A LIE THEORETICAL CONSTRUCTION OF A LANDAU–GINZBURG MODEL WITHOUT PROJECTIVE MIRRORS

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ABSTRACT. We describe the Fukaya–Seidel category of a Landau–Ginzburg model LG(2) for the semisimple adjoint orbit of $\mathfrak{sl}(2,\mathbb{C})$. We prove that this category is equivalent to a full triangulated subcategory of the category of coherent sheaves on the second Hirzebruch surface. We show that no projective variety can be mirror to LG(2), and that this remains so after compactification.

CONTENTS

1. INTRODUCTION

We describe the Fukaya–Seidel category corresponding to a Landau–Ginzburg model for the semisimple adjoint orbit of $\mathfrak{sl}(2,\mathbb{C})$. This is the simplest application of the following general result:

Theorem 1.1. [8, Thm. 3.1] *Let $\mathfrak{h}$ be the Cartan subalgebra of a complex semisimple Lie algebra $\mathfrak{g}$. Given $H_0 \in \mathfrak{h}$ and $H \in \mathfrak{h}_\mathbb{R}$ with H a regular element, the height function $f_H : \mathcal{O}(H_0) \to \mathbb{C}$ defined by*

$$f_H(x) = \langle H, x \rangle, \qquad x \in \mathcal{O}(H_0)$$

has a finite number ($= |\mathcal{W}|/|\mathcal{W}_{H_0}|$) of isolated singularities and gives $\mathcal{O}(H_0)$ the structure of a symplectic Lefschetz fibration.

Here $\mathcal{O}(H_0)$ denotes the adjoint orbit of H_0 viewed as a symplectic submanifold of $\mathfrak{sl}(2,\mathbb{C})$ with the symplectic form

$$(1.1) \qquad\qquad \Omega = \operatorname{im}\mathcal{H},$$

where $\mathcal{H}$ is the Hermitian form on $\mathfrak{g}$ defined by

$$\mathcal{H}(u,v) = \langle u, Jv \rangle,$$

for J any almost complex structure and $\langle \cdot, \cdot \rangle$ denoting the Cartan–Killing form. In the following we will take J to be multiplication by i coordinatewise.

In the language of mirror symmetry, f_H is called a superpotential.

Notation 1.2. Let us denote by LG(2) the Landau–Ginzburg model formed by the pair (X, f_H) where $X := \mathcal{O}(H_0)$ is the semisimple orbit of $\mathfrak{sl}(2, \mathbb{C})$ for $H_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ considered as a symplectic manifold with the symplectic form as in (1.1) and given the structure of a symplectic Lefschetz fibration by the superpotential $f_H \colon X \to \mathbb{C}$ for the choice $H = H_0$.

We calculate the category of Lagrangian vanishing cycles for LG(2) and obtain:

Theorem 3.1. *The Fukaya–Seidel category* Fuk(LG(2)) *is generated by two Lagrangians* L_0 *and* L_1 *with morphisms:*

$$\mathrm{Hom}(L_i, L_j) \simeq \begin{cases} \mathbb{Z} \oplus \mathbb{Z}[-1] & i < j \\ \mathbb{Z} & i = j \\ 0 & i > j \end{cases}$$

and the products m_k *all vanish except for* $m_2(\cdot, \mathrm{id})$ *and* $m_2(\mathrm{id}, \cdot)$.

We then consider the question of finding a mirror to LG(2). That is, we look for an algebraic variety Y such that its derived category of coherent sheaves $D^b(\mathrm{Coh}\, Y)$ is equivalent to the Fukaya–Seidel category of LG(2). We first obtain a negative result.

Theorem 4.1. LG(2) *has no projective mirrors.*

This came to us as a surprise and brought along the question of whether the absence of projective mirrors might have resulted of the noncompactness of LG(2). We then compactified LG(2) to a new model $\overline{\mathrm{LG}}(2)$ where we extend the potential to a map with target $\mathbb{P}^1$. However, for the compactified $\overline{\mathrm{LG}}(2)$, the absence of projective mirrors persists:

Theorem 7.6. $\overline{\mathrm{LG}}(2)$ *has no projective mirrors.*

The next best thing to do then is to find some projective variety Y such that a proper subcategory of $D^b(\mathrm{Coh}\, Y)$ is equivalent to Fuk(LG(2)). We find that an appropriate choice is $Y = F_2$, the second Hirzebruch surface.

Theorem 8.1. Fuk(LG(2)) *is equivalent to the full triangulated subcategory* $D^b(\eth\mathrm{J}(2)) := \langle \mathcal{O}_{F_2}, \mathcal{O}_{F_2}(-E) \rangle$ *of* $D^b(\mathrm{Coh}\, F_2)$, *where* F_2 *is the second Hirzebruch surface and* E *is the divisor with self-intersection* -2.

We also describe these categories using quivers in §9.

Remark 1.3. It turns out that for the choices made in 1.2, we obtain an example already studied by Khovanov and Seidel in [13], where they describe the Fukaya category of the Milnor fibration corresponding to an A_m singularity. More precisely, they consider deformations of the A_m singularities, and their case $m = 1$ happens to be algebraically and symplectically isomorphic to the adjoint orbit $\mathcal{O}(H_0)$ of $\mathfrak{sl}(2, \mathbb{C})$, see §2.

In our approach outlined above, we use Lie theory to define a potential on $\mathcal{O}(H_0)$, making it into a Landau–Ginzburg model, and then obtain information about the mirror category.

In future work we intend to consider the cases of adjoint orbits of the Lie algebras $\mathfrak{sl}(n, \mathbb{C})$ with $n > 2$. Then, the corresponding spaces will not be deformations of A_m singularities of Remark 1.3, since such adjoint orbits have dimension strictly greater than 2.

We hope that illustrating this simplest case using an alternative or complementary approach will lead to further study of symplectic Lefschetz fibrations and their Fukaya–Seidel categories using techniques from Lie theory. In light of Theorem 1.1, this approach can indeed be formulated for all semisimple Lie algebras and we hope that it will lead to further results in mirror symmetry.

2. THE STRUCTURES OF LG(2)

In this section we describe the Landau–Ginzburg model $LG(2) = (\mathcal{O}(H_0), f_H)$ defined in 1.2 corresponding to the choices $H_0 = H = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and symplectic form $\Omega(A, B) = \text{im}\langle A, iB\rangle$.

ALGEBRAIC STRUCTURE. Set $X := \mathcal{O}(H_0)$. Given $A = \left(\begin{smallmatrix} x & y \\ z & -x \end{smallmatrix}\right) \in X \subset \mathfrak{sl}(2, \mathbb{C})$, its eigenvalues are ± 1 and

$$(x - \lambda)(-x - \lambda) - yz = \det(A - \lambda I) = (\lambda + 1)(\lambda - 1) = \lambda^2 - 1.$$

Hence, X is the hypersurface in $\mathbb{C}^3$ cut out by the equation

$$(2.1) \qquad x^2 + yz - 1 = 0.$$

Since the derivatives of the polynomial $x^2 + yz - 1$ vanish simultaneously only at the origin which does not lie in X, it follows that X is a smooth complex surface.

We know that in the case of $\mathfrak{sl}(n, \mathbb{C})$, $\langle A, B\rangle$ is a constant multiple of $\text{tr}(AB)$. The choice of $H = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ gives the height function

$$f_H(A) = \text{tr}(HA) = \text{tr}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x & y \\ z & -x \end{pmatrix} = 2x.$$

So, we write the potential as

$$(2.2) \qquad \begin{aligned} f_H \colon X &\to \mathbb{C} \\ (x, y, z) &\mapsto 2x. \end{aligned}$$

SMOOTH STRUCTURE. X is not compact. In further generality, let $\mathfrak{u}$ be a real compact form of $\mathfrak{sl}(n, \mathbb{C})$, then [9, Thm. 2.1] proves that the semisimple adjoint orbit is diffeomorphic to the cotangent bundle of the generalized flag variety $\mathcal{O}(H_0) \cap i\mathfrak{u}$. For the orbit of $\mathfrak{sl}(2, \mathbb{C})$ the flag variety is $\mathbb{P}^1 \approx S^2$ and consequently we have the diffeomorphism $X \simeq T^* S^2$.

COMPLEX STRUCTURE. Let $Z_2 = \text{Tot}\,\mathcal{O}_{\mathbb{P}^1}(-2) = T^*\mathbb{P}^1$ with its canonical complex structure, and let $\tau \in H^1(Z_2, TZ_2)$ be a non-zero cohomology class. Denote by $\mathscr{L}_2(\tau)$ the complex deformation of Z_2 corresponding to τ, see [6, §4] or [5] for details. Observe that Z_2 is not an affine variety (as the nontrivial first cohomology shows), hence the complex structure of X cannot be isomorphic to that of Z_2. We claim that X is biholomorphic to $\mathscr{L}_2(\tau)$. In fact, the algebra of global functions of $\mathscr{L}_2(\tau)$ can be calculated via Čech cohomology using canonical transition functions as in [6], giving:[1]

$$\mathbb{C}[x, y, z]\big/\big((x+1)^2 - yz - 1\big).$$

The change of coordinates $(x, y) \mapsto (x - 1, -y)$ shows that $\mathscr{L}_2(\tau) \simeq X$ as affine varieties.

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[1]Details of the cohomology calculations are presented in [5]. Global functions on $\mathscr{L}_2(\tau)$ were also described by Tyurina in [17] in the context of resolutions of the A_1 singularity. The space $\mathscr{L}_2(\tau)$ had been studied earlier by Atiyah in [1].

SYMPLECTIC STRUCTURE. We have just shown that the diffeomorphism type of X is that of the cotangent bundle of a sphere. The next result shows that this sphere is a Lagrangian subvariety of X.

Lemma 2.1. *Consider the orbit X with the symplectic form Ω defined in (1.1), then $Y \subset X$ given by the real equation $p^2 + q^2 + r^2 = 1$ is a Lagrangian submanifold.*

Proof. Let $\mathfrak{u}$ be a real compact form of $\mathfrak{sl}(2,\mathbb{C})$. Here $\mathfrak{u}$ is the set of anti-Hermitian matrices with trace zero, thus $i\mathfrak{u}$ is the set of Hermitian matrices with trace zero. Note that the submanifold Y can be described as the intersection $Y = X \cap i\mathfrak{u}$. In fact, an arbitrary matrix $S \in i\mathfrak{u}$ has the form

$$S = \begin{pmatrix} r & -p+iq \\ -p-iq & -r \end{pmatrix},$$

with $p, q, r \in \mathbb{R}$. Since the orbit X consists of 2×2 complex matrices whose entries satisfy $x^2 + yz = 1$, we see that $S \in X$ if and only if its entries satisfy $p^2 + q^2 + r^2 = 1$.

The tangent space of Y at S is given by $T_S Y = \{[S, A] \mid A \in \mathfrak{u}\}$. Since $[i\mathfrak{u}, \mathfrak{u}] \subset i\mathfrak{u}$ and $\mathrm{tr}(MN)$ is real when $M, N \in i\mathfrak{u}$, we conclude that $\Omega_S([S, A], [S, B]) = 0$; thus Y is Lagrangian. $\square$

Remark 2.2. In greater generality, let $\mathfrak{u}$ be a real compact form of $\mathfrak{g}$. The intersection $\mathcal{O}(H_0) \cap i\mathfrak{u}$ is a generalized flag variety, and a similar argument shows that such a generalized flag variety is Lagrangian for the symplectic form Ω.

3. THE FUKAYA–SEIDEL CATEGORY OF LG(2)

In this section we prove:

Theorem 3.1. *The Fukaya–Seidel category of* LG(2) *is generated by two Lagrangians L_0 and L_1 with morphisms:*

$$(3.1) \qquad \mathrm{Hom}(L_i, L_j) \simeq \begin{cases} \mathbb{Z} \oplus \mathbb{Z}[-1] & i < j \\ \mathbb{Z} & i = j \\ 0 & i > j \end{cases}$$

where we think of $\mathbb{Z}$ as a complex concentrated in degree 0 and $\mathbb{Z}[-1]$ as its shift, concentrated in degree 1, and the products m_k all vanish except for $m_2(\cdot, \mathrm{id})$ and $m_2(\mathrm{id}, \cdot)$.

We will now describe the thimbles using branched covers. As described in §2 the orbit is $X = \{x^2 + yz = 1\}$ together with the potential

$$f_H \colon X \to \mathbb{C}$$
$$(x, y, z) \mapsto 2x.$$

For each regular value $c \in \mathbb{C}$ we have $f_H(A) = 2x = c$ and a corresponding regular fibre over c, to simplify notation we parametrize the regular fibres by $\lambda := c/2$, so

$$X_\lambda := \left\{ yz = 1 - \lambda^2 \right\}.$$

From the above description it is immediate that the singular fibres occur when $\lambda^2 = 1$. The singular fibres $X_{\pm 1} = f_H^{-1}(\pm 1) = \{yz = 0\}$ correspond to the critical points $(x, y, z) = (\pm 1, 0, 0)$ of the potential f_H.

We first consider the cut given by $y = z$ where we need to analyse the two branches of the square root $y = \pm\sqrt{1 - \lambda^2}$. We get the two curves

$$\left(\lambda, \pm\sqrt{1 - \lambda^2}, \pm\sqrt{1 - \lambda^2} \right) \xrightarrow{\lambda \to 1} (1, 0, 0).$$

Using these curves we want to write down the thimbles, that is, for each λ we wish to identify a circle in X parametrized by $\gamma(t)$ with $\gamma(0) = \left(\lambda, \sqrt{1 - \lambda^2}, \sqrt{1 - \lambda^2} \right)$ and $\gamma(\pi) =$

$\left(\lambda, -\sqrt{1-\lambda^2}, -\sqrt{1-\lambda^2}\right)$. For $0 \le t \le 2\pi$ we chose the thimble as:

$$\alpha_\lambda(t) = \left(\lambda, e^{it}\sqrt{1-\lambda^2}, e^{-it}\sqrt{1-\lambda^2}\right).$$

Thus, $\alpha_\lambda(t) \to (1,0,0)$ as $\lambda \to 1$ (so that $c \to 2$) and for a regular value λ the curve $\gamma(t) := \alpha_\lambda(t)$ is a Lagrangian circle on the fibre $f_H^{-1}(2\lambda)$. We fix the regular value $0 \in \mathbb{C}$, and consider the straight line joining 0 to the critical value 2; this is our choice of a *matching path*. Then the family of Lagrangian circles $\alpha_\lambda(t)$ is fibred over this matching path and produces the Lagrangian thimble. With a similar analysis we can produce the Lefschetz thimble associated to the critical value -2.

Consider now the thimbles over the union of the two matching paths (line joining the two critical values -2 and 2), the circles fibering over them result in a sphere Y inside the orbit X. As shown in Lemma 2.1 this sphere is Lagrangian in X.

We will now describe the Fukaya–Seidel category Fuk(LG(2)) associated to the Landau–Ginzburg model LG(2), which is the category of vanishing cycles defined as follows.

Definition 3.2. [3, Def. 3.1] A *directed category of vanishing cycles* associated to a Landau–Ginzburg model is an A_∞-category (over a coefficient ring R) with r objects $L_1, \ldots, L_r$ corresponding to the vanishing cycles (or more accurately, to the thimbles); the morphisms between the objects are given by

(3.2)
$$\mathrm{Hom}(L_i, L_j) = \begin{cases} CF^*(L_i, L_j) = R^{|L_i \cap L_j|} & \text{if } i < j \\ R \cdot \mathrm{id} & \text{if } i = j \\ 0 & \text{if } i > j \end{cases}$$

and the differential m_1, composition m_2 and higher order products m_k are defined in terms of Lagrangian Floer homology inside the regular fibre. See [3] for further details.

We fix the regular value $0 \in \mathbb{C}$ of our Landau–Ginzburg model and consider the line segments β and γ that join -2 to 0 and 0 to 2, respectively. The objects of the Fukaya–Seidel category are the two Lagrangian thimbles $L_0 := \alpha_{\beta(s)}(t)$ and $L_1 := \alpha_{\gamma(s)}(t)$ (abusing notation we consider as L_0 and L_1 only the vanishing cycles in the regular fibre X_0; in our case, both circles S^1).

Note that the vanishing cycles represent a single object in the Fukaya category of the regular fibre, but represent two distinct objects in the Fukaya–Seidel category of LG(2).

To specify the products in the category, we need to describe $CF^*(L_0, L_1)$. The regular fibre X_0 is homeomorphic to $\mathbb{C}^*$ and to the cylinder T^*S^1 via the map $g: \mathbb{C}^* \to T^*S^1$ given by

$$g(y) = \left(\frac{y}{|y|}, \ln|y|\right).$$

In the regular fibre the vanishing cycles can be parametrized by the curve $(0, e^{it}, e^{-it}) \in X_0$ by setting $\lambda = 0$ in the expressions for the thimbles. Moreover, Lemma 2.1 implies that L_0 (and L_1) is Lagrangian in X_0 and therefore by Weinstein's theorem we have that a tubular neighbourhood of L_0 is symplectomorphic to the cotangent bundle T^*S^1. In this situation the Floer homology is well known, see [2] and [7].

Lemma 3.3. $HF^*(L_0, L_1) \approx H^*(S^1; \mathbb{R})$.

We now fix a Morse function $f: S^1 \to \mathbb{R}$ with exactly two critical points. A critical point p of f with Morse index $\mathrm{ind}(p)$ defines a generator of degree $\deg(p) = n - \mathrm{ind}(p)$ in the Floer complex, where n is the dimension of the variety (in our case $\dim S^1 = 1$). Since we have chosen f with exactly two critical points, a minimum x_0 and a maximum x_1, the Morse indices are 0 and 1, respectively. We obtain:

Lemma 3.4. *There is a natural choice of grading such that* $\deg(x_0) = 0$ *and* $\deg(x_1) = 1$.

Since the product m_1 in the Fukaya–Seidel category is the differential of Floer homology, using Lemma 3.3, we obtain the following description of the products m_k:

Lemma 3.5. *The products m_k for the Fukaya–Seidel category of* LG(2) *all vanish, except for the trivial products $m_2(\mathrm{id}, \cdot)$ and $m_2(\cdot, \mathrm{id})$.*

Here, the strict unit id equals x_0, and the result follows from strict unitality and the degree considerations. Specifically, $m_2(x_1, x_1)$ has degree 2 and so it is zero. Moreover, strict unitality implies that the only possible non-zero products for $k > 2$ take only x_1 as argument, and $m_k(x_1, \ldots, x_1)$ is zero because it has degree $2 - k + k = 2$.

Remark 3.6. We compare with the mirror of $\mathbb{P}^1$. The Fukaya–Seidel category we just described is not isomorphic to the Fukaya–Seidel category of the mirror of $\mathbb{P}^1$ described in [3]. Indeed, although the number of objects, morphisms and products of the A_∞ structures coincide, the gradings are different, hence the categories are not equivalent. We give a more detailed argument for this in the proof of Theorem 4.1.

4. MIRROR CANDIDATES

We show that no projective variety is mirror to LG(2). In other words, suppose that we have a variety Y such that the bounded derived category $D^b(\mathrm{Coh}\, Y)$ of coherent sheaves on Y is equivalent to our Fukaya–Seidel category of Theorem 3.1. Thus, we would need to have that $D^b(\mathrm{Coh}\, Y)$ is generated by some $\mathscr{F}_0, \mathscr{F}_1 \in \mathrm{Coh}\, Y$ satisfying:

$$\mathrm{Hom}(\mathscr{F}_i, \mathscr{F}_j) \simeq \begin{cases} \mathbb{C} \oplus \mathbb{C}[-1] & i < j \\ \mathbb{C} & i = j \\ 0 & i > j. \end{cases}$$

We prove that any such variety Y cannot be projective. Hence:

Theorem 4.1. LG(2) *has no projective mirrors.*

Proof. We first argue that if $\dim Y = n > 1$ and Y is projective, then $D^b(\mathrm{Coh}\, Y)$ cannot be generated by two simple objects $\mathscr{L}_0, \mathscr{L}_1$ such that $\mathrm{Hom}(\mathscr{L}_i, \mathscr{L}_i) = \mathbb{C}$ for $i = 0, 1$.

We will use the following facts. First, if $\mathscr{C}$ is an abelian category, such as $\mathrm{Coh}\, Y$ for any scheme Y, then the Grothendieck group $K(\mathscr{C})$ of $\mathscr{C}$ is isomorphic to the Grothendieck group $K(D^b(\mathscr{C}))$ of the bounded derived category of $\mathscr{C}$. Recall that the Grothendieck group in either case is generated by the isomophism classes of objects in the respective category. The relations in the first case are given by short exact sequences[2], while in the latter by exact triangles, see [12, Ex. 1.27].

Second, if $\langle \mathscr{A}, \mathscr{B} \rangle$ is a semi-orthogonal decomposition of a triangulated category $\mathscr{D}$, e.g. $\mathscr{D} = D^b(\mathrm{Coh}\, Y)$, then $K(\mathscr{D}) = K(\mathscr{A}) \oplus K(\mathscr{B})$. Note that the Grothendieck group of a triangulated category is defined in the obvious way: the generators are the isomorphism classes of objects, the relations come from exact triangles.

Last, if $D^b(\mathrm{Coh}\, Y)$ admits a semi-orthogonal decomposition by sheaves $\mathscr{F}_1, \ldots, \mathscr{F}_m$ together with another factor $\mathscr{A}$, that is,

$$D^b(\mathrm{Coh}\, Y) = \langle \mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_m, \mathscr{A} \rangle,$$

then

$$G_0(Y) := K(\mathrm{Coh}\, Y) = K(D^b(\mathrm{Coh}\, Y)) = K(\mathscr{F}_1) \oplus \cdots \oplus K(\mathscr{F}_m) \oplus K(\mathscr{A}),$$

where by $K(\mathscr{F}_i)$ we mean the Grothendieck group of the full triangulated category generated by $\mathscr{F}_i$, each of which is isomorphic to $\mathbb{C}$. (We assume Y is a scheme over the complex numbers, but this works over any field.) Thus, $\dim G_0(Y) \geq m$, as claimed (or use [18, Prop. 2.1]). Since $\dim G_0(Y) \geq n + 1$, we get $n = 1$.

Assume now that $n = 1$. If the normalization Y' of Y has geometric genus ≥ 1, then [18, Prop. 4.6] gives that $G_0(Y)$ is not finitely generated. If $Y' = \mathbb{P}^1$ and $Y \neq \mathbb{P}^1$, then [18, Prop. 4.1] gives that a categorical resolution (in the sense of [14]) of $D^b(\mathrm{Coh}\, Y)$ has a full

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[2]namely, if $0 \to A \to B \to C \to 0$ is a short exact sequence, then $[B] = [A] + [C]$ in $K(\mathscr{C})$

exceptional collection, but the proof of [18, Prop. 4.1] gives that its length m is at least 3. Hence $G_0(Y) \neq \mathbb{Z}^2$.

Finally, we exclude the case $Y = \mathbb{P}^1$. Assume $Y = \mathbb{P}^1$ and that $\mathscr{L}_0$ and $\mathscr{L}_1$ are simple objects of $D^b(\mathrm{Coh}\,\mathbb{P}^1)$. Since $\mathbb{P}^1$ is a smooth curve, every coherent sheaf $\mathscr{F}$ on $\mathbb{P}^1$ is a direct sum of a torsion sheaf $\mathrm{Tors}(\mathscr{F})$ and a locally free sheaf $\mathscr{F}/\mathrm{Tors}(\mathscr{F})$. Every locally free sheaf is isomorphic to a direct sum of line bundles. Every torsion sheaf is a direct sum of skyscraper sheaves $\mathcal{O}_p$, $p \in \mathbb{P}^1$. Hence the only simple coherent sheaves on $\mathbb{P}^1$ are the line bundles $\mathcal{O}_{\mathbb{P}^1}(t)$, $t \in \mathbb{Z}$, and the sheaves $\mathcal{O}_p$, $p \in \mathbb{P}^1$. No pair of them, not even after a shift $\mathscr{L}_0[-i]$, $\mathscr{L}_1[-j]$ may be of this form: if $p, q \in \mathbb{P}^1$ and $p \neq q$, then $\mathrm{Ext}^i(\mathcal{O}_p, \mathcal{O}_q) = 0$ for all i, either $\mathrm{Hom}(\mathscr{R}, \mathscr{L}) = 0$ or $\mathrm{Ext}^1(\mathscr{L}, \mathscr{R}) = 0$ for any line bundles $\mathscr{L}, \mathscr{R}$, $\mathrm{Hom}(\mathcal{O}_p, \mathscr{L}) = 0$ and $\dim \mathrm{Ext}^1(\mathcal{O}_p, \mathscr{L}) = 1$ for all $p \in \mathbb{P}^1$ and any line bundle $\mathscr{L}$. Since $\mathbb{P}^1$ is a smooth curve, [10, Prop. 6.3] gives that every simple element of $D^b(\mathbb{P}^1)$ is isomorphic to some $\mathscr{F}[-i]$ with $\mathscr{F}$ a simple coherent sheaf on $\mathbb{P}^1$. $\qquad\square$

We now proceed to the task of compactifying our Landau–Ginzburg model and verifying the effect of compactification on the Fukaya–Seidel category.

5. COMPACTIFICATION OF THE ORBIT

Recall that the orbit X is an affine surface in $\mathbb{C}^3$, as described in (2.1). We will embed it into a projective surface $\overline{X}$, and see that the natural choice is $\overline{X} = \mathbb{P}^1 \times \mathbb{P}^1$. We compactify X by homogenizing equation (2.1). This produces the projective surface $\overline{X}$ cut out by $x^2 + yz - t^2 = 0$ in $\mathbb{P}^3$, that can be taken to the standard quadric equation by the change of coordinates $x \mapsto x - t$ and $t \mapsto x + t$, hence the surface is $\mathbb{P}^1 \times \mathbb{P}^1$. This compactification also works well from the symplectic point of view. Thus, we have:

Theorem 5.1. *The semisimple adjoint orbit (X, Ω) of $\mathfrak{sl}(2, \mathbb{C})$ compactifies holomorphically and symplectically to $\mathbb{P}^1 \times \mathbb{P}^1$.*

Proof. Recall from §2 that we may identify the complex structure of X with that of a nontrivial deformation $\mathfrak{X}_2(\tau)$ of $Z_2 = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-2))$. In fact, the deformation of Z_2 extends to a deformation of its natural compactification, the second Hirzebruch surface F_2 obtained from Z_2 by adding a line at infinity, an irreducible divisor with self-intersection $+2$. It is well known that the complex surface F_2 deforms to the Hirzebruch surface $F_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Identifying Z_2 as a subset of F_2, this deformation corresponds to a nontrivial element $\tau \in H^1(F_2, TF_2) \simeq H^1(Z_2, TZ_2)$.

Under deformation of F_2, the added line at infinity decomposes into the sum $E + F$ of two divisors E, F corresponding in the deformed surface $F_0 = \mathbb{P}^1 \times \mathbb{P}^1$ to the fibre and the zero section of F_0 (considered as the trivial $\mathbb{P}^1$-bundle over $\mathbb{P}^1$). The divisor $E + F$ is ample, and its complement is the affine variety $\mathfrak{X}_2(\tau) \simeq X$.

Thus, the complex structure of X agrees with the one inherited from F_0, and similarly the metric on X agrees with the Kähler metric inherited from F_0. These together imply that there exists a unique compatible symplectic structure on X fitting the compactification to F_0. On the other hand, it is clear from definition 1.1 that the symplectic structure Ω on X is compatible with the complex structure on $\mathfrak{sl}(2, \mathbb{C})$. Hence, the symplectic structure on F_0 restricts to Ω on X. $\qquad\square$

Let us identify the compactified fibres of the Landau–Ginzburg model and the divisor at infinity. As seen in (2.2) the potential on the open orbit X is $f_H(A) = 2x$ and it has critical values ± 2. Thus, 0 is a regular value, and we express the regular fibre over 0, X_0, as the affine variety in $\{(y, z) \in \mathbb{C}^2\}$ cut out by the equation

$$yz - 1 = 0$$

since it must satisfy equation (2.1) and $x = 0$. As with the orbit, we homogenize this equation and embed the fibre into the corresponding projective variety $\overline{X_0}$ cut out by

the equations $x = 0$ and $yz - t^2 = 0$ in $\mathbb{P}^3$. Here the complement of the orbit $\overline{X} \setminus X$ in the compactification is obtained by making $t = 0$, thus $x^2 - yz = 0$ inside a projective plane $\mathbb{P}^2$, hence a conic curve, that is, a $\mathbb{P}^1$.

Next we need to compactify the potential. We will first extend the potential as a rational map over $\overline{X}$ and this rational map will then give rise to a holomorphic map on a compactification $\overline{\Gamma}$. We shall choose the symplectic form on $\overline{\Gamma}$ such that it coincides with the original symplectic form on X on an open neighborhood of its thimbles, thus keeping the Lagrangians we used to build the Fukaya category.

6. THE POTENTIAL VIEWED AS A RATIONAL MAP

Our goal now is to extend the potential to the compactification. We will make use of another incarnation of the orbit, namely the adjoint orbit of $e_1 \otimes \varepsilon_1$ in $\mathbb{C}^2 \otimes (\mathbb{C}^2)^*$. The various incarnations of the orbits are described for the general case in [4, §4]. Here we will describe explicitly the isomorphism between two such incarnations for the case of $\mathfrak{sl}(2, \mathbb{C})$, then we will use the tensor product version of the orbit to show that the compactification naturally induces the Segre embedding into $\mathbb{P}^3$. Our extension of the potential to a rational map on $\mathbb{P}^1 \times \mathbb{P}^1$ factors through the Segre embedding. Note that the potential does not extend to a holomorphic map, not even if we change the target to $\mathbb{P}^1$. In [4, §6] it is shown how to extend the potential to a rational map for the cases when the orbit is diffeomorphic to $T^*\mathbb{P}^n$; all other cases remain open.

Let us first set up some notation. For this section we write $A \in \mathrm{SL}(2, \mathbb{C})$ as

$$(6.1) \qquad A = \begin{pmatrix} x & z \\ y & w \end{pmatrix},$$

with $wx - yz = 1$, and fix the following basis for the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$:

$$(6.2) \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad X_{-\alpha} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We consider the representation of the group $\rho \colon \mathrm{SL}(2, \mathbb{C}) \to \mathrm{GL}(\mathbb{C}^2)$ by left multiplication

$$\rho(A)v = Av$$

and its dual representation $\rho^* \colon \mathrm{SL}(2, \mathbb{C}) \to \mathrm{GL}(\mathbb{C}^2)^*$ given by

$$\rho^*(A)\varepsilon = \varepsilon \circ A^{-1}.$$

We denote by $\theta := d_e\rho$ the corresponding representation of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$.

Let α be the positive root of $\mathfrak{sl}(2, \mathbb{C})$, that is, $\alpha = \lambda_1 - \lambda_2$, where λ_i is the functional $\lambda_i(\mathrm{diag}(x_1, x_2)) = x_i$, $i = 1, 2$. The fundamental weight for $\theta \colon \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{gl}(\mathbb{C}^2)$ is $\mu = \frac{1}{2}\alpha$, and the corresponding element in the Cartan subalgebra is

$$H_\mu = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}.$$

Consider the canonical basis $\{e_1, e_2\}$ of $\mathbb{C}^2$. The weight spaces of the representation θ are: $V_1 = \mathrm{span}\{e_1\}$ and $V_{-1} = \mathrm{span}\{e_2\}$. Recall that $\theta(X_\alpha)$ maps V_{-1} to V_1 and that $\theta(X_{-\alpha})$ maps V_1 to V_{-1}. Explicitly,

$$\theta(X_\alpha)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}, \qquad \theta(X_{-\alpha})\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ a \end{pmatrix}.$$

We set $v_1 = (1, 0) \in \mathbb{C}^2$ and $\varepsilon_1 = (1, 0) \in (\mathbb{C}^2)^*$.

If $A \in \mathrm{SL}(2, \mathbb{C})$ is written as in (6.1), then

$$B = \mathrm{Ad}(A)H_\mu = AH_\mu A^{-1} = \begin{pmatrix} \frac{1}{2}(wx + yz) & -xz \\ yw & -\frac{1}{2}(wx + yz) \end{pmatrix}.$$

The eigenvectors of B are (x, y), associated to the eigenvalue $\frac{1}{2}$, and (z, w), associated to the eigenvalue $-\frac{1}{2}$.

Lemma 6.1. *The adjoint action on the tensor product expression of the orbit can be interpreted as the Segre embedding.*

Proof. We have the equality

$$A \cdot (v_1 \otimes \varepsilon_1) = \rho(A) v_1 \otimes \rho^*(A) \varepsilon_1,$$

where

$$\rho(A) v_1 = \begin{pmatrix} x & z \\ y & w \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

and

$$\rho^*(A) \varepsilon_1 = \varepsilon \circ \rho(A^{-1}) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} w & -z \\ -y & x \end{pmatrix} = \begin{pmatrix} w & -z \end{pmatrix}.$$

Therefore,

$$(6.3) \qquad A \cdot (v_1 \otimes \varepsilon_1) = \begin{pmatrix} xw & -xz \\ yw & -yz \end{pmatrix}.$$

Note that the eigenvalues of (6.3) are 0 (with associated eigenvector (z, w)) and 1 (with associated eigenvector (x, y)).

If we consider (x, y) and (z, w) as projective coordinates, then the action on the tensor product can be interpreted as the *Segre embedding* of $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^3$ (up to a sign), which is $([x : y], [z : w]) \mapsto [xz : xw : yz : yw]$. $\qquad \square$

The next lemma provides a diffeomorphism between the orbit $\mathrm{SL}(2, \mathbb{C}) \cdot (v_1 \otimes \varepsilon_1)$ and the adjoint orbit $\mathrm{Ad}(\mathrm{SL}(2, \mathbb{C})) H_\mu$.

Lemma 6.2. *The orbit* $\mathrm{SL}(2, \mathbb{C}) \cdot (v_1 \otimes \varepsilon_1)$ *is diffeomorphic to the adjoint orbit* $\mathrm{Ad}(\mathrm{SL}(2, \mathbb{C})) H_\mu$.

Proof. The diffeomorphism between the orbits of $\mathrm{SL}(2, \mathbb{C})$ will be written using the moment map

$$(6.4) \qquad M(v \otimes \varepsilon)(Z) = \varepsilon(\theta(Z) v),$$

where $v \in \mathbb{C}^2, \varepsilon \in (\mathbb{C}^2)^*, Z \in \mathfrak{sl}(2, \mathbb{C})$. Let $v = (x, y)$ and $\varepsilon = (z, w)$. To describe $M(v \otimes \varepsilon)$ in the base (6.2), we write:

$$\begin{aligned} \langle M(v \otimes \varepsilon), H \rangle &= \varepsilon(\theta(H) v) &= \varepsilon\left(\tfrac{1}{2} x, -\tfrac{1}{2} y\right) = \tfrac{1}{2}(xw + yz) \\ \langle M(v \otimes \varepsilon), X_\alpha \rangle &= \varepsilon(\theta(X_{-\alpha}) v) = \varepsilon(0, x) &= -xz \\ \langle M(v \otimes \varepsilon), X_{-\alpha} \rangle &= \varepsilon(\theta(X_\alpha) v) &= \varepsilon(y, 0) &= yw. \end{aligned}$$

Therefore,

$$(6.5) \qquad M(v \otimes \varepsilon) = \begin{pmatrix} \tfrac{1}{2}(wx + yz) & -xz \\ yw & -\tfrac{1}{2}(wx + yz) \end{pmatrix} = \mathrm{Ad}(A) H_\mu.$$

$\qquad \square$

Theorem 6.3. *The rational map* $R_H \colon \overline{X} = \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ *that extends the potential is*

$$R_H([x : y], [z : w]) = [xw + yz : xw - yz].$$

Proof. Choosing $H = \mathrm{diag}(1, -1)$ we wish to extend the potential f_H to a rational map on the compactification

$$(6.6) \qquad R_H \colon \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1.$$

The rational map R_H that we are looking for is the map to $\mathbb{P}^1$ associated to f_H, that is, the rational map defined on the compactification that coincides with f_H in the open orbit. We claim that the extension is given by

$$(6.7) \qquad R_H(v \otimes \varepsilon) = \frac{\mathrm{tr}((v \otimes \varepsilon) \theta(H))}{\mathrm{tr}(v \otimes \varepsilon)} = \frac{xw + yz}{xw - yz}.$$

Observe that:

- If $v \otimes \varepsilon$ belongs to the adjoint orbit, then $v \otimes \varepsilon$ has the form $A \cdot v_1 \otimes \epsilon_1$ for some $A \in \mathrm{SL}(2, \mathbb{C})$, that is, $v \otimes \varepsilon \in \mathbb{C}^2 \otimes (\mathbb{C}^2)^*$ is a matrix of the form (6.3).
- The previous item implies that $\mathrm{tr}(v \otimes \varepsilon) = 1$ if $v \otimes \varepsilon$ are in the orbit. Therefore $R_H = f_H$ on the orbit.
- The poles of R_H are vectors whose coordinates satisfy $xw = yz$. In other words, (x, y) is a multiple of (z, w). These are the pairs that are *not* in the adjoint orbit (formed by transversal lines).

Therefore, the map defined by formula (6.7) factors through the Segre embedding:

$$(6.8) \qquad ([x : y], [z : w]) \mapsto [xz : xw : yz : yw] \mapsto [xw + yz : xw - yz].$$

and coincides with f_H on the orbit, that is

$$([x : y], [z : w]) \mapsto [f_H : 1].$$

R_H is defined on points outside the orbit as

$$([x : y], [z : w]) \mapsto [2xw : 0],$$

except the points of the base locus $P_1 = ([1 : 0], [1 : 0])$ and $P_2 = ([0 : 1], [0 : 1])$, where the map is ill defined. $\qquad \square$

Remark 6.4. The rational map in Theorem 6.3 is defined outside the points P_1 and P_2. Observe that these points are associated to the nilpotent matrices:

$$([1 : 0], [1 : 0]) \mapsto [1 : 0 : 0 : 0] \simeq \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$([0 : 1], [0 : 1]) \mapsto [0 : 0 : 0 : 1] \simeq \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

7. THE COMPACTIFIED LG MODEL

In Theorem 6.3 we extended the potential to a rational map $R_H : \overline{X} = \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ as

$$(7.1) \qquad ([x : y], [z : w]) \mapsto [xw + yz : xw - yz].$$

However, the map R_H is ill defined at $P_1 = ([1 : 0], [1 : 0])$ and $P_2 = ([0 : 1], [0 : 1])$. We wish to extend R_H to a holomorphic map and will do so by blowing up.

Notation 7.1. We take coordinates $[r : s]$ on the target $\mathbb{P}^1$ and consider the graph Γ of R_H inside the product. We denote by $\overline{\Gamma}$ the closure of Γ in $\overline{X} \times \mathbb{P}^1$, hence $\overline{\Gamma}$ is the surface cut out inside $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ by

$$s(xw + yz) = r(xw - yz).$$

Lemma 7.2. $\overline{\Gamma}$ *is a holomorphic and symplectic compactification of* X.

Proof. By construction $\overline{\Gamma}$ is a complex hypersurface of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ obtained by blowing up points on $\overline{X}$. Hence is it clearly a holomorphic compactification of X. However, pulling back the symplectic form of $\overline{X}$ to $\overline{\Gamma}$ by the blow-up map gives rise to a form that is degenerate on the exceptional set. We will now fix the degeneracy.

As shown in Theorem 5.1 the symplectic structure on $\overline{X} = \mathbb{P}^1 \times \mathbb{P}^1$ is compatible with the one on X. In Theorem 6.3 the potential was extended to a rational map R_H on $\overline{X}$. We need to adapt the symplectic structure on $\overline{\Gamma}$ to fit the situation. We claim that we have arrived at the situation of [16, §3] where Seidel considers a holomorphic Morse function σ_0 / σ_1 defined on a smooth projective variety. In our case we have $\sigma_0 / \sigma_1 = xy + yz / xw - yz$ defined on $\mathbb{P}^1 \times \mathbb{P}^1$. In this situation, we then look at the Lefschetz fibration of hypersurfaces

$$Y_z = \left\{ p \in X \,\middle|\, \sigma_0(p) / \sigma_1(p) = z \right\}$$

for $z \in \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. Note that here Y_∞ is smooth, as required by [16]. Thus, we arrived directly at the second stage of his construction, where we already have a Lefschetz fibration together with a rational function on it (without having passed by a Lefschetz pencil beforehand). Following his method of patching in a correction to the symplectic form on a small neighborhood of the exceptional set we then arrive at the desired symplectic form. For our purposes it is important to take the neighborhood small enough so that it does not intersect the thimbles we had in X, but this can be done since the points P_1 and P_2 where the R_H was ill defined are far from the thimbles of f_H. $\qquad\square$

We will now use the projection to $[r : s]$ to extend the rational map R_H on $\overline{X}$ to a holomorphic map F_H on $\overline{\Gamma}$.

Theorem 7.3. *Let* $\pi_3 \colon \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ *be the projection onto the third factor and set*

$$F_H := \pi_3\big|_{\overline{\Gamma}}.$$

Then F_H *is a holomorphic extension of* f_H.

Proof. In fact, for points in $\overline{\Gamma}$ we have that

$$F_H([x : y], [z : w], [r : s]) = [r : s] = [xw + yz : xw - yz] = R_H([x : y], [z : w]).$$

Thus, F_H is an extension of R_H which in turn is an extension of f_H as shown in Theorem 6.3. $\qquad\square$

Corollary 7.4. *The critical points of* F_H *coincide with the critical points of* f_H.

Proof. For a fixed value $[r_0 : s_0]$ on the target $\mathbb{P}^1$, the fibre of F_H is cut out inside $\mathbb{P}^1 \times \mathbb{P}^1$ by the polynomial equation

$$s_0(xw + yz) - r_0(xw - yz) = 0.$$

This describes a singular conic only in the cases when $s_0 = \pm r_0$, thus the only critical values of F_H are $[1 : 1]$ and $[1 : -1]$ with corresponding critical points $([1 : 0], [0 : 1])$ and $([0 : 1], [1 : 0])$. These in turn correspond to the critical points $\pm H$ of f_H. We conclude that extending f_H to F_H does not produce any extra critical points. $\qquad\square$

Corollary 7.5. *The Fukaya–Seidel category of* $\overline{\mathrm{LG}}(2)$ *is the same as the one of* $\mathrm{LG}(2)$.

Proof. Observe that in Lemma 7.2 chose the symplectic form on the compactification $\overline{\Gamma}$ so that our original Lagrangian thimbles that generated $\mathrm{Fuk}(\mathrm{LG}(2))$ remain Lagrangian in the compactification. Moreover, Corollary 7.4 shows that no new critical points arise when we extend the potential to the compactification. Therefore the Fukaya–Seidel category corresponding to the compactification is the same as the one described in Theorem 3.1. $\qquad\square$

In particular, using the results of §4 we conclude that this compact LG model $\overline{\mathrm{LG}}(2)$ does not have a projective mirror either. Hence we obtain:

Theorem 7.6. $\overline{\mathrm{LG}}(2)$ *has no projective mirrors.*

8. Mirror category

Theorem 3.1 states that the Fukaya–Seidel category of $\mathrm{LG}(2)$ is generated by two Lagrangians L_0 and L_1 with the following morphisms

$$\mathrm{Hom}(L_i, L_j) \simeq \begin{cases} \mathbb{Z} \oplus \mathbb{Z}[-1] & i < j \\ \mathbb{Z} & i = j \\ 0 & i > j \end{cases}.$$

Theorems 4.1 and 7.6 show that no projective variety may be the mirror of either $\mathrm{LG}(2)$ or else $\overline{\mathrm{LG}}(2)$. However, we do have the following result, which in light of Lemma 9.1 below may be thought of as an instance of [15, Cor. 2.7].

Theorem 8.1. Fuk(LG(2)) *is equivalent to the full triangulated subcategory* $D^b(\text{ƆI}(2)) := \langle \mathcal{O}_{F_2}, \mathcal{O}_{F_2}(-E) \rangle$ *of* $D^b(\text{Coh } F_2)$, *where* F_2 *is the second Hirzebruch surface and* E *is the divisor with self-intersection* -2.

Proof. Let $[x_0 : x_1 : x_2]$ and $[y_0 : y_1]$ be the standard coordinates on $\mathbb{P}^2$ and $\mathbb{P}^1$. The second Hirzebruch surface is the hypersurface $F_2 \subset \mathbb{P}^2 \times \mathbb{P}^1$ cut out by the equation $x_0 y_0^2 - x_1 y_1^2$. The fibre F of the natural projection to $\mathbb{P}^1$ is a divisor with self-intersection 0; the exceptional fibre of the natural projection to $\mathbb{P}^2$ is a prime divisor E with self-intersection -2. The line bundles associated to E, F generate the Picard group $\text{Pic}(F_2)$ with relations

$$E^2 = -2, \quad E \cdot F = 1, \quad F^2 = 0.$$

Now consider the derived category generated by the line bundles $\mathcal{O}_{F_2}$ and $\mathcal{O}_{F_2}(-E)$; we denote this category by $D^b(\text{ƆI}(2))$, even though Theorem 4.1 shows that it is not the derived category of coherent sheaves on any projective variety $\text{ƆI}(2)$.

The Hom and Ext^k groups of line bundles on F_2 may be calculated via toric geometry, giving

$$\text{Hom}(\mathcal{O}, \mathcal{O}) \simeq \text{Hom}(\mathcal{O}(-E), \mathcal{O}(-E)) \simeq \mathbb{C}$$

(8.1)
$$\text{Hom}(\mathcal{O}(-E), \mathcal{O}) \simeq \mathbb{C}$$

$$\text{Ext}^1(\mathcal{O}(-E), \mathcal{O}) \simeq H^1(F_2, \mathcal{O}(E)) \simeq \mathbb{C}$$

all other Hom and Ext^k groups being zero.

Setting $\mathcal{L}_0 := \mathcal{O}_{F_2}(-E)$ and $\mathcal{L}_1 := \mathcal{O}_{F_2}$ we have in the derived category

$$\text{Hom}(\mathcal{L}_i, \mathcal{L}_j) \simeq \begin{cases} \mathbb{C} \oplus \mathbb{C}[-1] & i < j \\ \mathbb{C} & i = j \\ 0 & i > j \end{cases}$$

in agreement with (3.1). $\square$

9. QUIVERS

(8.1) shows that the collection $(\mathcal{L}_0, \mathcal{L}_1)$ has nonvanishing Ext^k groups for $k = 1$. Following [11] we apply a "partial mutation" (modifying some elements in the collection) to find a pair of locally free sheaves generating $D^b(\text{ƆI}(2))$, with *vanishing* Ext^k groups for $k > 0$.

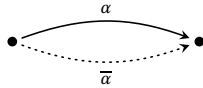Let $\mathcal{E}$ be a nontrivial extension of $\mathcal{O}(-E)$ by $\mathcal{O}$. We obtain a triangle

$$\mathcal{O} \to \mathcal{E} \to \mathcal{O}(-E) \otimes \text{Ext}^1(\mathcal{O}(-E), \mathcal{O}) \to \mathcal{O}[1].$$

By the results of [11] (or by direct verification) we have:

Lemma 9.1. *The pair* $(\mathcal{O}, \mathcal{E})$ *generates* $D^b(\text{ƆI}(2))$ *and has vanishing* Ext^k *groups for* $k > 0$. *In particular,* $\mathcal{O} \oplus \mathcal{E}$ *has no self-extensions and is a tilting bundle for* $D^b(\text{ƆI}(2))$.
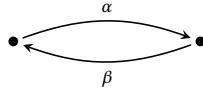
The fact that the collections $(\mathcal{O}, \mathcal{O}(-E))$ and $(\mathcal{O}, \mathcal{E})$ both generate $D^b(\text{ƆI}(2))$ means that $D^b(\text{ƆI}(2))$ is equivalent to the derived categories of (DG) modules over the corresponding (DG) quivers. These equivalences can be obtained by writing the objects of each collection as vertices and a basis for the morphisms between these objects as arrows.

The presence of nontrivial extensions means that the collection $(\mathcal{O}, \mathcal{O}(-E))$ gives rise to a *DG quiver* $\widetilde{Q}$

where α is of degree 0, $\overline{\alpha}$ of degree 1. The associated path algebra $\widetilde{A}$ is a DG algebra with differential $\partial\alpha = \overline{\alpha}$. Since $\widetilde{Q}$ contains no composable arrows, all products except for multiplication by scalars vanish; *cf.* Lemma 3.5. We have an equivalence of triangulated categories $D^b(\text{dg mod-}\widetilde{A}) \simeq D^b(\mho\mathrm{I}(2))$.

The absence of nontrivial extensions means that the collection $(\mathscr{O},\mathscr{E})$ gives rise to an ordinary quiver

with relation $\beta\alpha = 0$. The associated path algebra A is a noncommutative ordinary algebra, *i.e.* a DG algebra concentrated in degree 0. Again, we obtain an equivalence of triangulated categories $D^b(\text{dg mod-}A) \simeq D^b(\text{mod-}A) \simeq D^b(\mho\mathrm{I}(2))$.

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