mathematics

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Set Evincing the Ranks with Respect to an Embedded Variety (Symmetric Tensor Rank and Tensor Rank

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Received: 11 July 2018; Accepted: 8 August 2018; Published: 14 August 2018

Abstract: Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety. We study when a finite set $S \subset X$ evinces the X-rank of the general point of the linear span of S. We give a criterion when X is the order d Veronese embedding $X_{n,d}$ of $\mathbb{P}^n$ and $|S| \le \binom{n+\lfloor d/2\rfloor}{n}$. For the tensor rank, we describe the cases with $|S| \le 3$. For $X_{n,d}$, we raise some questions of the maximum rank for $d \gg 0$ (for a fixed n) and for $n \gg 0$ (for a fixed d).

Keywords: X-rank; symmetric tensor rank; tensor rank; veronese variety; segre variety

1. Introduction

Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety. For any $q \in \mathbb{P}^r$, the X-rank $r_X(q)$ of q is the minimal cardinality of a finite set $S \subset X$ such that $q \in \langle S \rangle$, where $\langle\ \rangle$ denotes the linear span. The definition of X-ranks captures the notion of tensor rank (take as X the Segre embedding of a multiprojective space) of rank decomposition of a homogeneous polynomial (take as X a Veronese embedding of a projective space) of partially symmetric tensor rank (take a complete linear system of a multiprojective space) and small variations of it may be adapted to cover other applications. See [1] for many applications and [2] for many algebraic insights. For the pioneering works on the applied side, see, for instance, [3–7]. The paper [7] proved that X-rank is not continuous and showed why this has practical importance. The dimensions of the secant varieties (i.e., the closure of the set of all $q \in \mathbb{P}^r$ with a prescribed rank) has a huge theoretical and practical importance. The Alexander–Hirschowitz theorem computes in all cases the dimensions of the secant varieties of the Veronese embeddings of a projective space ([8–14]). For the dimensions of secant varieties, see [15–17] for tensors and [18–27] for partially symmetric tensors (i.e., Segre–Veronese embeddings of multiprojective spaces). For the important problem of the uniqueness of the set evincing a rank (in particular for the important case of tensors) after the classical [28], see [29–38]. See [39–47] for other theoretical works.

Let $S \subset X$ be a finite set and $q \in \mathbb{P}^r$. We say that S *evinces the X-rank of q* if $q \in \langle S \rangle$ and $|S| = r_X(q)$. We say that S *evinces an X-rank* if there is $q \in \mathbb{P}^r$ such that S evinces the X-rank of q. Obviously, S may evince an X-rank only if it is linearly independent, but this condition is not a sufficient one, except in very trivial cases, like when $r_X(q) \le 2$ for all $q \in \mathbb{P}^r$. Call $r_{X,\max}$ the maximum of all integers $r_X(q)$. An obvious necessary condition is that $|S| \le r_{X,\max}$ and this is in very special cases a sufficient condition (see Propositions 1 for the rational normal curve). If S evinces the X-rank of $q \in \mathbb{P}^r$, then $q \in \langle S \rangle$ and $q \notin \langle S' \rangle$ for any $S' \subsetneq S$. For any finite set $S \subset \mathbb{P}^r$, set $\langle S \rangle' := \langle S \rangle \setminus (\cup_{S' \subsetneq S} \langle S' \rangle)$. Note that $\langle S \rangle' = \emptyset$ if and only if either $S = \emptyset$ or S is linearly dependent (when $|S| = 1$, $\langle S \rangle' = S$ and S evinces itself). In some cases, it is possible to show that some finite $S \subset X$ evinces the X-rank of all points of $\langle S \rangle'$. We say that S *evinces generically the*

X-ranks if there is a non-empty Zariski open subset U of $\langle S \rangle$ such that S evinces the X-ranks of all $q \in U$. We say that S *totally evinces the X-ranks* if S evinces the X-ranks of all $q \in \langle S \rangle'$. We first need an elementary and well-known bound to compare it with our results.

Let $\rho(X)$ be the maximal integer such that each subset of X with cardinality $\rho(X)$ is linearly independent. See ([43] Lemma 2.6, Theorem 1.18) and ([42] Proposition 2.5) for some uses of the integer $\rho(X)$. Obviously, $\rho(X) \le r+1$ and it is easy to check and well known that equality holds if and only if X is a Veronese embedding of $\mathbb{P}^1$ (Remark 1). If $|S| \le \lfloor (\rho(X)+1)/2 \rfloor$, then S totally evinces the X-ranks (as in [43] Theorem 1.18) while, for each integer $t > \lfloor (\rho(X)+1)/2 \rfloor$ with $t \le r+1$, there is a linearly independent subset of X with cardinality t and not totally evincing the X-ranks (Lemma 3). Thus, to say something more, we need to make some assumptions on S and these assumptions must be related to the geometry of X or the reasons for the interest of the X-ranks. We do this in Section 3 for the Veronese embeddings and in Section 4 for the tensor rank. For tensors, we only have results for $|S| \le 3$ (Propositions 3 and 4).

For all positive integers n, d let $\nu_{d,n} : \mathbb{P}^n \to \mathbb{P}^r$, $r = \binom{n+d}{n} - 1$, denote the Veronese embedding of $\mathbb{P}^n$, i.e., the embedding of $\mathbb{P}^n$ induced by the complete linear system $|\mathcal{O}_{\mathbb{P}^n}(d)|$. Set $X_{n,d} := \nu_{d,n}(\mathbb{P}^n)$. At least over an algebraically closed base field of characteristic 0 (i.e., in the set-up of this paper), for any $q \in \mathbb{P}^r$, the integer $r_{X_{n,d}}(q)$ is the minimal number of d-powers of linear forms in $n+1$ variables whose sum is the homogeneous polynomial associated to q.

We prove the following result, whose proof is elementary (see Section 3 for the proof). In its statement, the assumption "$h^1(\mathcal{I}_A(\lfloor d/2 \rfloor)) = 0$" just means that the vector space of all degree $\lfloor d/2 \rfloor$ homogeneous polynomials in $n+1$ variables vanishing on A has dimension $\binom{n+\lfloor d/2 \rfloor}{n} - |A|$, i.e., A imposes $|A|$ independent conditions to the homogeneous polynomials of degree $\lfloor d/2 \rfloor$ in $n+1$ variables.

Theorem 1. *Fix integers $n \ge 2$, $d > k > 2$ and a finite set $A \subset \mathbb{P}^n$ such that $h^1(\mathcal{I}_A(\lfloor d/2 \rfloor)) = 0$. Set $S := \nu_{d,n}(A)$. Then, S totally evinces the ranks for $X_{n,d}$.*

A general $A \subset \mathbb{P}^n$ satisfies the assumption of Theorem 1 if and only if $|A| \le \binom{n+\lfloor d/2 \rfloor}{n}$. For much smaller $|A|$, one can check the condition $h^1(\mathcal{I}_A(\lfloor d/2 \rfloor)) = 0$ if A satisfies some geometric conditions (e.g., if A is in linearly general position, it is sufficient to assume $|A| \le n\lfloor d/2 \rfloor + 1$).

We conclude the paper with some questions related to the maximum of the X-ranks when X is a Veronese embedding of $\mathbb{P}^n$.

2. Preliminary Lemmas

Remark 1. *Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety. Since any $r+2$ points of $\mathbb{P}^r$ are linearly dependent, we have $\rho(X) \le r+1$. If X is a rational normal curve, then $\rho(X) = r+1$ because any $r+1$ points of X spans $\mathbb{P}^r$. Now, we check that, if $\rho(X) = r+1$, then X is a rational normal curve. This is well known, but usually stated in the set-up of Veronese embeddings or the X-ranks of curves. Set $n := \dim X$ and $d := \deg(X)$. Assume $\rho(X) = r+1$. Let $H \subset \mathbb{P}^r$ be a general hyperplane. If $n > 1$, then $X \cap H$ has dimension $n-1 > 0$ and in particular it has infinitely many points. Any $r+1$ points of $X \cap H$ are linearly dependent. Now, assume $n = 1$. Since X is non-degenerate, we have $d \ge n$. By Bertini's theorem, $X \cap H$ contains d points of X. Since $\rho(X) = r+1$, $\dim H = r-1$ and $H \cap X \subset H$, we have $d \le r$. Hence, $d = r$, i.e., X is a rational normal curve.*

The following example shows, that in many cases, there are are sets evincing X-ranks, but not totally evincing X-ranks or even generically evincing X-ranks.

Example 1. *Let $X \subset \mathbb{P}^r$, $r \ge 3$, be a rational normal curve. Take $q \in \mathbb{P}^r$ with $r_X(q) = r$, i.e., take $q \in \tau(X) \setminus X$, where $\tau(X)$ is the tangential variety of X ([48]). Take $S \subset X$ evincing the X-rank of q. Thus, $|S| = r$ and S spans a hyperplane $\langle S \rangle$. Since $\dim \tau(X) = 2$ and $\tau(X)$ spans $\mathbb{P}^r$, $\langle S \rangle \cap \tau(X)$ is a* *proper closed algebraic subset of $\langle S\rangle$. Thus, for a general $p \in \langle S\rangle$, we have $r_X(p) < |S|$ and hence S does not generically evinces X-ranks.*

Lemma 1. *If $S \subset X$ is a finite set evincing the rank of some $q \in \mathbb{P}^r$, then each $S' \subset S$, $S' \neq \emptyset$, evinces the X-rank of some $q' \in \mathbb{P}^r$.*

Proof. We may assume $S' \neq S$. Write $S'' := S \setminus S'$. Since S evinces the rank of q, S is linearly independent, but $S \cup \{q\}$ is not linearly independent. Since $S' \neq \emptyset$ and $S'' \neq \emptyset$, there are unique $q' \in \langle S'\rangle$ and $q'' \in \langle S''\rangle$ such that $q \in \langle \{q', q''\}\rangle$. Since S evinces the rank of q, S' evinces the rank of q'. □

Lemma 2. *Every non-empty subset of a set evincing generically (resp. totally) X-ranks evinces generically (resp. totally) the X-ranks.*

Proof. Assume that S evinces generically the X-ranks and call U a non-empty open subset of $\langle S\rangle'$ such that $r_X(q) = |S|$ for all $q \in U$; if S evinces totally the X-ranks, take $U := \langle S\rangle'$. Fix $S' \subsetneq S$, $S' \neq 0$ and set $S'' := S \setminus S'$. Let E be the set of all $q \in \langle S\rangle'$ such that $\langle \{q\} \cup S''\rangle \cap U \neq \emptyset$. If $q \in E$, then $r_X(q) = |S'|$ because $r_X(q') = |S|$ for each $q' \in \langle \{q\} \cup S''\rangle \cap U$. Since $S' \cap S'' = \emptyset$ and $S' \cup S'' = S$ is linearly independent, E is a non-empty open subset of $\langle S\rangle'$ (a general element of $\langle S\rangle$ is contained in the linear span of a general element of $\langle S'\rangle$ and a general element of $\langle S'\rangle$). Now, assume $U = \langle S\rangle'$. Every element of $\langle S\rangle'$ is in the linear span of an element of $\langle S'\rangle'$ and an element of $\langle S''\rangle'$. □

Lemma 3. *Take a finite set $S \subset X$, $S \neq \emptyset$.*

(a) *If $|S| \le \lfloor(\rho(X)+1)/2\rfloor$, then S totally evinces the X-ranks.*
(b) *For each integer $t > \lfloor(\rho(X)+1)/2\rfloor$, there is $A \subset X$ such that $|A| = t$ and A does not totally evince the X-ranks.*

Proof. Take $q \in \langle S\rangle'$ and assume $r_X(q) < |S|$. Take $B \subset X$ evincing the X-rank of q. Since $|B| < |S|$, we have $B \neq S$. Since $q \in \langle S\rangle \cap \langle B\rangle$, but no proper subset of either B or S spans q, $S \cup B$ is linearly dependent. Since $|B| \le |S| - 1$, we have $|B \cup S| \le \rho(X)$, contradicting the definition of $\rho(X)$.

Now, we prove part (b). By Lemma 1, it is sufficient to do the case $t = \lfloor(\rho(X)+1)/2\rfloor + 1$. By the definition of the integer $\rho(X)$, there is a subset $D \subset X$ with $|D| = \rho(X) + 1$ and D linearly dependent. Write $D = A \sqcup E$ with $|A| = \lfloor(\rho(X)+1)/2\rfloor + 1$ and $|E| = \rho(X) + 1 - |A|$. Note that $|A| > |E|$. Since $|A| \le \rho(X)$ (remember that $\rho(X) \ge 2$), both A and E are linearly independent. Since $A \cup E$ is linearly dependent, there is $q \in \langle A\rangle \cap \langle E\rangle$. Since $|D| = \rho(X) + 1$, every proper subset of D is linearly independent. Hence, $\langle A'\rangle \cap \langle E\rangle = \emptyset$ for all $A' \subsetneq A$. Thus, $q \in \langle A\rangle'$. Since $|E| < |A|$, A does not evince the X-rank of q. □

Remark 2. *Take $X \subset \mathbb{P}^r$ such that $r_X(q) \le 2$ for all $q \in \mathbb{P}^r$ (e.g., by [49], we may take most space curves). Any set $S \subset X$ with $|S| = 2$ evinces its X-ranks if and only if X contains no line.*

3. The Veronese Embeddings of Projective Spaces

Let $\nu_{d,n} : \mathbb{P}^n \to \mathbb{P}^r$, $r := -1 + \binom{n+d}{n}$, denote the Veronese embedding of $\mathbb{P}^n$. Set $X_{n,d} := \nu_{d,n}(\mathbb{P}^n)$.

Proposition 1. *Let $X \subset \mathbb{P}^d$, $d \ge 2$, be the rational normal curve.*

(a) *A non-empty finite set $S \subset X$ evinces some rank of $\mathbb{P}^d$ if and only if $|S| \le d$.*
(b) *A non-empty finite set $A \subset X$ totally evinces the X-ranks if and only if $|A| \le \lfloor(d+2)/2\rfloor$.*

Proof. By a theorem of Sylvester's ([48]), every $q \in \mathbb{P}^d$ has X-rank at most d. Thus, the condition $|S| \le d$ is a necessary condition for evincing some rank. By Lemma 1 to prove part (a), it is sufficient to prove it when $|S| = d$. Take any connected zero-dimensional scheme $Z \subset X$ with $\deg(Z) = 2$ and $S \cap Z = \emptyset$. Thus, $\deg(Z \cup S) = d + 2$. Since $X \cong \mathbb{P}^1$, $\deg(\mathcal{O}_X(1)) = d$ and X is projectively normal, we have $h^1(\mathcal{I}_{S\cup Z}(1)) = 1$ and $h^1(\mathcal{I}_W(1)) = 0$ for each $W' \subsetneq S \cup Z$. This is equivalent to say that the line $\langle Z \rangle$ meets $\langle S \rangle$ at a unique point, q and $q \ne Z_{\mathrm{red}}$. By Sylvester's theorem, $r_X(q) = d$ ([48]). Since $q \in \langle S \rangle$ and $|S| = d$, S evinces the X-rank of q.

If $A \ne \emptyset$ and $|A| \le \lfloor (d+2)/2 \rfloor$, then A totally evinces the X-ranks by part (a) of Lemma 3 and the fact that $\rho(X) = d + 1$. Now, assume $d \ge |A| > \lfloor (d+2)/2 \rfloor$. Fix a set $E \subset X \setminus A$ with $|E| = d + 2 - |A|$. Adapt the proof of part (b) of Lemma 3. □

Proposition 2. *Fix a set $S \subset X_{n,d}$, $n \ge 2$, with $|S| = d + 1$. The following conditions are equivalent:*

1. *there is a line $L \subset \mathbb{P}^n$ such that $|S \cap L| > \lfloor (d+2)/2 \rfloor$;*
2. *S evinces no $X_{n,d}$-rank;*
3. *there is $q \in \langle S \rangle'$ such that S does not evince the $X_{n,d}$-rank of q.*

Proof. Obviously, (2) implies (3). If $X' \subset X$ is a subvariety and $q \in \langle X' \rangle$, we have $r_{X'}(q) \ge r_X(q)$. Thus, Sylvester's theorem ([48]) and Lemma 2 show that (1) implies (2).

Now, assume the existence of $q \in \langle S \rangle'$ such that S does not evince the X-rank of q, i.e., $r_X(q) \le d$. Take $A \subset \mathbb{P}^n$ such that $\nu(A) = S$ and take $B \subset \mathbb{P}^n$ such that $\nu_d(B)$ evinces the X-rank of q. Since $q \in \langle S \rangle'$, (Ref. [50] Lemma 1) gives $h^1(\mathbb{P}^n, \mathcal{I}_{A\cup B}(d)) > 0$. Since $|A \cup B| \le 2d + 1$, (Ref. [51] Lemma 34) gives the existence of a line $L \subset \mathbb{P}^n$ such that $|L \cap (A \cup B)| \ge d + 2$. Let $H \subset \mathbb{P}^n$ be a general hyperplane containing L. Since H is general and $A \cup B$ is a finite set, we have $H \cap (A \cup B) = L \cap (A \cup B)$. Since $|L \cap (A \cup B)| \ge d + 2$, we have $|A \cup B \setminus H \cap (A \cup B)| \le d - 1$ and hence $h^1(\mathbb{P}^n, \mathcal{I}_{A\cup B \setminus H\cap(A\cup B)}(d-1)) = 0$. By ([52] Lemma 5.2), we have $A \setminus A \cap H = B \setminus B \cap H$. □

See [53,54] for some results on the geometry of sets $S \subset X_{n,d}$ with controlled Hilbert function and that may be useful to extend Proposition 2.

Proof of Theorem 1: Set $k := \lfloor d/2 \rfloor$. Note that $h^1(\mathcal{I}_A(x)) = 0$ for all $x \ge k$ and in particular $h^1(\mathcal{I}_A(d-k)) = 0$. Fix $q \in \langle \nu_{d,n}(A) \rangle'$ and assume $r_{X_{n,d}}(q) < |A|$. Fix $B \subset \mathbb{P}^n$ such that $\nu_{d,n}(B)$ evinces the $X_{n,d}$-rank of q. Since $h^1(\mathcal{I}_A(k)) = 0$ and $|A| > |B|$, we have $h^0(\mathcal{I}_B(k)) > h^0(\mathcal{I}_A(k))$. Thus, there is $M \in |\mathcal{O}_{\mathbb{P}^n}(k)|$ containing B, but with $A \nsubseteq M$, i.e., $A \setminus A \cap M \ne \emptyset$, while $B \setminus B \cap M = \emptyset$. Since $h^1(\mathcal{I}_A(d-k)) = 0$, we have $h^1(\mathcal{I}_{A\setminus A\cap M}(d-k)) = 0$. Since $h^1(\mathcal{I}_A(d)) = 0$, $\nu_{d,n}(A)$ is linearly independent. Since $\nu_{d,n}(B)$ evinces a rank, it is linearly independent. Grassmann's formula gives $\dim \langle \nu_{d,k}(A) \rangle \cap \langle \nu_{d,b}(B) \rangle = |A \cap B| + h^1(\mathcal{I}_{A\cup B}(d)) - 1$. We have $A \cup B = ((A \cup B) \cap M) \cup (A \setminus A \cap M)$. Since $A \setminus A \cap B$ is a finite set, we have $h^2(\mathcal{I}_{A\setminus A\cap B}(d-k)) = h^2(\mathcal{O}_{\mathbb{P}^n}(d-k)) = 0$. Since $h^1(\mathcal{I}_{A\setminus A\cap M}(d-k)) = 0$, the residual exact sequence (also known as the Castelnuovo's sequence)

$$0 \to \mathcal{I}_{A\setminus A\cap B}(d-k) \to \mathcal{I}_{A\cup B}(d) \to \mathcal{I}_{M\cap(A\cup B),M}(d) \to 0$$

gives $h^1(\mathcal{I}_{A\cup B}(d)) = h^1(M, \mathcal{I}_{M\cap(A\cup B)}(d))$. Since M is projectively normal, $h^1(M, \mathcal{I}_{M\cap(A\cup B)}(d)) = h^1(\mathcal{I}_{A\cup B}(d))$. Thus, the Grassmann's formula gives $\dim \langle \nu_{d,n}(A \cap M) \rangle \cap \langle \nu_{d,n}(B \cap M) \rangle = |A \cap B \cap M| + h^1(\mathcal{I}_{A\cup B}(d)) - 1$. Since $B \subset M$, we get $\langle \nu_{d,n}(A \cap M) \rangle \cap \langle \nu_{d,n}(B \cap M) \rangle = \langle \nu_{d,k}(A) \rangle \cap \langle \nu_{d,b}(B) \rangle$. Since $A \cap M \supsetneq A$, we get $q \notin \langle \nu_{d,n}(A) \rangle'$, a contradiction. □

4. Tensors, i.e., the Segre Varieties

Fix an integer $k \ge 2$ and positive integers $n_1, \dots, n_k$. Set $Y := \prod_{i=1}^k \mathbb{P}^{n_i}$ (the Segre variety) and $N := -1 + \prod_{i=1}^k (n_i + 1)$. Let $\nu : Y \to \mathbb{P}^N$ denote the Segre embedding. Let $\pi_i : Y \to \mathbb{P}^{n_i}$ denote the projection on the i-th factor. For any $i \in \{1, \dots, k\}$, set $Y[i] := \prod_{h\ne i} \mathbb{P}^{n_h}$ and call $\eta_i : Y \to Y[i]$ the projection which forgets the i-th component. Let $\nu[i] : Y[i] \to \mathbb{P}^{N_i}$, $N_i := -1 + \prod_{h \neq i}(n_h + 1)$ denote the Segre embedding of $Y[i]$. A key difficulty is that $\rho(\nu(Y)) = 2$ because $\nu(Y)$ contains lines.

Lemma 4. *Let $S \subset Y$ be any finite set such that there is $i \in \{1, \dots k\}$ with $\eta_{i|S}$ not injective. Then, $\nu(S)$ evinces no rank.*

Proof. By Lemma 1, we reduce to the case $|S| = 2$, say $S = \{a, b\}$ with $a = (a_1, \dots, a_k)$, $b = (b_1, \dots, b_k)$ with $a_i = b_i$ if and only if $i > 1$. Since all lines of Y are contained in one of the factors of Y and all lines of $\nu(Y)$ are images of lines of Y, we get $S \subset \nu(Y)$. Thus, each element of $\langle \nu(S) \rangle$ is contained in $\nu(Y)$ and hence it has rank 1. Since $|S| > 1$, $\nu(S)$ evinces no rank. □

Lemma 5. *Let $S \subset Y$ such that there are $S' \subseteq S$ and $i \in \{1, \dots, k\}$ with $|S'| = 3$, $\nu_i(\eta_i(S'))$ linearly dependent and $\pi_i(S') \subset \mathbb{P}^{n_i}$ linearly dependent. Then, $\nu(S)$ evinces no rank.*

Proof. Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface. Q is projectively equivalent to the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ and each point of $\mathbb{P}^3$ has at most Q-rank 2 by [47] (Proposition 5.1). By Lemma 1, we may assume $S' = S$. By Lemma 4, we may assume that $\eta_{i|S}$ is injective. Thus, $|\eta_{i|S}| = 3$. Since $\nu_i(\eta_i(S))$ is not linearly independent and it has cardinality 3, it is contained in a line of $\nu_i(Y[i])$. Thus, $\eta_i(S)$ is contained in a line of one of the factors of $Y[i]$. By assumption, $\pi_i(S)$ is contained in a line of $\mathbb{P}^{n_i}$. Thus, S is contained in a subscheme of Y isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Since each point of $\mathbb{P}^3$ has Q-rank ≤ 2 and $|S| = 3$, $\nu(S)$ evinces no rank. □

Remark 3. *Fix a finite set $A \subset Y$ such that $S := \nu(A)$ is linearly independent. S evinces no tensor rank if there is a multiprojective subspace $Y' \subset Y$ such that $A \subset Y'$ and $|S|$ is larger than the maximum tensor rank of $\nu(Y')$.*

Note that Lemmas 4 and 5 may be restated as a way to check for very low $|S|$ if there is some Y' as in Lemma 3 exists.

Proposition 3. *Take $S \subset \nu(Y)$ with $|S| = 2$. Let Y' be the minimal multiprojective subspace of Y containing S. The following conditions are equivalent:*

1. *S evinces no rank;*
2. *S does not generically evince ranks;*
3. *S does not totally evince ranks;*
4. *$Y' \cong \mathbb{P}^1$.*

Proof. Since any two distinct points of $\mathbb{P}^N$ are linearly independent (i.e., $\langle S \rangle$ is a line) and $\nu(Y)$ is the set of all points with $\nu(Y)$-rank 1, S evinces no rank if and only if $\langle S \rangle \subset \nu(Y)$. Use the fact that the lines of $\nu(Y)$ are contained in one of the factors of $\nu(Y)$. Since $\nu(Y)$ is cut out by quadrics, if $\langle S \rangle \nsubseteq \nu(Y)$, then $|\langle S \rangle \cap \nu(Y)| \leq 2$. Since $S \subset \langle S \rangle \cap \nu(Y)$, we see that all points of $\langle S \rangle \setminus S$ have rank 2 □

Proposition 4. *Take $S \subset \nu(Y)$ with $|S| = 3$ and $\nu(S)$ linearly independent. Write $S = \nu(A)$ with $A \subset Y'$. Let Y' be the minimal multiprojective subspace of Y containing A. Write $Y' = \mathbb{P}^{m_1} \times \cdots \mathbb{P}^{m_s}$ with $s \geq 1$ and $m_1 \geq \cdots \geq m_s > 0$. We have $m_1 \leq 2$.*

If $\eta_{i|A}$ is injective for all i and either $m_2 = 2$ or $s \geq 4$ or $m_1 = 2$ and $s = 3$, then S totally evinces its ranks. In all other cases for a general $E \in Y'$ with $|E| = 3$, $\nu(E)$ does not generically evince its ranks.

Proof. If $\eta_{i|A}$ is not injective for some i, then S evinces no rank by Lemma 4. Thus, we may assume that each $\eta_{i|A}$ is injective for all i. Each factor of Y' is the linear span of $\pi_i(A)$ in $\mathbb{P}^{n_i}$. Hence, $m_1 \leq 2$.

Omitting all factors which are points, we get the form of Y' we use. If $Y' = \mathbb{P}^1$ (resp. $\mathbb{P}^2$, resp. $\mathbb{P}^1 \times \mathbb{P}^1$), then each point of $\langle S\rangle$ has rank 1 (resp. 1, resp. ≤ 2). Thus, in these cases, S evinces no rank. If either $Y' = \mathbb{P}^2 \times \mathbb{P}^1$ or $Y' = (\mathbb{P}^1)^3$, then $\sigma_2(\mathbb{P}^2 \times \mathbb{P}^1) = \mathbb{P}^5$ and $\sigma_2((\mathbb{P}^1)^3) = \mathbb{P}^7$ ([23,26]). Thus, the last assertion of the proposition is completed.

(a) Assume $s \geq 2$ and $m_2 = 2$. Taking a projection onto the first two factors, we reduce to the case $s = 2$ (this reduction step is used only to simplify the notation). Take a $H \in |\mathcal{O}_{Y'}(1,0)|$ containing B (this is possible because $h^0(\mathcal{O}_{\mathbb{P}^2}(1)) = 3 > |B|$). Since Y' is the minimal multiprojective subspace of Y containing A, we have $A \setminus A \cap H \neq \emptyset$. Since $B \setminus B \cap H = \emptyset$, (Ref. [52] Lemma 5.1) gives $h^1(\mathcal{I}_{A\setminus A\cap H}(0,1)) > 0$. Thus, either there is $A' \subset A$ with $|A'| = 2$ and $\eta_{1|A'}$ not injective (we excluded this possibility) or $|A \setminus A \cap H| = 3$ (i.e., $A \cap H = \emptyset$) and $\eta_1(A) \subset \mathbb{P}^2$ is contained in a line R. Set $M := \mathbb{P}^2 \times R$. We get $A \subset M$ and hence A is a contained in a proper multiprojective subspace, contradicting the definition of Y'.

(b) Assume $s \geq 3$ and $m_1 = 2$. By part (a), we may assume $m_2 = 1$. Taking a projection, we reduce to the case $s = 3$, i.e., $Y' = \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$. Take H as in step (a). As in step (a), we get $A \cap H = \emptyset$ and $\eta_1(A)$ contained in a line R of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$, contradicting the definition of Y'.

(c) Assume $s \geq 4$. By step (b), we may assume $m_1 = 1$. Taking a projection onto the first four factors of Y', we reduce to the case $Y' = (\mathbb{P}^1)^4$. Fix any $H \in |\mathcal{O}_{Y'}(1,1,0,0)|$ containing B. Assume for the moment $A \nsubseteq H$. By ([52] Lemma 5.1), we have $h^1(\mathcal{I}_{A\setminus A\cap H}(0,0,1,1)) > 0$, i.e., either there are $a = (a_1, a_2, a_3, a_4) \in A$, $b = (b_1, b_2, b_3, b_4) \in A$ with $a \neq b$ and $(a_3, a_4) = (b_3, b_4)$ of $A \cap H = \emptyset$ and the projection of A onto the last 2 factors of Y' is contained in a line. The last possibility is excluded by the minimality of Y'. Thus, $a, b \in A$ exists. Set $A := \{a, b, c\}$ and write $c = (c_1, c_2, c_3, c_4)$. Permuting the factors of Y', we see that, for each $E \subset \{1, 2, 3, 4\}$, there is $A_E \subset A$ with $|A_E| = 2$ and $\pi_E(A_E)$ is a singleton, where $\pi_E : Y' \to \mathbb{P}^1 \times \mathbb{P}^1$ denote the projection onto the factors of Y' corresponding to E. Since the cardinality of the set $\mathcal{S}$ of all subset of $\{1, 2, 3, 4\}$ with cardinality 2 is larger than the cardinality of the set of all subsets of A with cardinality 3, there are $E, F \in \mathcal{S}$ such that $E \neq F$ and $A_E = A_F$. If $E \cap F \neq \emptyset$, say $E \cap F = \{i\}$, then $\eta_{i|A}$ is not injective, contradicting our assumption. If $E \cap F = \emptyset$, we have $E \cup F = \{1, 2, 3, 4\}$. Since $A_E = A_F$, we get $|A_E| = 1$, a contradiction. □

Remark 4. *Take a finite $S \subset \nu(Y)$ and fix $q \in \langle \nu(S)\rangle'$. Let $A \subset Y$ be the subset with $\nu(A) = S$. It is easier to prove that S evinces the rank of q if we know that the minimal multiprojective subspace of Y containing A is the minimal multiprojective subspace Y'' of Y with $q \in \langle \nu(Y'')\rangle$. Note that this is always true if $Y'' = Y$, i.e., if the tensor q is concise.*

5. Questions on the Case of Veronese Varieties

Let $r_{\max}(n, d)$ denote the maximum of all $X_{n,d}$-ranks (in [55,56] it is denoted with $r_{\max}(n + 1, d)$). The integer $r_{\max}(n, d)$ depends on two variables, n and d. In this section, we ask some question on the asymptotic behavior of $r_{\max}(n, d)$ when we fix one variable, while the other one goes to $+\infty$.

Let $r_{\text{gen}}(n, d)$ denote the $X_{n,d}$-rank of a general $q \in \mathbb{P}^r$. These integers do not depend on the choice of the algebraically closed base field $\mathbb{K}$ with characteristic 0. The diagonalization of quadratic forms gives $r_{\max}(n, 2) = r_{\text{gen}}(n, 2) = n + 1$. The integers $r_{\text{gen}}(n, d)$, $d > 2$, are known by an important theorem of Alexander and Hirschowitz ([8–13]); with four exceptional cases, we have $r_{\text{gen}}(n, d) = \lceil \binom{n+d}{n}/(n + 1)\rceil$. An important theorem of Blekherman and Teitler gives $r_{\max}(n, d) \leq 2r_{\text{gen}}(n, d)$ (and even $r_{\max}(n, d) \leq 2r_{\text{gen}}(n, d) - 1$ with a few obvious exceptions) ([57,58]). In particular, for a fixed n, we have

$$\frac{1}{(n+1)!} \leq \liminf_{d\to+\infty} r_{\max}(n, d)/d^n \leq \limsup_{d\to+\infty} r_{\max}(n, d)/d^n \leq \frac{2}{(n+1)!}.$$

It is reasonable to ask if $\liminf_{d\to+\infty} r_{\max}(n,d)/d^n$ exists and its value. Of course, it is tempting also to ask a more precise information about $r_{\max}(n,d)$ for $d \gg 0$. In the case $n = 2$, De Paris proved in [55,56] that $r_{\max}(2,d) \geq \lfloor (d^2+2d+5)/4 \rfloor$ ([56] Theorem 3), which equality holds if d is even ([56] (Proposition 2.4)) and suggested that equality holds for all d. Since $r_{\max}(2,d+1) \geq r_{\max}(2,d)$ even for odd d, the integer $r_{\max}(2,d)$ grows like $d^2/4$. Thus, there is an interesting interval between the general upper bound of [57] (which, in this case, has order $d^2/3$) and $r_{\max}(2,d)$. There are very interesting upper bounds for the dimensions of the set of all points with rank bigger than the generic one ([59]).

What are

$$\limsup_{n\to+\infty} \frac{(n+1)!r_{\max}(n,d)}{d^n} \text{ and } \liminf_{n\to+\infty} \frac{(n+1)!r_{\max}(n,d)}{d^n} \ ?$$

For all $d \geq 3$, study $r_{\max}(n,d) - r_{\max}(n,d-1)$ and compare for $d \gg 0$ $r_{\max}(n,d) - r_{\max}(n,d-1)$ with $r_{\max}(n-1,d)$ and $r_{\text{gen}}(n-1,d)$. Of course, this is almost exactly known when $n = 2$ by Sylvester's theorem ([48]) and De Paris ([55,56]), but $r_{\max}(2,d) - r_{\max}(2,d-1)$ for $d \gg 0$ is both $\sim r_{\text{gen}}(1,d)$ and $\sim r_{\max}(1,d)/2$ and so we do not have any suggestion for the case $n > 2$.

Funding: The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

Conflicts of Interest: The author declares no conflict of interest.

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