

# FANO FIVEFOLDS OF INDEX TWO WITH BLOW-UP STRUCTURE

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## Abstract

We classify Fano fivefolds of index two which are blow-ups of smooth manifolds along a smooth center.

## 1 Introduction

A smooth complex projective variety  $X$  is called *Fano* if its anticanonical bundle  $-K_X$  is ample; the *index*  $r_X$  of  $X$  is the largest natural number  $m$  such that  $-K_X = mH$  for some (ample) divisor  $H$  on  $X$ , while the *pseudoindex*  $i_X$  is the minimum anticanonical degree of rational curves on  $X$ .

By a theorem of Kobayashi and Ochiai [15],  $r_X = \dim X + 1$  if and only if  $(X, L) \simeq (\mathbb{P}^{\dim X}, \mathcal{O}_{\mathbb{P}}(1))$ , and  $r_X = \dim X$  if and only if  $(X, L) \simeq (\mathbb{Q}^{\dim X}, \mathcal{O}_{\mathbb{Q}}(1))$ , where  $\mathbb{Q}^{\dim X}$  is a quadric hypersurface in  $\mathbb{P}^{\dim X+1}$ . Fano manifolds of index equal to  $\dim X - 1$  and to  $\dim X - 2$ , which are called *del Pezzo* and *Mukai* manifolds respectively, have been classified, mainly by Fujita, Mukai and Mella (see [11, 18, 17]). In case of index equal to  $\dim X - 3$ , the classification has been completed for Fano manifolds of Picard number  $\rho_X$  greater than one and dimension greater or equal than six (see [29]).

For Fano manifolds of dimension five and index two it was proved in [1] that the Picard number is less than or equal to five, equality holding only for a product of five copies of  $\mathbb{P}^1$ . Then, in [9], the structure of the possible Mori cones of curves of those manifolds, i.e., the number and type of their extremal contractions, was described. A first step in going from the table of the cones given in [9] to the actual classification of Fano fivefolds

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of index two has been done in [19], where ruled Fano fivefolds of index two, i.e., fivefolds of index two with a  $\mathbb{P}^1$ -bundle structure over a smooth fourfold, were classified.

In this paper we classify Fano fivefolds of index two which are blow-ups of smooth manifolds along smooth centers. In section 3 we recall the structure of the cones of curves of these manifolds, as described in [9], and we summarize the known results. Using previous results we are reduced to the following cases:

$\rho_X = 2$  and the two extremal rays of  $\text{NE}(X)$  correspond respectively to the blow-up of a smooth variety  $X'$  along a smooth surface  $S$  and to a fiber type contraction  $\vartheta : X \rightarrow Y$ .

$\rho_X = 3$ . In this case  $\text{NE}(X)$  has three extremal rays: one of them is associated to the blow-up of a smooth variety along a smooth surface, one corresponds to a fiber type contraction, and the last one is associated either to another blow-up contraction or to another fiber type contraction.

The hardest case, which is the heart of the paper and is dealt with in Section 4, is when  $\rho_X = 2$ . In this case it is easy to show that the pseudoindex of  $X'$  is equal either to six or to four: if  $i_{X'} = 6$  then  $X' \simeq \mathbb{P}^5$  by results in [14], and the classification of  $S$  follows observing that  $S$  cannot have proper trisecants. In case  $i_{X'} = 4$  we prove that also  $r_{X'} = 4$ , i.e., that  $X'$  is a del Pezzo manifold and that  $S$  is a del Pezzo surface. The classification of  $(X', S)$  then follows studying the possible conormal bundles  $N_{S/X'}^*$ .

In Section 5 we study the case  $\rho_X = 3$ ; apart from one case, the target of the birational contraction is a Fano manifold, which is either a product with  $\mathbb{P}^1$  as a factor or a  $\mathbb{P}^3$ -bundle over a surface; the classification of the center follows.

Our results are summarized in the following

**THEOREM 1.1.** *Let  $X$  be a Fano fivefold of index two which is the blow-up of a smooth variety  $X'$  along a smooth subvariety  $S$ . Then  $(X', S)$  is as in the following table, where, in the last column,  $F$  denotes a fiber type extremal ray,  $D_i$  denotes a birational extremal ray whose associated contraction contracts a divisor to an  $i$ -dimensional variety and  $S$  denotes a ray whose associated contraction is small.*

$\rho_X$	No.	$X'$	$S$	$NE(X)$
2	(a1)	$\mathbb{P}^5$	a point	$\langle F, D_0 \rangle$
	(b1)	$\mathbb{P}^5$	a linear $\mathbb{P}^2$	$\langle F, D_2 \rangle$
	(b2)	$\mathbb{P}^5$	the complete intersection of three quadrics	$\langle F, D_2 \rangle$
	(b3)	$\mathbb{P}^5$	$\mathbb{P}^1 \times \mathbb{P}^1$ embedded by $\mathcal{O}(1, 2)$	$\langle F, D_2 \rangle$
	(b4)	$\mathbb{P}^5$	$\mathbb{F}_2$ embedded by $C_0 + 3f$	$\langle F, D_2 \rangle$
	(b5)	$\mathbb{P}^5$	the blow-up of $\mathbb{P}^2$ in four points $x_1, \dots, x_4$ such that the line bundle $\mathcal{O}_{\mathbb{P}^2}(3) - \sum E_i$ is very ample	$\langle F, D_2 \rangle$
	(b6)	$\mathbb{P}^5$	the blow-up of $\mathbb{P}^2$ in seven points $x_0, \dots, x_6$ such that the line bundle $\mathcal{O}_{\mathbb{P}^2}(4) - 2E_0 - \sum_{i=1}^6 E_i$ is very ample	$\langle F, D_2 \rangle$
	(b7)	$V_d$ (*)	the complete intersection of three general members of $ \mathcal{O}_{V_d}(1) $	$\langle F, D_2 \rangle$
	(b8)	$V_3$	$\mathbb{P}^2$ with $(\mathcal{O}_{V_3}(1))_{\mathbb{P}^2} \simeq \mathcal{O}_{\mathbb{P}^2}(1)$	$\langle F, D_2 \rangle$
	(b9)	$V_4$	$\mathbb{P}^2$ with $(\mathcal{O}_{V_4}(1))_{\mathbb{P}^2} \simeq \mathcal{O}_{\mathbb{P}^2}(1)$	$\langle F, D_2 \rangle$
	(b10)	$V_4$	$\mathbb{Q}^2$ with $(\mathcal{O}_{V_4}(1))_{\mathbb{Q}} \simeq \mathcal{O}_{\mathbb{Q}}(1)$	$\langle F, D_2 \rangle$
	(b11)	$V_5$	a plane of bidegree (1, 0) (**)	$\langle F, D_2 \rangle$
	(b12)	$V_5$	a quadric of bidegree (1, 1)	$\langle F, D_2 \rangle$
	(b13)	$V_5$	a surface $\mathbb{F}_1$ of bidegree (2, 1) not contained in a $\mathbb{G}(1, 3)$	$\langle F, D_2 \rangle$
	(c1)	$\mathbb{P}^5$	a Veronese surface	$\langle D_2, D_2 \rangle$
	(c2)	$\mathbb{P}^5$	$\mathbb{F}_1$ embedded by $C_0 + 2f$	$\langle D_2, D_2 \rangle$
	(c3)	$V_5$	a plane of bidegree (0, 1)	$\langle D_2, D_2 \rangle$
	(d1)	$\mathbb{P}^5$	$\mathbb{Q}^2$ with $(\mathcal{O}_{\mathbb{P}}(1))_{\mathbb{Q}} \simeq \mathcal{O}_{\mathbb{Q}}(1)$	$\langle D_2, S \rangle$
3	(e1)	$\mathbb{P}^1 \times \mathbb{Q}^4$	$\mathbb{P}^1 \times l$ with $l$ a line in $\mathbb{Q}^4$	$\langle F, F, D_2 \rangle$
	(e2)	$\mathbb{P}^1 \times \mathbb{Q}^4$	$\mathbb{P}^1 \times \Gamma$ with $\Gamma \subset \mathbb{Q}^4$ a conic not contained in a plane $\Pi \subset \mathbb{Q}^4$	$\langle F, F, D_2 \rangle$
	(e3)	$X' \in  \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^4}(1, 1) $	$\mathbb{P}^2$ , a fiber of the projection $X' \rightarrow \mathbb{P}^4$	$\langle F, F, D_2 \rangle$
	(e4)	$X' \in  \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^4}(1, 1) $	$\mathbb{F}_1$ , the complete intersection of $X'$ and three general members of the linear system $ \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^4}(0, 1) $	$\langle F, F, D_2 \rangle$
	(f1)	$\mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(1)^{\oplus 3})$	$\mathbb{P}^2$ , a section corresponding to the surjection $\mathcal{O} \oplus \mathcal{O}(1)^{\oplus 3} \rightarrow \mathcal{O}$	$\langle F, D_2, D_2 \rangle$
	(f2)	$\text{Bl}_{\pi}(\mathbb{P}^5)$ (***)	$\mathbb{P}^2$ , a non trivial fiber of $\text{Bl}_{\pi}(\mathbb{P}^5) \rightarrow \mathbb{P}^5$	$\langle F, D_2, D_2 \rangle$
	(f3)	$\text{Bl}_p(\mathbb{P}^5)$	$\mathbb{F}_1$ , the strict transform of a plane in $\mathbb{P}^5$ through $p$	$\langle F, D_2, D_2 \rangle$
	(f4)	$\text{Bl}_{\pi}(\mathbb{P}^5)$	$\mathbb{P}^2$ , the strict transform of a plane in $\mathbb{P}^5$ not meeting $\pi$	$\langle F, D_2, D_2 \rangle$
4	(g1)	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^3$	$\mathbb{P}^1 \times \mathbb{P}^1 \times \{p\}$	$\langle F, F, F, D_2 \rangle$

In [4], Fano manifolds  $X$  obtained by blowing up a smooth variety  $Y$  along a center  $T$  of dimension  $\dim T \leq i_X - 1$  were classified; the results in this paper show that the case  $\dim T = i_X$  will be far more complicated.

(\*)  $V_d$  denotes a del Pezzo fivefold of degree  $d$ .

(\*\*)  $V_5$  is a hyperplane section of  $\mathbb{G}(1, 4)$ . The bidegree of  $S$  is the bidegree of  $S$  in  $\mathbb{G}(1, 4)$ .

(\*\*\*)  $\text{Bl}_{\pi}(\mathbb{P}^5)$  (resp.  $\text{Bl}_p(\mathbb{P}^5)$ ) denotes the blow-up of  $\mathbb{P}^5$  along a 2-plane  $\pi$  (resp. along a point  $p$ ).

## 2 Preliminaries

### 2.1 Fano-Mori contractions and rational curves

Let  $X$  be a smooth Fano variety of dimension  $n$  and  $K_X$  its canonical divisor. By Mori's *Cone Theorem* the cone  $\text{NE}(X)$  of effective 1-cycles, which is contained in the  $\mathbb{R}$ -vector space  $N_1(X)$  of 1-cycles modulo numerical equivalence, is polyhedral; a face  $\tau$  of  $\text{NE}(X)$  is called an *extremal face* and an extremal face of dimension one is called an *extremal ray*. To every extremal face  $\tau$  one can associate a morphism  $\varphi : X \rightarrow Z$  with connected fibers onto a normal variety; the morphism  $\varphi$  contracts those curves whose numerical class lies in  $\tau$ , and is usually called the *Fano-Mori contraction* (or the *extremal contraction*) associated to the face  $\tau$ . A Cartier divisor  $D$  such that  $D = \varphi^*A$  for an ample divisor  $A$  on  $Z$  is called a *supporting divisor* of the map  $\varphi$  (or of the face  $\tau$ ).

An extremal ray  $R$  is called *numerically effective*, or of *fiber type*, if  $\dim Z < \dim X$ , otherwise the ray is *non nef* or *birational*. We usually denote with  $E = E(\varphi) := \{x \in X \mid \dim \varphi^{-1}(\varphi(x)) > 0\}$  the *exceptional locus* of  $\varphi$ ; if  $\varphi$  is of fiber type then of course  $E = X$ . If the exceptional locus of a birational ray  $R$  has codimension one, the ray and the associated contraction are called *divisorial*, otherwise they are called *small*.

DEFINITION 2.1. An elementary fiber type extremal contraction  $\varphi : X \rightarrow Z$  is called a *scroll* (resp. a *quadric fibration*) if there exists a  $\varphi$ -ample line bundle  $L \in \text{Pic}(X)$  such that  $K_X + (\dim X - \dim Z + 1)L$  (resp.  $K_X + (\dim X - \dim Z)L$ ) is a supporting divisor of  $\varphi$ . An elementary fiber type extremal contraction  $\varphi : X \rightarrow Z$  onto a smooth variety  $Z$  is called a  $\mathbb{P}$ -*bundle* if there exists a vector bundle  $\mathcal{E}$  of rank  $\dim X - \dim Z + 1$  on  $Z$  such that  $X \simeq \mathbb{P}_Z(\mathcal{E})$ ; every equidimensional scroll is a  $\mathbb{P}$ -bundle by [10, Lemma 2.12].

DEFINITION 2.2. Let  $\text{Ratcurves}^n(X)$  be the normalized space of rational curves in  $X$  in the sense of [16]; a *family of rational curves* will be an irreducible component  $V \subset \text{Ratcurves}^n(X)$ . Given a rational curve  $f : \mathbb{P}^1 \rightarrow X$  we call a *family of deformations* of  $f$  any irreducible component  $V \subset \text{Ratcurves}^n(X)$  containing the equivalence class of  $f$ .

We define  $\text{Locus}(V)$  to be the subset of  $X$  of points through which there is a curve parametrized by  $V$ ; we say that  $V$  is a *dominating family* if  $\overline{\text{Locus}(V)} = X$ . Moreover, for every point  $x \in \text{Locus}(V)$ , we will denote by  $V_x$  the subscheme of  $V$  parametrizing rational curves passing through  $x$ .

DEFINITION 2.3. Let  $V$  be a family of rational curves on  $X$ . We say that  $V$  is *unsplit* if it is proper and that  $V$  is *locally unsplit* if every component of  $V_x$  is proper for the general  $x \in \text{Locus}(V)$ .

PROPOSITION 2.4. [16, IV.2.6] *Let  $X$  be a smooth projective variety,  $V$  a family of rational curves and  $x \in \text{Locus}(V)$  such that every component of  $V_x$  is proper. Then*

- (a)  $\dim X - K_X \cdot V \leq \dim \text{Locus}(V) + \dim \text{Locus}(V_x) + 1$ ;
- (b)  $-K_X \cdot V \leq \dim \text{Locus}(V_x) + 1$ .

In case  $V$  is the unsplit family of deformations of a minimal extremal rational curve, Proposition 2.4 gives the *fiber locus inequality*:

PROPOSITION 2.5. [13, 30] *Let  $\varphi$  be a Fano-Mori contraction of  $X$  and  $E$  its exceptional locus. Let  $F$  be an irreducible component of a (non trivial) fiber of  $\varphi$ . Then*

$$\dim E + \dim F \geq \dim X + l - 1$$

where  $l = \min\{-K_X \cdot C \mid C \text{ is a rational curve in } F\}$ .

If  $\varphi$  is the contraction of a ray  $R$ , then  $l$  is called the *length of the ray*.

DEFINITION 2.6. We define a *Chow family of rational curves*  $\mathcal{V}$  to be an irreducible component of  $\text{Chow}(X)$  parametrizing rational and connected 1-cycles.

If  $V$  is a family of rational curves, the closure of the image of  $V$  in  $\text{Chow}(X)$  is called the *Chow family associated to  $V$* .

DEFINITION 2.7. Let  $X$  be a smooth variety,  $\mathcal{V}^1, \dots, \mathcal{V}^k$  Chow families of rational curves on  $X$  and  $Y$  a subset of  $X$ . We denote by  $\text{Locus}(\mathcal{V}^1, \dots, \mathcal{V}^k)_Y$  the set of points  $x \in X$  that can be joined to  $Y$  by a connected chain of  $k$  cycles belonging respectively to the families  $\mathcal{V}^1, \dots, \mathcal{V}^k$ . We denote by  $\text{ChLocus}_m(\mathcal{V}^1, \dots, \mathcal{V}^k)_Y$  the set of points  $x \in X$  that can be joined to  $Y$  by a connected chain of at most  $m$  cycles belonging to the families  $\mathcal{V}^1, \dots, \mathcal{V}^k$ .

DEFINITION 2.8. Let  $V^1, \dots, V^k$  be unsplit families on  $X$ . We will say that  $V^1, \dots, V^k$  are *numerically independent* if their numerical classes  $[V^1], \dots, [V^k]$  are linearly independent in the vector space  $N_1(X)$ . When moreover  $C \subset X$  is a curve, we will say that  $V^1, \dots, V^k$  are numerically independent from  $C$  if the class of  $C$  in  $N_1(X)$  is not contained in the vector subspace generated by  $[V^1], \dots, [V^k]$ .

LEMMA 2.9. [1, Lemma 5.4] *Let  $Y \subset X$  be a closed subset and  $V$  an unsplit family. Assume that curves contained in  $Y$  are numerically independent from curves in  $V$ , and that  $Y \cap \text{Locus}(V) \neq \emptyset$ . Then for a general  $y \in Y \cap \text{Locus}(V)$*

- (a)  $\dim \text{Locus}(V)_Y \geq \dim(Y \cap \text{Locus}(V)) + \dim \text{Locus}(V_y)$ ;
- (b)  $\dim \text{Locus}(V)_Y \geq \dim Y - K_X \cdot V - 1$ .

Moreover, if  $V^1, \dots, V^k$  are numerically independent unsplit families such that curves contained in  $Y$  are numerically independent from curves in  $V^1, \dots, V^k$ , then either  $\text{Locus}(V^1, \dots, V^k)_Y = \emptyset$  or

$$(c) \dim \text{Locus}(V^1, \dots, V^k)_Y \geq \dim Y + \sum(-K_X \cdot V^i) - k.$$

DEFINITION 2.10. We define on  $X$  a relation of *rational connectedness with respect to*  $\mathcal{V}^1, \dots, \mathcal{V}^k$  in the following way:  $x$  and  $y$  are in  $\text{rc}(\mathcal{V}^1, \dots, \mathcal{V}^k)$ -relation if there exists a chain of rational curves in  $\mathcal{V}^1, \dots, \mathcal{V}^k$  which joins  $x$  and  $y$ , i.e. if  $y \in \text{ChLocus}_m(\mathcal{V}^1, \dots, \mathcal{V}^k)_x$  for some  $m$ . If all the points of  $X$  are in  $\text{rc}(\mathcal{V}^1, \dots, \mathcal{V}^k)$ -relation we say that  $X$  is  $\text{rc}(\mathcal{V}^1, \dots, \mathcal{V}^k)$ -connected.

To the  $\text{rc}(\mathcal{V}^1, \dots, \mathcal{V}^k)$ -relation we can associate a fibration, at least on an open subset. (see [16, IV.4.16]); we will call it  $\text{rc}(\mathcal{V}^1, \dots, \mathcal{V}^k)$ -fibration.

DEFINITION 2.11. Let  $\mathcal{V}$  be the Chow family associated to a family of rational curves  $V$ . We say that  $V$  is *quasi-unsplit* if every component of any reducible cycle in  $\mathcal{V}$  is numerically proportional to  $V$ .

*Notation:* Let  $T$  be a subset of  $X$ . We write  $N_1^X(T) = \langle V^1, \dots, V^k \rangle$  if the numerical class in  $X$  of every curve  $C \subset T$  can be written as  $[C] = \sum_i a_i [C_i]$ , with  $a_i \in \mathbb{Q}$  and  $C_i \in V^i$ . We write  $\text{NE}^X(T) = \langle V^1, \dots, V^k \rangle$  (or  $\text{NE}^X(T) = \langle R_1, \dots, R_k \rangle$ ) if the numerical class in  $X$  of every curve  $C \subset T$  can be written as  $[C] = \sum_i a_i [C_i]$ , with  $a_i \in \mathbb{Q}_{\geq 0}$  and  $C_i \in V^i$  (or  $[C_i]$  in  $R_i$ ).

PROPOSITION 2.12. [1, Corollary 4.2], [9, Corollary 2.23] *Let  $V$  be a family of rational curves and  $x$  a point in  $\text{Locus}(V)$ .*

(a) *If  $V$  is quasi-unsplit, then  $\text{NE}^X(\text{ChLocus}_m(V)_x) = \langle V \rangle$  for every  $m \geq 1$ ;*

(b) *if  $V_x$  is unsplit, then  $\text{NE}^X(\text{Locus}(V_x)) = \langle V \rangle$ .*

*Moreover, if  $\tau$  is an extremal face of  $\text{NE}(X)$ ,  $F$  is a fiber of the associated contraction and  $V$  is unsplit and independent from  $\tau$ , then*

(c)  $\text{NE}^X(\text{ChLocus}_m(V)_F) = \langle \tau, [V] \rangle$  for every  $m \geq 1$ .

## 2.2 Fano bundles

DEFINITION 2.13. Let  $\mathcal{E}$  be a vector bundle on a smooth complex projective variety  $Z$ . We say that  $\mathcal{E}$  is a *Fano bundle* if  $X = \mathbb{P}_Z(\mathcal{E})$  is a Fano manifold. By [27, Theorem 1.6] if  $\mathcal{E}$  is a Fano bundle over  $Z$  then  $Z$  is a Fano manifold.

M. Szurek and J. Wiśniewski have classified Fano bundles over  $\mathbb{P}^2$  ([26, 28]) and Fano bundles of rank two on surfaces [28]. What follows is a characterization of Fano bundles of rank  $r \geq 2$  over del Pezzo surfaces, which generalizes some results in [28].

PROPOSITION 2.14. *Let  $S_k$  be a del Pezzo surface obtained by blowing up  $k > 0$  points in  $\mathbb{P}^2$ , and let  $\mathcal{E}$  be a Fano bundle of rank  $r \geq 2$  over  $S_k$ ; then, up to twist  $\mathcal{E}$  with a suitable line bundle, the pair  $(S_k, \mathcal{E})$  is one of the following:*

- (i)  $(S_k, \oplus \mathcal{O}^{\oplus r})$ ;
- (ii)  $(S_1, \theta^*(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus(r-1)}))$ ;
- (iii)  $(S_1, \theta^*(T\mathbb{P}^2(-1) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus(r-2)}))$ ,

where  $\theta : S_1 \rightarrow \mathbb{P}^2$  is the blow-up of  $\mathbb{P}^2$  at one point.

PROOF. Let  $\mathcal{E}$  be a Fano bundle of rank  $r \geq 2$  over  $S_k$  and let  $X = \mathbb{P}_{S_k}(\mathcal{E})$ ; by [19, Proposition 3.4] there is a one-to-one correspondence between the extremal rays of  $\text{NE}(S_k)$  and the extremal rays of  $\text{NE}(X)$  spanning a two-dimensional face with the ray  $R_{\mathcal{E}}$  corresponding to the projection  $p : X \rightarrow S_k$ .

Let  $R_{\theta_1} \subset \text{NE}(S_k)$  be an extremal ray of  $S_k$  associated to a blow-up  $\theta_1 : S_k \rightarrow S_{k-1}$ , and call  $E_{\theta_1}$  the exceptional divisor of  $\theta_1$ ; let  $R_{\vartheta_1}$  be the corresponding ray in  $\text{NE}(X)$ , with associated extremal contraction  $\vartheta_1 : X \rightarrow X_1$ .

By [19, Lemma 3.5]  $\vartheta_1$  is birational and has one-dimensional fibers, hence by [3, Theorem 5.2] we have that  $X_1$  is smooth and  $\vartheta_1$  is the blow-up of a smooth subvariety of codimension two in  $X_1$ ; moreover, by [19, Lemma 3.5] and reasons of dimension,  $\text{Exc}(R_{\vartheta_1}) = p^{-1}(E_{\theta_1})$ . The divisor  $E_{\vartheta_1} := \text{Exc}(R_{\vartheta_1})$  has two projective bundle structures: a  $\mathbb{P}^1$  bundle structure over the center of the blow-up and a  $\mathbb{P}^{r-1}$ -bundle structure over  $E_{\theta_1}$ ; by [24, Main theorem] we have that  $E_{\vartheta_1} \simeq \mathbb{P}^1 \times \mathbb{P}^{r-1}$ .

It follows that  $\mathcal{E}|_{E_{\vartheta_1}} \simeq \mathcal{O}^{\oplus r}$ , hence by [2, Lemma 2.9] there exists a vector bundle of rank  $r$  on  $S_{k-1}$  such that  $\mathcal{E} = \theta_1^* \mathcal{E}_1$ . It is now easy to prove that the induced map  $\mathbb{P}_{S_k}(\theta_1^* \mathcal{E}_1) = X \rightarrow \mathbb{P}_{S_{k-1}}(\mathcal{E}_1)$  is nothing but  $\vartheta_1$ , hence  $X_1 = \mathbb{P}_{S_{k-1}}(\mathcal{E}_1)$ .

Since  $\text{NE}(E_{\vartheta_1}) = \langle R_{\mathcal{E}}, R_{\vartheta_1} \rangle$ , the divisor  $E_{\vartheta_1}$  cannot contain the exceptional locus of another extremal ray of  $X$ ; it follows that  $X_1$  is a Fano manifold by [30, Proposition 3.4].

We iterate the argument  $k$  times, until we find a Fano bundle  $\mathcal{E}_k$  over  $\mathbb{P}^2$  such that, denoted by  $\theta$  and  $\vartheta$  the composition of the contractions  $\theta_i$  and  $\vartheta_i$  respectively,  $\mathcal{E} = \theta^* \mathcal{E}_k$ . We have a commutative diagram

$$\begin{array}{ccc} \mathbb{P}_{S_k}(\mathcal{E}) = X & \xrightarrow{\vartheta} & X_k = \mathbb{P}_{\mathbb{P}^2}(\mathcal{E}_k) \\ \downarrow p & & \downarrow p_k \\ S_k & \xrightarrow{\theta} & \mathbb{P}^2 \end{array}$$

Up to considering the tensor product of  $\mathcal{E}_k$  with a suitable line bundle, we can assume that  $0 \leq c_1(\mathcal{E}_k) \leq r - 1$ ; by [26, Proposition 2.2] we have that  $\mathcal{E}_k$  is nef.

Let  $l$  be a line in  $\mathbb{P}^2$ ; the restriction of  $\mathcal{E}_k$  to  $l$  decomposes as a sum of nonnegative line bundles, hence we can write  $(\mathcal{E}_k)|_l \simeq \bigoplus_{i=0}^{r-1} \mathcal{O}(a_i)$ , with  $a_0 = 0$  and  $a_i \geq 0$ . Let  $\tilde{l}$  be the strict transform of  $l$  in  $S_k$ ; since  $\theta_{\tilde{l}}: \tilde{l} \rightarrow l$  is an isomorphism we have  $\mathcal{E}_{\tilde{l}} \simeq (\mathcal{E}_k)|_l$ ; let  $C_0 \subset X$  be a section of  $p$  over  $\tilde{l}$  corresponding to a surjection  $\mathcal{E}_{\tilde{l}} \rightarrow \mathcal{O} \rightarrow 0$ ; we have

$$(1) \quad 0 < -K_X \cdot C_0 = ra_0 - K_{S_k} \cdot \tilde{l} - \sum_{i=0}^{r-1} a_i = -K_{S_k} \cdot \tilde{l} - c_1(\mathcal{E}_k).$$

Now if  $l$  passes through a point blown up by  $\theta$ , by equation (1) we have  $c_1(\mathcal{E}_k) \leq 1$ . In this case, by the classification in [26], either  $\mathcal{E}_k$  is trivial, or  $\mathcal{E}_k \simeq \mathcal{O}(1) \oplus \mathcal{O}^{\oplus(r-1)}$ , or  $\mathcal{E}_k \simeq T\mathbb{P}^2(-1) \oplus \mathcal{O}^{\oplus(r-2)}$ .

Assume that  $k \geq 2$  and let  $l$  be a line in  $\mathbb{P}^2$  joining two of the blown-up points; again by equation (1) we have  $c_1(\mathcal{E}_k) = 0$ , so only the first case occurs.  $\square$

**PROPOSITION 2.15.** *Let  $\mathcal{E}$  be a Fano bundle of rank  $r \geq 2$  over  $\mathbb{P}^1 \times \mathbb{P}^1$ ; then, up to twist  $\mathcal{E}$  with a suitable line bundle,  $\mathcal{E}$  is one of the following:*

- (i)  $\mathcal{O}^{\oplus r}$ ;
- (ii)  $\mathcal{O}(1, 0) \oplus \mathcal{O}^{\oplus(r-1)}$ ;
- (iii)  $\mathcal{O}(1, 1) \oplus \mathcal{O}^{\oplus(r-1)}$ ;
- (iv)  $\mathcal{O}^{\oplus(r-2)} \oplus \mathcal{O}(1, 0) \oplus \mathcal{O}(0, 1)$ ;
- (v) a vector bundle fitting in the exact sequence  $0 \rightarrow \mathcal{O}(-1, -1) \rightarrow \mathcal{O}^{\oplus(r+1)} \rightarrow \mathcal{E} \rightarrow 0$ .

*In all cases the cone of curves of  $X = \mathbb{P}(\mathcal{E})$  is generated by the ray corresponding to the bundle projection and by other two extremal rays; in case (i) the other rays are of fiber type, in case (ii) one of them is of fiber type and the other correspond to a smooth blow-up while in cases (iii)-(v) both the other rays correspond to smooth blow-ups.*

**PROOF.** We will show the result by induction on  $r$ , the case  $r = 2$  having been established in [28, Main Theorem]. Let  $X = \mathbb{P}(\mathcal{E})$ ; first of all we prove that  $\text{NE}(X)$  is generated by three extremal rays. Let  $R_{\mathcal{E}} \subset \text{NE}(X)$  be the extremal ray corresponding to the projection  $p: X \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ ; since  $\rho_X = 3$  it is enough to prove that any other extremal ray of  $\text{NE}(X)$  lies in a two-dimensional face with  $R_{\mathcal{E}}$ .

Let  $R_{\vartheta}$  be another extremal ray of  $X$  with associated contraction  $\vartheta$  and let  $F$  be a nontrivial fiber of  $\vartheta$ . We claim that  $\dim F = 1$ : in fact, since curves contained in  $F$  are not contracted by  $p$ , we have  $\dim F \leq 2$ , and, if  $\dim F = 2$ , we would have  $X = p^{-1}(p(F))$  and  $\text{NE}(X) = \langle R, R_{\mathcal{E}} \rangle$  by Proposition 2.12 (c), against the fact that  $\rho_X = 3$ . In particular, by Proposition 2.5,  $\vartheta$  cannot be a small contraction.

Let  $V_{\vartheta}$  be a family of rational curves of minimal degree (with respect to some fixed ample line bundle) among the families which dominate the exceptional locus of  $R_{\vartheta}$  and



whose class is in  $R_\vartheta$ . Such a family is quasi-unsplit by the extremality of  $R_\vartheta$  and locally unsplit by the assumptions on its degree.

We claim that  $V_\vartheta$  is horizontal and dominating with respect to  $p$ . This is clear if the contraction  $\vartheta$  associated to  $R_\vartheta$  is of fiber type. Assume that  $\vartheta$  is divisorial, with exceptional locus  $E$ : we cannot have  $E \cdot R_\mathcal{E} = 0$ , otherwise  $E = p^*D$  for some effective divisor  $D$  in  $\mathbb{P}^1 \times \mathbb{P}^1$ ; but every effective divisor on  $\mathbb{P}^1 \times \mathbb{P}^1$  is nef and so  $E$  would be nef, against the fact that  $E \cdot R_\vartheta < 0$ . It follows that  $E \cdot R_\mathcal{E} > 0$ , so  $E$  dominates  $\mathbb{P}^1 \times \mathbb{P}^1$  and thus  $V_\vartheta$  is horizontal and dominating with respect to  $p$ , and the claim is proved.

We can now apply [9, Lemma 2.4] and conclude that  $[V_\vartheta]$  and  $R_\mathcal{E}$  lie in a two-dimensional extremal face of  $\text{NE}(X)$ .

We have thus proved that every extremal ray different from  $R_\mathcal{E}$  lies in a two-dimensional face with  $R_\mathcal{E}$ ; therefore  $\text{NE}(X)$  is generated by three extremal rays. We will call  $R_{\vartheta_1}$  and  $R_{\vartheta_2}$  the two rays different from  $R_\mathcal{E}$ , i.e.,  $\text{NE}(X) = \langle R_\mathcal{E}, R_{\vartheta_1}, R_{\vartheta_2} \rangle$ .

By [19, Proposition 3.4], for every  $i = 1, 2$  we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\vartheta_i} & Z_i \\ p \downarrow & \searrow \psi_i & \downarrow \\ \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{\theta_i} & \mathbb{P}^1 \end{array}$$

where  $\psi_i$  is the contraction of the face of  $\text{NE}(X)$  spanned by  $R_\mathcal{E}$  and  $R_{\vartheta_i}$ .

Let  $x \in \mathbb{P}^1$  and let  $f_x^i$  be the fiber of  $\theta_i$  over  $x$ ; since we can factor  $\psi_i$  as  $\psi_i = \theta_i \circ p$ , the fiber of  $\psi_i$  over  $x$  is  $\mathbb{P}(\mathcal{E}|_{f_x^i})$ . By the smoothness of  $\psi_i$  and adjunction,  $\mathbb{P}(\mathcal{E}|_{f_x^i})$  is a Fano manifold, hence either  $\mathcal{E}|_{f_x^i} \simeq \mathcal{O}(a)^{\oplus r}$  or  $\mathcal{E}|_{f_x^i} \simeq \mathcal{O}(a+1) \oplus \mathcal{O}(a)^{\oplus(r-1)}$ .

Since the degree of  $\mathcal{E}$  does not change as  $x$  varies in  $\mathbb{P}^1$  we have that, for a fixed  $i = 1, 2$ , the splitting type of  $\mathcal{E}$  along the fibers of  $\theta_i$  is constantly  $(a, \dots, a)$  or  $(a+1, a, \dots, a)$ . Up to twist  $\mathcal{E}$  with a line bundle we can assume that its splitting type along the fibers of  $\theta_i$  is constantly  $(0, \dots, 0)$  or  $(1, 0, \dots, 0)$

If for some  $i = 1, 2$  the splitting type of  $\mathcal{E}$  on the fibers of  $\theta_i$  is  $(0, \dots, 0)$  then  $\mathcal{E} \simeq \theta_i^* \mathcal{E}'$ , with  $\mathcal{E}'$  a vector bundle on  $\mathbb{P}^1$ ; hence  $\mathcal{E}$  is decomposable and we are in case (i) or (ii).

Assume now that the splitting type of  $\mathcal{E}$  on the fibers of  $\theta_i$  is  $(1, 0, \dots, 0)$  for  $i = 1$  and  $i = 2$ , and thus  $c_1(\mathcal{E}) = (1, 1)$ . We claim that in this case the contractions  $\vartheta_i : X \rightarrow Z_i$  are birational.

Assume by contradiction that for some  $i$ , say  $i = 1$ , the contraction  $\vartheta_1$  is of fiber type. Let  $x \in \mathbb{P}^1$  be a general point; the fiber of  $Z_1 \rightarrow \mathbb{P}^1$  has dimension strictly smaller than the dimension of  $\psi_1^{-1}(x)$ . It follows that both the restrictions of  $\vartheta_1$  and  $p$  to  $\psi_1^{-1}(x)$  are of fiber type, yet  $\psi_1^{-1}(x) \simeq \text{Bl}_{\mathbb{P}^{r-2}}(\mathbb{P}^r)$ , so it has only one fiber type contraction.

We have already proved that the nontrivial fibers of the contractions  $\vartheta_i$  are one dimensional, hence for every  $i = 1, 2$  the variety  $Z_i$  is smooth and  $\vartheta_i$  is the blow-up of a smooth subvariety of codimension two in  $Z_i$  by [3, Theorem 5.2].

Consider one of the birational contractions of  $X$ , say  $\vartheta_1 : X \rightarrow Z_1$ , and let  $E_1$  be its exceptional locus. For every fiber  $f_x$  of  $\theta_1$  the restriction of  $E_1$  to  $\mathbb{P}_{f_x}(\mathcal{E}|_{f_x})$  is a non nef divisor, hence it is the exceptional divisor of the contraction  $\mathbb{P}_{f_x}(\mathcal{E}|_{f_x}) \rightarrow \mathbb{P}^r$ . In particular  $E_1 \cdot R_{\mathcal{E}} = 1$  and  $E_1$  does not contain any fiber of  $p$ . By [10, Lemma 2.12] the restriction of  $p$  makes  $E_1$  a projective bundle over  $\mathbb{P}^1 \times \mathbb{P}^1$ , that is  $E_1 = \mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{E}')$  with  $\mathcal{E}'$  a rank  $r - 1$  vector bundle over  $\mathbb{P}^1 \times \mathbb{P}^1$ . We will now split the proof in two cases, depending on the sign of the intersection number of  $E_1$  with  $R_{\vartheta_2}$ .

**Case 1.**  $E_1 \cdot R_{\vartheta_2} \leq 0$ .

In this case the line bundle  $-K_X - E_1$  is ample on  $X$ ; therefore its restriction to  $E_1$  is ample,  $E_1$  is a Fano manifold and  $\mathcal{E}'$  is a Fano bundle of rank  $r - 1$  over  $\mathbb{P}^1 \times \mathbb{P}^1$ .

Note also that  $E_1$  has a fiber type contraction different from the bundle projection onto  $\mathbb{P}^1 \times \mathbb{P}^1$ , coming by the blow-up contraction  $\vartheta_1$ , so, by induction, either  $\mathcal{E}'$  is trivial or  $\mathcal{E}' \simeq \mathcal{O}(1, 0) \oplus \mathcal{O}^{\oplus(r-2)}$ . The injection  $\mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{E}') \hookrightarrow \mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{E})$  gives an exact sequence of bundles on  $\mathbb{P}^1 \times \mathbb{P}^1$

$$0 \longrightarrow \mathcal{O}(a, b) \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}' \longrightarrow 0,$$

with  $E_1 = \xi_{\mathcal{E}} + p^*\mathcal{O}(-a, -b)$ . Computing the intersection numbers of  $E_1$  with  $R_{\vartheta_1}$  and  $R_{\vartheta_2}$  and recalling the splitting type of  $\mathcal{E}$  we have the following possibilities:

$$0 \longrightarrow \mathcal{O}(0, 1) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}^{\oplus(r-2)} \oplus \mathcal{O}(1, 0) \longrightarrow 0;$$

$$0 \longrightarrow \mathcal{O}(1, 1) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}^{\oplus(r-1)} \longrightarrow 0.$$

Both these sequences split, so we are in cases (iii) or (iv).

**Case 2.**  $E_1 \cdot R_{\vartheta_2} > 0$ .

By [30, Proposition 3.4]  $Z_1$  is a Fano manifold.  $Z_1$  has a fiber type elementary contraction onto  $\mathbb{P}^1$ . For a general  $x \in \mathbb{P}^1$  the fiber  $\psi_1^{-1}(x) = \mathbb{P}(\mathcal{E}|_{f_x^i})$  is isomorphic to  $\text{Bl}_{\mathbb{P}^{r-2}}(\mathbb{P}^r)$ , hence the fiber of  $Z_1 \rightarrow \mathbb{P}^1$  over  $x$  is isomorphic to  $\mathbb{P}^r$ . It follows that  $Z_1$  has a projective bundle structure over  $\mathbb{P}^1$  (cfr. [19, Lemma 2.17]), so either  $Z_1 \simeq \mathbb{P}^1 \times \mathbb{P}^r$  or  $Z_1 \simeq \text{Bl}_{\mathbb{P}^{r-1}}(\mathbb{P}^{r+1})$ .

The second case cannot happen: in fact, let  $\psi : X \rightarrow \mathbb{P}^{r+1}$  be the contraction of the face spanned by  $R_{\vartheta_1}$  and  $R_{\vartheta_2}$ . Denoting by  $E$  the exceptional divisor of the contraction  $Z_1 \rightarrow \mathbb{P}^r$ , by  $\tilde{E}$  its strict transform in  $X$ , and applying twice the canonical bundle formula for blow-ups we have

$$K_X = \vartheta_1^* K_{Z_1} + E_1 = \psi^* K_{\mathbb{P}^{r+1}} + \vartheta_1^* E + E_1 = \psi^* K_{\mathbb{P}^{r+1}} + \tilde{E} + kE_1.$$

Since  $K_X \cdot R_{\vartheta_2} = -1$  and  $\psi^* K_{\mathbb{P}^{r+1}} \cdot R_{\vartheta_2} = 0$  we have  $\tilde{E} \cdot R_{\vartheta_2} < 0$ . This implies that  $\tilde{E} = E_2$ , and thus  $\tilde{E} \cdot R_{\vartheta_2} = -1$ , yielding  $E_1 \cdot R_{\vartheta_2} = 0$ , a contradiction.

Note that the minimal extremal curves contracted by  $\vartheta_i$  are the minimal sections (those corresponding to the trivial summands) of  $p : \mathbb{P}(\mathcal{E}|_{f_x^i}) \rightarrow \mathbb{P}^1$  along the fibers of  $\theta_i$ ; therefore  $\xi_{\mathcal{E}} \cdot R_{\vartheta_i} = 0$  for  $i = 1, 2$ . Being trivial on the face spanned by  $R_{\vartheta_1}$  and  $R_{\vartheta_2}$  and positive on  $R_{\mathcal{E}}$  the line bundle  $\xi_{\mathcal{E}}$  is nef.

Let  $\psi$  be the contraction of the face spanned by  $R_{\vartheta_1}$  and  $R_{\vartheta_2}$ ; this contraction factors through  $Z_1 \simeq \mathbb{P}^1 \times \mathbb{P}^r$  and therefore is onto  $\mathbb{P}^r$ , since it does not contract curves in  $R_{\mathcal{E}}$ . The line bundle  $\xi_{\mathcal{E}}$  restricts to  $\mathcal{O}(1)$  on the fibers of  $p$ , hence  $\xi_{\mathcal{E}} = \psi^* \mathcal{O}_{\mathbb{P}^r}(1)$ . Therefore  $\xi_{\mathcal{E}}$  (and so  $\mathcal{E}$ ) is spanned and we have an exact sequence on  $\mathbb{P}^1 \times \mathbb{P}^1$ :

$$0 \longrightarrow \mathcal{O}(a, b) \longrightarrow \mathcal{O}^{\oplus(r+1)} \longrightarrow \mathcal{E} \longrightarrow 0,$$

Computing the first Chern class we have  $a = -1, b = -1$  and we are in case (v). In this case  $X = \mathbb{P}(\mathcal{E})$  is a divisor in the linear system  $\mathcal{O}(1, 1, 1)$  in  $\mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O}^{\oplus(r+1)}) \simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^r$ .  $\square$

### 2.3 Surfaces in $\mathbb{G}(1, 4)$

Let  $\mathbb{G}(r, n)$  be the Grassmann variety of projective  $r$ -spaces in  $\mathbb{P}^n$ , embedded in  $\mathbb{P}^N$  via the Plücker embedding. We will denote a point in  $\mathbb{G}(r, n)$  by a capital letter, and the corresponding linear space in  $\mathbb{P}^n$  by the same small letter.

Consider the Schubert cycles  $\Omega_1 := \Omega(0, 1, \dots, r-1, r+2)$  and  $\Omega_2 := \Omega(0, 1, \dots, r-2, r, r+1)$ ; the cohomology class of a surface  $S \subset \mathbb{G}(r, n)$  can be written as  $\alpha\Omega_1 + \beta\Omega_2$ . Recalling that the class of an hyperplane section of  $\mathbb{G}(r, n)$  is the class of the Schubert cycle  $\Omega_H := \Omega(n-r-1, n-r, \dots, n-2, n)$  we obtain that the degree of  $S$  as a subvariety of  $\mathbb{P}^N$  is given by

$$\deg(S) = \alpha\Omega_1\Omega_H^2 + \beta\Omega_2\Omega_H^2 = \alpha + \beta.$$

The integer  $\alpha$  is the number of linear spaces parametrized by  $S$  which meet a general  $(n-r-2)$ -space in  $\mathbb{P}^n$ , as one can see intersecting with the Schubert cycle  $\Omega(n-r-2, n-r+1, n-r+2, \dots, n)$ ; it is called the *order* of  $S$  and denoted by  $\text{ord}(S)$ .

The integer  $\beta$  is the number of linear spaces parametrized by  $S$  which meet a general  $n-r$  space in a line, as one can see intersecting with the Schubert cycle  $\Omega(n-r-1, n-r, n-r+2, \dots, n)$ ; it is called the *class* of  $S$  and denoted by  $\text{cl}(S)$ .

**DEFINITION 2.16.** The *bidegree* of  $S$  is the pair  $(\text{ord}(S), \text{cl}(S))$ . By the discussion above we have that  $\deg S = \text{ord}(S) + \text{cl}(S)$ .

REMARK 2.17. A 2-plane  $\Lambda_\pi^2$  in  $\mathbb{G}(1, 4)$  which parametrizes the family of lines which are contained in a given 2-plane  $\pi \subset \mathbb{P}^4$ , classically called a  $\rho$ -plane, has bidegree  $(0, 1)$ . Moreover, given a point  $L \in \mathbb{G}(1, 4)$  there exists a line in  $\mathbb{G}(1, 4)$  joining  $\Lambda_\pi^2$  and  $L$  if and only if the corresponding line  $l \subset \mathbb{P}^4$  has nonempty intersection with  $\pi$ .

REMARK 2.18. The family of lines through a given point  $p$  in  $\mathbb{P}^4$  is parametrized by a three-dimensional linear space  $\Lambda_p^3 \subset \mathbb{G}(1, 4)$ , classically called a  $\Sigma$ -solid.

A two-dimensional linear subspace of a  $\Sigma$ -solid, classically called a  $\sigma$ -plane, parametrizes the family of lines through a given point in  $\mathbb{P}^4$  which lie in a given hyperplane  $H$ , and has bidegree  $(1, 0)$ ; we will denote it by  $\Lambda_{p,H}^2$ .

Given a  $\sigma$ -plane  $\Lambda_{p,H}^2$  and a point  $L \in \mathbb{G}(1, 4)$  there exists always a line in  $\mathbb{G}(1, 4)$  joining  $\Lambda_{p,H}^2$  and  $L$ . This is clear if  $L$  is contained in the  $\Sigma$ -solid  $\Lambda_p^3$ ; otherwise, let  $\pi$  be the plane  $\subset \mathbb{P}^4$  spanned by  $l$  and  $p$  and let  $q$  be  $l \cap H$  if  $l \not\subset H$  or any point of  $l$  if  $l \subset H$ : the pencil of lines in  $\pi$  with center  $q$  is represented by a line in  $\mathbb{G}(1, 4)$  passing through  $L$  and meeting  $\Lambda_{p,H}^2$ .

EXAMPLE 2.19. If  $\Lambda_\pi^2$  is a 2-plane of bidegree  $(0, 1)$  (a  $\rho$ -plane) then the blow-up of  $\mathbb{G}(1, 4)$  along  $\Lambda_\pi^2$  is a Fano manifold whose other contraction is the blow-up of  $\mathbb{P}^6$  along a cubic threefold contained in a hyperplane (see [25, Theorem XLI]).

If else  $\Lambda_{p,H}^2$  is a 2-plane of bidegree  $(1, 0)$  (a  $\sigma$ -plane) the linear system  $|\mathcal{O}_{\mathbb{G}}(1) \otimes \mathcal{I}_{\Lambda_{p,H}^2}|$  defines a rational map  $\mathbb{G} \dashrightarrow \mathbb{P}^6$  whose image is a quadric cone in  $\mathbb{P}^6$  with zero-dimensional vertex; the blow-up of  $\mathbb{G}(1, 4)$  along  $\Lambda_{p,H}^2$  is a Fano manifold whose other contraction is of fiber type onto this quadric cone. This can be checked by direct computation.

LEMMA 2.20. *Let  $S$  be a surface in  $\mathbb{G}(1, 4)$ . If  $\text{ord}(S) = 0$ , then  $S$  is a plane of bidegree  $(0, 1)$ , while if  $\text{cl}(S) = 0$ , then  $S$  is contained in a  $\Sigma$ -solid.*

PROOF. Let  $I \subset \mathbb{G}(1, 4) \times \mathbb{P}^4$  be the incidence variety. Denote by  $p_1 : I \rightarrow \mathbb{G}(1, 4)$  and  $p_2 : I \rightarrow \mathbb{P}^4$  the projections and let  $\text{Locus}(S) = p_2(p_1^{-1}(S))$ . If  $\text{ord}(S) = 0$ , then the general line of  $\mathbb{P}^4$  does not meet  $\text{Locus}(S)$ ; therefore  $\text{Locus}(S)$  is two-dimensional. Moreover, since  $p_1^{-1}(S)$  is irreducible, also  $\text{Locus}(S)$  is irreducible. Therefore  $\text{Locus}(S)$  is an irreducible surface in  $\mathbb{P}^4$  which contains a two-parameter family of lines. It is easy to prove that  $\text{Locus}(S)$  is a plane, hence  $S$  is the  $\rho$ -plane which parametrizes the lines of  $\text{Locus}(S)$ .

Assume now that  $\text{cl}(S) = 0$ . Since we can identify  $\mathbb{G}(1, 4)$  with the Grassmannian  $\mathbb{G}(2, 4)$  of planes in the dual space  $\mathbb{P}^{4*}$ ,  $S$  can be viewed as a surface which parametrizes a two-dimensional family of planes in  $\mathbb{P}^{4*}$ . The duality exchanges order and class, so  $S$ , as a subvariety of  $\mathbb{G}(2, 4)$ , has order zero, i.e., through a general point of  $\mathbb{P}^{4*}$  there are no planes parametrized by  $S$ .

Denote by  $I^* \subset \mathbb{G}(2, 4) \times \mathbb{P}^{4*}$  the incidence variety, by  $p_1^* : I^* \rightarrow \mathbb{G}(2, 4)$  and  $p_2^* : I^* \rightarrow \mathbb{P}^{4*}$

the projections and define  $\text{Locus}^*(S) = p_2^*(p_1^{*-1}(S))$ . Then  $\dim \text{Locus}^*(S) \leq 3$ .

Therefore  $\text{Locus}^*(S) \subset \mathbb{P}^{4*}$  is an irreducible threefold which contains a two-parameter family of planes. It is easy to prove that in this case  $\text{Locus}^*(S)$  is a hyperplane of  $\mathbb{P}^{4*}$ .

It follows that  $S$  parametrizes a family of planes in  $\mathbb{P}^{4*}$  contained in a hyperplane, and hence, by duality,  $S$  parametrizes a two-dimensional family of lines passing through a point of  $\mathbb{P}^4$ , and it is therefore contained in a  $\Sigma$ -solid.  $\square$

LEMMA 2.21. *Let  $S$  be a surface in  $\mathbb{G}(1,3) \subset \mathbb{P}^5$ . If  $\text{ord}(S) \geq 2$  or  $\text{cl}(S) \geq 2$ , then there exist proper secant lines of  $S$  which are contained in  $\mathbb{G}(1,3)$ .*

PROOF. Let  $p \in \mathbb{P}^3$  be a general point. The order of  $S$  is the number of lines parametrized by  $S$  which pass through  $p$ . Hence, if  $\text{ord}(S) \geq 2$ , there exist at least two lines  $l_1, l_2$  parametrized by  $S$  containing  $p$ . The pencil of lines generated by  $l_1$  and  $l_2$  corresponds to a line in  $\mathbb{G}(1,3)$  joining the points  $L_1, L_2 \in S$ . Since  $p$  is general, the general member of the pencil is not a line parametrized by  $S$ , and hence the corresponding secant is not contained in  $S$ .

Let  $\pi \subset \mathbb{P}^3$  be a general plane; the class of  $S$  is the number of lines parametrized by  $S$  contained in  $\pi$ . So if  $\text{cl}(S) \geq 2$  there exist  $l_1, l_2 \subset \pi$ , and the pencil of lines generated by  $l_1$  and  $l_2$  corresponds to a line in  $\mathbb{G}(1,3)$  joining the points  $L_1$  and  $L_2$ . Since  $\pi$  is general, the general member of the pencil is not a line parametrized by  $S$ , and hence the corresponding secant is not contained in  $S$ .  $\square$

COROLLARY 2.22. *If  $S \subset \mathbb{G}(1,3)$  and  $\deg S \geq 3$  then there exist proper secant lines of  $S$  which are contained in  $\mathbb{G}(1,3)$ .*

PROPOSITION 2.23. *Let  $\mathcal{Q} \subset \mathbb{G}(1,4) \subset \mathbb{P}^9$  be a two-dimensional smooth quadric such that no proper secant of  $\mathcal{Q}$  is contained in  $\mathbb{G}(1,4)$ ; then  $\mathcal{Q}$  is contained in a  $\mathbb{G}(1,3)$  and has bidegree  $(1,1)$ . In particular,  $\mathcal{Q}$  parametrizes the family of lines which lie in a hyperplane  $H \subset \mathbb{P}^4$  and meet two skew lines  $r, s \subset H$ .*

PROOF. We have  $2 = \deg(\mathcal{Q}) = \text{ord}(\mathcal{Q}) + \text{cl}(\mathcal{Q})$ ; by Lemma 2.20 we cannot have  $\text{ord}(\mathcal{Q}) = 0$ . If  $\text{ord}(\mathcal{Q}) = 2$  then  $\text{cl}(\mathcal{Q}) = 0$  and the same Lemma yields that  $\mathcal{Q}$  is contained in a  $\Sigma$ -solid, and in this case all the lines in the  $\Sigma$ -solid meet  $\mathcal{Q}$  and are contained in  $\mathbb{G}(1,4)$ . Therefore  $\text{ord}(\mathcal{Q}) = 1$  and the statement follows by [22, Main Theorem].  $\square$

PROPOSITION 2.24. *Let  $S \subset \mathbb{G}(1,4)$  be a surface of degree three such that no proper secant of  $S$  is contained in  $\mathbb{G}(1,4)$ ; then the bidegree of  $S$  is  $(2,1)$  and  $S$  is not contained in any  $\mathbb{G}(1,3)$ .*

PROOF. We have  $3 = \deg(S) = \text{ord}(S) + \text{cl}(S)$ ; we cannot have  $\text{ord}(S) = 0$  by Lemma 2.20. By the same lemma, if  $\text{ord}(S) = 3$  then  $S$  is contained in a  $\Sigma$ -solid, and in this case

all the lines in the  $\Sigma$ -solid are secant to  $S$  and lie in  $\mathbb{G}(1, 4)$ .

If  $S \subset \mathbb{G}(1, 3)$  then  $S$  has proper secants contained in  $\mathbb{G}(1, 3)$  by Lemma 2.21. Moreover if  $\text{ord}(S) = 1$  then  $S \subset \mathbb{G}(1, 3)$  by [22, Main Theorem].  $\square$

**PROPOSITION 2.25.** *Let  $\mathcal{S} \subset \mathbb{G}(1, 4)$  be a surface of bidegree  $(2, 1)$  not contained in a subgrassmannian  $\mathbb{G}(1, 3)$ . Then  $\mathcal{S}$  parametrizes lines which are contained in a family  $F_1$  of planes of a quadric cone  $\mathcal{C} \subset \mathbb{P}^4$  with zero-dimensional vertex and meet a given line  $m$  which lies in a plane  $\pi_m \in F_2$ , where  $F_2$  is the other family of planes of  $\mathcal{C}$ .*

**PROOF.** Identifying  $\mathbb{G}(1, 4)$  with the Grassmannian  $\mathbb{G}(2, 4)$  of planes in the dual space  $\mathbb{P}^{4*}$ ,  $\mathcal{S}$  can be viewed as a surface which parametrizes a two-dimensional family of planes in  $\mathbb{P}^{4*}$ . The duality exchanges order and class, so  $\mathcal{S}$ , as a subvariety of  $\mathbb{G}(2, 4)$ , has bidegree  $(1, 2)$ . We apply [22, Main theorem] and we have the following description of  $\mathcal{S}$ :

let  $\beta : \text{Bl}_{M^*}(\mathbb{P}^{4*}) \rightarrow \mathbb{P}^{4*}$  be the blow-up of  $\mathbb{P}^{4*}$  along a plane  $M^* \subset \mathbb{P}^{4*}$ . We can write  $\text{Bl}_{M^*}(\mathbb{P}^{4*}) = \mathbb{P}_{\mathbb{P}^1}(\mathcal{E})$ , where  $\mathcal{E} := \mathcal{O}_{\mathbb{P}^1}^3 \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ ; denote by  $p$  the projection  $\text{Bl}_{M^*}(\mathbb{P}^{4*}) \rightarrow \mathbb{P}^1$ . Let  $\mathcal{F}$  be a quotient of  $\mathcal{E}$  with  $\text{rk}(\mathcal{F}) = \text{deg } \mathcal{F} = 2$  and denote by  $p_0 := p|_{\mathbb{P}(\mathcal{F})}$ .

$$\begin{array}{ccc} & \text{Bl}_{M^*}(\mathbb{P}^{4*}) & \\ \beta \swarrow & & \searrow p \\ \mathbb{P}^{4*} & & \mathbb{P}^1 \end{array}$$

Then

$$\mathcal{S} = \mathcal{S}(M^*, \mathcal{F}) := \{\pi \in \mathbb{G}(2, 4) \mid \beta(p_0^{-1}(x)) \subset \pi \subset \beta(p^{-1}(x)) \text{ for some } x \in \mathbb{P}^1\}.$$

Since  $\mathcal{E}$  is nef also  $\mathcal{F}$  is, so  $\mathcal{F} = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$  with  $a, b \geq 0$  and  $a + b = 2$ . Therefore two cases can occur:

- (i)  $a = 1, b = 1$ , i.e.,  $\mathbb{P}(\mathcal{F}) \simeq \mathbb{P}^1 \times \mathbb{P}^1$ . In this case the tautological bundle  $\xi_{\mathcal{E}}$  restricts to  $\mathcal{F}$  as  $\mathcal{O}(1, 1)$ , so the image  $\beta(\mathbb{P}(\mathcal{F})) \subset \mathbb{P}^{4*}$  is a smooth quadric  $\mathcal{Q}$ . The plane  $M^*$  contains a line in one ruling of the quadric, and  $\mathcal{S}(M^*, \mathcal{F})$  parametrizes planes in  $\mathbb{P}^{4*}$  which intersect  $M^*$  along this line and contain a line belonging to the other ruling of  $\mathcal{Q}$ .

Passing to the dual we have the claimed description of  $\mathcal{S}$ , where  $m$  is the dual line to the plane  $M^*$ .

- (ii)  $a = 0, b = 2$ , i.e.,  $\mathbb{P}(\mathcal{F}) \simeq \mathbb{F}_2$ . In this case the tautological bundle  $\xi_{\mathcal{E}}$  restricts to  $\mathcal{F}$  as  $C_0 + 2f$ , so the image  $\beta(\mathbb{P}(\mathcal{F})) \subset \mathbb{P}^{4*}$  is a quadric cone whose vertex is a point  $h^* \in M^*$ , therefore all the planes parametrized by  $\mathcal{S}$  pass through  $h^*$ .

It follows that all the lines parametrized by  $\mathcal{S} \subset \mathbb{G}(1, 4)$  are contained in the hyperplane  $H$ , dual to  $h^*$ ; in particular,  $\mathcal{S}$  is contained in  $\mathbb{G}_H(1, 3)$ . This contradicts our hypothesis and thus exclude this case.  $\square$

### 3 Getting started

REMARK 3.1. Let  $X$  be a Fano fivefold with Picard number  $\rho_X \geq 2$  and index  $r_X = 2$ ; then  $X$  has pseudoindex two. In fact, by [1], the generalized Mukai conjecture

$$\rho_X(i_X - 1) \leq \dim X$$

holds for a Fano fivefold, hence we have that  $i_X$  cannot be a multiple of  $r_X = 2$ .

LEMMA 3.2. *Let  $X$  be a Fano fivefold of index two and  $\sigma : X \rightarrow X'$  a birational extremal contraction of  $X$  which contracts a divisor to a surface. Then  $\sigma$  is a smooth blow-up.*

PROOF. Let  $R_\sigma$  be the extremal ray in  $\text{NE}(X)$  corresponding to  $\sigma$ . From the fiber locus inequality we have  $l(R_\sigma) = 2$ , since the general fiber of  $\sigma$  is two-dimensional. Let  $A'$  be a very ample line bundle on  $X'$ ; the line bundle  $A = H \otimes \sigma^*A'$  is relatively ample and  $K_X + 2A = 2\sigma^*A'$  is a supporting divisor for  $\sigma$ . We can thus apply [5, Corollary 5.8.1] to get that  $\sigma$  is equidimensional and the statement then follows from [3, Theorem 5.2].  $\square$

PROPOSITION 3.3. *Let  $X$  be a Fano fivefold of index two which is the blow-up of a smooth variety  $X'$  along a smooth center  $T$ ; then the cone of curves of  $X$  is one among those listed in the following table, where  $F$  denotes a fiber type extremal ray,  $D_i$  denotes a birational extremal ray whose associated contraction contracts a divisor to an  $i$ -dimensional variety and  $S$  denotes a ray whose associated contraction is small:*

$\rho_X$	$R_1$	$R_2$	$R_3$	$R_4$	
2	$F$	$D_0$			(a)
	$F$	$D_2$			(b)
	$D_2$	$D_2$			(c)
	$D_2$	$S$			(d)
3	$F$	$F$	$D_2$		(e)
	$F$	$D_2$	$D_2$		(f)
4	$F$	$F$	$F$	$D_2$	(g)

PROOF. The result will follow from the list in [9, Theorem 1.1], once we have proved that  $X$  has no contractions of type  $D_1$ .

Let  $\sigma : X \rightarrow X'$  be the blow-up of  $X'$  along  $T$ , let  $E$  be the exceptional divisor and let

$l$  be a line in a fiber of  $\sigma$ . Let  $H$  be the fundamental divisor of  $X$ ; from the canonical bundle formula

$$-2H = K_X = \sigma^*K_{X'} + (\text{codim } T - 1)E$$

we know that  $-2H \cdot l = (\text{codim } T - 1)E \cdot l$ , so the codimension of  $T$  is odd. It follows that either  $T$  is a surface in  $X'$  or  $T$  is a point.  $\square$

In this paper we will deal with cases (b), (e) and (f), since the other cases have already been classified; in particular:

- in case (a)  $X' \simeq \mathbb{P}^5$  by [8, Théorème 1].
- As noted in the introduction of [9], for a Fano fivefold of pseudoinde $x$  2 possessing a quasi-unsplit locally unsplit dominating family of rational curves is equivalent to have a fiber type elementary contraction, so, in cases (c) and (d), we can apply [9, Theorem 1.2] and see that either  $X' \simeq \mathbb{P}^5$  and  $T$  is

(c1) a Veronese surface,

(c2)  $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(1) \oplus \mathcal{O}(2))$  embedded in a hyperplane of  $\mathbb{P}^5$  by the tautological bundle (a cubic scroll),

(d1) a two-dimensional smooth quadric (a section of  $\mathcal{O}(2)$  in a linear  $\mathbb{P}^3 \subset \mathbb{P}^5$ ),

or  $X'$  is a del Pezzo manifold of degree five and  $T$  is a plane of bidegree  $(0, 1)$ . This corresponds to case (c3) which arises as the other extremal contraction of case (c2); for a detailed description see [9, Section 3, Example e1].

- In case (g)  $X' \simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^3$  and  $T \simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \{p\}$  by [19, Corollary 5.3].

## 4 Case (b)

### 4.1 Classification of $X'$

We will now prove that if  $X$  is as in case (b) then  $X'$  is either the projective space of dimension five or a del Pezzo manifold of degree  $\leq 5$ .

Assume throughout the section that  $X$  is a Fano fivefold of index two with  $-K_X = 2H$  and Mori cone  $\text{NE}(X) = \langle R_\vartheta, R_\sigma \rangle$ , where  $\vartheta : X \rightarrow Y$  is a fiber type contraction and  $\sigma : X \rightarrow X'$  is a blow-down with center a smooth surface  $S \subset X'$  and exceptional divisor  $E$ . By [7, Theorem 1] we know that  $X'$  is a smooth Fano variety with  $\rho_{X'} = 1$  and  $i_{X'} \geq 2$ ; moreover by the canonical bundle formula

$$K_X = \sigma^*K_{X'} + 2E$$



we have that  $r_{X'}$  is even.

LEMMA 4.1. *Let  $V'$  be a minimal dominating family for  $X'$ ,  $V$  a family of deformations of the strict transform of a general curve in  $V'$  and  $\mathcal{V}$  the Chow family associated to  $V$ . Then  $E \cdot V = 0$ , the family  $\mathcal{V}$  is not quasi-unsplit and  $-K_{X'} \cdot V' = 4$  or  $6$ .*

PROOF. By [16, II.3.7], the general curve in  $V'$  does not intersect  $S$ , so  $E \cdot V = 0$ . It follows that

$$(2) \quad -K_X \cdot V = -K_{X'} \cdot V' \leq \dim X' + 1 = 6.$$

The family  $V$  is dominating and it is not extremal, otherwise  $E$  would be non positive on the whole cone of  $X$ . This implies by [9, Lemma 2.4] that  $X$  is  $\text{rc}\mathcal{V}$ -connected; in particular, since  $\rho_X = 2$ , the family  $\mathcal{V}$  is not quasi-unsplit. Therefore  $-K_{X'} \cdot V' = -K_X \cdot V \geq 4$  so, recalling that  $r_{X'}$  is even, the lemma is proved.  $\square$

If the anticanonical degree of the minimal dominating family  $V'$  is equal to  $6 = \dim X' + 1$  then  $X' \simeq \mathbb{P}^5$  by [14, Theorem 1.1] (Note that the assumptions of the quoted result are different, but the proof actually works in our case since for a very general  $x' \in X'$  the pointed family  $(V')_{x'}$  has the properties 1-3 in [14, Theorem 2.1]).

We are thus left with the case  $-K_{X'} \cdot V' = 4$ , which requires some more work. First of all we will analyze the families of rational curves on  $X$ ; as a consequence we will prove that the exceptional divisor  $E$  of the blow-up is a Fano manifold and that the fiber type extremal contraction of  $X$  restricts to an extremal contraction of  $E$  with the same target  $Y$ . Using the classification of Fano bundles over a surface, given in [26] and [28] and completed in Section 2.2 of the present paper, we will find a line bundle on  $Y$  whose pullback to  $X$  has degree one on the fibers of the blow-up, and this implies the existence of a line bundle on  $X'$  which has degree one on the rational curves of minimal degree in  $X'$ . In this way we will be able to show that  $X'$  is a del Pezzo manifold.

LEMMA 4.2. *Let  $D$  be an effective divisor of  $X$ ; then  $D$  contains curves whose numerical class is in  $R_\sigma$ .*

PROOF. We can assume that  $D \neq E$ , otherwise the statement is trivial. The image of  $D$  via  $\sigma$  is an effective divisor in  $X'$ , hence it is ample since  $\rho_{X'} = 1$ ; therefore  $\sigma(D) \cap S \neq \emptyset$  and so  $D \cap E \neq \emptyset$ . Let  $x$  be a point in  $D \cap E$  and let  $F_x$  be the fiber of  $\sigma$  through  $x$ ; since  $\dim F_x = 2$  then  $D \cap F_x$  contains a curve in  $F_x$ .  $\square$

LEMMA 4.3. *Let  $W$  be an unsplit family of rational curves on  $X$  such that  $\text{Locus}(W) \subseteq E$ ; then  $[W] \in R_\sigma$ .*

PROOF. Let  $F$  be a fiber of  $\sigma$  such that  $F \cap \text{Locus}(W) \neq \emptyset$ ; we have  $\text{Locus}(W)_F \subseteq \text{Locus}(W) \subseteq E$ . Assume that  $[W] \notin R_\sigma$ ; we can apply Lemma 2.9 to get  $\dim \text{Locus}(W)_F = 4$ , so in this case  $E = \text{Locus}(W)_F = \text{Locus}(W)$  and  $\text{NE}^X(E) = \langle [W], R_\sigma \rangle$  by Proposition 2.12 (c).

It follows that  $E$  contains two independent unsplit dominating families, and it is easy to prove that their degree with respect to  $-K_E$  is equal to three; we can therefore apply [20, Theorem 1] and obtain that  $E \simeq \mathbb{P}^2 \times \mathbb{P}^2$ .

The effective divisor  $E$ , being negative on  $R_\sigma$ , must be positive on  $R_\vartheta$ , so  $E$  dominates  $Y$ ; since  $\mathbb{P}^2 \times \mathbb{P}^2$  is a toric variety, by [21, Theorem 1] we have that  $Y \simeq \mathbb{P}^4$ .

Moreover  $\vartheta : X \rightarrow \mathbb{P}^4$  is a  $\mathbb{P}^1$ -bundle by [19, Corollary 2.15]; by [19, Theorem 1.2] it must be  $X \simeq \mathbb{P}_{\mathbb{P}^4}(\mathcal{O} \oplus \mathcal{O}(a))$  with  $a = 1$  or  $a = 3$ , and in these cases  $X$  is not a blow-up along a surface, a contradiction.  $\square$

LEMMA 4.4. *There does not exist on  $X$  any unsplit family of rational curves  $W$  which satisfies all the following conditions:*

- (i)  $-K_X \cdot W = 2$ ;
- (ii)  $[W]$  is not extremal in  $\text{NE}(X)$ ;
- (iii)  $D_W := \text{Locus}(W)$  has dimension 4;
- (iv)  $\text{NE}^X(D_W) \subset \langle R_\sigma, [W] \rangle$ .

PROOF. Assume by contradiction that such a family exists. In this case we have  $D_W \cdot R_\sigma \geq 0$  (otherwise we would have  $D_W = E$  and  $[W] \in R_\sigma$  by Lemma 4.3, against assumption (ii)) and  $D_W \cdot R_\vartheta > 0$  (otherwise  $D_W$  would contain curves in  $R_\vartheta$ , against assumption (iv)); this implies that  $D_W$  is nef, and that it possibly vanishes only on  $R_\sigma$ . By [19, Corollary 2.15] the contraction  $\vartheta : X \rightarrow Y$  is a  $\mathbb{P}^1$ -bundle, i.e.,  $X = \mathbb{P}_Y(\mathcal{E} = \vartheta_* H)$ ; by the classification in [19, Theorem 1.3] (note that we are in case  $\rho_X = 2$ ) this is possible only if  $Y$  is a Fano manifold of index one and pseudoindex two or three; in fact in none of the other cases of [19, Theorem 1.3]  $X$  is the blow-up of a smooth variety along a (smooth) surface.

Let  $V_Y$  be a family of rational curves on  $Y$  with  $-K_Y \cdot V_Y = i_Y$  and let  $\nu : \mathbb{P}^1 \rightarrow Y$  be the normalization of a curve in  $V_Y$ ; the pull-back  $\nu^* \mathcal{E}$  splits as  $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$  in case  $i_Y = 2$ , and as  $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$  in case  $i_Y = 3$ . We have a commutative diagram

$$\begin{array}{ccc} S := \mathbb{P}(\nu^* \mathcal{E}) & \xrightarrow{\bar{\nu}} & X \\ p \downarrow & & \downarrow \vartheta \\ \mathbb{P}^1 & \xrightarrow{\nu} & Y \end{array}$$

Let  $C \subset S$  be a section corresponding to a surjection  $\nu^*\mathcal{E} \rightarrow \mathcal{O}_{\mathbb{P}^1}(1) \rightarrow 0$ , and let  $V_C$  be the family of deformations of  $\bar{\nu}(C)$ ; since  $H \cdot \bar{\nu}(C) = \mathcal{O}_{\mathbb{P}(\nu^*\mathcal{E})}(1) \cdot C = 1$  the family  $V_C$  has anticanonical degree two and is unsplit.

We claim that the numerical class of  $W$  lies in the interior of the cone spanned by  $[V_C]$  and  $R_\vartheta$ ; this is trivial if  $[V_C] \in R_\sigma$ , so we can assume that this is not the case.

The cone of curves of  $S$  is generated by the numerical class of a fiber and the numerical class of  $C$ , i.e.,  $\text{NE}(S) = \langle [C], [f] \rangle$ . The morphism  $\bar{\nu}$  induces a map  $N_1(S) \rightarrow N_1(X)$  which allows us to identify  $\text{NE}(S)$  with the subcone of  $\text{NE}(X)$  generated by  $[V_C]$  and  $R_\vartheta$ . The divisor  $D_W$  is positive on this subcone, hence the effective divisor  $\Gamma = \bar{\nu}^*D_W$  is ample on  $S$ . It follows that  $\Gamma$  lies in the interior of  $\text{NE}(S)$ , hence  $\bar{\nu}(\Gamma)$ , which is a curve in  $D_W$ , lies in the interior of the cone generated by  $[V_C]$  and  $R_\vartheta$ . Therefore also  $[W]$  lies in the interior of the cone generated by  $[V_C]$  and  $R_\vartheta$  by assumption (iv), and we can write

$$[W] = a[C_\vartheta] + b[V_C], \quad \text{with } a, b > 0,$$

where  $C_\vartheta$  is a minimal curve in  $R_\vartheta$ .

Intersecting with  $H$  we get  $a + b = 1$ , and intersecting with  $-\vartheta^*K_Y$  we have

$$-\vartheta^*K_Y \cdot W = bi_Y < i_Y;$$

therefore if  $C_W$  is a curve in  $W$  we have  $-K_Y \cdot \vartheta_*(C_W) < i_Y$ , a contradiction.  $\square$

**PROPOSITION 4.5.** *Let  $V'$  be a minimal dominating family for  $X'$ ,  $V$  a family of deformations of the strict transform of a curve in  $V'$  and  $\mathcal{V}$  the Chow family associated to  $V$ . Assume that  $-K_{X'} \cdot V' = 4$ . Then any irreducible component of a reducible cycle in  $\mathcal{V}$  which is not numerically proportional to  $V$  is a minimal extremal curve.*

**PROOF.** Let  $\Gamma = \sum \Gamma_i$  be a reducible cycle in  $\mathcal{V}$  with  $[\Gamma_1] \neq \lambda[V]$ ; since  $r_X = 2$ ,  $\Gamma$  has exactly two irreducible components. Denote by  $W$  and  $\bar{W}$  their families of deformations, which have anticanonical degree two and so are unsplit.

Since by Lemma 4.1  $E \cdot V = 0$ , we can assume that  $E \cdot W < 0$ , hence by Lemma 4.3 we have that  $[W] \in R_\sigma$ .

As a consequence, note that if  $\Gamma' = \Gamma'_1 + \Gamma'_2$  is another reducible cycle in  $\mathcal{V}$ , then either  $\Gamma'_1$  and  $\Gamma'_2$  are numerically proportional to  $V$  or, denoted by  $W'$  and  $\bar{W}'$  their families of deformations, we can assume that  $[W'] = [W]$  and  $[\bar{W}'] = [\bar{W}]$ .

We claim that  $[\bar{W}]$  is extremal.

**Case 1**  $V$  is not locally unsplit.

Let  $\{\bar{W}^i\}_{i=1,\dots,n}$  be the families of deformations of the irreducible components of cycles in  $\mathcal{V}$  such that  $[\bar{W}^i] = [\bar{W}]$ ; since  $V$  is not locally unsplit, for some index  $i$  the family  $\bar{W}^i$  is dominating. We can then apply [9, Lemma 2.4].

**Case 2**  $V$  is locally unsplit.

Assume by contradiction that  $[\overline{W}]$  is not extremal. By the argument in the proof of Case 1 we have that  $\overline{W}^i$  is not dominating for every  $i$ . By inequality 2.4 (a) we have that  $\dim \text{Locus}(\overline{W}^i) = 3$  or  $4$ ; we distinguish two cases:

(i) There exists an index  $i$  such that  $\dim \text{Locus}(\overline{W}^i) = 4$ .

Let  $D = \text{Locus}(\overline{W}^i)$ ; if  $D \cdot V = 0$  then  $D$  is negative on an extremal ray of  $\text{NE}(X)$ , hence on  $R_\sigma$ , but this implies  $D = E$ , against Lemma 4.3.

Therefore  $D \cdot V > 0$ , hence  $D \cap \text{Locus}(V_x) \neq \emptyset$  for a general  $x \in X$ . Since we are assuming that  $V$  is locally unsplit we have that  $\dim \text{Locus}(V_x) \geq 3$  and  $\text{NE}^X(\text{Locus}(V_x)) = \langle V \rangle$  by Proposition 2.12 (b), so  $\dim \text{Locus}(\overline{W}^i)_{\text{Locus}(V_x)} \geq 4$  by Lemma 2.9 (b) and  $D = \text{Locus}(\overline{W}^i)_{\text{Locus}(V_x)}$ . It follows by [20, Lemma 1] that every curve in  $D$  can be written as  $aC_V + bC_{\overline{W}^i}$  with  $a \geq 0$ ,  $C_V$  a curve contained in  $\text{Locus}(V_x)$  and  $C_{\overline{W}^i}$  a curve in  $\overline{W}^i$ . Therefore  $\text{NE}^X(D) \subset \langle R_\sigma, [\overline{W}^i] \rangle$ , but this is excluded by Lemma 4.4.

(ii) For every  $i$  we have  $\dim \text{Locus}(\overline{W}^i) = 3$ .

By inequality 2.4 (a) we have  $\dim \text{Locus}(\overline{W}_x) = 3$  for every  $x \in \text{Locus}(\overline{W})$ .

Let  $\Omega = \cup_i (\text{Locus}(W^i) \cup \text{Locus}(\overline{W}^i)) = E \cup_i \text{Locus}(\overline{W}^i)$ , and take a point  $y$  outside  $\Omega$ ; since  $X$  is  $\text{rc}\mathcal{V}$ -connected we can join  $y$  and  $\Omega$  with a chain of cycles in  $\mathcal{V}$ . Let  $C$  be the first irreducible component of these cycles which meets  $\Omega$ .

Clearly  $C$  cannot belong to any family  $W^i$  or  $\overline{W}^i$  because it is not contained in  $\Omega$ , so it belongs either to  $V$  or to a family  $\lambda V$  which is numerically proportional to  $V$ ; by [1, Lemma 9.1] we have that either  $C \subset \text{Locus}(V_z)$  for some  $z$  such that  $V_z$  is unsplit or  $C \subset \text{Locus}(\lambda V)$ .

Moreover, since  $E \cdot V = 0$  the intersection  $C \cap \Omega$  is contained in  $\Omega \setminus E$ . Let  $t$  be a point in  $C \cap \Omega$  and let  $\Omega_j = \text{Locus}(\overline{W}^j)$  be the irreducible component of  $\Omega$  which contains  $t$ . If  $C \subset \text{Locus}(V_z)$  we have  $\dim(\text{Locus}(V_z) \cap \Omega_j) \geq 1$ , against the fact that  $N_1^X(V_z) = \langle [V] \rangle$  and  $N_1^X(\Omega_j) = \langle [\overline{W}^j] \rangle$ .

If else  $C \subset \text{Locus}(\lambda V)$  we have that  $\dim \text{Locus}(\lambda V)_{\Omega_j} \geq 4$  by Lemma 2.9 (b) and that  $\text{NE}^X(\text{Locus}(\lambda V)_{\Omega_j}) \subset \langle [\lambda V], R_\vartheta \rangle$  by [20, Lemma 1]; this is clearly impossible if  $\text{Locus}(\lambda V)_{\Omega_j} = X$ , and it contradicts Lemma 4.2 if  $\dim \text{Locus}(\lambda V)_{\Omega_j} = 4$ .

Finally, since  $-K_X \cdot W^i = -K_X \cdot \overline{W}^i = 2$  we also have that the curves of  $W^i$  and  $\overline{W}^i$  are minimal in  $R_\sigma$  and  $R_\vartheta$  respectively.  $\square$

**COROLLARY 4.6.** *In the assumptions of Proposition 4.5, denoting as usual by  $C_\sigma$  and  $C_\vartheta$  minimal rational curves in the rays  $R_\sigma$  and  $R_\vartheta$ , we have, in  $\text{NE}(X)$ ,  $[V] = [C_\sigma] + [C_\vartheta]$ ; in particular we have  $H \cdot C_\vartheta = 1$ .*

PROPOSITION 4.7. *Let  $V'$  be a minimal dominating family for  $X'$ , let  $V$  be a family of deformations of the strict transform of a curve in  $V'$  and assume that  $-K_{X'} \cdot V' = 4$ . Then  $E$  is a Fano manifold and  $X'$  is a del Pezzo manifold.*

PROOF. By Lemma 4.1 we have  $E \cdot V = 0$ , hence  $E \cdot C_\vartheta = -E \cdot C_\sigma = 1$  by Corollary 4.6; It follows that

$$(-K_X - E) \cdot C_\sigma = 2 + 1 = 3$$

$$(-K_X - E) \cdot C_\vartheta = 2 - 1 = 1,$$

hence  $-K_X - E$  is ample on  $X$  by Kleiman criterion. By adjunction  $-K_E = (-K_X - E)|_E$  is ample on  $E$  and  $E$  is a Fano manifold.

We note that  $E$  contains curves of  $R_\vartheta$ : otherwise the fiber type contraction  $\vartheta$  would be a  $\mathbb{P}^1$ -bundle by [19, Lemma 2.13], and since  $E \cdot C_\vartheta = 1$  it follows that  $E$  would be a section of  $\vartheta$ , against the fact that  $\rho_Y = 1$  and  $\rho_E = \rho_S + 1 \geq 2$ .

Consider the divisor  $D = H - E$ : it is nef and vanishes on  $R_\vartheta$ , so it is a supporting divisor for  $\vartheta$ . The restriction  $D|_E$  is nef but not ample, since  $E$  contains curves of  $R_\vartheta$ , so  $D|_E$  is associated to an extremal face of  $\text{NE}(E)$  and to an extremal contraction  $\vartheta_E : E \rightarrow Z$  and we have a commutative diagram:

$$\begin{array}{ccc} E & \longrightarrow & X \\ \vartheta_E \downarrow & \searrow \vartheta|_E & \downarrow \vartheta \\ Z & \longrightarrow & Y \end{array}$$

We will prove that, for every  $m \in \mathbb{N}$  the restriction map  $H^0(X, mD) \rightarrow H^0(E, mD|_E)$  is an isomorphism, hence  $\vartheta|_E = \vartheta_E$  and  $Z = Y$ . Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X(mD - E) \longrightarrow \mathcal{O}_X(mD) \longrightarrow \mathcal{O}_E(mD|_E) \longrightarrow 0.$$

Since  $E$  is not contracted by  $\vartheta$  we have that  $h^0(mD - E) = 0$ ; moreover, we can write

$$mD - E = K_X + (m - 1)D + 3H - 2E.$$

By Kleiman criterion  $3H - 2E$  is ample on  $X$  and, being  $(m - 1)D$  nef, the divisor  $(m - 1)D + 3H - 2E$  is ample, too. By the Kodaira Vanishing Theorem  $h^1(mD - E) = 0$ . We have proved that  $E$  is a Fano manifold, and we know that it has a  $\mathbb{P}^2$ -bundle structure over  $S$ , i.e.,  $E \simeq \mathbb{P}_S(\mathcal{E})$  with  $\mathcal{E}$  a Fano bundle of rank three over  $S$ . This implies that  $S$  is a del Pezzo surface.

Let  $L_Y$  be the ample generator of  $\text{Pic}(Y)$ ; by Proposition 2.14, Proposition 2.15 and the classification in [26], the pull-back of  $L_Y$  has degree one on the fibers of the  $\mathbb{P}^2$ -bundle.

The line bundle  $H - E$  has degree two on the fibers of the  $\mathbb{P}^2$ -bundle and is trivial on the fibers of  $\vartheta$ , hence  $H - E = 2\vartheta^*L_Y$  and so  $H - \vartheta^*L_Y$  is trivial on the fibers of  $\sigma$ , i.e.,  $H - \vartheta^*L_Y = \sigma^*H_{X'}$  for some  $H_{X'} \in \text{Pic}(X')$ . By the canonical bundle formula we have

$$(3) \quad -\sigma^*K_{X'} = -K_X + 2E = 2(H + E) = 4H - 4\vartheta^*L_Y = 4\sigma^*H_{X'},$$

i.e.,  $r_{X'} = 4$  and so  $X'$  is a del Pezzo fivefold.  $\square$

**COROLLARY 4.8.** *By the classification of del Pezzo manifolds given by Fujita [11], denoting by  $d := H_{X'}^5$ , the degree of  $X'$  and recalling that  $\rho_{X'} = 1$ , we have the following possibilities:*

- (i) if  $d = 1$  then  $X' \simeq V_1$  is a degree six hypersurface in the weighted projective space  $\mathbb{P}(3, 2, 1, \dots, 1)$ ;
- (ii) if  $d = 2$  then  $X' \simeq V_2$  is a double cover of  $\mathbb{P}^5$  branched along a smooth quartic hypersurface;
- (iii) if  $d = 3$  then  $X' \simeq V_3$  is a cubic hypersurface in  $\mathbb{P}^6$ ;
- (iv) if  $d = 4$  then  $X' \simeq V_4$  is the complete intersection of two quadrics in  $\mathbb{P}^7$ ;
- (v) if  $d = 5$  then  $X' \simeq V_5$  is a linear section of the grassmannian  $\mathbb{G}(1, 4) \subset \mathbb{P}^9$ .

## 4.2 Classification of $S$

**THEOREM 4.9.** *If  $X' \simeq \mathbb{P}^5$  then  $S$  is as in Theorem 1.1, cases (b1)-(b6).*

**PROOF.** Let  $H$  be a hyperplane of  $\mathbb{P}^5$ , let  $\tilde{H} \subset X$  be its strict transform via  $\sigma$  and let  $\mathcal{H} = \sigma^*H$ . We know that  $\tilde{H}$  is an effective divisor different from  $E$ , hence it is nef; moreover if  $S \subset H$  we can write  $\tilde{H} = \mathcal{H} - kE$  with  $k > 0$ .

Let  $\Gamma$  be a proper bisecant of  $S$ , and let  $\tilde{\Gamma}$  be its strict transform; if  $S \subset H$  we have

$$0 \leq \tilde{H} \cdot \tilde{\Gamma} \leq 1 - 2k;$$

it follows that  $S$  has no proper bisecants, i.e.,  $S$  is a linear subspace of  $\mathbb{P}^5$  and we are in case (b1). If else  $S$  is not contained in any hyperplane, note that  $S$  cannot be the Veronese surface, since the blow-up of  $\mathbb{P}^5$  along a Veronese surface has two birational contractions; therefore the secant variety of  $S$  fills  $\mathbb{P}^5$ .

Let  $l$  be a line in  $\mathbb{P}^5$  not contained in  $S$  and  $\tilde{l}$  its strict transform; we have

$$-K_X \cdot \tilde{l} = \sigma^*\tilde{\mathcal{O}}_{\mathbb{P}^5}(6) \cdot \tilde{l} - 2E \cdot \tilde{l} = 6 - 2(\#(S \cap l)) > 0;$$

therefore if  $l$  is a proper bisecant of  $S$  we have  $-K_X \cdot \tilde{l} = 2$ ; moreover  $S$  cannot have (proper) trisecant lines.

In the notation of [6], the condition on the trisecants is equivalent to the fact that the trisecant variety of  $S$  (which consists of all lines contained in  $S$  and of the proper trisecants) is contained in  $S$ , so by the description in [6] (see in particular Theorem 7, Section 4 and Appendix A2) we have the possibilities (b2)-(b6).

We now show that in all these cases the blow-up of  $X'$  along  $S$  is a Fano manifold with the prescribed cone of curves.

The linear system  $\mathcal{L} = |\mathcal{O}_{\mathbb{P}^5}(2) \otimes \mathcal{I}_S|$  of the quadrics in  $\mathbb{P}^5$  containing  $S$  has  $S$  as its base locus scheme (see [12]), so  $\sigma^*\mathcal{L}$  defines a morphism  $\vartheta : X \rightarrow \mathbb{P}(\mathcal{L})$ .

Since  $2\mathcal{H} - E$  is nef and vanishes on the strict transforms of the bisecants of  $S$  it follows that the numerical class of these curves is extremal in  $\text{NE}(X)$ , and since  $-K_X$  is positive on these curves we can conclude that  $X$  is a Fano manifold.

Moreover since  $S$  is neither degenerate nor the Veronese surface, the bisecants to  $S$  cover  $\mathbb{P}^5$  and so  $\vartheta$  is of fiber type.  $\square$

LEMMA 4.10. *Assume that  $X'$  is a del Pezzo fivefold. Let  $H_{X'} = \mathcal{O}_{X'}(1)$  and  $H_S = (H_{X'})|_S$ . Then*

- (i) *If  $\dim Y = 2$  then  $H_S^2 = \deg X' = -K_S \cdot H_S$ .*
- (ii) *If  $\dim Y = 3$  then  $\deg X' = -K_S \cdot H_S$  and  $\deg X' - H_S^2 \geq 2$ .*
- (iii) *If  $\dim Y = 4$  then  $\deg X' > -K_S \cdot H_S$ .*

PROOF. Denote by  $\mathcal{N}$  the normal bundle of  $S$  in  $X'$  and by  $\mathcal{N}^*$  the conormal bundle; let  $C = \det \mathcal{N}^* \in \text{Pic}(S)$ . Recall that  $E = \mathbb{P}_S(\mathcal{N}^*)$  and that  $-E|_E = \xi_{\mathcal{N}^*}$ .

Let  $\mathcal{H} = \sigma^*H_{X'}$ ; we have

$$\mathcal{H}^5 = (H_{X'})^5 = \deg X' =: d,$$

and since the intersection of three or more sections of a very ample multiple of  $H_{X'}$  does not meet  $S$  we have also

$$\mathcal{H}^4 E = \mathcal{H}^3 E^2 = 0.$$

Then we have

$$\begin{aligned} K_S &= (K_{X'} + \det \mathcal{N})|_S = -4H_S - C, \\ \mathcal{H}^2 E^3 &= (\mathcal{H}^2 E^2)|_E = H_S^2, \\ \mathcal{H} E^4 &= (\mathcal{H} E^3)|_E = (-\mathcal{H} \xi_{\mathcal{N}^*}^3)|_E = -C \cdot H_S, \end{aligned}$$

Let  $L := \mathcal{H} - E$ ; from the above equalities it follows that

$$(4) \quad L^4 \mathcal{H} = \mathcal{H}^5 - 4\mathcal{H}^2 E^3 + \mathcal{H} E^4 = d + K_S \cdot H_S;$$

$$(5) \quad L^3 \mathcal{H}^2 = \mathcal{H}^5 - \mathcal{H}^2 E^3 = d - H_S^2.$$

By Corollary 4.6 we have that  $H \cdot C_\vartheta = 1$ ; then equation (3) yields that  $\mathcal{H} \cdot R_\vartheta = E \cdot R_\vartheta = 1$ , hence  $L$  is trivial on the fibers of  $\vartheta$  and therefore  $L = \vartheta^*L_Y$ .

(i) If  $\dim Y = 2$  we have  $L^4\mathcal{H} = L^3\mathcal{H}^2 = 0$ , so it follows from (4) and (5) that

$$0 = d + K_S \cdot H_S = d - H_S^2.$$

(ii) If  $\dim Y = 3$  then  $L^4\mathcal{H} = 0$ , and so by 4 we have

$$d + K_S \cdot H_S = 0.$$

The contraction  $\vartheta$  is a quadric fibration (see Definition 2.1) and  $\mathcal{H}|_F = \mathcal{O}_F(1)$  for a general fiber  $F$  of  $\vartheta$ ; hence  $L^3\mathcal{H}^2 = (L_Y^3)(\mathcal{H}_F^2) \geq 2$ , and 5 yields that

$$d - H_S^2 \geq 2.$$

(iii) Finally, if  $\dim Y = 4$  the general fiber  $F$  of  $\vartheta$  is one-dimensional and  $\mathcal{H} \cdot F = 1$ , hence  $L^4\mathcal{H} = L_Y^4 > 0$ ; again by 4 we have that

$$d + K_S \cdot H_S > 0.$$

□

LEMMA 4.11. *If  $\dim Y > 2$  then  $S$  is  $\mathbb{P}^2$ , a smooth quadric  $\mathcal{Q}$  or the ruled surface  $\mathbb{F}_1$ , i.e. the blow-up of  $\mathbb{P}^2$  at a point.*

PROOF. By Proposition 4.7  $E$  is a Fano manifold and, by the proof of the same Proposition, we know that the restriction  $\vartheta|_E : E \rightarrow Y$  is an extremal contraction of  $E$ . Moreover, by the classification in Proposition 2.14 we know that for every del Pezzo surface  $S_k$  with  $k \geq 2$  the exceptional divisor  $E$  is isomorphic to  $S_k \times \mathbb{P}^2$ , and in this case  $E$  has no maps on a variety with Picard number one and dimension greater than two. □

THEOREM 4.12. *If  $X'$  is a del Pezzo fivefold then the pairs  $(X', S)$  are as in Theorem 1.1, cases (b7)-(b13).*

PROOF. The contraction  $\vartheta : X \rightarrow Y$  is supported by  $\mathcal{H} - E$ , and is the resolution of the rational map  $\theta : X' \dashrightarrow Y$  defined by the linear system  $\mathcal{L} := \sigma_*|\vartheta^*L_Y|$ , where  $L_Y$  is the ample generator of  $\text{Pic}(Y)$ ; since  $|\vartheta^*L_Y|$  is base point free we have  $\text{Bs } \mathcal{L} \subseteq S$ ; on the other hand  $\mathcal{L} \subseteq |H_{X'} \otimes \mathcal{I}_S|$ , therefore  $\text{Bs } \mathcal{L} \supseteq S$  and so  $\text{Bs } \mathcal{L} = S$ .

It follows that the strict transforms of curves of degree one with respect to  $H_{X'}$  which meet  $S$  are contracted by  $\vartheta$ . Moreover, since  $\mathcal{H} - E$  is nef, no curves of degree one with respect to  $H_{X'}$  and not contained in  $S$  can meet  $S$  in more than one point.

- If  $\dim Y = 2$  then  $\vartheta$  is equidimensional and by [5, Corollary 1.4] we have that  $Y$  is smooth; moreover  $\rho_Y = 1$  and  $Y$  is dominated by a Fano manifold, so  $Y \simeq \mathbb{P}^2$ .

Therefore  $\dim \mathcal{L} = 3$ , so  $S$  is the complete intersection of three general sections in  $|H_{X'}|$  and we are in case (b7).



- In case  $\dim Y = 3$ , if  $S \simeq \mathbb{P}^2$  then  $H_S \simeq \mathcal{O}_{\mathbb{P}^2}(a)$ , with  $a > 0$ . By Lemma 4.10 (ii) we have  $d = -K_{\mathbb{P}^2} \cdot H_{\mathbb{P}^2} = 3a$ ; recalling that  $d \leq 5$  we find  $H_S = \mathcal{O}_{\mathbb{P}^2}(1)$  and  $d = 3$  (case (b8)).

If  $S \simeq \mathbb{P}^1 \times \mathbb{P}^1$  then  $H_S \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a, b)$ , with  $a, b > 0$ . By Lemma 4.10 (ii) we have  $d = -K_{\mathbb{P}^1 \times \mathbb{P}^1} \cdot H_{\mathbb{P}^2} = 2a + 2b$ ; recalling that  $d \leq 5$  we find  $H_S = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$  and  $d = 4$  (case (b10)).

For  $S \simeq \mathbb{F}_1$  we have  $-K_{\mathbb{F}_1} \cdot C \geq 5$  for every ample  $C \in \text{Pic}(\mathbb{F}_1)$ , equality holding if and only if  $C = C_0 + 2f$ ; hence, by Lemma 4.10 (ii) we have  $d = -K_{\mathbb{F}_1} \cdot H_{\mathbb{F}_1} = 5$  and  $H_S = C_0 + 2f$ . Since all the bisecants of  $S$  which are contained in  $\mathbb{G}(1, 4)$  are also contained in a linear section  $V_5$ , it follows by Proposition 2.24 that  $S$  is as in case (b13).

- Finally, in case  $\dim Y = 4$  we can apply Lemma 4.10 (iii) and get: if  $S \simeq \mathbb{P}^2$  then  $H_S = \mathcal{O}(1)$  and  $H_S^2 = 1$ , so  $d = 4$  (case (b9)) or  $d = 5$ ; in the latter case, being  $\vartheta$  of fiber type, we exclude the case of a plane of bidegree  $(0, 1)$  in view of Remark 2.19 and we are in case (b11).

If  $S \simeq \mathbb{P}^1 \times \mathbb{P}^1$  the bound  $-K_S \cdot H_S \geq 4$  gives  $H_S = \mathcal{O}(1, 1)$  and  $H_S^2 = 2$ , hence  $d = 5$ ; in this case  $S$  has bidegree  $(1, 1)$  by Proposition 2.23 and we are in case (b12).

The center of the blow-up cannot be  $\mathbb{F}_1$  since  $-K_{\mathbb{F}_1} \cdot H_{\mathbb{F}_1} \geq 5$ , which contradicts Lemma 4.10 (iii).

We show now that in all these cases the blow-up of  $X'$  along  $S$  is a Fano manifold with the prescribed cone of curves.

Let  $(X', S)$  be a pair as in the theorem and denote by  $H_{X'}$  the fundamental divisor of  $X'$ . We claim that the linear system  $|H_{X'} \otimes \mathcal{I}_S|$  has  $S$  as its base locus scheme; this is clear apart from cases (b10), which is described in Proposition 4.13, and (b12) and (b13), which are treated in Proposition 4.14.

Therefore the linear system  $|\sigma^* H_{X'} - E|$  defines a morphism  $\vartheta : X \rightarrow \mathbb{P}(|\sigma^* H_{X'} - E|)$ . Since  $\sigma^* H_{X'} - E$  is nef and vanishes on the strict transforms of the rational curves of degree one in  $X'$  which meet  $S$ , it follows that the numerical class of these curves is extremal in  $\text{NE}(X)$ . Being  $-K_X$  positive on these curves, we can conclude that  $X$  is a Fano manifold. Finally, since the curves of degree one with respect to  $H_{X'}$  which meet  $S$  cover  $X'$  we have that  $\vartheta$  is a fiber type contraction.  $\square$

**PROPOSITION 4.13.** *Let  $\mathcal{Q}$  be a smooth two-dimensional quadric in  $V_4 \subset \mathbb{P}^7$ . Then  $\mathcal{Q}$  is the intersection of  $V_4$  and the hyperplanes of  $\mathbb{P}^7$  which contain  $\mathcal{Q}$ .*

**PROOF.** Let  $\mathcal{Q}$  be a smooth two-dimensional quadric in  $V_4 = \mathbb{Q} \cap \mathbb{Q}' \subset \mathbb{P}^7$ , and let  $\Lambda_{\mathcal{Q}}^3$  be the three-dimensional linear subspace of  $\mathbb{P}^7$  which contains  $\mathcal{Q}$ .

We claim that  $\Lambda_{\mathcal{Q}}^3$  is contained in one of the two quadrics  $\mathcal{Q}, \mathcal{Q}'$ . From [23, Proposition 2.1] we know that the intersection of two quadrics is smooth if and only if there exist coordinates in  $\mathbb{P}^n$  such that

$$\mathcal{Q} = \left\{ \sum x_i^2 = 0 \right\} \quad \mathcal{Q}' = \left\{ \sum \lambda_i x_i^2 = 0 \right\}$$

with  $\lambda_i \neq \lambda_j$  for every  $i \neq j$ . So assume by contradiction that  $\Lambda_{\mathcal{Q}}^3 \not\subset \mathcal{Q} \cup \mathcal{Q}'$ ; in this case  $\Lambda_{\mathcal{Q}}^3 \cap \mathcal{Q} = \Lambda_{\mathcal{Q}}^3 \cap \mathcal{Q}' = \mathcal{Q}$ , so it must be

$$\left( \sum (1 - \lambda_i) x_i^2 \right) |_{\Lambda_{\mathcal{Q}}^3} \equiv 0.$$

But there is at most one index  $i$  such that  $\lambda_i = 1$ , so the kernel of the quadratic form  $\sum (1 - \lambda_i) x_i^2$  is at most one-dimensional and we reach a contradiction.  $\square$

**PROPOSITION 4.14.** *Let  $S$  be a smooth two-dimensional quadric of bidegree  $(1, 1)$  or a surface of bidegree  $(2, 1)$  not contained in a  $\mathbb{G}(1, 3)$ , in  $V_5 \subset \mathbb{P}^8$ . Then  $S$  is the intersection of  $V_5$  and the hyperplanes of  $\mathbb{P}^8$  which contain  $S$ .*

**PROOF.** Since  $V_5$  is an hyperplane section of  $\mathbb{G}(1, 4)$  we will show that  $S \subset \mathbb{G}(1, 4) \subset \mathbb{P}^9$  is the intersection of  $\mathbb{G}(1, 4)$  and the hyperplanes of  $\mathbb{P}^9$  which contain  $S$ , by finding explicitly its equations.

By Proposition 2.23, if  $S$  is a quadric of bidegree  $(1, 1)$ , then it parametrizes lines in  $\mathbb{P}^4$  which meet two given skew lines  $r, s$ . Up to a change of coordinates in  $\mathbb{P}^4$ , we can assume that  $r$  and  $s$  have equations

$$r : x_0 = x_1 = x_2 = 0, \quad s : x_0 = x_3 = x_4 = 0,$$

so  $H$  is the hyperplane of equation  $x_0 = 0$ ; in this case the equations of  $S$  in  $\mathbb{G}$  are

$$\begin{cases} y_0 = \dots = y_4 = y_9 = 0 \\ y_5 y_8 = y_6 y_7 \end{cases}$$

and  $S$  is the intersection of  $\mathbb{G}$  with the three-dimensional linear subspace  $\Lambda^3 \subset \mathbb{P}^9$  of equations

$$y_0 = \dots = y_4 = y_9 = 0.$$

Let now  $S \subset \mathbb{G}$  be a surface of bidegree  $(2, 1)$  not contained in a  $\mathbb{G}(1, 3)$ , as described in Proposition 2.25. Up to a coordinate change in  $\mathbb{P}^4$ , assume that  $\mathcal{C}$  is the cone of vertex  $(0 : 0 : 0 : 0 : 1)$  on the quadric of equations

$$x_0 x_2 = x_1 x_3, \quad x_4 = 0,$$

and that  $m$  is the line of equations  $x_0 = x_1 = x_4 = 0$ . The two families of planes contained in  $\mathcal{C}$  have equations

$$F_1 = \begin{cases} \lambda x_0 = \mu x_1 \\ \lambda x_3 = \mu x_2 \end{cases}, \quad F_2 = \begin{cases} \lambda x_0 = \mu x_3 \\ \lambda x_1 = \mu x_2 \end{cases},$$

and  $m$  lies in the plane  $\pi_m \in F_2$  of equations  $x_0 = x_1 = 0$ . The equations of the scroll  $S \subset \mathbb{G}$  are

$$\begin{cases} y_0 = y_3 = y_6 = y_7 = 0 \\ y_1 = y_5 \\ y_1^2 = y_2 y_4 \\ y_1 y_8 = y_4 y_9 \\ y_1 y_9 = y_2 y_8 \end{cases}$$

In particular,  $S$  is the intersection of  $\mathbb{G}$  with the four-dimensional linear space  $\Lambda_S^4$  of equations  $y_0 = y_3 = y_6 = y_7 = 0$ ,  $y_1 = y_5$ .  $\square$

## 5 Cases (e)-(f)

**Setup.** Throughout the section, let  $X$  be a Fano fivefold whose cone of curves is as in cases (e)-(f), and let  $\sigma : X \rightarrow X'$  be an extremal contraction of  $X$  which is the blow-up of  $X'$  along a smooth surface.

**PROPOSITION 5.1.** *Let  $X$  be as above. Then either  $X = \mathbb{P}_{\mathbb{P}^2 \times \mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(1, 1))$  or  $X'$  is a Fano manifold of even index.*

**PROOF.** Let  $E$  be the exceptional locus of  $\sigma$ ; by [30, Proposition 3.4]  $X'$  is a Fano manifold unless  $E$  contains the exceptional locus of another extremal ray; this is clearly possible only if  $X$  has another birational contraction, i.e., in case (f). Note that in this case both the birational contractions of  $X$  are smooth blow-ups by Lemma 3.2.

Let  $\bar{\sigma}$  be the other blow-up contraction of  $X$ , denote by  $R_\sigma$  and  $R_{\bar{\sigma}}$  the extremal rays corresponding to  $\sigma$  and  $\bar{\sigma}$  and by  $R_\vartheta$  the extremal ray corresponding to the fiber type contraction  $\vartheta : X \rightarrow Y$ .

Let  $F$  be a fiber of  $\sigma$ ; by Lemma 2.9 (a) we have  $\dim \text{Locus}(R_{\bar{\sigma}})_F \geq 4$ , hence  $E = \text{Locus}(R_{\bar{\sigma}})_F$  and  $\text{NE}^X(E) = \langle R_\sigma, R_{\bar{\sigma}} \rangle$  by Proposition 2.12. Moreover  $E \cdot R_\sigma < 0$  and  $E \cdot R_{\bar{\sigma}} < 0$ , hence  $E \cdot R_\vartheta > 0$  and  $\vartheta$  is a  $\mathbb{P}^1$ -bundle by [19, Corollary 2.15]. We can thus apply [19, Theorem 1.1], noting that the only Fano manifold in the list given in that result with two birational contractions with the same exceptional locus is  $X = \mathbb{P}_{\mathbb{P}^2 \times \mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(1, 1))$ . The claim about the index of  $X'$  follows from the canonical bundle formula for  $\sigma$ .  $\square$

LEMMA 5.2. *Let  $X$  be a Fano fivefold whose cone of curves is as in case (f); denote by  $R_\sigma$  and  $R_{\bar{\sigma}}$  the divisorial extremal rays of  $\text{NE}(X)$ , by  $R_\vartheta$  the fiber type extremal ray and by  $E$  (resp.  $\bar{E}$ ) the exceptional locus of  $R_\sigma$  (resp.  $R_{\bar{\sigma}}$ ). Then either  $E \cdot R_\vartheta > 0$ , or  $\bar{E} \cdot R_\vartheta > 0$ .*

PROOF. Consider a minimal horizontal dominating family  $V$  for  $\vartheta$ .

**Claim.** *The numerical class of  $V$  belongs to a two-dimensional extremal face of  $\text{NE}(X)$  which contains  $R_\vartheta$ .*

If  $V$  is unsplit, since  $\rho_X = 3$  the claim follows from [9, Lemma 2.4].

Denote by  $V_\vartheta$  the family of deformations of a minimal curve in  $R_\vartheta$ . If  $V$  is not unsplit, for a general  $x \in \text{Locus}(V)$  we have that  $\dim \text{Locus}(V_x) \geq 3$  by Proposition 2.4,  $\text{NE}^X(\text{Locus}(V_x)) = \langle V \rangle$  by Proposition 2.12 and  $\dim \text{Locus}(V_\vartheta, V)_x \geq 4$  by Lemma 2.9 (c). Call  $D = \text{Locus}(V_\vartheta, V)_x$ : then  $N_1^X(D) = \langle R_\vartheta, V \rangle$  by [20, Lemma 1], so  $D$  is a divisor since  $\rho_X = 3$ .

It cannot be  $D \cdot R_\vartheta > 0$ , otherwise we could write  $X = \text{ChLocus}(V_\vartheta, V)_x$  and we would have  $\rho_X = 2$ ; so it must be  $D \cdot R_\vartheta = 0$ . This implies that  $D$  is positive on a birational ray, say  $R_\sigma$ , hence  $\dim(D \cap F) \geq 1$  for every fiber  $F$  of  $\sigma$ ; since  $N_1^X(D) = \langle R_\vartheta, V \rangle$  and  $\text{NE}^X(F) = \langle R_\sigma \rangle$ , the claim is proved.

It follows that  $E \cdot R_\vartheta > 0$ : in fact, if  $E \cdot R_\vartheta = 0$  then  $E \cdot V < 0$ , since curves of  $V$  are not contracted by  $\vartheta$  and so they do not belong to  $R_\vartheta$ . But then we would have  $\text{Locus}(V) \subset E$  and  $V$  would not be dominating for  $\vartheta$ , a contradiction.  $\square$

PROPOSITION 5.3. *Let  $X$  be a Fano fivefold whose cone of curves is as in cases (e) – (f), and let  $\sigma : X \rightarrow X'$  be the blow-up of  $X'$  along a smooth surface; assume that  $E$  is positive on a fiber type extremal ray of  $X$ .*

*If  $X'$  is a Fano manifold, then either  $X' \simeq \mathbb{P}^1 \times \mathbb{Q}^4$ , and in this case either  $S \simeq \mathbb{P}^1 \times l$  with  $l$  a line in  $\mathbb{Q}^4$  or  $S \simeq \mathbb{P}^1 \times \Gamma$  with  $\Gamma$  a conic not contained in a plane  $\pi \subset \mathbb{Q}^4$ , or  $X'$  is a  $\mathbb{P}^3$ -bundle over  $\mathbb{P}^2$  and  $S$  dominates  $\mathbb{P}^2$  via the bundle projection.*

PROOF. Let  $R_\vartheta$  be the extremal ray on which  $E$  is positive, and let  $\vartheta : X \rightarrow Y$  be its associated contraction; let  $\psi : X \rightarrow W$  be the contraction of the face spanned by  $R_\sigma$  and  $R_\vartheta$ . Then  $\psi$  factors through  $\sigma$  and a morphism  $\theta : X' \rightarrow W$ , and we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\vartheta} & Y \\ \sigma \downarrow & \searrow \psi & \downarrow \\ X' & \xrightarrow{\theta} & W \end{array}$$

The contractions  $\sigma$  and  $\psi$  have connected fibers, so the same is true for  $\theta$ ; moreover  $W$  is a normal variety with  $\rho_W = \rho_{X'} - 1$  and  $\dim W < \dim X'$ .

It follows that  $\theta$  is an extremal elementary fiber type contraction of the Fano manifold  $X'$ ; denote by  $R_\theta$  the corresponding extremal ray in  $\text{NE}(X')$ .

Let  $V'_\theta$  be a dominating family of rational curves whose numerical class belongs to  $R_\theta$  and whose degree with respect to some ample line bundle is minimal among the degrees of the families with this property. In particular, by the minimality assumption, such a family is locally unsplit. Let  $V$  be the family of deformations of the strict transform in  $X$  of a general curve in  $V'_\theta$ .

Since curves of  $V$  are contracted by  $\psi$ , the numerical class of  $V$  in  $\text{NE}(X)$  lies in the face spanned by  $R_\sigma$  and  $R_\vartheta$ .

By [16, II.3.7], the general curve in  $V'_\theta$  does not intersect the center  $S$  of the blow-up, so  $E \cdot V = 0$ ; it follows that  $[V] \notin R_\vartheta$ . Clearly we cannot have  $[V] \in R_\sigma$ , being  $E \cdot R_\sigma < 0$ , so the class  $[V]$  does not generate an extremal ray of  $X$ . In particular, since  $V$  is dominating and  $X$  has no small contractions,  $V$  cannot be unsplit in view of [9, Lemma 2.29], hence

$$4 \leq -K_X \cdot V = -K_{X'} \cdot V'_\theta.$$

For a general  $x \in X'$  we have, by Proposition 2.4 (b) that  $\dim \text{Locus}(V'_\theta)_x \geq 3$ , so a general fiber of  $\theta$  is at least three-dimensional and  $\dim W \leq 2$ .

If  $\dim W = 1$  then the contraction of the extremal ray of  $X$  different from  $R_\sigma$  and  $R_\vartheta$  is a  $\mathbb{P}^1$ -bundle by [19, Corollary 2.15] (take a fiber of  $\psi$  for  $D$ ). Now we apply [19, Lemma 4.1], to get that  $X$  is a product with  $\mathbb{P}^1$  as a factor; looking at the classification table in [19, Appendix] we find that the only products with  $\rho_X = 3$  and a blow-down contraction of type  $D_2$  are  $X \simeq \mathbb{P}^1 \times \text{Bl}_l(\mathbb{Q}^4)$  or  $X \simeq \mathbb{P}^1 \times \text{Bl}_\Gamma(\mathbb{Q}^4)$ ; the description of  $X'$  and  $S$  follows.

If  $\dim W = 2$  then  $X'$  is a  $\mathbb{P}^3$ -bundle over  $\mathbb{P}^2$  by [19, Lemma 2.18];  $E$  is positive on the fiber type ray  $R_\vartheta$ , so the image via  $\sigma$  of every curve in  $R_\vartheta$  is a curve contracted by  $\theta$  which meets  $S$ . Since  $\vartheta$  is a fiber type contraction, we know that curves in  $R_\vartheta$  dominate  $X$ , hence curves contracted by  $\theta$  which meet  $S$  dominate  $X'$ . Therefore  $S$  dominates  $\mathbb{P}^2$ .  $\square$

**THEOREM 5.4.** *Let  $X$  be a Fano fivefold whose cone of curves is as in cases (e)-(f), and let  $\sigma : X \rightarrow X'$  be the blow-up of  $X'$  along a smooth surface  $S$ . Then the pairs  $(X', S)$  are as in Theorem 1.1, cases (e1)-(e4) or (f1)-(f4).*

**PROOF.** By Proposition 5.1, either  $X \simeq \mathbb{P}_{\mathbb{P}^2 \times \mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(1, 1))$  and therefore  $(X', S)$  is as in case (f1) or we can apply Proposition 5.3: in fact, in case (e) the positivity of  $E$  on a fiber type ray of  $\text{NE}(X)$  is trivial, otherwise it follows from Lemma 5.2.

Therefore either  $(X', S)$  is as in cases (e1)-(e2) or, up to exchange  $\sigma$  with  $\bar{\sigma}$ , we have that

$X'$  is a  $\mathbb{P}^3$ -bundle over  $\mathbb{P}^2$ .

In this case, the classification in [26] yields that  $X'$  is either the blow-up of  $\mathbb{P}^5$  along a plane  $\pi_1$  or  $X' \simeq \mathbb{P}_{\mathbb{P}^2}(T\mathbb{P}^2(-1) \oplus \mathcal{O}^{\oplus 2})$ . Considering the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^2}^{\oplus 5} \longrightarrow T\mathbb{P}^2(-1) \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \longrightarrow 0$$

we see that  $X' = \mathbb{P}_{\mathbb{P}^2}(T\mathbb{P}^2(-1) \oplus \mathcal{O}^{\oplus 2})$  embeds in  $\mathbb{P}^2 \times \mathbb{P}^4$  as a section of  $\mathcal{O}(1, 1)$ .

Let  $l \subset X'$  be a line in a fiber of the  $\mathbb{P}^3$ -bundle not contained in  $S$ , and let  $\tilde{l} \subset X$  be its strict transform; by the canonical bundle formula

$$(6) \quad -K_X \cdot \tilde{l} = -\sigma^* K_{X'} \cdot \tilde{l} - 2E \cdot \tilde{l} \leq 4 - 2\#(S \cap l);$$

since  $X$  is Fano it must be  $\#(S \cap l) \leq 1$ .

Let  $R_{\bar{\theta}} \subset \text{NE}(X')$  be the extremal ray of  $X'$  not associated to the  $\mathbb{P}^3$ -bundle contraction. Let  $C$  be a minimal extremal curve in  $R_{\bar{\theta}}$  not contained in  $S$  and let  $\tilde{C} \subset X$  be its strict transform. Again by the canonical bundle formula

$$-K_X \cdot \tilde{C} = -\sigma^* K_{X'} \cdot \tilde{C} - 2E \cdot \tilde{C} \leq 2 - 2\#(S \cap C),$$

hence  $S \cap C = \emptyset$ . Therefore, if  $S$  meets a two-dimensional fiber  $F_{\bar{\theta}}$  of  $\bar{\theta}$  then  $S = F_{\bar{\theta}}$ .

- In case  $X' \simeq \text{Bl}_{\pi_1}(\mathbb{P}^5)$ , the map  $\bar{\theta}$  is the blow-up map, so denoted by  $E'$  the exceptional divisor of  $\bar{\theta}$  we have that either  $S$  is a fiber of  $\bar{\theta}$  and we are in case (f2), or  $S \cap E' = \emptyset$ ; in particular  $S$  cannot meet a fiber of the  $\mathbb{P}^3$ -bundle in a curve.

In the first case,  $X$  has another blow-down contraction  $\bar{\sigma} : X \rightarrow \text{Bl}_p(\mathbb{P}^5)$ , whose center is the strict transform of a plane passing through  $p$ ; this corresponds to case (f3). In fact,  $X$  can be described as follows: let  $Y$  be the blow-up of  $\mathbb{P}^4$  along a line, let  $E_Y$  be the exceptional divisor, let  $H_Y$  be the pullback of  $\mathcal{O}_{\mathbb{P}^4}(1)$  and let  $\mathcal{E} = (2H_Y + E_Y) \oplus (3H_Y + E_Y)$ . Then  $X = \mathbb{P}_Y(\mathcal{E})$ , and the following diagram shows the extremal contractions of  $X$ :

$$\begin{array}{ccccc}
 & & \mathbb{P}^2 & \xleftarrow{\theta} & \text{Bl}_{\pi_1}(\mathbb{P}^5) \\
 & & \uparrow & & \nearrow \bar{\theta} \\
 & & \text{Bl}_l(\mathbb{P}^4) & \xleftarrow{\vartheta} & X \\
 & & \downarrow & & \searrow \bar{\sigma} \\
 & & \mathbb{P}^4 & \xleftarrow{\quad} & \text{Bl}_p(\mathbb{P}^5) \\
 & & & & \nearrow \\
 & & & & \mathbb{P}^5
 \end{array}$$

In case  $S \cap E' = \emptyset$ , equation (6) yields that  $S$  is a section of the  $\mathbb{P}^3$ -bundle contraction of  $X'$ ; therefore it corresponds to a surjection  $\mathcal{O}^3 \oplus \mathcal{O}(1) \rightarrow \mathcal{O}(1)$ , the image of  $S$  in  $\mathbb{P}^5$  is a plane  $\pi_2$  not meeting  $\pi_1$  and we are in case (f4). In this case  $X \simeq \mathbb{P}_{\mathbb{P}^2 \times \mathbb{P}^2}(\mathcal{O}(0, 1) \oplus \mathcal{O}(1, 0))$ .

- If  $X' \simeq \mathbb{P}_{\mathbb{P}^2}(T\mathbb{P}^2(-1) \oplus \mathcal{O}^{\oplus 2})$  the contraction  $\bar{\theta}$  is of fiber type; it follows that  $S$  is the union of all the fibers of  $\bar{\theta}$  which have nonempty intersection with  $S$  itself.

In particular, either  $S$  is a two-dimensional fiber of  $\bar{\theta}$ , i.e., a section corresponding to a surjection  $T\mathbb{P}^2(-1) \oplus \mathcal{O}^{\oplus 2} \rightarrow \mathcal{O}$ , and we are in case (e3), or  $\bar{\theta}$  is a  $\mathbb{P}^1$ -bundle and  $S$  contains a one-parameter family of fibers isomorphic to  $\mathbb{P}^1$ .

In this last case, the restriction of  $\bar{\theta}$  to  $S$  is a morphism from  $S$  to a curve, and therefore  $S \not\simeq \mathbb{P}^2$ ; so  $S$  cannot be a section of the natural projection  $p : X' \rightarrow \mathbb{P}^2$ . By equation (6) the restriction of  $p$  to  $S$  is a birational morphism  $p|_S : S \rightarrow \mathbb{P}^2$ , and the only surface which is birational to  $\mathbb{P}^2$  and has a morphism on a curve all whose fibers are  $\simeq \mathbb{P}^1$  is the Hirzebruch surface  $\mathbb{F}_1$ . In particular, the exceptional curve of  $S$  is a line in a fiber of  $p$ , therefore  $\bar{\theta}(\mathbb{F}_1) = \bar{\theta}(C_0)$  is a line  $l \subset \mathbb{P}^4$  and  $S$  is the intersection of the pullback of three hyperplanes in  $\mathbb{P}^4$  meeting along  $l$  (case (e4)).

To conclude, we prove the effectiveness of  $X$  in these last two cases: in case (e3) let  $Y$  be a general member of  $|\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^3}(1, 1)|$ , and let  $\mathcal{E} = \mathcal{O}_Y(1, 1) \oplus \mathcal{O}_Y(1, 2)$ ; then  $X \simeq \mathbb{P}_Y(\mathcal{E})$ , as proved in [19, Proposition 7.3], and  $X$  is a  $\mathbb{P}^1$ -ruled Fano manifold. In case (e4)  $X$  can be realized as follows: let  $Z = \text{Bl}_l(\mathbb{P}^4)$ , and let  $H_Z$  be the pullback of  $\mathcal{O}_{\mathbb{P}^4}(1)$ ; then  $X$  is a general section in the linear system  $|p_1^* \mathcal{O}_{\mathbb{P}^2}(1) + p_2^* H|$  in  $\mathbb{P}^2 \times Z$ , where  $p_1$  and  $p_2$  denote the projections onto the factors.  $\square$

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## References

- [1] M. ANDREATTA, E. CHIERICI AND G. OCCHETTA, Generalized Mukai conjecture for special Fano varieties, *Cent. Eur. J. Math.* 2 (2004), 272–293.
- [2] M. ANDREATTA AND G. OCCHETTA, Extending extremal contractions from an ample section, *Adv. Geom.* 2 (2002), 133–146.
- [3] M. ANDREATTA AND G. OCCHETTA, Special rays in the Mori cone of a projective variety, *Nagoya Math. J.* 168 (2002), 127–137.
- [4] M. ANDREATTA AND G. OCCHETTA, Fano manifolds with long extremal rays, *Asian J. Math.* 9 (2005), 523–544.

- [5] M. ANDREATTA AND J. A. WIŚNIEWSKI, On contractions of smooth varieties, *J. Algebraic Geom.* 7 (1998), 253–312.
- [6] I. BAUER, The classification of surfaces in  $\mathbf{P}^5$  having few trisecants, *Rend. Sem. Mat. Univ. Politec. Torino* 56 (1998), 1–20.
- [7] L. BONAVERO, Pseudo-index of Fano manifolds and smooth blow-ups, *Geom. Dedicata* 114 (2005), 79–86.
- [8] L. BONAVERO, F. CAMPANA AND J. A. WIŚNIEWSKI, Variétés complexes dont l'éclatée en un point est de Fano, *C. R. Math. Acad. Sci. Paris* 334 (2002), 463–468.
- [9] E. CHERICI AND G. OCCHETTA, The cone of curves of Fano varieties of coindex four, *Internat. J. Math.* 17 (2006), 1195–1221.
- [10] T. FUJITA, On polarized manifold whose adjoint bundles are not semipositive, *Algebraic geometry, Sendai, Adv. Stud. in Pure Math.* 10 (1987), 167–178.
- [11] T. FUJITA, Classification theories of polarized varieties, volume 155 of *London Mathematical Society Lecture Note Series*, Cambridge University Press, Cambridge, 1990.
- [12] P. IONESCU, Embedded projective varieties of small invariants, In *Algebraic geometry, Bucharest 1982*, volume 1056 of *Lecture Notes in Math.* Springer-Verlag, 1984, 142–186.
- [13] P. IONESCU, Generalized adjunction and applications. *Math. Proc. Camb. Phil. Soc.*, 99:457–472, 1986.
- [14] S. KEBEKUS, Characterizing the projective space after Cho, Miyaoka and Shepherd-Barron, In *Complex geometry (Göttingen, 2000)*, Springer, Berlin, 2002, 147–155.
- [15] S. KOBAYASHI AND T. OCHIAI, Characterizations of complex projective spaces and hyperquadrics, *J. Math. Kyoto Univ.* 13 (1973), 31–47.
- [16] J. KOLLÁR, Rational curves on algebraic varieties, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, Berlin, 1996.
- [17] M. MELLA, Existence of good divisors on Mukai varieties, *J. Algebraic Geom.* 8 (1999), 197–206.
- [18] S. MUKAI, Biregular classification of Fano 3-folds and Fano manifolds of coindex 3, *Proc. Nat. Acad. Sci. U.S.A.* 86 (1989), 3000–3002.



- [19] C. NOVELLI AND G. OCCHETTA, Ruled Fano fivefolds of index two, *Indiana Univ. Math. J.* 56 (2007), 207-241.
- [20] G. OCCHETTA, A characterization of products of projective spaces, *Canad. Math. Bull.* 49 (2006), 270–280.
- [21] G. OCCHETTA AND J. A. WIŚNIEWSKI, On Euler-Jaczewski sequence and Remmert-Van de Ven problem for toric varieties, *Math. Z.* 241 (2002), 35–44.
- [22] Z. RAN, Surfaces of order 1 in Grassmannians, *J. Reine Angew. Math.* 368 (1986), 119–126.
- [23] M. REID, The intersection of two quadrics, PhD thesis, Cambridge University, 1972. [www.maths.warwick.ac.uk/~miles/3folds/qu.ps](http://www.maths.warwick.ac.uk/~miles/3folds/qu.ps).
- [24] E. SATO, Varieties which have two projective space bundle structures, *J. Math. Kyoto Univ.* 25 (1985), 445–457.
- [25] J. G. SEMPLE AND L. ROTH, Introduction to algebraic geometry, Reprint of the 1949 original, Oxford Science Publications, The Clarendon Press Oxford University Press, New York, 1985.
- [26] M. SZUREK AND J. A. WIŚNIEWSKI, On Fano manifolds, which are  $\mathbf{P}^k$ -bundles over  $\mathbf{P}^2$ , *Nagoya Math. J.* 120 (1990), 89–101.
- [27] M. SZUREK AND J. A. WIŚNIEWSKI, Fano bundles over  $\mathbf{P}^3$  and  $\mathbf{Q}^3$ , *Pacific J. Math.* 141 (1990), 197–208.
- [28] M. SZUREK AND J. A. WIŚNIEWSKI, Fano bundles of rank 2 on surfaces, *Compositio Math.* 76 (1990), 295–305.
- [29] J. A. WIŚNIEWSKI, On Fano manifolds of large index, *Manuscripta Math.* 70 (1991), 145–152.
- [30] J. A. WIŚNIEWSKI, On contractions of extremal rays of Fano manifolds, *J. Reine Angew. Math.* 417 (1991), 141–157.

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