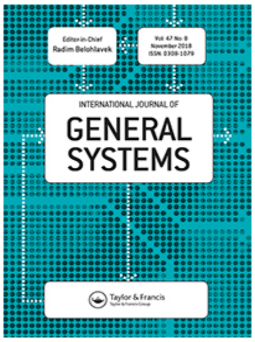

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A survey of inconsistency indices for pairwise comparisons

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ABSTRACT
Pairwise comparisons are a well-established tool to compare weights of criteria and alternatives or, more in general, any entities. Their ultimate goal is to facilitate the search for a suitable weight vector. In this context, the concepts of inconsistency and inconsistency index have emerged. This manuscript has two goals. Firstly, it surveys the most relevant inconsistency indices by means of a self-contained exposition. Secondly, by analyzing recent trends and milestones it presents some conclusions and a discussion on possible directions of future research.

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Pairwise comparison matrix; preference relation; consistency; inconsistency index; multi-criteria decision-making

Man is a rational animal: so at least I have been told. Throughout a long life, I have looked diligently for evidence in favour of this statement, but so far I have not had the good fortune to come across it. (Bertrand Russell)

1. Introduction

Very often, in operations research, economics, engineering, and other fields of science, it is necessary to estimate a positive normalized vector, i.e. a "weight" vector $w = (w_1, \dots, w_n)$ such that $w_i > 0$ and $w_1 + \cdots + w_n = 1$. Simple examples of weight vectors could be

$$w = (0.4, 0.2, 0.3, 0.1) \quad \text{and} \quad w = \left(\frac{1}{3}, \frac{1}{6}, \frac{1}{10}, \frac{1}{4}, \frac{3}{20}\right).$$

Components of w are often called weights or priorities and, among other uses, they have been employed (i) as coefficients of convex linear combinations of n vectors, (ii) weights of the n arguments of linear functions, (iii) relative weights of criteria in multi-criteria decision making problems, (iv) subjective probabilities, and (v) proportions in which to split a given quantity.

Problems arise when weights need to be determined subjectively by an expert or a decision maker. Namely, when each weight w_i is estimated by asking its value directly to an expert, the cognitive burden could be excessive and, especially when n is large, the likelihood of incurring in cognitive biases could be a risk.[1] A common strategy to avoid these pitfalls is to divide the problem into subproblems and then tackle these latter ones. In this context, a pairwise comparison a_{ij} is a value, usually provided by an expert, expressing his estimation of the ratio between weights w_i and w_j, i.e. $a_{ij} \approx w_i/w_j$. For sake of example, the

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possible value $a_{24} = 3$ indicates that the expert considers w_2 three times greater than w_4. Of course, since all the weights are positive, we also know that, in general, $a_{ij} > 0$.

Besides splitting the problem into smaller and more tractable subproblems, pairwise comparisons give the analyst the possibility of estimating how irrational the judgments of a given experts are. We can consider the example in Figure 1(a), with $a_{12} = 2$, $a_{23} = 3$, and $a_{13} = 6$, which is a set of non-contradictory pairwise comparisons. A set of non-contradictory pairwise comparisons is called *consistent*. If a_{13} is replaced by $\dot{a}_{13} = 4$, the set of comparisons, now depicted in Figure 1(b), becomes *inconsistent*. If, instead, $a_{13} = 4$ is replaced by $\ddot{a}_{13} = 1/5$, as shown in Figure 1(c), it is sensible to say that the new set of comparisons is even more *inconsistent* than the one in Figure 1(b).

Figure 1 evidences how sets of pairwise comparisons can either be consistent or inconsistent but also that the condition of consistency can be violated to different extents, and therefore the concept of inconsistency should be a matter of degree.

Although a subset of $(n-1)$ pairwise comparisons a_{ij}, chosen so that they induce a spanning tree on the set of weights, would be sufficient to determine the weight vector w, for sake of robustness, in Multi Attribute Value Theory (MAVT) Keeney and Raiffa (1976, 123) stated that "it may be desirable to ask additional questions thereby getting an over-determined system of equations, fully expecting that the set of responses would be inconsistent in practice". Even more so, in the Analytic Hierarchy Process (AHP), Saaty (1977) required that all the $n(n-1)/2$ pairwise comparisons be elicited. Of course, as reminded by Keeney and Raiffa, redundancy of information may lead to inconsistencies, since it is hardly ever possible to be consistent. Even in this case, both proponents of MAVT and AHP agree: Saaty (1977) stated that pairwise comparisons should be "close" to

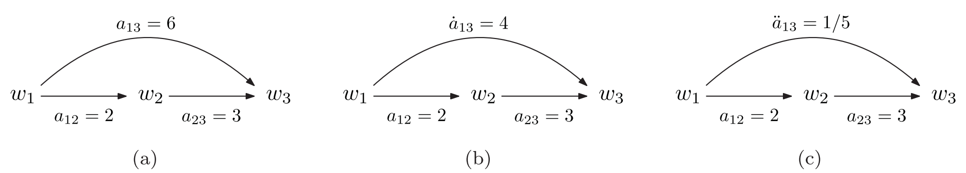

Figure 1. Example of sets of pairwise comparisons and their consistency. (a) A set of consistent pairwise comparisons, (b) a set of inconsistent pairwise comparisons and (c) another set of inconsistent pairwise comparisons.

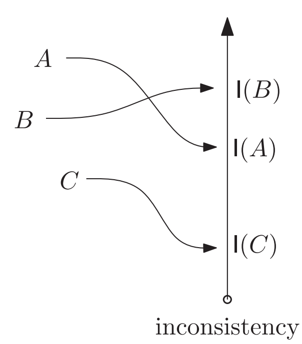

Figure 2. Visual interpretation of an inconsistency index. Each index represents a mapping of PCMs to the real line.

consistency and Keeney and Raiffa (1976, 205) recommended that inconsistencies be kept at a "nominal level". The role of inconsistency indices is precisely that of quantifying the deviation of a set of pairwise comparisons from being consistent. The goal of this paper is to offer an overview of inconsistency indices and their ramifications.

This manuscript is organized as follows. The next section introduces the concepts of pairwise comparison matrix and inconsistency index. Section 3 is a self-contained presentation of some notable inconsistency indices. Section 4 completes the presentation of the inconsistency indices with a summary of recent comparative studies. Section 5 elaborates on other uses and ramifications of the concept of inconsistency index. Section 6 draws some conclusions.

2. Pairwise comparison matrices and inconsistency

A pairwise comparison matrix is a mathematical structure collecting pairwise comparisons. More formally, a *pairwise comparison matrix* (PCM) is a positive square matrix $A = (a_{ij})_{n\times n}$ where $a_{ij} > 0$ is the subjective estimation of the ratio between w_i and w_j. Note that, it is reasonably assumed that $a_{ii} = 1\ \forall i$ and $a_{ij} = 1/a_{ji}\ \forall i,j$. Hence, a PCM has the following structure:

$$A = \begin{array}{c} \\ w_1 \\ w_2 \\ \vdots \\ w_n \end{array} \begin{array}{c} \begin{array}{cccc} w_1 & w_2 & \cdots & w_n \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \end{array} = \begin{array}{c} \\ w_1 \\ w_2 \\ \vdots \\ w_n \end{array} \begin{array}{c} \begin{array}{cccc} w_1 & w_2 & \cdots & w_n \end{array} \\ \begin{pmatrix} 1 & a_{12} & \cdots & a_{1n} \\ 1/a_{12} & 1 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 1/a_{1n} & 1/a_{2n} & \ldots & 1 \end{pmatrix} \end{array}.$$

Among the many methods to estimate a suitable weight vector w from a PCM A, two gained a special importance. The first is the *eigenvector method*, for which the weight vector is the normalized solution of the system $Aw = \lambda_{\max} w$ where $\lambda_{\max}$ is the Perron-Frobenius eigenvalue of A (Saaty 1977) and where w is considered a column vector. The second is the *geometric mean method*, according to which each weight w_i is the geometric mean of the elements on the ith row of A (Crawford and Williams 1985). Also in this case, the vector w is further processed so that its components sum up to 1.

A pairwise comparison matrix is *consistent*, if and only if,

$$a_{ik} = a_{ij}a_{jk} \quad \forall\, i,j,k. \tag{1}$$

Condition (1) is a formalization and generalization of the idea of consistency exemplified in Figure 1(a). Its interpretation is that each pairwise comparison a_{ik} is indirectly supported by all comparisons between w_i and w_k passing through $w_j \forall j$.

An inconsistency index associates pairwise comparison matrices to real numbers. That is, if $\mathcal{A}$ is the set of all the pairwise comparison matrices, an *inconsistency index* [2] is a function $\mathsf{I} : \mathcal{A} \rightarrow \mathbb{R}$. The usual semantic of an inconsistent index is that the greater $\mathsf{I}(A) \in \mathbb{R}$, the greater is the inconsistency of A.

It was shown that the consistency condition (1) is equivalent to any of the following:

- There exists a positive vector $w = (w_1, \ldots, w_n)$ such that $a_{ij} = w_i/w_j \forall\, i,j$
- The Perron-Frobenius eigenvalue of A is equal to n
- The matrix A has rank 1.

It is easy to envision that such a variety of conditions has been a fertile ground for the development of different inconsistency indices.

3. Inconsistency indices

The first and by far the most popular inconsistency index was introduced by Saaty (1977) in his theory of the Analytic Hierarchy Process (AHP) and is an affine transformation of the Perron-Frobenius eigenvalue of A, here denoted as $\lambda_{\max}$:

$$\mathsf{CI}(A) = \frac{\lambda_{\max} - n}{n-1}.$$

To make this quantification of the inconsistency more intelligible, Saaty (1977) showed that it can alternatively be expressed as

$$\mathsf{CI}(A) = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(a_{ij} \frac{w_j}{w_i} - 1 \right),$$

where w is the Perron-Frobenius eigenvector of A. Moreover, Saaty (1977) proposed to divide CI by the so-called Random Index RI_n, the average CI calculated on a large number of PCMs of order n with entries sampled from the scale $\{1/9, 1/8, \ldots, 8, 9\}$, thus obtaining the Consistency Ratio $\mathsf{CR} = \mathsf{CI}/RI_n$. Moreover, still Saaty proposed the threshold 0.1 for CR so that only matrices with $\mathsf{CR}(A) \leq 0.1$ should be classified as sufficiently consistent. Alonso and Lamata (2006) computed the values of RI_n of large datasets of random PCMs and Aupetit and Genest (1993) provided a tight upper bound for the value of CI in the cases when entries of A are expressed on a bounded scale.

Essentially for historical reasons, CR has been the dominating inconsistency index for the last 40 years. Nevertheless, other inconsistency indices have gained prominence since the inception of CI and CR. Crawford and Williams (1985) proposed to use the geometric mean method to estimate the weight vector $w = (w_1, \ldots, w_n)$ and formulated the Geometric Consistency Index,

$$\mathsf{GCI}(A) = \frac{2}{(n-1)(n-2)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \ln^2 \left(a_{ij} \frac{w_j}{w_i} \right).$$

Thresholds and formal properties of GCI were studied by Crawford (1987) and Aguarón and Moreno-Jiménez (2003). Brunelli, Critch, and Fedrizzi (2013, Proposition 2) showed an equivalent formulation of GCI and proved that it is proportional to another inconsistency index (Fedrizzi and Giove 2007, formula (10)).

Koczkodaj (1993) proposed a measure of inconsistency for a PCM of order three. Duszak and Koczkodaj (1994) generalized the original proposal to $n \geq 3$, and presented the inconsistency index,

$$\mathsf{K}(A) = \max_{i<j<k} \min \left\{ \left| 1 - \frac{a_{ik}}{a_{ij}a_{jk}} \right|, \left| 1 - \frac{a_{ij}a_{jk}}{a_{ik}} \right| \right\}.$$

Csató (2018b) provided a full characterization of index K by means of six axioms.

Grzybowski (2016) considered the index K and replaced the max operator with the arithmetic mean. The validity of the obtained inconsistency index, i.e.

$$\mathsf{ATI}(A) = \frac{1}{\binom{n}{3}} \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} \min \left\{ \left| 1 - \frac{a_{ik}}{a_{ij}a_{jk}} \right|, \left| 1 - \frac{a_{ij}a_{jk}}{a_{ik}} \right| \right\}$$

was related by Grzybowski (2016) to the reliability of the priority vector.

Golden and Wang (1989) studied an inconsistency index based on the dissimilarity between the columns of A and the priority vector w. First they transformed the original PCM A in $\bar{A} = (\bar{a}_{ij})_{n\times n}$ whose entries are normalized so that each column add up to one. At this point, their index is

$$\mathsf{GW}(A) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left| \bar{a}_{ij} - w_i \right|.$$

Shiraishi, Obata, and Daigo (1998) proposed to take the coefficient c_3 of the characteristic polynomial of A as a measure of inconsistency. They found an analytic formula for c_3 which can be used as an inconsistency index

$$\mathsf{c}_3(A) = \frac{1}{\binom{n}{3}} \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} \left(2 - \frac{a_{ik}}{a_{ij}a_{jk}} - \frac{a_{ij}a_{jk}}{a_{ik}} \right).$$

It is worth noting that the interpretation of c_3 is reversed. Namely $\mathsf{c}_3 \leq 0$ and equal to 0 if and only if A is consistent. Brunelli, Critch, and Fedrizzi (2013) proved that c_3 is proportional to another index proposed by Peláez and Lamata (2003).

Stein and Mizzi (2007) considered s_j to be the sum of the entries on the jth column of A and defined $h(A) = (n/\sum_{i=1}^{n} s_j^{-1})$ to be their harmonic mean. They proved that $h(A) \geq n$ and equal to n if and only if A is consistent, and defined the Harmonic Consistency Index:

$$\mathsf{HCI}(A) = \frac{(h(A) - n)(n+1)}{n(n-1)}.$$

Cavallo and D'Apuzzo (2009, 2010, 2012) generalized the idea of PCM from the algebraic point of view using commutative linearly ordered groups and proposed and justified a new inconsistency index,

$$\mathsf{I}_{\mathsf{CD}}(A) = \prod_{i=1}^{n-2} \prod_{j=i+1}^{n-1} \prod_{k=j+1}^{n} \left(\max \left\{ \frac{a_{ik}}{a_{ij}a_{jk}}, \frac{a_{ij}a_{jk}}{a_{ik}} \right\} \right)^{1/\binom{n}{3}}.$$

The index I_{CD} is equal to 1 in the case of consistent matrices and strictly greater than 1, otherwise. In addition, Brunelli (2016b, Proposition 1) showed that I_{CD} is functionally related to another inconsistency index (Brunelli 2016b, formula (2)).

While attempting to formulate a method for increasing the rationality of preferences, Takeda (1993) proposed the following inconsistency index

$$\mathsf{MC}(A) = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \left(\prod_{k=1}^{n} a_{ij} a_{jk} a_{ki} \right)^{1/n}.$$

It is straightforward to show that $\mathsf{MC} \geq 1$ and that it is related to another index proposed, later on, by Wu and Xu (2012).

Salo and Hämäläinen (1995) considered the sets $R_{ij} = \{a_{ik}a_{kj} | k = 1, \ldots, n\}$. Each set contains all the theoretical values of a_{ij} obtained indirectly through alternative k. Clearly, if A is consistent, each set R_{ij} is a singleton. At this point, Salo and Hämäläinen (1995) proposed the following inconsistency index

$$\mathsf{AI}(A) = \frac{2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{\max R_{ij} - \min R_{ij}}{(1 + \max R_{ij})(1 + \min R_{ij})},$$

which can be interpreted as a normalized sum of the lengths of the intervals induced by the sets R_{ij}'s. Index AI, which is the acronym of Ambiguity Index, was used to evaluate the inconsistency of pairwise comparisons in decision processes related to energy production (Salo and Hämäläinen 1995) and traffic planning (Hämäläinen and Pöyhönen 1996).

Fedrizzi and Ferrari (2018) based their inconsistency index on an interpretation of a PCM as a contingency table. First they defined a chi-square-based consistent approximation of the matrix A and called it $E = (e_{ij})_{n \times n}$ and then they defined their inconsistency index as

$$\chi^2(A) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{(a_{ij} - e_{ij})^2}{e_{ij}}.$$

If A is consistent, its rank is equal to 1 and its columns are collinear vectors. Employing the concept of cosine of angles between vectors, Kou and Lin (2014) proposed an inconsistency index related to a quantification of the lack of collinearity between the columns of A. The analytic formula of the Cosine Consistency Index is

$$\mathsf{CCI}(A) = \sqrt{\sum_{i=1}^{n} \left(\sum_{j=1}^{n} b_{ij} \right)^2},$$

with $b_{ij} = a_{ij} / \sqrt{\sum_{k=1}^{n} a_{kj}^2}$.

Gass and Rapcsák (2004) used the Singular Value Decomposition to decompose A in the form $A = UDV^{\mathrm{T}}$ where D is a diagonal matrix with diagonal entries $\alpha_1, \ldots, \alpha_n$. Then, they find the rank one approximation $A_{[1]} = u\alpha_1 v^{\mathrm{T}}$ where u and v are the first columns of U and V, respectively. Finally, their inconsistency index is

$$\mathsf{SVD}(A) = \|A - A_{[1]}\|,$$

where $\| \cdot \|$ denotes the Frobenius norm.

Kułakowski (2015) considered the condition of order preservation (COP) proposed by Bana e Costa and Vansnick (2008) and suggested the following inconsistency index

$$\mathsf{E}(A) = \max_{i,j} \left\{ a_{ij} \frac{w_j}{w_i} - 1 \right\}.$$

Small values of E guarantee that COP is preserved. Although it was presented for the case of w estimated as the maximum eigenvector of A, this index can be extended to the case of w estimated by means of other methods.

Barzilai (1998) used the logarithm as group isomorphism to pass from the multiplicative to the additive approach and proposed the following as a measure of inconsistency

$$\mathsf{RE}(A) = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} \left(\log a_{ij} - \log \frac{w_i}{w_j} \right)^2}{\sum_{i=1}^{n} \sum_{j=1}^{n} (\log a_{ij})^2},$$

where w is estimated using the geometric mean method. RE is normalized since its values always range in the interval $[0, 1]$.

Dixit (2018) drew a parallel between pairwise comparison matrices and time-irreversible Markov chains and introduced

$$\mathsf{S}(A) = \frac{1}{\lambda_{\max}} \sum_{i=1}^{n} \sum_{j=1}^{n} u_i w_j a_{ij} \log a_{ij},$$

where w and u are the principal right and left eigenvectors of A, respectively. Besides acting as an inconsistency index, S has an interpretation in statistical physics and it can be adapted to work with matrices with missing entries. This issue will be discussed more in detail later in the paper.

From the presentation of the inconsistency indices proposed so far, it is possible to informally study some of their basic properties. Table 1 summarizes some of them. The table should be taken as a rough analysis since these properties have not been thoroughly formalized in the literature.

Figure 3 presents the timeline of inconsistency indices whose definitions have been recalled in this section. Other notable approaches, often variants or generalizations, have been proposed, among others, by Siraj, Mikhailov, and Keane (2015),

Table 1. Some characteristics of inconsistency indices.

	Index															
Property	CI	GCI	K	ATI	GW	c_3	HCI	I_{CD}	MC	AI	χ^2	CCI	SVD	E	RE	S
Weight vector[a]	Y	Y	N	N	Y	N	N	N	N	N	Y	Y	N	Y	Y	Y
Maxitive[b]	N	N	Y	N	N	N	N	N	N	N	N	N	N	Y	N	N
Closed form[c]	N	Y	Y	Y	N	Y	Y	Y	Y	Y	Y	Y	N	N	Y	N

Note: Y = yes, N = no.
[a]The formulation of the inconsistency index makes explicit reference to the weight vector w or it has been shown that it can be rewritten as a function of the components of w.
[b]The inconsistency index focuses on the most inconsistent triad or element of the matrix.
[c]The inconsistency index can be formulated in a closed form or has an analytic solution. For some indices, e.g. GW and E, this depends on the method used to estimate the weight vector.

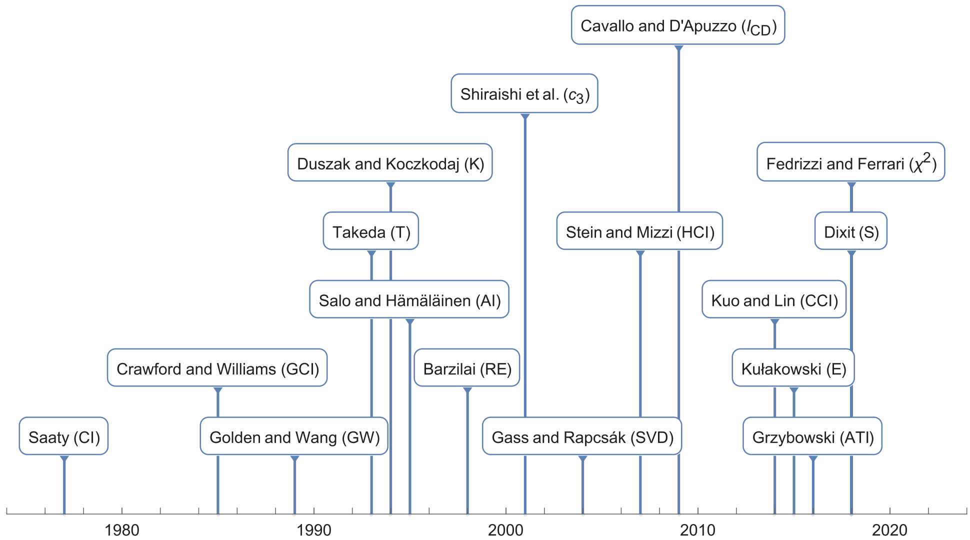

Figure 3. Timeline of the main developments of inconsistency indices.

Kułakowski and Szybowski (2014), Szybowski (2016), Osei-Bryson (2006), Wan, Chen, and Zhang (2013), Peláez, Martínez, and Vargas (2018), Brunelli and Fedrizzi (2018), and Mizuno (2019).

4. Comparative studies

Two main streams of comparative studies stand out from the literature. The first, mostly based on numerical simulations, has investigated whether different inconsistency indices are similar or dissimilar. The second stream, based on formal studies, has aimed at understanding the meaning of different indices.

4.1. Similarities between indices

The first comparative studies appeared already in some of the papers introducing specific indices. For instance, Obata et al. (1999) compared their index c_3 with the better known CI. Comparisons of CI with single indices were made, among other, by Stein and Mizzi (2007) for HCI and Fedrizzi and Ferrari (2018) for χ^2. More comprehensive numerical studies were proposed by Brunelli, Canal, and Fedrizzi (2013), Kazibudzki (2016) and Cavallo (2018).

Formal studies have also been proposed to study relations between indices. Bozóki and Rapcsák (2008) compared CI and K and derived formal results for the case of PCMs of orders 3 and 4. In particular, CI and K are equivalent in the case of order 3. Brunelli, Critch, and Fedrizzi (2013) and Brunelli (2016b) proved that some indices which were independently introduced are, instead, functionally related, and therefore equivalent. Cavallo (2018) showed that, in fact, there are many indices which are functionally related in the case of PCMs of order 3.

As a corollary to the numerical studies, we present a summary study based on a dataset of perturbed PCM of order 6. A perturbed PCM is obtained by perturbing the entries of

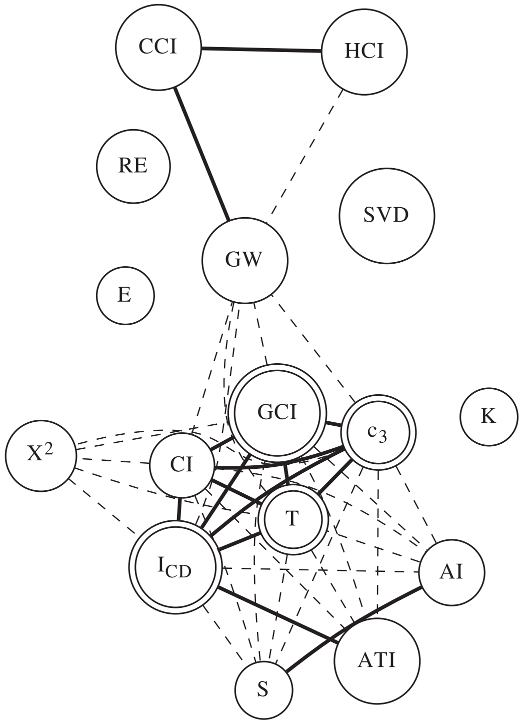

Figure 4. Degrees of similarity between inconsistency indices, tested on perturbed PCMs of order $n = 6$ obtained with $\sigma = 0.43$. Thick solid lines denote a value of the Spearman index greater than 0.95. Thin dashed lines denote a value of the Spearman index greater than 0.9. Double circled inconsistency indices have been found functionally related to some others proposed in the literature.

a consistent PCM with independent random noises ϵ_{ij}'s. More formally, given a consistent PCM $A^C = (a^c_{ij})_{n\times n}$ a perturbed PCM $A^P = (a^p_{ij})_{n\times n}$ is a PCM with entries

$$a^p_{ij} = a^c_{ij}\epsilon_{ij} \quad i < j,$$

where $\epsilon_{ij} \sim \text{Lognormal}(0, \sigma^2)$ and the lower triangular entries are obtained by reciprocity. Figure 4 shows that there is a kernel of similar indices, but also some outliers.

Furthermore, since some of them are functionally related they also have a double interpretation. On one hand, an analyst might be tempted to rely on indices from this kernel. On the other hand, when evaluating inconsistency, it might be prudent to use two dissimilar indices in order to consider separate evidence. It must be noted that the fact that an index is dissimilar to the others is not *per se* a negative feature.

4.2. Soundness of indices

The second stream of comparative studies has focused on formal properties of inconsistency indices. That is, the goal of these studies has been the definition of some basic reasonable properties which should be satisfied by any theoretically sound estimation of inconsistency. Some early attempts of formalization were proposed already by proponents of single inconsistency indices. For example, Barzilai (1998) claimed that his index RE satisfies a number of good properties. Only recently full-fledged formal studies have appeared in the literature. Brunelli and Fedrizzi (2015a) proposed five properties and tested them on various inconsistency indices, showing that they are quite restrictive. A sixth property was added later on by Brunelli (2017) and two more by Csató (2018a). Koczkodaj and Szwarc (2014) independently proposed another set of axioms and showed that index K satisfies these properties. Koczkodaj et al. (2017) claimed the necessity of normalization of any inconsistency index. Koczkodaj and Urban (2018) proposed a new set of axioms to improve those proposed by Koczkodaj and Szwarc. At present, it seems that the major divergence between different axiomatization attempts is on whether to consider the inconsistency level of the most inconsistent triad of comparisons as representative of the inconsistency of the entire pairwise comparison matrix (as done by index K) or to allow indices to average local inconsistencies (as done by the vast majority of indices). Commentaries on different attempts of studying formal properties of inconsistency indices were written by Koczkodaj et al. (2016) and Brunelli (2016a).

5. Other uses of inconsistency indices

The use of inconsistency indices is not limited to the mere quantification of inconsistency. In the years, they have proven useful for other tasks. Among the most notable uses, one might recall those presented in this section.

5.1. Inconsistency reduction

As also recalled by Keeney and Raiffa (1976, 123), "[inconsistencies] can be used by the analyst to 'force' the decision maker to rethink through his preferences", which means that when a decision maker expresses pairwise comparisons which are too inconsistent he is invited to revise them. One of the first attempts to support analysts and decision makers was presented by Harker (1987a), where he analyzed the sensitivity of the index CI to the entries of the matrix A and suggested to rethink the entries to which the value of CI is the most sensitive. Similar approaches, which consider a measure of global consistency and indicate the contribution of each entry to suggest which ones ought to be revised, were adopted by Ishizaka and Lusti (2004), Ergu et al. (2011), Siraj, Mikhailov, and Keane (2012), and Kou, Ergu, and Shang (2014). Pragmatically, Bozóki, Fülöp, and Poesz (2011) proposed to define a maximal tolerable level of inconsistency, say δ, and then find the "closest" matrix to the original A such that its inconsistency, measured by different indices, is smaller or equal to δ. A similar approach was used by Xu and Cuiping (1999) to find an approximation of the original matrix by another matrix with CR lower than 0.1. Bozóki, Fülöp, and Poesz (2015) presented some optimization problems to find the minimum number of entries to be changed to transform a generic PCM in a consistent one. Li and Ma (2007) exploited the idea proposed by Genest and Zhang (1996) and employed Gower plots for the visual analysis of inconsistency and its determinants. Temesi (2018) presented an analysis of different techniques for inconsistency reduction with a special attention on the connection between entries of the PCM and verbal statements.

There is not an agreement on whether automated procedures for inconsistency reduction have a positive effective on the reliability of the final result. Benítez et al. (2012)

and Finan and Hurley (1997) claimed that there is a positive effect. Conversely, Gaul and Gastes (2012) and Gastes and Gaul (2012) questioned the validity of automated approaches for the improvement of inconsistency. This said, an involvement of the decision maker in the revision process is always deemed auspicable.

5.2. Incomplete pairwise comparison matrices

The use of inconsistency indices has gone beyond the mere quantification of inconsistency. One of the most important uses has been that of dealing with PCMs with missing comparisons. An incomplete pairwise comparison matrix is a PCM with some entries missing. The following is an example of incomplete PCM with entries a_{13} and a_{14} missing:

$$A = \begin{pmatrix} 1 & 2 & x & y \\ 1/2 & 1 & 1/3 & 1 \\ 1/x & 3 & 1 & 2 \\ 1/y & 1 & 1/2 & 1 \end{pmatrix}. \quad (2)$$

Given an incomplete PCM one can follow two approaches: estimate the weight vector $w = (w_1, \ldots, w_n)$ only from the information coming from the given entries (Harker 1987b) or try to estimate the given entries. If one follows the second strategy, a widely accepted approach is to seek for the values of the missing entries which seem the most coherent with the known entries of the matrix. Considering the previous example, values of the missing entries are the solutions of the optimization problem

$$\arg\min_{x,y>0} \mathsf{I}(A),$$

Various inconsistency indices have been used as objective functions. Shiraishi and Obata (2002) proposed to maximize the index c_3, bearing in mind that the interpretation of this index is reversed: the greater its value, the smaller the inconsistency. Peláez and Lamata (2003) proposed something similar with their index. However, since their index is proportional to c_3, this approach is equivalent to the one proposed by Shiraishi and Obata (2002). Koczkodaj, Herman, and Orlowski (1999) and Bozóki, Fülöp, and Koczkodaj (2011) followed the same approach for index K. Similar techniques were implemented by Bozóki, Fülöp, and Rónyai (2010) for CI and Khatwani and Kar (2016) for CCI. The proposal by Bozóki, Fülöp, and Rónyai (2010) is particularly interesting because it regards the most popular inconsistency index, CI, and optimizes a non-trivial objective function by means of an algorithm based on cyclic coordinates.

Another research direction regarded the extensions of inconsistency indices to the calculation of inconsistency of incomplete PCMs and possibly the real-time prediction, while filling the entries, of the inconsistency of the complete PCM (Wedley 1993). For example, $\lambda_{\max}$ always exists for a PCM, but deeper results were necessary to extend it to incomplete PCMs (Oliva, Setola, and Scala 2017). The capacity of an inconsistency index to be extensible to incomplete PCMs should be considered a good property. Recently, Ureña et al. (2015) offered a comprehensive survey on incomplete preferences and related topics.

5.3. Extensions to other representations of pairwise comparisons

It is safe to say that the theory of pairwise comparison matrices — and the development of inconsistency indices with it — has been extended in two distinct directions. Although a full exposition of this topic is well beyond the scope of this manuscript (see instead Xu 2007), it is important to underline the fundamental concepts.

(1) The first type of extensions regards pairwise comparisons expressed on *different representation domains*. For example, additive preferences are expressed on the entire real line and reciprocal preference relations, also called fuzzy preference relations, are expressed on the unit interval $]0, 1[$ (sometimes $[0, 1]$). Consistency indices for reciprocal preference relations have been introduced, among others, by Fedrizzi and Giove (2007) and Herrera-Viedma et al. (2007). Nonetheless, the number of inconsistency indices proposed for these approaches is small when compared to those proposed for PCMs. In fact, other representations of pairwise comparisons can be transformed into PCMs by means of suitable morphisms resulting in different approaches being *de facto* isomorphic (Cavallo and D'Apuzzo 2009). Oftentimes such isomorphisms have revealed that inconsistency indices proposed separately for different approaches were instead equivalent. The existence of transformations between different representation domains implies that inconsistency indices normally used for PCMs can straightforwardly be extended to many other representations of preferences.

(2) The second direction includes approaches which generalize the concept of pairwise comparison matrix to introduce *uncertainty in the judgments* by employing probability theory, fuzzy sets theory, interval arithmetics and so forth. When the judgments are expressed as fuzzy numbers one could use, for instance, the inconsistency index proposed by Ramík and Korviny (2010), Ramík and Perzina (2010) and commented by Brunelli (2011), or the indices proposed by Fedrizzi and Marques Pereira (1995), Ohnishi et al. (2008), Kubler et al. (2017). In the case of preferences expressed as real intervals, one could, instead, refer to the indices proposed by Li, Wang, and Tong (2016), Cavallo and Brunelli (2018), Zhang (2017), and Liu et al. (2018). Other representations, like those based on hesitant fuzzy sets, have been objects of proposals (Zhang and Wu 2014; Zhu and Xu 2014; Liu, Xu, and Liao 2016). A monograph on non-conventional representations of preferences and the measurement of their inconsistency with a special emphasis on fuzzy sets was recently written by Krejčí (2018). By observing the indices proposed in these extended frameworks, one can however note that they are generalizations of well-known indices used for PCMs, or at the very least they have a very clear interpretation in the real valued case. Hence, when the general representations collapse into real numbers, then also the proposed inconsistency indices collapse in well-known ones.

Although both types of generalizations have been a fertile ground for the introduction of inconsistency indices, one can argue that their originality is limited, since they often descend from indices already proposed for PCMs. For this reason, the analysis of indices proposed for PCMs does not seem limiting and results on these indices can be extended to their generalizations.

5.4. Group decision-making

Since most of real-world decisions are not made by individuals, but by groups and committees, the relation between inconsistency and group decision-making has attracted a great deal of attention. The fundamental premise is that, given m individual PCMs $A_1, \ldots, A_m$, a group PCM A^* can be obtained as the entry-wise weighted geometric mean of the individual PCMs,

$$A^* = \left(a_{ij}^*\right)_{n \times n} = \left(\prod_{k=1}^{m} \left(a_{ij}^{(k)}\right)^{\omega_k}\right)_{n \times n}, \tag{3}$$

where $\omega_1, \ldots, \omega_m$ with $\omega_k \geq 0$ and $\omega_1 + \cdots + \omega_m = 1$ are the degrees of importance (or voting powers) of different decision makers in the preference aggregation process. It has been proven that this aggregation method (3) is the only one which satisfies a number of reasonable properties (Aczél and Saaty 1983).

A research question regarded possible implications between the inconsistency of individual PCMs $A_1, \ldots, A_m$ and the inconsistency of the group PCM A^*. Initially, Xu (2000) claimed the proof that the CI of the group preferences cannot be greater than the CI of the most inconsistent individual PCM, i.e. $\mathsf{CI}(A^*) \leq \max\{\mathsf{CI}(A_1), \ldots, \mathsf{CI}(A_m)\}$. Lin et al. (2008) found a fallacy in Xu's proof, thus leaving it as a conjecture. Liu, Zhang, and Wang (2012) finally proved Xu's conjecture and Grošelj and Stirn (2012) showed that it comes from more general results by Elsner, Johnson, and Dias da Silva (1988). Escobar, Aguarón, and Moreno-Jiménez (2004) provided similar results for the case of index GCI. Brunelli and Fedrizzi (2015b) used a more general approach and related the property of being below a given threshold with some general properties of a given inconsistency index.

A second research direction has tried to formulate algorithms able to account for consensus and consistency within the same mathematical model. There is a vast literature on this subject, but a common goal seems to be that of reaching a sufficiently consensual solution keeping the inconsistency of preferences under a given threshold (Chiclana et al. 2008; Wu and Xu 2012; Escobar, Aguarón, and Moreno-Jiménez 2015).

In spite of the relevance of these results, the inconsistency of the group pairwise comparisons has to be interpreted with some caution. In fact, a low level of inconsistency of the group preferences should not be taken as a guarantee of their reliability, since the group preferences could have been obtained by fusing extremely inconsistent individual pairwise comparisons. Seemingly, some deeper reflection on the interpretation of the (in)consistency of a group PCM and its implications is due.

5.5. Weaker rationality conditions

It is nowadays accepted that consistency, albeit desirable, is an utopic condition which can hardly ever be reached in practice. Out of the many possible conditions of rationality, consistency is the most restrictive. Then, some authors (Basile and D'Apuzzo 2002, 2006) preferred to investigate conditions weaker than consistency as, for instance, transitivity,

$$a_{ij} \geq 1 \text{ and } a_{jk} \geq 1 \Rightarrow a_{ik} \geq 1 \quad \forall\, i, j, k,$$

restricted max –max transitivity

$$a_{ij} \geq 1 \text{ and } a_{jk} \geq 1 \Rightarrow a_{ik} \geq \max\{a_{ij}, a_{jk}\} \quad \forall\, i, j, k,$$

rank-order consistency (Finan and Hurley 1996), weak consistency (Cavallo, D'Apuzzo, and Basile 2016), and pairwise dominance (Saaty and Vargas 2012). Unlike inconsistency indices, which are based on a concept of "cardinal" distance, so far indices used to capture the violation of these weaker conditions have been proposed as simple counts of how many times the condition is violated in a given PCM (Kułakowski 2018). Further research could help bridge this gap.

6. Discussion

In a recent a bibliographical study on the publications about pairwise comparisons in multi-criteria decision-making in the years 2010–2015, Kou et al. (2016) stated that "The most popular research topic is the inconsistency issue". The scope of this survey is precisely that of offering an overview on inconsistency indices and their connections with related topics.

Oppositely to some studies questioning the connection between rationality of pairwise comparisons and reliability of decision (Linares 2009; Temesi 2011), the vast majority of the literature seems to agree on the fact that a high degree of inconsistency of pairwise comparisons could be symptomatic of a lack of quality of these latter. This, in final analysis, could negatively affect the quality of the weight vector $w = (w_1, \ldots, w_n)$ obtained from the pairwise comparisons. Remarkably, this point of view, albeit more studied in the AHP, is relevant for other methodologies, such as MAVT.

In spite of a well-established definition of consistency, there is not a meeting of mind on a unique definition of inconsistency. In fact, each inconsistency index can be seen as an implicit definition of inconsistency. In this paper, we explicitly considered 16 inconsistency indices – each reflecting a different perspective on the quantification of inconsistency – and reviewed their developments also in connection with other aspects of pairwise comparisons. It is also interesting to reckon that most of the methods for deriving a weight vector w from A also implicitly define an inconsistency metric. In fact, the derived w can be associated to a consistent matrix $(w_i/w_j)_{n\times n}$ and so what the method for deriving the vector really does is finding the "closest" (with respect to an implicit metric) matrix $(w_i/w_j)_{n\times n}$ to A. Of course, the metric quantification of "closest" can be interpreted as the level of inconsistency of A.

Within the study of inconsistency indices some promising research niches have manifested themselves and are worth few additional remarks.

- Studying formal properties of inconsistency indices has made some formal order and improved our understanding of what the indices really do. It has also shifted the discussion on the general idea of inconsistency index and on what we really mean with this definition. Although there is not (and there might never be) a complete agreement on what basic properties ought to be satisfied by an inconsistency index, it is safe to say that analyzing formal properties of inconsistency indices is a necessary step in order to make them intelligible and communicable.
- Algebraic studies have appeared in the recent literature (Cavallo and D'Apuzzo 2009; Koczkodaj, Szybowski, and Wajch 2016; Cavallo and Brunelli 2018). These studies employ concepts from abstract algebra and they have helped clarify some controversial aspects of the pairwise comparisons technique. Similarly, they helped define and

study mathematically sound inconsistency indices. Hence, they should be welcome in the literature, as long as they are not an end unto themselves.

- Most of the studies on inconsistency indices have been based on formal methods or on Monte Carlo simulations. Needless to say, real-world pairwise comparisons presents regularities and patterns and are far from being random. Results on inconsistency indices, and on preference relations in general, are as realistic as they are based on real data. This was recently acknowledged, among others, by Bozóki et al. (2013) and Cavallo et al. (2018) in some studies on pairwise comparisons and by Regenwetter et al. (2006) in a critique of the random culture in computational social choice. Due to these reasons, and considering the scarcity of empirical studies in the literature, it is foreseeable that new studies of this type could be valuable contributions.
- A concrete problem, with practical implications, is the determination of the so-called thresholds. Inconsistency is a matter of degree but it is very often used together with some sharp thresholds, as the widely accepted rule that matrices with $\mathsf{CR} \leq 0.1$ are not too inconsistent and thus acceptable. That is, a crisp decision rule must be stipulated on the fuzzy concept of "too inconsistent". In spite of this *impasse*, it is still safe so say that deeper statistical studies (Lin, Kou, and Ergu 2013, 2014) on threshold values, also in relation to the index on which they are used, could improve the decision analysis process. Another viable solution would be that of favoring indices whose values have a clear interpretation.
- Many inconsistency indices have been proposed thus far. This does not mean that there is not space to introduce new ones. Rather, this should entail that, to make a valuable contribution, when introducing a new index scholars need to account for the existing indices, pointing out the novelties in the newly introduced one and *clearly* show its properties and discuss its advantages.

All in all, the study of the inconsistency of pairwise comparisons has attracted the attention of scholars from the most disparate fields of science, such as mathematics, behavioral sciences, and economics, just to name a few, and the use of different tools of scientific inquiry, like formal methods and behavioral studies. It is the author's opinion that this ought to be regarded as a paragon of how different fields can build on each other.

Notes

1. Consider, for example, the "subadditivity effect" bias (Hilbert 2012).
2. Inconsistency indices have often been equivalently called consistency indices in the literature.

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No potential conflict of interest was reported by the author.

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References

Aczél, Janos, and Thomas L. Saaty. 1983. "Procedures for Synthesizing Ratio Judgements." *Journal of Mathematical Psychology* 27 (1): 93–102.

Aguarón, Juan, and José María Moreno-Jiménez. 2003. "The Geometric Consistency Index: Approximated Thresholds." *European Journal of Operational Research* 147 (1): 137–145.

Alonso, José, and Maria Teresa Lamata. 2006. "Consistency in the Analytic Hierarchy Process: A New Approach." *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* 14 (4): 445–459.

Aupetit, Bernard, and Christian Genest. 1993. "On Some Useful Properties of the Perron Eigenvalue of a Positive Reciprocal Matrix in the Context of the Analytic Hierarchy Process." *European Journal of Operational Research* 70 (2): 263–268.

Bana e Costa, Carlos A., and J.-C. Vansnick. 2008. "A Critical Analysis of the Eigenvalue Method Used to Derive Priorities in AHP." *European Journal of Operational Research* 187 (3): 1422–1428.

Barzilai, Jonathan. 1998. "Consistency Measures for Pairwise Comparison Matrices." *Journal of Multi-Criteria Decision Analysis* 7 (3): 123–132.

Basile, Luciano, and Livia D'Apuzzo. 2002. "Weak Consistency and Quasi-Linear Means Imply the Actual Ranking." *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* 10 (3): 227–240.

Basile, Luciano, and Livia D'Apuzzo. 2006. "Transitive Matrices, Strict Preference Order and Ordinal Evaluation Operators." *Soft Computing* 10 (10): 933–940.

Benítez, Joaquín, Xitlali Delgado-Galván, Joaquín Izquierdo, and Rafael Pérez-García. 2012. "Improving Consistency in AHP Decision-Making Processes." *Applied Mathematics and Computation* 219 (5): 2432–2441.

Bozóki, Sándor, Linda Dezsö, Attila Poesz, and József Temesi. 2013. "Analysis of Pairwise Comparison Matrices: An Empirical Research." *Annals of Operations Research* 211 (1): 511–528.

Bozóki, Sándor, János Fülöp, and Waldemar W. Koczkodaj. 2011. "An LP-Based Inconsistency Monitoring of Pairwise Comparison Matrices." *Mathematical and Computer Modelling* 54 (1–2): 789–793.

Bozóki, Sándor, János Fülöp, and Attila Poesz. 2011. "On Pairwise Comparison Matrices that Can be Made Consistent by the Modification of a Few Elements." *Central European Journal of Operations Research* 19 (2): 157–175.

Bozóki, Sándor, János Fülöp, and Attila Poesz. 2015. "On Reducing Inconsistency of Pairwise Comparison Matrices Below an Acceptance Threshold." *Central European Journal of Operations Research* 23 (4): 849–866.

Bozóki, Sándor, János Fülöp, and Lajos Rónyai. 2010. "On Optimal Completion of Incomplete Pairwise Comparison Matrices." *Mathematical and Computer Modelling* 52 (1): 318–333.

Bozóki, Sándor, and Tamás Rapcsák. 2008. "On Saaty's and Koczkodaj's Inconsistencies of Pairwise Comparison Matrices." *Journal of Global Optimization* 42 (2): 157–175.

Brunelli, Matteo. 2011. "A Note on the Article 'Inconsistency of Pair-Wise Comparison Matrix with Fuzzy Elements Based on Geometric Mean'. [Fuzzy Sets and Systems 161 (2010) 1604–1613]." *Fuzzy Sets and Systems* 176 (1): 76–78.

Brunelli, Matteo. 2016a. "Recent Advances on Inconsistency Indices for Pairwise Comparisons — A Commentary." *Fundamenta Informaticae* 144 (3–4): 321–332.

Brunelli, Matteo. 2016b. "A Technical Note on Two Inconsistency Indices for Preference Relations: A Case of Functional Relation." *Information Sciences* 357: 1–5.

Brunelli, Matteo. 2017. "Studying a Set of Properties of Inconsistency Indices for Pairwise Comparisons." *Annals of Operations Research* 248 (1): 143–161.

Brunelli, Matteo, Luisa Canal, and Michele Fedrizzi. 2013. "Inconsistency Indices for Pairwise Comparison Matrices: A Numerical Study." *Annals of Operations Research* 211 (1): 493–509.

Brunelli, Matteo, Andrew Critch, and Michele Fedrizzi. 2013. "A Note on the Proportionality between Some Consistency Indices in the AHP." *Applied Mathematics and Computation* 219 (14): 7901–7906.

Brunelli, Matteo, and Michele Fedrizzi. 2015a. "Axiomatic Properties of Inconsistency Indices for Pairwise Comparisons." *Journal of the Operational Research Society* 66 (1): 1–15.

Brunelli, Matteo, and Michele Fedrizzi. 2015b. "Boundary Properties of the Inconsistency of Pairwise Comparisons in Group Decisions." *European Journal of Operational Research* 230 (3): 765–773.

Brunelli, Matteo, and Michele Fedrizzi. 2018. "A General Formulation for Some Inconsistency Indices of Pairwise Comparisons." *Annals of Operations Research*. doi:10.1007/s10479-018-2936-6.

Cavallo, Bice. 2018. "Functional Relations and Spearman Correlation between Consistency Indices." *Journal of the Operational Research Society*. doi:10.1080/01605682.2018.156178.

Cavallo, Bice, and Matteo Brunelli. 2018. "A General Unified Framework for Interval Pairwise Comparison Matrices." *International Journal of Approximate Reasoning* 93: 178–198.

Cavallo, Bice, and Livia D'Apuzzo. 2009. "A General Unified Framework for Pairwise Comparison Matrices in Multicriterial Methods." *International Journal of Intelligent Systems* 24 (4): 377–398.

Cavallo, Bice, and Livia D'Apuzzo. 2010. "Characterizations of Consistent Pairwise Comparison Matrices over Abelian Linearly Ordered Groups." *International Journal of Intelligent Systems* 25 (10): 1035–1059.

Cavallo, Bice, and Livia D'Apuzzo. 2012. "Investigating Properties of the $\odot$-Consistency Index." *IPMU* 4: 315–327.

Cavallo, Bice, Livia D'Apuzzo, and Luciano Basile. 2016. "Weak Consistency for Ensuring Priority Vectors Reliability." *Journal of Multi-Criteria Decision Analysis* 23 (3–4): 126–138.

Cavallo, Bice, Alessio Ishizaka, Maria Grazia Olivieri, and Massimo Squillante. 2018. "Comparing Inconsistency of Pairwise Comparison Matrices Depending on Entries." *Journal of the Operational Research Society*. doi:10.1080/01605682.2018.1464427.

Chiclana, Francisco, Francisco Mata, Luis Martínez, Enrique Herrera-Viedma, and Sergio Alonso. 2008. "Integration of a Consistency Control Module within a Consensus Model." *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* 16 (supp01): 35–53.

Crawford, Gordon. 1987. "The Geometrical Mean Procedure for Estimating the Scale of a Judgment Matrix." *Mathematical Modelling* 9 (3–5): 327–334.

Crawford, Gordon, and Cindy Williams. 1985. "A Note on the Analysis of Subjective Judgment Matrices." *Journal of Mathematical Psychology* 29 (4): 387–405.

Csató, László. 2018a. "Axiomatizations of Inconsistency Indices for Triads." arXiv preprint arXiv:1801.03355.

Csató, László. 2018b. "Characterization of an Inconsistency Ranking for Pairwise Comparison Matrices." *Annals of Operations Research* 261 (1–2): 155–165.

Dixit, Purushottam D. 2018. "Entropy Production Rate as a Criterion for Inconsistency in Decision Theory." *Journal of Statistical Mechanics: Theory and Experiment* 2018 (5): 053408.

Duszak, Zbigniew, and Waldemar W. Koczkodaj. 1994. "Generalization of a New Definition of Consistency for Pairwise Comparisons." *Information Processing Letters* 52 (5): 273–276.

Elsner, Ludwig, Charles R. Johnson, and J. A. Dias da Silva. 1988. "The Perron Root of a Weighted Geometric Mean of Nonnegative Matrices." *Linear and Multilinear Algebra* 24 (1): 1–13.

Ergu, Daji, Gang Kou, Yi Peng, and Yong Shi. 2011. “A Simple Method to Improve the Consistency Ratio of the Pair-Wise Comparison Matrix in ANP.” *European Journal of Operational Research* 213 (1): 246–259.

Escobar, María Teresa, Juan Aguarón, and José María Moreno-Jiménez. 2004. “A Note on AHP Group Consistency for the Row Geometric Mean Prioritization Procedure.” *European Journal of Operational Research* 153 (2): 318–322.

Escobar, María Teresa, Juan Aguarón, and José María Moreno-Jiménez. 2015. “Some Extensions of the Precise Consistency Consensus Matrix.” *Decision Support Systems* 74: 67–77.

Fedrizzi, Michele, and Fabio Ferrari. 2018. “A Chi-Square-Based Inconsistency Index for Pairwise Comparison Matrices.” *Journal of the Operational Research Society* 69 (7): 1125–1134.

Fedrizzi, Michele, and Silvio Giove. 2007. “Incomplete Pairwise Comparisons and Consistency Optimization.” *European Journal of Operational Research* 183 (1): 303–313.

Fedrizzi, M., and R. A. Marques Pereira. 1995. “Positive Fuzzy Matrices, Dominant Eigenvalues, and an Extension of Saaty’s Analytical Hierarchy Process.” In *IFSA 1995*, 245–247. Instituto Nacional de Pesquisa Espacial INPE.

Finan, James S., and William J. Hurley. 1997. “The Analytic Hierarchy Process: Does Adjusting a Pairwise Comparison Matrix to Improve the Consistency Ratio Help?” *Computers & Operations Research* 24 (8): 749–755.

Finan, J. S., and W. J. Hurley. 1996. “A Note on a Method to Ensure Rank-Order Consistency in the Analytic Hierarchy Process.” *International Transactions in Operational Research* 3 (1): 99–103.

Gass, Saul I., and Tamás Rapcsák. 2004. “Singular Value Decomposition in AHP.” *European Journal of Operational Research* 154 (3): 573–584.

Gastes, Dominic, and Wolfgang Gaul. 2012. “The Consistency Adjustment Problem of AHP Pairwise Comparison Matrices.” In *Quantitative Marketing and Marketing Management*, 51–62. Springer.

Gaul, Wolfgang, and Dominic Gastes. 2012. “A Note on Consistency Improvements of AHP Paired Comparison Data.” *Advances in Data Analysis and Classification* 6 (4): 289–302.

Genest, Christian, and Shuang-Shuang Zhang. 1996. “A Graphical Analysis of Ratio-Scaled Paired Comparison Data.” *Management Science* 42 (3): 335–349.

Golden, B. L., and Q. Wang. 1989. “An Alternate Measure of Consistency.” In *The Analythic Hierarchy Process, Applications and Studies*, edited by B. L. Golden, E. A. Wasil, and P. T. Harker, 68–81. Springer-Verlag.

Grošelj, Petra, and Lidija Zadnik Stirn. 2012. “Acceptable Consistency of Aggregated Comparison Matrices in Analytic Hierarchy Process.” *European Journal of Operational Research* 223 (2): 417–420.

Grzybowski, Andrzej Z. 2016. “New Results on Inconsistency Indices and their Relationship with the Quality of Priority Vector Estimation.” *Expert Systems with Applications* 43: 197–212.

Hämäläinen, Raimo P., and Mari Pöyhönen. 1996. “On-Line Group Decision Support by Preference Programming in Traffic Planning.” *Group Decision and Negotiation* 5 (4–6): 485–500.

Harker, Patrick T. 1987a. “Derivatives of the Perron Root of a Positive Reciprocal Matrix: With Application to the Analytic Hierarchy Process.” *Applied Mathematics and Computation* 22 (2): 217–232.

Harker, Patrick Timothy. 1987b. “Alternative Modes of Questioning in the Analytic Hierarchy Process.” *Mathematical Modelling* 9: 353–360.

Herrera-Viedma, Enrique, Sergio Alonso, Francisco Chiclana, and Francisco Herrera. 2007. “A Consensus Model for Group Decision Making With Incomplete Fuzzy Preference Relations.” *IEEE Transactions on Fuzzy Systems* 15 (5): 863–877.

Hilbert, Martin. 2012. “Toward a Synthesis of Cognitive Biases: How Noisy Information Processing Can Bias Human Decision Making.” *Psychological Bulletin* 138 (2): 211.

Ishizaka, Alessio, and Markus Lusti. 2004. “An Expert Module to Improve the Consistency of AHP Matrices.” *International Transactions in Operational Research* 11 (1): 97–105.

Kazibudzki, Paul Thaddeus. 2016. “An Examination of Performance Relations among Selected Consistency Measures for Simulated Pairwise Judgments.” *Annals of Operations Research* 244 (2): 525–544.

Keeney, R. L., and H. Raiffa. 1976. *Decisions with Multiple Objectives: Preferences and Value Tradeoffs*. New York: Wiley.
Khatwani, Gaurav, and Arpan Kumar Kar. 2016. "Improving the Cosine Consistency Index for the Analytic Hierarchy Process for Solving Multi-Criteria Decision Making Problems." *Applied Computing and Informatics* 13 (2): 118–129.
Koczkodaj, Waldemar W. 1993. "A New Definition of Consistency of Pairwise Comparisons." *Mathematical and Computer Modelling* 18 (7): 79–84.
Koczkodaj, Waldemar W., Michael W. Herman, and Marian Orlowski. 1999. "Managing Null Entries in Pairwise Comparisons." *Knowledge and Information Systems* 1 (1): 119–125.
Koczkodaj, Waldemar W., J.-P. Magnot, J. Mazurek, James F. Peters, H. Rakhshani, Michael Soltys, D. Strzałka, Jacek Szybowski, and Arturo Tozzi. 2017. "On Normalization of Inconsistency Indicators in Pairwise Comparisons." *International Journal of Approximate Reasoning* 86: 73–79.
Koczkodaj, Waldemar W., Ludmil Mikhailov, Grzegorz Redlarski, Michael Soltys, Jacek Szybowski, Gaik Tamazian, Eliza Wajch, and Kevin Kam Fung Yuen. 2016. "Important Facts and Observations about Pairwise Comparisons (the special issue edition)." *Fundamenta Informaticae* 144 (3–4): 291–307.
Koczkodaj, Waldemar W., and R. Szwarc. 2014. "On Axiomatization of Inconsistency Indicators for Pairwise Comparisons." *Fundamenta Informaticae* 132 (4): 485–500.
Koczkodaj, Waldemar W., Jacek Szybowski, and Eliza Wajch. 2016. "Inconsistency Indicator Maps on Groups for Pairwise Comparisons." *International Journal of Approximate Reasoning* 69: 81–90.
Koczkodaj, W. W., and R. Urban. 2018. "Axiomatization of Inconsistency Indicators for Pairwise Comparisons." *International Journal of Approximate Reasoning* 94: 18–29.
Kou, Gang, Daji Ergu, Changsheng Lin, and Yang Chen. 2016. "Pairwise Comparison Matrix in Multiple Criteria Decision Making." *Technological and Economic Development of Economy* 22 (5): 738–765.
Kou, Gang, Daji Ergu, and Jennifer Shang. 2014. "Enhancing Data Consistency in Decision Matrix: Adapting Hadamard Model to Mitigate Judgment Contradiction." *European Journal of Operational Research* 236 (1): 261–271.
Kou, Gang, and Changsheng Lin. 2014. "A Cosine Maximization Method for the Priority Vector Derivation in AHP." *European Journal of Operational Research* 235 (1): 225–232.
Krejčí, Jana. 2018. *Pairwise Comparison Matrices and their Fuzzy Extension: Multi-criteria Decision Making with a New Fuzzy Approach*. Vol. 366 of *Studies in Fuzziness and Soft Computing*. Springer.
Kubler, Sylvain, William Derigent, Alexandre Voisin, Jérémy Robert, and Yves Le Traon. 2017. "Knowledge-Based Consistency Index for Fuzzy Pairwise Comparison Matrices." In *2017 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)*, 1–7.
Kułakowski, Konrad. 2015. "Notes on Order Preservation and Consistency in AHP." *European Journal of Operational Research* 245 (1): 333–337.
Kułakowski, Konrad. 2018. "Inconsistency in the Ordinal Pairwise Comparisons Method with and Without Ties." *European Journal of Operational Research* 270 (1): 314–327.
Kułakowski, Konrad, and Jacek Szybowski. 2014. "The New Triad Based Inconsistency Indices for Pairwise Comparisons." *Procedia Computer Science* 35: 1132–1137.
Li, Han-Lin, and Li-Ching Ma. 2007. "Detecting and Adjusting Ordinal and Cardinal Inconsistencies through a Graphical and Optimal Approach in AHP Models." *Computers & Operations Research* 34 (3): 780–798.
Li, Kevin W., Zhou-Jing Wang, and Xiayu Tong. 2016. "Acceptability Analysis and Priority Weight Elicitation for Interval Multiplicative Comparison Matrices." *European Journal of Operational Research* 250 (2): 628–638.
Lin, Changsheng, Gang Kou, and Daji Ergu. 2013. "An Improved Statistical Approach for Consistency Test in AHP." *Annals of Operations Research* 211 (1): 289–299.
Lin, Changsheng, Gang Kou, and Daji Ergu. 2014. "A Statistical Approach to Measure the Consistency Level of the Pairwise Comparison Matrix." *Journal of the Operational Research Society* 65 (9): 1380–1386.

Lin, Robert, Jennifer Shu-Jen Lin, Jason Chang, Didos Tang, Henry Chao, and Peter C. Julian. 2008. "Note on Group Consistency in Analytic Hierarchy Process." *European Journal of Operational Research* 190 (3): 672–678.

Linares, Pedro. 2009. "Are Inconsistent Decisions Better? An Experiment with Pairwise Comparisons." *European Journal of Operational Research* 193 (2): 492–498.

Liu, Haifeng, Zeshui Xu, and Huchang Liao. 2016. "The Multiplicative Consistency Index of Hesitant Fuzzy Preference Relation." *IEEE Transactions on Fuzzy Systems* 24 (1): 82–93.

Liu, Fang, Qin Yu, Witold Pedrycz, and Wei-Guo Zhang. 2018. "A Group Decision Making Model Based on an Inconsistency Index of Interval Multiplicative Reciprocal Matrices." *Knowledge-Based Systems* 145: 67–76.

Liu, Fang, Wei-Guo Zhang, and Zhong-Xing Wang. 2012. "A Goal Programming Model for Incomplete Interval Multiplicative Preference Relations and its Application in Group Decision-Making." *European Journal of Operational Research* 218 (3): 747–754.

Mizuno, Takafumi. 2019. "A Link Diagram for Pairwise Comparisons." In *Intelligent Decision Technologies 2018*, edited by Ireneusz Czarnowski, Robert J. Howlett, Lakhmi C. Jain, and Ljubo Vlacic, Cham, 181–186. Springer International Publishing.

Obata, Tsuneshi, Shunsuke Shiraishi, Motomasa Daigo, and Nobuyuki Nakajima. 1999. "Assessment for an Incomplete Comparison Matrix and Improvement of an Inconsistent Comparison: Computational Experiments." In *ISAHP*, 200–205.

Ohnishi, Shin-Ichi, Didier Dubois, Henri Prade, and Takahiro Yamanoi. 2008. "A Fuzzy Constraint-Based Approach to the Analytic Hierarchy Process." In *Uncertainty and Intelligent Information Systems*, 217–227. World Scientific.

Oliva, Gabriele, Roberto Setola, and Antonio Scala. 2017. "Sparse and Distributed Analytic Hierarchy Process." *Automatica* 85: 211–220.

Osei-Bryson, Kweku-Muata. 2006. "An Action Learning Approach for Assessing the Consistency of Pairwise Comparison Data." *European Journal of Operational Research* 174 (1): 234–244.

Peláez, Jose Ignacio, and Maria Teresa Lamata. 2003. "A New Measure of Consistency for Positive Reciprocal Matrices." *Computers & Mathematics with Applications* 46 (12): 1839–1845.

Peláez, José Ignacio, Eustaquio A. Martínez, and Luis G. Vargas. 2018. "Consistency in Positive Reciprocal Matrices: An Improvement in Measurement Methods." *IEEE access* 6: 25600–25609.

Ramík, Jaroslav, and Petr Korviny. 2010. "Inconsistency of Pair-Wise Comparison Matrix with Fuzzy Elements Based on Geometric Mean." *Fuzzy Sets and Systems* 161 (11): 1604–1613.

Ramík, Jaroslav, and Radomír Perzina. 2010. "A Method for Solving Fuzzy Multicriteria Decision Problems with Dependent Criteria." *Fuzzy Optimization and Decision Making* 9 (2): 123–141.

Regenwetter, Michel, Bernard Grofman, Ilia Tsetlin, and A. A. J. Marley. 2006. *Behavioral Social Choice: Probabilistic Models, Statistical Inference, and Applications*. New York: Cambridge University Press.

Saaty, Thomas L. 1977. "A Scaling Method for Priorities in Hierarchical Structures." *Journal of Mathematical Psychology* 15 (3): 234–281.

Saaty, Thomas L., and Luis G. Vargas. 2012. "The Possibility of Group Choice: Pairwise Comparisons and Merging Functions." *Social Choice and Welfare* 38 (3): 481–496.

Salo, Ahti A., and Raimo Hämäläinen. 1995. "Preference Programming through Approximate Ratio Comparisons." *European Journal of Operational Research* 82 (3): 458–475.

Shiraishi, Shunsuke, and Tsuneshi Obata. 2002. "On a Maximization Problem Arising from a Positive Reciprocal Matrix in the AHP." *Bulletin of Informatics and Cybernetics* 34: 91–96.

Shiraishi, Shunsuke, Tsuneshi Obata, and Motomasa Daigo. 1998. "Properties of a Positive Reciprocal Matrix and their Application to AHP." *Journal of the Operations Research Society of Japan* 41 (3): 404–414.

Siraj, Sajid, Ludmil Mikhailov, and John Keane. 2012. "A Heuristic Method to Rectify Intransitive Judgments in Pairwise Comparison Matrices." *European Journal of Operational Research* 216 (2): 420–428.

Siraj, Sajid, Ludmil Mikhailov, and John A. Keane. 2015. "Contribution of Individual Judgements toward Inconsistency in Pairwise Comparisons." *European Journal of Operational Research* 242 (2): 557–567.

Stein, William E., and Philip J. Mizzi. 2007. “The Harmonic Consistency Index for the Analytic Hierarchy Process.” *European Journal of Operational Research* 177 (1): 488–497.

Szybowski, Jacek. 2016. “The Cycle Inconsistency Index in Pairwise Comparisons Matrices.” *Procedia Computer Science* 96: 879–886.

Takeda, Eiji. 1993. “A Note on Consistent Adjustments of Pairwise Comparison Judgments.” *Mathematical and Computer Modelling* 17 (7): 29–35.

Temesi, József. 2011. “Pairwise Comparison Matrices and the Error-Free Property of the Decision Maker.” *Central European Journal of Operations Research* 19 (2): 239–249.

Temesi, József. 2018. “An Interactive Approach to Determine the Elements of a Pairwise Comparison Matrix.” *Central European Journal of Operations Research*. doi:10.1007/s10100-018-0539-6.

Ureña, Raquel, Francisco Chiclana, Juan Antonio Morente-Molinera, and Enrique Herrera-Viedma. 2015. “Managing Incomplete Preference Relations in Decision Making: A Review and Future Trends.” *Information Sciences* 302: 14–32.

Wan, Zhong, Ming Chen, and Ling Zhang. 2013. “New Consistency Index for Comparison Matrices and its Properties.” *International Journal of Applied Mathematics and Statistics* 42 (12): 206–218.

Wedley, William C. 1993. “Consistency Prediction for Incomplete AHP Matrices.” *Mathematical and Computer Modelling* 17 (4–5): 151–161.

Wu, Zhibin, and Jiuping Xu. 2012. “A Consistency and Consensus Based Decision Support Model for Group Decision Making with Multiplicative Preference Relations.” *Decision Support Systems* 52 (3): 757–767.

Xu, Zeshui. 2000. “On Consistency of Weighted Geometric Mean Complex Judgment Matrix in AHP.” *European Journal of Operational Research* 126: 683–687.

Xu, Zeshui. 2007. “A Survey of Preference Relations.” *International Journal of General Systems* 36 (2): 179–203.

Xu, Zeshui, and Wei Cuiping. 1999. “A Consistency Improving Method in the Analytic Hierarchy Process.” *European Journal of Operational Research* 116 (2): 443–449.

Zhang, Zhiming. 2017. “Logarithmic Least Squares Approaches to Deriving Interval Weights, Rectifying Inconsistency and Estimating Missing Values for Interval Multiplicative Preference Relations.” *Soft Computing* 21 (14): 3993–4004.

Zhang, Zhiming, and Chong Wu. 2014. “A Decision Support Model for Group Decision Making with Hesitant Multiplicative Preference Relations.” *Information Sciences* 282: 136–166.

Zhu, Bin, and Zeshui Xu. 2014. “Consistency Measures for Hesitant Fuzzy Linguistic Preference Relations.” *IEEE Transactions on Fuzzy Systems* 22 (1): 35–45.