Some Sphere Theorems in Linear Potential Theory

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Abstract

In this paper we analyze the capacitary potential due to a charged body in order to deduce sharp analytic and geometric inequalities, whose equality cases are saturated by domains with spherical symmetry. In particular, for a regular bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$, we prove that if the mean curvature H of the boundary obeys the condition

$$- \left[\frac{1}{\mathrm{Cap}(\Omega)} \right]^{\frac{1}{n-2}} \leq \frac{H}{n-1} \leq \left[\frac{1}{\mathrm{Cap}(\Omega)} \right]^{\frac{1}{n-2}} ,$$

then Ω is a round ball.

1 Introduction and statement of the main result

In this work we are interested in studying the analytic and geometric properties of a solution (Ω, u) to the following problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n \backslash \overline{\Omega} \, , \\ u = 1 & \text{on } \partial\Omega \, , \\ u(x) \to 0 & \text{as } |x| \to +\infty \, . \end{cases} \tag{1.1}$$

In classical potential theory, the bounded domain Ω is interpreted as a uniformly charged body and the function u is the electrostatic (or capacitary) potential due to Ω. The basic properties of solutions to (1.1), such as existence, uniqueness, regularity, validity of maximum and minimum principles as well as asymptotic estimates, are of course very well known and for a systematic treatment, we refer the reader to [34, 31]. In order to introduce our main results, we specify our assumptions on the regularity of both the domain Ω and the function u with the following definition.

1.1 Definition. *A pair (Ω, u) is said to be a* regular solution *to problem* (1.1) *if*

- $\Omega \subset \mathbb{R}^n$, $n \geq 3$, *is a bounded open domain with smooth boundary. Without loss of generality, we also assume that Ω contains the origin and $\mathbb{R}^n \backslash \Omega$ is connected.*

- *The function $u \colon \mathbb{R}^n \backslash \Omega \to (0,1]$ is a solution to problem* (1.1)*, it is $\mathcal{C}^\infty$ in $\mathbb{R}^n \backslash \Omega$ and analytic in $\mathbb{R}^n \backslash \overline{\Omega}$.*

Before proceeding, we point out that the assumption about the connectedness of $\mathbb{R}^n \backslash \Omega$ is not restrictive. In fact, since Ω is bounded, the set $\mathbb{R}^n \backslash \Omega$ has one and only one unbounded connected component. The other ones are then forced to be compact so that u has to be constant on them. In other words, these connected components are completely irrelevant to the analysis of problem (1.1) and the above definition allows us to simply ignore them.

In order to fix some notation as well as some convention, we recall that the *electrostatic capacity* associated to the set Ω is defined by

$$\mathrm{Cap}(\Omega) = \inf\left\{ \frac{1}{(n-2)|S^{n-1}|} \int_{\mathbb{R}^n} |Dw|^2 \, dv \;\middle|\; w \in C_c^\infty(\mathbb{R}^n),\; w \equiv 1 \text{ in } \Omega \right\},$$

where $|S^{n-1}|$ is the area of the unit sphere S^{n-1} in $\mathbb{R}^n$ and $d\sigma$ is the hypersurface measure in $\mathbb{R}^n$. It is well known that the capacity of Ω can be represented in terms of the *capacitary potential u* by means of the following formula

$$\mathrm{Cap}(\Omega) \;=\; \frac{1}{(n-2)|S^{n-1}|} \int_{\partial\Omega} |Du| \, d\sigma, \tag{1.2}$$

We recall that if Ω is a round ball centered at the origin, then the capacitary potential u is radially symmetric and it is given by the explicit formula $u(x) = \mathrm{Cap}(\Omega)|x|^{2-n}$. For more general domains, u satisfies the following asymptotic expansions as $|x| \to \infty$

$$
\begin{aligned}
u &= \mathrm{Cap}(\Omega)|x|^{2-n} \cdot (1 + o(1)), \\
D_i u &= -(n-2)\,\mathrm{Cap}(\Omega)|x|^{-n} x_i \cdot (1 + o(1)), \\
D_{ij}^2 u &= (n-2)\,\mathrm{Cap}(\Omega)|x|^{-2-n}\big(n x_i x_j - |x|^2 g_{ij}^{\mathbb{R}^n}\big) \cdot (1 + o(1)).
\end{aligned}
\tag{1.3}
$$

We point out that, using these estimates, one easily proves that the sets $\{u = \varepsilon\}$ for small values of $\varepsilon \geqslant 0$ are diffeomorphic to a sphere, and that their hypersurface area has the following behavior

$$\int_{\{u=\varepsilon\}} d\sigma = \left[\frac{\mathrm{Cap}(\Omega)}{\varepsilon}\right]^{\frac{n-1}{n-2}} |S^{n-1}| + o(\varepsilon^{-\frac{n-1}{n-2}}). \tag{1.4}$$

To introduce our main result, we recall that in the recent work [2] the authors prove the monotonicity along the level sets of u of suitably chosen functions U_p, $p \geqslant 2 - 1/(n-1)$, and use it to prove the following sharp inequality

$$|S^{n-1}|^{\frac{1}{p}} \left[\mathrm{Cap}(\Omega)\right]^{1 - \frac{(p-1)(n-1)}{p(n-2)}} \leqslant \left\|\frac{Du}{n-2}\right\|_{L^p(\partial\Omega)}.$$

An interesting feature of this inequality is that it comes together with a rigidity statement. In fact, as soon as the equality is fulfilled, everything is rotationally symmetric. Finally, letting $p \to +\infty$, one deduces

$$\left[\mathrm{Cap}(\Omega)\right]^{-\frac{1}{n-2}} \leqslant \max_{\partial\Omega} \left|\frac{Du}{n-2}\right|.$$

However, *a priori* it is not clear if the rigidity statement holds true for this inequality. Our main result answers in the affirmative to this question.

1.2 Theorem. *Let (Ω, u) be a regular solution to problem (1.1) in the sense of Definition 1.1. Then the following inequality holds*

$$\frac{1}{\mathrm{Cap}(\Omega)} \leqslant \left(\max_{\partial\Omega} \left|\frac{Du}{n-2}\right|\right)^{n-2}. \tag{1.5}$$

Moreover, the equality is fulfilled if and only if Ω is a ball and u is rotationally symmetric.

The proof of a slightly extended version of Theorem 1.2 (see Theorem 2.6) will be presented in full details in Section 6. However, before proceeding, it is important to remark that the new rigidity statement is obtained by exploiting the peculiar geometric features of the new method (*spherical ansatz*), in contrast with the method employed in [2] (*cylindrical ansatz*), for which this result was clearly out of reach. For further comments about the way the rigidity is proven and for a more detailed comparison between the two methods, we refer the reader to Section 2.

It is worth pointing out that in dimension $n = 3$, a proof of Theorem 1.2 has already been provided by Payne and Philippin in [41, Formula (3.24)]. This 3-dimensional proof is based on [41, Theorem in Section 2] and the well known Poincaré-Faber-Szegö inequality

$$|\Omega| \leqslant \frac{4\pi}{3}[\mathrm{Cap}(\Omega)]^3, \tag{1.6}$$

which holds on every bounded domain of $\mathbb{R}^3$, see [42, Section 1.12]. The proof in [41] can be extended to any dimension $n \geqslant 3$ using the following generalized version of (1.6) (see for instance [33])

$$\left(\frac{|\Omega|}{|B^n|}\right)^{\frac{n-2}{n}} \leqslant \mathrm{Cap}(\Omega)\,, \tag{1.7}$$

where $B^n \subset \mathbb{R}^n$ is the n-dimensional unit ball. This inequality has also been extended to conformally flat manifolds in [25, Theorem 1]. However, this is not enough to generalize the approach of Payne and Philippin beyond the Euclidean setting, since the computations in [41] make use of integral identities that exploit the peculiar structure of $\mathbb{R}^n$.

In contrast with this, we emphasize that the proof proposed in the present work is self–contained, provides a clear geometric interpretation of the result (see Theorem 2.5), and does not depend heavily on the structure of $\mathbb{R}^n$. In view of these considerations, it is reasonable to believe that the method described in this work could be successfully employed to study a much wider family of problems. For instance, it should be possible to adapt with minor changes the present techniques to treat the exterior problem on a complete nonparabolic Riemannian manifold with nonnegative Ricci tensor (here the nonparabolicity of the ends is needed in order to guarantee the existence of an exterior solution to the analogue of system (1.1)). Another promising direction of research would consist in using the *spherical ansatz* to investigate other types of elliptic problems. As a relevant example, we recall that the *cylindrical ansatz* introduced in [2] to study problem (1.1), has been employed in [4] to establish some relevant geometric inequalities in the context of static metrics in general relativity. These and other natural questions will be the object of future investigations.

From the previous discussion, it is clear that one of the main focuses of this paper is on the method employed. For this reason, in the next section we will explain in more details the strategy of the proof, postponing to Section 8 the discussion of the consequences of Theorem 1.2. Here we limit to single out one of the results that we believe of particular interest, which is the following completely geometric sphere theorem.

1.3 Theorem. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and $n \geqslant 3$. Suppose that*

$$-\left[\frac{1}{\mathrm{Cap}(\Omega)}\right]^{\frac{1}{n-2}} \leqslant \frac{H}{n-1} \leqslant \left[\frac{1}{\mathrm{Cap}(\Omega)}\right]^{\frac{1}{n-2}}\,.$$

Then $\partial\Omega$ is a sphere.

This result can be interpreted as a version of the celebrated Alexandrov's Soap Bubble Theorem, where the constancy of the mean curvature is replaced by a pinching condition. Further comments, together with the proof of this result, can be found in Section 8, see Corollary 8.6.

2 Geometric methods and strategy of the proof

In this section we outline the strategy of the proof of the previously mentioned results and we set up some notations. This gives us the opportunity to describe in details the analogies and differences between our method and the one introduced in [1, 2], hoping that this comparison will be useful for further applications.

2.1 Spherical ansatz vs cylindrical ansatz

The method presented in [1, 2] relies on a geometric construction called *cylindrical ansatz*. One starts from the simple observation that the Euclidean metric $g_{\mathbb{R}^n}$ can be written as

$$g_{\mathbb{R}^n} \;=\; d|x| \otimes d|x| + |x|^2 g_{S^{n-1}} \;=\; |x|^2 \left(|x|^{-2} d|x| \otimes d|x| + g_{S^{n-1}}\right) \;=\; |x|^2 g_{\mathrm{cyl}} \;=\; u_0^{-\frac{2}{n-2}} g_{\mathrm{cyl}}\,,$$

where g_{cyl} is the cylindrical metric on $\mathbb{R}^n \backslash \{0\}$ and we have denoted by $u_0(x) = |x|^{2-n}$ the rotationally symmetric solution to problem (1.1) when Ω is a ball centered at the origin with radius 1. With this in mind, given a generic solution (Ω, u) to problem (1.1), one is lead to consider the ansatz metric

$$g \;=\; u^{\frac{2}{n-2}} g_{\mathbb{R}^n}\,,$$

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and study under which conditions g is actually cylindrical. To proceed, one sets $\varphi = \log(u)$ and notices that, when (Ω, u) is a rotationally symmetric solution, then $\nabla \varphi$ represents the splitting direction of the cylinder. For this reason, the strategy of the proof in [1, 2] is to find suitable conditions under which some *splitting principle* applies. When this is the case, one recovers the rotational symmetry of the solution (Ω, u).

In this work, we introduce a different geometric construction, called *spherical ansatz*, and we aim to investigate its consequences along the same lines as the ones just described for the *cylindrical ansatz*. The main difference is that this time the rigidity statements will be deduced from a suitable *sphere theorem* instead of a *splitting principle*. To be more concrete, we indicate with $S^n = \{(z^1, \ldots, z^{n+1}) \in \mathbb{R}^{n+1} \; : \; |z|^2 = 1\}$ the unit sphere sitting in the Euclidean space $\mathbb{R}^{n+1}$ and with N its north pole $(0, \ldots, 0, 1)$. We let then π be the stereographic projection that maps conformally $S^n \backslash \{N\}$ onto the n-dimensional plane $\{(z^1, \ldots, z^{n+1}) \in \mathbb{R}^{n+1} \; : \; z^{n+1} = 0\}$. We set $z' = (z^1, \ldots, z^n)$ and recall the precise formulæ:

$$\pi : S^n \backslash \{N\} \longrightarrow \mathbb{R}^n \qquad\qquad \pi^{-1} : \mathbb{R}^n \longrightarrow S^n \backslash \{N\}$$

$$(z', z^{n+1}) \longmapsto \frac{z'}{1 - z^{n+1}} \qquad\qquad x \longmapsto \left(\frac{2x}{|x|^2 + 1}, \frac{|x|^2 - 1}{|x|^2 + 1} \right).$$

Pulling back the standard round metric g_{S^n} (i.e., the one induced by the Euclidean metric $g_{\mathbb{R}^{n+1}}$ on the unit sphere S^n) one obtains on $\mathbb{R}^n$ the metric $g_{\mathrm{sph}} = (\pi^{-1})^* g_{S^n}$, so that $(\mathbb{R}^n, g_{\mathrm{sph}})$ is isometric to a sphere, except for a missing point. The explicit form of such metric is

$$g_{\mathrm{sph}} = (\pi^{-1})^* g_{S^n} = \left(\frac{2}{1 + |x|^2} \right)^2 g_{\mathbb{R}^n} = \left(\frac{2}{1 + u_0^{-\frac{2}{n-2}}} \right)^2 g_{\mathbb{R}^n},$$

where we recall that $u_0(x) = |x|^{2-n}$. Therefore, starting from a generic regular solution (Ω, u) to problem (1.1), we are lead to consider the *ansatz* metric

$$g = \left(\frac{2}{1 + u^{-\frac{2}{n-2}}} \right)^2 g_{\mathbb{R}^n}. \tag{2.1}$$

In order to obtain rigidity statements, we need to investigate under which conditions the Riemannian manifold $(\mathbb{R}^n \backslash \overline{\Omega})$ is isometric to an hemisphere. This will imply in turn the rotational symmetry of both Ω and u. As anticipated, sphere theorems will play here the same role as the splitting principles in the *cylindrical ansatz* situation.

2.1 Remark. *It should be noticed that if the domain Ω is actually a ball of radius ρ, then the solution u is rotationally symmetric and it is explicitly given by $u = \rho^{n-2}|x|^{2-n}$. On the other hand, from (1.3), one has that*

$$u = \mathrm{Cap}(\Omega)|x|^{2-n},$$

so that the radius ρ can be computed in terms of Ω according to the formula

$$\rho = [\mathrm{Cap}(\Omega)]^{\frac{1}{n-2}}.$$

In this case, formula (2.1) rewrites as

$$g = \left(\frac{2}{1 + [\mathrm{Cap}(\Omega)]^{-\frac{2}{n-2}} |x|^2} \right)^2 \cdot dx^1 \otimes dx^1 + \ldots + dx^n \otimes dx^n =$$

$$= [\mathrm{Cap}(\Omega)]^{\frac{2}{n-2}} \left(\frac{2}{1 + |\xi|^2} \right)^2 \cdot d\xi^1 \otimes d\xi^1 + \ldots + d\xi^n \otimes d\xi^n = [\mathrm{Cap}(\Omega)]^{\frac{2}{n-2}} g_{\mathrm{sph}},$$

where in the second equality we have used the change of coordinates $\xi = [\mathrm{Cap}(\Omega)]^{\frac{1}{n-2}} x$, and in the last equality we have used the definition of g_{sph}. This means that, in general, in what concerns the rigidity statements, we expect the manifold $(\mathbb{R}^n \backslash \overline{\Omega}, g)$ to be isometric to a round hemisphere of radius $\rho = [\mathrm{Cap}(\Omega)]^{1/(n-2)}$.

In the case under consideration, it seems reasonable to proceed in the direction of the classical Obata rigidity theorem [40, Theorem A], according to which a necessary and sufficient condition for a closed manifold to be isometric to a unit-radius sphere is to admit a function f satisfying the condition $\nabla^2 f = -fg$. On the standard sphere $S^n \subset \mathbb{R}^{n+1}$ such condition is verified by the restriction to the sphere itself of any coordinate function of $\mathbb{R}^{n+1}$. So in our case it is natural to supplement the ansatz metric g with a function f that is the candidate to trigger the rigidity theorem. In order to find such a function, we turn ourselves again to the model case and consider the function

$$f_0(x) = \left((\pi^{-1})^* z^{n+1}\right)(x) = z^{n+1} \circ \pi^{-1}(x) = \frac{|x|^2 - 1}{|x|^2 + 1} = \frac{u_0^{-\frac{2}{n-2}} - 1}{u_0^{-\frac{2}{n-2}} + 1}. \tag{2.2}$$

Given a generic solution (Ω, u) to (1.1), it will then sound reasonable to set

$$f = \frac{u^{-\frac{2}{n-2}} - 1}{u^{-\frac{2}{n-2}} + 1} = -\tanh\left(\frac{\log u}{n-2}\right) \tag{2.3}$$

and study the equations it solves. In light of the above formulæ, and depending on the context, it will be convenient to consider f either as a function of u or as a function of the point x, hopefully without confusing the reader. Similarly we will possibly consider u as a function of either f or x, depending on the context. In order to understand the domain of definition as well as the range of f, we observe that by the classical strong maximum principle (see for instance [24]), when the point x varies in $\mathbb{R}^n \backslash \overline{\Omega}$, the function $u(x)$ takes values in $(0, 1)$. On the other hand, it is clear by formula (2.3) that f is strictly decreasing as a function of u. Moreover it is equal to 0 on $\partial\Omega$, tends to 1 as $|x| \to \infty$ and takes values in $(0, 1)$ when its argument is ranging in $\mathbb{R}^n \backslash \overline{\Omega}$. A consequence of this fact is that the level sets of both u and f are compact, which in turn implies that the set of regular values is open in the codomain. By Sard's lemma it is also a set of full measure, so in particular it is dense. Also, by construction, f and u share the same level sets (although with opposite choices of sublevel sets) and each of them is regular for either both or none of u and f.

As we will show in Section 3, the relation between the ansatz metric g and the Euclidean metric $g_{\mathbb{R}^n}$ can be expressed in terms of f as

$$g = (1-f)^2 g_{\mathbb{R}^n},$$

so that the original problem (1.1) can be rewritten in terms of f and g as

$$\begin{cases} \Delta_g f = -n\left(\frac{|\nabla f|_g^2}{1-f^2}\right) f & \text{in } \mathbb{R}^n \backslash \overline{\Omega}, \\[2mm] \mathrm{Ric} = \frac{n-2}{1-f} \nabla^2 f + \frac{n-1-f}{1-f}\left(\frac{|\nabla f|_g^2}{1-f^2}\right) g & \text{in } \mathbb{R}^n \backslash \overline{\Omega}, \\[2mm] f = 0 & \text{on } \partial\Omega, \\[2mm] f(x) \to 1 & \text{as } |x| \to +\infty. \end{cases} \tag{2.4}$$

It is important to notice that the second equation in the above system, corresponds to the identity $\mathrm{Ric}_{g_{\mathbb{R}^n}} = 0$, that was hidden in the original formulation of problem (1.1).

In what follows we will focus our analysis on system (2.4), and then we will translate our results in the original formalism in order to prove Theorem 1.2. It should be mentioned that the study of problem (2.4) has some interests on its own. In fact, similar systems are obtained through the application of a cylindrical ansatz argument to other problems, as in the already cited [1, 2, 4] and also [14, 15]. The second equation in (2.4) shares also some analogies with the formulæ arising in the study of quasi-Einstein metrics in the sense of the Bakry-Émery Ricci tensor [8, 48]. Finally, we take the opportunity to point out that works such as [23, 13] already introduced the idea of rescaling the potential u in order to obtain a new function, satisfying an equation analogue to the first in (2.4), which is then shown to verify some spherical rigidity property. The novelty in the approach presented in this paper and in [1, 2] is the combination of such a rescaling with an opportune conformal change of the metric.

In order to simplify the references for the forthcoming analysis, we observe that in the new conformal framework the counterpart of Definition 1.1 is given by the following

2.2 Definition. *A triple (Ω, f, g) is said to be a* regular solution *to system* (2.4) *if*

- $\Omega \subset \mathbb{R}^n$, $n \geqslant 3$, *is a bounded open domain. Without loss of generality, we also assume that Ω contains the origin and $\mathbb{R}^n \backslash \Omega$ is connected.*

- *The function $f \colon \mathbb{R}^n \backslash \Omega \to [0,1)$ and the Riemannian metric g defined on $\mathbb{R}^n \backslash \Omega$ satisfy system (2.4). The function f and the coefficients of the metric g –whenever the latter are expressed with respect to standard Euclidean coordinates– are C^∞ in $\mathbb{R}^n \backslash \Omega$, and analytic in $\mathbb{R}^n \backslash \overline{\Omega}$.*

In order to make the new formulation more familiar, we observe that taking the trace of the second equation in (2.4) one finds that the scalar curvature R of g verifies

$$\mathrm{R} = n(n-1) \frac{|\nabla f|_g^2}{1 - f^2}. \tag{2.5}$$

As already noticed, when Ω is a ball, we have that g is the spherical metric, hence R is constant on the whole $\mathbb{R}^n \backslash \Omega$. This suggests that, when looking for rigidity results, it is natural to investigate under which conditions one can conclude that the quantity

$$\frac{|\nabla f|_g^2}{1 - f^2} = u^{-2\frac{n-1}{n-2}} \left| \frac{Du}{n-2} \right|^2$$

is constant. In fact, this quotient will play the role of what is called a *P-function* in the literature about overdetermined boundary value problems.

2.2 Outline of the proof

We are now in the position to outline the analysis of the regular solutions of system (2.4), focusing on the key points. This gives us the opportunity to present some intermediate results that we believe of some independent interest. It is important to keep in mind that these results are based on Definition 2.2 and have no "knowledge" of the original function u.

Combining the Bochner formula with the equations in (2.4), we prove in Section 4 that the vector field

$$X = \frac{1}{(1-f)(1+f)^{n-1}} \left(\nabla |\nabla f|_g^2 - \frac{2}{n} \Delta_g f \nabla f \right) = \frac{1}{(1+f)^{n-2}} \nabla \left(\frac{|\nabla f|_g^2}{1 - f^2} \right) \tag{2.6}$$

has nonnegative divergence. This fact admits two different interpretations, which we leverage to obtain two different classes of results, one based on the divergence theorem and the other based on the strong maximum principle.

Monotonicity via divergence theorem. On one hand, one can integrate $\mathrm{div}_g X$ between two different level sets of f and obtain by Stokes' theorem that the quantity $\int_{\{f=s\}} \langle X | \nu_g \rangle_g \, d\sigma_g$ is monotonic. In turn, this fact leads to the monotonicity of the function

$$[0,1) \ni s \longmapsto \Phi(s) = \frac{1}{(1-s^2)^{\frac{n+2}{2}}} \int_{\{f=s\}} |\nabla f|_g^3 \, d\sigma_g, \tag{2.7}$$

which is the content of the following theorem (notice that the fact that Φ is well defined is an easy consequence of the analyticity of f and g, as we will prove in Section 5).

2.3 Theorem. *Let (Ω, f, g) be a regular solution to problem (2.4) in the sense of Definition 2.2. Then the function Φ defined by (2.7) is differentiable and, for any $s \in [0,1)$, it holds*

$$(1-s^2)^{\frac{n+2}{2}} \Phi'(s) = -2 \left[\int_{\{f=s\}} |\nabla f|_g^2 H_g \, d\sigma_g + (n-1) \frac{s}{1-s^2} \int_{\{f=s\}} |\nabla f|_g^3 \, d\sigma_g \right] \leqslant 0, \tag{2.8}$$

where H_g is the mean curvature of the hypersurface $\{f = s\}$ computed with respect to the normal pointing towards $\{f \geqslant s\}$ and the metric g. In fact, the left hand side is increasing in s and it tends to zero as $s \to 1$. Moreover, $\Phi'(s) = 0$ for some $s \in [0,1)$ if and only if Ω is a ball and f is rotationally symmetric.

Translating Theorem 2.3 back in terms of u and $g_{\mathbb{R}^n}$, one obtains the following result.

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2.4 Theorem. *Let (Ω, u) be a regular solution to problem (1.1) in the sense of Definition 1.1. Then, for all $t \in (0,1]$, it holds*

$$\int_{\{u=t\}} |Du|^2 \left[\frac{H}{n-1} - \frac{|Du|}{(n-2)u} \right] d\sigma \geq 0.$$

(2.9)

Moreover, equality is fulfilled for some $t \in (0,1]$ if and only if Ω is a ball and u is rotationally symmetric.

The proof of formula (2.8) and its equivalence with (2.9) is treated in Section 5. The equality case is shown to imply that $\mathrm{div}_g X = 0$, which in turn implies that $\nabla^2 f = \lambda g$, for a certain function $\lambda \in \mathcal{C}^\infty(M)$. This can be used as the start to trigger a rigidity argument, which is fully discussed in Section 7. It should be noticed that Theorem 2.4 is not a new result. The original proof appeared in [1] as an application of the *cylindrical ansatz*, and a second proof of the same result, based on some basic inequalities for symmetric functions, is shown in [11]. We also remark that Theorem 2.4 has been vastly extended in [2]. In fact, our function $\Phi(s)$, rewritten in terms of u and $g_{\mathbb{R}^n}$, coincides, up to a constant, with the function $U_3(t)$ defined in [2]. As stated in the introduction, in [2] the *cylindrical ansatz* is used to prove the monotonicity of an entire family of functions $U_p(t)$ indexed on $p \geq 2 - 1/(n-1)$, while it seems that the *spherical ansatz* helps to recover the monotonicity only for $p = 3$.

Rigidity via strong maximum principle. On the other hand, the divergence of X can be seen as a positive definite elliptic differential operator of the second order in divergence form, applied to the function

$$\frac{|\nabla f|_g^2}{1 - f^2} = \frac{\mathrm{R}}{n(n-1)}.$$

More explicitly, recalling the definition of X given in (2.6), the inequality $\mathrm{div}_g X \geq 0$ translates in the following elliptic inequality for the scalar curvature R of g

$$\Delta_g \mathrm{R} - (n-2) \left\langle \frac{\nabla f}{1+f} \,\middle|\, \nabla \mathrm{R} \right\rangle_g \geq 0.$$

(2.10)

This is the starting point of our second approach, that exploits a peculiar feature of the *spherical ansatz*, namely the one point compactification. In fact, the present geometric construction allows one to compactify the space $\mathbb{R}^n \backslash \Omega$ adding the north pole N to its preimage via the stereographic projection. Before setting up some notations, we recall from Remark 2.1 that, in the rigidity statements, we ultimately expect $\mathbb{R}^n \backslash \overline{\Omega}$ to be an hemisphere of radius $\rho = [\mathrm{Cap}(\Omega)]^{1/(n-2)}$, so that it is natural to consider the stereographic projection

$$\pi_\rho : S_\rho^n \backslash \{N\} \longrightarrow \mathbb{R}^n \qquad (z', z^{n+1}) \longmapsto \frac{\rho z'}{\rho - z^{n+1}}.$$

where S_ρ^n is the standard sphere of radius $\rho = [\mathrm{Cap}(\Omega)]^{1/(n-2)}$ and $N = (0, \ldots, 0, \rho) \in \mathbb{R}^{n+1}$ is its north pole. The space

$$M = \pi_\rho^{-1}(\mathbb{R}^n \backslash \Omega) \cup \{N\} \subset S_\rho^n$$

(2.11)

is now compact. On M we define the function $\varphi : M \to \mathbb{R}$ and the metric γ respectively as the pull-back of f and g, suitably extended through the north pole. More precisely, we set

$$\varphi(y) = \begin{cases} f \circ \pi_\rho(y) & \text{if } y \in \pi_\rho^{-1}(\mathbb{R}^n \backslash \Omega), \\ 1 & \text{if } y = N, \end{cases} \qquad \gamma_{|y} = \begin{cases} (\pi_\rho^* g)_{|y} & \text{if } y \in \pi_\rho^{-1}(\mathbb{R}^n \backslash \Omega), \\ (g_{S_\rho^n})_{|N} & \text{if } y = N. \end{cases}$$

(2.12)

Using the asymptotic estimates on the function f and its derivative, it is easy to deduce (see Section 6) that $\varphi \in \mathcal{C}^2(M)$, whereas the metric γ has $\mathcal{C}^2$ regularity in a neighborhood of the north pole. In particular, its scalar curvature R_γ is well-defined and continuous on the whole M. It is then possible to extend the validity of (2.10) at the north pole. In fact, we will show that R_γ satisfies the elliptic inequality

$$\Delta_\gamma \mathrm{R}_\gamma - (n-2) \left\langle \frac{\nabla \varphi}{1+\varphi} \,\middle|\, \nabla \mathrm{R}_\gamma \right\rangle_\gamma \geq 0$$

in the $\mathcal{W}^{1,2}(M)$–sense. This allows to invoke a strong maximum principle, which leads to the following result.

2.5 Theorem. *Let* (Ω, f, g) *be a regular solution to problem* (2.4) *in the sense of Definition* 2.2, *and let* (M, γ) *be defined by* (2.11) *and* (2.12), *respectively. Then, for every* $y \in M$, *it holds*

$$\mathrm{R}_\gamma(y) \leqslant \max_{\partial M} \mathrm{R}_\gamma \,. \tag{2.13}$$

Moreover, if the equality is fulfilled for some $y \in M \backslash \partial M = \pi^{-1}(\mathbb{R}^n \backslash \overline{\Omega}) \cup \{N\}$, *then* (M, γ) *is isometric to an hemisphere of radius* $[\mathrm{Cap}(\Omega)]^{1/(n-2)}$.

Translating Theorem 2.5 in terms of u and $g_{\mathbb{R}^n}$, we obtain the following extended version of Theorem 1.2 in the introduction.

2.6 Theorem. *Let* (Ω, u) *be a regular solution to problem* (1.1) *in the sense of Definition* 1.1. *Then, the following two inequalities hold*

$$\frac{1}{\mathrm{Cap}(\Omega)} \;\leqslant\; \left(\max_{\partial \Omega} \left|\frac{Du}{n-2}\right|\right)^{n-2} \quad and \quad \frac{|Du|}{u^{\frac{n-1}{n-2}}}(x) \;\leqslant\; \max_{\partial \Omega} |Du| \,, \tag{2.14}$$

for every $x \in \mathbb{R}^n \backslash \overline{\Omega}$. *Moreover, if the equality is fulfilled in either the first or the second inequality, for some* $x \in \mathbb{R}^n \backslash \overline{\Omega}$, *then* Ω *is a ball and* u *is rotationally symmetric.*

The detailed proof of inequality (2.13) and its equivalence with formulæ (2.14) in Theorem 2.6 can be found in Section 6. Again, the equality case is shown to imply that $\nabla^2 f = \lambda g$ for a certain function λ, and this fact is used in Section 7 to prove the rigidity statements.

3 The conformally equivalent formulation

We will shortly detail the study of the conformally equivalent formulation of the problem that we proposed in Section 2. Before that, let us recall a number of notational conventions.

Given $\Omega \subset \mathbb{R}^n$ bounded open set with smooth boundary and containing the origin, consider the unique solution of system (1.1), then build the metric g and the function f as in (2.1) and (2.3). In the following we will use the same conventions as in [1], using the symbols $\langle \cdot | \cdot \rangle$, $|\cdot|$, D, D^2 and Δ respectively as the scalar product, norm, covariant derivative, Hessian and Laplacian with respect to the metric $g_{\mathbb{R}^n}$ and $\langle \cdot | \cdot \rangle_g$, $|\cdot|_g$, ∇, ∇^2 and Δ_g as the same objects with respect to the metric g. For the metric g we also define Γ^k_{ij} and Ric_{ij} to be the Christoffel symbols and the Ricci tensor, whose sign is chosen so that the standard sphere has positive scalar curvature. The notations Du and ∇f will also denote the gradient vector of the corresponding functions, i.e. the composition of the covariant derivative with the inverse of the metric.

Once the framework is set up, all we have to do is to compute the relationships between the geometric objects built on $g_{\mathbb{R}^n}$ and u and those built on g and f, which we shall readily do. The reader wishing to read interesting mathematical content is suggested to skip to the next section and come back here only to verify formulæ as they desire so.

Conformal change and derivation of system (2.4). From (2.3) we have that

$$u^{-\frac{2}{n-2}} = \frac{1+f}{1-f}.$$

Rewriting g in terms of f therefore gives

$$g = \left(\frac{2}{1 + \frac{1+f}{1-f}}\right)^2 g_{\mathbb{R}^n} = (1-f)^2 g_{\mathbb{R}^n}, \tag{3.1}$$

from which it is easy to compute the Christoffel symbols:

$$\Gamma^k_{ij} = -\frac{1}{1-f}(D_i f \delta^k_j + D_j f \delta^k_i - D_h f g^{kh}_{\mathbb{R}^n} g^{\mathbb{R}^n}_{ij}).$$

Computing the covariant derivatives of f (and recalling that $\Delta u = 0$) we obtain

$$Df = -\frac{4}{n-2} \cdot \frac{u^{-\frac{n}{n-2}}}{\left(u^{-\frac{2}{n-2}}+1\right)^2} Du$$

$$= -\frac{1}{n-2} \left(\frac{1+f}{1-f}\right)^{\frac{n-2}{2}} (1-f^2)Du,$$

$$D^2 f = \left(\frac{n}{2} \cdot \frac{1}{1+f} + \frac{n-4}{2} \cdot \frac{1}{1-f}\right) Df \otimes Df - \frac{1}{n-2} \left(\frac{1+f}{1-f}\right)^{\frac{n-2}{2}} (1-f^2)D^2 u,$$

$$\Delta f = \frac{1}{1-f^2} \cdot (n-2-2f)|Df|^2,$$

for the Euclidean case. Concerning the geometrical case, we first notice that $\nabla f = Df$, as covariant derivation of functions does not actually depend on the metric (however $|\nabla f|_g^2 = (1-f)^{-2}|Df|^2$ and, in particular, $|\nabla f|_g^2 g = |Df|^2 g_{\mathbb{R}^n}$). Recalling the standard formula for the hessian of a smooth function $w \colon \mathbb{R}^n\backslash\overline{\Omega} \to \mathbb{R}$, that is $\nabla^2_{ij} w = D^2_{ij} w - \Gamma^k_{ij} D_k w$, taking $w = f$ we compute

$$\nabla^2_{ij} f = \frac{n}{1-f^2} D_i f D_j f - \frac{1}{1-f}|Df|^2 g^{\mathbb{R}^n}_{ij} - \frac{1}{n-2}\left(\frac{1+f}{1-f}\right)^{\frac{n-2}{2}}(1-f^2)D^2_{ij} u$$

$$= \frac{4}{(n-2)^2} \frac{u^{-2\frac{n-1}{n-2}}}{(u^{-\frac{2}{n-2}}+1)^2}\left[nD_i u D_j u - 2\frac{u^{-\frac{2}{n-2}}}{u^{-\frac{2}{n-2}}+1}|Du|^2 g^{\mathbb{R}^n}_{ij} - (n-2)uD^2_{ij}u\right],$$

$$\Delta_g f = \frac{1}{(1-f)^2}\Delta f + \frac{1}{(1-f)^3}(2-n)|Df|^2$$

$$= (-nf)\frac{|\nabla f|_g^2}{1-f^2}.$$

The above relations, together with formulæ (1.3), can be used to derive the following asymptotic estimates for f

$$f = 1 - 2\operatorname{Cap}(\Omega)^{\frac{2}{n-2}}|x|^{-2} \cdot (1+o(1)),$$

$$\nabla_i f = 4 \cdot \operatorname{Cap}(\Omega)^{\frac{2}{n-2}}|x|^{-4}x_i \cdot (1+o(1)), \tag{3.2}$$

$$\nabla^2_{ij} f = -4 \cdot \operatorname{Cap}(\Omega)^{\frac{2}{n-2}}|x|^{-4}g^{\mathbb{R}^n}_{ij} \cdot (1+o(1)).$$

Recalling the definition (3.1) of the metric g, from (3.2) one can also deduce the following formulæ, which will prove useful in the following

$$|\nabla f|_g = 2|x|^{-1} \cdot (1+o(1))$$

$$H_g = -\frac{n-1}{2}\operatorname{Cap}(\Omega)^{-\frac{2}{n-2}}|x| \cdot (1+o(1)),$$

$$dv_g = 2^n \cdot \operatorname{Cap}(\Omega)^{-2\frac{n}{n-2}}|x|^{-2n} \cdot (1+o(1)) \cdot dv, \tag{3.3}$$

$$\nu^i_g = \left[2 \cdot \operatorname{Cap}(\Omega)^{\frac{2}{n-2}}\right]^{-1}|x|x_j g^{ij}_{\mathbb{R}^n} \cdot (1+o(1)),$$

$$g_{ij} = 4 \cdot \operatorname{Cap}(\Omega)^{\frac{4}{n-2}}|x|^{-4} \cdot (1+o(1)).$$

We also mention that from formula (1.4) we obtain the following estimate on the area of the level sets of the function f

$$\int_{\{f=1-\varepsilon\}} d\sigma_g < C\,\varepsilon^{\frac{n-1}{2}} \tag{3.4}$$

for some constant $C > 0$ and for $0 < \varepsilon < 1$ small enough. Of course one can derive a more precise estimate, but (3.4) is enough for the purposes of this work. For future convenience, we also write down the relation

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between the norm of the gradients of f and u

$$\frac{|\nabla f|_g^2}{1 - f^2} = u^{-2\frac{n-1}{n-2}} \left| \frac{Du}{n-2} \right|^2, \tag{3.5}$$

from which the following estimate is obtained for $|x| \to \infty$:

$$\frac{|\nabla f|_g^2}{1 - f^2} = [\mathrm{Cap}(\Omega)]^{-\frac{2}{n-2}} + o(1). \tag{3.6}$$

Another equation that we need is the one that relates the Ricci tensor of the metric g with the hessian of the function f. Starting from formula (d) given in [10, Theorem 1.159] (who uses the opposite convention as us for the sign of the Laplacian), one computes

$$
\begin{aligned}
\mathrm{Ric} &= \frac{n-2}{1-f} \cdot D^2 f + \frac{2(n-2)}{(1-f)^2} \cdot Df \otimes Df + \frac{1}{1-f} \cdot \Delta f \, g_{\mathbb{R}^n} - \frac{n-3}{(1-f)^2} \cdot |Df|^2 g_{\mathbb{R}^n} \\
&= \frac{n-2}{1-f} \nabla^2 f + \frac{1}{1-f} \Delta f \, g_{\mathbb{R}^n} - \frac{1}{(1-f)^2} |Df|^2 g_{\mathbb{R}^n} \\
&= \frac{n-2}{1-f} \nabla^2 f + \frac{n-1-f}{1-f} \cdot \frac{|\nabla f|_g^2}{1-f^2} g.
\end{aligned}
$$

In particular, we have recovered system (2.4) stated in the introduction.

Relevant geometric quantities on the level sets. We pass now to analyze the geometry of the level sets of u and f. Let us start from the case in which our level set is regular, for either f or u (we have already noticed that the two notions are equivalent). We define the normal vector fields ν and ν_g, second fundamental forms h and h_g and the mean curvatures H and H_g (already mentioned in the introduction) with respect to the two metrics. The normal vectors are chosen to be directed outwards, namely, on a level set $\{u = t\} = \{f = s\}$, both ν and ν_g point towards the exterior set $\{u \leqslant t\} = \{f \geqslant s\}$. The second fundamental forms are chosen with respect to said normal vectors and stipulating that the standard sphere immersed in the Euclidean space has positive mean curvature with respect to the normal pointing away from the origin. The exact formulæ are (in this case i and j only run along the tangential coordinates to the level set):

$$\nu = -\frac{Du}{|Du|}, \qquad h_{ij} = \langle D_i \nu | \partial_j \rangle, \qquad H = g_{\mathbb{R}^n}^{ij} h_{ij},$$

$$\nu_g = \frac{\nabla f}{|\nabla f|_g}, \qquad (h_g)_{ij} = \langle \nabla_i \nu_g | \partial_j \rangle_g, \qquad H_g = g^{ij} (h_g)_{ij}.$$

If $\{f = s\}$ is a regular level set and p is one of its points, then the function f is a local coordinate, which can be completed to a full set of local coordinates $(f, x^2, \ldots, x^n)$ around p such that at any point of the level set the vector fields ∂_{x^k} are tangent to the level set for $k = 2, \ldots, n$ and ∂_f is orthogonal to it; the coordinates can also be taken to be all orthogonal to each other at the single point p. The coordinates $x^2, \ldots, x^n$ are first chosen as the usual normal coordinates on $\{f = s\}$ and then extended to a small open set around p through the flow of the vector field ∇f. The following relationship is then valid at any point of $\{f = s\}$:

$$\nabla f = |\nabla f|_g \cdot \nu_g = |\nabla f|_g^2 \cdot \partial_f,$$

since ∂_f and ∇f are by construction parallel and

$$df(\partial_f) = \partial_f(f) = 1, \qquad df(\nabla f) = \langle \nabla f | \nabla f \rangle_g = |\nabla f|_g^2.$$

With this choice of coordinates, the metric can be locally written as

$$|\nabla f|_g^{-2} df \otimes df + g_{\alpha\beta}(f, x) dx^\alpha \otimes dx^\beta,$$

where Greek indices range over tangential coordinates. In particular

$$|\nabla f|_g \cdot dv_g = d\sigma_g \cdot df. \tag{3.7}$$

where v_g is the measure induced by g on $\mathbb{R}^n \backslash \Omega$ and σ_g is the measure induced by g on the level set $\{f = s\}$. For a regular level set $\{f = s\}$ we also have

$$(h_g)_{ij} = \langle \nabla_i \nu_g | \partial_j \rangle_g = \left\langle \nabla_i \frac{\nabla f}{|\nabla f|_g} \Big| \partial_j \right\rangle_g = \frac{1}{|\nabla f|_g} \langle \nabla_i \nabla f | \partial_j \rangle_g = \frac{\nabla^2_{ij} f}{|\nabla f|_g},$$

$$H_g = g^{ij}(h_g)_{ij} = \frac{\Delta_g f - g^{ff} \nabla^2_{ff} f}{|\nabla f|_g} = \frac{\Delta_g f - \nabla^2_{vv} f}{|\nabla f|_g} = \frac{\Delta_g f}{|\nabla f|_g} - \frac{\nabla^2 f(\nabla f, \nabla f)}{|\nabla f|_g^3}, \tag{3.8}$$

so

$$\nabla^2 f(\nabla f, \nabla f) = |\nabla f|_g^2 \Delta_g f - |\nabla f|_g^3 H_g = -\frac{nf}{1-f^2} |\nabla f|_g^4 - |\nabla f|_g^3 H_g. \tag{3.9}$$

The corresponding formula for $g_{\mathbb{R}^n}$, thanks to the harmonicity of u, simplifies to

$$D^2 u(Du, Du) = |Du|^3 H.$$

Let us now pass to the analysis of the nonregular level sets of u and f. First of all, since u is harmonic, it follows from [30, Theorem 1.7] (see also [35]) that the $(n-1)$-dimensional Hausdorff measure of the level sets of u (thus also of f) is locally finite. The properness of u and f then implies that all the level sets of u, f have finite $\mathcal{H}^{n-1}$-measure. Moreover, from [20, Theorem 1.17] we know that the Minkowski dimension of the set of the critical points of u is bounded above by $(n-2)$. In particular, also the Hausdorff dimension is bounded by $(n-2)$ (the bound on the Hausdorff dimension was proved before, see for instance [29]). This implies that, for all $t \in (0,1]$, the normal ν to the level set $\{u = t\}$ is defined $\mathcal{H}^{n-1}$-almost everywhere on $\{u = t\}$, and, consequently, so are the second fundamental form h and mean curvature H. Since the critical points of f are the same as the ones of u (as one deduces immediately from (3.5)), the same is true for the normal ν_g, the second fundamental form h_g and the mean curvature H_g with respect to g. In particular, formulæ (3.7), (3.8) and (3.9) hold $\mathcal{H}^{n-1}$-almost everywhere on any level set of f.

4 Pointwise identities via Bochner formula

In this section we will abstract completely from the original formulation of the problem (which is system (1.1)) and work only with triples (Ω, f, g) satisfying system (2.4), that we recall here:

$$\begin{cases} \Delta_g f = -n \left(\frac{|\nabla f|_g^2}{1-f^2} \right) f & \text{in } \mathbb{R}^n \backslash \overline{\Omega}, \\ \mathrm{Ric} = \frac{n-2}{1-f} \nabla^2 f + \frac{n-1-f}{1-f} \left(\frac{|\nabla f|_g^2}{1-f^2} \right) g & \text{in } \mathbb{R}^n \backslash \overline{\Omega}, \\ f = 0 & \text{on } \partial\Omega, \\ f(x) \to 1 & \text{as } |x| \to +\infty. \end{cases} \tag{4.1}$$

We also recall that, in the spherical setting, we ultimately expect that the function

$$x \longmapsto \frac{|\nabla f|_g^2}{1-f^2}(x)$$

will be relevant in the rigidity statements, as we need to prove it is constant. Also, we expect the metric g to be isometric to the north round half-sphere deprived of the north pole, where the function f plays the role of the z^{n+1} coordinate (consistently, it has value zero on $\partial\Omega$, which will be the equator, and goes to 1 at infinity, which will be the north pole).

We will make use of the well-known Bochner formula, that we repeat here for the convenience of the reader: for any smooth function f on a Riemannian manifold there holds

$$\frac{1}{2} \Delta_g |\nabla f|_g^2 = |\nabla^2 f|_g^2 + \mathrm{Ric}(\nabla f, \nabla f) + \langle \nabla \Delta_g f | \nabla f \rangle_g.$$

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The formula is easy to prove rearranging and commuting derivatives, but see also [26, Proposition 4.15]. In our case, using (4.1) the Bochner formula rewrites as

$$\frac{1}{2}\Delta_g|\nabla f|_g^2 = |\nabla^2 f|_g^2 + \frac{n-2}{1-f}\nabla^2 f(\nabla f, \nabla f) + \frac{n-1-f}{1-f}\cdot\frac{|\nabla f|_g^4}{1-f^2} + \langle\nabla\Delta_g f|\nabla f\rangle.$$

We can exploit this formula to find a monotonic quantity for the function f. Let us consider the vector field X which was defined in (2.6) and which we report here again for the reader's convenience:

$$X = \frac{1}{(1-f)(1+f)^{n-1}}\left(\nabla|\nabla f|_g^2 - \frac{2}{n}\Delta_g f\nabla f\right) = \frac{1}{(1+f)^{n-2}}\nabla\left(\frac{|\nabla f|_g^2}{1-f^2}\right). \tag{4.2}$$

4.1 Lemma. *Let (Ω, f, g) be a regular solution to problem (4.1) in the sense of Definition 2.2, and let X be the vector field defined by (4.2). Then $\mathrm{div}_g X \geqslant 0$, where div_g denotes the divergence with respect to the metric g (in coordinates: $\mathrm{div}_g X = \nabla_i X^i$). Moreover, at any point, $\mathrm{div}_g X = 0$ if and only if $\nabla^2 f = \lambda g$ for some real number λ.*

Proof. We start by computing

$$\begin{aligned}\mathrm{div}_g X &= \frac{1}{(1-f)(1+f)^{n-1}}\left[\Delta_g|\nabla f|_g^2 - \frac{2}{n}(\Delta_g f)^2 - \frac{2}{n}\langle\nabla\Delta_g f|\nabla f\rangle_g\right.\\ &\qquad\left. -\frac{-nf+n-2}{1-f^2}\left\langle\nabla|\nabla f|_g^2 - \frac{2}{n}\Delta_g f\nabla f\,\bigg|\,\nabla f\right\rangle_g\right]\\ &= \frac{2}{(1-f)(1+f)^{n-1}}\left[|\nabla^2 f|_g^2 - \frac{1}{n}(\Delta_g f)^2 - \frac{1}{n}\langle\nabla\Delta_g f|\nabla f\rangle_g\right.\\ &\qquad + \frac{n-2}{1-f}\nabla^2 f(\nabla f, \nabla f) + \frac{n-1-f}{1-f}\cdot\frac{|\nabla f|_g^4}{1-f^2} + \langle\nabla\Delta_g f|\nabla f\rangle_g\\ &\qquad\left. -\frac{1}{2}\cdot\frac{-nf+n-2}{1-f^2}\langle\nabla|\nabla f|_g^2|\nabla f\rangle_g - \frac{1}{n}\cdot\frac{-nf+n-2}{1-f^2}\Delta_g f|\nabla f|_g^2\right].\end{aligned}$$

In order to simplify this formula, we single out a couple of computations. First of all, we have

$$\langle\nabla|\nabla f|_g^2|\nabla f\rangle_g = g_{ij}\cdot\nabla_i(g_{k\ell}\nabla_k f\nabla_\ell f)\cdot\nabla_j f = 2g_{ij}g_{k\ell}\nabla_{ik}^2 f\cdot\nabla_\ell f\cdot\nabla_j f = 2\nabla^2 f(\nabla f, \nabla f). \tag{4.3}$$

Moreover, using the equations in (4.1) we compute

$$\langle\nabla\Delta_g f|\nabla f\rangle_g = -n\left\langle\nabla\left(f\frac{|\nabla f|_g^2}{1-f^2}\right)\bigg|\nabla f\right\rangle_g = -n\left[\frac{1+f^2}{(1-f^2)^2}|\nabla f|_g^4 + \frac{f}{1-f^2}\langle\nabla|\nabla f|_g^2|\nabla f\rangle_g\right].$$

Putting these informations inside the formula for $\mathrm{div}_g X$, with some straightforward computations we obtain

$$\mathrm{div}_g X = \frac{2}{(1-f)(1+f)^{n-1}}\left[|\nabla^2 f|_g^2 - \frac{1}{n}(\Delta_g f)^2\right] \geqslant 0.$$

The last inequality follows from the arithmetic-quadratic mean inequality, since if $\lambda_1, \ldots \lambda_n$ are the eigenvalues of $\nabla^2 f$ (i.e., the principal curvatures), then $|\nabla^2 f|_g^2 = \sum_i \lambda_i^2$ and $(\Delta_g f)^2 = (\sum_i \lambda_i)^2$. In particular, when equality is fulfilled all λ_i are identical and $\nabla^2 f = \lambda g$. □

We conclude this section showing that the inequality $\mathrm{div}_g X$ can be rewritten as a second order elliptic inequality for the scalar curvature R of g. To this end, recalling the definition of X given in (4.2) we compute

$$\begin{aligned}\mathrm{div}_g X &= \mathrm{div}_g\left[\frac{1}{(1+f)^{n-2}}\nabla\left(\frac{|\nabla f|_g^2}{1-f^2}\right)\right]\\ &= \frac{1}{(1+f)^{n-2}}\left[\Delta_g\left(\frac{|\nabla f|_g^2}{1-f^2}\right) - (n-2)\left\langle\frac{\nabla f}{1+f}\bigg|\nabla\left(\frac{|\nabla f|_g^2}{1-f^2}\right)\right\rangle_g\right]\end{aligned}$$

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Since we have already observed in (2.5) that the scalar curvature R is a multiple of $|\nabla f|_g^2/(1-f^2)$, the inequality $\mathrm{div}_g\, X$ can be rewritten as

$$\Delta_g R - (n-2)\left\langle \frac{\nabla f}{1+f} \,\Big|\, \nabla R\right\rangle_g \;\geqslant\; 0\,. \tag{4.4}$$

Therefore, R satisfies a second order elliptic inequality, as anticipated. Inequality (4.4) is the starting point of the maximum principle argument that will be discussed in Section 6.

5 Integral identities

In this section we will study the properties of the function Φ, which was defined in Subsection 2.2 and which we recall here:

$$\Phi(s) \;=\; \frac{1}{(1-s^2)^{\frac{n+2}{2}}} \int_{\{f=s\}} |\nabla f|_g^3\, d\sigma_g. \tag{5.1}$$

More precisely, we will show that Φ is differentiable and monotonic for all $s \in [0,1)$. As a byproduct of this analysis, we will prove inequality (2.8) and its equivalence with (2.9).

We point out that the proof of the results presented in the Introduction do not depend on the content of this section. However, it is our opinion that the computations here should help the reader to compare the *spherical ansatz* with the *cylindrical ansatz*, shedding some lights on the advantages and disadvantages of both methods.

Well-definedness and continuity. First of all, let us notice that $\Phi(s)$ is indeed well-defined for all values $s \in [0,1)$.

5.1 Lemma. *Let (Ω, f, g) be a regular solution to problem (4.1) in the sense of Definition 2.2. Then the function $\Phi : [0,1) \to \mathbb{R}$ introduced in (5.1) is well-defined and it holds* $\limsup_{s\to 1^-} \Phi(s) < +\infty$.

Proof. We have already observed in Section 3 that, from the analyticity and properness of f, it follows that the level sets of f have finite $(n-1)$-dimensional Hausdorff measure. Moreover, from (3.3), it follows that $|\nabla f|_g \to 0$ as $|x| \to \infty$. In particular, since $|\nabla f|_g$ is smooth, it is uniformly bounded on $\mathbb{R}^n\backslash\Omega$. This proves that $\Phi(s)$ is well-defined.

In order to prove that it is bounded as $s \to 1^-$, we use the estimates (3.2) and (3.3) to write

$$\Phi(s) \;<\; C_1 \int_{\{f=s\}} |x|^{n-1}\, d\sigma_g \;<\; C_2 \int_{\{f=s\}} (1-s)^{-\frac{n-1}{2}}\, d\sigma_g$$

for $0 < s < 1$ sufficiently close to 1 and for some constants $0 < C_1 < C_2$. The bound on $\Phi(s)$ as $s \to 1^-$ now follows from estimate (3.4). $\qquad\square$

We pass now to discuss the continuity of Φ.

5.2 Lemma. *Let (Ω, f, g) be a regular solution to problem (4.1) in the sense of Definition 2.2. Then the function $\Phi : [0,1) \to \mathbb{R}$ defined by (5.1) is continuous and, for all $s \in [0,1)$, it holds*

$$(1-s^2)\Phi(s) \;=\; -\int_{\{f>s\}} \frac{\nabla^2 f(\nabla f, \nabla f)}{2(1-f^2)^{\frac{n}{2}}}\, dv_g\,. \tag{5.2}$$

Proof. We start with the following computation

$$\mathrm{div}_g\left[\frac{|\nabla f|_g^2}{(1-f^2)^{\frac{n}{2}}}\cdot \nabla f\right] \;=\; \frac{\left\langle \nabla|\nabla f|_g^2 \,\Big|\, \nabla f\right\rangle_g}{(1-f^2)^{\frac{n}{2}}} + \frac{|\nabla f|_g^2}{(1-f^2)^{\frac{n}{2}}}\cdot \Delta_g f + \frac{nf|\nabla f|_g^4}{(1-f^2)^{\frac{n+2}{2}}}$$

$$=\; \frac{\nabla^2 f(\nabla f, \nabla f)}{2(1-f^2)^{\frac{n}{2}}}\,, \tag{5.3}$$

where the second equality is a consequence of the second formula in system (4.1) and of identity (4.3). Now let $0 \leqslant s < 1$ be a regular level set of f and let $s < S < 1$. It follows from the asymptotic estimates (3.2) that

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if we choose S large enough, then necessarily the level set $\{f = S\}$ is regular. Integrating formula (5.3) in $\{s < f < S\}$, and applying the divergence theorem, we find

$$
\begin{aligned}
\int_{\{s<f<S\}} \frac{\nabla^2 f(\nabla f, \nabla f)}{2(1-f^2)^{\frac{n}{2}}} \, dv_g &= \int_{\{f=S\}} \left\langle \frac{|\nabla f|_g^2}{(1-f^2)^{\frac{n}{2}}} \cdot \nabla f \,\middle|\, \nu_g \right\rangle_g d\sigma_g - \int_{\{f=s\}} \left\langle \frac{|\nabla f|_g^2}{(1-f^2)^{\frac{n}{2}}} \cdot \nabla f \,\middle|\, \nu_g \right\rangle_g d\sigma_g \\
&= \int_{\{f=S\}} \frac{|\nabla f|_g^3}{(1-f^2)^{\frac{n}{2}}} \, d\sigma_g - \int_{\{f=s\}} \frac{|\nabla f|_g^3}{(1-f^2)^{\frac{n}{2}}} \, d\sigma_g \\
&= (1-S^2)\Phi(S) - (1-s^2)\Phi(s) \, .
\end{aligned}
\tag{5.4}
$$

Moreover, from Lemma 5.1, we know that Φ is bounded for values sufficiently close to 1, thus, taking the limit of (5.4) as $S \to 1^-$, we obtain (5.2).

In the case where the level set $\{f = s\}$ is critical, the argument for the proof of (5.2) becomes slightly more technical: for every $\varepsilon > 0$, one lets

$$
U_\varepsilon = B_\varepsilon(\mathrm{Crit}(f)) = \bigcup_{x \in \mathrm{Crit}(f)} B_\varepsilon(x)
$$

be the ε-neighborhood of the critical points, integrates (5.3) in $\{s < f < S\} \backslash U_\varepsilon$, and then applies the divergence theorem. Taking the limit as $\varepsilon \to 0$, since the set of the critical points of f has Minkowski dimension bounded above by $(n-2)$ (as observed in Section 3), the set U_ε shrinks fast enough and one recovers (5.4). For a more careful explanation of the technical details, we address the interested reader to the proof of [2, Proposition 4.1], which uses the same technique to deduce a similar integral identity.

To conclude the proof, it is enough to show that the right hand side of (5.2) is continuous. To this end, we first show that the integrand in (5.2) stays in $L^1(\mathbb{R}^n \backslash \Omega)$. Using the estimates in (3.2), we compute

$$
\begin{aligned}
\int_{\{f>s\}} \left| \frac{\nabla^2 f(\nabla f, \nabla f)}{2(1-f^2)^{\frac{n}{2}}} \right| dv_g &< C_1 \int_{\{f>s\}} \frac{|x|^{-2}}{|x|^{-n}} \cdot |x|^{-2n} \, dv \\
&= C_1 \int_{\{f>s\}} |x|^{-n-2} |x|^{n-1} \, d\sigma_{g_{\mathbb{S}^{n-1}}} d|x| \\
&< C_2 \int_{\{f>s\}} |x|^{-3} \, d|x| \, ,
\end{aligned}
$$

for some constants $0 < C_1 < C_2$. Using again the estimates (3.2), we see that the domain of integration $\{f > s\} = \{1 - f < 1 - s\}$ is contained in $\{\frac{1}{C_3}|x|^{-2} < 1 - s\} = \{|x| > [C_3(1-s)]^{-\frac{1}{2}}\}$ for some other constant C_3. Hence, we get

$$
\int_{\{f>s\}} \left| \frac{\nabla^2 f(\nabla f, \nabla f)}{2(1-f^2)^{\frac{n}{2}}} \right| dv_g < C_2 \int_{[C_3(1-s)]^{-\frac{1}{2}}}^{+\infty} |x|^{-3} \, d|x| < C_4(1-s) < C_4 \, ,
\tag{5.5}
$$

where C_4 is, yet again, another constant, possibly bigger than C_1, C_2, C_3. This proves that the integrand in (5.2) stays in L^1.

Now let φ^+, φ^- be the positive and negative part of the integrand in (5.2), so that

$$
\frac{\nabla^2 f(\nabla f, \nabla f)}{2(1-f^2)^{\frac{n}{2}}} = \varphi^+ - \varphi^- \, ,
$$

and consider the two measures μ^+, μ^- on $\mathbb{R}^n \backslash \Omega$, defined as

$$
\mu^+(E) = \int_E \varphi^+ \, dv_g \, , \qquad \mu^-(E) = \int_E \varphi^- \, dv_g \, ,
$$

for every v_g-measurable set $E \subseteq \mathbb{R}^n \backslash \Omega$. Clearly μ^+, μ^- are finite (because of the bound (5.5)) positive measures, and they are absolutely continuous with respect to v_g. In particular, since the Hausdorff dimension of the level sets of f is bounded by $(n-1)$ (as already noticed in Section 3), we have that the v_g-measure of the level sets of f is zero, hence also $\mu^+(\{f = s\}) = \mu^-(\{f = s\}) = 0$ for all $s \in [0, 1)$. It follows then from [7,

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Proposition 2.6] that the functions $s \mapsto \mu^+(\{f > s\}), s \mapsto \mu^-(\{f > s\})$ are continuous for all $s \in [0, 1)$. This implies that

$$\Phi(s) = -\frac{\mu^+(\{f > s\}) - \mu^-(\{f > s\})}{1 - s^2}.$$

is continuous, as wished. $\qquad\square$

Differentiability and monotonicity. Before addressing the issue of the differentiability of Φ, it is useful to prove the following intermediate result.

5.3 Lemma. *Let (Ω, f, g) be a regular solution to problem (4.1) in the sense of Definition 2.2, and let X be the vector field defined by (4.2). Then, for every $s \in [0, 1)$, it holds*

$$\int_{\{f>s\}} \mathrm{div}_g X \, dv_g = -\int_{\{f=s\}} \langle X | \nu_g \rangle_g \, d\sigma_g$$

$$= \frac{2}{(1-s)(1+s)^{n-1}} \left[\int_{\{f=s\}} |\nabla f|_g^2 H_g \, d\sigma_g + (n-1)\frac{s}{1-s^2} \int_{\{f=s\}} |\nabla f|_g^3 \, d\sigma_g \right].$$

Proof. The second equality is just a computation: using (4.2), (4.3), (3.8) and (4.1) we have

$$-\int_{\{f=s\}} \langle X | \nu_g \rangle_g \, d\sigma_g = -\int_{\{f=s\}} \frac{1}{|\nabla f|_g} \langle X | \nabla f \rangle_g \, d\sigma_g$$

$$= -\frac{1}{(1-s)(1+s)^{n-1}} \int_{\{f=s\}} \left(2\frac{\nabla^2 f(\nabla f, \nabla f)}{|\nabla f|_g} - \frac{2}{n}\Delta_g f|\nabla f|_g \right) d\sigma_g$$

$$= \frac{2}{(1-s)(1+s)^{n-1}} \left[\int_{\{f=s\}} |\nabla f|_g^2 H_g \, d\sigma_g + (n-1)\frac{s}{1-s^2} \int_{\{f=s\}} |\nabla f|_g^3 \, d\sigma_g \right].$$

It remains to prove the identity

$$\int_{\{f>s\}} \mathrm{div}_g X \, dv_g = \int_{\{f=s\}} \langle X | \nu_g \rangle_g \, d\sigma_g. \tag{5.6}$$

Again, as in the proof of Lemma 5.2, let us start from the case where $s > 0$ is a regular value, and let $s < S < 1$ be a value big enough so that the level set $\{f = S\}$ is regular. As an application of the divergence theorem, it holds

$$\int_{\{s<f<S\}} \mathrm{div}_g X \, dv_g = \int_{\{f=S\}} \langle X | \nu_g \rangle_g \, d\sigma_g - \int_{\{f=s\}} \langle X | \nu_g \rangle_g \, d\sigma_g. \tag{5.7}$$

From Cauchy-Schwarz, we have

$$\int_{\{f=S\}} \langle X | \nu_g \rangle_g \, d\sigma_g \leq \int_{\{f=S\}} |X|_g \, d\sigma_g,$$

and using the asymptotic estimates 3.3 and (3.4), we conclude

$$\lim_{S \to 1^-} \int_{\{f=S\}} \langle X | \nu_g \rangle_g \, d\sigma_g = 0.$$

Therefore, taking the limit of identity (5.7), we obtain (5.6) for all the regular values $0 < s < 1$.

If s is a critical value, one proceeds as in the proof of Lemma 5.2, that is, one integrates $\mathrm{div}_g X$ on $\{s < f < S\} \backslash U_\varepsilon$, where U_ε is the ε-neighborhood of the critical points, and then takes the limit as $\varepsilon \to 0$ to recover (5.6) (again, we refer to [2] for the technical details). $\qquad\square$

We are finally ready to prove the main result of this section.

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5.4 Proposition. *Let (Ω, f, g) be a regular solution to problem (4.1) in the sense of Definition 2.2. Then the function $\Phi : [0, 1) \to \mathbb{R}$ defined by (5.1) is differentiable and, for all $s \in [0, 1)$, it holds*

$$(1 - s^2)^{\frac{n+2}{2}} \Phi'(s) = -2 \left[\int_{\{f=s\}} |\nabla f|_g^2 H_g \, d\sigma_g + (n-1) \frac{s}{1-s^2} \int_{\{f=s\}} |\nabla f|_g^3 \, d\sigma_g \right] \leqslant 0. \tag{5.8}$$

In fact, the left hand side is increasing in s and it tends to zero as $s \to 1$. Moreover, if $\Phi'(s) = 0$ for some $s \in [0, 1)$, then it holds $\nabla^2 f = \lambda g$ on the whole $\mathbb{R}^n \backslash \Omega$, for some $\lambda \in \mathcal{C}^\infty(\mathbb{R}^n \backslash \Omega)$.

Proof. Applying the coarea formula to (5.2), we obtain the following:

$$(1 - s^2)\Phi(s) = -\int_s^1 \left[\int_{\{f=\tau\}} \frac{\nabla^2 f(\nabla f, \nabla f)}{2(1 - f^2)^{\frac{n}{2}}} \cdot \frac{1}{|\nabla f|_g} \, d\sigma_g \right] d\tau. \tag{5.9}$$

The statement of the Riemannian coarea formula is in [19, Exercise III.12, (d)]. The missing proof follows from Fubini-Tonelli's theorem and (3.7). By the fundamental theorem of calculus, we have that, if the function

$$\tau \mapsto \int_{\{f=\tau\}} \frac{\nabla^2 f(\nabla f, \nabla f)}{2(1 - f^2)^{\frac{n}{2}}} \cdot \frac{1}{|\nabla f|_g} \, d\sigma_g \tag{5.10}$$

is continuous, then Φ is differentiable. To prove the continuity of (5.10), we first compute

$$
\begin{aligned}
\int_{\{f=\tau\}} \frac{\nabla^2 f(\nabla f, \nabla f)}{2(1 - f^2)^{\frac{n}{2}}} \cdot \frac{1}{|\nabla f|_g} \, d\sigma_g &= \int_{\{f=\tau\}} \frac{\langle \nabla |\nabla f|_g^2 | \nabla f \rangle_g}{(1 - f^2)^{\frac{n}{2}}} \cdot \frac{1}{|\nabla f|_g} \, d\sigma_g \\
&= \int_{\{f=\tau\}} \frac{\left\langle (1-f)(1+f)^{n-1} X + \frac{2}{n} \Delta_g f \nabla f \, \middle| \, \nabla f \right\rangle_g}{(1 - f^2)^{\frac{n}{2}}} \cdot \frac{1}{|\nabla f|_g} \, d\sigma_g \\
&= \int_{\{f=\tau\}} \left[\left(\frac{1+f}{1-f} \right)^{\frac{n-2}{2}} \langle X | \nu_g \rangle - 2f \frac{|\nabla f|_g^3}{(1 - f^2)^{\frac{n+2}{2}}} \right] d\sigma_g \\
&= -\left(\frac{1+\tau}{1-\tau} \right)^{\frac{n-2}{2}} \int_{\{f>\tau\}} \operatorname{div}_g X \, dv_g - 2\tau \Phi(\tau), \tag{5.11}
\end{aligned}
$$

where in the latter equality we have used Lemma 5.3. Since we have already proved the continuity of Φ in Lemma 5.2, it remains to discuss the continuity of the function

$$\tau \mapsto \int_{\{f>\tau\}} \operatorname{div}_g X \, dv_g.$$

This can be done following the exact same scheme as in the proof of the continuity of Φ in Lemma 5.2, namely, one defines a positive finite measure μ on $\mathbb{R}^n \backslash \Omega$ as

$$\mu(E) = \int_E \operatorname{div}_g X \, dv_g$$

and shows that $\tau \mapsto \mu(\{f > \tau\})$ is continuous using [7, Proposition 2.6].

We have thus proved the differentiability of Φ. Now, taking the derivative of (5.9), and using (5.11), we obtain

$$(1 - s^2)\Phi'(s) = -\left(\frac{1+s}{1-s} \right)^{\frac{n-2}{2}} \int_{\{f>s\}} \operatorname{div}_g X \, dv_g. \tag{5.12}$$

Identity (5.8) follows from (5.12) and Lemma 5.3. The monotonicity of Φ is a consequence of the nonnegativity of the divergence of X, proved in Lemma 4.1.

Suppose now $\Phi'(s) = 0$ for some $s \in [0, 1)$. Comparing (5.8) and Lemma 5.3 we have that $\operatorname{div}_g X = 0$ on $\{f > s\}$. Since g and f are analytic, so is $\operatorname{div}_g X$, hence, by unique continuation, it holds $\operatorname{div}_g X = 0$ on the whole $\mathbb{R}^n \backslash \Omega$. The theorem then follows from Lemma 4.1. $\qquad\square$

With Proposition 5.4, we have almost completed the proof of Theorem 2.3 stated in Section 2. It only remains to show that the condition $\nabla^2 f = \lambda g$, $\lambda \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \Omega)$, implies the spherical symmetry of the solution. This will be the object of Section 7.

Theorem 2.3 implies Theorem 2.4. To conclude the section, we show that inequality (5.8), proved above, is equivalent to inequality (2.9) in Theorem 2.4 (as pointed out above, the rigidity statements will be discussed later, in Section 7). This is a rather straightforward computation, using the formulæ developed in Section 3

$$
\begin{aligned}
|\nabla f|_g H_g &= \Delta_g f - \frac{\nabla_{ij}^2 f \nabla_i f \nabla_j f}{|\nabla f|_g^2} \\
&= -(n-1)\frac{|Df|^2}{(1-f)^3} + \frac{1}{n-2}\left(\frac{1+f}{1-f}\right)^{\frac{n}{2}} D_{ij}^2 u \frac{D_i f D_j f}{|Df|^2} \\
&= \frac{n-1}{n-2}\left[-\frac{1}{n-2}\left(\frac{1+f}{1-f}\right)^{n-1}(1+f)|Du|^2 + \frac{1}{n-1}\left(\frac{1+f}{1-f}\right)^{\frac{n}{2}} D_{ij}^2 u \frac{D_i u D_j u}{|Du|^2} \right],
\end{aligned}
$$
$$
d\sigma_g = (1-f)^{n-1}\, d\sigma.
$$

We are now able to rewrite the summands of (2.9) in terms of u on a level set $\{f = s\} = \{u = t\}$. The first one is:

$$
\begin{aligned}
-2\int_{\{f=s\}} |\nabla f|_g^2 H_g\, d\sigma_g &= -\frac{2}{n-2}(1-s)(1+s)^{n-1}\int_{\{f=s\}}\left(\frac{1+f}{1-f}\right)^{-(n-1)+\frac{n-2}{2}+1} |Du|\cdot|\nabla f|_g H_g\, d\sigma \\
&= -\frac{2(n-1)}{(n-2)^2}(1-s)(1+s)^{n-1}\int_{\{u=t\}} |Du|^2 \cdot\left[\frac{H}{n-1}\left(\frac{1+f}{1-f}\right) - \frac{|Du|}{n-2}\left(\frac{1+f}{1-f}\right)^{\frac{n}{2}}(1+f)\right] d\sigma.
\end{aligned}
$$

The second summand is:

$$
-2\cdot\frac{(n-1)s}{1-s^2}\int_{\{f=s\}} |\nabla f|_g^3\, d\sigma_g = -\frac{2(n-1)}{(n-2)^2}(1-s)(1+s)^{n-1}\int_{\{u=t\}}\left(\frac{1+f}{1-f}\right)^{\frac{n}{2}}\cdot f\cdot|Du|^3\, d\sigma.
$$

From (2.3) we see that $\frac{1+f}{1-f} = u^{-\frac{2}{n-2}}$, therefore:

$$
\begin{aligned}
&-2\int_{\{f=s\}} |\nabla f|_g^2 H_g\, d\sigma_g - 2\cdot\frac{(n-1)s}{1-s^2}\int_{\{f=s\}} |\nabla f|_g^3\, d\sigma_g \\
&= -\frac{2(n-1)}{(n-2)^2}(1-s)(1+s)^{n-1}\int_{\{u=t\}} |Du|^2 \cdot\left(\frac{H}{n-1}u^{-\frac{2}{n-2}} - \frac{|Du|}{n-2}u^{-\frac{n}{n-2}}\right) d\sigma,
\end{aligned}
$$

which is (up to a positive factor) the quantity that appears in (2.9).

6 Geometric inequalities via strong maximum principle

The aim of this section is to prove inequality (2.13) in Theorem 2.5 and to show how this is used to recover (1.5) in Theorem 1.2.

Definition of γ and regularity of R_γ. We start by recalling some notations that were anticipated in Section 2. Given a regular solution (Ω, u) of problem (1.1), we set $\rho = [\mathrm{Cap}(\Omega)]^{1/(n-2)}$. We have already noticed that, in the case where Ω is a ball, then ρ is equal to its radius. We denote by S_ρ^n the sphere of radius ρ with its standard embedding in $\mathbb{R}^{n+1}$, and we define the stereographic projection

$$
\pi_\rho : S_\rho^n \setminus \{N\} \longrightarrow \mathbb{R}^n, \qquad (z', z^{n+1}) \longmapsto \frac{\rho\, z'}{\rho - z^{n+1}}.
$$

Let f and g be defined by (2.3) and (2.1), respectively. As anticipated in the introduction, we consider the compact space

$$
M := \pi_\rho^{-1}(\mathbb{R}^n \setminus \Omega) \cup \{N\} \subset S_\rho^n
$$

endowed with the metric

$$
\gamma_{|_y} = \begin{cases} \left(\pi_\rho^* g\right)_{|_y} & \text{if } y \in \pi_\rho^{-1}(\mathbb{R}^n \backslash \Omega)\,, \\[2mm] \left(g_{S_\rho^n}\right)_{|_N} & \text{if } y = N\,. \end{cases}
\tag{6.1}
$$

We observe that the metric γ can be seen as a conformal modification of the standard metric $g_{S_\rho^n}$. In fact, recalling the definition of the spherical metric $g_{\text{sph}} = (1 - f_0)^2 g_{\mathbb{R}^n}$ on $\mathbb{R}^n$, where f_0 is as in (2.2), we compute

$$
\pi_\rho^* g = (1 - f \circ \pi_\rho)^2 \pi_\rho^* g_{\mathbb{R}^n} = \left(\frac{1 - f \circ \pi_\rho}{1 - f_0 \circ \pi_\rho}\right)^2 \pi_\rho^* g_{\text{sph}} = \left(\frac{1 - f \circ \pi_\rho}{1 - f_0 \circ \pi_\rho}\right)^2 [\text{Cap}(\Omega)]^{-\frac{2}{n-2}} g_{S_\rho^n}\,.
$$

In particular, using the asymptotic estimates (3.2) we see that the conformal factor in the rightmost hand side tends to 1 at the north pole N, so that the metric γ is in fact continuous at N (and of course it is smooth on $M \backslash \{N\}$). In order to show that γ is also $\mathcal{C}^2$ in a neighborhood of N, it is sufficient to prove that the conformal factor has the same regularity. Again, this readily follows from (3.2). Since γ is $\mathcal{C}^2$, its scalar curvature R_γ is well-defined and continuous on the whole M. Moreover, R_γ is clearly $\mathcal{C}^\infty$ in $M \backslash \{N\}$. The following lemma tells us more about the regularity of R_γ at the north pole.

6.1 Lemma. *Let (Ω, f, g) be a regular solution to problem* (4.1) *in the sense of Definition* 2.2. *Consider the compact domain $M = \pi_\rho^{-1}(\mathbb{R}^n \backslash \Omega) \cup \{N\} \subset S_\rho^n$, equipped with the metric γ defined in* (6.1). *Then the scalar curvature R_γ of the metric γ belongs to $\mathcal{W}^{1,2}(M) \cap \mathcal{C}^0(M)$.*

Proof. Since R_γ is continuous and M is compact, clearly $R_\gamma \in L^2(M)$. We want to show that also its gradient ∇R_γ belongs to $L^2(M)$. Since R_γ is smooth on $M \backslash \{N\}$ and M is compact, it is enough to prove that the function ∇R_γ belongs to L^2 near the north pole. We observe that, from (2.5) and (4.2), we have

$$
\nabla R_\gamma(y) = n(n-1)(1+f)^{n-2} X(\pi_\rho(y))
$$

for all $y \in M \backslash \{N\}$. In particular, it holds $|\nabla R_\gamma|_\gamma^2(y) = n^2(n-1)^2(1+f)^{2(n-2)}|X|_g^2(\pi_\rho(y))$, and using the estimates in (3.3):

$$
|\nabla R_\gamma|_\gamma^2(y) = |\pi_\rho(y)|^2 \cdot o(1),
$$

as $y \to N$ (or equivalently, as $|\pi_\rho(y)| \to \infty$). Thus, for all $T > 0$, we obtain:

$$
\begin{aligned}
\int_{M \cap \{|\pi_\rho(y)| > T\}} |\nabla R_\gamma|_\gamma^2 \, dv_\gamma &= \int_{\mathbb{R}^n \cap \{|x| > T\}} |x|^2 o(1) \cdot |x|^{-2n} \, dv \\
&= o(1) \cdot \int_{\mathbb{R}^n \cap \{|x| > T\}} |x|^{-2n+2} |x|^{n-1} \, d\sigma_{g_{S^{n-1}}} d|x| \\
&= o(1) \cdot T^{2-n}.
\end{aligned}
$$

Since $n \geqslant 3$, this last term goes to zero as T goes to infinity.

Now it only remains to prove that ∇R_γ is the weak derivative of R_γ, that is, we want to show

$$
\int_M \psi \nabla_i R_\gamma \, dv_\gamma = -\int_M R_\gamma \nabla_i \psi \, dv_\gamma, \qquad \text{for all } \psi \in \mathcal{C}_c^\infty(M),\ i = 1, \ldots, n.
\tag{6.2}
$$

Fix $\psi \in \mathcal{C}_c^\infty(M)$. For all $T > 0$, we can write $\psi = \alpha_T + \beta_T$, where α_T, β_T are functions with support contained in $\{|\pi_\rho(y)| > T\} \ni N$ and $\{|\pi_\rho(y)| < 2T\}$, respectively. It is important to notice that, from the estimates on g in (3.3), it easily follows that the distance between the sets $\mathbb{R}^n \cap \{|x| < T\}$ and $\mathbb{R}^n \cap \{|x| > 2T\}$ is $O(T^{-1})$ as $T \to +\infty$. Pulling back on the sphere, we have that also the distance between $M \cap \{|\pi_\rho(y)| < T\}$ and $M \cap \{|\pi_\rho(y)| > 2T\}$ is $O(T^{-1})$ as $T \to +\infty$. From this follows that we can choose the function α_T in such a way that it holds

$$
|\nabla \alpha_T|_\gamma \leqslant CT
$$

on the whole M, for some constant $C > 0$.

Notice that the support of β_T does not contain the north pole, hence, from the smoothness of R_γ in $M \backslash \{N\}$, it immediately follows

$$
\int_M \beta_T \nabla_i R_\gamma \, dv_\gamma = -\int_M R_\gamma \nabla_i \beta_T \, dv_\gamma\,.
$$

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Therefore, in order to prove (6.2), it is enough to show that both $\int_M \alpha_T \nabla_i R_\gamma \, dv_\gamma$ and $\int_M R_\gamma \nabla_i \alpha_T \, dv_\gamma$ go to zero as $T \to +\infty$. Using again $|\nabla R_\gamma|^2_\gamma(y) = |\pi_\rho(y)|^2 \cdot o(1)$, we obtain, for all $T > 0$:

$$\left| \int_M \alpha_T \nabla_i R_\gamma \, dv_\gamma \right| \leqslant \max_M |\alpha_T| \cdot \int_{M \cap \{|\pi_\rho(y)| > T\}} |\nabla R_\gamma|_\gamma \, dv_\gamma$$

$$= \int_{M \cap \{|\pi_\rho(y)| > T\}} |\pi_\rho(y)| \cdot o(1) \, dv_\gamma$$

$$= \int_{\mathbb{R}^n \cap \{|x| > T\}} |x| \cdot o(1) \, dv_g$$

where in the last equality we have pulled back the integral on $\mathbb{R}^n$. Recalling the relation between the densities dv and dv_g shown in (3.3), we compute

$$\left| \int_M \alpha_T \nabla_i R_\gamma \, dv_\gamma \right| \leqslant \int_{\mathbb{R}^n \cap \{|x| > T\}} |x|^{-2n+1} \cdot o(1) \, dv$$

$$= \int_{\mathbb{R}^n \cap \{|x| > T\}} |x|^{-2n+1} \cdot o(1) \cdot |x|^{n-1} \, d\sigma_{g_{S^{n-1}}} d|x|$$

$$= o(1) \cdot \int_T^{+\infty} |x|^{-n} d|x|$$

$$= o(1) \cdot T^{1-n},$$

hence this integral goes to zero as $T \to +\infty$.

The estimate of the second integral is done in a similar fashion. Recalling $|\nabla \alpha_T|_\gamma \leqslant CT$ for some constant C, and using again the estimates (3.3):

$$\left| \int_M R_\gamma \nabla \alpha_T \, dv_\gamma \right| \leqslant \max_M (R_\gamma) \cdot \int_{M \cap \{|\pi_\rho(y)| > T\}} |\nabla \alpha_T|_\gamma \, dv_\gamma$$

$$\leqslant C_1 T \int_{\mathbb{R}^n \cap \{|x| > T\}} dv_g$$

$$\leqslant C_2 T^{1-n},$$

where $0 < C_1 < C_2$ are constants and we have used the fact that R_γ is bounded on M (because R_γ is continuous and M is compact). Therefore also this term goes to zero as $T \to +\infty$. This proves that ∇R_γ is indeed the weak derivative of R_γ in M. $\qquad\square$

Elliptic inequality for R_γ and maximum principle. Pulling back the quantities in formula (4.4), we find that the scalar curvature R_γ satisfies the following second order elliptic inequality

$$\Delta_\gamma R_\gamma - (n-2) \left\langle \frac{\nabla \varphi}{1 + \varphi} \,\middle|\, \nabla R_\gamma \right\rangle_\gamma \geqslant 0 \tag{6.3}$$

pointwise on $M \backslash \{N\}$, where we have denoted by φ the completion of the pull-back of f. Namely, $\varphi : M \to \mathbb{R}$ is defined by

$$\varphi(y) = \begin{cases} f \circ \pi_\rho(y) & \text{if } y \in \pi_\rho^{-1}(\mathbb{R}^n \backslash \Omega), \\ 1 & \text{if } y = N. \end{cases}$$

Notice that φ is smooth in $M \backslash \{N\}$, and it is also $\mathcal{C}^2$ near N, as it can be easily deduced from the asymptotic estimates (3.2). Having this in mind, we are going to prove that the differential inequality (6.3) holds in a weak sense on the whole M. Combining this with a weak version of the strong maximum principle, we deduce the following theorem.

6.2 Proposition. *Let (Ω, f, g) be a regular solution to problem (4.1) in the sense of Definition 2.2. Consider the domain $M = \pi_\rho^{-1}(\mathbb{R}^n \backslash \Omega) \cup \{N\} \subset S_\rho^n$, equipped with the metric γ defined by (6.1). Then, for any $y \in M$, it holds*

$$R_\gamma(y) \leqslant \max_{\partial M} R_\gamma. \tag{6.4}$$

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Moreover, the equality is fulfilled for some $y \in M \backslash \partial M = \pi_\rho^{-1}(\mathbb{R}^n \backslash \overline{\Omega}) \cup \{N\}$, if and only if $\nabla^2 f = \lambda g$ on the whole $\mathbb{R}^n \backslash \Omega$, for some $\lambda \in \mathcal{C}^\infty(\mathbb{R}^n \backslash \Omega)$.

Proof. The strategy of the proof consists in showing that inequality (6.3) holds in the weak sense on the whole M, so that we can use a strong maximum principle (see for example [28, Theorem 8.19]) to deduce the thesis. This amounts to prove that for every nonnegative $\psi \in \mathcal{C}_c^\infty(M)$ it holds

$$\int_M \left[\langle \nabla \psi | \nabla \mathrm{R}_\gamma \rangle_\gamma + (n-2)\psi \left\langle \frac{\nabla \varphi}{1+\varphi} \,\middle|\, \nabla \mathrm{R}_\gamma \right\rangle_\gamma \right] dv_\gamma \ \leqslant \ 0. \tag{6.5}$$

Fix a nonnegative $\psi \in \mathcal{C}_c^\infty(M)$. As in the proof of Lemma 6.1, for all $T > 0$ we write $\psi = \alpha_T + \beta_T$, where α_T, β_T are nonnegative functions with support contained in $\{|\pi_\rho(y)| > T\} \ni N$ and $\{|\pi_\rho(y)| < 2T\}$, respectively. As already noticed in the proof of Lemma 6.1, the function α_T can be chosen in such a way that $|\nabla \alpha_T|_\gamma \leqslant CT$, for some constant $C > 0$. Furthermore, it holds $|\nabla \mathrm{R}_\gamma|_\gamma^2 (y) = |\pi_\rho(y)|^2 \cdot o(1)$ as y approaches the north pole. We also observe that $|\nabla \varphi|_\gamma = |\pi_\rho(y)| \cdot (1 + o(1))$, as it follows from 3.3.

Since the support of β_T does not contains the north pole, and R_γ is smooth in $M \backslash \{N\}$, we immediately have the following estimate

$$\int_M \left[\langle \nabla \beta_T | \nabla \mathrm{R}_\gamma \rangle_\gamma + (n-2)\beta_T \left\langle \frac{\nabla \varphi}{1+\varphi} \,\middle|\, \nabla \mathrm{R}_\gamma \right\rangle_\gamma \right] dv_\gamma \ =$$

$$= \ -\int_{M \cap \{|\pi_\rho(y)| < 2T\}} \beta_T \left[\Delta_\gamma \mathrm{R}_\gamma - (n-2) \left\langle \frac{\nabla \varphi}{1+\varphi} \,\middle|\, \nabla \mathrm{R}_\gamma \right\rangle_\gamma \right] dv_\gamma \leqslant 0, \tag{6.6}$$

where in the last inequality we have used formula (6.3), that holds outside of N.

The right-hand side of (6.6) being independent of T, in order to prove (6.5) it is enough to show that

$$\lim_{T \to +\infty} \int_M \left[\langle \nabla \alpha_T | \nabla \mathrm{R}_\gamma \rangle_\gamma + (n-2)\alpha_T \left\langle \frac{\nabla \varphi}{1+\varphi} \,\middle|\, \nabla \mathrm{R}_\gamma \right\rangle_\gamma \right] dv_\gamma \ = \ 0. \tag{6.7}$$

Converting as usual to $\mathbb{R}^n$ with the metric g, from estimates (3.2) and (3.3) we obtain:

$$\left| \int_{M \cap \{|\pi_\rho(y)| > T\}} \left[\langle \nabla \alpha_T | \nabla \mathrm{R}_\gamma \rangle_\gamma + (n-2)\alpha_T \left\langle \frac{\nabla \varphi}{1+\varphi} \,\middle|\, \nabla \mathrm{R}_\gamma \right\rangle_\gamma \right] dv_\gamma \right| \leqslant$$

$$\leqslant \int_{M \cap \{|\pi_\rho(y)| > T\}} \left[|\nabla \alpha_T|_\gamma + (n-2)|\alpha_T| \cdot \left| \frac{\nabla \varphi}{1+\varphi} \right| \right] \cdot |\nabla \mathrm{R}_\gamma|_\gamma \, dv_\gamma$$

$$= \int_{M \cap \{|\pi_\rho(y)| > T\}} |\pi_\rho(y)|^2 \cdot o(1) \, dv_\gamma$$

$$= \int_{\mathbb{R}^n \cap \{|x| > T\}} |x|^2 \cdot o(1) \cdot |x|^{-2n} \cdot |x|^{n-1} \, d\sigma_{g_{S^{n-1}}} d|x|$$

$$= T^{2-n} \cdot o(1). \tag{6.8}$$

Since $n \geqslant 3$, estimate (6.8) proves (6.7), which in turn implies (6.5), as discussed. Therefore R_γ indeed solves (6.3) in the $\mathcal{W}^{1,2}(M)$–sense.

We are now in the position to apply [28, Theorem 8.19]. The version that we need of this result states that, if a continuous $\mathcal{W}^{1,2}$ function satisfies an elliptic inequality of the second order in the $\mathcal{W}^{1,2}$–sense (with some standard hypotheses on the coefficients, that are trivially satisfied in our case), then the strong maximum principle applies. We have already proved that the function R_γ belongs to $\mathcal{C}^0(M) \cap \mathcal{W}^{1,2}(M)$ and that it satisfies the elliptic inequality (6.3) in the weak sense, hence we have $\mathrm{R}_\gamma \leqslant \max_{\partial M} \mathrm{R}_\gamma$ as wished. Moreover, if equality holds in (6.4), then R_γ is constant on M. Therefore, from (2.5) and (4.2) we deduce that the field X is everywhere vanishing, and so is its divergence. We can now conclude using Lemma 4.1. $\square$

In order to conclude the proof of Theorem 2.5, it only remains to show that the identity $\nabla^2 f = \lambda g$ forces Ω and f to be rotationally symmetric (the fact that (M, γ) is isometric to a sphere follows easily). This will be the subject of Section 7.

Theorem 2.5 implies Theorem 2.6. We conclude this section showing that inequality (6.4) implies formulæ (2.14). Evaluating (6.4) at $y = N$ and recalling from (2.5), (3.6) that

$$\text{R}_\gamma = n(n-1)\frac{|\nabla\varphi|^2}{1-\varphi^2}, \quad \text{and} \quad \frac{|\nabla\varphi|^2}{1-\varphi^2}(N) = [\text{Cap}(\Omega)]^{-\frac{2}{n-2}}, \tag{6.9}$$

we obtain

$$\left[\frac{1}{\text{Cap}(\Omega)}\right]^{\frac{2}{n-2}} \leq \max_{\partial M}\frac{|\nabla\varphi|_\gamma^2}{1-\varphi^2}. \tag{6.10}$$

Since $\varphi = f \circ \pi_\rho$, $\partial M = \pi_\rho^{-1}(\partial\Omega)$, $f = 0$ on $\partial\Omega$, $u = 1$ on $\partial\Omega$, using formula (3.5) we find

$$\max_{\partial M}\frac{|\nabla\varphi|_\gamma^2}{1-\varphi^2} \leq \max_{\partial\Omega}|\nabla f|_g^2 = \max_{\partial\Omega}\left|\frac{Du}{n-2}\right|^2. \tag{6.11}$$

Comparing formulæ (6.10) and (6.11), we obtain the first inequality in (2.14). In order to prove the second inequality in (2.14), we evaluate (6.4) at a point $y \in \pi_\rho^{-1}(\mathbb{R}^n\backslash\Omega)$. Recalling (6.11) and (6.9), we have

$$\frac{|\nabla\varphi|_\gamma^2}{1-\varphi^2}(y) \leq \max_{\partial\Omega}\left|\frac{Du}{n-2}\right|^2.$$

Since $\varphi = f \circ \pi_\rho$, calling $x = \pi_\rho(y)$ and using (3.5), we obtain

$$u^{-2\left(\frac{n-1}{n-2}\right)}\left|\frac{Du}{n-2}\right|^2(x) \leq \max_{\partial\Omega}\left|\frac{Du}{n-2}\right|^2,$$

for all $x \in \mathbb{R}^n\backslash\overline{\Omega}$. An easy algebraic manipulation of this inequality leads to the second formula in (2.14).

7 Proof of the rigidity statements

In the two previous sections we obtained the equation $\nabla^2 f = \lambda g$, $\lambda \in \mathcal{C}^\infty(\mathbb{R}^n\backslash\Omega)$, as a byproduct of the equality case. Such condition is very strong, and in this section we show that it induces a very rigid geometric structure on the problem.

In order to simplify the formulæ, *throughout this section we will assume* $\text{Cap}(\Omega) = 1$. The case of a general capacity can be addressed following the exact same strategy, at the cost of slightly more cumbersome computations. Alternatively, one can always reduce himself to study the case $\text{Cap}(\Omega) = 1$ via an homothety of Ω, and an appropriate rescaling of u so that the boundary condition in (1.1) remains satisfied. It is easy to check that the new f, g, obtained from the rescaled u via (2.3), (2.1), still satisfy $\nabla^2 f = \lambda g$, for some (possibly different) function λ.

7.1 Proposition. *Let (Ω, f, g) be a regular solution to problem* (4.1) *in the sense of Definition 2.2, with* $\text{Cap}(\Omega) = 1$. *Suppose that the equation $\nabla^2 f = \lambda g$ is satisfied on $\mathbb{R}^n\backslash\Omega$, where $\lambda \in \mathcal{C}^\infty(\mathbb{R}^n\backslash\Omega)$. Then:*

- *The function f is regular everywhere and in particular $|\nabla f|_g^2 = 1 - f^2 \neq 0$ on $\mathbb{R}^n\backslash\Omega$.*

- *The region $\mathbb{R}^n\backslash\Omega$ is isometric to $[0, \frac{\pi}{2}) \times \partial\Omega$ with the warped product metric*

$$dr \otimes dr + \cos^2(r) \cdot g_{\alpha\beta}^{\partial\Omega} \cdot dx^\alpha \otimes dx^\beta,$$

where $g_{\alpha\beta}^{\partial\Omega}$ is the restriction of the metric g to $\partial\Omega$.

- *The above isometry sends the level sets of f to level sets of r; in particular, it sends $\{f = s\}$ to $\{r = \arcsin s\}$ for all $s \in [0, 1)$. This implies that a point tends to ∞ in $\mathbb{R}^n$ if and only if its image have $r \to \frac{\pi}{2}$.*

Proof. Let $s \in [0, 1)$ be a regular value for f and take a vector field Y tangent to $\{f = s\}$ at a point p. Then

$$\nabla_Y|\nabla f|_g^2 = 2\nabla^2 f(Y, \nabla f) = 2\lambda\langle Y|\nabla f\rangle_g = 0,$$

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so $|\nabla f|_g$ is locally constant on every regular level set of f. In particular it can be locally written as a function of f. Let us define S to be the supremum of all $s \in [0, 1)$ such that $|\nabla f|_g \neq 0$ on the region $\{f < s\}$. By Hopf's lemma (see [31, Lemma 10.55]), 0 is a regular value of f and since the set of regular values is always open this implies that $S > 0$. All the computations from now on will be done in the region $\{f < S\}$. By standard Morse theory (see for example [32, Theorem 2.2]) the region $\{f < S\}$ is diffeomorphic to $[0, 1) \times \{f = 0\} = [0, 1) \times \partial\Omega$, so local coordinates on $\partial\Omega$ can be extended, together with f, on the whole region $\{f < S\}$.

On the region $\{f < S\}$, the function $|\nabla f|_g(f)$ is well-defined and it does not vanish, so we can use it to define the function $r(f)$ given by the ordinary differential equation:

$$\begin{cases} \frac{dr}{df} = \frac{1}{|\nabla f|_g} & \text{for } 0 \leqslant f < S, \\ r(0) = 0. \end{cases}$$

It is easy to see that r can be used as a local coordinate instead of f and that the metric can be written as

$$g = dr \otimes dr + g_{\alpha\beta}(r, x) dx^\alpha \otimes dx^\beta.$$

From now on we use Greek letters for indices that must range over tangential directions. We denote by f' and f'' the derivatives of f with respect to the new variable r. From the definition of r we have that $f' = \frac{df}{dr} = |\nabla f|_g$. The Christoffel symbols of g have the following form:

$$\Gamma_{rr}^r = \Gamma_{rr}^\gamma = \Gamma_{\alpha r}^r = 0, \qquad \Gamma_{\alpha\beta}^r = -\frac{1}{2}\partial_r g_{\alpha\beta}, \qquad \Gamma_{\alpha r}^\gamma = \frac{1}{2}g^{\gamma\delta}\partial_r g_{\alpha\delta}.$$

Computing the Hessian with these coordinates:

$$\nabla^2 f = f'' dr \otimes dr + f' \nabla^2 r = f'' dr \otimes dr + \frac{1}{2}f' \partial_r g_{\alpha\beta} \, dx^\alpha \otimes dx^\beta.$$

Since $\nabla^2 f = \lambda g$ we obtain by comparison:

$$f'' = \lambda, \qquad \frac{1}{2}f'\partial_r g_{\alpha\beta} = \lambda g_{\alpha\beta}. \tag{7.1}$$

In particular $\partial_r \log g_{\alpha\beta} = 2\partial_r \log f'$, so

$$g_{\alpha\beta}(x, r) = \left[\frac{f'(r)}{f'(0)}\right]^2 g_{\alpha\beta}(x, 0)$$

(at least if $g_{\alpha\beta}(x, 0) > 0$; if it is negative we can use the same argument for $-g_{\alpha\beta}$; if it is zero, then it remains identically zero; in any case, formula (7.1) holds). In other words, g is a warped metric defined over the base space $\{f = 0\}$.

Let us now recall that f satisfies system (2.4), so it must hold

$$\lambda = -f\frac{|\nabla f|_g^2}{1 - f^2}.$$

Expanding in (7.1), we have $f''(1 - f^2) = -f|\nabla f|_g^2 = -(f')^2 f$. This implies that

$$\frac{1}{2}\frac{\partial}{\partial r}\left[\frac{(f')^2}{1 - f^2}\right] = \frac{f'f''(1 - f^2) + (f')^2 ff'}{(1 - f^2)^2} = \frac{-(f')^3 f + (f')^3 f}{(1 - f^2)^2} = 0,$$

and comparing with estimate (3.6), we deduce $(f')^2 = (1 - f^2)$ on the whole $\mathbb{R}^n \backslash \Omega$. It follows that the equation for f can be rewritten as

$$\begin{cases} f'' = -f, \\ f(0) = 0, \\ f'(0) = 1, \end{cases}$$

whose solution is $f(r) = \sin(r)$. This implies that f is defined and regular for $r \in [0, \frac{\pi}{2})$, which coincides with the region $f \in [0, 1)$. This proves that $S = 1$, and the construction above gives an isometry of the whole space $\mathbb{R}^n \backslash \Omega$ with $[0, \frac{\pi}{2}) \times \partial\Omega$. The proposition is therefore established. $\qquad\square$

These results are expected in view of the thesis we are aiming to, according to which the metric g is an hemisphere and the hypersurface $\{f = 0\}$ is its equator. In order to conclude, it is enough to show that the metric on the base space $\partial\Omega$ is isometric to a round sphere (so far we have not even proved it is homeomorphic to a sphere).

The last key clue lies in the fact that, as seen in (6.1) and in the following discussion, if the end of $\mathbb{R}^n$ is compactified by adding the point at infinity, then the metric g can be extended at the infinity and becomes isometric to γ, which is also $\mathcal{C}^1$ at infinity. The warped structure of g then converts to a set of normal coordinates for γ at the north pole, allowing us to trigger the following lemma taken from [18, Claim 4 at page 133].

7.2 Lemma. *Let (M^n, γ) be a complete Riemannian manifold and N a point in its interior, such that the metric γ is smooth on $M \backslash \{N\}$ and $\mathcal{C}^1$ at N; call $r\colon M \to \mathbb{R}_{\geqslant 0}$ the distance function from N. Suppose there exists $\varepsilon > 0$, a function $w\colon [0, \varepsilon) \to \mathbb{R}$ and a closed Riemannian manifold (Σ^{n-1}, g_Σ) such that the punctured ball $\{x \in M \mid 0 < r(x) < \varepsilon\}$ is isometric to the manifold $((0, \varepsilon) \times \Sigma, g)$, where*

$$g(r, \theta) = dr \otimes dr + w^2(r) g_\Sigma(\theta), \qquad r \in (0, \varepsilon), \theta \in \Sigma,$$

and that the isometry sends geodesic spheres on Σ slices. Suppose moreover that $w(0) = 0$ and $w'(0) \neq 0$. Then (Σ, g_Σ) is isometric to the standard round sphere S^{n-1} with its canonical metric, up to a constant factor.

The proof of the rigidity statement is thus complete.

8 Sphere theorems

In this section we discuss the main consequences of Theorem 1.2, distinguishing between analytic and geometric results.

8.1 Analytic consequences

As a first implication of Theorem 1.2 we obtain the following sphere theorem, which roughly speaking says that if the strength of the electric field $|Du|$ measured on the surface of the uniformly charged body Ω is bounded above by the inverse of the so called surface radius $(|\partial\Omega|/|S^{n-1}|)^{1/(n-1)}$, then the charged body must be rotationally symmetric.

8.1 Corollary (Sphere Theorem I). *Let (Ω, u) be a regular solution to problem (1.1) in the sense of Definition 1.1, and suppose that*

$$\left| \frac{Du}{n-2} \right| \leqslant \left(\frac{|S^{n-1}|}{|\partial\Omega|} \right)^{\frac{1}{n-1}} \quad \text{on } \partial\Omega. \tag{8.1}$$

Then $\partial\Omega$ is a sphere and u is rotationally symmetric.

Proof. Dividing both sides of (8.1) by $|S^{n-1}|$, integrating over $\partial\Omega$ and recalling formula (1.2) for the capacity of Ω, we get

$$\mathrm{Cap}(\Omega) \ = \ \frac{1}{(n-2)|S^{n-1}|} \int_{\partial\Omega} |Du| \, d\sigma \ \leqslant \ \left(\frac{|S^{n-1}|}{|\partial\Omega|} \right)^{-\frac{n-2}{n-1}}.$$

Combining this inequality with (1.5) and using again the hypothesis (8.1), we obtain the following chain of inequalities

$$\left(\frac{|S^{n-1}|}{|\partial\Omega|} \right)^{\frac{n-2}{n-1}} \ \leqslant \ \frac{1}{\mathrm{Cap}(\Omega)} \ \leqslant \ \left(\max_{\partial\Omega} \left| \frac{Du}{n-2} \right| \right)^{n-2} \ \leqslant \ \left(\frac{|S^{n-1}|}{|\partial\Omega|} \right)^{\frac{n-2}{n-1}}.$$

In particular, the second inequality holds with the equality sign and the conclusion follows by the rigidity statement of Theorem 1.2. $\qquad\square$

Corollary 8.1 should be compared with [45, Theorem 2], where a similar sphere theorem is proven. In terms of our notation, Reichel's Theorem reads as follows:

8.2 Theorem (Reichel [45]). *Let (Ω, u) be a regular solution to problem (1.1) in the sense of Definition 1.1 and suppose that*

$$|Du| \equiv c \quad on \; \partial\Omega, \tag{8.2}$$

for some positive constant $c > 0$. Then $\partial\Omega$ is a sphere and u is rotationally symmetric.

8.3 Remark. *Reichel's Theorem has been generalized in a number of directions. A different proof of Theorem 8.2 has appeared in [13], and similar results have been proved for other classes of elliptic inequalities [44], for annulus-type domains [43] and for the p-laplacian [27, 12]. The case of a disconnected Ω –with possibly different Dirichlet and Neumann condition on the different boundary components– is analyzed in [47]. It seems interesting to ask if our Corollary 8.1 admits similar generalizations. We just point out that our main result extends without effort to the case in which Ω is not connected, but assuming that we have the same Dirichlet condition $u = 1$ on the whole $\partial\Omega$.*

To understand the relationship between Reichel's overdetermining assumption (8.2) and our assumption (8.1) we observe that in virtue of our Theorem 1.2, the constant c that appears in Reichel's hypothesis is no longer allowed to be any positive constant, but it must obey the *a priori* constraint

$$(n-2) \left[\frac{1}{\mathrm{Cap}(\Omega)} \right]^{\frac{1}{n-2}} \leqslant c. \tag{8.3}$$

Using again Reichel's overdetermining condition (8.2), one can express the capacity of Ω as

$$\mathrm{Cap}(\Omega) = \frac{c}{n-2} \left(\frac{|\partial\Omega|}{|S^{n-1}|} \right)$$

so that, up to some algebra, inequality (8.3) can be written as

$$\left(\frac{|S^{n-1}|}{|\partial\Omega|} \right)^{\frac{1}{n-1}} \leqslant \frac{c}{n-2} \equiv \left| \frac{Du}{n-2} \right| \quad on \; \partial\Omega.$$

Having this in mind, we can rephrase our Corollary 8.1 as follows.

8.4 Corollary (Sphere Theorem I - Revisited). *If (Ω, u) is a regular solution to problem (1.1) in the sense of Definition 1.1, and we suppose that its gradient $|Du|$ at $\partial\Omega$ is bounded from above by the least admissible constant Neumann data in Reichel's overdetermining prescription (8.2), then $\partial\Omega$ is a sphere and u is rotationally symmetric.*

8.2 Geometric consequences

In order to state some purely geometric consequence of Theorem 1.2, we recall from [2, formula (2.6)] the inequality

$$\max_{\partial\Omega} \left| \frac{Du}{n-2} \right| \leqslant \max_{\partial\Omega} \left| \frac{H}{n-1} \right|, \tag{8.4}$$

which holds for regular solutions of (1.1). Here H is the mean curvature of the hypersurface $\partial\Omega$ with respect to the normal pointing towards the interior of Ω and the standard flat metric of $\mathbb{R}^n$. We precise that, in our convention, the mean curvature H is defined as the sum of the principal curvatures. Combining (8.4) with (1.5) we obtain a geometric inequality (see (8.5) below) involving only the capacity of Ω and the mean curvature of its boundary, whose equality case is characterized in terms of the rotational symmetry of Ω. This latter rigidity statement answers to a question raised in [2] as well as [3].

8.5 Theorem. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and $n \geqslant 3$. Then the following inequality holds*

$$\frac{1}{\mathrm{Cap}(\Omega)} \leqslant \left(\max_{\partial\Omega} \left| \frac{H}{n-1} \right| \right)^{n-2}. \tag{8.5}$$

Moreover, the equality is fulfilled if and only if Ω is a round ball.

In other words, Theorem 8.5 tells us that among regular bounded domains for which the maximum of the mean curvature of the boundary is fixed, the capacity is minimized by spheres. As an immediate corollary of Theorem 8.5, we obtain the following sphere theorem, that should be compared with the classical Alexandrov Theorem for compact connected constant mean curvature hypersurfaces in Euclidean space. Here we draw the same conclusion under a pointwise pinching assumption for the mean curvature.

8.6 Corollary (Sphere Theorem II). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and $n \geqslant 3$. Suppose that*

$$- \left[\frac{1}{\mathrm{Cap}(\Omega)} \right]^{\frac{1}{n-2}} \leqslant \frac{H}{n-1} \leqslant \left[\frac{1}{\mathrm{Cap}(\Omega)} \right]^{\frac{1}{n-2}} . \tag{8.6}$$

Then $\partial\Omega$ is a sphere.

8.7 Remark. *This result was already provided in [41, Formula (3.29)] in dimension $n = 3$. In the case of starshaped domains, a simple proof, which requires only the second inequality in (8.6), can be obtained combining inequality (1.7) with Minkowski's identity*

$$\int_{\partial\Omega} \frac{H}{n-1} \langle x | \nu \rangle \, d\sigma = |\partial\Omega| ,$$

in order to recover the equality case of the standard isoperimetric inequality. We also mention that, in the case where Ω is convex and $n = 3$, another easy proof of the same result has been given in [9].

We observe that the relation between our Sphere Theorem II and the celebrated Alexandrov Theorem is the perfect analogous of the one described in the previous subsection between our Sphere Theorem I and Reichel's result. In order to see this, let us rephrase the Alexandrov Theorem in our setting.

8.8 Theorem (Alexandrov [6]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and $n \geqslant 3$. Suppose that*

$$H \equiv c \quad on \ \partial\Omega , \tag{8.7}$$

for some constant $c > 0$. Then $\partial\Omega$ is a sphere.

As for Reichel's Theorem, also Alexandrov's Theorem has been generalized in several ways. A similar result for the k-mean curvature has been proved in [39, 46], and quantitative versions of Alexandrov's Theorem appeared in [21, 22, 36, 37]. Also, a characterization of the constant mean curvature hypersurfaces have been found in other space-forms [5] and in warped product manifolds (see [16, 17, 38]). Corollary 8.6 follows yet another direction, and can be interpreted as a rigidity result under a pinching condition on the mean curvature. Since the technique used in this work does not seem to depend deeply on the structure of the Euclidean space, it is natural to ask if also our result extends to other manifolds. It should be mentioned that a different pinching condition has been proven in [36, formula (1.9)], where it is shown that if

$$\frac{|\partial\Omega|}{n \, |\Omega|} \leqslant \frac{H}{n-1} ,$$

then Ω is a ball.

In Theorem 8.8 we have dropped any attempt of writing the optimal regularity assumptions. Moreover, we observe that by the compactness of $\partial\Omega$ the hypothesis on the sign of the constant c is not restrictive. On the other hand, in virtue of Theorem 8.5, the constant c in (8.7) is *a priori* forced to obey the constraint

$$\left[\frac{1}{\mathrm{Cap}(\Omega)} \right]^{\frac{1}{n-2}} \leqslant \frac{c}{n-1} \equiv \left| \frac{H}{n-1} \right| \quad on \ \partial\Omega .$$

Hence, it is clear that our Corollary 8.6 can be rephrased as follows.

8.9 Corollary (Sphere Theorem II - Revisited). *If a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary and $n \geqslant 3$ is such that the absolute value of the mean curvature of its boundary is bounded from above by the least admissible constant in (8.7), then $\partial\Omega$ is a sphere.*

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