A SMALL TRIVIA ABOUT MONIC POLYNOMIALS OF SECOND DEGREE WITH POSITIVE INTEGER COEFFICIENTS

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Abstract. In this paper we investigate the problem of simultaneous factorization of second degree polynomials with positive integer coefficients.

1. Introduction

If we consider a second degree polynomial with integer coefficients

$$p(x) = x^2 + qx + p$$

which is reducible in $\mathbb{Z}[x]$ it is well possible that even the polynomial

$$q(x) = x^2 + px + q$$

is reducible in $\mathbb{Z}[x]$ as well. For instance, if $p > 0$ then

$$p(x) = x^2 - (p+1)\,x + p$$
$$q(x) = x^2 + px - (p+1)$$

are both reducible in $\mathbb{Z}[x]$. But what about a polynomial

$$p(x) = x^2 + qx + p$$

where both p and q are positive and $p < q$? We will prove that, with this condition, only the polynomial

$$p(x) = x^2 + 6x + 5$$

has the required property.

2. The result

Theorem 1. *If p, q are positive integers then*

$$p(x) = x^2 + qx + p$$
$$q(x) = x^2 + px + q$$

are both reducible in $\mathbb{Z}[x]$ if and only if $p = 5, q = 6$.

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Proof. The "if" is, of course, trivial. For the "only if", let us consider the polynomial

$$p(x) = x^2 + qx + p$$

If $p(x)$ is reducible, then for a suitable divisor d of p we must have that

$$q = d + \frac{p}{d}.$$

We can always assume that $1 \le d \le \sqrt{p}$. We split the proof in two cases

(1) For first we consider the case $d = 1$. We have that

$$p(x) = x^2 + (p+1)\,x + p$$
$$q(x) = x^2 + px + (p+1)$$

thus, from $q(x)$ we have that

$$p = d' + \frac{p+1}{d'}$$

where d' is a suitable divisor of $p+1$ and we can suppose, with generality, that

$$2 \leqslant d' \leqslant \sqrt{p+1}.$$

But, for each p and each d' it is

$$d' + \frac{p+1}{d'} \leqslant \sqrt{p+1} + \frac{p+1}{2}$$

and

$$\sqrt{p+1} + \frac{p+1}{2} < p$$

is true as soon as $p \ge 7$. Hence, we have only to check the values $p = 1, 2, 3, 4, 5, 6$ and among them we find that the only acceptable value is $p = 5$.

(2) Now we assume that $2 \le d \le \sqrt{p}$. In this case, from $p(x)$ we have that

$$q = d + \frac{p}{d} \leqslant \sqrt{p} + \frac{p}{2}$$

while, from $q(x)$, it must be

$$p = d' + \frac{q}{d'}$$

for a suitable divisor of q. Thus, it must be

$$p = d' + \frac{q}{d'} \leqslant \sqrt{q} + q \leqslant \sqrt{\sqrt{p} + \frac{p}{2}} + \sqrt{p} + \frac{p}{2}$$

which is false as soon as $p > 15$. Hence, we have to check only the cases $p = 1, ..., 14$ and, among them, we cannot find any further polynomial. This proves the result.

$\square$

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