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The effects of recharge on flow nonuniformity and macrodispersion

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Abstract. The spatial, statistical structure of the fluid velocity field in the case of uniformly recharged heterogeneous aquifers is investigated, and the spatial covariances of the velocity field are derived. This information is necesary for an investigation of flow and macrodispersion using the Lagrangian formalism proposed by Dagan (1984). The resulting first two moments of the velocity are nonstationary. They are functions of a parameter β which characterizes the degree of flow nonuniformity and is related to the recharge. The displacement variances are computed and tested favorably using numerical simulations. Simple relations are developed which relate the transport parameters found for the case of uniform-in-the-average flows to nonuniform flows using a simple, nonlinear transformation of the travel time, based on β.

1. Theoretical Development

The effects of distributed recharge on the motion of passive solutes in heterogeneous porous media are investigated using the Lagrangian formalism suggested by *Dagan* [1989]. Our goal is to relate the kinetics of fluid particles in a spatially variable velocity field to the evolution of the concentration field over time and space. Given the spatial variability in the velocity, this goal is hard to attain, yet the Lagrangian approach, posed in a stochastic framework, offers the possibility to bypass that difficulty.

Our starting point is the specification of the concentration field associated with a single solute particle [*Dagan*, 1982, 1984, 1989]

$$\Delta C(\mathbf{x},\ t,\ \mathbf{x}_0,\ t_0)=\frac{\Delta M}{n}\ \delta[\mathbf{x}-\mathbf{X}(t,\ \mathbf{x}_0,\ t_0)] \qquad (1)$$

where ΔC denotes the incremental contribution to the concentration at time t and at the space coordinate $\mathbf{x}$ due to a particle that was released at time t_0 from $\mathbf{x}_0$. The mass of the particle is given by ΔM, and $\mathbf{X}(t;\ \mathbf{x}_0,\ t_0)$ denotes its displacement in space as a function of time (the terms in parenthesis will be omitted for brevity subsequently). Here, δ is the Dirac delta, and n is porosity. Here and subsequently, boldface letters denote vectors, capital letters denote random functions, and lower case letters denote realizations of random functions or constants.

Equation (1) is a general model and in order to make it site specific, $\mathbf{X}$ should reflect the specific flow dynamics prevailing at a particular site. Under the conditions generally met in practice, $\mathbf{X}$ cannot be accurately specified since the conductivity field is heterogeneous, and its variations cannot be mapped deterministically. An alternative is to model $\mathbf{X}$ as a random function. Such an approach allows one to treat the problems of heterogeneity and data uncertainty quantitatively and has a long, respectable tradition in the field of groundwater hydrology [see *Gelhar*, 1986; *Dagan*, 1989; *Neuman et al.*, 1987; *Graham and McLaughlin*, 1989a, b; *Sposito et al.*, 1986].

It is common for the stochastic-Lagrangian methods [*Dagan*, 1989; *Rubin*, 1990] to determine the velocity field by solving the flow equation as a function of the boundary conditions and a random function model representing the heterogeneous conductivity field. The randomness of the conductivity leads to a definition of the velocity as a space random function (SRF). The velocity SRF can then be related to the statistics of $\mathbf{X}$ through a simple kinematic relationship.

Dagan [1984] and *Rubin* [1990] reported on the spatial covariances of the velocity and the displacement $\mathbf{X}$ for the case of two-dimensional, steady flow in the horizontal plane, where the conductivity is modeled as a stationary SRF with an exponential, isotropic covariance function. *Rubin and Dagan* [1992b] and *Zhang and Neuman* [1992] reported on the velocity statistics in the case of three-dimensional, steady flow for an anisotropic, axisymmetric correlation structure of the logconductivity. *Zhang and Neuman* [1992] suggested some simple expressions for the case of three-dimensional isotropic logconductivity covariances. Recharge was not considered in these studies.

To our knowledge the effects of natural recharge on the displacement statistics have not been investigated so far using a stochastic framework, and this is the task undertaken here.

In our study, recharge is assumed to be uniformly distributed, and flow is considered at the regional scale [*Dagan*, 1986], which justifies the shallow flow approximation. Hence flow is taken as essentially horizontal, and the vertical component of the flow induced by the recharge is assumed negligible.

Toward this goal we define the fluid velocity at the space coordinate $\mathbf{x}=(x_1, x_2)$ as $U_k(\mathbf{x})=V_k(\mathbf{x})+u_k(\mathbf{x})$, $k=1$, 2, where $V_k(\mathbf{x})=\langle U_k(\mathbf{x})\rangle$, $u_k(\mathbf{x})$ is the local deviation of the velocity from its mean, and angle brackets denote the expectation operator. An alternative to a deterministic approach is to derive the velocity SRF model given the spatial

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structure of the hydraulic conductivity field by solving the flow equation coupled with Darcy's law

$$U_k(\mathbf{x}) = -\frac{1}{n} K(\mathbf{x}) \frac{\partial H(\mathbf{x})}{\partial x_k} \qquad k = 1, 2 \tag{2}$$

where H denotes the hydraulic head and K is the hydraulic conductivity. Subsequently, we model K as a lognormal SRF, that is, $K(\mathbf{x}) = \exp[Y(\mathbf{x})]$, where Y is normal. Y in this study is assumed to have a constant mean, that is, $Y = m_Y + Y'$, where m_Y is the expected value of Y and Y' is the local deviation from the mean. Using $H(\mathbf{x}) = \langle H(\mathbf{x})\rangle + h(\mathbf{x})$ where $\langle H(\mathbf{x})\rangle$ represents a deterministic trend and h stands for the local deviation, then under a perturbation expansion, truncated at first order, (2) simplifies to

$$U_k(\mathbf{x}) = \frac{1}{n} e^{m_Y}[1 + Y'(\mathbf{x})]\left[J_k(\mathbf{x}) - \frac{\partial h(\mathbf{x})}{\partial x_k}\right] \tag{3}$$

where $J_k(\mathbf{x}) = -\partial\langle H(\mathbf{x})\rangle/\partial x_k$. Equation (3) is the key to the derivation of the velocity SRF model as it relates U_k to the local variations Y and H. Previous works (*Dagan* [1982], *Gelhar and Axness* [1983], and many others) assumed J_k to be uniform. In the present study the presence of recharge precludes such an option, and J_k becomes a function of the space coordinates.

For a spatially correlated logconductivity and through (3) the velocity becomes a spatially correlated random function. The expected value of the velocity can be obtained by taking expected value over (3), yielding $V_k(\mathbf{x}) = \exp(m_Y)J_k(\mathbf{x})/n$, where $\langle Y'(\mathbf{x})\partial h(\mathbf{x})/\partial x_k\rangle$ is ignored. After defining the velocity random residual by subtracting the expected value from U_k, (3), the velocity spatial covariance tensor is obtained as

$$\frac{u_{jk}(\mathbf{x}, \mathbf{x}')}{V_j(\mathbf{x})V_k(\mathbf{x}')} = C_Y(\mathbf{x}, \mathbf{x}') - \frac{1}{J_j(\mathbf{x}')}\frac{\partial}{\partial x_j'} C_{YH}(\mathbf{x}, \mathbf{x}') - \frac{1}{J_k(\mathbf{x})}\frac{\partial}{\partial x_k} C_{YH}(\mathbf{x}', \mathbf{x}) - \frac{1}{J_j(\mathbf{x}')J_k(\mathbf{x})}\frac{\partial^2}{\partial x_j'\partial x_k}\Gamma(\mathbf{x}, \mathbf{x}'); \qquad j, k = 1, 2 \tag{4}$$

where the summation convention for repeating indices is not applicable. Equation (4) is an expansion of the relationship developed by *Dagan* [1984] for a constant J_k to the more general case of a nonuniform flows. In (4), $C_Y(\mathbf{x}, \mathbf{x}') = \langle Y'(\mathbf{x})Y'(\mathbf{x}')\rangle$ denotes the spatial covariance of the logconductivity, $C_{YH}(\mathbf{x}, \mathbf{x}') = \langle Y'(\mathbf{x})h(\mathbf{x}')\rangle$ is the log conductivity head cross covariance, and $\Gamma(\mathbf{x}, \mathbf{x}')$ is the head residuals variogram, defined as $0.5\langle[h(\mathbf{x}) - h(\mathbf{x}')]^2\rangle$.

C_{YH} and Γ depend on the Y spatial structure and the average flow conditions. *Dagan* [1985] derived explicit expressions for these functions for two-dimensional flow with a constant J_k and for a logconductivity field with an exponential, isotropic covariance, and later [*Dagan*, 1989] for three-dimensional flow conditions and anisotropic covariances. *Rubin and Dagan* [1987] derived those functions for two-dimensional steady state flow conditions and for an isotropic exponential covariance but in the presence of a uniformly distributed recharge. Under these conditions the mean head gradient cannot be considered as constant in both magnitude and orientation. Substituting the explicit expressions derived for C_{YH} [*Rubin and Dagan*, 1987, equations (16), (17), and (18)] and Γ [*Rubin and Dagan*, 1987, equations (19), (20), and (21)] for the case of distributed recharge into (4) leads to explicit expressions for the velocity covariances.

The general expressions obtained are cumbersome and not very informative, and hence are not repeated here. Results for particular flow configurations will be given subsequently. In general, both $u_{11}(\mathbf{x}, \mathbf{x}')$ and $u_{22}(\mathbf{x}, \mathbf{x}')$ are nonstationary. In the case where the recharge is equal to zero the mean head gradient is not a function of the space coordinate, and the covariances simplify to the expressions which were derived by *Rubin* [1990] and were later tested numerically by *Bellin et al.* [1992].

The derivation of u_{ij} is the first, though the major step, toward defining the displacement statistics. This goal is pursued in section 2 for the particular case of nonuniform, unidirectional flow.

2. The Case of Nonuniform, Unidirectional Flow

The case of nonuniform, unidirectional flow offers a good opportunity to consider macrodispersion in nonuniform flows. The transmissivity field is heterogeneous but statistically stationary, and the only source of nonuniformity in the mean flow is the recharge. Such a flow regime prevails in a uniformly recharged domain bounded by two parallel head boundaries and two parallel no-flow boundaries.

Consider a domain of dimension L between head boundaries H_1 and H_2, and a large, unspecified width between no-flow boundaries. The no-flow boundaries are parallel to the x_1 axis and the origin is on the H_1 boundary. This domain is uniformly recharged at a constant rate R. Under these conditions the expected value of the head $H(\mathbf{x})$ is given by

$$\langle H(\mathbf{x})\rangle = -\frac{Rx_1^2}{2T_G} + \left[\frac{H_2 - H_1}{L} + \frac{RL}{2T_G}\right]x_1 + H_1 \tag{5}$$

The mean head gradient is now a function of the x_1 coordinates only, that is, $\mathbf{J}(\mathbf{x}) = -\nabla\langle H\rangle = (J_1(x_1), 0)$. From (5), J_1 and the recharge are related through

$$J_1(x_1) = J_0[1 + \beta(x_1 - x_0)/I] \tag{6}$$

where $J_0 = J_1(x_0)$ is the gradient at x_0, $\beta = RI/T_GJ_0$, R is the recharge, I is the integral scale of Y and $T_G = \exp(m_Y)$ is the geometric mean of the transmissivity.

Next, consider the covariance u_{22} for separation distances along the mean flow direction

$$\frac{u_{22}(x_1, x_2'; x_1', x_2')}{V_0^2\sigma_Y^2} = \frac{\langle u_2(x_1, x_2')u_2(x_1', x_2')\rangle}{V_0^2\sigma_Y^2} = \frac{1}{4}[(1 + \beta x_1/I)^2 + (1 + \beta x_1'/I)^2] \cdot \left[\frac{18(e^r - r - 1) - 8r^2 - e^r r^2 - 2r^3}{e^r r^4} + \frac{\beta^2(12(1 - e^r + r) + 7r^2 + 7e^r r^2 + 3r^3}{32e^r r^2}\right] \tag{7}$$

where $V_0 = V_1(x_0)$, $r = |x'_1 - x_1|/I$ and $x_0 = 0$ was taken. As mentioned already, this result is obtained by coupling the expressions taken from *Rubin and Dagan* [1987] with (4). By taking $x_1 = x'_1$, the variance of U_2 assumes the nonstationary form

$$\frac{u_{22}(x_1, x_2)}{V_0^2\sigma_Y^2} = \frac{1}{8}\left[(1 + \beta x_1/I)^2 + 2\beta^2\right] \tag{8}$$

The variance $u_{22}(x_1, x_2)$, (8), changes with distance from the origin. The application of (7) and (8) for negative βs at large distances is, of course, limited because of the reversal of the flow direction at distance I/β from x_0. Note that by taking $\beta = 0$, the mean velocity V_1, as well as the variance $u_2(x_1, x_2)$ becomes stationary, with $u_{22}(x_1, x_2)$ equal to the value found by *Dagan* [1984] for the case of zero-recharge, uniform flow.

The covariance of U_1 as a function of separation distance along the mean flow direction is given by

$$\begin{aligned}\frac{u_{11}(x_1, x'_2; x'_1, x'_2)}{V_0^2\sigma_Y^2} &= (1 + \beta x_1/I)(1 + \beta x'_1/I)e^{-r} \\ &+ \frac{1}{4}\left[(1 + \beta x_1/I)^2 + (1 + \beta x'_1/I)^2\right] \\ &\cdot \left[e^{-r}\left(-2 + \frac{6}{r^2} + \frac{18}{r^3} + \frac{18}{r^4}\right) - \frac{18}{r^4} - \frac{3}{r^2}\right] \\ &+ \frac{\beta^2}{32e^r r^2}\left[30e^r r^2 - 62r^2 - 30r^3\right. \\ &\left. - 3r^4 + 96(e^r - 1 - r)\right]\end{aligned} \tag{9}$$

By taking the $r \to 0$ limit of u_{11}, the nonstationary U_1 variance is given by

$$\frac{u_{11}(x_1, x_2)}{V_0^2\sigma_Y^2} = \frac{3}{8}(1 + \beta x_1/I)^2 + \frac{\beta^2}{2} \tag{10}$$

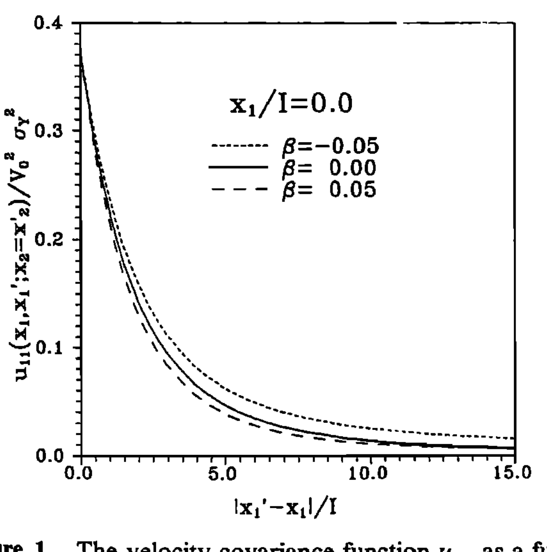

Figure 1. The velocity covariance function u_{11} as a function of the distance $|x'_1 - x_1|$ along the mean flow direction for $x_1 = 0$.

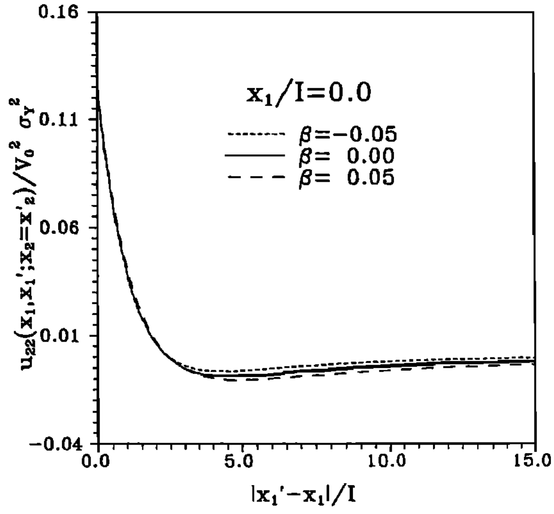

Figure 2. The velocity covariance function u_{22} as a function of the distance $|x'_1 - x_1|$ along the mean flow direction for $x_1 = 0$.

For $\beta = 0$ the variance becomes stationary and is equal to the value obtained by *Dagan* [1984] for uniform flows.

The theoretical limitations of the above results emanate from the assumptions employed in their derivations. First and foremost is the assumption of small variability in Y, which was essential for the linearized solution of the flow equation. This assumption is followed by an assumption of flow in an infinite domain, which translates to the requirement that the flow domain under investigation is sufficiently removed from physical flow boundaries. Previous studies [*Rubin and Dagan*, 1988, 1989] have shown that a distance of about 2 integral scales from the boundary is sufficient to define a nonimpacted area. This assumption is adopted for the sake of deriving a simplified, closed-form analytical solutions. An additional assumption made here is that expected values of primed terms are negligible. The derivations contained in this section will be utilized in section 3 to demonstrate the effects of distributed recharge on solute transport.

3. Case of Nonuniform, Unidirectional Flow: Applications and Numerical Evaluation

Figure 1 depicts u_{11} as a function of separation distance along the mean flow direction using (9) for various β. Positive recharge leads to smaller correlation compared to the cases of zero or negative recharge. In all three cases the covariance decays to zero, indicating the existence of a finite integral scale for u_{11}. In the case of $\beta = 0$, previous works [*Gelhar and Axness*, 1983; *Dagan*, 1984] showed that a finite integral scale for the velocity leads to constant macrodispersion coefficients at sufficiently long travel times. Here, however, the nonstationarity of the covariance, as evidenced by the constant increase in the velocity variance, is a new factor to reckon with and will be investigated later in this work.

Unlike the case of u_{11}, the effect of recharge on u_{22} is hardly noticeable at small lags (Figure 2). Negative correlations are displayed for medium to large separation distances. This behavior exists for all types of β. It indicates that a positive deviation in U_2 is expected to prevail over certain

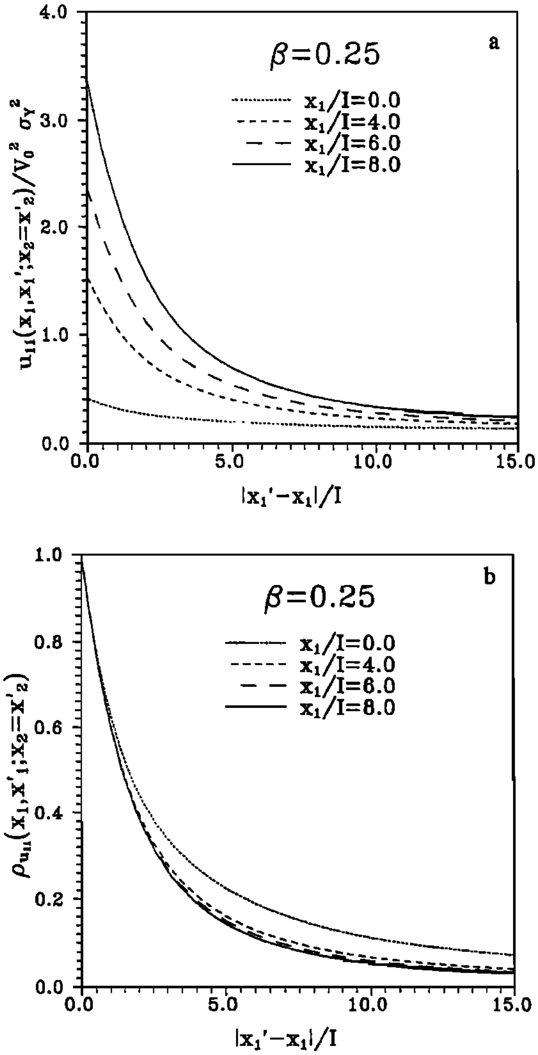

Figure 3. (a) The velocity covariance function u_{11}, and (b) the correlation function $\rho_{u_{11}}$ as a function of the distance $|x_1' - x_1|$ along the mean flow direction ($x_1/I = 0.0$; $x_1/I = 4.0$; $x_1/I = 6.0$; $x_1/I = 8.0$).

distances and be followed by a negative deviation at distances larger than $2.5I$, and thereby ensuring a flow which is unidirectional in the average. In cases of transport this type of behavior leads to symmetry of the plume around the mean flow direction.

The scale dependency and scale invariance of the spatial correlation structure of the velocity become evident upon inspecting Figures 3 and 4, where the covariances and the respective correlation functions are depicted as functions of lag distance for different reference points x. These figures clearly demonstrate the increase in the velocity variance with distance from the reference point, and this is in agreement with (8) and (10). While the velocity covariances are scale-dependent, the velocity correlation functions, defined as $u_{ii}(x_1, x_1')/(\sigma_i(x_1)\sigma_i(x_1'))$ where $\sigma_i(x)$ denotes the standard deviation of the velocity $U_i(x)$, approach a stationary behavior at distances of about $4I$ from the reference point. From a physial point of view, this result suggests that the increase in the mean velocity and its variance do not lead to change in the integral scales of the velocity field. From a practical point of view, this result implies that an increase in the mean velocity and its variability do not require an increase in the density of the mesh used for numerical modeling, which is determined by the magnitude of the velocity integral scales [*Bellin et al.*, 1992].

The previous relationships were evaluated for accuracy using a numerical Monte Carlo study. The numerical scheme employed here is the one presented and analyzed in great detail by its developers [*Bellin et al.*, 1992], hence its description is omitted for brevity. The numerical code combines a flow equation solver with a particle-tracking scheme. Random fields are generated using the Gutjahr method [*Gutjahr*, 1989; *Bellin*, 1991]. The numerical results presented here are based on 1500 replicates of the velocity field. To ensure convergence, the number of blocks of constant transmissivity per integral scale was set at four, and the finite element grid was obtained by dividing each squared integral scale into 32 triangular elements. A linear shape function was employed for the elements.

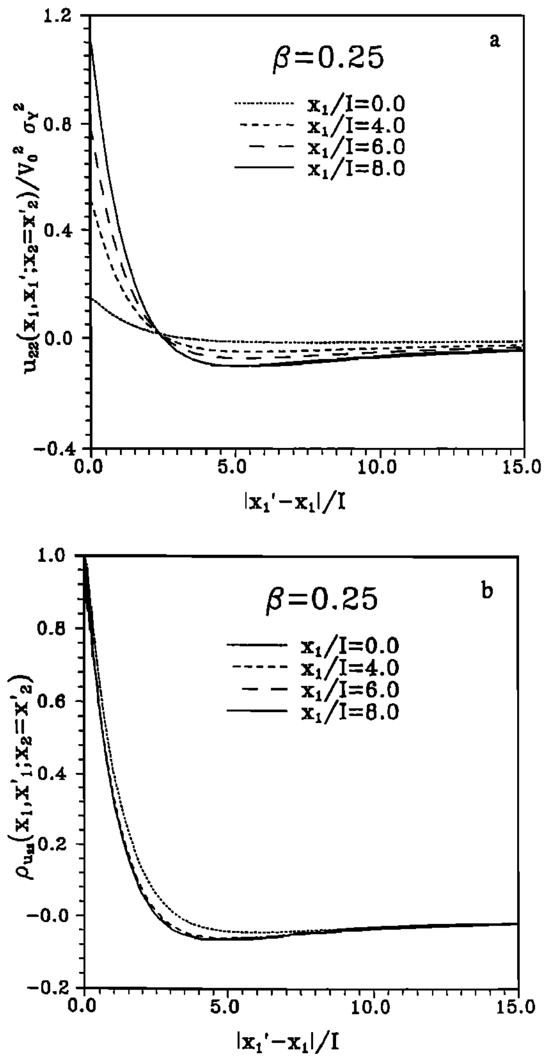

Figure 4. (a) The velocity covariance function u_{22} and (b) the correlation function $\rho_{u_{22}}$ as a function of the distance $|x_1' - x_1|$ along the mean flow direction ($x_1/I = 0.0$; $x_1/I = 4.0$; $x_1/I = 6.0$; $x_1/I = 8.0$).

From (3) we get the following relation for the mean velocity:

$$V_1(x_1) = [1 + \beta(x_1 - x_0)/I]V_1(x_0) \tag{11}$$

which is evaluated for $\beta = 0.05$ and $\sigma_Y^2 = 0.8$ in Figure 5. Numerical and analytical results are in excellent agreement.

Since the flow is unidirectional in the average, the mean displacement $\langle \mathbf{X} \rangle = \langle X_1 \rangle$ is obtained by solving the kinematic relation

$$\frac{d\langle X_1 \rangle}{dt} = V_1(\langle X_1 \rangle) \tag{12}$$

leading to

$$\frac{1}{I}\langle X_1(\tau)\rangle = \frac{1}{\beta}[e^{\beta\tau} - 1] \tag{13}$$

where $\tau = tV_0/I$ denotes nondimensional time and $V_0 = V_1(0)$. This relationship is evaluated in Figure 6 for $\beta = 0.05$, and $\sigma_Y^2 = 0.8$, and also shows good agreement between theoretical and numerical results.

Figure 7 depicts a series of covariances for the case of $\sigma_Y^2 = 0.8$ and $\beta = 0.05$, with x_1 denoting distance from the reference point x_0. Analytical (equation (9)) and numerical results are compared. Good agreement between numerical and analytical results is observed at all lags. The numerical results are slightly lower than the analytical results at nonzero lags, and the analytical variance is lower than the numerical variance. This type of behavior was also observed by *Bellin et al.* [1992] and *Fiorotto* [1992] in their investigation of uniform-in-the-average flows, the former using a finite element method and the latter using a spectral method.

The second moments tensor of the displacement $\mathbf{X}$ is denoted here by $X_{jk}(t) = \langle X_j'(t)X_k'(t)\rangle$, $j, k = 1, 2$, where $X_j'(t) = X_j(t) - \langle X_j(t)\rangle$, that is, the deviation of the actual displacement of the solute particle from its expected value. Using a Lagrangian approach, the displacement covariances can be related to the velocity statistics [*Dagan*, 1984] by the following integration:

$$X_{jk}(t) = \int_0^t \int_0^t u_{jk}^L(t', t'')\, dt'\, dt'' \qquad j, k = 1, 2 \tag{14}$$

where u_{jk}^L is the tensor of the Lagrangian velocity covariances. The computation of X_{jk} in (14) requires the transformation of the previously derived Eulerian covariances into Lagrangian covariances using

$$u_{jk}^L(t', t'') = u_{jk}[\mathbf{x}(t'), \mathbf{x}(t'')] \tag{15}$$

Since $\mathbf{X}$ is known only through its moments, the substitution of (15) into (14) leads to an implicit integrodifferential equation. Following a procedure suggested by *Dagan* [1984], we replace $\mathbf{X}$ by its mean $\langle \mathbf{X} \rangle$. This procedure is consistent with the first-order approximation adopted at the previous steps

$$u_{jk}(t', t'') = u_{jk}[\langle \mathbf{X}(t')\rangle, \langle \mathbf{X}(t'')\rangle] \tag{16}$$

This approximation was tested successfully in previous studies [*Bellin et al.*, 1992]. Combining (16) and (14), we get the following expression for X_{jk}:

$$X_{jk}(t) = \int_0^{\langle X_1(t)\rangle} \int_0^{\langle X_1(t)\rangle} \frac{u_{jk}[x_1', x_1'']}{V_1(x_1')V_1(x_1'')}\, dx_1'\, dx_1'' \tag{17}$$

which is evaluated here by numerical integration using (13) for $\langle X_1(t)\rangle$ and taking $\langle X_2(t)\rangle = 0$.

Figure 8 shows the nondimensional X_{11} for positive, zero, and negative βs. Negative βs lead to a smaller X_{11} as a result of the reduction in the velocity variance of the fluid particles, while an increase in the mean velocity ($\beta > 0$) leads to a substantial growth in X_{11}. The time growth rate of X_{11}, in the form of a macrodispersion coefficient D_{11}, is inspected in Figure 9 ($D_{ij} = dX_{ij}/(2\, d\tau)$. Constant growth rates for X_{11} are attained only for $\beta = 0$, and a zero asymptotic limit is attained for $\beta < 0$. In the case of a positive β, X_{11} does not approach a constant growth rate regime, one that would justify the use of a constant dispersion coefficient, even at exceedingly large travel distances.

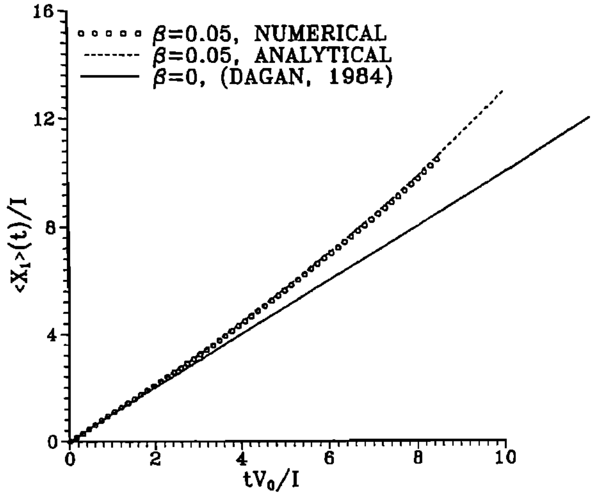

Figure 6. Lagrangian mean trajectory ($\sigma_Y^2 = 0.80$).

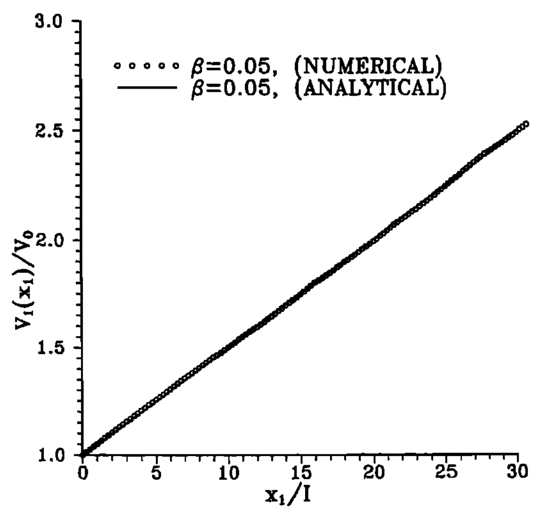

Figure 5. Mean longitudinal velocity as a function of distance from the reference point along the flow domain ($\sigma_Y^2 = 0.80$). Solid line (analytical) is overlain by circles (numerical).

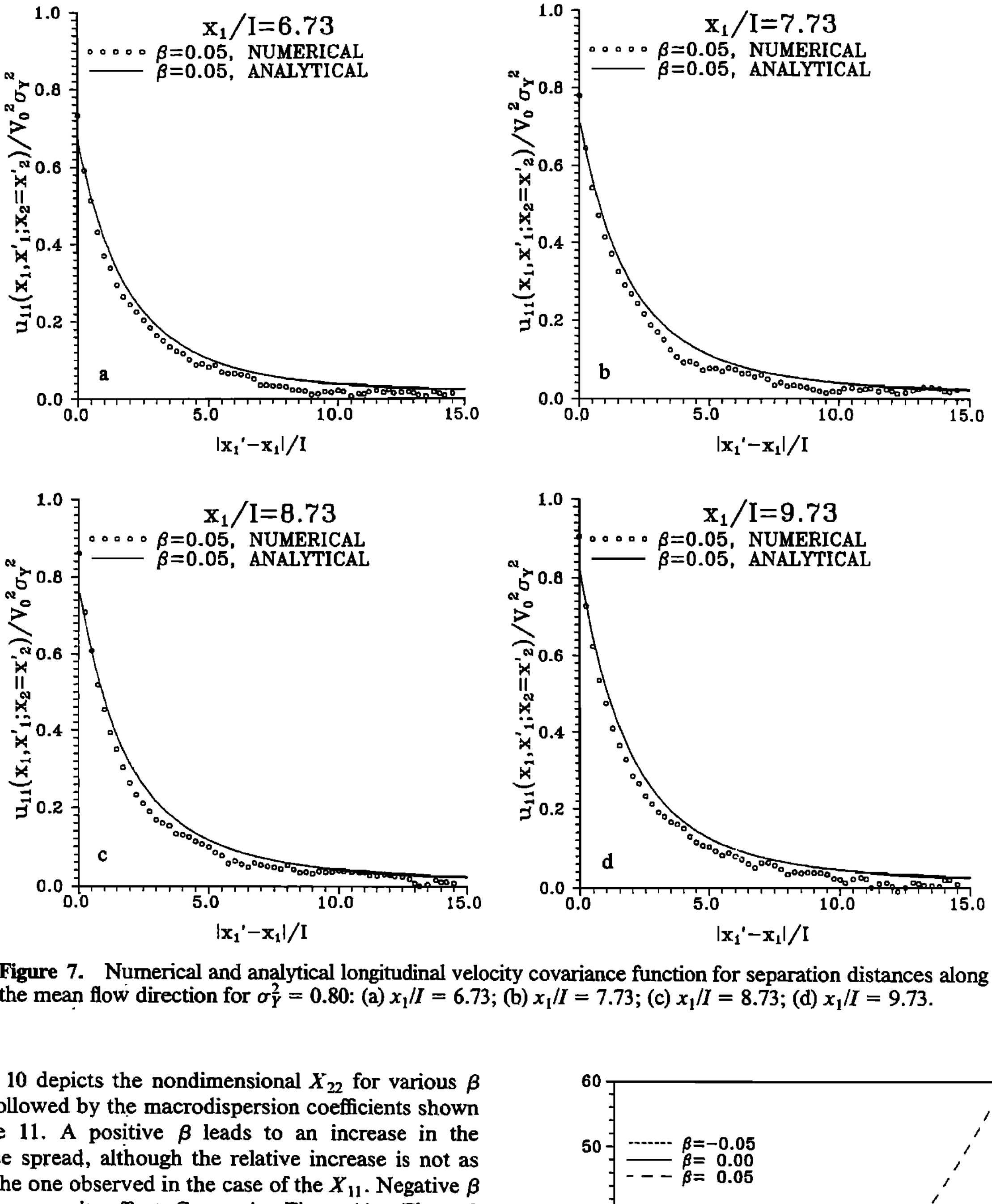

Figure 7. Numerical and analytical longitudinal velocity covariance function for separation distances along the mean flow direction for $\sigma_Y^2 = 0.80$: (a) $x_1/I = 6.73$; (b) $x_1/I = 7.73$; (c) $x_1/I = 8.73$; (d) $x_1/I = 9.73$.

Figure 10 depicts the nondimensional X_{22} for various β values, followed by the macrodispersion coefficients shown in Figure 11. A positive β leads to an increase in the transverse spread, although the relative increase is not as large as the one observed in the case of the X_{11}. Negative β leads to an opposite effect. Comparing Figure 11 to Figure 9 suggests that the lateral spread is not as sensitive to recharge as the longitudinal spread, and asymptotic, constant dispersion coefficients are attained. Numerical testing of the results for X_{11} and X_{22} are depicted on Figures 12 and 13 for $\sigma_Y^2 = 0.2$ and $\beta = 0.02$ and show a favorable agreement. Good agreement was also found for $\sigma_Y^2 = 0.8$.

A particularly useful application of the displacement moments is the derivation of the cumulative probability distribution function (cdf) of the travel time between an injection point and any plane in space. For regulatory purposes this plane represents the accessible environment. Discussion of such an approach with application to the case of uniform-in-the-average flow were given by *Dagan and Nguyen* [1989], *Rubin and Dagan* [1992*a*], and *Cvetkovic et al.*, [1992]. The crossing time cdf F for a plane normal to the mean flow direction and a Gaussian $\mathbf{X}$ is given generally by

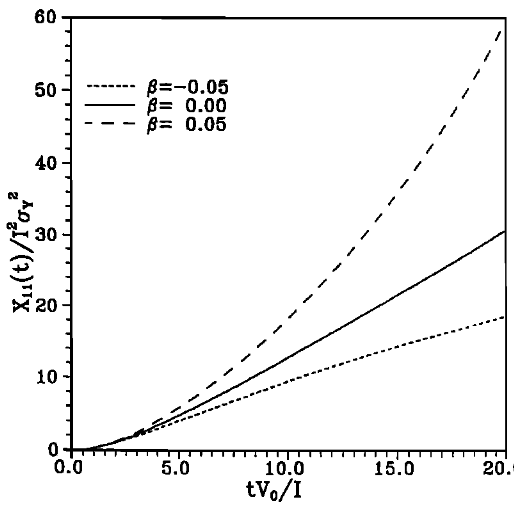

Figure 8. Longitudinal displacement variance X_{11} for negative and positive β compared with the uniform case ($\beta = 0$). X_{11} is evaluated numerically from equation (17).

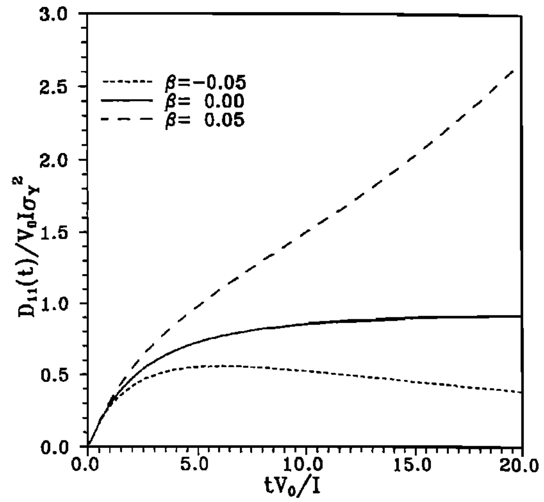

Figure 9. Longitudinal macrodispersion coefficient D_{11} for negative and positive β. D_{11} is evaluated numerically from equation (17).

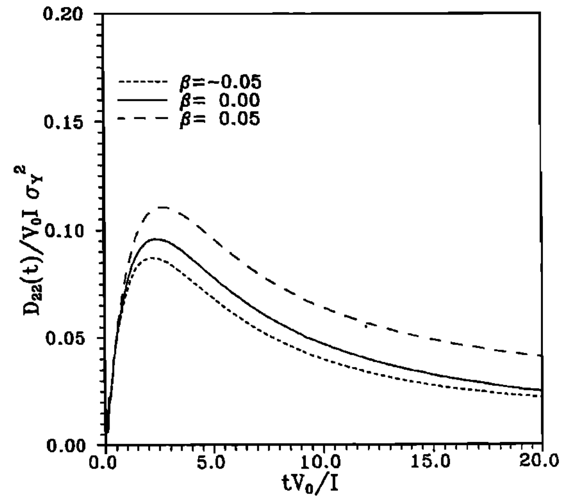

Figure 11. Transverse macrodispersion coefficient D_{22} for negative and positive β. D_{22} is evaluated numerically from equation (17).

$$F(\tau) = \frac{1}{2}\ erfc \left[\frac{X_{cp} - \langle X_1(\tau)\rangle}{[2X_{11}(\tau)]^{1/2}}\right] \tag{18}$$

where X_{cp} denotes the distance of the accessible environment from the source, and τ is nondimensional time.

Consider the case of a positive β. The constant increase in the mean velocity should lead to earlier crossings, but on the other hand, the increase in the X_{11} may lead to an increase in the total crossing time. Yet an examination of $F(\tau)$ in Figure 14 indicates that the overall result is an earlier breakthrough and shorter time span between the earliest and latest crossing; hence the effect of the increase in the mean velocity overwhelms the effect of larger longitudinal spread.

4. Discussion

In Figures 3 and 4, the covariances u_{11} and u_{22} are depicted with their respective correlation functions. While the covariances are found to be nonstationary, the velocity correlation functions

$$\rho_{ii}(\mathbf{x}, \mathbf{x}') = \frac{u_{ii}(\mathbf{x}, \mathbf{x}')}{[u_{ii}(\mathbf{x}, \mathbf{x})u_{ii}(\mathbf{x}', \mathbf{x}')]^{1/2}} \qquad i = 1, 2 \tag{19}$$

(no summation over repeated indices implied) are shown to be nearly stationary. Recall now that in the Lagrangian formalism, the displacement variances are computed using (17). Substituting (7) and (9) in turns into (17) yields

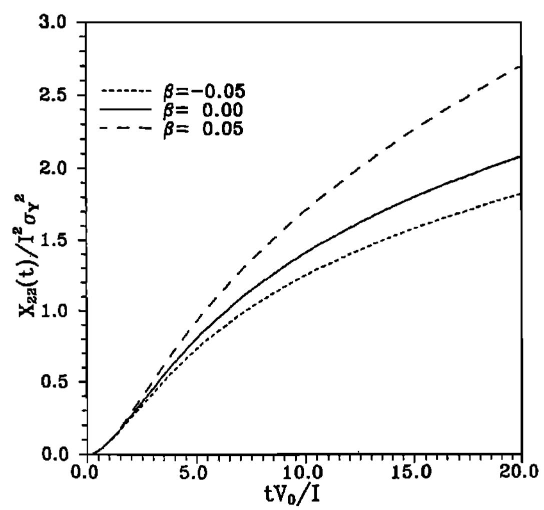

Figure 10. Transverse displacement variance X_{22} for negative and positive β compared with the uniform case ($\beta = 0$). X_{22} is evaluated numerically from equation (17).

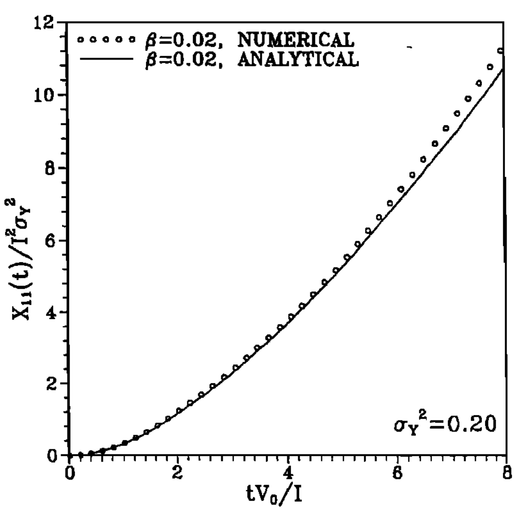

Figure 12. Numerical and semianalytical longitudinal displacement variance versus dimensionless time tV_0/I, ($\sigma_Y^2 = 0.20$).

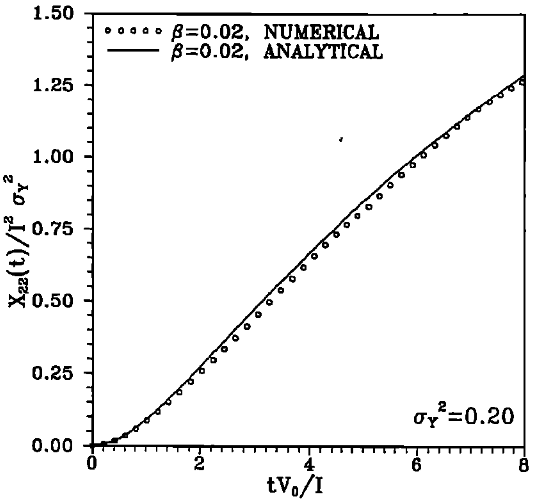

Figure 13. Numerical and semianalytical transverse displacement variance versus dimensionless time tV_0/I, ($\sigma_Y^2 = 0.20$).

$$X_{ii}(t) = \sigma_Y^2 \int_0^{\langle X_1(t)\rangle} \int_0^{\langle X_1(t)\rangle} \{\delta_{i1} e^{-r} - \left[\frac{(1+\beta x_1/I)}{1+\beta x_1'/I} + \frac{1+\beta x_1'/I}{(1+\beta x_1/I)}\right] F_{ii}(r) + \frac{\beta^2}{(1+\beta x_1/I)(1+\beta x_1'/I)} F_{ii}^1(r)\}\, dx_1\, dx_1' \tag{20}$$

where F_{ii} and F_{ii}^1 are stationary functions which can be defined by inspection of (7) and (9), δ is the Kronecker delta, and $i = 1, 2$. Expanding (20) in Taylor series with respect to β yields

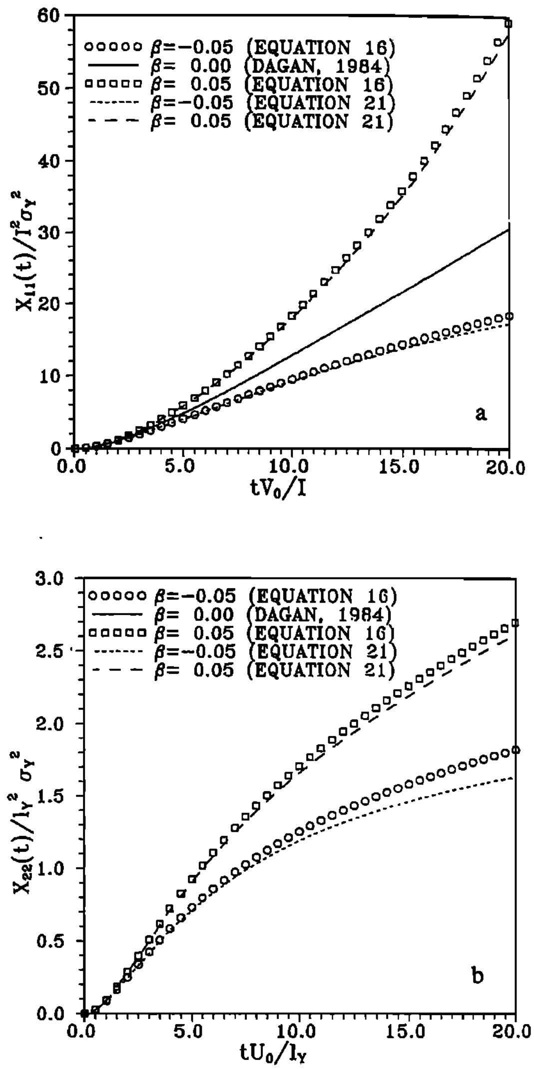

Figure 15. Semianalytical (squares and circles) and first-order analytical (dashed lines) (a) longitudinal and (b) transverse displacement variances. ($\beta = -0.05$; 0.0; 0.05).

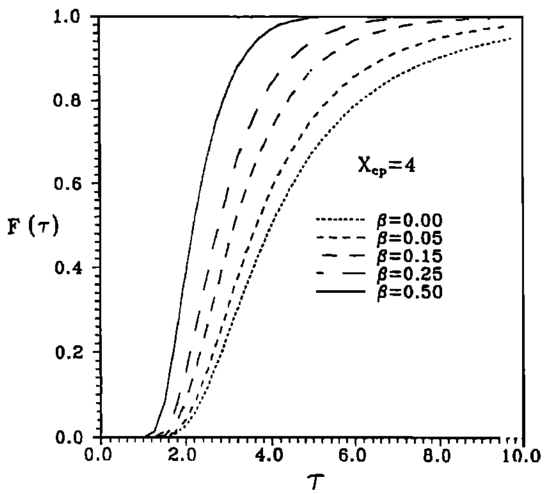

Figure 14. Cumulative probability distribution function of the travel time at $X_{cp} = 4I$.

$$X_{ii}(t) = \sigma_Y^2 \int_0^{\langle X_1(t)\rangle} \int_0^{\langle X_1(t)\rangle} \cdot \left\{\delta_{i1} e^{-r} - 2F_{ii}(r)\right\} dx_1\, dx_1' + O(\beta^2) \tag{21}$$

which leads, upon inspection of the velocity covariances obtained by *Rubin* [1990] for the case of $\beta = 0$, to the result

$$X_{ii}(\tau) = X_{ii}^u[\tau' = (e^{\beta\tau} - 1)/\beta] + O(\beta^2) \tag{22}$$

where X_{ii}^u are the displacement variances applicable for the case of $\beta = 0$ [see *Dagan*, 1984]. While u_{ii}^u are the zero-order approximations of u_{ii}, X_{ii} are $O(\beta)$ accurate because the first-order terms in β cancel out during the Taylor expansion.

Equation (22) shows that the displacement variances at time τ for nonzero βs can be approximated by the displacement variances obtained for $\beta = 0$ but at time $\tau' = (\exp(\beta\tau) - 1)/\beta$ (see (13)). The dependence on space coordinates appears only in terms of order β^2. Figures 15a and 15b depict $X_{11}(\tau)$ and $X_{22}(\tau)$ for positive and negative βs, where the

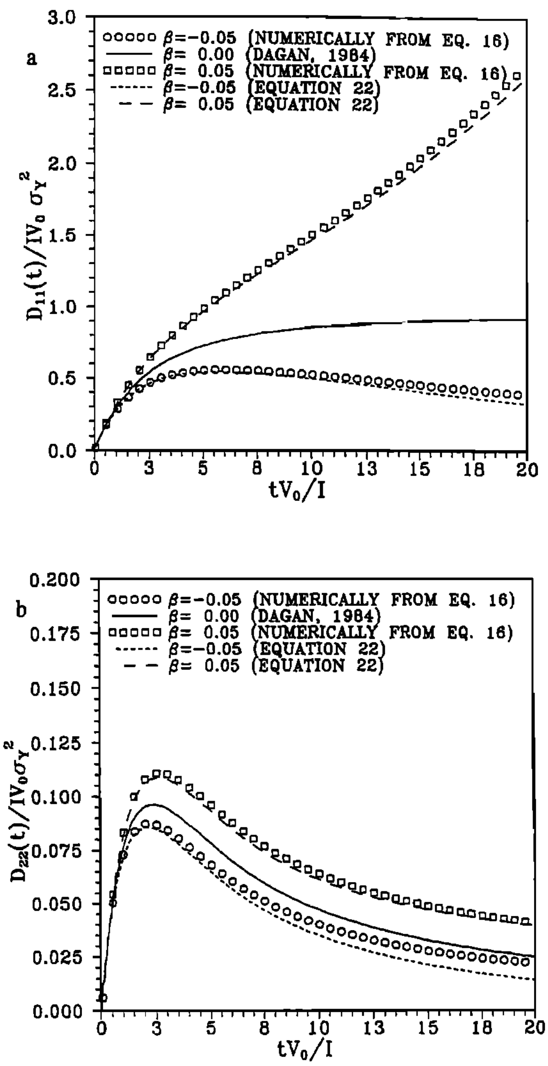

Figure 16. Semianalytical (squares and circles) and first-order analytical (dashed lines) (a) longitudinal and (b) transverse macrodispersion coefficients ($\beta = -0.05$; 0.0; 0.05).

semianalytical results obtained in section 3 are compared with their approximations obtained by (22).

The above development allows the following relationship for the macrodispersion coefficient:

$$D_{ii}(\tau) = e^{\beta\tau} D_{ii}^{u}(\tau') \tag{23}$$

with $\tau' = (\exp(\beta\tau) - 1)/\beta$, and where $D_{ii}^{u}(\tau')$ are the macrodispersion coefficients obtained for the case of $\beta = 0$ [see *Dagan*, 1984]. Here, τ and τ' are both nondimensional time, defined as in section 3. Figure 16 repeats Figure 15 only for D_{11} and D_{22}. Both X_{ii} and D_{ii}, $i = 1, 2$, computed through (22) and (23), respectively, are in good agreement with the theoretical results. Although the approximation introduced in the evaluation of u_{ii} is of the same order for both u_{11} and u_{22}, X_{11} and D_{11} show a better agreement with the theoretical results. This can be attributed to the sensitivity of both X_{22} and D_{22} to the slightest deviations in u_{22}.

While *Dagan* [1984] showed that D_{11}^{u} asymptotically approaches a constant value, (22) shows that for $\beta > 0$, D_{11} does not reach a constant limit. At small time, D_{11} is very close to the steady state coefficient, but with time there is an increasing departure between these two parameters.

From *Dagan* [1984, equation (4.8)], we get $D_{22}^{u}(\tau' \to \infty) = V_0 I \sigma_Y^2 / 2\tau'$. Using the transformation $\tau' = (\exp(\beta\tau) - 1)/\beta$, and substituting in (23), we get

$$D_{22}(\tau \to \infty) = V_0 I \sigma_Y^2 \beta \tag{24}$$

Hence unlike the $\beta = 0$ case, D_{22} does not approach zero. Large time implies only a constant growth rate of the displacement variance.

Again using equation (4.8) from *Dagan* [1984] leads to $D_{11}^{u}(\tau' \to \infty) = V_0 I \sigma_Y^2$, and hence from (23)

$$D_{11}(\tau) = e^{\beta\tau} V_0 I \sigma_Y^2 \tag{25}$$

which can also be expressed in terms of the travel distance L, using (13), as

$$D_{11}(L) = V_0 I \left[1 + \frac{\beta L}{I}\right] \sigma_Y^2 \tag{26}$$

The effects of flow nonuniformity on the longitudinal spread of passive solute were discussed by *Gelhar and Collins* [1971] and more recently by *Adams and Gelhar* [1992] (hereafter referred to as AG) and *Serrano* [1992]. The AG study considers flow nonuniformity resulting from a trend in the mean logconductivity, such as that observed in the Columbus field experiment.

In order to explain the observed macrodispersivity, AG modeled the ensuing mean velocity field as $V_1 = V_0(1 + \beta x_1/I)$ and $V_2 = V_0 \beta x_2/I$. The factor β indicates the degree of flow nonuniformity, and in this sense, is perfectly analogous to the β we use here (see (6)) although in our study, β represents flow nonuniformity coming from a different source. The macrodispersion coefficients are then assumed by AG to be proportional to $V_1(\mathbf{x})$ and $V_2(\mathbf{x})$ and are given by $D_{11} = A_{11} V_1(x_1)$ and $D_{22} = A_{22} V_2(x_1)$, where A_{11} and A_{22} are constant dispersivities.

Hence in the case of D_{11}, recalling (25) and (6), the AG conjecture is found valid given that A_{11} is taken equal to $\sigma_Y^2 I$. In the case of D_{22} a comparison is valid only in the subdomains where V_2 is much smaller than V_1, and can be neglected altogether. The AG conjecture for that case is in disagreement with the present results, which show a persistent growth in X_{22}, that is, a nonzero D_{22}, even if $V_2 = 0$.

Serrano [1992] also investigated the effect of recharge on transport using a conjecture similar to that of AG. Our study establishes under what conditions this conjecture is valid and applicable for both Serrano's and AG's studies.

5. Summary

The fluid velocity in a heterogeneous, uniformly recharged aquifer is derived as a space random function (SRF). This model includes the expected value and the variance-covariance tensor. The recharge leads to nonstationarity of these two moments, which constantly increase with distance from any arbitrary reference point.

In the first part of our study the velocity SRF is used to derive the single-particle displacement statistics, which are needed for estimation of the ensuring concentration field, as a function of σ_Y^2, the logconductivity variance, and β, a parameter which characterizes the flow nonuniformity. We found that while the velocity covariances are nonstationarity, the velocity correlation functions, and hence its integral scales, are practically stationary. The results compared favorably with a numerical Monte Carlo study.

In the second part of this paper we found that the near stationarity of the velocity correlation functions leads to simple relationships between the displacement covariance tensor and macrodispersion coefficients derived for uniform flows and their analogs pertaining to nonuniform flows through a nonlinear transformation of the travel time scale. This opens the opportunity for a simplified analysis of solute dispersion in nonuniform flows in heterogeneous media.

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