

A NOTE ON THE CACTUS RANK FOR SEGRE-VERONESE VARIETIES

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ABSTRACT. We give an upper bound for the cactus rank of any multi-homogeneous polynomial.

1. INTRODUCTION

The notion of cactus rank for homogeneous polynomials was first introduced in the literature by A. Iarrobino and V. Kanev in their famous book [14, Definition 5.1, Definition 5.66] with the name of “scheme length”. The expression “cactus rank” was first used in [5, 17] after the pioneering work of W. Buczyńska and J. Buczyński [6] where the authors introduced the notion of cactus varieties.

Let \mathbb{K} be an algebraically closed field and let $S = \mathbb{K}_{DP}[x_0, \dots, x_n]$ be the divided power ring of polynomials in $n + 1$ variables over \mathbb{K} (if $\text{char}(\mathbb{K}) = 0$ one can work with the standard polynomial ring – we refer to [14, Appendix A], [4, Section 2.1], [9, Appendix 2] for the theory on divided power rings). Denote by S_d the component of degree d in S and let ν_d denote the d -th Veronese embedding $\nu_d : \mathbb{P}^n \rightarrow \mathbb{P}(S_d) = \mathbb{P}^{\binom{n+d}{d}-1}$ given by $[L] \mapsto [L^{[d]}]$ (for $f \in S$ homogeneous, $[f]$ denotes its class in the corresponding projective space), namely the embedding of \mathbb{P}^n defined by the line bundle $\mathcal{O}(d)$; its image $\nu_d(\mathbb{P}^n) := \{[L^{[d]}] : L \in S_1\} \subseteq \mathbb{P}S_d$ is the so-called Veronese variety. Given a homogeneous polynomial $f \in S_d$ (here we assume $d > \text{char}(\mathbb{K})$ if \mathbb{K} has positive characteristic), the *cactus rank* of f is the minimum length of a zero-dimensional scheme Z contained in $\nu_d(\mathbb{P}^n)$ such that $[f] \in \langle Z \rangle$.

Equivalently, the cactus rank of f is the minimum length of a zero-dimensional scheme $\Gamma \subseteq \mathbb{P}^n$ that is *apolar* to f , in the sense of [14, Def. 1.11]. In this setting $Z = \nu_d(\Gamma)$. The scheme Γ is called a minimal apolar scheme to f and it has been extensively studied in the recent years (see [14, 6, 5, 17, 4, 3, 7]). In particular in [5, 4] the authors explore the behaviors for the cactus rank in the case of cubics polynomials, while in [17] the monomial case is studied.

The notion of cactus rank can be defined with respect to an arbitrary algebraic variety (see e.g. [11]). In this work, we focus on the setting of partially symmetric tensors, which has been studied also in [1, 12, 18, 13], where the authors used a natural extension of the notion of apolarity, and in [10, 11], where the author widened this notion to an even more general setting, and provided equations for cactus varieties via vector bundle techniques (see e.g. [16, 18]).

Given positive integers s, n_1, \dots, n_s , let $R = \mathbb{K}_{DP}[x_{i,j} : j = 1, \dots, s, i = 0, \dots, n_j]$ be a polynomial ring in $\sum(n_j + 1)$ variables. Given a multidegree $\mathbf{d} = (d_1, \dots, d_s)$, write $R_{\mathbf{d}}$ for the multihomogeneous component of multidegree \mathbf{d} , i.e. $f \in R_{\mathbf{d}}$ if, for every j , f is homogeneous of degree $d = d_1 + \dots + d_s$ in R and in addition it is homogeneous of degree d_j in the j -th set of variables $\{x_{i,j} : i = 0, \dots, n_j\}$.

Let $Y = \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_s}$, where we may regard $\mathbb{P}^{n_j} = \mathbb{P}(R_{e_j})$ with $e_j = (0, \dots, 0, 1, 0, \dots, 0)$ (1 being at the j -th entry); the Segre-Veronese embedding of multidegree $\mathbf{d} = (d_1, \dots, d_s)$ is the

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map $\nu_{\mathbf{d}} : Y \rightarrow \mathbb{P}R_{\mathbf{d}} = \mathbb{P}^r$ where $r + 1 = \prod_{i=1}^s \binom{n_i + d_i}{n_i}$, namely the embedding of Y defined by the line bundle $\mathcal{O}_Y(\mathbf{d})$ (notice $R_{\mathbf{d}} = H^0(\mathcal{O}_Y(\mathbf{d}))$). The notion of cactus rank extends to this setting as follows.

Definition 1.1. Let $X := \nu_{\mathbf{d}}(Y) \subset \mathbb{P}^r$ be the Segre-Veronese variety, $r + 1 = \prod_{i=1}^s \binom{n_i + d_i}{n_i}$ (here $\sum d_i < \text{char}(\mathbb{K})$ if we work in positive characteristic). For any $f \in \mathbb{P}^r$ the cactus rank $cr_{\mathbf{d}}(f)$ is the minimal length of a zero-dimensional scheme $Z \subset X$ such that $f \in \langle Z \rangle$.

In this work, we give an upper bound on the maximum possible value of $cr_{\mathbf{d}}$ in terms of d_1, \dots, d_s and n_1, \dots, n_s , or equivalently on the value of $cr_{\mathbf{d}}$ for every $f \in R_{\mathbf{d}}$ (Theorem 1.2).

Possible approaches to this problem include the use of the notion of Hankel operators as it is applied in [2] to partially symmetric tensors, or of extended notions of apolarity as in [13, 12, 10]. Our approach is based on the one followed by M. V. Catalisano, A. V. Geramita and A. Gimigliano in [8], where the authors show that the Hilbert function of a certain number of double fat points on the Segre-Veronese variety coincides with the Hilbert function of a scheme contained in \mathbb{P}^n consisting of fat points and linear spaces, for $n = n_1 + \dots + n_s$. This will allow us to give an explicit upper bound in the case of Segre-Veronese varieties.

Following [8] we construct an embedding

$$u : R_{\mathbf{d}} = H^0(\mathcal{O}_Y(d_1, \dots, d_s)) \rightarrow H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$$

with $n := n_1 + \dots + n_s$ and $d := d_1 + \dots + d_s$, which will allow us to extend the results on the cactus rank of homogeneous polynomials to the case of multi-homogeneous polynomials.

Our main result is the following:

Theorem 1.2. *For any multi-homogeneous polynomial $f \in \mathbb{P}^r \simeq \mathbb{P}R_{\mathbf{d}}$ with $r+1 = \prod_{i=1}^s \binom{n_i + d_i}{n_i}$, $n := n_1 + \dots + n_s$ and $d := d_1 + \dots + d_s$, we have that*

- if d is even, say $d = 2k + 2$ for some $k \in \mathbb{N}$, then $cr_{\mathbf{d}}(f) \leq \binom{n+k}{k} + \binom{n+k+1}{k+1}$,
- if $d = 2k + 1$ is odd we have $cr_{\mathbf{d}}(f) \leq 2\binom{n+k}{k}$.

Remark 1.3. We compare our result with some known results on this topic. For $s = 1$, we obtain an upper bound for the cactus rank in the symmetric case: for $d = 3$, Thm. 1.2 recovers the result of [5] on cubic forms; for higher values of d , Thm. 1.2 does not give an improvement with respect to the bound provided by [17].

In the tensor case (when $d_i = 1$ for $i = 1, \dots, s$), we obtain more interesting results: if s is odd and $m = n_i$ for every i , Thm. 1.2 provides the upper bound $2\binom{m + \lfloor s/2 \rfloor}{\lfloor s/2 \rfloor}$ which grows as a polynomial of degree $\lfloor s/2 \rfloor$ in ms ; a similar growth is attained in the case where s is even. In particular, in the case $s = 3$, we obtain that the maximum value of $cr_{1,1,1}$ grows with leading term $6m$ which was already known to Buczyński (personal communication, 2016); we observe that a bound with the same leading term can be obtained as a straightforward application of the result of [5]. For every fixed value of s , Thm. 1.2 shows that the maximum possible cactus rank grows as a polynomial of degree $s/2$ in m (a lower bound with this growth is immediate via flattening techniques), whereas the maximum possible value for the tensor rank (see e.g. [15]) grows as a polynomial of degree $s - 1$ in m .

We point out that during the referee process of this paper we have been informed that M. Gałazka is currently working on the same topic and he independently obtained bounds similar to the ones of Thm. 1.2 using different techniques.

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2. MAP CONSTRUCTION

Consider the set of variables $\{z_0\} \cup \{z_{i,j}\}_{1 \leq j \leq s, 1 \leq i \leq n_j}$ and regard them, as homogeneous coordinates on \mathbb{P}^n , with $n = n_1 + \dots + n_s$; let $S := \mathbb{K}_{\text{DP}}[z_0, z_{i,j}]_{1 \leq j \leq s, 1 \leq i \leq n_j}$ be the corresponding polynomial ring and let S_d be its component of degree d . For any fixed $j = 1, \dots, s$, let $M_j \subset \mathbb{P}^n$ be the linear subspace with equations $z_0 = 0$ and $z_{i,h} = 0$ for all i and for $h \in \{1, \dots, s\} \setminus \{j\}$. Note that the only non-zero variables on M_j are $z_{1,j}, \dots, z_{n_j,j}$ and so $\dim M_j = n_j - 1$. For every j , let $L_j := (d - d_j)M_j \subset \mathbb{P}^n$ be the scheme defined by the $(d - d_j)$ -th power $(I(M_j))^{(d - d_j)}$ of the ideal of M_j . The equations of L_j are all monomials of degree $d - d_j$ in the variables $\{z_0, z_{i,h}\}_{h \in \{1, \dots, s\} \setminus \{j\}}$.

Let

$$\mathcal{A} := H^0(\mathbb{P}^n, \mathcal{I}_{L_1 \cup \dots \cup L_s}(d)).$$

Following [8], we construct an injective linear map $u : R_{\mathbf{d}} \rightarrow H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = S_d$ whose image is \mathcal{A} . The map u is the evaluation map defined by

$$(1) \quad \begin{aligned} u(x_{0,j}) &= z_0 & \text{for } j = 1, \dots, s \\ u(x_{i,j}) &= z_{i,j} & \text{for } j = 1, \dots, s, i = 1, \dots, n_s. \end{aligned}$$

In particular, $u : R_{\mathbf{d}} \rightarrow S_{d_1 + \dots + d_s}$ is obviously linear on the graded components of interest.

3. PROOF OF THE MAIN THEOREM

Lemma 3.1. *The evaluation $u : R_{\mathbf{d}} \rightarrow S_d$ is injective.*

Proof. Take $f \in R_{\mathbf{d}}$ and write

$$f = \sum_{0 \leq a_1 \leq d_1, \dots, 0 \leq a_s \leq d_s} x_{0,1}^{a_1} \cdots x_{0,s}^{a_s} F_{a_1, \dots, a_s},$$

with $F_{a_1, \dots, a_s} \in R_{d_1 - a_1, \dots, d_s - a_s}$ and no $x_{0,j}$ appearing in F_{a_1, \dots, a_s} , so that, for every j , F_{a_1, \dots, a_s} is homogeneous of degree $d_j - a_j$ in $\{x_{i,j}\}_{i=1, \dots, n_j}$. Therefore

$$u(f) = \sum_{0 \leq a_1 \leq d_1, \dots, 0 \leq a_s \leq d_s} z_0^{a_1 + \dots + a_s} F_{a_1, \dots, a_s}(z_{ij}),$$

where $F_{a_1, \dots, a_s}(z_{ij})$ is just the image of the evaluation $x_{i,j} \mapsto z_{i,j}$. Now, if $u(f) = 0$, then for every $\rho \in \{0, \dots, d\}$, we have $\sum_{a_1 + \dots + a_s = \rho} F_{a_1, \dots, a_s}(z_{ij}) = 0$; this is a sum of linearly independent terms, because the summands have different multidegree in the variables $\{z_{i,j}\}$. This shows $F_{a_1, \dots, a_s}(z_{ij}) = 0$ for every choice of (a_1, \dots, a_s) and therefore $f = 0$. \square

Lemma 3.2. *The image of $u : R_{\mathbf{d}} \rightarrow S_d$ is \mathcal{A} .*

Proof. Recall $\mathcal{A} = H^0(\mathbb{P}^n, \mathcal{I}_{L_1 \cup \dots \cup L_s}(d))$. We first prove that $u(R_{\mathbf{d}}) \subseteq \mathcal{A}$. Fix $f \in R_{\mathbf{d}}$ and $j \in \{1, \dots, s\}$. It is sufficient to prove that $u(f) \in H^0(\mathbb{P}^n, \mathcal{I}_{L_j}(d))$ for every j . Since f has degree d_j with respect to the variables $\{x_{0,j}, x_{1,j}, \dots, x_{n_j,j}\}$, every monomial appearing with non-zero coefficient in f has degree at most d_j with respect to $\{x_{1,j}, \dots, x_{n_j,j}\}$; therefore each monomial appearing with non-zero coefficient in $u(f)$ has degree at most d_j with respect to $\{z_{1,1}, \dots, z_{n_j,j}\}$ and hence degree at least $d - d_j$ with respect to the other homogeneous coordinates of \mathbb{P}^n . This is equivalent to saying that $u(f) \in H^0(\mathbb{P}^n, \mathcal{I}_{L_j}(d))$ by the definitions of M_j and of L_j . The inclusion $u(H^0(\mathcal{O}_Y(\mathbf{d}))) \supseteq \mathcal{A}$, now follows from the fact that $u(H^0(\mathcal{O}_Y(\mathbf{d})))$ and \mathcal{A} have the same dimension, which is a consequence of the injectivity of u (Lemma 3.1), and of the fact that $R_{\mathbf{d}}$ and \mathcal{A} have the same dimension (case $Z = W = \emptyset$ of [8, Theorem 1.1]). \square

Remark 3.3. The map u induces an isomorphism between the open set $\mathbb{A}^n \subseteq \mathbb{P}^n$ with $z_0 \neq 0$ and the open set $U \subseteq Y$ with $x_{0,j} \neq 0$. Explicitly, this isomorphism is

$$U \rightarrow \mathbb{A}^n \\ ((1, \xi_{1,1}, \dots, \xi_{n_1,1}), \dots, (1, \xi_{1,s}, \dots, \xi_{n_s,s})) \mapsto (\xi_{i,j} : i = 1, \dots, n_j, j = 1, \dots, s),$$

where $(\xi_{1,1}, \dots, \xi_{n_j,1})$ are coordinates on the open set $\mathbb{A}^{n_j} \subseteq \mathbb{P}^{n_j}$ of the j -th factor of Y . In particular, any zero-dimensional scheme $W \subset \mathbb{A}^n = U \subset \mathbb{P}^n$ corresponds isomorphically to a zero-dimensional scheme $Z \subset Y$ of the same degree.

Proof of Theorem 1.2. Let $\nu_d : \mathbb{P}^n \rightarrow \mathbb{P}^N$, $N = \binom{n+d}{n} - 1$, be the order d Veronese embedding. The point f is (the class of) a multi-homogeneous polynomial; let $p = u(f) \in \mathbb{P}^N$ be its image via u , so that, from Lemma 3.2, $p \in \mathbb{P}\mathcal{A}$. Following [5], let $N_d = \binom{n+k}{k} + \binom{n+k+1}{k+1}$ if $d = 2k + 2$ is even) and $N_d = 2\binom{n+k}{k}$ if $d = 2k + 1$ is odd. Theorem 3 in [5] shows that $cr(p) \leq N_d$ and in fact that there is a punctual (i.e. connected) zero-dimensional scheme $W \subset \mathbb{A}^n \subseteq \mathbb{P}^n$ such that $\deg(W) \leq N_d$ and $p \in \langle \nu_d(W) \rangle$. If \mathbb{K} has positive characteristic, (in this case we assume $\text{char}(\mathbb{K}) > d$ as in Definition 1.1) we follow [4, Section 2.1 and Section 4]. Let $Z = u^{-1}(W) \subset \mathbb{A}^n \subset Y$ be the corresponding zero-dimensional scheme in Y obtained via the isomorphism induced by u . We have $p = u(f) \in \langle \nu_d(W) \rangle$ and therefore $f \in \langle \nu_d(Z) \rangle$. \square

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