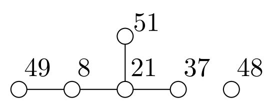
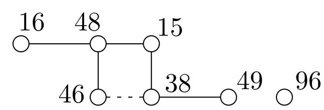
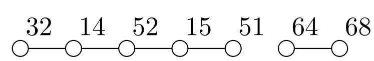
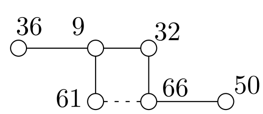
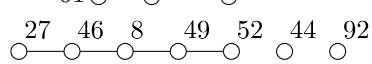
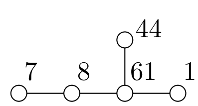
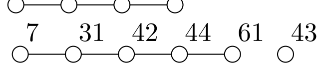
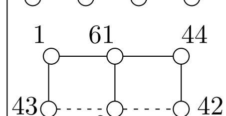
label	dim	characteristic	sheet diagram	rk	representative
E_8	240	$2\,2\,\overset{2}{2}\,2\,2\,2\,2$	$2\,2\,\overset{2}{2}\,2\,2\,2\,2$	8	1 — 3 — 4 — 5 — 6 — 7 — 8, 4 — 2

Induced orbits in type E_8					
$E_8(a_1)$	238	2 2 $\overset{2}{0}$ 2 2 2 2	2 2 $\overset{2}{0}$ 2 2 2 2	7	
$E_8(a_2)$	236	2 2 $\overset{2}{0}$ 2 0 2 2	2 2 $\overset{2}{0}$ 2 0 2 2	6	
$E_8(a_3)$	234	2 0 $\overset{0}{2}$ 0 2 2 2	2 0 $\overset{0}{2}$ 0 2 2 2 0 0 $\overset{2}{2}$ 2 2 2 2	5 6	 
E_7	232	2 1 $\overset{1}{0}$ 1 2 2 2	2 1 $\overset{1}{0}$ 1 2 2 2	4	
$E_8(a_4)$	232	2 0 $\overset{0}{2}$ 0 2 0 2	2 0 $\overset{0}{2}$ 0 2 0 2 0 0 $\overset{0}{2}$ 2 2 2 2	4 5	 
$E_8(b_4)$	230	2 0 $\overset{0}{2}$ 0 0 2 2	2 1 $\overset{1}{0}$ 1 2 0 2 2 0 $\overset{0}{2}$ 0 0 2 2	3 4	 
$E_7(a_1)$	228	2 1 $\overset{1}{0}$ 1 0 2 2	0 0 $\overset{2}{0}$ 2 2 2 2	5	
$E_8(a_5)$	228	2 0 $\overset{0}{2}$ 0 0 2 0	2 0 $\overset{0}{2}$ 0 0 2 0 0 0 $\overset{2}{2}$ 0 0 2 2	3 4	 

Induced orbits in type E_8					
$E_8(b_5)$	226	$0\,0\,\overset{0}{2}\,0\,0\,2\,2$	$0\,0\,\overset{0}{2}\,0\,0\,2\,2$	3	
			$0\,2\,\overset{0}{0}\,0\,2\,2\,2$	4	
			$2\,0\,\overset{0}{1}\,0\,2\,2\,2$	4	
D_7	226	$2\,1\,\overset{1}{0}\,1\,1\,0\,1$	$2\,1\,\overset{1}{0}\,1\,2\,0\,0$	2	
$E_7(a_2)$	224	$0\,1\,\overset{1}{0}\,1\,0\,2\,2$	$1\,0\,\overset{0}{1}\,0\,2\,2\,2$	3	
$E_8(a_6)$	224	$0\,0\,\overset{0}{2}\,0\,0\,2\,0$	$0\,0\,\overset{0}{2}\,0\,0\,2\,0$	2	
			$0\,2\,\overset{0}{0}\,0\,2\,0\,2$	3	
			$2\,0\,\overset{0}{1}\,0\,2\,0\,2$	3	
$D_7(a_1)$	222	$2\,0\,\overset{0}{0}\,2\,0\,0\,2$	$1\,0\,\overset{0}{1}\,0\,2\,0\,2$	2	
			$2\,0\,\overset{0}{0}\,2\,0\,0\,2$	3	
E_6+A_1	222	$1\,0\,\overset{0}{1}\,0\,1\,2\,2$	$1\,0\,\overset{0}{1}\,0\,1\,2\,2$	2	
$E_7(a_3)$	220	$2\,0\,\overset{0}{1}\,0\,1\,0\,2$	$0\,0\,\overset{0}{0}\,2\,2\,2\,2$	4	

Induced orbits in type E_8

$E_8(b_6)$	220	$\begin{smallmatrix} & & 0 & & & & \\ 0 & 0 & 2 & 0 & 0 & 0 & 2 \end{smallmatrix}$	$\begin{smallmatrix} & & 0 & & & & \\ 1 & 0 & 1 & 0 & 1 & 2 & 0 \end{smallmatrix}$	1	
			$\begin{smallmatrix} & & 0 & & & & \\ 0 & 0 & 2 & 0 & 0 & 0 & 2 \end{smallmatrix}$	2	
			$\begin{smallmatrix} & & 0 & & & & \\ 2 & 0 & 1 & 0 & 2 & 0 & 0 \end{smallmatrix}$	2	
$E_6(a_1)+A_1$	218	$\begin{smallmatrix} & & 0 & & & & \\ 1 & 0 & 1 & 0 & 1 & 0 & 2 \end{smallmatrix}$	$\begin{smallmatrix} & & 0 & & & & \\ 0 & 0 & 0 & 2 & 0 & 2 & 2 \end{smallmatrix}$	3	
A_7	218	$\begin{smallmatrix} & & 0 & & & & \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 \end{smallmatrix}$	$\begin{smallmatrix} & & 0 & & & & \\ 1 & 0 & 1 & 0 & 2 & 0 & 0 \end{smallmatrix}$	1	
E_6	216	$\begin{smallmatrix} & & 0 & & & & \\ 2 & 0 & 0 & 0 & 2 & 2 & 2 \end{smallmatrix}$	$\begin{smallmatrix} & & 0 & & & & \\ 2 & 0 & 0 & 0 & 2 & 2 & 2 \end{smallmatrix}$	4	
$D_7(a_2)$	216	$\begin{smallmatrix} & & 0 & & & & \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{smallmatrix}$	$\begin{smallmatrix} & & 0 & & & & \\ 0 & 0 & 0 & 2 & 0 & 2 & 0 \end{smallmatrix}$	2	
			$\begin{smallmatrix} & & 2 & & & & \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \end{smallmatrix}$	2	
D_6	216	$\begin{smallmatrix} & & 1 & & & & \\ 2 & 1 & 0 & 0 & 0 & 1 & 2 \end{smallmatrix}$	$\begin{smallmatrix} & & 1 & & & & \\ 2 & 1 & 0 & 0 & 0 & 1 & 2 \end{smallmatrix}$	2	
$E_6(a_1)$	214	$\begin{smallmatrix} & & 0 & & & & \\ 2 & 0 & 0 & 0 & 2 & 0 & 2 \end{smallmatrix}$	$\begin{smallmatrix} & & 0 & & & & \\ 2 & 0 & 0 & 0 & 2 & 0 & 2 \end{smallmatrix}$	3	
			$\begin{smallmatrix} & & 0 & & & & \\ 0 & 1 & 0 & 0 & 2 & 2 & 2 \end{smallmatrix}$	3	
D_5+A_2	214	$\begin{smallmatrix} & & 0 & & & & \\ 0 & 0 & 0 & 2 & 0 & 0 & 2 \end{smallmatrix}$	$\begin{smallmatrix} & & 0 & & & & \\ 2 & 0 & 1 & 0 & 1 & 0 & 0 \end{smallmatrix}$	1	
			$\begin{smallmatrix} & & 0 & & & & \\ 0 & 0 & 0 & 2 & 0 & 0 & 2 \end{smallmatrix}$	2	

Induced orbits in type E_8					
$E_7(a_4)$	212	$0\ 0\ \overset{0}{1}\ 0\ 1\ 0\ 2$	$0\ 1\ \overset{0}{0}\ 0\ 2\ 0\ 2$	2	
			$2\ 0\ \overset{0}{0}\ 1\ 0\ 1\ 2$	2	
A_6+A_1	212	$1\ 0\ \overset{0}{1}\ 0\ 1\ 0\ 0$	$0\ 0\ \overset{0}{2}\ 0\ 0\ 0\ 0$	1	
$D_6(a_1)$	210	$0\ 1\ \overset{1}{0}\ 0\ 0\ 1\ 2$	$0\ 0\ \overset{2}{0}\ 0\ 0\ 2\ 2$	3	
A_6	210	$2\ 0\ \overset{0}{0}\ 0\ 2\ 0\ 0$	$2\ 0\ \overset{0}{0}\ 0\ 2\ 0\ 0$	2	
$E_8(a_7)$	208	$0\ 0\ \overset{0}{0}\ 2\ 0\ 0\ 0$	$0\ 0\ \overset{0}{0}\ 2\ 0\ 0\ 0$	1	
			$0\ 1\ \overset{0}{0}\ 0\ 2\ 0\ 0$	1	
			$0\ 2\ \overset{0}{0}\ 0\ 0\ 0\ 2$	2	
			$2\ 0\ \overset{0}{1}\ 0\ 0\ 0\ 2$	2	
D_5+A_1	208	$1\ 0\ \overset{0}{0}\ 1\ 0\ 1\ 2$	$0\ 0\ \overset{0}{1}\ 0\ 0\ 2\ 2$	2	
$E_7(a_5)$	206	$0\ 0\ \overset{0}{1}\ 0\ 1\ 0\ 0$	$1\ 0\ \overset{0}{1}\ 0\ 0\ 0\ 2$	1	
			$0\ 0\ \overset{0}{1}\ 0\ 0\ 2\ 0$	1	
$D_6(a_2)$	204	$0\ 1\ \overset{1}{0}\ 0\ 0\ 1\ 0$	$2\ 0\ \overset{0}{1}\ 0\ 0\ 0\ 1$	1	
$E_6(a_3)+A_1$	204	$1\ 0\ \overset{0}{0}\ 1\ 0\ 1\ 0$	$0\ 1\ \overset{0}{0}\ 0\ 1\ 0\ 2$	1	
D_5	200	$2\ 0\ \overset{0}{0}\ 0\ 0\ 2\ 2$	$2\ 0\ \overset{0}{0}\ 0\ 0\ 2\ 2$	3	
$E_6(a_3)$	198	$2\ 0\ \overset{0}{0}\ 0\ 0\ 2\ 0$	$2\ 0\ \overset{0}{0}\ 0\ 0\ 2\ 0$	2	
			$2\ 0\ \overset{0}{0}\ 0\ 1\ 0\ 2$	2	

Induced orbits in type E_8

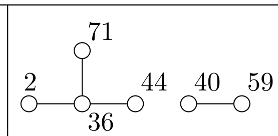
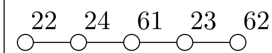
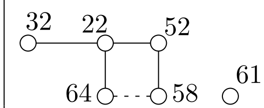
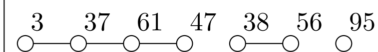
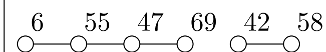
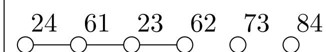
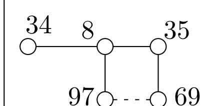
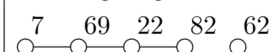
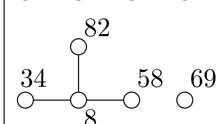
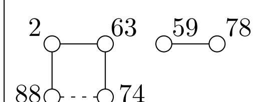
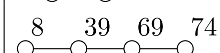
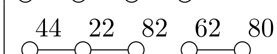
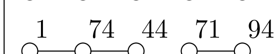
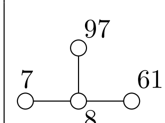
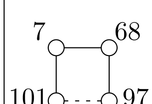
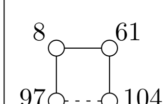
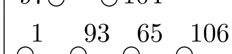
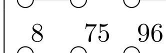
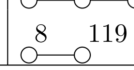
$D_4{+}A_2$	198	0 0 $\overset{2}{0}$ 0 0 0 2	0 0 $\overset{2}{0}$ 0 0 0 2	2	
A_5	196	2 0 $\overset{0}{0}$ 0 1 0 1	2 0 $\overset{0}{0}$ 0 1 0 1	1	
$D_5(a_1){+}A_1$	196	0 0 $\overset{0}{1}$ 0 0 0 2	0 0 $\overset{0}{1}$ 0 0 0 2	1	
$A_4{+}A_2{+}A_1$	196	0 1 $\overset{0}{0}$ 0 1 0 0	0 2 $\overset{0}{0}$ 0 0 0 0	1	
$A_4{+}A_2$	194	0 0 $\overset{0}{0}$ 0 2 0 0	0 0 $\overset{0}{0}$ 0 2 0 0	1	
$A_4{+}2A_1$	192	0 0 $\overset{0}{1}$ 0 0 0 1	2 0 $\overset{0}{0}$ 1 0 0 0	1	
$D_5(a_1)$	190	1 0 $\overset{0}{0}$ 0 1 0 2	0 0 $\overset{1}{0}$ 0 0 2 2	2	
$A_4{+}A_1$	188	1 0 $\overset{0}{0}$ 0 1 0 1	0 0 $\overset{1}{0}$ 0 0 2 0	1	
$D_4{+}A_1$	184	0 0 $\overset{1}{0}$ 0 0 1 2	0 0 $\overset{1}{0}$ 0 0 1 2	1	
$D_4(a_1){+}A_2$	184	0 0 $\overset{2}{0}$ 0 0 0 0	0 0 $\overset{2}{0}$ 0 0 0 0	1	
A_4	180	2 0 $\overset{0}{0}$ 0 0 0 2	2 0 $\overset{0}{0}$ 0 0 0 2	2	
$A_3{+}A_2$	178	1 0 $\overset{0}{0}$ 0 1 0 0	0 1 $\overset{0}{0}$ 0 0 0 2	1	
			2 0 $\overset{0}{0}$ 0 0 1 0	1	
D_4	168	0 0 $\overset{0}{0}$ 0 0 2 2	0 0 $\overset{0}{0}$ 0 0 2 2	2	
$D_4(a_1)$	166	0 0 $\overset{0}{0}$ 0 0 2 0	0 0 $\overset{0}{0}$ 0 0 2 0	1	
			0 0 $\overset{0}{0}$ 0 1 0 2	1	
$2A_2$	156	2 0 $\overset{0}{0}$ 0 0 0 0	2 0 $\overset{0}{0}$ 0 0 0 0	1	
A_3	148	1 0 $\overset{0}{0}$ 0 0 0 2	0 0 $\overset{1}{0}$ 0 0 0 2	1	
A_2	114	0 0 $\overset{0}{0}$ 0 0 0 2	0 0 $\overset{0}{0}$ 0 0 0 2	1	

Table 9: Induced nilpotent orbits in the Lie algebra of type G_2.

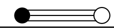
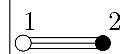
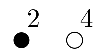
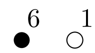
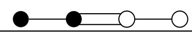
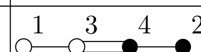
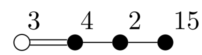
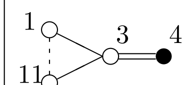
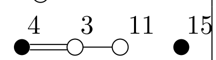
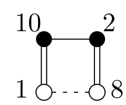
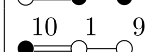
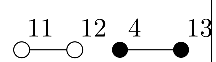
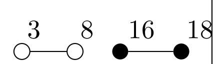
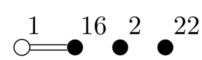
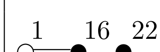
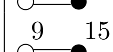
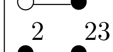
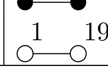
label	dim	characteristic	sheet diagram	rk	representative
					
G_2	12	2 2	2 2	2	
$A_1+\widetilde{A}_1$	10	2 0	2 0	1	
			0 2	1	

Table 10: Induced nilpotent orbits in the Lie algebra of type F_4.

label	dim	characteristic	sheet diagram	rk	representative
					
F_4	48	2 2 2 2	2 2 2 2	4	
$F_4(a_1)$	46	2 2 0 2	2 2 2 0	3	
			0 2 2 2	3	
$F_4(a_2)$	44	0 2 0 2	0 2 2 0	2	
			2 0 1 2	2	
B_3	42	2 2 0 0	2 2 0 0	2	
C_3	42	1 0 1 2	0 0 2 2	2	
$F_4(a_3)$	40	0 2 0 0	0 2 0 0	1	
			0 0 2 0	1	
			2 0 0 2	2	
$C_3(a_1)$	38	1 0 1 0	0 1 0 2	1	
B_2	36	2 0 0 1	2 0 0 1	1	
A_2	30	2 0 0 0	2 0 0 0	1	
$\widetilde{A}_2$	30	0 0 0 2	0 0 0 2	1	

5. Concluding Remarks

5.1 We remark that from the tables several things can be read off. For example, a sheet is said to be a *Dixmier sheet* if it contains semisimple elements. We recall that an orbit which is induced from the zero orbit of a Levi subalgebra is called a *Richardson orbit*. In other words, for a Richardson orbit Ge there exists a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$, with nilradical $\mathfrak{n}$ such that $Ge \cap \mathfrak{n}$ is dense in $\mathfrak{n}$. In this case elements of $Ge \cap \mathfrak{n}$ are called *Richardson elements* of $\mathfrak{n}$. It is known (cf. [3]) that a sheet is Dixmier if and only if it corresponds to a Richardson orbit. In other words, a sheet is Dixmier if and only if the sheet diagram does not contain any labels 1.

5.2 Let $x \in \mathfrak{g}$ be nilpotent. A parabolic subgroup $P \subset G$ is said to be a *polarization* of x if x is a Richarson element of the nilradical of the Lie algebra $\mathfrak{p}$ of P. In [14] and [16] all polarizations of the Richardson orbits are determined, in Lie algebras of classical type. Our tables give all polarizations of the Richardson orbits of the Lie algebras of exceptional type.

5.3 Furthermore, we note that the dimension of a sheet is easily determined from our tables. Indeed, from [4], §5.7, Korollar (c), it follows that the dimension of the sheet is equal to the dimension of the induced orbit plus the rank of the sheet. So we get the dimension of the sheet by adding the numbers in the second and fifth columns.

5.4 In [5], [7] the diagram of a nilpotent element is called admissible if

- the roots corresponding to the nodes are linearly independent,
- every cycle in the diagram has an even number of nodes.

By inspection it can be seen that all diagrams in our tables are admissible, except in five cases: one for E_7, three for E_8 and one for F_4 (in which cases the diagram has a cycle with an odd number of nodes).

5.5 In [1], [2] it was shown that every Richardson orbit in a Lie algebra of classical type (over a field of good characteristic) has a representative x, lying in the nilradical of a parabolic subalgebra, with

$$x = \sum_{\alpha \in \Gamma} x_\alpha,$$

where $\Gamma \subset \Phi$. Furthermore, it was shown that the representative x can be chosen such that the size of Γ is equal to rank $\mathfrak{g}$ minus the dimension of a maximal torus of $C_{\mathfrak{g}}(x)$ (which is the minimal size Γ can have). By going through the tables given here it is readily verified that this same statement holds for all induced nilpotent orbits (and hence for all Richardson orbits) of Lie algebras of exceptional type, in characteristic 0. We believe that this also holds for all induced orbits in Lie algebras of classical type (and some hand and computer calculations support this). A proof of this is beyond the scope of the present paper, and will be a theme for further research.

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