Maximum principle for an optimal control problem associated to a SPDE with nonlinear boundary conditions

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Abstract

We study a control problem where the state equation is a nonlinear partial differential equation of the calculus of variation in a bounded domain, perturbed by noise. We allow the control to act on the boundary and set stochastic boundary conditions that depend on the time derivative of the solution on the boundary. This work provides necessary and sufficient conditions of optimality in the form of a maximum principle. We also provide a result of existence for the optimal control in the case where the control acts linearly.
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1 Introduction

Let $\mathcal{O} \subseteq \mathbb{R}^n$ be a bounded smooth domain with regular boundary $\Gamma := \partial\mathcal{O}$; we also fix a terminal time $T > 0$. We fix a nonlinear operator $\operatorname{div} \mathbf{a}(x, \nabla y)$ of Leray–Lions type, and we consider the following controlled nonlinear diffusion equation with dynamical boundary conditions:

$$(1) \qquad \begin{cases} dy(t,x) = \operatorname{div} \mathbf{a}(x, \nabla y)\, dt + b\, dw(t,x), & (t,x) \in (0,T) \times \mathcal{O}; \\ dy(t,\xi) = [-\mathbf{a}(\xi, \nabla y) \cdot \nu - \gamma(\xi, y(t,\xi), u(t,\xi))]\, dt + \tilde{b}\, d\tilde{w}(t,\xi), & (t,\xi) \in (0,T) \times \Gamma; \\ y(0,x) = y_0(x). & x \in \bar{\mathcal{O}}. \end{cases}$$

w and $\tilde{w}$ are infinite dimensional Wiener processes with values in $L^2(\mathcal{O})$ and $L^2(\partial\mathcal{O})$, respectively. We assume that u is an admissible control acting on the boundary and we study the problem of minimizing the cost functional

$$J(u) := \mathbb{E}\left\{\int_0^T \left[\int_{\mathcal{O}} \ell(x, y(s,x))\, dx + \int_\Gamma \bar{\ell}(\xi, y(s,\xi), u(s,\xi))\, d\xi\right] ds\right\} + \mathbb{E}\left[\int_{\mathcal{O}} \psi(x, y(T,x)) dx + \int_\Gamma \bar{\psi}(\xi, y(T,\xi))\, d\xi\right] \tag{2}$$

Equations of the form (1), called fully parabolic boundary value problem in the seminal paper of Escher [17], have been considered also in the stochastic setting, see *e.g.* Chueshov and Schmalfuß [10], Bonaccorsi and Ziglio [8] and Barbu, Bonaccorsi and Tubaro [4]. Such problems are used to describe a wide variety of physical processes, among which we mention heat propagation in a plasma gas, population dynamics and other nonlinear diffusive phenomena (e.g., see [11]). It should be noticed that boundary conditions of the form prescribed in (1) are of a non-standard type; nevertheless, dynamical boundary conditions, *i.e.* involving a time derivative of the solution on the boundary are used as a model in several physical systems , see the paper [20] for a derivation and a physical interpretation in the case of the heat equation; further applications are given to heat transfer in a solid imbedded in a moving fluid [33, §7.2], surface gravity waves in oceanic models ([13], [14], [29]), as well as in fluid dynamics [32], phase separation phenomena [15], and this list is far from being exhaustive.

In our setting, existence for the solution of equation (1) is proven in [8] or [4] via an operatorial approach which allows to rewrite the system as a stochastic differential equation in the product space $H^1(\mathcal{O}) \times L^2(\Gamma)$. A similar approach was recently developed for a class of deterministic parabolic equation with Wentzell boundary conditions in [5].

Our objective is to control such a system trough the boundary, considering that in practice it is easier to implement boundary control than distributed parameter controls (see [12] for a discussion about the subject). Such control problems have been widely studied in the deterministic literature (see [26]) and have been addressed in the stochastic case as well (see [16], [21], [25], [28], [12]). With regard to dynamical boundary conditions, we mention that an associated control problem have already been addressed by Bonaccorsi, Confortola, Mastrogiacomo [7], following the backward SDEs (BSDEs, for short) approach introduced by Fuhrman and Tessitore in [19] in an abstract setting. We emphasize that in general the above papers concern one-dimensional domains.

The present article deals with the control problem from a different point of view. We will follow the maximum principle approach, which has been introduced by Pontryagin and his group in the 1950's in order to establish necessary conditions of optimality for deterministic controlled systems. Towards the extension to the stochastic controlled systems one difficulty is that the adjoint equation becomes a linear BSDE, especially for stochastic PDEs (SPDEs), in which case the respective BSDE can be seen as a backward SPDE (BSPDE, for short). Several papers are devoted to the study of maximum principles for SPDEs; see, e.g., [6], [24], [31]. Stochastic maximum principle for SPDEs with noise and control on the boundary were established by Guatteri [22] and Guatteri and Masiero [23], in the case of an interval in $\mathbb{R}$. Their treatment, based on semigroup theory, is different from ours; in this paper we deal with variational solutions for the controlled system, as well as for the adjoint equation.

The paper is organized as follows. In section 2, we introduce some notations and recall some preliminary results concerning the well-posedness of the state equation. Section 3 is devoted to the derivation of necessary and sufficient optimality conditions in the form of a maximum principle. In order to achieve this, we use the duality between the adjoint equation and the variation equation. We will first analyze the adjoint equation, for which we give an existence theorem based on a result of Márquez-Durán and Real [27] concerning BSDEs in a variational framework. Then, the variation equation is obtained by using a linear perturbation of the control. In section 4, we prove directly the existence of an optimal control under the assumption that the coefficient γ depends linearly on the control.

2 Preliminaries

Let $\mathcal{O} \subseteq \mathbb{R}^n$ be a bounded domain which is sufficiently regular (see, e.g. [1], Remark 7.45). On $\mathcal{O}$ we introduce the standard Sobolev space $H^1(\mathcal{O})$; on the boundary $\Gamma := \partial\mathcal{O}$ we consider the fractional order Sobolev space

$$H^{\frac{1}{2}}(\Gamma) := \left\{ \bar{y} \in L^2(\Gamma) \mid \int_\Gamma \int_\Gamma \frac{|\bar{y}(\xi) - \bar{y}(\xi')|^2}{|\xi - \xi'|^n} d\xi d\xi' < +\infty \right\}.$$

The following result of compactness of the injection will be useful later:

$$H^{\frac{1}{2}}(\Gamma) \hookrightarrow L^2(\Gamma), \text{ compactly.}$$

It is well-known that for a smooth domain $\mathcal{O}$, the *trace* operator $\tau : H^1(\mathcal{O}) \to L^2(\Gamma)$, with the property that $\tau(y) = y|_\Gamma$, $\forall y \in H^1(\mathcal{O}) \cap C(\bar{\mathcal{O}})$, is well-defined. Moreover, the range of τ is actually $H^{\frac{1}{2}}(\Gamma)$ and

$$\|\tau(y)\|_{H^{\frac{1}{2}}(\Gamma)} \leq K \|y\|_{H^1(\mathcal{O})}, \ \forall y \in H^1(\mathcal{O})$$

for some constant K depending only on $\mathcal{O}$.

In what follows we suppose that the domain $\mathcal{O}$ is bounded and smooth. We introduce the "pivot" space $H := L^2(\mathcal{O}) \times L^2(\Gamma)$ endowed with the natural inner product

$$\langle (y, \bar{y}), (y', \bar{y}') \rangle_H := \langle y, y' \rangle_{L^2(\mathcal{O})} + \langle \bar{y}, \bar{y}' \rangle_{L^2(\Gamma)}, \ (y, \bar{y}), (y', \bar{y}') \in H$$

and norm $\|\cdot\|_H$. Let us consider the Banach space

$$V := \left\{ (y, \bar{y}) \in H^1(\mathcal{O}) \times H^{\frac{1}{2}}(\Gamma) \mid \bar{y} = \tau(y) \right\};$$

endowed with the norm

$$\|(y, \bar{y})\|_V := \|\nabla y\|_{L^2(\mathcal{O})} + \|\bar{y}\|_{L^2(\Gamma)}.$$

The embedding $V \hookrightarrow H$ is compact. Furthermore, the space V is isomorphic to $H^1(\mathcal{O})$ and it is densely embedded in H. Let V^* be the dual space of V, with the dualization denoted ${}_{V^*}\langle \cdot, \cdot \rangle_V$. We fix the Gelfand triple $V \subseteq H \subseteq V^*$ (the last formal inclusion implies that ${}_{V^*}\langle \mathbf{z}, \mathbf{y} \rangle_V = \langle \mathbf{z}, \mathbf{y} \rangle_H$ for every $\mathbf{y} \in V$ and $\mathbf{z} \in H$).

Let U be a convex, closed subset of an Euclidian space $\mathbb{R}^m$. On the coefficients of the equation we impose the following conditions:

($\mathbf{A}_0$) $\mathbf{a} : \mathcal{O} \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function† with $\mathbf{a}(x, \cdot) \in C^1(\mathbb{R}^n; \mathbb{R}^n)$, dx-a.e. on $\mathcal{O}$;

there exist constants $\delta, c_0 > 0$ and positive functions $\rho \in L^2(\mathcal{O})$, $\tilde{\rho} \in L^2(\Gamma)$ such that:

($\mathbf{A}_1$) for almost all $x \in \mathcal{O}$ and all $\zeta \in \mathbb{R}^n$:

$$|\mathbf{a}(x, \zeta)| \leq c_0(\rho(x) + |\zeta|),$$
$$|D_\zeta \mathbf{a}(x, \zeta)| \leq c_0;$$

†i.e., $\mathbf{a}(x, \cdot)$ is continuous for every $x \in \mathcal{O}$ and $\mathbf{a}(\cdot, \zeta)$ is measurable for every $\zeta \in \mathbb{R}^n$

(A_2) for almost all $x \in \mathcal{O}$ and all $\zeta \in \mathbb{R}^n$:

$$(\mathbf{a}(x,\zeta) - \mathbf{a}(x,\eta)) \cdot (\zeta - \eta) \geq \delta \left|\zeta - \eta\right|^2;$$

(B) $b : L^2(\mathcal{O}) \to L^2(\mathcal{O})$ and $\tilde{b} : L^2(\Gamma) \to L^2(\Gamma)$ are trace-class linear operators;

(C_0) $\gamma : \Gamma \times \mathbb{R} \times U \to \mathbb{R}$ is a Carathéodory function with $\gamma(\xi, \cdot) \in C^1(\mathbb{R} \times U)$, $d\xi$-a.e. on Γ;

(C_1) for almost all $\xi \in \Gamma$ and all $(\bar{y}, u) \in \mathbb{R} \times U$:

$$\begin{aligned} |\gamma(\xi,\bar{y},u)| &\leq c_0 \left(\tilde{\rho}(\xi) + |\bar{y}| + |u|\right), \\ |D_{\bar{y}}\gamma(\xi,\bar{y},u)| &\leq c_0, \\ |D_u\gamma(\xi,\bar{y},u)| &\leq c_0 \left(\tilde{\rho}(\xi) + |\bar{y}|\right); \end{aligned}$$

(C_2) for almost all $\xi \in \Gamma$ and all $(\bar{y}, u) \in \mathbb{R} \times U$:

$$(\gamma(\xi,\bar{y},u) - \gamma(\xi,\bar{y}',u))\,(\bar{y} - \bar{y}') \geq \delta \left|\bar{y} - \bar{y}'\right|^2.$$

In order to give a functional setting for our equation, we set $A : V \times L^2(\Gamma; U) \to V^*$ by

$${}_{V^*}\langle A(y,\bar{y},u),(z,\bar{z})\rangle_V := -\int_{\mathcal{O}} \mathbf{a}(x,\nabla y)\cdot\nabla z\,dx - \int_\Gamma \gamma(\xi,\bar{y},u)\bar{z}\,d\xi,$$

which is well-defined, by the above hypotheses.

We also set $B := \begin{pmatrix} b & 0 \\ 0 & \tilde{b} \end{pmatrix} \in L_2(H)$, and consider a H-cylindrical Wiener process, formally written

$$W(t) = \begin{pmatrix} w(t) \\ \tilde{w}(t) \end{pmatrix} := \begin{pmatrix} \sum_{k=1}^\infty \beta_k^1(t) g_k^1 \\ \sum_{k=1}^\infty \beta_k^2(t) g_k^2 \end{pmatrix},$$

where $\left\{g_k^1\right\}$ and $\left\{g_k^2\right\}$ are orthonormal bases in $L^2(\mathcal{O})$ and $L^2(\Gamma)$, respectively, $\left\{\beta_k^i\right\}_{k=1,\infty}^{i=1,2}$ is a sequence of independent Brownian motions on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\{\mathcal{F}_t\}_{t\geq 0}$ is the filtration generated by $\left\{\beta_k^i\right\}_{k=1,\infty}^{i=1,2}$, augmented by the null sets of $\mathcal{F}$.

Then, for $\boldsymbol{y}_0 = (y_0, \bar{y}_0) \in H$, the state equation (1) can be written as

$$\mathbf{Y}(t) = \mathbf{y}_0 + \int_0^t A(\mathbf{Y}(s), u(s))\,ds + \int_0^t B\,dW(s),\ t \in [0,T]. \tag{3}$$

Here we assume that u is an *admissible control* (or simply, control), *i.e.* a progressively measurable process $u \in L^2\left(\Omega \times [0,T]; L^2(\Gamma; U)\right)$. We will denote by $\mathcal{U}$ the space of all admissible controls.

Theorem 2.1. *Under hypotheses (A_0)–(A_2), (B), (C_0)–(C_2), for every control u, there exists a unique solution $\mathbf{Y}^u = (Y^u, \bar{Y}^u) \in L^2(\Omega \times [0,T]; V)$ of equation (3) such that $\mathbf{Y}^u$ is a continuous, adapted process with values in H. Moreover,*

$$\mathbb{E} \sup_{t\in[0,T]} \|\mathbf{Y}^u(t)\|_H^2 < +\infty.$$

For the proof of this result the reader can refer to the book of Prévôt and Röckner [30], where a general result of existence and uniqueness for variational solutions was given. The task of verifying that the above hypotheses are sufficient to place ourselves into their framework was already carried in [8].

Concerning the cost functional (2), on its coefficients we impose the following hypotheses (the functions ρ and $\tilde{\rho}$ were already introduced for the previous set of conditions):

($\mathbf{F}_0$) $\psi : \mathcal{O}\times\mathbb{R}$ and $\bar{\psi} : \Gamma\times\mathbb{R}$ are Carathéodory functions with $\psi(x,\cdot) \in C^1(\mathbb{R})$, dx-a.e. on $\mathcal{O}$ and $\bar{\psi}(\xi,\cdot) \in C^1(\mathbb{R})$, $d\xi$-a.e. on Γ;

there exist constants $c_1, c_2 > 0$ such that:

($\mathbf{F}_1$) for almost all $x \in \mathcal{O}$ and all $y \in \mathbb{R}$:

$$|\psi(x,y)| \le c_1(\rho(x)^2 + |y|^2),$$
$$|D_y\psi(x,y)| \le c_1(\rho(x) + |y|);$$

for almost all $\xi \in \Gamma$ and all $\bar{y} \in \mathbb{R}$:

$$\left|\bar{\psi}(\xi,\bar{y})\right| \le c_1(\tilde{\rho}(\xi)^2 + |\bar{y}|^2),$$
$$\left|D_{\bar{y}}\bar{\psi}(\xi,\bar{y})\right| \le c_1(\tilde{\rho}(\xi) + |\bar{y}|);$$

($\mathbf{L}_0$) $\ell : \mathcal{O}\times\mathbb{R}$ and $\bar{\ell} : \Gamma\times\mathbb{R}\times U$ are Carathéodory functions with $\ell(x,\cdot) \in C^1(\mathbb{R})$, dx-a.e. on $\mathcal{O}$ and $\bar{\ell}(\xi,\cdot,\cdot) \in C^1(\mathbb{R}\times U)$, $d\xi$-a.e. on Γ;

($\mathbf{L}_1$) for almost all $x \in \mathcal{O}$ and all $y \in \mathbb{R}$:

$$|\ell(x,y)| \le c_2(\rho(x)^2 + |y|^2),$$
$$|D_y\ell(x,y)| \le c_2(\rho(x) + |y|),$$

for almost all $\xi \in \Gamma$ and all $(\bar{y},u) \in \mathbb{R}\times U$:

$$\left|\bar{\ell}(\xi,\bar{y},u)\right| \le c_2(\tilde{\rho}(\xi)^2 + |\bar{y}|^2 + |u|^2),$$
$$\left|D_{\bar{y}}\bar{\ell}(\xi,\bar{y},u)\right| \le c_2(\tilde{\rho}(\xi) + |\bar{y}| + |u|),$$
$$\left|D_u\bar{\ell}(\xi,\bar{y},u)\right| \le c_2(\tilde{\rho}(\xi) + |\bar{y}| + |u|).$$

The cost functional can then be written as

$$J(u) := \mathbb{E}\left[\int_0^T L(\mathbf{Y}^u(t), u(t))dt + \Psi(\mathbf{Y}^u(T))\right], \tag{4}$$

where $L : H \times L^2(\Gamma;U) \to \mathbb{R}$ and $\Psi : H \to \mathbb{R}$ are defined by

$$L(y,\bar{y},u) := \int_O \ell(x,y(x))dx + \int_\Gamma \bar{\ell}(\xi,\bar{y}(\xi),u(\xi))d\xi;$$
$$\Psi(y,\bar{y}) := \int_O \psi(x,y(x))dx + \int_\Gamma \bar{\psi}(\xi,\bar{y}(\xi))d\xi.$$

From now on, we will assume that conditions (A_0)–(A_2), (B), (C_0)–(C_2), (F_0), (F_1), (L_0) and (L_1) are in force.

It is easy to show that Ψ and L are Gâteaux differentiable in $\mathbf{y} = (y, \bar{y}) \in H$, with

$$D_{\mathbf{y}}\Psi(\mathbf{y}) = \left(D_y\psi(\cdot, y(\cdot)), D_{\bar{y}}\bar{\psi}(\cdot, \bar{y}(\cdot))\right);$$
$$D_{\mathbf{y}}L(\mathbf{y}, u) = \left(D_y\ell(\cdot, y(\cdot)), D_{\bar{y}}\bar{\ell}(\cdot, \bar{y}(\cdot), u(\cdot))\right).$$

Also, A is Gâteaux differentiable in $\mathbf{y} = (y, \bar{y}) \in V$, with

$${}_{V^*}\langle (D_{\mathbf{y}}A)(\mathbf{y}, u)(p, \bar{p}), (z, \bar{z})\rangle_V = -\int_O D_\zeta \mathbf{a}(x, \nabla y)\nabla p \cdot \nabla z \, dx - \int_\Gamma D_{\bar{y}}\gamma(\xi, \bar{y}, u)\bar{p}\bar{z}\, d\xi, \ \ (p, \bar{p}), (z, \bar{z}) \in V.$$

3 Maximum principle

3.1 The adjoint equation

We consider the following linear BSDE in V^*:

(5) $$\mathbf{P}^u(t) = D_{\mathbf{y}}\Psi(\mathbf{Y}^u(T)) + \int_t^T (D_{\mathbf{y}}A)^*(\mathbf{Y}^u(s), u(s))\mathbf{P}^u(s)\, ds + \int_t^T D_{\mathbf{y}}L(\mathbf{Y}^u(s), u(s))\, ds - \int_t^T Q^u(s)\, dW(s).$$

Theorem 3.1. *For every control u, there exists a unique solution $(\mathbf{P}^u, Q^u) = (P^u, \bar{P}^u, Q^u) \in L^2(\Omega \times [0,T]; V) \times L^2(\Omega \times [0,T]; L_2(H))$ such that $\mathbf{P}^u$ is a continuous, adapted process with values in H.*

Proof. In order to prove this theorem, we will use a result of Márquez-Durán and Real [27] which asserts existence and uniqueness for general (non-linear) BSDEs in a variational setting. Let us now verify that the hypotheses of Theorem 2.2 in [27] are fulfilled for the coefficients of our BSDE.

1. *Final condition.* The fact that $D_{\mathbf{y}}\Psi(\mathbf{Y}^u(T)) \in L^2(\Omega, \mathcal{F}_T, P; H)$ is clearly implied by linear growth condition on $D_y\psi$ and $D_{\bar{y}}\bar{\psi}$.

2. *Measurability.* Of course,
$$(D_{\mathbf{y}}A)^*(\mathbf{Y}^u, u)\mathbf{p} + D_{\mathbf{y}}L(\mathbf{Y}^u, u)$$
is a progressively measurable process with values in V^* for every $(\mathbf{p}, q) \in V \times L_2(H)$.

3. *Hemicontinuity.* The mapping
$$\lambda \longmapsto {}_{V^*}\left\langle (D_{\mathbf{y}}A)^*(\mathbf{Y}^u(t), u(t))(\mathbf{p} + \lambda\mathbf{p}'), \mathbf{z}\right\rangle_V$$
is continuous, for every $(t, \mathbf{p}, \mathbf{p}') \in [0,T] \times V \times V$ and $\mathbf{z} \in V$. Indeed, for $\mathbf{p} = (p, \bar{p})$, $\mathbf{p}' = (p', \bar{p}')$ and $\mathbf{z} = (z, \bar{z})$, we have
$$\begin{aligned} {}_{V^*}\left\langle (D_{\mathbf{y}}A)^*(\mathbf{Y}^u(t), u(t))(\mathbf{p} + \lambda\mathbf{p}'), \mathbf{z}\right\rangle_V = &-\int_O D_\zeta\mathbf{a}(x, \nabla Y^u(t))\nabla z \cdot (\nabla p + \lambda\nabla p')\, dx \\ &- \int_\Gamma D_{\bar{y}}\gamma(\xi, \bar{Y}^u(t), u(t))\bar{z}\,(\bar{p} + \lambda\bar{p}')\, d\xi \end{aligned}$$
and the conclusion follows from the Lebesgue's dominated convergence theorem, by (A_1) and (C_1).

4. *Boundedness.* By (L$_1$), $D_{\mathbf{y}}L(\mathbf{Y}^u(\cdot),u(\cdot)) \in L^2(\Omega\times[0,T];H)$. Moreover, by (A$_1$) and (C$_1$), for every $(t,\mathbf{p},q)\in[0,T]\times V\times L_2(H)$, $\left\|(D_{\mathbf{y}}A)^*(\mathbf{Y}^u(t),u(t))\mathbf{p}\right\|_{V^*}$ is bounded by c_0.
5. *Monotonicity.* We have that

$$_{V^*}\left\langle (D_{\mathbf{y}}A)^*(\mathbf{Y}^u(t),u(t))\mathbf{p},\mathbf{p}\right\rangle_V = -\int_O D_\zeta \mathbf{a}(x,\nabla Y^u)\nabla p\cdot\nabla p\,dx - \int_\Gamma D_{\bar y}\gamma(\xi,\bar Y^u,u(t))\,|\bar p|^2\,d\xi \le 0,$$

for every $(\mathbf{p},q)=(p,\bar p,q)\in V\times L_2(H)$, $d\mathbb{P}\times dt$ a.e., by assumptions (A$_2$) and (C$_2$).
6. *Coercivity.* There exist $\alpha>0$, $\lambda\in\mathbb{R}$ and a progressively measurable process $C(\cdot)\in L^1(\Omega\times[0,T])$ such that

$$-\,_{V^*}\left\langle (D_{\mathbf{y}}A)^*(\mathbf{Y}^u(t),u(t))\mathbf{p},\mathbf{p}\right\rangle_V - \langle D_{\mathbf{y}}L(\mathbf{Y}^u(t),u(t)),\mathbf{p}\rangle_H + \lambda\|\mathbf{p}\|_H^2 + C(t) \ge \alpha\|\mathbf{p}\|_V^2,$$

for every $(\mathbf{p},q)\in V\times L_2(H)$, $d\mathbb{P}\times dt$ a.e. Indeed, for $\mathbf{p}=(p,\bar p)$, we have

$$\begin{aligned}
-\,_{V^*}\left\langle (D_{\mathbf{y}}A)^*(\mathbf{Y}^u(t),u(t))\mathbf{p},\mathbf{p}\right\rangle_V - \langle D_{\mathbf{y}}L(\mathbf{Y}^u(t),u(t)),\mathbf{p}\rangle_H &= \int_O D_\zeta\mathbf{a}(x,\nabla Y^u(t))\nabla p\cdot\nabla p\,dx\\
&\quad+\int_\Gamma D_{\bar y}\gamma(\xi,\bar Y^u(t),u(t))\,|\bar p|^2\,d\xi - \langle D_{\mathbf{y}}L(\mathbf{Y}^u(t),u(t)),\mathbf{p}\rangle_H\\
&\ge\delta\left(\|\nabla p\|^2_{L^2(\mathcal{O})}+\|\bar p\|^2_{L^2(\Gamma)}\right)-\frac12\|D_{\mathbf{y}}L(\mathbf{Y}^u(t),u(t))\|_H^2-\frac12\|\mathbf{p}\|_H^2\\
&\ge\delta\|\mathbf{p}\|_V^2-\frac12\|\mathbf{p}\|_H^2-\frac32 c_2^2\left(\|(\rho,\tilde\rho)\|_H^2+\|\mathbf{Y}^u(t)\|_H^2+\|u(t)\|^2_{L^2(\Gamma)}\right).
\end{aligned}$$

□

3.2 The variation equation

We define the operator $\mathcal{G}: L^2(\Gamma)\times L^2(\Gamma;U)\times L^\infty(\Gamma;U)\to H$ by

$$\mathcal{G}(\bar y,u,\bar u) := (0,-D_u\gamma(\cdot,\bar y,u)\cdot\bar u).$$

Let now u and v be two controls such that $v-u$ is bounded; let, for $\theta\in[0,1]$, $u^\theta:=(1-\theta)u+\theta v$. Let us denote, for simplicity, $\mathbf{Y}^\theta$, Y^θ and $\bar Y^\theta$ instead of $\mathbf{Y}^{u^\theta}$, Y^{u^θ} and $\bar Y^{u^\theta}$, respectively.

Proposition 3.2. *The equation*

$$\mathbf{Z}(t)=\int_0^t D_{\mathbf{y}}A(\mathbf{Y}^u(s),u(s))\mathbf{Z}(s)ds+\int_0^t\mathcal{G}(\bar Y^u(s),u(s),v(s)-u(s))ds,\ t\in[0,T] \tag{6}$$

has a unique variational solution $\mathbf{Z}\in L^2(\Omega\times[0,T];V)$. Moreover, $\frac1\theta\left(\mathbf{Y}^\theta-\mathbf{Y}^0\right)$ and $\frac1\theta\left(\mathbf{Y}^\theta(T)-\mathbf{Y}^0(T)\right)$ converge weakly to $\mathbf{Z}$ and $\mathbf{Z}(T)$ in $L^2(\Omega\times[0,T];V)$, respectively in $L^2(\Omega;H)$.

Proof. We have, by Itô's formula,

$$\begin{aligned}
\mathbb{E}\left\|\mathbf{Y}^\theta(t)\right\|_H^2 &= \|\mathbf{y}_0\|_H^2 - 2\mathbb{E}\left[\int_0^t\int_O \mathbf{a}(x,\nabla Y^\theta(s))\cdot\nabla Y^\theta(s)\,dx\,ds\right]\\
&\quad-2\mathbb{E}\left[\int_0^t\int_\Gamma\gamma(\xi,\bar Y^\theta(s),u^\theta(s))\bar Y^\theta(s)\,d\xi\,ds\right]+t\|B\|^2_{L_2(H)},\ t\in[0,T],
\end{aligned}$$

therefore, by (A_1), (A_2), (C_1) and (C_2),

$$\sup_{\theta\in[0,1]}\left[\sup_{t\in[0,T]}\mathbb{E}\left\|\mathbf{Y}^\theta(t)\right\|_H^2+\mathbb{E}\int_0^T\left\|\mathbf{Y}^\theta(t)\right\|_V^2dt\right]<+\infty. \tag{7}$$

Since, for any $t\in[0,T]$,

$$\begin{aligned}\left\|\left(\mathbf{Y}^\theta(t)-\mathbf{Y}^0(t)\right)\right\|_H^2=&-2\int_0^t\int_O\left[\mathbf{a}(x,\nabla Y^\theta(s))-\mathbf{a}(x,\nabla Y^0(s))\right]\cdot\left[\nabla Y^\theta(s)-\nabla Y^0(s)\right]\,dx\,ds\\&-2\int_0^t\int_\Gamma\left[\gamma(\xi,\bar Y^\theta(s),u^\theta(s))-\gamma(\xi,\bar Y^0(s),u(s))\right]\left[\bar Y^\theta(s)-\bar Y^0(s)\right]\,d\xi\,ds,\end{aligned}$$

we have, by the assumptions on $\mathbf{a}$ and γ,

$$\begin{aligned}&\left\|\mathbf{Y}^\theta(t)-\mathbf{Y}^0(t)\right\|_H^2+2\delta\int_0^t\left\|\mathbf{Y}^\theta(s)-\mathbf{Y}^0(s)\right\|_V^2\,ds\\&\le-2\int_0^t\int_\Gamma\left[\gamma(\xi,\bar Y^0(s),u^\theta(s))-\gamma(\xi,\bar Y^0(s),u(s))\right]\left(\bar Y^\theta(s)-\bar Y^0(s)\right)\,d\xi\,ds\\&=-2\theta\int_0^t\int_\Gamma\left[\int_0^1D_u\gamma(\xi,\bar Y^0(s),u^{\lambda\theta}(s))d\lambda\right](v(s)-u(s))\left(\bar Y^\theta(s)-\bar Y^0(s)\right)\,d\xi\,ds\\&\le C\theta\int_0^t\int_\Gamma\left[\tilde\rho(\xi)+\left|\bar Y^0(s)\right|\right]\left(\bar Y^\theta(s)-\bar Y^0(s)\right)\,d\xi\,ds\\&\le C\theta^2\int_0^t\left(\tilde\rho(\xi)^2+\left\|\mathbf{Y}^0(s)\right\|_V^2\right)ds+\delta\int_0^t\left\|\mathbf{Y}^\theta(s)-\mathbf{Y}^0(s)\right\|_V^2\,ds,\end{aligned}$$

where $C>0$ is a constant whose value is allowed to change from line to line. Hence

$$\mathbb{E}\left[\sup_{\theta\in[0,1]}\sup_{t\in[0,T]}\left\|\tfrac{1}{\theta}(\mathbf{Y}^\theta(t)-\mathbf{Y}^0(t))\right\|_H^2\right]+\mathbb{E}\left[\sup_{\theta\in[0,1]}\int_0^T\left\|\tfrac{1}{\theta}(\mathbf{Y}^\theta(t)-\mathbf{Y}^0(t))\right\|_V^2\,ds\right]<+\infty. \tag{8}$$

Then there exists a progressively measurable process $\tilde{\mathbf{Z}}\in L^2(\Omega\times[0,T];V)$ such that, at least on a subsequence:

- $\frac{1}{\theta}(\mathbf{Y}^\theta-\mathbf{Y}^0)$ converges weakly to $\tilde{\mathbf{Z}}$ as $\theta\to0$ in $L^2(\Omega\times[0,T];V)$;
- ∇Y^θ converges to ∇Y^0 a.e. as $\theta\to0$ on $\Omega\times[0,T]\times\mathcal{O}$;
- $\bar Y^\theta$ converges to $\bar Y^0$ a.e. as $\theta\to0$ on $\Omega\times[0,T]\times\Gamma$.

For $\mathbf{z} = (z, \bar{z}) \in V$ with $z \in C_b^1(\mathcal{O})$, we have

$$
\begin{aligned}
(9)\quad \left\langle \frac{\mathbf{Y}^\theta(t) - \mathbf{Y}^0(t)}{\theta}, \mathbf{z} \right\rangle_H =& - \int_0^t \int_O \frac{\mathbf{a}(x, \nabla Y^\theta(s)) - \mathbf{a}(x, \nabla Y^0(s))}{\theta} \cdot \nabla z \, dx \, ds \\
& - \int_0^t \int_\Gamma \frac{\gamma(\xi, \bar{Y}^\theta(s), u^\theta(s)) - \gamma(\xi, \bar{Y}^0(s), u(s))}{\theta} \bar{z} \, d\xi \, ds \\
=& - \int_0^t \int_O T^{1,\theta}(s) \frac{\nabla(Y^\theta - Y^0)(s)}{\theta} \cdot \nabla z dx ds - \int_0^t \int_\Gamma T^{2,\theta}(s) \frac{(\bar{Y}^\theta - \bar{Y}^0)(s)}{\theta} \bar{z} \, d\xi \, ds \\
& - \int_0^t \int_\Gamma T^{3,\theta}(s)(v(s) - u(s)) \bar{z} d\xi ds - \int_\Gamma D_u \gamma(\xi, \bar{Y}^0(s), u(s))(v(s) - u(s)) \bar{z} \, d\xi \\
& - \int_0^t \int_O D_\zeta \mathbf{a}(x, \nabla Y^0(s))) \frac{\nabla(Y^\theta - Y^0)(s)}{\theta} \cdot \nabla z \, dx \, ds \\
& - \int_0^t \int_\Gamma D_{\bar{y}} \gamma(\xi, \bar{Y}^0(s), u(s)) \frac{(\bar{Y}^\theta - \bar{Y}^0)(s)}{\theta} \bar{z} \, d\xi \, ds,
\end{aligned}
$$

where, for the sake of simplicity, we have denoted

$$
\begin{aligned}
T^{1,\theta}(s) &:= \int_0^1 \left[D_\zeta \mathbf{a}(x, \nabla Y^0(s) + \lambda \nabla(Y^\theta - Y^0)(s)) - D_\zeta \mathbf{a}(x, \nabla Y^0(s)) \right] d\lambda; \\
T^{2,\theta}(s) &:= \int_0^1 \left[D_{\bar{y}} \gamma(\xi, \bar{Y}^0(s) + \lambda(\bar{Y}^\theta - \bar{Y}^0)(s), u^{\lambda\theta}(s)) - D_{\bar{y}} \gamma(\xi, \bar{Y}^0(s), u(s)) \right] d\lambda; \\
T^{3,\theta}(s) &:= \int_0^1 \left[D_u \gamma(\xi, \bar{Y}^0(s) + \lambda(\bar{Y}^\theta - \bar{Y}^0)(s), u^{\lambda\theta}(s)) - D_u \gamma(\xi, \bar{Y}^0(s), u(s)) \right] d\lambda.
\end{aligned}
$$

By the dominated convergence theorem and (8), since $T^{1,\theta}$ and $T^{2,\theta}$ are bounded, we have that

$$
\lim_{\theta \to 0} \mathbb{E} \left[\int_0^T \int_O \left| T^{1,\theta}(s) \right| \left| \frac{\nabla(Y^\theta - Y^0)(s)}{\theta} \right| |\nabla z| \, dx \, ds \right] = 0
$$

and

$$
\lim_{\theta \to 0} \mathbb{E} \left[\int_0^t \int_\Gamma \left| T^{2,\theta}(s) \right| \left| \frac{(\bar{Y}^\theta - \bar{Y}^0)(s)}{\theta} \right| |\bar{z}| \, d\xi \, ds \right] = 0.
$$

We also have that

$$
\begin{aligned}
\mathbb{E} \left[\int_0^t \int_\Gamma \left| T^{3,\theta}(s) \right| |(v(s) - u(s))| \, |\bar{z}| \, d\xi \, ds \right] \leq & C \mathbb{E} \left[\int_0^T \int_\Gamma \left| T^{3,\theta}(s) \right| \rho_1 \left((\bar{Y}^\theta - \bar{Y}^0)(s) \right) \, d\xi \, ds \right] \\
& + C \mathbb{E} \left[\int_0^T \int_\Gamma \left| T^{3,\theta}(s) \right| (1 - \rho_1) \left((\bar{Y}^\theta - \bar{Y}^0)(s) \right) \, d\xi \, ds \right],
\end{aligned}
$$

where ρ_1 is a smooth function defined on $\mathbb{R}$ such that $0 \leq \rho_1 \leq 1$, $\rho_1(\bar{y}) = 1$ for $|\bar{y}| \leq 1$ and $\rho_1(\bar{y}) = 0$ for $|\bar{y}| \geq 2$. Since, by (C_1),

$$
\left| T^{3,\theta}(s) \right| \rho_1 \left((\bar{Y}^\theta - \bar{Y}^0)(s) \right) \leq C \left(\tilde{\rho}(\xi) + \left| \bar{Y}^0(s) \right| \right),
$$

we have, by the dominated convergence theorem, that

$$\lim_{\theta\to 0}\mathbb{E}\left[\int_0^T\int_\Gamma \left|T^{3,\theta}(s)\right|\rho_1\left((\bar{Y}^\theta-\bar{Y}^0)(s)\right)\,d\xi\,ds\right]=0.$$

On the other hand, by (7) and (8),

$$\begin{aligned}&\mathbb{E}\left[\int_0^T\int_\Gamma \left|T^{3,\theta}(s)\right|(1-\rho_1)\left((\bar{Y}^\theta-\bar{Y}^0)(s)\right)\,d\xi\,ds\right]\\ &\leq C\mathbb{E}\left[\int_0^T\int_\Gamma\left(\tilde{\rho}(\xi)+\left|\bar{Y}^0(s)\right|+\left|\bar{Y}^\theta(s)\right|\right)\mathbf{1}_{\left\{\left|(\bar{Y}^\theta-\bar{Y}^0)(s)\right|\geq 1\right\}}\,d\xi\,ds\right]\\ &\leq C\left(\mathbb{E}\left[\int_0^T\int_\Gamma\mathbf{1}_{\left\{\left|(\bar{Y}^\theta-\bar{Y}^0)(s)\right|\geq 1\right\}}d\xi\right]\right)^{1/2}ds\leq C\left(\mathbb{E}\left[\int_0^T\int_\Gamma\left|(\bar{Y}^\theta-\bar{Y}^0)(s)\right|^2\,d\xi\,ds\right]\right)^{1/2}\leq C\theta.\end{aligned}$$

Therefore,

$$\lim_{\theta\to 0}\mathbb{E}\left[\int_0^T\int_\Gamma\left|T^{3,\theta}(s)\right|\left|v(s)-u(s)\right|\left|\bar{z}\right|\,d\xi\,ds\right]=0.$$

Let $\mathbf{Z}\in C([0,T];L^2(\Omega;V^*))$ be defined by

$$\mathbf{Z}(t)=\int_0^t D_{\mathbf{y}}A(\mathbf{Y}^u(s),u(s))\tilde{\mathbf{Z}}(s)\,ds+\int_0^t\mathcal{G}(\bar{Y}^u(s),u(s),v(s)-u(s))\,ds,\ t\in[0,T].$$

By the weak convergence of $\frac{1}{\theta}(\mathbf{Y}^\theta-\mathbf{Y}^0)$ to $\tilde{\mathbf{Z}}$ in $L^2(\Omega\times[0,T];V)$, the boundedness of $\frac{\mathbf{Y}^\theta(t)-\mathbf{Y}^0(t)}{\theta}$ in $L^2(\Omega;H)$ and the density of $\left\{(z,\bar{z})\in V\mid z\in C_b^1(\mathcal{O})\right\}$ in V, we can pass to the limit in relation (9) and obtain that, for every $t\in[0,T]$, $\mathbf{Z}(t)\in L^2(\Omega;H)$ and $\frac{\mathbf{Y}^\theta(t)-\mathbf{Y}^0(t)}{\theta}$ converges weakly to $\mathbf{Z}(t)$ in $L^2(\Omega;H)$. This allows the identification

$$\mathbf{Z}(t)=\tilde{\mathbf{Z}}(t),\ \text{a.e. } t\in[0,T],$$

from which we can infer that $\mathbf{Z}$ is a variational solution of equation (6).

The uniqueness of the solution of (6) is obtained by applying Theorem 4.2.4 in [30], for instance. A consequence of the uniqueness is that the weak convergences stated inside this argument hold not only on a subsequence, but on a whole right neighborhood of 0. □

3.3 Necessary conditions of optimality

In this section we will derive, in the form of a maximum principle, necessary conditions for an admissible control to be optimal. Let us define the *Hamiltonian* $\mathcal{H}:V\times L^2(\Gamma;U)\times V\times L_2(H)\to\mathbb{R}$ by

$$\mathcal{H}(\mathbf{y},u,\mathbf{p},q):={}_{V^*}\langle A(\mathbf{y},u),\mathbf{p}\rangle_V+L(\mathbf{y},u)+\mathrm{tr}(qB).$$

Theorem 3.3. *Let u^* be an optimal control. Then, a.s., $d\xi\,dt$-a.e.,*

$$\left[\bar{P}^{u^*}(t,\xi)D_u\gamma(\xi,\bar{Y}^{u^*}(t,\xi),u^*(t,\xi))-D_u\bar{\ell}(\xi,\bar{Y}^{u^*}(t,\xi),u^*(t,\xi))\right]\cdot(v-u^*(t,\xi))\leq 0,\ \forall v\in U. \tag{10}$$

Remark. This inequality is equivalent to

$$\mathcal{H}_u(\mathbf{Y}^{u^*}(t), u^*(t), \mathbf{P}^{u^*}(t), Q^{u^*}(t); v(t) - u^*(t)) \geq 0,\ \mathbb{P}dt\text{-a.e.}, \forall v \in \mathcal{U}^\infty_{u^*},$$

where $\mathcal{U}^\infty_{u^*}$ is the set of admissible controls v such that $v - u^* \in L^\infty(\Gamma; \mathbb{R}^m)$, $\mathbb{P}dt$-a.e. and $\mathcal{H}_u(\mathbf{y}, u, \mathbf{p}, q; w)$ denotes the directional derivative of $\mathcal{H}$ with respect to u in the direction w (which exists if $w \in L^\infty(\Gamma; \mathbb{R}^m)$ and $u + w \in L^2(\Gamma; U)$). This is known as the *local form* of the maximum principle.

Proof. As in the previous section, we will take first an arbitrary control v such that $v - u^*$ is bounded and we will use the same notations u^θ, $\mathbf{Y}^\theta$, $\mathbf{Z}$, for $\theta \in [0,1]$. We will also write $\mathbf{P}$, Q instead of $\mathbf{P}^{u^*}$, Q^{u^*}, respectively. Let us apply Itô's formula to $\mathbf{P} \cdot \mathbf{Z}$:

$$\begin{aligned}
\langle \mathbf{P}(t), \mathbf{Z}(t) \rangle_H = &- \int_0^t {}_{V^*}\left\langle (D_\mathbf{y} A)^*(\mathbf{Y}^0(s), u^*(s))\mathbf{P}(s), \mathbf{Z}(s) \right\rangle_V \, ds \\
&- \int_0^t D_\mathbf{y} L(\mathbf{Y}^0(s), u^*(s))\mathbf{Z}(s)\, ds \\
&+ \int_0^t {}_{V^*}\left\langle D_\mathbf{y} A(\mathbf{Y}^0(s), u^*(s))\mathbf{Z}(s) + \mathcal{G}(\bar{Y}^0(s), u^*(s), v(s) - u^*(s)), \mathbf{P}(s) \right\rangle_V \, ds \\
&- \int_0^t \langle \mathbf{Z}(s), Q(s)\, dW(s) \rangle_H\,,\ \forall t \in [0,T].
\end{aligned}$$

Therefore, letting $t = T$ and taking expectation, we get

$$\begin{aligned}
\text{(11)} \qquad \mathbb{E}\left\langle D_\mathbf{y}\Psi(\mathbf{Y}^0(T)), \mathbf{Z}(T) \right\rangle_H = &\mathbb{E}\left[\int_0^T {}_{V^*}\left\langle \mathcal{G}(\bar{Y}^0(s), u^*(s), v(s) - u^*(s)), \mathbf{P}(s) \right\rangle_V \, ds\right] \\
&- \mathbb{E}\left[\int_0^T D_\mathbf{y} L(\mathbf{Y}^0(s), u^*(s))\mathbf{Z}(s)\, ds\right].
\end{aligned}$$

On the other hand, since u^* is an optimal control, $J(u^*) \leq J(u^\theta)$ for any $\theta \in (0,1)$, *i.e.*

$$\mathbb{E}\left[\int_0^T \left(L(\mathbf{Y}^\theta(t), u^\theta(t)) - L(\mathbf{Y}^0(t), u^*(t))\right) dt + \Psi(\mathbf{Y}^\theta(T)) - \Psi(\mathbf{Y}^0(T))\right] \geq 0,$$

which is equivalent to

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \int_0^1 D_\mathbf{y} L(\mathbf{Y}^0(t) + \lambda(\mathbf{Y}^\theta - \mathbf{Y}^0)(t), u^{\lambda\theta}(t)) d\lambda \frac{(\mathbf{Y}^\theta - \mathbf{Y}^0)(t)}{\theta}\, dt\right] \\
&+ \mathbb{E}\left[\int_0^T \int_0^1 L_u(\mathbf{Y}^0(t) + \lambda(\mathbf{Y}^\theta - \mathbf{Y}^0)(t), u^{\lambda\theta}(t); v(t) - u^*(t)) d\lambda\, dt\right] \\
&+ \mathbb{E}\left[\int_0^1 D_\mathbf{y}\Psi(\mathbf{Y}^0(T) + \lambda(\mathbf{Y}^\theta - \mathbf{Y}^0)(T))\, d\lambda \frac{(\mathbf{Y}^\theta - \mathbf{Y}^0)(T)}{\theta}\right] \geq 0.
\end{aligned}$$

Here, $L_u(\mathbf{y}, u; w)$ denotes the directional derivative of L with respect to u in the direction w. Passing to the limit as $\theta \to 0$, by the weak convergence property stated in Proposition 3.2 and similar arguments as in its proof, we obtain

$$\mathbb{E}\left\{\int_0^T \left[D_{\mathbf{y}} L(\mathbf{Y}^0(t), u^*(t))\mathbf{Z}(t) + L_u(\mathbf{Y}^0(t), u^*(t); v(t) - u^*(t))\right] dt\right\} \geq -\mathbb{E}\left\langle D_y \Psi(\mathbf{Y}^0(T)), \mathbf{Z}(T)\right\rangle_H.$$

Combining this inequality with relation (11), we derive

$$\mathbb{E}\left\{\int_0^T \left[{}_{V^*}\left\langle \mathcal{G}(\bar{Y}^0(s), u^*(s), v(s) - u^*(s)), \mathbf{P}(s)\right\rangle_V + L_u(\mathbf{Y}^0(t), u^*(t); v(t) - u^*(t))\right] ds\right\} \geq 0,$$

i.e.

$$\mathbb{E}\left\{\int_0^T \int_\Gamma \left[\bar{P}(t,\xi) D_u \gamma(\xi, \bar{Y}^0(t,\xi), u^*(t,\xi)) - D_u \bar{\ell}(\xi, \bar{Y}^0(t,\xi), u^*(t,\xi))\right] \cdot (v(t,\xi) - u^*(t,\xi))\, d\xi\, dt\right\} \leq 0.$$

Since the control v such that $v - u^*$ is bounded is chosen arbitrarily, we can infer easily that a.s., $d\xi dt$-a.e.

$$\left[\bar{P}(t,\xi) D_u \gamma(\xi, \bar{Y}^0(t,\xi), u^*(t,\xi)) - D_u \bar{\ell}(\xi, \bar{Y}^0(t,\xi), u^*(t,\xi))\right] \cdot (v - u^*(t,\xi)) \leq 0, \ \forall v \in U.$$

□

3.4 Sufficient conditions of optimality

In this section we show that condition (10) is, under some supplementary assumptions, sufficient for the optimality of a given control.

Theorem 3.4. *Let u^* be a control satisfying (10). If the mappings Ψ and*

$$\begin{aligned} &V \times L^2(\Gamma; U) \to \mathbb{R} \\ &(\mathbf{y}, u) \mapsto \mathcal{H}(\mathbf{y}, u, \mathbf{P}^{u^*}(t), Q^{u^*}(t)) \end{aligned} \tag{12}$$

are convex a.s., dt-a.e., then u^* is optimal.

Remark. Under the above convexity hypothesis, (10) becomes equivalent to

$$u^*(t) \in \operatorname{argmin} \mathcal{H}(\mathbf{Y}^{u^*}, \cdot, \mathbf{P}^{u^*}(t), Q^{u^*}(t)), \ \mathbb{P}dt\text{-a.e.},$$

which is the *global form* of the maximum principle.

Proof. For an admissible control v such that $v - u^*$ is bounded, let us apply Itô's formula to $\mathbf{P}^{u^*} \cdot (\mathbf{Y}^v - \mathbf{Y}^{u^*})$:

$$\begin{aligned} \left\langle \mathbf{P}^{u^*}(t), \mathbf{Y}^v(t) - \mathbf{Y}^{u^*}(t)\right\rangle_H = &- \int_0^t {}_{V^*}\left\langle (D_{\mathbf{y}} A)^*(\mathbf{Y}^{u^*}(s), u^*(s)) \mathbf{P}^{u^*}(s), \mathbf{Y}^v(s) - \mathbf{Y}^{u^*}(s)\right\rangle_V ds \\ &- \int_0^t D_{\mathbf{y}} L(\mathbf{Y}^{u^*}(s), u^*(s))(\mathbf{Y}^v(s) - \mathbf{Y}^{u^*}(s))\, ds \\ &+ \int_0^t {}_{V^*}\left\langle A(\mathbf{Y}^v(s), v(s)) - A(\mathbf{Y}^{u^*}(s), u^*(s)), \mathbf{P}^{u^*}(s)\right\rangle_V ds \\ &- \int_0^t \left\langle \mathbf{Y}^v(s) - \mathbf{Y}^{u^*}(s), Q^{u^*}(s)\, dW(s)\right\rangle_H. \end{aligned} \tag{13}$$

Since the map $\mathcal{H}(\cdot,\cdot,\mathbf{P}^{u^*}(t),Q^{u^*}(t))$ is convex, we have

$$
\begin{aligned}
&\mathcal{H}(\mathbf{Y}^v(t),v(t),\mathbf{P}^{u^*}(t),Q^{u^*}(t)) - \mathcal{H}(\mathbf{Y}^{u^*}(t),u^*(t),\mathbf{P}^{u^*}(t),Q^{u^*}(t)) \geq \\
&\qquad \mathcal{H}_{(\mathbf{y},u)}\Big(\mathbf{Y}^{u^*}(t),u^*(t),\mathbf{P}^{u^*}(t),Q^{u^*}(t);(\mathbf{Y}^v(t)-\mathbf{Y}^{u^*}(t),v(t)-u^*(t))\Big) \\
&\quad = {}_{V^*}\Big\langle (D_{\mathbf{y}}A)^*(\mathbf{Y}^{u^*}(t),u^*(t))\mathbf{P}^{u^*}(t),\mathbf{Y}^v(t)-\mathbf{Y}^{u^*}(t)\Big\rangle_V + D_{\mathbf{y}}L(\mathbf{Y}^{u^*}(t),u^*(t))(\mathbf{Y}^v(t)-\mathbf{Y}^{u^*}(t)) \\
&\qquad - \int_\Gamma \bar{P}^{u^*}(t,\xi)D_u\gamma(\xi,\bar{Y}^{u^*}(t,\xi),u^*(t,\xi))\cdot(v(t,\xi)-u^*(t,\xi))\,d\xi \\
&\qquad\qquad + L_u(\mathbf{Y}^{u^*}(t),u^*(t);v(t)-u^*(t)),
\end{aligned}
$$

where $\mathcal{H}_{(\mathbf{y},u)}(\mathbf{y},u,\mathbf{p},q;(\mathbf{w},w))$ denotes the directional derivative of $\mathcal{H}$ with respect to $(\mathbf{y},u)$ in the direction $(\mathbf{w},w)$. We make the remark that

$$
\int_\Gamma \bar{P}^{u^*}(t,\xi)D_u\gamma(\xi,\bar{Y}^{u^*}(t,\xi),u^*(t,\xi))\cdot(v(t,\xi)-u^*(t,\xi))\,d\xi
$$

may be infinite, but exists, by (10). From relation (13) we get

$$
\begin{aligned}
\mathbb{E}\Big\langle D_{\mathbf{y}}\Psi(\mathbf{Y}^{u^*}(T)),\mathbf{Y}^v(T)-\mathbf{Y}^{u^*}(T)\Big\rangle_H \geq{}& \mathbb{E}\left[\int_0^T\int_\Gamma \bar{P}^{u^*}(t,\xi)D_u\gamma(\xi,\bar{Y}^{u^*}(t,\xi),u^*(t,\xi))\cdot(v(t,\xi)-u^*(t,\xi))\,d\xi\,dt\right] \\
&- \mathbb{E}\left[\int_0^T\Big[L(t,\mathbf{Y}^v(t),v(t))-L(t,\mathbf{Y}^{u^*}(t),u^*(t))\Big]\,dt\right] \\
&+ \mathbb{E}\left[\int_0^T L_u(\mathbf{Y}^{u^*}(t),u^*(t);v(t)-u^*(t))\,dt\right].
\end{aligned}
$$

The convexity of Ψ implies that

$$
\mathbb{E}\Big[\Psi(\mathbf{Y}^v(T))-\Psi(\mathbf{Y}^{u^*}(T))\Big] \geq \mathbb{E}\Big\langle D_{\mathbf{y}}\Psi(\mathbf{Y}^u(T)),\mathbf{Y}^v(T)-\mathbf{Y}^{u^*}(T)\Big\rangle_H;
$$

consequently

$$
\begin{aligned}
&J(v)-J(u^*) \geq \\
&\quad \mathbb{E}\left\{\int_0^T\int_\Gamma\Big[-\bar{P}^{u^*}(t,\xi)D_u\gamma(\xi,\bar{Y}^{u^*}(t,\xi),u^*(t,\xi))+D_u\bar{\ell}(\xi,\bar{Y}^{u^*}(t,\xi),u^*(t,\xi))\Big]\cdot(v(t,\xi)-u^*(t,\xi))\,d\xi\,dt\right\}.
\end{aligned}
$$

By relation (10), the right-hand side of the above inequality is positive, so $J(v)\geq J(u^*)$.

If $v-u^*$ is not bounded, we can take, for $n\geq 1$,

$$
v_n(t,\xi) := \begin{cases} v(t,\xi), & |v(t,\xi)-u^*(t,\xi)|\leq n; \\ u^*(t,\xi), & |v(t,\xi)-u^*(t,\xi)|>n. \end{cases}
$$

Applying Itô's formula to $\mathbf{Y}^{v_n}(t) - \mathbf{Y}^v(t)$, we get, by the properties of $\mathbf{a}$ and γ,

$$\begin{aligned}
&\|\mathbf{Y}^{v_n}(t) - \mathbf{Y}^v(t)\|_H^2 + 2\delta \int_0^t \|\mathbf{Y}^{v_n}(s) - \mathbf{Y}^v(s)\|_V^2 \, ds \\
&\le 2 \int_0^t \int_\Gamma \left[\gamma(\xi, \bar{Y}^v(s), v_n(s)) - \gamma(\xi, \bar{Y}^v(s), v(s))\right] \left(\bar{Y}^{v_n}(s) - \bar{Y}^v(s)\right) d\xi \, ds \\
&\le C \int_0^T \int_\Gamma \left|\gamma(\xi, \bar{Y}^v(s), v_n(s)) - \gamma(\xi, \bar{Y}^v(s), v(s))\right|^2 d\xi \, ds + \delta \int_0^T \int_\Gamma \left|\bar{Y}^{v_n}(s) - \bar{Y}^v(s)\right|^2 d\xi \, ds.
\end{aligned}$$

Therefore, by the dominated convergence theorem,

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|\mathbf{Y}^{v_n}(t) - \mathbf{Y}^v(t)\|_H^2 = 0.$$

This implies that $\lim_{n\to\infty} J(v_n) = J(v)$; hence $J(v) \ge J(u^*)$. □

Example. The convexity hypothesis for $\mathcal{H}(\cdot, \cdot, \mathbf{P}^{u^*}(t), Q^{u^*}(t))$ is hard to verify in practice, since the direction of ∇P^{u^*} and the sign of $\bar{P}^{u^*}$ are not *a priori* determinable. However, under convexity assumptions on the coefficients, we just need to strengthen condition (10) in order to derive a sufficient optimality condition.

We will take $\mathbf{a}(x, \zeta) = \zeta, \ (x, \zeta) \in \mathcal{O} \times \mathbb{R}^n$ (or, more general, linear with respect to ζ). Moreover, the functions $\ell(x, \cdot)$, $\psi(x, \cdot)$ and $\bar{\psi}(\xi, \cdot)$ are supposed to be convex, dx-a.e. on $\mathcal{O}$, respectively $d\xi$-a.e. on Γ. For $\sigma \in \{-1, 1\}$, on γ and $\bar{\ell}$ we impose that:

- $(\bar{y}, u) \mapsto -\sigma\gamma(\xi, \bar{y}, u)$ is convex, $d\xi$-a.e. on Γ;
- $(\bar{y}, u) \mapsto \bar{\ell}(\xi, \bar{y}, u)$ is convex, $d\xi$-a.e. on Γ.

Let, for $(\xi, \bar{y}, u) \in \Gamma \times \mathbb{R} \times U$,

$$S(\xi, \bar{y}, u) := \left\{\alpha \in \mathbb{R} \mid \alpha D_u\gamma(\xi, \bar{y}, u) - D_u\bar{\ell}(\xi, \bar{y}, u) \in \mathcal{N}_U(u)\right\},$$

where $\mathcal{N}_U(u)$ is the exterior normal cone to U in u if $u \in \partial U$ and $\mathcal{N}_U(u) = \{0\}$ if $u \in \operatorname{int} U$.

A sufficient condition of optimality for an admissible control u^* is then

$$\bar{P}^{u^*}(t, \xi) \in S(\xi, \bar{Y}^{u^*}(t, \xi), u^*(t, \xi)) \cap \sigma\mathbb{R}_+, \ d\xi dt\text{-a.e.} \tag{14}$$

This condition is obviously equivalent to (10) when $S(\xi, \bar{y}, u) \cap \sigma\mathbb{R}_-^* = \emptyset, \ \forall \bar{y} \in \mathbb{R}, \ d\xi$-a.e.

4 Existence of an optimal control

Let now study the problem of the existence of a maximum control under the convexity conditions on the coefficients of the cost functional and linearity of control.

Assume that U is bounded and:

($\mathbf{C}_3$) $\gamma(\xi, \bar{y}, u) = \tilde{\gamma}(\xi, \bar{y}) + \beta(\xi) \cdot u$, where $\tilde{\gamma}$ satisfies conditions (C_0)–(C_2) and $\beta \in L^\infty(\Gamma; \mathbb{R}^m)$;

($\mathbf{F}_2$) $\psi(x, \cdot)$ and $\bar{\psi}(\xi, \cdot)$ are convex, dx-a.e. on $\mathcal{O}$, respectively $d\xi$-a.e. on Γ;

($\mathbf{L}_2$) $\ell(x, \cdot)$ and $\bar{\ell}(\xi, \cdot)$ are convex, dx-a.e. on $\mathcal{O}$, respectively $d\xi$-a.e. on Γ.

Remark. Notice that our assumptions, although stringent, cover most of the cases in the literature. For instance, Debussche, Fuhrman and Tessitore [12], Fabbri and Goldys [18] and Bonaccorsi, Confortola and Mastrogiacomo [7] consider linear control problems on the boundary (for Neumann, Dirichlet and dynamic boundary conditions, respectively), and all those papers are concerned with the one-dimensional problem. These papers, further, consider linear quadratic term in the cost functional, that hence satisfy assumptions (F_2) and (L_2).

On the other hand, in this paper we do not consider the structure condition that is necessary to apply the forward-backward approach of Fuhrman and Tessitore [19], i.e., the condition that the control and the noise enters the equation with the same operator in front of them.

Theorem 4.1. *Under the above assumptions, there exists at least an optimal control.*

The necessary condition (10) provides more information about the optimal control whose existence is guaranteed by the above result. In fact, it can be written as

$$\beta(\xi)\bar{P}^{u^*}(t,\xi) - D_u\bar{\ell}(\xi, \bar{Y}^{u^*}(t,\xi), u^*(t,\xi)) \in \mathcal{N}_U(u^*(t,\xi)),\ d\xi dt\text{-a.e.}$$

(recall that $\mathcal{N}_U(u)$ is the exterior normal cone to U in u if $u \in \partial U$ and $\mathcal{N}_U(u) = \{0\}$ if $u \in \operatorname{int} U$).

Proof. By Itô's formula applied to $\|Y^u\|_H^2$, it is clear that

$$\mathbb{E} \sup_{t\in[0,T]} \|Y^u(t)\|_H^2 \le C,$$

for every control u (we recall that we use the generic term C for positive constants, whose values can change from one place to another). Since U is bounded, J is bounded, too. Let (u_n) be a sequence of controls such that $J(u_n) \searrow \inf_{u\in\mathcal{U}} J(u)$. There exists $u^* \in L^2\left(\Omega\times[0,T]; L^2(\Gamma;\mathbb{R}^m)\right)$ such that a subsequence of (u_n) converges weakly to u^*. Without restricting the generality, we can suppose that the whole sequence converges to u^*.

For the sake of simplicity, let us denote $\mathbf{Y}^n := \mathbf{Y}^{u^n}$. Let us show that $\mathbf{Y}^n$ converges to $\mathbf{Y}^{u^*}$. We have that, exactly as in (7), that

$$\sup_{n\in\mathbb{N}} \left[\sup_{t\in[0,T]} \mathbb{E}\,\|\mathbf{Y}^n(t)\|_H^2 + \mathbb{E}\int_0^T \|\mathbf{Y}^n(t)\|_V^2\,dt \right] < +\infty.$$

Consequently, the sequences $(\mathbf{a}(\cdot,\nabla Y^n))_{n\ge1}$ and $\left(\tilde{\gamma}(\cdot,\bar{Y}^n)\right)_{n\ge1}$ are also bounded in $L^2\left(\Omega\times[0,T]; L^2(\mathcal{O})\right)$, respectively in $L^2\left(\Omega\times[0,T]; L^2(\Gamma)\right)$. Therefore, at least on a subsequence:

- $\mathbf{Y}^n$ converges weakly in $L^2(\Omega\times[0,T]; V)$ to a process $\mathbf{Y}^* = (Y^*, \bar{Y}^*)$;
- $\mathbf{Y}^n(t)$ converges weakly in $L^2(\Omega; H)$ to $\mathbf{Y}^*(t)$ for every $t\in[0,T]$;
- $\mathbf{a}(\cdot,\nabla Y^n)$ converges weakly in $L^2\left(\Omega\times[0,T]; L^2(\mathcal{O})\right)$ to a process χ;
- $\tilde{\gamma}(\cdot,\bar{Y}^n)$ converges weakly in $L^2\left(\Omega\times[0,T]; L^2(\Gamma)\right)$ to a process $\varkappa$.

If $\mathbf{z} = (z,\bar{z}) \in V$, then

$$\langle \mathbf{Y}^n(t), \mathbf{z}\rangle_H = \langle \mathbf{y}_0, \mathbf{z}\rangle_H - \int_0^t \left[\langle \mathbf{a}(\cdot,\nabla Y^n(s)), \nabla z\rangle_{L^2(\mathcal{O})} + \left\langle \tilde{\gamma}(\cdot,\bar{Y}^n(s)), \bar{z}\right\rangle_{L^2(\Gamma)} \right] ds$$
$$- \int_0^t \int_\Gamma \beta(\xi)\bar{z}\cdot u^n(s,\xi)\,d\xi\,ds + \left\langle \int_0^t B\,dW(s), \mathbf{z}\right\rangle_H,\ t\in[0,T].$$

Passing to the limit in this relation, we obtain, for $t \in [0,T]$,

$$\langle \mathbf{Y}^*(t), \mathbf{z}\rangle_H = \langle \mathbf{y}_0, \mathbf{z}\rangle_H - \int_0^t \left[\langle \chi(s), \nabla z\rangle_{L^2(\mathcal{O})} + \langle \varkappa(s), \bar{z}\rangle_{L^2(\Gamma)} \right] ds - \int_0^t \int_\Gamma \beta(\xi) \bar{z} \cdot u^*(s,\xi)\, d\xi\, ds + \left\langle \int_0^t B\, dW(s), \mathbf{z} \right\rangle_H ,$$

meaning that $\mathbf{Y}^*$ satisfies the relation

$$\mathbf{Y}^*(t) = \mathbf{y}_0 + \int_0^t \tilde{A}(s)\, ds + \int_0^t B\, dW(s),\ \ t \in [0,T],$$

where the V^*-valued, square-integrable process $\tilde{A}$ is defined by

$${}_{V^*}\left\langle \tilde{A}(s), (z, \bar{z}) \right\rangle_V := - \int_{\mathcal{O}} \chi(s) \cdot \nabla z\, dx - \int_\Gamma \left[\varkappa(s) + \beta(\xi) u^*(s) \right] \bar{z}\, d\xi,\ \ (z, \bar{z}) \in V.$$

In order to assert that $\mathbf{Y}^* = \mathbf{Y}^{u^*}$, we have to prove the identification $\tilde{A}(s) = A(\mathbf{Y}^*(s), u^*(s))$, dt-a.s. For that, we will use some results from the theory of maximal monotone operators (see [3], for example).

We have that, $\mathbb{P}$-a.s., $\mathbf{Y}^n(\cdot) - \mathbf{Y}^1(\cdot) \in W^{1,2}(0,T;V^*)$ and

$$\frac{d}{dt}\left(\mathbf{Y}^n(t) - \mathbf{Y}^1(t) \right) = A(\mathbf{Y}^n(t), u^n(t)) - A(\mathbf{Y}^1(t), u^1(t)),\ \ dt\text{-a.e.}$$

Moreover, we have

$$\left\| \mathbf{Y}^n(t) - \mathbf{Y}^1(t) \right\|_H^2 + \delta \int_0^t \left\| \mathbf{Y}^n(s) - \mathbf{Y}^1(s) \right\|_V^2 ds \le C \left\| \beta \right\|_{L^\infty(\Gamma;\mathbb{R}^m)},\ \ \forall t \in [0,T]$$

and

$$\begin{aligned} \int_0^T \left\| A(\mathbf{Y}^n(t), u^n(t)) - A(\mathbf{Y}^1(t), u^1(t)) \right\|_{V^*}^2 dt &\le C \left(1 + \int_0^T \left(\left\| \mathbf{Y}^n(t) \right\|_V^2 + \left\| \mathbf{Y}^1(t) \right\|_V^2 \right) dt \right) \\ &\le C \left(1 + \int_0^T \left(\left\| \mathbf{Y}^n(t) - \mathbf{Y}^1(t) \right\|_V^2 + \left\| \mathbf{Y}^1 \right\|_V^2 \right) dt \right). \end{aligned}$$

Consequently, the sequence $\left(\mathbf{Y}^n(\cdot) - \mathbf{Y}^1(\cdot) \right)_{n\ge 1}$ is bounded in $L^2(0,T;V) \cap W^{1,2}(0,T;V^*)$, $\mathbb{P}$-a.s. By a well-known result of Aubin (see, for example, Theorem 1.20 in [3]), since the inclusion $V \subseteq H$ is compact, $\left(\mathbf{Y}^n(\cdot) - \mathbf{Y}^1(\cdot) \right)_{n\ge 1}$ is relatively compact in $L^2(0,T;H)$.

As we already have that $\left(\mathbf{Y}^n - \mathbf{Y}^1 \right)_{n\ge 1}$ converges weakly to $\mathbf{Y}^* - \mathbf{Y}^1$ in $L^2(\Omega \times [0,T]; V)$, we infer that $\left(\mathbf{Y}^n(\cdot) - \mathbf{Y}^1(\cdot) \right)_{n\ge 1}$ converges strongly to $\mathbf{Y}^*(\cdot) - \mathbf{Y}^1(\cdot)$ in $L^2(0,T;H)$, $\mathbb{P}$-a.s. By the dominated convergence theorem $\mathbf{Y}^n$ converges strongly to $\mathbf{Y}^*$ in $L^2(\Omega \times (0,T); H)$.

Let us define the operator $\mathcal{A}$ on $L^2(\Omega \times (0,T) \times \mathcal{O}) \times L^2(\Omega \times (0,T) \times \Gamma)$ by

$$\mathcal{A}(\zeta, \bar{y}) := (\mathbf{a}(\cdot, \zeta(\cdot)), \tilde{\gamma}(\cdot, \bar{y}))$$

Since $\mathcal{A}$ is hemicontinuous and monotone, by Theorem 2.4 in [3], $\mathcal{A}$ is a maximal monotone operator.

Itô's formula applied to $\mathbf{Y}^n$, respectively $\mathbf{Y}^*$, yield

$$2\mathbb{E}\left[\int_0^T\int_{\mathcal{O}}\mathbf{a}(x,\nabla Y^n(t))\cdot\nabla Y^n(t)\,dx\,dt\right]+2\mathbb{E}\left[\int_0^T\int_\Gamma\tilde{\gamma}(\xi,\bar{Y}^n(t))\bar{Y}^n(t)\,d\xi\,dt\right]$$
$$=-\mathbb{E}\left\|\mathbf{Y}^n(T)\right\|_H^2-2\mathbb{E}\left[\int_0^T\beta(\xi)\bar{Y}^n(t)\cdot u^n(t)\,d\xi\,dt\right]+\left\|\mathbf{y}_0\right\|_H^2+T\left\|B\right\|_{L_2(H)}^2$$

and

$$2\mathbb{E}\left[\int_0^T\int_{\mathcal{O}}\chi(t)\cdot\nabla Y^*(t)\,dx\,dt\right]+2\mathbb{E}\left[\int_0^T\int_\Gamma\varkappa(t)\bar{Y}^*(t)\,d\xi\,dt\right]$$
$$=-\mathbb{E}\left\|\mathbf{Y}^*(T)\right\|_H^2-2\mathbb{E}\left[\int_0^T\beta(\xi)\bar{Y}^*(t)\cdot u^*(t)\,d\xi\,dt\right]+\left\|\mathbf{y}_0\right\|_H^2+T\left\|B\right\|_{L_2(H)}^2.$$

Consequently, since the norm in H is lower-semicontinuous with respect to the weak topology, u^n converges weakly to u^* in $L^2\left(\Omega\times[0,T];L^2(\Gamma;\mathbb{R}^m)\right)$ and $\bar{Y}^n$ converges strongly to $\mathbf{Y}^*$ in $L^2\left(\Omega\times[0,T];L^2(\Gamma)\right)$, we get

$$\limsup_{\varepsilon\to 0}\left\langle\mathcal{A}(\nabla Y^n,\bar{Y}^n),(\nabla Y^n,\bar{Y}^n)\right\rangle\le\left\langle(\chi,\varkappa),(\nabla Y^*,\bar{Y}^*)\right\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L^2\left(\Omega\times(0,T)\times\mathcal{O}\right)\times L^2\left(\Omega\times(0,T)\times\Gamma\right)$. By Corollary 2.4 in [3],

$$(\chi,\varkappa)=\mathcal{A}(\nabla Y^*,\bar{Y}^*),$$

i.e.

$$\chi(t)=\mathbf{a}(\cdot,\nabla Y^*(t)),\ \mathbb{P}\text{-a.s.}\times dx\,dt\text{-a.e.};$$
$$\varkappa(t)=\tilde{\gamma}(\cdot,\bar{Y}^*(t)),\ \mathbb{P}\text{-a.s.}\times d\xi\,dt\text{-a.e.}$$

By the uniqueness of the solution of equation (3), we have that $\mathbf{Y}^*=\mathbf{Y}^{u^*}$.

The functional $\mathcal{J}:L^2\left(\Omega\times[0,T];H\right)\times L^2\left(\Omega\times[0,T];L^2(\Gamma;\mathbb{R}^m)\right)\to\mathbb{R}$, defined by

$$\mathcal{J}(\mathbf{Y},u):=\mathbb{E}\left[\int_0^T L(\mathbf{Y}(t),u(t))dt+\Psi(\mathbf{Y}(T))\right]$$

is strongly continuous. By conditions (F_2) and (L_2), it is also convex and therefore weakly lower semi-continuous. As a consequence, $\liminf_{n\to\infty}\mathcal{J}(\mathbf{Y}^n,u^n)\ge\mathcal{J}(\mathbf{Y}^*,u^*)$. Since $\mathcal{J}(\mathbf{Y}^n,u^n)=J(u^n)$, $\mathcal{J}(\mathbf{Y}^*,u^*)=J(u^*)$ and $J(u^n)\to\inf_{u\in\mathcal{U}}J(u)$, u^* has to be an optimal control. □

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