STRUCTURE OF TANGENCIES TO DISTRIBUTIONS VIA THE IMPLICIT FUNCTION THEOREM

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Abstract. We investigate the structure and the dimension of the tangency set to a C^1 smooth distribution of n-dimensional vector subspaces of $\mathbb{R}^{n+m}$, by an argument based on the implicit function theorem.

1. Introduction

Let a C^1 smooth distribution $\mathcal{D}$ of n-dimensional vector subspaces of $\mathbb{R}^{n+m}$ be assigned on an open subset U of $\mathbb{R}^{n+m}$. Then one can pose the problem of describing the structure of the set $\mathcal{T}$ of points at which any given C^2 smooth n-submanifold Γ of U is tangent to $\mathcal{D}$. This problem becomes particularly interesting in sub-Riemannian contexts such as Carnot groups or Hörmander vector fields, compare [2] (where the relationship with the Alberti's result [1] is shown) and [3].

The simple idea behind our work is to attack this problem by applying the implicit function theorem. In order to give a more detailed account of this idea, we first assume that $\mathcal{D}$ is given as the intersection of the kernels of m linearly independent differential one-forms $\theta^{(1)}, \dots, \theta^{(m)}$ of class C^1 in U, that is

$$\mathcal{D}(z) := \ker(\theta^{(1)}_z) \cap \cdots \cap \ker(\theta^{(m)}_z), \qquad z \in U.$$

Moreover we suppose that Γ is the graph of a function $f \in C^2(\Omega, \mathbb{R}^m)$, where Ω is an open subset of $\mathbb{R}^n$, that is $\Gamma = F(\Omega)$ with $F : \Omega \to \mathbb{R}^{n+m}$ defined by $F(x) := (x, f(x))$. Then we can easily find a function $\Psi \in C^1(\Omega, \mathbb{R}^{nm})$ such that

$$\Psi^{-1}(0) = F^{-1}(\mathcal{T}) = \{x \in \Omega \,|\, (x, f(x)) \in \mathcal{T}\}$$

compare Proposition 3.1 below. Now a trivial application of the implicit function theorem shows that if $p \in \{1, \dots, n\}$ then the set

$$\{x \in \Psi^{-1}(0) \,|\, \mathrm{rank}(D\Psi(x)) \geq p\}$$

can be covered by a finite union of C^1 submanifolds of dimension less or equal to $n-p$. In particular, the set $\{x \in \Omega \,|\, (x, f(x)) \in \mathcal{T}\}$ (hence $\mathcal{T}$ itself) can be covered by a finite union of C^1 submanifolds of dimension less or equal to $n - r_0$, with $r_0 := \min\{\mathrm{rank}(D\Psi(x)) \,|\, x \in$

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$\Psi^{-1}(0)\}$. In view of this simple fact, looking for results which relate the rank of $D\Psi$ to the properties of the $\theta^{(j)}$ becomes a natural issue. Our main result, Theorem 4.1, provides an explicit formula for $d\theta^{(j)}$ on Γ (see also Proposition 4.1) and is actually a step in this direction. As an application of this machinery we give a new and considerably simplified proof of two well-known theorems concerning the Hausdorff dimension of the tangency set of a submanifold with respect to a (non-involutive) distribution and in particular the main result of [3], where this subject is developed by a different and more geometric approach.

2. General notation

We will often have to deal with maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ and with their graphs. The standard basis of $\mathbb{R}^{n+m}$ and the corresponding coordinates are denoted by $e_1, \dots, e_{n+m}$ and $(x_1, \dots, x_n, y_1, \dots, y_m)$, respectively. We may write $\mathbb{R}^n_x$ in place of $\mathbb{R}^n$ and $\mathbb{R}^m_y$ in place of $\mathbb{R}^m$. Let $\pi : \mathbb{R}^n_x \times \mathbb{R}^m_y \to \mathbb{R}^n_x$ be the orthogonal projection

$$\pi(x_1, \dots, x_n, y_1, \dots, y_m) := (x_1, \dots, x_n).$$

As one expects, the dual basis of $e_1, \dots, e_{n+m}$ is indicated with

$$dx_1, \dots, dx_n, dy_1, \dots, dy_m.$$

Also we need the trivial isomorphism $J : \mathbb{R}^n \times \mathbb{R}^m \to (\mathbb{R}^n \times \mathbb{R}^m)^*$ mapping every e_i to its corresponding member in the dual basis, i.e.

$$J(e_i) = \begin{cases} dx_i & \text{if } i = 1, \dots, n \\ dy_{i-n} & \text{if } i = n+1, \dots, n+m. \end{cases}$$

The Grassmannian of k-planes in $\mathbb{R}^{n+m}$ is denoted by $G(n+m, k)$. If A is a $n \times n$ matrix with real entries, we define

$$W_A := \sum_{i=1}^n Ae_i \wedge e_i = \sum_{i,p=1}^n A_{pi}\, e_p \wedge e_i = \sum_{\substack{i,p=1 \\ p<i}}^n (A_{pi} - A_{ip})\, e_p \wedge e_i$$

and

$$\omega_A := (\Lambda^2 J) W_A = \sum_{i,p=1}^n A_{pi}\, dx_p \wedge dx_i = \sum_{\substack{i,p=1 \\ p<i}}^n (A_{pi} - A_{ip})\, dx_p \wedge dx_i.$$

Observe that the maps $A \mapsto W_A$ and $A \mapsto \omega_A$ are linear and the identities

$$W_A = -W_{A^t}, \qquad \omega_A = -\omega_{A^t} \tag{2.1}$$

hold. Moreover

$$\omega_A(u, v) = u \cdot (Av) - v \cdot (Au)$$

for all $u, v \in \mathbb{R}^n$. In particular one has

$$\omega_A|_{\ker A \times \ker A} = 0. \tag{2.2}$$

Also observe that A is symmetric if and only if $W_A = 0$ (A is symmetric if and only if $\omega_A = 0$).

If m, r are positive integers with $r \le m$ then $I(m,r)$ is the set of integer multi-indices $(\alpha_1, \dots, \alpha_r)$ such that $1 \le \alpha_1 < \dots < \alpha_r \le m$, while $\tilde{I}(m,r)$ denotes the set of integer multi-indices $(\beta_1, \dots, \beta_r)$ such that $1 \le \beta_1 \le \dots \le \beta_r \le m$. Moreover the symmetric group of degree k is denoted by S_k.

If E is a subset of $\mathbb{R}^{n+m}$, then $\dim_H(E)$ denotes the Hausdorff dimension of E. Recall that $\dim_H$ is monotone and stable with respect to countable unions, namely

$$\dim_H(E) \le \dim_H(F), \qquad \dim_H\left(\bigcup_i E_i\right) \le \sup_i \Big(\dim_H(E_i)\Big) \tag{2.3}$$

whenever $E \subset F \subset \mathbb{R}^{n+m}$ and $E_i \subset \mathbb{R}^{n+m}$ for $i = 1, 2, \dots$, compare [7, Section 4.8].

3. Structure and dimension of the tangency set. Role of the implicit function theorem

Consider an open subset U of $\mathbb{R}^n_x \times \mathbb{R}^m_y$ and a family of m linearly independent differential one-forms of the type

$$\theta^{(j)} = \sum_{i=1}^{n} a_i^{(j)} dx_i - dy_j \qquad (j = 1, \dots, m) \tag{3.1}$$

with $a_i^{(j)} \in C^1(U)$. Denote by $\mathcal{D}$ the distribution determined by the family $\theta^{(1)}, \dots, \theta^{(m)}$, namely (for all $z \in U$)

$$\begin{aligned}\mathcal{D}(z) &:= \ker(\theta_z^{(1)}) \cap \dots \cap \ker(\theta_z^{(m)}) \\ &= \Big[\mathrm{span}\{J^{-1}(\theta_z^{(1)})\}\Big]^\perp \cap \dots \cap \Big[\mathrm{span}\{J^{-1}(\theta_z^{(m)})\}\Big]^\perp \\ &= \Big[\mathrm{span}\{J^{-1}(\theta_z^{(1)}), \dots, J^{-1}(\theta_z^{(m)})\}\Big]^\perp\end{aligned}$$

that is

$$\mathcal{D}(z) = \Big[\mathrm{span}\{a^{(j)}(z) - e_{n+j} \,|\, j = 1, \dots, m\}\Big]^\perp \qquad (z \in U) \tag{3.2}$$

where

$$a^{(j)} := (a_1^{(j)}, \cdots, a_n^{(j)})^t.$$

Moreover let Ω be an open subset of $\mathbb{R}^n$ and Γ be the graph of a function $f \in C^2(\Omega, \mathbb{R}^m)$ such that $\Gamma \subset U$. Consider the tangency set

$$\mathcal{T} := \{z \in \Gamma \,|\, T_z\Gamma = \mathcal{D}(z)\}.$$

We want to study the structure of $\mathcal{T}$ and the first step in this direction is to find a function of class C^1 whose zero set is

$$\pi(\mathcal{T}) = \{x \in \Omega \,|\, T_{(x,f(x))}\Gamma = \mathcal{D}(x, f(x))\}.$$

To this aim, for $j = 1, \dots, m$, define

$$\psi_j(x) := a^{(j)}(x, f(x)) - \nabla f_j(x), \quad x \in \Omega.$$

Moreover set

$$\Psi := \left(\psi_1^t, \dots, \psi_m^t\right)^t \in C^1(\Omega, \mathbb{R}^{nm}) \tag{3.3}$$

and let Ψ_q be the q-th component of Ψ, so that $\Psi = (\Psi_1, \dots, \Psi_{nm})$.

If define $F \in C^1(\Omega, \mathbb{R}^{n+m})$ as

$$F(x) := (x, f(x)), \qquad x \in \Omega$$

then, for all $x \in \Omega$, the tangent space of Γ at $(x, f(x))$ is the image of dF_x. Since the matrix of dF_x is

$$DF(x) = \begin{pmatrix} I \\ Df(x) \end{pmatrix} \tag{3.4}$$

we find

$$T_{(x,f(x))}\Gamma = \operatorname{span}\left\{\tau_i(x) \,\middle|\, i = 1, \dots, n\right\} \tag{3.5}$$

with

$$\tau_i(x) := dF_x(e_i) = e_i + \sum_{k=1}^m D_i f_k(x) e_{n+k}.$$

Hence the vectors

$$\nu_h(x) := -e_{n+h} + \nabla f_h(x) \quad (h = 1, \dots, m) \tag{3.6}$$

form a basis of $(T_{(x,f(x))}\Gamma)^\perp$. Observe that

$$\Psi^{-1}(0) = \bigcap_{j=1}^m \left\{x \in \Omega \,\middle|\, \theta^{(j)}_{(x,f(x))} = J\left(\nu_j(x)\right)\right\}. \tag{3.7}$$

Proposition 3.1. *The following identity holds:*

$$\pi(\mathcal{T}) = \{x \in \Omega \,|\, T_{(x,f(x))}\Gamma = \mathcal{D}(x, f(x))\} = \Psi^{-1}(0).$$

Proof. For all $x \in \Omega$ one has

$$\mathcal{D}(x, f(x)) = \left[\operatorname{span}\{a^{(j)}(x, f(x)) - e_{n+j} \,|\, j = 1, \dots, m\}\right]^\perp$$

by (3.2). As a consequence, for $x \in \Omega$, the identity

$$T_{(x,f(x))}\Gamma = \mathcal{D}(x, f(x))$$

occurs if and only if

$$\left(e_i + \sum_{k=1}^m D_i f_k(x) e_{n+k}\right) \cdot \left(a^{(j)}(x, f(x)) - e_{n+j}\right) = 0$$

which is equivalent to

$$a_i^{(j)}(x, f(x)) - D_i f_j(x) = 0$$

for all $i = 1, \dots, n$ and $j = 1, \dots, m$, that is $\Psi(x) = 0$. □

From Proposition 3.1 we got the idea to apply the implicit function theorem to investigate the structure of the tangency set $\mathcal{T}$. In order to make more clear this idea, for $l \in \{1, \ldots, n\}$ and $\gamma \in I(nm, l)$, we put

$$\Psi_\gamma := (\Psi_{\gamma_1}, \ldots, \Psi_{\gamma_l})^t$$

and observe that

$$\Sigma_\gamma := \{x \in \Omega \,|\, \Psi_\gamma(x) = 0,\ \mathrm{rank}(D\Psi_\gamma(x)) = l\}$$

is a $(n-l)$-dimensional regularly imbedded C^1 submanifolds of Ω, by the implicit function theorem (e.g. compare [6, Theorem 4.3.1] or [4, Ch. 1, Theorem 3.2]).

If $1 \le p \le n$ then

$$\begin{aligned}\{x \in \Psi^{-1}(0) \,|\, \mathrm{rank}(D\Psi(x)) \ge p\} &= \\ &= \bigcup_{l=p}^{n} \{x \in \Omega \,|\, \Psi(x) = 0,\ \mathrm{rank}(D\Psi(x)) = l\} \\ &\subset \bigcup_{l=p}^{n} \left(\bigcup_{\gamma \in I(nm,l)} \{x \in \Omega \,|\, \Psi_\gamma(x) = 0,\ \mathrm{rank}(D\Psi_\gamma(x)) = l\} \right)\end{aligned}$$

namely

$$\{x \in \Psi^{-1}(0) \,|\, \mathrm{rank}(D\Psi(x)) \ge p\} \subset \bigcup_{l=p}^{n} \left(\bigcup_{\gamma \in I(nm,l)} \Sigma_\gamma \right). \tag{3.8}$$

As we shall see, this simple inclusion is the basis for the applications below.

Remark 3.1. Let

$$r_0 := \min \left\{ \mathrm{rank}(D\Psi(x)) \,\middle|\, x \in \Psi^{-1}(0) \right\}$$

and assume $r_0 \ge 1$. Then

$$\pi(\mathcal{T}) = \Psi^{-1}(0) \subset \bigcup_{l=r_0}^{n} \left(\bigcup_{\gamma \in I(nm,l)} \Sigma_\gamma \right)$$

by Proposition 3.1 and (3.8), hence

$$\mathcal{T} \subset \bigcup_{l=r_0}^{n} \left(\bigcup_{\gamma \in I(nm,l)} F(\Sigma_\gamma) \right).$$

By recalling (2.3) we also obtain

$$\dim_H(\mathcal{T}) \le n - r_0.$$

4. The main result

Let us assume the notation of Section 3. Moreover define

$$M_j := [Da^{(j)}]^t \qquad (j = 1, \dots, m).$$

For simplicity, given $z \in U$, let us denote $(d\theta^{(j)})_z$ by $d\theta_z^{(j)}$ and observe that

$$d\theta_z^{(j)} = \sum_{i=1}^{n} \left(\sum_{k=1}^{n} D_{x_k} a_i^{(j)}(z)\, dx_k + \sum_{h=1}^{m} D_{y_h} a_i^{(j)}(z)\, dy_h \right) \wedge dx_i = \sum_{i=1}^{n} J(M_j(z)e_i) \wedge J(e_i)$$

that is

$$d\theta_z^{(j)} = (\Lambda^2 J) \left(\sum_{i=1}^{n} M_j(z) e_i \wedge e_i \right) \tag{4.1}$$

for all $j = 1, \dots, m$ and $z = (x, y) \in U$.

Theorem 4.1. *For all $j = 1, \dots, m$ and $x \in \Omega$, one has*

$$\sum_{i=1}^{n} M_j(x, f(x)) e_i \wedge e_i = -W_{D\psi_j(x)} - \sum_{h=1}^{m} \nu_h(x) \wedge (D_{y_h} a^{(j)})(x, f(x)).$$

Proof. If define

$$\begin{aligned} N_j(x) :&= [(D_x a^{(j)})(x, f(x)) + (D_y a^{(j)})(x, f(x)) Df(x)]^t \\ &= [(D_x a^{(j)})(x, f(x))]^t + [Df(x)]^t [(D_y a^{(j)})(x, f(x))]^t \end{aligned} \tag{4.2}$$

then

$$[D\psi_j(x)]^t = N_j(x) - D^2 f_j(x).$$

Moreover

$$W_{D^2 f_j(x)} = 0$$

since $D^2 f_j(x)$ is symmetric. Then, recalling also (2.1), one has

$$W_{N_j(x)} = W_{[D\psi_j(x)]^t} = -W_{D\psi_j(x)}. \tag{4.3}$$

By (4.2) and (4.3) we obtain

$$
\begin{aligned}
&\sum_{i=1}^{n} M_j(x, f(x))e_i \wedge e_i = \\
&\quad = \sum_{i=1}^{n} \left([(D_x a^{(j)})(x, f(x))]^t e_i + \sum_{h=1}^{m} (D_{y_h} a_i^{(j)})(x, f(x)) e_{n+h} \right) \wedge e_i \\
&\quad = \sum_{i=1}^{n} \left(N_j(x) - [Df(x)]^t [(D_y a^{(j)})(x, f(x))]^t \right) e_i \wedge e_i + \\
&\qquad + \sum_{i=1}^{n} \sum_{h=1}^{m} (D_{y_h} a_i^{(j)})(x, f(x)) e_{n+h} \wedge e_i \\
&\quad = -W_{D\psi_j(x)} - \sum_{i=1}^{n} \left([Df(x)]^t [(D_y a^{(j)})(x, f(x))]^t \right) e_i \wedge e_i + \\
&\qquad + \sum_{h=1}^{m} e_{n+h} \wedge (D_{y_h} a^{(j)})(x, f(x)).
\end{aligned} \tag{4.4}
$$

Observe that

$$
\begin{aligned}
\left([Df(x)]^t [(D_y a^{(j)})(x, f(x))]^t \right) e_i &= \sum_{k=1}^{n} \left[\left([Df(x)]^t [(D_y a^{(j)})(x, f(x))]^t \right) e_i \cdot e_k \right] e_k \\
&= \sum_{k=1}^{n} \left(e_i \cdot [(D_y a^{(j)})(x, f(x))][Df(x)] e_k \right) e_k \\
&= \sum_{k=1}^{n} \left(\sum_{h=1}^{m} D_k f_h(x)\, e_i \cdot [(D_y a^{(j)})(x, f(x))] e_h \right) e_k \\
&= \sum_{k=1}^{n} \left(\sum_{h=1}^{m} D_k f_h(x) (D_{y_h} a_i^{(j)})(x, f(x)) \right) e_k
\end{aligned}
$$

hence

$$
\begin{aligned}
\sum_{i=1}^{n} \left([Df(x)]^t [(D_y a^{(j)})(x, f(x))]^t \right) e_i \wedge e_i &= \sum_{i=1}^{n} \sum_{k=1}^{n} \sum_{h=1}^{m} D_k f_h(x) (D_{y_h} a_i^{(j)})(x, f(x))\, e_k \wedge e_i \\
&= \sum_{h=1}^{m} \left(\sum_{k=1}^{n} D_k f_h(x) e_k \right) \wedge \sum_{i=1}^{n} (D_{y_h} a_i^{(j)})(x, f(x)) e_i \\
&= \sum_{h=1}^{m} \nabla f_h(x) \wedge (D_{y_h} a^{(j)})(x, f(x))
\end{aligned}
$$

which, combined with (4.4), yields

$$
\sum_{i=1}^{n} M_j(x, f(x)) e_i \wedge e_i = -W_{D\psi_j(x)} + \sum_{h=1}^{m} (e_{n+h} - \nabla f_h(x)) \wedge (D_{y_h} a^{(j)})(x, f(x)).
$$

The conclusion follows by recalling the definition (3.6) of $\nu_h(x)$. □

The following simple corollary of Theorem 4.1, which will be useful in the next section, shows the strict relationship occuring between $d\theta^{(j)}_{F(x)}$ and the 2-form associated to $D\psi_j(x)$, provided $F(x) \in \mathcal{T}$. In the statement below, $F^\#$ denotes the pull-back operator induced by F. Recall that $F^\#$ and the exterior differentiation commute, compare [5, Theorem 6.2.9].

Proposition 4.1. *Let $x \in \Psi^{-1}(0)$ and $j \in \{1, \dots, m\}$. Then*

$$d\theta^{(j)}_{(x,f(x))} = -\omega_{D\psi_j(x)} - \sum_{h=1}^{m} \theta^{(h)}_{(x,f(x))} \wedge J\Big((D_{y_h} a^{(j)})(x, f(x))\Big).$$

Moreover

$$d(F^\# \theta^{(j)})(x) = F^\#(d\theta^{(j)})(x) = -\omega_{D\psi_j(x)}$$

i.e.

$$d\theta^{(j)}_{(x,f(x))}(dF_x(u), dF_x(v)) = -\omega_{D\psi_j(x)}(u, v)$$

for all $u, v \in \mathbb{R}^n$.

Proof. By combining Theorem 4.1, (4.1) and (3.7) we get at once the first identity. The second identity follows from the first one by recalling that $\mathrm{Im}(dF_x) = T_{(x,f(x)}\Gamma$, hence $\theta^{(h)}_{(x,f(x))}|_{\mathrm{Im}(dF_x)} = 0$ (for all $h = 1, \dots, m$). □

5. Structure and dimension of the tangency set. Applications of the main result.

Assume the notation of the previous sections.

5.1. First application of the main result.

5.1. First application of the main result. in order to state and prove the next results, we need to consider the following sets

$$A_k := \Big\{ z \in U \,\Big|\, \text{there exists } X \in G(n+m, k) \text{ s.t. } \theta^{(j)}_z|_X = 0 \text{ and } d\theta^{(j)}_z|_{X\times X} = 0 \text{ for all } j = 1, \dots, m \Big\}$$

for $k = 1, \dots, n+m$, compare [3].

Theorem 5.1. *Let $k \in \{1, \dots, n\}$. Then one has*

$$\pi(\mathcal{T} \setminus A_{k+1}) \subset \{x \in \Psi^{-1}(0) \,|\, \mathrm{rank}(D\Psi(x)) \geq n - k\} \subset \bigcup_{l=n-k}^{n} \left(\bigcup_{\gamma \in I(nm,l)} \Sigma_\gamma \right). \tag{5.1}$$

Hence

$$\mathcal{T} \setminus A_{k+1} \subset \bigcup_{l=n-k}^{n} \left(\bigcup_{\gamma \in I(nm,l)} F(\Sigma_\gamma) \right) \tag{5.2}$$

and

$$\dim_H\Big(\mathcal{T} \setminus A_{k+1}\Big) \leq k. \tag{5.3}$$

Proof. If
$$x \in \pi(\mathcal{T} \setminus A_{k+1})$$
then
$$x \in \Omega, \qquad T_{(x,f(x))}\Gamma = \mathcal{D}(x, f(x))$$
and
$$(x, f(x)) \notin A_{k+1}. \tag{5.4}$$
From Proposition 3.1 we get
$$x \in \Psi^{-1}(0)$$
and we want to prove that $\operatorname{rank}(D\Psi(x)) \geq n - k$, i.e.
$$\dim(\ker D\Psi(x)) \leq k. \tag{5.5}$$
To this aim, we proceed by contradiction assuming that it does not hold. Then there exists a family of linearly independent vectors
$$v_1, \dots, v_{k+1} \in \ker D\Psi(x) \subset \mathbb{R}^n$$
and one has
$$X := \operatorname{span}\Big\{dF_x(v_i)\,\Big|\, i = 1, \dots, k+1\Big\} \in G(n+m, k+1).$$
by (3.4). Observe that
$$X \subset \operatorname{Im}(dF_x) = T_{(x,f(x))}\Gamma = \mathcal{D}(x, f(x))$$
thus
$$\theta^{(j)}_{(x,f(x))}|_X = 0, \text{ for all } j = 1, \dots, m.$$
On the other hand, one obviously has
$$\ker D\Psi(x) \subset \bigcap_{j=1}^{m} \ker D\psi_j(x)$$
hence
$$v_1, \dots, v_{k+1} \in \ker D\psi_j(x), \text{ for all } j = 1, \dots, m.$$
Then, by (2.2) and the second identity of Proposition 4.1, we obtain
$$d\theta^{(j)}_{(x,f(x))}|_{X\times X} = 0, \text{ for all } j = 1, \dots, m.$$
So $(x, f(x)) \in A_{k+1}$, which is in contradiction with (5.4). This concludes the proof of (5.5) and of the first inclusion in (5.1). The second inclusion in (5.1) follows from (3.8). Now (5.2) follows at once from (5.1). Finally, (5.2) and (2.3) yield (5.3). □

Corollary 5.1. *One has*
$$\mathcal{T} \subset \bigcup_{k=1}^{n} \left[(A_k \setminus A_{k+1}) \cap \bigcup_{l=n-k}^{n} \left(\bigcup_{\gamma\in I(nm,l)} F(\Sigma_\gamma) \right) \right]$$
and
$$\dim_H(\mathcal{T}) \leq \max_{1\leq k\leq n} \Big\{ \min\{\dim_H(A_k \setminus A_{k+1}), k\} \Big\}. \tag{5.6}$$

Proof. Observe that

$$A_n \subset A_{n-1} \subset \ldots \subset A_2 \subset A_1 = U, \qquad A_{n+1} = \ldots = A_{n+m} = \emptyset$$

compare [3]. Thus one has the disjoint decomposition

$$\mathcal{T} = \bigcup_{k=1}^{n} \mathcal{T} \cap (A_k \setminus A_{k+1}) = \bigcup_{k=1}^{n} (\mathcal{T} \setminus A_{k+1}) \cap (A_k \setminus A_{k+1}). \tag{5.7}$$

The conclusion follows from Theorem 5.1, (5.7) and (2.3). □

5.2. **Second application of the main result.** First we need the following simple technical lemma.

Proposition 5.1. *Let $A^{(1)}, \ldots, A^{(k)}$ be $n \times n$ matrices with real entries, with $k \leq n$. Moreover, for all*

$$h \in \{1, \ldots, k\}, \qquad i, j, j_1, \ldots, j_k \in \{1, \ldots, n\}, \qquad \alpha = (\alpha_1, \ldots, \alpha_k) \in I(n, k),$$

let us define

$$A^{(h)}_{ji} := (A^{(h)} e_i) \cdot e_j, \qquad D(j_1, \ldots, j_k; \alpha) := \det \begin{pmatrix} A^{(1)}_{j_1\alpha_1} & \ldots & A^{(1)}_{j_1\alpha_k} \\ \vdots & \ddots & \vdots \\ A^{(k)}_{j_k\alpha_1} & \ldots & A^{(k)}_{j_k\alpha_k}. \end{pmatrix}$$

Then this identity holds:

$$\begin{aligned} &W_{A^{(1)}} \wedge \cdots \wedge W_{A^{(k)}} \\ &\quad = (-1)^{\frac{k(k-1)}{2}} \sum_{j_1, \ldots, j_k = 1}^{n} \sum_{\alpha \in I(n,k)} D(j_1, \ldots, j_k; \alpha)\, e_{j_1} \wedge \cdots \wedge e_{j_k} \wedge e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_k}. \end{aligned}$$

Proof. One has

$$\begin{aligned} W_{A^{(1)}} \wedge \cdots \wedge W_{A^{(k)}} &= (-1)^{\frac{k(k-1)}{2}} \sum_{i_1, \ldots, i_k = 1}^{n} A^{(1)} e_{i_1} \wedge \cdots \wedge A^{(k)} e_{i_k} \wedge e_{i_1} \wedge \cdots \wedge e_{i_k} \\ &= (-1)^{\frac{k(k-1)}{2}} \sum_{j_1, \ldots, j_k = 1}^{n} e_{j_1} \wedge \cdots \wedge e_{j_k} \wedge \left(\sum_{i_1, \ldots, i_k = 1}^{n} A^{(1)}_{j_1 i_1} \cdots A^{(k)}_{j_k i_k} e_{i_1} \wedge \cdots \wedge e_{i_k} \right) \end{aligned}$$

where

$$\begin{aligned} &\sum_{i_1, \ldots, i_k = 1}^{n} A^{(1)}_{j_1 i_1} \cdots A^{(k)}_{j_k i_k} e_{i_1} \wedge \cdots \wedge e_{i_k} \\ &\qquad = \sum_{\alpha \in I(n,k)} \left(\sum_{\sigma \in S_k} \mathrm{sign}(\sigma) A^{(1)}_{j_1 \alpha_{\sigma(1)}} \cdots A^{(k)}_{j_k \alpha_{\sigma(k)}} \right) e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_k} \\ &\qquad = \sum_{\alpha \in I(n,k)} D(j_1, \ldots, j_k; \alpha)\, e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_k}. \end{aligned}$$

□

Theorem 5.2. *For $\alpha \in I(m,r)$ and $\beta \in \tilde{I}(m,s)$, with $r+s \geq m+1$, consider the subset of Ω defined as*

$$R(\alpha,\beta) := \left\{ x \in \Psi^{-1}(0) \,|\, \theta^{(\alpha_1)}_{(x,f(x))} \wedge \cdots \wedge \theta^{(\alpha_r)}_{(x,f(x))} \wedge d\theta^{(\beta_1)}_{(x,f(x))} \wedge \cdots \wedge d\theta^{(\beta_s)}_{(x,f(x))} \neq 0 \right\}.$$

Then

$$\begin{aligned} R(\alpha,\beta) &\subset \{x \in \Psi^{-1}(0) \,|\, \mathrm{rank}(D\Psi(x)) \geq r+s-m\} \\ &\subset \bigcup_{l=r+s-m}^{n} \left(\bigcup_{\gamma \in I(nm,l)} \Sigma_\gamma \right). \end{aligned} \tag{5.8}$$

Hence

$$\dim_H \Big(R(\alpha,\beta) \Big) \leq n+m-(r+s). \tag{5.9}$$

Proof. Let $x \in \Omega$. From Theorem 4.1 and (4.1) we obtain

$$\theta^{(\alpha_1)}_{(x,f(x))} \wedge \cdots \wedge \theta^{(\alpha_r)}_{(x,f(x))} \wedge d\theta^{(\beta_1)}_{(x,f(x))} \wedge \cdots \wedge d\theta^{(\beta_s)}_{(x,f(x))} = (\Lambda^{r+2s} J)\eta(x)$$

with

$$\begin{aligned} \eta(x) := \nu_{\alpha_1}(x) \wedge \cdots \wedge \nu_{\alpha_r}(x) \wedge \\ \wedge \left(-W_{D\psi_{\beta_1}(x)} - \sum_{h_1=1}^{m} \nu_{h_1}(x) \wedge (D_{y_{h_1}} a^{(\beta_1)})(x,f(x)) \right) \wedge \\ \wedge \cdots \wedge \left(-W_{D\psi_{\beta_s}(x)} - \sum_{h_s=1}^{m} \nu_{h_s}(x) \wedge (D_{y_{h_s}} a^{(\beta_s)})(x,f(x)) \right). \end{aligned}$$

We can develop this wedge product into a sum of monomials

$$\eta(x) = \sum_{k=0}^{s} \eta_k(x)$$

where k indicates the number of factors of the type

$$W_{D\psi_{\beta_j}(x)}$$

figuring in the corresponding monomial $\eta_k(x)$.

Observe that $\eta_k(x)$ includes the wedge product of $r+s-k$ vectors of the family

$$\nu_1(x), \ldots, \nu_m(x)$$

which is a basis of $(T_{(x,f(x))}\Gamma)^\perp$. It follows that $\eta_k(x) = 0$ whenever $r+s-k \geq m+1$, i.e. $k \leq r+s-m-1$. Thus

$$\eta(x) = \eta_{r+s-m}(x) + \eta_{r+s-m+1}(x) + \ldots + \eta_s(x).$$

From this identity and Proposition 5.1, we infer that if $\eta(x) \neq 0$ then the rank of $D\Psi(x)$ has to be at least $r+s-m$, namely

$$R(\alpha,\beta) \subset \{x \in \Psi^{-1}(0) \,|\, \mathrm{rank}(D\Psi(x)) \geq r+s-m\}.$$

We complete the proof of (5.8) by recalling (3.8) with $p = r + s - m$. The inequality (5.9) follows from (5.8) and (2.3). $\square$

Corollary 5.2. *Let $\alpha \in I(m, r)$ and $\beta \in \tilde{I}(m, s)$, with $r + s \geq m + 1$, be such that*

$$\theta_z^{(\alpha_1)} \wedge \cdots \wedge \theta_z^{(\alpha_r)} \wedge d\theta_z^{(\beta_1)} \wedge \cdots \wedge d\theta_z^{(\beta_s)} \neq 0, \text{ for all } z \in \Gamma.$$

Then

$$\mathcal{T} \subset \bigcup_{l=r+s-m}^{n} \left(\bigcup_{\gamma \in I(nm,l)} F(\Sigma_\gamma) \right) \tag{5.10}$$

hence

$$\dim_H(\mathcal{T}) \leq n + m - r - s. \tag{5.11}$$

Proof. First of all, by definition, one has $R(\alpha, \beta) = \Psi^{-1}(0)$. Then, from Proposition 3.1 and Theorem 5.2 it follows that

$$\pi(\mathcal{T}) = \Psi^{-1}(0) = R(\alpha, \beta) \subset \bigcup_{l=r+s-m}^{n} \left(\bigcup_{\gamma \in I(nm,l)} \Sigma_\gamma \right)$$

hence (5.10). Finally (5.10) and (2.3) imply (5.11). $\square$

6. Extension to submanifolds

We can easily extend the inequalities in Corollary 5.1 and Corollary 5.2 to the case when Γ is a C^2 smooth n-submanifold of an open subset U of $\mathbb{R}^{n+m}$ and $\theta^{(1)}, \dots, \theta^{(m)}$ is a general family of linearly independent differential one-forms of class C^1 in U. This can be done by first recalling that Γ is locally the graph of a C^1 function and then by applying the two corollaries above. More precisely, let $\mathcal{D}$ and $\mathcal{T}$ be defined as in Section 3 and consider an arbitrary point

$$z_0 \in \mathcal{T} = \{z \in \Gamma \,|\, T_z\Gamma = \mathcal{D}(z)\}.$$

Using the argument in the proof of [8, Proposition 2.11.7], we may choose the coordinate system so that the differential forms $\theta^{(j)}$ are of the special type (3.1) in a neighbourhood U_{z_0} of z_0. It follows that

$$T_{z_0}\Gamma = \mathcal{D}(z_0) = \left[\operatorname{span}\{a^{(j)}(z_0) - e_{n+j} \,|\, j = 1, \dots, m\}\right]^\perp$$

by (3.2), hence the family of vectors

$$e_i - \sum_{k=1}^{m} a_i^{(k)}(z_0) e_{n+k}, \qquad (i = 1, \dots, n)$$

has to be a basis of $T_{z_0}\Gamma$. In consequence of this fact, we can assume that there exist an open subset Ω of $\mathbb{R}^n_x$ and $f \in C^2(\Omega, \mathbb{R}^m_y)$ such that

$$\Gamma_{z_0} := \{(x, f(x)) \,|\, x \in \Omega\}$$

is a neighbourhood of z_0 with respect to the induced topology of Γ, with $\Gamma_{z_0} \subset U_{z_0}$. Observe that

$$\{z \in \Gamma_{z_0} \,|\, T_z\Gamma_{z_0} = \mathcal{D}(z)\} = \mathcal{T} \cap \Gamma_{z_0}. \tag{6.1}$$

Now we are in position to extend the corollaries very easily:

- One has

$$\dim_H(\mathcal{T}) \leq \max_{1\leq k\leq n} \Big\{ \min\{\dim_H(A_k \setminus A_{k+1}), k\}\Big\}. \tag{6.2}$$

Proof. From (5.6) and (6.1) we obtain

$$\dim_H(\mathcal{T} \cap \Gamma_{z_0}) \leq \max_{1\leq k\leq n} \Big\{ \min\{\dim_H([A_k \cap U_{z_0}] \setminus [A_{k+1} \cap U_{z_0}]), k\}\Big\}.$$

But

$$\begin{aligned}\dim_H([A_k \cap U_{z_0}] \setminus [A_{k+1} \cap U_{z_0}]) &= \dim_H([A_k \setminus A_{k+1}] \cap U_{z_0}) \\ &\leq \dim_H(A_k \setminus A_{k+1})\end{aligned}$$

by (2.3), hence

$$\dim_H(\mathcal{T} \cap \Gamma_{z_0}) \leq \max_{1\leq k\leq n} \Big\{ \min\{\dim_H(A_k \setminus A_{k+1}), k\}\Big\}.$$

By the arbitrariness of z_0, we get (6.2).

□

- Let $\alpha \in I(m,r)$ and $\beta \in \tilde{I}(m,s)$, with $r+s \geq m+1$, be such that

$$\theta_z^{(\alpha_1)} \wedge \cdots \wedge \theta_z^{(\alpha_r)} \wedge d\theta_z^{(\beta_1)} \wedge \cdots \wedge d\theta_z^{(\beta_s)} \neq 0$$

for all $z \in \Gamma$. Then

$$\dim_H(\mathcal{T}) \leq n + m - (r+s). \tag{6.3}$$

Proof. From (5.11) and (6.1) we obtain

$$\dim_H(\mathcal{T} \cap \Gamma_{z_0}) \leq n + m - r - s.$$

Hence (6.3) follows by the arbitrariness of z_0. □

Remark 6.1. The inequalities (6.2) and (6.3) have been proved in [3] by a different and very geometric approach. They correspond to [3, Theorem 1.3] (i.e. the main result) and [3, Corollary 6.8], respectively.

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