BOUNDS ON THE TENSOR RANK

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ABSTRACT. We give a sufficient criterion for a lower bound of the cactus rank of a tensor. Then we refine that criterion in order to be able to give an explicit sufficient condition for a non-redundant decomposition of a tensor to be minimal and unique.

1. INTRODUCTION

The study of minimal decompositions of a given tensor T as a linear combination of rank one tensors is a hot topic in many areas, ranging from pure algebraic geometry to applications to signal processing, big data analysis, quantum information...

Vectors $v_{j,i} \in \mathbb{C}^{n_j+1}$, $j = 1, \dots, k$, $i = 1, \dots, r$, such that

$$(1) \qquad T = \sum_{i=1}^{r} \lambda_i v_{1,i} \otimes \cdots \otimes v_{k,i}$$

for some $\lambda_i \in \mathbb{C} \setminus \{0\}$ determine a *decomposition* of T. We will say that the decomposition is *non-redundant* (cf. Definition 2.1) if we cannot extract any proper subset of $\{v_{1,i} \otimes \cdots \otimes v_{k,i}\}_{i=1,\dots,r}$ which generates T.

Since we will use geometric arguments through the paper, we use a geometric notation. Thus we identify (up to scalar multiplication) a tensor T with a point in the projective space $\mathbb{P}(\mathbb{C}^{n_1+1} \otimes \cdots \otimes \mathbb{C}^{n_k+1})$ and a decomposition of T with a finite subset S of the Segre embedding of the abstract product $\mathbb{P}(\mathbb{C}^{n_1+1}) \times \cdots \times \mathbb{P}(\mathbb{C}^{n_k+1}) = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, such that T belongs to the linear span of S.

Cleary, having a non-redundant decomposition of a given tensor T does not imply that such a decomposition is minimal, i.e. it has a minimal number of addenda (so that r is the rank $\mathrm{rk}(T)$ of T, cf. Definition 2.3).

In this paper we give a criterion that certifies if a non-redundant decomposition of a general tensor T is also minimal, thus that it computes the rank of T (cf. Theorem 3.1). Moreover, under certain conditions, we can also show that a decomposition is the unique minimal decomposition of T (cf. Theorem 4.6).

These two facts rely on our main result Theorem 3.1, where we give a criterion to find a lower bound for the cactus rank (cf. Definition 2.3). The idea is geometrically quite simple: Assume one has a non-redundant decomposition S of a tensor $T \in \mathbb{P}(\mathbb{C}^{n_1+1} \otimes \cdots \otimes \mathbb{C}^{n_k+1})$, then one can flatten the product $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ in a partition of two factors and study the geometry of the two projections of S, to get the result.

One can easily compare our result with the Kruskal's result on the identifiability of tensors ([K77]), which implies a criterion for the minimality of a decomposition.

It turns out that our criterion is geometrically simpler, and it applies in a wider range of numerical cases.

Theorem 3.1 has the consequence that if a non-redundant decomposition projects onto two linearly independents subset in the flattening, then it is also a minimal one (cf. Example 3.6). This can be considered as a step towards the celebrated Strassen's Conjecture on the rank of a block tensor (cf. [S73]).

We show also that Theorem 3.1 provides new evidences towards the Comon's Conjecture, i.e. the equality between the rank and the symmetric rank of a symmetric tensor T. In a wide numerical range, if we know the existence of a symmetric decomposition of T with sufficiently general geometric properties, then the conjecture holds for T (cf. Corollary 3.10). For instance, the Corollary applies for general tensors in $\mathbb{P}(\mathrm{Sym}^6(\mathbb{C}^3))$ and $\mathbb{P}(\mathrm{Sym}^8(\mathbb{C}^3))$, i.e. for general forms of degree 6 and 8 in three variables.

Another consequence of Theorem 3.1 is described in Theorem 3.8. Given a minimal decomposition with r addenda, then for any integer r' such that $r \leq r' < -1 + \Pi(n_i + 1)$, it is always possible to find a non-redundant decomposition with r' addenda. We notice that this happens to be false for symmetric decompositions of a symmetric tensor, even in the case of $S^d(\mathbb{C}^2)$ with $d \geq 4$ (cf. Sylvester Theorem in e.g. [CS11, BGI11]).

In Section 4 we focus on the identifiability of a minimal decomposition, meaning the uniqueness of a decomposition of a tensor T with exactly $\mathrm{rk}(T)$ addenda. The main result of this section is Theorem 4.6 (that is again a consequence of Theorem 3.1), where we point out a sufficient condition for a non-redundant decomposition to be minimal and unique. Again we compare our result with Kruskal's bound ([K77]). Since Kruskal's bound is sharp ([SB00], [D13]) and since our geometric assumptions are weaker, and then easier to verify, than Kruskal's ones, we cannot hope to produce applications outside Kruskal's numerical range. There are few cases in which our the numerical range of application matches with Kruskal's range. One of them e.g. is given by tensors of type $3 \times 2 \times 2 \times 2$.

2. NOTATION AND PRELIMINARIES

For a subscheme $Z \subset \mathbb{P}^m$, we indicate with $\langle Z \rangle$ the linear span of Z and with $\deg(Z)$ its length (when it is finite). If Z is finite and reduced, we indicate with $\sharp Z$ the cardinality of Z.

For any product of projective spaces $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ call ν the Segre map

$$\nu : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \to \mathbb{P}^M, \qquad M = -1 + \Pi(n_i + 1).$$

In order to have a more compact notation we will always write

$$Y := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$$

for the abstract product, and

$$X := \nu(Y) \subset \mathbb{P}^M$$

for the Segre variety.

For any $i = 1, \ldots, k$ call π_i the projection of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ to the i-th factor.

We can generalize this notation by setting, for any collection of sub-indices $\mathbf{u} = \{u_1, \ldots, u_i\} \subset \{1, \ldots, k\}$,

$$\pi_{\mathbf{u}} = \text{projection to the product of the factors } u_1, \ldots, u_i.$$

In particular $\pi_{\{1,\ldots,i\}}$ is the projection to the product of the first i factors.

For any subset $\mathbf{u} = \{u_1, \ldots, u_i\} \subset \{1, \ldots, k\}$, we will denote with $\mathcal{O}(\mathbf{u})$ the line bundle on $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ pull back of the hyperplane bundle on $\nu(\pi_{\mathbf{u}}(Y))$.

We will write simply $\mathcal{O}(1)$ when $\mathbf{u} = \{1, \ldots, k\}$, i.e. $\mathcal{O}(1)$ is the pull back of the hyperplane bundle in the Segre embedding of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$.

For any subset Z of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ we will write consequently

$$\mathcal{I}_Z(\mathbf{u}) = \mathcal{I}_Z \otimes \mathcal{O}(\mathbf{u}), \quad \mathcal{O}_Z(\mathbf{u}) = \mathcal{O}_Y \otimes \mathcal{O}(\mathbf{u})$$

and write $\mathcal{I}_Z(1), \mathcal{O}_Z(1)$ when $\mathbf{u} = \{1, \ldots, k\}$.

Notice that the dimension $h^0(\mathcal{I}_Z(\mathbf{u}))$ corresponds then to the co-dimension of the linear span of $\nu(\pi_{\mathbf{u}}(Z))$. Obviously, if Z is zero-dimensional and $h^1(\mathcal{I}_Z(\mathbf{u})) = 0$, then also $h^1(\mathcal{I}_Z(\mathbf{u}')) = 0$ for all $\mathbf{u}' \supseteq \mathbf{u}$.

We will say that a finite subset $S \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ has *different coordinates* if for all $i = 1, \ldots, k$ the projection π_i to the i-th factor is an embedding of S into $\mathbb{P}^{n_i}$.

We will need the process of *residuation* with respect to a divisor. Let X be a variety For any zero-dimensional scheme $Z \subset X$ and for any effective divisor D on X the " *residue of Z w.r.t. D* " is the scheme $Res_D(Z)$ defined by the ideal sheaf $\mathcal{I}_Z : \mathcal{I}_D$, where $\mathcal{I}_Z, \mathcal{I}_D$ are the ideal sheaves of Z and D respectively. The multiplication by local equations of D defines the exact sequence of sheaves:

$$(2) \qquad 0 \to \mathcal{I}_{Res_D(Z)}(-D) \to \mathcal{I}_Z \to \mathcal{I}_{D \cap Z, D} \to 0$$

where the rightmost sheaf is the ideal sheaf of $D \cap Y$ in D.

We will identify elements $T \in \mathbb{P}^M$, which is the space of embedding of $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, as tensors of type $(n_1 + 1) \times \cdots \times (n_k + 1)$ (modulo scalars).

Definition 2.1. *A finite reduced subset $S \subset Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is a* decomposition *of T if $T \in \langle \nu(S) \rangle$ (with an abuse of notation sometime we will say that also $\nu(S) \subset X$ is a decomposition of T). If moreover $T \notin \langle \nu(S') \rangle$ for any $S' \subsetneq S$, the decomposition S is said to be* not-redundant. *Finally, if $\sharp S = \min\{\sharp S' \mid S' \subset Y \text{ and } T \in \langle \nu(S') \rangle\}$ then S is called a* minimal *decomposition of T.*

Remark 2.2. *Clearly " not-redundant " does not imply " minimal ". As we will detail in Theorem 3.8, it is alway possible to build a non-minimal non-redundant decomposition.*

Our target is to study the identifiability of a tensor $T \in \mathbb{P}^M$ (i.e. when T has only one minimal decomposition) by means of the knowledge of the numbers $h^0(\mathcal{I}_A(\mathbf{u}))$, for all $\mathbf{u} \subset \{1, \ldots, k\}$, where $A \subset Y$ is a finite set which corresponds to a decomposition of T.

We will use the following notions for the rank of T.

Definition 2.3. *The* rank*, $\mathrm{rk}(T)$, of T is the minimum r for which there exists a minimal decomposition of T.*

The cactus rank *of T is the minimum r for which there exists a zero-dimensional subscheme $\Gamma \subset X$ with $\deg(\Gamma) = r$ and $T \in \langle \Gamma \rangle$.*

Clearly:

$$\text{rank of } (T) \geq \text{cactus rank of } (T).$$

In analogy to the rank case, we will say that a zero-dimensional scheme $Z \subset Y$ is a " minimal cactus decomposition " of $T \in \mathbb{P}^M$ if Z is of minimal degree among the zero-dimensional schemes $Z' \subset Y$ such that $T \in \langle \nu(Z') \rangle$.

If σ_r is the r-secant variety of X, then all tensors of rank r belong to σ_r.

For any tensor T of rank r, let $\mathcal{S}(T)$ denote the set of all (reduced) finite subsets $S \in Y$ of cardinality r such that $T \in \langle \nu(S) \rangle$. Of course for all $S \in \mathcal{S}(T)$ the image $\nu(S)$ is linearly independent, for otherwise T is contained in the span of a subset of cardinality $r' < r$, thus it has rank smaller than r.

A tensor T is *identifiable* when $\mathcal{S}(T)$ is a singleton. The abstract product $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is " *generically identifiable* in rank r " if the general $T \in \sigma_r$ is identifiable.

The main tool for our analysis of the identifiability of a tensor $T \in \mathbb{P}^M$ relies in the following proposition that is an immediate consequence of [BB12, Lemma 1] if we consider the residue exact sequence of $A \cup B$ cut by a linear space containing A.

Proposition 2.4. *Consider linearly independent zero-dimensional schemes $A, B \subset Y$. Then the linear spans of the images $\nu(A), \nu(B)$ in the Segre map satisfy*

$$\dim(\langle \nu(A) \rangle \cap \langle \nu(B) \rangle) = \dim(\langle \nu(A \cap B) \rangle + h^1(\mathcal{I}_{A \cup B}(\mathbf{1})).$$

3. Rank

If we know a decomposition $T = T_1 + \cdots + T_r$ of T in terms of tensors T_i of rank 1, in general we cannot directly conclude that r is the rank of T.

Our analysis will prove that, for small values of r, the rank of T is r provided that the summands correspond to points in a suitably general geometric position. We will give a criterion to find a lower bound for the cactus rank (and therefore also for the rank).

Theorem 3.1. *Fix a partition $E \sqcup F = \{1, \ldots, k\}$ of the k factors of the abstract product $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, i.e. $E = \{a_1, \ldots, a_{k-h}\}$ and $F = \{b_1, \ldots, b_h\}$ for some fixed $0 < h < k$. Let $M_F := \prod_{i=1}^{h}(n_{b_i} + 1)$ be the affine dimension of the ambient space of the Segre embedding of the F-factors $Y_F := \mathbb{P}^{b_1} \times \cdots \times \mathbb{P}^{b_h}$. Now fix $0 < c < M_F$ and let $A \subset Y$ be a zero-dimensional scheme which satisfies $h^1(\mathcal{I}_A(E)) = 0$ and $h^0(\mathcal{I}_A(F)) < M_F - c$. Take any $T \in \langle \nu(A) \rangle$ such that $T \notin \langle \nu(A') \rangle$ for any $A' \subsetneq A$. Then there are no zero-dimensional schemes $B \subset Y$ with $\deg(B) \le c$ such that $T \in \langle \nu(B) \rangle$.*

Proof. Notice that the Segre embedding of the projection π_F maps Y to $\mathbb{P}^{M_F - 1}$.

Since $h^1(\mathcal{I}_A(E)) = 0$, we have $h^1(\mathcal{I}_A(1)) = 0$. The condition $h^0(\mathcal{I}_A(F)) < M_F - c$ implies that $\deg(A) > c$. Assume that the theorem fails and take $B \subset Y$ with $\deg(B) \le c$ and $T \in \langle \nu(B) \rangle$. Since $\deg(A) > \deg(B)$ and $T \notin \langle \nu(A') \rangle$ for any $A' \subsetneq A$, we have $B \nsubseteq A$. Moreover $h^1(\mathcal{I}_{A \cup B}(1)) > 0$, by Proposition 2.4. More precisely, since $T \in \langle \nu(B) \rangle$ and $T \notin \langle \nu(A') \rangle$ for any $A' \subsetneq A$, we have

$$(3) \qquad\qquad h^1(\mathcal{I}_{A \cup B}(1)) > h^1(\mathcal{I}_{A' \cup B}(1))$$

for all $A' \subset A$ with $A \cup B \neq A' \cup B$.

Since $\deg(B) \le c < M_F$, we have $h^0(\mathcal{I}_B(F)) > 0$. Take a general divisor $D \in |\mathcal{I}_B(F)|$. In other words, D is the inverse image in Y of a hyperplane in the Segre embedding of Y_F and $B \subset D$. Since $h^0(\mathcal{I}_A(F)) < M_F - c \le h^0(\mathcal{I}_B(F))$, then $A \nsubseteq D$, so that $(D \cap A) \cup B$ is strictly contained in $A \cup B$. Hence by (3) we

get $h^1(\mathcal{I}_{(D \cap A) \cup B}(1)) < h^1(\mathcal{I}_{A \cup B}(1))$. The residual exact sequence (2) applied to D gives $h^1(\mathcal{I}_{\mathrm{Res}_D(A \cup B)}(E)) > 0$. Since $\mathrm{Res}_D(A \cup B) \subseteq A$, we get a contradiction. $\quad \square$

Observe that the condition $h^1(\mathcal{I}_A(E)) = 0$ can be satisfied only when $\deg(A) \leq \prod_{i=1}^{k-h}(n_{a_i} + 1)$, the affine dimension of the ambient space of the Segre embedding of the E-factors $Y_E = \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_{k-h}}$.

We can rephrase Theorem 3.1 to produce results on the rank of tensors.

Corollary 3.2. *With the previous notation, if T sits in the linear span of a scheme $A \subset Y$ which satisfies the assumptions of Theorem 3.1, then the cactus rank of T is a least $c + 1$. Hence also the rank of T cannot be smaller than $c + 1$.*

Corollary 3.3. *Let $A \subset Y$ be a zero dimensional scheme of $\deg(A) = c + 1$ (resp. a finite set with $\sharp(A) = c + 1$). With the Notation of Theorem 3.1, assume that $h^1(\mathcal{I}_A(E)) = 0$ and $h^1(\mathcal{I}_A(F)) = 0$. Then any $T \in \langle \nu(A) \rangle$ such that $T \notin \langle \nu(A') \rangle$ for any $A' \subsetneq A$ has cactus rank (resp. cactus rank and rank) equal to $c + 1$.*

Proof. It is straightforward from Theorem 3.1 $\quad \square$

Let we point out an application to the case of 3-way tensors.

Proposition 3.4. *Consider $k = 3$ and let T be a tensor of type $(n_1 + 1) \times (n_2 + 1) \times (n_3 + 1)$, which has a not-redundant decomposition $T = T_1 + \cdots + T_r$, where the T_i's are tensors of rank 1. Identify each T_i with a point in $X = \nu(\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \mathbb{P}^{n_3})$ and set $A = \{T_1, \ldots, T_r\} \subset X$. Call A_E (resp. A_F) the projection of A to $Y_E = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ (resp. $Y_F = \mathbb{P}^{n_3}$).*

Assume that A has different coordinates, A_E is linearly independent and A_F is contained in no hyperplanes of $\mathbb{P}^{n_3}$. Then the rank of T is at least $n_3 + 1$.

Proof. In the notation of Theorem 3.1, take $E = \{1, 2\}$ and $F = \{3\}$. Then our assumptions on A_E, A_F imply that A satisfies $h^1(\mathcal{I}_A(E)) = h^0(\mathcal{I}_A(F)) = 0$. Thus T cannot have a decomposition with $M_F - 1 = n_3$ summands. $\quad \square$

Example 3.5. *Kruskal's Theorem for the identifiability of tensors ([K77]) provides results similar to the previous proposition for the rank. We notice that the numerical range of application of Proposition 3.4 is sometimes wider than Kruskal's range.*

For instance, consider a tensor T of type $3 \times 4 \times 6$ having a decomposition with 6 summands. If the decomposition determines a subset A satisfying the assumptions of Proposition 3.4, we can conclude that the rank of T is 6. We cannot get the same conclusion directly with Kruskal's Theorem because we are outside Kruskal's numerical range, since $6 > (3 + 4 + 6 - 2)/2$.

The following example should be considered as a step towards the Strassen's Conjecture on the rank of a block tensor (see [S73]).

Example 3.6. *Proposition 3.4 can give results on the rank of a sum of tensors, when we have some information on the decompositions of the summands.*

For instance, consider again tensors of type $3 \times 4 \times 6$ and take a tensor T which is the sum $T = T' + T''$ of two tensors of rank 3. Consider a decomposition of T' (resp. T'') in a sum of three tensors of rank 1 and call S' (resp. S'') the set of cardinality 3 in the product $\mathbb{P}^2 \times \mathbb{P}^3 \times \mathbb{P}^5$ determined by the decomposition.

If the set $S = S' \cup S''$ has cardinality 6 and satisfies the assumptions of Proposition 3.4 (i.e. both $\pi_{\{1,2\}}(\nu(S))$ and $\pi_{\{3\}}(\nu(S))$ are linearly independent), then we can conclude that the rank of T is 6.

We show below that, on the contrary, if we increase the cardinality of a decomposition, we can always construct new non-redundant decompositions of a tensor T.

Example 3.7. *Fix $P \in Y$ and write $P = (p_1, \ldots, p_k)$ with $p_i \in \mathbb{P}^{n_i}$. Assume $n_i > 0$. Take two points $b_i, c_i \in \mathbb{P}^{n_i}$ such that $p_i \neq b_i, c_i$ but p_i is contained in the line of $\mathbb{P}^{n_i}$ spanned by b_i and c_i. Let $O_i := (u_1, \ldots, u_k)$, $Q_i := (v_1, \ldots, v_k)$ be the points of Y with $u_j = v_j$ for all $j \neq i$, $u_i = b_i$ and $v_i = c_i$. We have $\nu(P) \in \langle \{\nu(O_i), \nu(Q_i)\} \rangle$, and of course $\nu(P) \notin \langle \nu(S') \rangle$ for any $S' \subsetneq \{O_i, Q_i\}$.*

We show that indeed the previous construction can yield a *non-redundant* decomposition. Moreover the following also show that having found a non-redundant decomposition does not imply that it is a minimal one.

Theorem 3.8. *Assume $n_i > 0$ for at least one i. Take a finite set $A \subset X$ of cardinality $r \leq M$, such that $\nu(A) \subset X$ is linearly independent. Take a general $T \in \langle \nu(A) \rangle$. Then there exists a non-redundant decomposition $S \subset Y$ of T of cardinality $r + 1$.*

Proof. Fix $P = (p_1, \ldots, p_k) \in A$ and take Q_1, O_1 as in Example 3.7, with $i = 1$, and with the additional condition that $O_1, Q_1 \notin (A \setminus \{P\})$. We may take O_1 to be a general point of $\mathbb{P}^{n_1} \times \{p_2\} \times \cdots \times \{p_k\}$. Hence we may take $O_1 = (a_1, p_2, \ldots, p_k)$, with a_1 general. Set $A' := A \setminus \{P\}$.

(a) Assume $\nu(O_1) \notin \langle \nu(A) \rangle$. This is always possible unless $\langle \nu(A) \rangle$ contains $\mathbb{P}^{n_1} \times \{p_2\} \times \cdots \times \{p_k\}$. Since $P \in A$, this is equivalent to $\nu(Q_1) \notin \langle \nu(A) \rangle$. Set $S := (A \setminus \{P\}) \cup \{O_1, Q_1\}$. We have $\sharp(S) = r + 1$ and $\langle \nu(S) \rangle \supseteq \langle \nu(A) \rangle$ so that $T \in \langle \nu(S) \rangle$. The set S does not depend on Q_1, but only on A and P. To prove that S satisfies the claim, it is sufficient to prove that for a general $T \in \langle \nu(A) \rangle$ there is no $S' \subsetneq S$ with $T \in \langle \nu(S') \rangle$. It is sufficient to test all subsets of S with cardinality r. Take $S' \subset S$ with $\sharp(S') = r$. If $S' \supset A'$, i.e. if either $S' = A' \cup \{O_1\}$ or $S' = A' \cup \{Q_1\}$, then $\langle \nu(S') \rangle \cap \langle \nu(A) \rangle = \langle \nu(A') \rangle$, because $\{\nu(O_1), \nu(Q_1)\} \cap \langle \nu(A) \rangle = \emptyset$. Thus T cannot stay in $\langle \nu(S') \rangle$ because it cannot stay in $\langle \nu(A') \rangle$ for a proper subset A' of A. Hence $S' \not\supseteq A'$. Since $\sharp(S') = r$, we have $S' = A'' \cup \{O_1, Q_1\}$ with $A'' \subset A'$ and $\sharp(A'') = r - 2$. If $T \in \langle \nu(S') \rangle$, then $\langle \nu(S') \rangle \cap \langle \nu(A) \rangle \supseteq \langle \nu(A'') \cup \{T\} \rangle$. Since $\nu(O_1) \notin \langle \nu(A) \rangle$, we get:

$$\langle \nu(S') \rangle \cap \langle \nu(A) \rangle = \langle \nu(A'') \cup \{T\} \rangle.$$

The left hand side of this equality does not depends on the choice of T. Varying $T \in \langle \nu(A) \rangle \setminus \langle \nu(A') \rangle$ we get $\langle \nu(S') \rangle \supseteq \langle \nu(A) \rangle$. Since $\sharp(S') = \sharp(A)$ and $\nu(A)$ is linearly independent, we get $\langle \nu(S') \rangle = \langle \nu(A) \rangle$ and hence $\nu(O_1) \in \langle \nu(A) \rangle$, a contradiction.

(b) If $n_j = 0$ for all $j > 1$ we are done. Assume for instance that $n_2 > 0$ and that $\langle \nu(A) \rangle \supset \mathbb{P}^{n_1} \times \{p_2\} \times \cdots \times \{p_k\}$, so that we cannot take $\nu(O_1) \notin \langle \nu(A) \rangle$. Since $\nu(P) \in \langle \{\nu(O_1), \nu(Q_1)\} \rangle$, we get $\langle \nu(\{O_1\} \cup A') \rangle = \langle \nu(A) \rangle$. Replace thus A with $A' \cup \{O_1\}$ and P by O_1. Then take the construction of Example 3.7 with $i = 2$. If the new general point O_2 satisfies $\nu(O_2) \notin \langle \nu(A) \rangle$, then we conclude as in step (a). Otherwise we have $\langle \nu(A) \rangle \supset \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \{p_3\} \times \cdots \times \{p_k\}$ Then continue with $i = 3$ in the construction of Example 3.7, and so on. After at most k steps we must conclude by step (a), because $\langle A \rangle$ cannot contain the whole product Y, since $\sharp(A) \leq M$. $\square$

Remark 3.9. *Take $P = (p_1, \ldots, p_k)$, $i \in \{1, \ldots, k\}$ with $n_i > 0$ and b_i, c_i as in Example 3.7. Notice that in the previous proof we can choose for c_i any point*

(different from b_i) in the line spanned by b_i, p_i. Thus, in Theorem 3.8 we get a positive-dimensional family of sets $S \subset Y$ such that $\sharp(S) = r + 1$, $T \in \langle \nu(S) \rangle$ and $T \notin \langle \nu(S') \rangle$ for any $S' \subsetneq S$.

We end this section with a discussion on some consequence of Theorem 3.1 on symmetric tensors.

Assume $n := n_1 = \cdots = n_k$ so that Y is a product of k copies of $\mathbb{P}^n$. The Segre map restricted to the diagonal $\Delta \subset Y$ can be identified with the *Veronese map* $v_k : \mathbb{P}^n \to \mathbb{P}^D$ where $D + 1 = \binom{k+n}{n}$. The space $\mathbb{P}^D$ parameterizes symmetric tensors T, which in turn can be identified with homogeneous polynomials (forms) of degree k in $n + 1$ variables. The *symmetric rank* is the minimum r for which there exists a finite subset $A \subset \mathbb{P}^n$ of cardinality r with $T \in \langle v_k(A) \rangle$.

A conjecture raised in [CGLM08] and well known as *Comon's Conjecture* predicts that the symmetric rank of a symmetric tensor T always coincides with the rank of T as a normal tensor in the span of $\nu(Y)$.

The following corollary of Theorem 3.1 implies that, if some assumptions on a minimal symmetric decomposition A of a symmetric tensor T are satisfied, that the symmetric rank of T coincides with its rank (and cactus rank).

Corollary 3.10. *With the previous notation, consider a zero-dimensional scheme $A \subset \mathbb{P}^n$ of degree $\deg(A) = c + 1$. Call $\mathcal{J}_A$ the ideal sheaf of A in $\mathbb{P}^n$ and assume that $h^1(\mathcal{J}_A(e)) = 0$ for some $e \leq k/2$. Take $T \in \langle v_k(A) \rangle$ such that $T \notin \langle v_k(A') \rangle$ for any $A' \subsetneq A$. Then T has cactus rank equal to $c + 1$. If A is reduced (i.e. it is a finite set of points), then T has also rank $c + 1$.*

Proof. Consider any subset $E \subset \{1, \ldots, k\}$ of cardinality e and take $F = \{1, \ldots, k\} \setminus E$. Notice that $f := \sharp F \geq e$, so that also $h^1(\mathcal{J}_A(f)) = 0$. This implies that, considering A as a subset of $\Delta \subset Y$, in the notation of Theorem 3.1, we have $h^1(\mathcal{I}_A(E)) = 0$ and $h^0(\mathcal{I}_A(F)) < M_F - c$. The claim follows from Theorem 3.1. $\square$

We show that Corollary 3.10 provides new evidences for the Comon's Conjecture, sometimes even in a numerical range larger than the ones considered in previous works on the topic (e.g. [F16] and [ZHQ16]).

Remark 3.11. *Assume $k = 2e$ even. Then by Corollary 3.10, the Comon's conjecture holds for symmetric tensors T having a minimal symmetric decomposition A with $h^1(\mathcal{J}_A(e)) = 0$.*

The condition $h^1(\mathcal{J}_A(e)) = 0$ holds for general subsets $A \subset \mathbb{P}^n$, as soon as $c + 1 = \sharp A$ satisfies

$$(4) \qquad\qquad c + 1 \leq r_0 := \binom{n + e}{e}.$$

So Comon's Conjecture holds for general tensors whose symmetric rank $c + 1$ is bounded by r_0.

When $k = 2e + 1$ is odd, a similar conclusion holds with $r_0 := 1 + \binom{n+e}{e}$.

Example 3.12. *After the Alexander-Hirshowitz classification of defective Veronese varieties ([AH95]), a general symmetric tensor in the span of $v_k(\mathbb{P}^n)$ is known to have symmetric rank*

$$r_g = \lceil \frac{\binom{n+k}{k}}{n + 1} \rceil$$

except for a list of few exceptional cases.

In general r_g is bigger than our bound r_0 of Remark 3.10, since, for fixed e, r_g grows asymptotically as $(2^n e^n)/(n+1)!$ while r_0 grows like $e^n/n!$.

Nevertheless, there are cases in which r_g and r_0 coincide. This happens for $(k,n) = (6,2)$ or $(k,n) = (8,2)$. Since these two cases are not in the list of exceptional Veronese varieties, we can conclude that Comon's Conjecture holds for general forms of degree 6 and degree 8 in 3 variables.

Example 3.13. *Take* $(k,n) = (4,3)$. *Then Remark 3.11 tells us that the Comon's conjecture holds for general symmetric tensors in the span of* $v_4(\mathbb{P}^3)$ *with a decomposition with* 10 *summands. On the other hand, in this case the number* r_g *is* 9, *smaller than* r_0.

Indeed $(k,n) = (4,3)$ *is in the list of exceptional cases in the Alexander-Hirschowitz theorem, so that a general form of degree 4 in* $\mathbb{P}^3$ *has symmetric rank* $10 > r_g$. *Thus Comon's conjecture also holds for such general forms.*

In some sense, Theorem 3.1 provides a new heuristic reason why the case $(k,n) = (4,3)$ *is exceptional: a general tensor with a non-redundant decomposition with* 10 *summands cannot have a decomposition with* 9 *summands. A similar remark holds for the other exceptional cases of even degree: quadrics in any* $\mathbb{P}^n$ *and quartics in* $\mathbb{P}^2$ *and* $\mathbb{P}^4$.

4. Identifiability

In order to get results on the identifiability of a tensor T, we need to refine the previous analysis, and we are going to do that in this section.

We will need the following terminology for the *Segre function* of a finite subset of the product, introduced in [CS16].

Definition 4.1. *For any set of points* $S \subset Y$, *the* Segre function $\mathrm{SF}_S : \{1, \ldots, k\} \to \mathbb{N}$ *is defined by:*

$\mathrm{SF}_S(i) = 1+$ *the dimension of the linear span of* $\nu(\pi_{\{1,\ldots,i\}}(S))$.

Remark that the knowledge of the sequence $h^0(\mathcal{I}_S(\{1,\ldots,i\}))$, $i = 1, \ldots, k$, is equivalent to the knowledge of the Segre function SF_S.

More precisely, the definition of Segre function depends on the ordering of the factors of the product. The knowledge of $h^0(\mathcal{I}_S(\mathbf{u}))$, for all possible $\mathbf{u} \subset \{1, \ldots, k\}$, is equivalent to the knowledge of the Segre functions of S under all possible rearrangements of the factors.

Let us recall the following definition for a minimal dependent set of point (the same can be found in [CS16, Definition 2.9], while where in [GKZ, Chap.7 Sec.1] a minimal dependent set of points is called a *circuit.*).

Definition 4.2. *A set of points* $S \subset \mathbb{P}^m$ *is* minimally dependent, *if* S *is linearly dependent, but any proper subset of* S *is linearly independent.*

We need now a list of results on the cohomology of $\mathcal{I}_S$, for finite sets S in a product of projective spaces.

Definition 4.3. *We say that a finite subset* $S \subset Y$ *is* degenerate *if there exists an index* i *such that* $\pi_i(S)$ *does not span* $\mathbb{P}^{n_i}$, *i.e. there is a hyperplane* $H \subset \mathbb{P}^{n_i}$ *such that*

$$S \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_{i-1}} \times H \times \mathbb{P}^{n_{i+1}} \times \cdots \times \mathbb{P}^{n_k}.$$

We say that $T \in \mathbb{P}^M$ is degenerate *if there exists a degenerate subset $A \subset \nu(Y)$ such that $T \in \langle A \rangle$.*

Notice that if $S \subset Y$ is non-degenerate, then necessarily $\sharp S > \max\{n_i\}$.

The two results below are mainly an extension to the non-symmetric case of results in [BGL13] and [BB12].

Lemma 4.4. *Fix a finite set $S \subset Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ such that $x := \sharp(S) \leq k+1$, $h^1(\mathcal{I}_S(1)) > 0$ and $h^1(\mathcal{I}_{S'}(1)) = 0$ for each $S' \subsetneq S$. Then there is $E \subset \{1, \ldots, k\}$ such that $\sharp(E) = k + 2 - x$ and $\sharp(\pi_i(S)) = 1$ for all $i \in E$.*

Proof. The lemma is trivial if $x = 2$, because $\mathcal{O}_Y(1)$ is very ample and so the assumptions of the lemma are never satisfied in this case. Thus we may assume $x > 2$ and use induction on the integer x.

The case $x = 3$ is also true, because $\nu(Y)$ is cut out by quadrics and each line contained in $\nu(Y)$ is the image by ν of a line in one of the k factors of Y.

Thus we assume $x > 3$ (and so $k \geq 3$).
Assume $\sharp(\pi_1(S)) \geq 2$, so that the Segre function of S satisfies $\mathrm{SF}_S(1) \geq 2$. Notice that the assumptions on S imply that $\mathrm{SF}_S(k) = x-1$. Take points $P = (a_1, \ldots, a_k)$, $Q = (b_1, \ldots, b_k) \in S$ with $P \neq Q$. Since $h^1(\mathcal{I}_{S'}(1)) = 0$ for all $S' \subsetneq S$ and $x > 3$, we may find $A \supset \{P, Q\}$ with $\sharp(A) = x - 1$ and $h^1(\mathcal{I}_A(1)) = 0$. Since $x \leq k+1$, there is a minimal integer $i \in \{2, \ldots, k\}$ such that $\mathrm{SF}_S(i-1) = \mathrm{SF}_S(i)$. By [CS16, Proposition 2.5] for every minimally dependent $S' \subseteq S$ with respect to the line bundle $\mathcal{O}(\{1, \ldots, i\})$ we have $\sharp(\pi_i(S')) = 1$. Since $i \geq 2$ and $P \neq Q$, we have $h^1(\mathcal{I}_{\{P,Q\}}(\{1, \ldots, i\})) = 0$. Hence we may find a minimally dependent set containing $\{P, Q\}$. Thus $a_i = b_i$. Take any $C = (c_1, \ldots, c_k) \in S \setminus \{P, Q\}$. If $c_1 \neq a_1$ we may take minimally dependent $S'' \subseteq S$ with respect to the line bundle $\mathcal{O}(\{1, \ldots, i-1\})$ containing $\{P, C\}$ and hence $c_i = a_i$. If $c_1 = a_1$ we may take minimally dependent $S'' \subseteq S$ with respect to the line bundle $\mathcal{O}(\{1, \ldots, i-1\})$ containing $\{Q, C\}$ and hence $c_i = b_i = a_i$. Thus $\pi_i(S) = \{a_i\}$. The lemma is now proved when $x = k+1$, by setting $E = \{i\}$.

If $x < k+1$, we apply the same proof to the projection $\pi_{\mathbf{u}}(X)$ where $\mathbf{u} = \{1, \ldots, k\} \setminus \{i\}$ and conclude by descending induction on the integer $k+1-x$. $\square$

Lemma 4.5. *Fix a finite set $S \subset Y$ such that $x := \sharp(S) \leq k + n_1$, $\langle \pi_1(S) \rangle = \mathbb{P}^{n_1}$, $h^1(\mathcal{I}_S(1)) > 0$ and $h^1(\mathcal{I}_{S'}(1)) = 0$ for each $S' \subsetneq S$. Then there is $E \subset \{2, \ldots, k\}$ such that $\sharp(E) = k + 2 - x$ and $\sharp(\pi_i(S)) = 1$ for all $i \in E$.*

Proof. Take $n_1 + 1$ points $A_1, \ldots, A_{n_i+1}$, say $A_j = (a_{j,1}, \ldots, a_{j,k})$, $1 \leq j \leq n_1 + 1$, with the property that the set $\{a_{1,1}, \ldots, a_{n_1+1,1}\}$ spans $\mathbb{P}^{n_1}$. Then just repeat the proof of Lemma 4.4. $\square$

Now, with the aid of Proposition 2.4, we are ready to prove the following theorem where we explicit a sufficient condition for a non-redundant decomposition to be minimal and unique.

Theorem 4.6. *Fix an element $T \in \mathbb{P}^M$ and a non-redundant decomposition $S \subset Y$ of T and let $\sharp(S) = r$. If $m := \max\{n_1, \ldots, n_k\}$, then*
(a) If $2r \leq k + m$, then the rank of T is r.
(b) If moreover $2r < k + m$, then $\mathcal{S}(T) = \{S\}$, i.e. T is identifiable.

Proof. By permuting the factors of Y we may assume that $n_1 = m$. The assumptions on T imply in particular that $\nu(S)$ is linearly independent, i.e. $h^1(\mathcal{I}_S(1)) = 0$.

Let $S' \subset Y$ be a minimal decomposition of T such that $\sharp S' \leq r$ and $S' \neq S$. Then, by Proposition 2.4, $h^1(\mathcal{I}_{S \cup S'}(1)) > 0$.

Let $\tilde{S} \subseteq S \cup S'$ be a minimal subset of $S \cup S'$ containing S and with $h^1(\mathcal{I}_{\tilde{S}}(1)) > 0$. Since $\tilde{S} \supseteq S$, and X is non-degenerate, the set $\pi_1(\tilde{S})$ spans $\mathbb{P}^{n_1}$. By Lemma 4.5 we have $\sharp(\tilde{S}) \geq k + n_i + 1$ and hence $\sharp(S') + r \geq k + m + 1$.

If $2r = k + m$ we get $\sharp(S') \geq r$. This proves that T has rank r. If $2r < k + m$ we get a contradiction. $\qquad\square$

Corollary 4.7. *Set $m := \max\{n_1, \ldots, n_k\}$. Take a non-degenerate $T \in \mathbb{P}^M$. If $2\mathrm{rk}(T) < k + m$, then T is identifiable.*

Proof. Take $S \in \mathcal{S}(T)$. Since $S \subset Y$ and $T \in \langle \nu(S) \rangle$, then X chi e' X? is the minimal multiprojective space containing S. Then apply part (b) of Theorem 4.6. $\qquad\square$

Remark 4.8. *One cannot give a result on the identifiability of tensors without comparing it with the celebrated Kruskal's bound ([K77]), which is known to be sharp ([SB00], [D13]).*

Our condition on the decomposition S of the tensor T is weaker than the condition imposed by Kruskal, which requires to compute the span of any subset of $\pi_i(S)$ up to cardinality $n_i + 1$ (in order to determine the Kruskal's rank), while we only need to check that $\pi_i(S)$ generates $\mathbb{P}^{n_i}$. Since the Kruskal's rank of the projections of S in principle can be even 1 (when π_i is not injective), for low values of the rank our result determines the identifiability of T under weaker assumptions.

Of course, as our assumptions are weaker than Kruskal's ones, we cannot give applications outside Kruskal's numerical range. There are few cases in which the numerical range of application of our result matches with Kruskal's range. One of them e.g. is given by tensors of type $3 \times 2 \times 2 \times 2$.

Next, we show that under some condition on the decomposition S of a tensor T, we can prove that any other decomposition S' of cardinality smaller or equal than $\sharp S$ must have projections in special position.

Definition 4.9. *A zero-dimensional scheme $Z \subset Y$ is said to be* curvilinear *if each connected component of Z has embedding dimension 1, i.e. there exists a smooth curve in Y containing Z.*

We will indeed prove the result even when S' is non-reduced, provided that S' is *curvilinear*.

Notice that, of course, any *reduced* finite subset of Y is curvilinear.

Theorem 4.10. *Set $m' := \min\{n_1, \ldots, n_k\}$. Fix integers $s > 0$, $0 < x < k$ such that $(m' + 1)^{k-x} \geq r$. Let $B \subset Y$ be zero-dimensional curvilinear scheme and $S \subset Y$ be a finite set with different coordinates such that $\sharp(S) = r$, $\deg(B) = x$ and $h^1(\mathcal{I}_S(\mathbf{u})) = 0$ for any subset $\mathbf{u} \subset \{1, \ldots, k\}$ of cardinality $k - x$.*

Assume that each projection π_i is an isomorphism when restricted to B (when B is reduced this is equivalent to say that also B has different coordinates).

Then, $h^1(\mathcal{I}_{S \cup B}(1)) = 0$.

Proof. Set $Z := S \cup B$ and assume $h^1(\mathcal{I}_Z(1)) > 0$. Taking $S \setminus (B \cap S)$ instead of S we reduce to the case $S \cap B = \emptyset$. Fix $\mathbf{u} = \{2, \ldots, k\}$ so that $\pi_{\mathbf{u}}$ is the projection to the last $k - 1$ factors and write $Y_{\mathbf{u}} = \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_k}$

(a) First assume $k = 2$ and hence $x = 1$, $\mathbf{u} = \{2\}$ and $\sharp(S) \leq m + 1$. Write $B = \{O\}$ with $O = (O_1, O_2)$. Take a general hyperplane $H \subset \mathbb{P}^{n_1}$ such that $O_1 \in H_1$ and set $W_1 := H_1 \times \mathbb{P}^{n_2}$. Since H_1 is a general hyperplane containing O_1 and S has different coordinates, either $W_1 \cap S = \emptyset$ or $W_1 \cap S$ is the unique point whose image by π_1 is O_1. Hence $\sharp(W_1 \cap Z) \leq 2$.

Since $\mathcal{O}(1)$ is very ample, then $h^1(\mathcal{I}_{W_1 \cap Z}(1)) = 0$, so $h^1(W_1, \mathcal{I}_{W_1 \cap Z, W_1}(1)) = 0$. The residual exact sequence of W_1 in Y gives $h^1(\mathcal{I}_{Z \setminus (Z \cap W_1)}(\mathbf{u})) > 0$. Since $Z \setminus (Z \cap W_1) \subseteq S$, we get $h^1(\mathcal{I}_S(\mathbf{u})) > 0$, a contradiction.

(b) Now assume $k \geq 3$ and assume that the claim holds for multiprojective spaces with $k - 1$ factors. Write $\{P(1), \dots, P(y)\}$ for the points of the reduced set B_{red}, where $1 \leq y \leq x$. For any $i \in \{1, \dots, k\}$ let $\mathcal{A}_i$ be the set of all pairs $(P(\mathbf{u}), Q(\mathbf{u})) \in B_{\mathrm{red}} \times S$ such that all the coordinates of $P(\mathbf{u})$ and $Q(\mathbf{u})$ are the same, except the i-th one (which is different, because $S \cap B = \emptyset$). Assume the existence of $(P(\mathbf{u}), Q(\mathbf{v})) \in \mathcal{A}_i$ and $(P(\mathbf{u}), Q(\mathbf{v})) \in \mathcal{A}_j$. Since $S \cap B = \emptyset$, $Q(\mathbf{u})$ (resp. $Q(\mathbf{v})$) has all coordinates equal to the one of $P(\mathbf{u})$, except the i-th (resp. j-th) one, which is different. Since $k \geq 3$ and S has different coordinates, we get $Q(\mathbf{u}) = Q(\mathbf{v})$. Hence $i = j$. Since $y \leq x < k$, there is $h \in \{1, \dots, k\}$ with $\mathcal{A}_h = \emptyset$. Permuting the factors of Y we may assume $h = 1$, i.e. $\mathcal{A}_1 = \emptyset$. This is equivalent to the injectivity of the map $\pi_{1|S \cup B_{\mathrm{red}}}$. Since $\pi_{1|B}$ is an embedding, we get that $\pi_{1|Z}$ is an embedding.

Fix $P \in B_{\mathrm{red}}$ and call $P_1, \dots, P_k$ its components. Take a general hyperplane H_1 of $\mathbb{P}^{n_1}$ containing P_1. Since $\pi_1(B)$ is curvilinear and H_1 is general, we have $\pi_1(B) \cap H_1 = \{P_1\}$ (scheme-theoretic intersection). Set $W_1 := H_1 \times Y_{\mathbf{u}}$. W_1 is an element of $|\mathcal{O}(\{1\})|$. Since $\pi_{1|B}$ is an embedding and $\pi_1(B) \cap H_1 = \{P_1\}$ (scheme-theoretic intersection), we have $B \cap W_1 = \{P\}$ (scheme-theoretic intersection). Since H_1 is general and S has different coordinates, either $W_1 \cap S = \emptyset$ or $W_1 \cap S$ is the unique point of S with P_1 as its first coordinate. Hence $Z_1 := Z \cap W_1$ is always reduced, it contains P and at most another point, which is in S. Set $Z_{\mathbf{u}} := \mathrm{Res}_{W_1}(Z)$, $B_1 := \mathrm{Res}_{W_1}(B)$ and $S_1 := \mathrm{Res}_{W_1}(S) = S \setminus (S \cap W_1)$. We have $Z_{\mathbf{u}} = B_1 \cup S_1$, $\deg(B_1) = x - 1$, $B_1 \subset B$, $S_1 \subseteq S$. Since $\sharp(Z_1) \leq 2$ and $\mathcal{O}(1)$ is very ample, we have $h^1(\mathcal{I}_{Z_1}(1)) = 0$. Hence $h^1(W_1, \mathcal{I}_{Z_1}(1)) = 0$. The residual exact sequence of W_1 in X gives $h^1(\mathcal{I}_{Z_{\mathbf{u}}}(\mathbf{u})) > 0$. Since $\pi_{\mathbf{u}}1|Z$, is an embedding, then $\pi_{\mathbf{u}|Z_{\mathbf{u}}}$ is an embedding. Hence $h^1(\mathcal{I}_{Z_{\mathbf{u}}}(\mathbf{u})) = h^1(Y_{\mathbf{u}}, \mathcal{I}_{Z_{\mathbf{u}}}(1))$. The inductive assumption on the number of factors of the multiprojective space gives $h^1(Y_{\mathbf{u}}, \mathcal{I}_{Z_{\mathbf{u}}}(1)) = 0$, a contradiction. $\qquad\square$

Mixing the previous theorem with Proposition 2.4, we get the following.

Corollary 4.11. *Set $m' := \min\{n_1, \dots, n_k\}$ and fix integers $s > 0$, $0 < x < k$ such that $(m' + 1)^{k-x} \geq r$. Take a tensor $T \in \mathbb{P}^M$ with a non-redundant decomposition $S \subset Y$ such that $\sharp S = r$ and S has different coordinates. Assume $h^1(\mathcal{I}_S(\mathbf{u})) = 0$ for any subset $\mathbf{u} \subset \{1, \dots, k\}$ of cardinality $k - x$.*

Then any other non-redundant decomposition $S' \subset X$ of T of cardinality $\leq x$ cannot have different coordinates.

The statement of Theorem 3.1 cannot produce corollaries on the identifiability, because the assumptions do not include the case $\sharp A = \sharp B$.

We can however modify the proof of Theorem 3.1, adding some assumptions on the decomposition S, which produces results which bound different the decompositions of a tensor T.

Definition 4.12. *Let $W \subset \mathbb{P}^r$ be an integral and non-degenerate projective variety. A finite set (resp. zero-dimensional scheme) $S \subset W$ is said to be set-theoretically quasi-general (resp. scheme-theoretically quasi-general) if the set-theoretic (resp. scheme-theoretic) intersection $W \cap \langle S \rangle$ is set-theoretically (resp. scheme-theoretical) equal to S.*

Lemma 4.13. *Let $W \subset \mathbb{P}^M$ be an integral and non-degenerate variety of dimension n. Fix a general reduced subset $S \subset W$ with $\sharp S \leq M - n - 1$. Then S is the scheme-theoretic base locus of the linear system on Y chi e' Y? induced by $H^0(\mathcal{I}_S(1))$, i.e S is scheme-theoretically quasi-general.*

Proof. Let $L \subset \mathbb{P}^M$ be a general linear space of dimension $M - r - 1$. By Bertini's theorem the scheme $L \cap W$ is a finite set of $\deg(Y)$ points, moreover the set $L \cap W$ is in linearly general position in L. Hence for any $S \subset L \cap W$ with $\sharp S \leq M - r - 1$, the restriction of $U := H^0(\mathcal{I}_S(1))$ to $L \cap W$ has S as its set-theoretic base locus. Since L is a linear space, the restriction of U to W has base locus contained in $L \cap W$. Thus S is the base locus of the restriction of U to W. Since W is integral and non-degenerate, a general subset $A \subset W$ with cardinality at most $M - n$ spans a general subspace of $\mathbb{P}^M$ with dimension $\sharp(A) - 1$. The claim follows. $\qquad\square$

Proposition 4.14. *Fix a partition $E \sqcup F = \{1, \dots, k\}$ with $E = \{a_1, \dots, a_{n-h}\}$ and $F = \{b_1, \dots, b_h\}$ for some $0 < h < k$, and a positive integer $c < M_F = \prod_{i=1}^{h}(n_{b_i} + 1)$. Let $Z \subset Y$ be a zero-dimensional scheme such that $\sharp(Z) = c$, $h^1(\mathcal{I}_Z(E)) = 0$ and $h^1(\mathcal{I}_Z(F)) = 0$. Assume that $\pi_F(Z)$ is set-theoretically (resp. scheme-theoretically) quasi-general. Take any $T \in \langle \nu(Z) \rangle$ such that $T \notin \langle \nu(Z') \rangle$ for any $Z' \subsetneq Z$.*

If S is a finite set (resp. zero-dimensional scheme) such that $S \neq Z$, $\deg(S) \leq b$ and $T \in \langle \nu(S) \rangle$, then $\deg(S) = b$ and $\pi_F(S) = \pi_F(Z)$.

Proof. By Theorem 3.1 it is sufficient to do the case $\deg(S) = c$ and $h^0(\mathcal{I}_S(F)) = h^0(\mathcal{O}_Y(F)) - b$. In particular π_F induces an embedding of S into $Y_F == \mathbb{P}^{b_1} \times \cdots \times \mathbb{P}^{b_k}$.

The proof of Theorem 3.1 works verbatim, if there is $H \in |\mathcal{O}_Y(F)|$ containing S, but not containing Z. Since $h^0(\mathcal{I}_S(F)) = h^0(\mathcal{I}_Z(F))$, this is equivalent to require that $H^0(\mathcal{I}_S(F)) \neq H^0(\mathcal{I}_Z(F))$, i.e. that $\pi_F(S)$ is not contained in the base locus of $\pi_F(Z)$. Since $\pi_F(Z)$ is the set-theoretic (resp. scheme-theoretic) quasi-general, the base locus of $|\mathcal{O}_{Y_F}(1)|$ is $\pi_F(Z)$. Thus we get $\pi_F(S) = \pi_F(Z)$. $\qquad\square$

Corollary 4.15. *For each $i \in \{1, \dots, k\}$ fix a set $F_i \subset \{1, \dots, k\}$ such that $i \in F_i$ and set $E_i := \{1, \dots, k\} \setminus F_i$. Let $S \subset Y$ be a finite set such that $\sharp(S) = r$, where:*

$$r < \prod_{j \in F_i}(n_j + 1) \quad and \quad r \leq \prod_{h \in E_i}(n_h + 1).$$

Take any $T \in \langle \nu(S) \rangle$, such that $T \notin \langle \nu(S') \rangle$ for any $S' \subsetneq S$. Assume that:

$$h^0(\mathcal{I}_S(F_i)) = h^0(\mathcal{O}_Y(F_i)) - r \quad and \quad h^0(\mathcal{I}_S(E_i)) = h^0(\mathcal{O}_Y(E_i)) - r$$

for all $i = \{1, \dots, k\}$. Assume moreover that each $\pi_{F_i}(S)$ is set-theoretically (resp. scheme-theoretical) quasi-general.

If $S' \neq S$ is a zero-dimensional subscheme of degree $\leq r$ such that $\langle \nu(S') \rangle$ contains T, then S' is a finite set, $\sharp(S') = r$ and $\pi_i(S') = \pi_i(S)$ for all i.

Notice that (unfortunately) we are not able to conclude that $S = S'$: they can be different even if $\pi_i(S') = \pi_i(S)$ for all i. Namely the points can differ by a rearrangement of the coordinates.

Proof. Since $h^0(\mathcal{I}_S(F_i)) = h^0(\mathcal{O}_Y(F_i)) - \sharp(S)$, each $\pi_{F_i|S}$ is injective. Assume that S' exists. By Proposition 4.14 applied to F_i, S' is a finite set with $\sharp(S') = r$, $\pi_{F_i|S'}$ injective and $\pi_{F_i}(S') = \pi_{F_i}(S)$. Thus $\pi_i(S') = \pi_i(S)$ for all i. $\qquad\square$

Remark 4.16. *Corollary 4.15 does not provide the identifiability of a tensor T: it simply bounds strictly the locus where different decompositions of the same tensor T could lie. We observe that, on the other hand, the numerical range of application of Corollary 4.15 is wider than the range of Kruskal's criterion of identifiability.*

Just to give an example, consider tensors of type $3 \times 3 \times 6$, corresponding (mod scalars) to points in the space $\mathbb{P}^{53}$ which contains the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^5$. Kruskal's criterion for the identifiability applies only when the rank r is bounded by $r \leq (3 + 3 + 6 - 2)/2 = 5$. Our Corollary 4.15 applies, taking $F_1 = F_2 = \{1, 2\}$, $F_3 = \{3\}$ and checking the geometric assumptions, even for $r = 6$.

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