The quaternionic Gauss-Lucas Theorem

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Abstract

The classic Gauss-Lucas Theorem for complex polynomials of degree $d \geq 2$ has a natural reformulation over quaternions, obtained via rotation around the real axis. We prove that such a reformulation is true only for $d = 2$. We present a new quaternionic version of the Gauss-Lucas Theorem valid for all $d \geq 2$, together with some consequences.

1 Introduction

Let p be a complex polynomial of degree $d \geq 2$ and let p' be its derivative. The Gauss-Lucas Theorem asserts that the zero set of p' is contained in the convex hull $\mathcal{K}(p)$ of the zero set of p. The classic proof uses the logarithmic derivative of p and it strongly depends on the commutativity of $\mathbb{C}$.

This note deals with a quaternionic version of such a classic result. We refer the reader to [3] for the notions and properties concerning the algebra $\mathbb{H}$ of quaternions we need here. The ring $\mathbb{H}[X]$ of quaternionic polynomials is defined by fixing the position of the coefficients w.r.t. the indeterminate X (e.g. on the right) and by imposing commutativity of X with the coefficients when two polynomials are multiplied together (see e.g. [5, §16]). Given two polynomials $P, Q \in \mathbb{H}[X]$, let $P \cdot Q$ denote the product obtained in this way. If P has real coefficients, then $(P \cdot Q)(x) = P(x)Q(x)$. In general, a direct computation (see [5, §16.3]) shows that if $P(x) \neq 0$, then

$$(P \cdot Q)(x) = P(x)Q(P(x)^{-1}xP(x)), \tag{1}$$

while $(P \cdot Q)(x) = 0$ if $P(x) = 0$. In this setting, a (left) root of a polynomial $P(X) = \sum_{h=0}^{d} X^h a_h$ is an element $x \in \mathbb{H}$ such that $P(x) = \sum_{h=0}^{d} x^h a_h = 0$.

Given $P(X) = \sum_{k=0}^{d} X^k a_k \in \mathbb{H}[X]$, consider the polynomial $P^c(X) = \sum_{k=0}^{d} X^k \bar{a}_k$ and the *normal polynomial* $N(P) = P \cdot P^c = P^c \cdot P$. Since $N(P)$ has real coefficients, it can be identified with a polynomial in $\mathbb{R}[X] \subset \mathbb{C}[X]$. We recall that a subset A of $\mathbb{H}$ is called *circular* if, for each $x \in A$, A contains the whole set (a 2-sphere if $x \notin \mathbb{R}$, a point if $x \in \mathbb{R}$)

$$\mathbb{S}_x = \{pxp^{-1} \in \mathbb{H} \mid p \in \mathbb{H}^*\}, \tag{2}$$

where $\mathbb{H}^* := \mathbb{H} \setminus \{0\}$. In particular, for any imaginary unit $I \in \mathbb{H}$, $\mathbb{S}_I = \mathbb{S}$ is the 2-sphere of all imaginary units in $\mathbb{H}$. For every subset B of $\mathbb{H}$ we define its *circularization* as the set $\bigcup_{x \in B} \mathbb{S}_x$. It is well-known ([3, §3.3]) that the zero set $V(N(P)) \subset \mathbb{H}$ of the normal polynomial is the circularization of the zero set $V(P)$, which consists of isolated points or isolated 2-spheres of the form (2) if $P \neq 0$.

Let the degree d of P be at least 2 and let $P'(X) = \sum_{k=1}^{d} X^{k-1} k a_k$ be the derivative of P. It is known (see e.g. [2]) that the Gauss-Lucas Theorem does not hold directly for quaternionic polynomials. For example, the polynomial $P(X) = (X-i)\cdot(X-j) = X^2 - X(i+j) + k$ has zero set $V(P) = \{i\}$, while P' vanishes at $x = (i+j)/2$.

Since the zero set $V(P)$ of P is contained in the set $V(N(P))$, a natural reformulation in $\mathbb{H}[X]$ of the classic Gauss-Lucas Theorem is the following: $V(N(P')) \subset \mathcal{K}(N(P))$ or equivalently

$$V(P') \subset \mathcal{K}(N(P)), \tag{3}$$

where $\mathcal{K}(N(P))$ denotes the convex hull of $V(N(P))$ in $\mathbb{H}$. This set is equal to the circularization of the convex hull of the zero set of $N(P)$ viewed as a polynomial in $\mathbb{C}[X] \subset \mathbb{H}[X]$. Recently two proofs of the above inclusion (3) were presented in [7, 2].

Our next two propositions prove that inclusion (3) is correct in its full generality only when $d = 2$.

2 Gauss-Lucas polynomials

Definition 1. *Given a polynomial $P \in \mathbb{H}[X]$ of degree $d \geq 2$ we say that P is a* Gauss-Lucas polynomial *if P satisfies* (3).

Proposition 2. *If P is a polynomial in $\mathbb{H}[X]$ of degree 2, then $V(N(P)) = \mathbb{S}_{x_1} \cup \mathbb{S}_{x_2}$ for some $x_1, x_2 \in \mathbb{H}$ (possibly with $\mathbb{S}_{x_1} = \mathbb{S}_{x_2}$) and*

$$V(P') \subset \bigcup_{y_1 \in \mathbb{S}_{x_1}, y_2 \in \mathbb{S}_{x_2}} \left\{ \frac{y_1 + y_2}{2} \right\}.$$

In particular every polynomial $P \in \mathbb{H}[X]$ of degree 2 is a Gauss-Lucas polynomial.

Proof. Let $P(X) = X^2 a_2 + X a_1 + a_0 \in \mathbb{H}[X]$ with $a_2 \neq 0$. Since $P \cdot a_2^{-1} = P a_2^{-1}$, $(P \cdot a_2^{-1})' = P' \cdot a_2^{-1} = P' a_2^{-1}$, we can assume $a_2 = 1$. Consequently, $P(X) = (X - x_1)\cdot(X - x_2) = X^2 - X(x_1 + x_2) + x_1 x_2$ for some $x_1, x_2 \in \mathbb{H}$. Then $x_1 \in V(P)$ and $\bar{x}_2 \in V(P^c)$, since $P^c(X) = (X - \bar{x}_2)\cdot(X - \bar{x}_1)$. Therefore $x_1, x_2 \in V(N(P))$. On the other hand $V(P') = \{(x_1 + x_2)/2\}$ as desired. □

Remark 3. Let $P(X) = \sum_{k=0}^{d} X^k a_k \in \mathbb{H}[X]$ of degree $d \geq 2$ and let $Q := P \cdot a_d^{-1}$ be the corresponding monic polynomial. Since $V(P) = V(Q)$ and $V(P') = V(Q')$, P is a Gauss-Lucas polynomial if and only if Q is.

Proposition 4. *Let $P \in \mathbb{H}[X]$ of degree $d \geq 3$. Suppose that $N(P)(X) = X^{2e} \cdot (X^2 + 1)^{d-e}$ for some $e < d$ and that $N(P')(X) = \sum_{k=0}^{2d-2} X^k b_k$ contains a unique monomial of odd degree, that is, $b_k \neq 0$ for a unique odd k. Then P is not a Gauss-Lucas polynomial.*

Proof. Since $V(N(P)) \subset \{0\} \cup \mathbb{S}$, $\mathcal{K}(N(P)) \subset \mathrm{Im}(\mathbb{H}) = \{x \in \mathbb{H} \mid \mathrm{Re}(x) = 0\}$. Then it suffices to show that $N(P')$ has at least one root in $\mathbb{H} \setminus \mathrm{Im}(\mathbb{H})$. Let $G, L \in \mathbb{R}[X]$ be the unique real polynomials such that $G(t) + iL(t) = N(P')(it)$ for every $t \in \mathbb{R}$. Since $N(P')$ contains a unique monomial of odd degree, say $X^{2\ell+1} b_{2\ell+1}$, then $L(t) = (-1)^\ell b_{2\ell+1} t^{2\ell+1}$ and hence $V(N(P')) \cap i\mathbb{R} \subset \{0\}$. Being $V(N(P'))$ a circular set, it holds $V(N(P')) \cap \mathrm{Im}(\mathbb{H}) \subset \{0\}$. Since $N(P')$ contains at least two monomials, namely those of degrees $2\ell + 1$ and $2d - 2$, we infer that it must have a nonzero root. Therefore $V(N(P')) \not\subset \{0\}$ and hence $V(N(P')) \not\subset \mathrm{Im}(\mathbb{H})$, as desired.

□

Corollary 5. *Let $d \geq 3$ and let*

$$P(X) = X^{d-3} \cdot (X - i) \cdot (X - j) \cdot (X - k).$$

Then $N(P)(X) = X^{2d-6} \cdot (X^2 + 1)^3$ and $N(P')$ contains a unique monomial of odd degree, namely $-4X^{2d-5}$. In particular P is not a Gauss-Lucas polynomial.

Proof. By a direct computation we obtain:

$$P(X) = X^d - X^{d-1}(i + j + k) + X^{d-2}(i - j + k) + X^{d-3}, \tag{4}$$

$$P'(X) = dX^{d-1} - (d - 1)X^{d-2}(i + j + k) + (d - 2)X^{d-3}(i - j + k) + (d - 3)X^{d-4}, \tag{5}$$

$$N(P')(X) = d^2X^{2d-2} + 3(d - 1)^2X^{2d-4} - 4X^{2d-5} + 3(d - 2)^2X^{2d-6} + (d - 3)^2X^{2d-8}. \tag{6}$$

Proposition 4 implies the thesis. □

Let $I \in \mathbb{S}$ and let $\mathbb{C}_I \subset \mathbb{H}$ be the complex plane generated by 1 and I. Given a polynomial $P \in \mathbb{H}[X]$, we will denote by $P_I : \mathbb{C}_I \to \mathbb{H}$ the restriction of P to $\mathbb{C}_I$. *If P_I is not constant, we will denote by $\mathcal{K}_{\mathbb{C}_I}(P)$ the convex hull in the complex plane $\mathbb{C}_I$ of the zero set $V(P_I) = V(P) \cap \mathbb{C}_I$. If P_I is constant, we set $\mathcal{K}_{\mathbb{C}_I}(P) = \mathbb{C}_I$.*

If $\mathbb{C}_I$ contains every coefficient of $P \in \mathbb{H}[X]$, then we say that P is a *$\mathbb{C}_I$-polynomial.*

Proposition 6. *The following holds:*

(1) Every $\mathbb{C}_I$-polynomial of degree ≥ 2 is a Gauss-Lucas polynomial.

(2) Let $d \geq 3$, let $\mathbb{H}_d[X] = \{P \in \mathbb{H}[X] \mid \deg(P) = d\}$ and let $E_d[X]$ be the set of all elements of $\mathbb{H}_d[X]$ that are not Gauss-Lucas polynomials. Identify each $P(X) = \sum_{k=0}^{d} X^k a_k$ in $\mathbb{H}_d[X]$ with $(a_0, \ldots, a_d) \in \mathbb{H}^d \times \mathbb{H}^* \subset \mathbb{R}^{4d+4}$ and endow $\mathbb{H}_d[X]$ with the relative Euclidean topology. Then $E_d[X]$ is a nonempty open subset of $\mathbb{H}_d[X]$. Moreover $E_d[X]$ is not dense in $\mathbb{H}_d[X]$, being $X^d - 1$ an interior point of its complement.

Proof. If P is a $\mathbb{C}_I$-polynomial, then P_I can be identified with an element of $\mathbb{C}_I[X]$. Consequently, the classic Gauss-Lucas Theorem gives $V(P') \cap \mathbb{C}_I = V(P'_I) \subset \mathcal{K}_{\mathbb{C}_I}(P)$. The zero set of the $\mathbb{C}_I$-polynomial P' has a particular structure (see [3, Lemma 3.2]): $V(P')$ is the union of $V(P') \cap \mathbb{C}_I$ with the set of spheres $\mathbb{S}_x$ such that $x, \bar{x} \in V(P'_I)$. It follows that

$$V(P') \subset \mathcal{K}(N(P)).$$

This proves (1).

Now we prove (2). By Corollary 5 we know that $E_d[X] \neq \emptyset$. If $P \in E_d[X]$, then $V(N(P')) \not\subset \mathcal{K}(N(P))$. $N(P)$ and $N(P')$ are polynomials with real coefficients. Since the roots of $N(P)$ and of $N(P')$ depend continuously on the coefficients of P and $\mathcal{K}(N(P))$ is closed in $\mathbb{H}$, for every $Q \in \mathbb{H}_d[X]$ sufficiently close to P, $V(N(Q'))$ is not contained in $\mathcal{K}(N(Q))$, that is $Q \in E_d[X]$.

To prove the last statement, observe that $P(X) = X^d - 1$ is not in $E_d[X]$ from part (1). Since $V(P') = V(N(P')) = \{0\}$ is contained in the interior of the set $\mathcal{K}(N(P))$, for every $Q \in \mathbb{H}_d[X]$ sufficiently close to P, $V(Q')$ is still contained in $\mathcal{K}(N(Q))$. □

3 A quaternionic Gauss-Lucas Theorem

Let $P(X) = \sum_{k=0}^{d} X^k a_k \in \mathbb{H}[X]$ of degree $d \geq 2$. For every $I \in \mathbb{S}$, let $\pi_I : \mathbb{H} \to \mathbb{H}$ be the orthogonal projection onto $\mathbb{C}_I$ and $\pi_I^{\perp} = id - \pi_I$. Let $P^I(X) := \sum_{k=1}^{d} X^k a_{k,I}$ be the $\mathbb{C}_I$-polynomial with coefficients $a_{k,I} := \pi_I(a_k)$.

Definition 7. *We define the* Gauss-Lucas snail of P *as the following subset* $\mathfrak{sn}(P)$ *of* $\mathbb{H}$*:*

$$\mathfrak{sn}(P) := \bigcup_{I \in \mathbb{S}} \mathcal{K}_{\mathbb{C}_I}(P^I).$$

Our quaternionic version of the Gauss-Lucas Theorem reads as follows.

Theorem 8. *For every polynomial* $P \in \mathbb{H}[X]$ *of degree* ≥ 2*,*

$$V(P') \subset \mathfrak{sn}(P). \tag{7}$$

Proof. Let $P(X) = \sum_{k=0}^{d} X^k a_k$ in $\mathbb{H}_d[X]$ with $d \geq 2$. We can decompose the restriction of P to $\mathbb{C}_I$ as $P_I = \pi_I \circ P_I + \pi_I^\perp \circ P_I = P^I{}_{|\mathbb{C}_I} + \pi_I^\perp \circ P_I$. If $x \in \mathbb{C}_I$, then $P^I(x) \in \mathbb{C}_I$ while $(\pi_I^\perp \circ P_I)(x) \in \mathbb{C}_I^\perp$. The same decomposition holds for P'. This implies that $V(P') \cap \mathbb{C}_I \subset V((P^I)') \cap \mathbb{C}_I$. The classic Gauss-Lucas Theorem applied to P^I on $\mathbb{C}_I$ gives $V(P') \cap \mathbb{C}_I \subset \mathcal{K}_{\mathbb{C}_I}(P^I)$. Since $V(P') = \bigcup_{I \in \mathbb{S}} (V(P') \cap \mathbb{C}_I)$, the inclusion (7) is proved. □

If P is monic Theorem 8 has the following equivalent formulation: *For every monic polynomial* $P \in \mathbb{H}[X]$ *of degree* ≥ 2*, it holds*

$$\mathfrak{sn}(P') \subset \mathfrak{sn}(P). \tag{8}$$

Remark 9. If P is a nonconstant monic polynomial in $\mathbb{H}[X]$, then two properties hold:

(a) $\mathcal{K}_{\mathbb{C}_I}(P^I)$ is a compact subset of $\mathbb{C}_I$ for every $I \in \mathbb{S}$.

(b) $\mathcal{K}_{\mathbb{C}_I}(P^I)$ depends continuously on I.

Let $I \in \mathbb{S}$. Since P is monic, also P^I is a monic, nonconstant polynomial and then $\mathcal{K}_{\mathbb{C}_I}(P^I)$ is a compact subset of $\mathbb{C}_I$. This prove property (a). To see that (b) holds, one can apply the Continuity theorem for monic polynomials (see e.g. [6, Theorem 1.3.1]). The roots of P^I depend continuously on the coefficients of P^I, which in turn depend continuously on I. Therefore the convex hull $\mathcal{K}_{\mathbb{C}_I}(P^I)$ depends continuously on I. Observe that (a) and (b) can not hold for polynomials that are not monic. For example, let $P(X) = X^2 i$. Then, given $I = \alpha_1 i + \alpha_2 j + \alpha_3 k \in \mathbb{S}$, $\mathcal{K}_{\mathbb{C}_I}(P^I) = \{0\}$ if $\alpha_1 \neq 0$ and $\mathcal{K}_{\mathbb{C}_I}(P^I) = \mathbb{C}_I$ if $\alpha_1 = 0$, since in this case P^I is constant.

Remark 10. If P is a monic polynomial in $\mathbb{H}[X]$ of degree ≥ 2, then its Gauss-Lucas snail is a closed subset of $\mathbb{H}$. To prove this fact, consider $q \in \mathbb{H} \setminus \mathfrak{sn}(P)$ and choose $I \in \mathbb{S}$ such that $q \in \mathbb{C}_I$. Write $q = \alpha + I\beta \in \mathbb{C}_I$ for some $\alpha, \beta \in \mathbb{R}$ and define $z := \alpha + i\beta \in \mathbb{C}$. Since P is monic, $\mathfrak{sn}(P) \cap \mathbb{C}_I = \mathcal{K}_{\mathbb{C}_I}(P^I)$ is a compact subset of $\mathbb{C}_I$. Moreover, $\mathcal{K}_{\mathbb{C}_I}(P^I)$ depends continuously on I, and then there exist an open neighborhood U_I of z in $\mathbb{C}$ and an open neighborhood W_I of I in $\mathbb{S}$ such that the set

$$[U_I, W_I] := \textstyle\bigcup_{J \in W_I} \{a + Jb \in \mathbb{C}_J \mid a + ib \in U_I\}$$

is an open neighborhood of q in $\bigcup_{J \in W_I} \mathbb{C}_J$, and it is disjoint from $\mathfrak{sn}(P)$. If $q \notin \mathbb{R}$ then q is an interior point of $\mathbb{H} \setminus \mathfrak{sn}(P)$, because $[U_I, W_I]$ is a neighborhood of q in $\mathbb{H}$ as well. Now assume that $q \in \mathbb{R}$. Since $\mathbb{S}$ is compact there exist $I_1, \ldots, I_n \in \mathbb{S}$ such that $\bigcup_{\ell=1}^{n} W_{I_\ell} = \mathbb{S}$. It follows that $[\bigcap_{\ell=1}^{n} U_\ell, \mathbb{S}]$ is a neighborhood of q in $\mathbb{H}$, which is disjoint from $\mathfrak{sn}(P)$. Consequently q is an interior point of $\mathbb{H} \setminus \mathfrak{sn}(P)$ also in this case. This proves that $\mathfrak{sn}(P)$ is closed in $\mathbb{H}$.

In Proposition 13 below we will show that the Gauss-Lucas snail of a monic polynomial in $\mathbb{H}[X]$ of degree ≥ 2 is also a compact subset of $\mathbb{H}$.

If all the coefficients of P are real, then $\mathfrak{sn}(P)$ is a circular set. In general, $\mathfrak{sn}(P)$ is neither closed nor bounded nor circular, as shown in the next example.

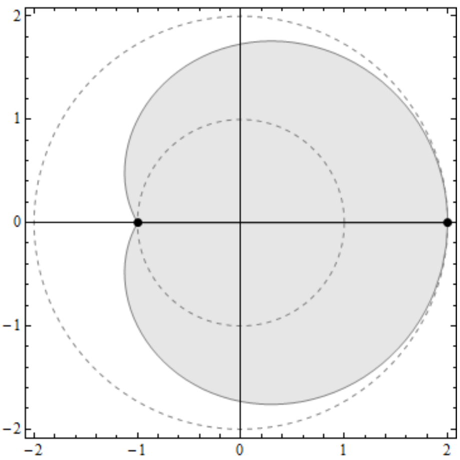

Figure 1: Cross-sections of $\mathfrak{sn}(P)$ (gray), of $V(P')$ and $\mathcal{K}(N(P))$ (dashed).

Example 11. *Let $P(X) = X^2 i + X$. Given $I = \alpha_1 i + \alpha_2 j + \alpha_3 k \in \mathbb{S}$, $P^I(x) = X^2 I\alpha_1 + X$ and then $\mathcal{K}_{\mathbb{C}_I}(P^I) = \{0\}$ if $\alpha_1 = 0$ while $\mathcal{K}_{\mathbb{C}_I}(P^I)$ is the segment from 0 to $I{\alpha_1}^{-1}$ if $\alpha_1 \neq 0$. It follows that $\mathfrak{sn}(P) = \{x_1 i + x_2 j + x_3 k \in \mathrm{Im}(\mathbb{H}) \,|\, 0 < x_1 \leq 1\} \cup \{0\}$. Finally, observe that the monic polynomial $Q(X) = -P(X) \cdot\ i = X^2 - Xi$ corresponding to P has compact Gauss-Lucas snail $\mathfrak{sn}(Q) = \{x_1 i + x_2 j + x_3 k \in \mathrm{Im}(\mathbb{H}) \,|\, (x_1 - 1/2)^2 + x_2^2 + x_3^2 \leq 1/4\}$.*

Remark 12. Even for $\mathbb{C}_I$-polynomials, the Gauss-Lucas snail of P can be strictly smaller than the circular convex hull $\mathcal{K}(N(P))$. For example, consider the $\mathbb{C}_i$-polynomial $P(X) = X^3 + 3X + 2i$, with zero sets $V(P) = \{-i, 2i\}$ and $V(P') = \mathbb{S}$. The set $\mathcal{K}(N(P))$ is the closed three-dimensional disc in $\mathrm{Im}(\mathbb{H})$, with center at the origin and radius 2. The Gauss-Lucas snail $\mathfrak{sn}(P)$ is the subset of $\mathrm{Im}(\mathbb{H})$ obtained by rotating around the i-axis the following subset of the coordinate plane $L = \{x = x_1 i + x_2 j \in \mathrm{Im}(\mathbb{H}) \,|\, x_1, x_2 \in \mathbb{R}\}$:

$$\{x = \rho\cos(\theta)i + \rho\sin(\theta)j \in L \mid 0 \leq \theta \leq \pi,\ 0 \leq \rho \leq 2\cos(\theta/3)\}.$$

Therefore $\mathfrak{sn}(P)$ is a proper subset of $\mathcal{K}(N(P))$ (the boundaries of the two sets intersect only at the point $2i$). Its boundary is obtained by rotating a curve that is part of the *limaçon trisectrix* (see Figure 1).

3.1 Estimates on the norm of the critical points

Let $p(z) = \sum_{k=0}^{d} a_k z^k$ be a complex polynomial of degree $d \geq 1$. The norm of the roots of p can be estimated making use of the norm of the coefficients $\{a_k\}_{k=0}^{d}$ of p. There are several classic results in this direction (see e.g. [6, §8.1]). For instance the estimate [6, (8.1.2)] (with $\lambda = 1, p = 2$) asserts that

$$\max_{z\in V(p)} |z| \leq |a_d|^{-1}\sqrt{\textstyle\sum_{k=0}^{d} |a_k|^2}\ . \tag{9}$$

Proposition 13. *For every monic polynomial $P \in \mathbb{H}[X]$ of degree $d \geq 2$, the Gauss-Lucas snail $\mathfrak{sn}(P)$ is a compact subset of $\mathbb{H}$.*

Proof. Since $P = \sum_{k=0}^{d} X^k a_k$ is monic, every polynomial P^I is monic. From (9) it follows that $\max_{x \in V(P^I)} |x|^2 \leq \sum_{k=0}^{d} |\pi_I(a_k)|^2 \leq \sum_{k=0}^{d} |a_k|^2$ and hence $\mathfrak{sn}(P) \subset \{x \in \mathbb{H} \,|\, |x|^2 \leq \sum_{k=0}^{d} |a_k|^2\}$ is bounded. Since $\mathfrak{sn}(P)$ is closed in $\mathbb{H}$, as seen in Remark 10, it is also a compact subset of $\mathbb{H}$. □

Define a function $C : \mathbb{H}[X] \to \mathbb{R} \cup \{+\infty\}$ as follows: $C(a) := +\infty$ if a is a quaternionic constant and

$$C(P) := |a_d|^{-1} \sqrt{\textstyle\sum_{k=0}^{d} |a_k|^2} \qquad \text{if } P(X) = \textstyle\sum_{k=0}^{d} X^k a_k \text{ with } d \geq 1 \text{ and } a_d \neq 0.$$

Proposition 14. *For every polynomial $P \in \mathbb{H}[X]$ of degree $d \geq 1$, it holds*

$$\max_{x \in V(P)} |x| \leq C(P). \tag{10}$$

Proof. We follow the lines of the proof of estimate (9) for complex polynomials given in [6]. Let $P(X) = \sum_{k=0}^{d} X^k a_k$ with $d \geq 1$ and $a_d \neq 0$. We can assume that $P(X)$ is not the monomial $X^d a_d$, since in this case the thesis is immediate. Let $b_k = |a_k a_d^{-1}|$ for every $k = 0, \ldots, d-1$. The real polynomial $h(z) = z^d - \sum_{k=0}^{d-1} b_k z^k$ has exactly one positive root ρ and is positive for real $z > \rho$ (see [6, Lemma 8.1.1]). Let $S := \sum_{k=0}^{d-1} b_k^2 = C(P)^2 - 1$. From the Cauchy-Schwartz inequality, it follows that

$$\left(\sum_{k=0}^{d-1} b_k C(P)^k \right)^2 \leq S \sum_{k=0}^{d-1} C(P)^{2k} = (C(P)^2 - 1) \frac{C(P)^{2d} - 1}{C(P)^2 - 1} < C(P)^{2d}.$$

Therefore $h(C(P)) > 0$ and then $C(P) > \rho$. Let $x \in V(P)$. It remains to prove that $|x| \leq \rho$. Since $x^d = -\sum_{k=0}^{d-1} x^k a_k a_d^{-1}$, it holds

$$|x|^d \leq \sum_{k=0}^{d-1} |x|^k \left| a_k a_d^{-1} \right| = \sum_{k=0}^{d-1} |x|^k b_k.$$

This means that $h(|x|) \leq 0$, which implies $|x| \leq \rho$. □

From Proposition 14 it follows that for every polynomial $P \in \mathbb{H}[X]$ of degree $d \geq 2$, it holds

$$\max_{x \in V(P')} |x| \leq C(P'). \tag{11}$$

Theorem 8 allows to obtain a new estimate.

Proposition 15. *Given any polynomial $P \in \mathbb{H}[X]$ of degree $d \geq 2$, it holds:*

$$\max_{x \in V(P')} |x| \leq \sup_{I \in \mathbb{S}} \{C(P^I)\}. \tag{12}$$

Proof. If $x \in V(P') \cap \mathbb{C}_I$, Theorem 8 implies that $x \in \mathcal{K}_{\mathbb{C}_I}(P^I)$. Therefore

$$\max_{x \in V(P') \cap \mathbb{C}_I} |x| \leq C(P^I) \qquad \text{for every } I \in \mathbb{S} \text{ with } V(P') \cap \mathbb{C}_I \neq \emptyset,$$

from which inequality (12) follows. □

Our estimate (12) can be strictly better than classic estimate (11), as explained below.

Remark 16. Let $d \geq 3$ and let $P(X) = X^{d-3} \cdot (X - i) \cdot (X - j) \cdot (X - k)$. Using (5), by a direct computation we obtain:

$$C(P') = d^{-1}\sqrt{8d^2 - 24d + 24}.$$

Moreover, given $I = \alpha_1 i + \alpha_2 j + \alpha_3 k \in \mathbb{S}$ for some $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ with $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$, we have $\pi_I(i+j+k) = \langle I, i+j+k\rangle I = (\alpha_1+\alpha_2+\alpha_3)I$ and $\pi_I(i-j+k) = \langle I, i-j+k\rangle I = (\alpha_1-\alpha_2+\alpha_3)I$ and hence

$$C(P^I) = \sqrt{4 + 4\alpha_1\alpha_3} \leq \sqrt{4 + 2(\alpha_1^2 + \alpha_3^2)} \leq \sqrt{6}.$$

This implies that

$$\sup_{I \in \mathbb{S}}\{C(P^I)\} \leq \sqrt{6}.$$

For every $d \geq 11$ it is easy to verify that $\sqrt{6} < C(P')$ so

$$\sup_{I \in \mathbb{S}}\{C(P^I)\} < C(P'),$$

as announced.

Remark 17. Some of the results presented here can be generalized to real alternative *-algebras, a setting in which polynomials can be defined and share many of the properties valid on the quaternions (see [4]). The polynomials given in Corollary 5 can be defined every time the algebra contains an Hamiltonian triple i, j, k. This property is equivalent to say that the algebra contains $\mathbb{H}$ as a subalgebra (see [1, §8.1]). For example, this is true for the algebra of octonions and for the Clifford algebras with signature $(0, n)$, with $n \geq 2$. Therefore in all such algebras there exist polynomials for which the zero set $V(P')$ (as a subset of the *quadratic cone*) is not included in the circularization of the convex hull of $V(N(P))$ viewed as a complex polynomial.

References

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