ON THE RANKS OF THE THIRD SECANT VARIETY OF SEGRE-VERONESE EMBEDDINGS

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Abstract. We give an upper bound for the rank of the border rank 3 partially symmetric tensors. In the special case of border rank 3 tensors $T \in V_1 \otimes \cdots \otimes V_k$ (Segre case) we can show that all ranks among 3 and $k-1$ arise and if $\dim V_i \geq 3$ for all i's, then also all the ranks between k and $2k-1$ arise.

Introduction

In this paper we deal with the problem of finding a bound for the minimum integer $r(T)$ needed to write a given tensor T as a linear combination of $r(T)$ decomposable tensors. Such a minimum number is now known under the name of *rank of* T. In order to be as general as possible we will consider the tensor T to be partially symmetric, i.e.

$$T \in S^{d_1}V_1 \otimes \cdots \otimes S^{d_k}V_k \tag{1}$$

where the d_i's are positive integers and V_i's are finte dimensional vector spaces defined over an algebraically closed field K. The decomposition that will give us the rank of such a tensor will be of the following type:

$$T = \sum_{i=1}^{r(T)} \lambda_i v_{1,i}^{\otimes d_1} \otimes \cdots \otimes v_{k,i}^{\otimes d_k} \tag{2}$$

where $\lambda_i \in K$ and $v_{j,i} \in V_j$, $i = 1, \ldots, r(T)$ and $j = 1, \ldots, k$.

Another very interesting and useful notion of " rank " is the minimum $r(T)$ such that a tensor T can be written as a limit of a sequence of rank $r(T)$ tensors. This last integer is called the *border rank of* T (Definition 1.5) and clearly it can be strictly smaller than the rank of T (Remark 1.6). It has become a common technique to fix a class of tensors of given border rank and then study all the possible ranks arising in that family (cf. [7, 3, 10, 14, 6]). The rank of tensors of border rank 2 is well known (cf. [7] for symmetric tensors, [2] for tensors without any symmetry, [4] for partially symmetric tensors). The first not completely classified case is the one of border rank 3 tensors. In [7, Theorem 37] the rank of any symmetric order d tensor of border rank 3 has been computed and it is shown that the maximum rank reached is $2d-1$. In the present paper, Theorem 1.7, we prove that the rank of partially symmetric tensors T as in (1) of border rank 3 can be at most

$$r(T) \leq -1 + \sum_{i=1}^{k} 2d_i.$$

In [10, Theorem 1.8] J. Buczyński and J.M. Landsberg described the cases in which the inequality in Theorem 1.7 is an equality: when $k = 3$ and $d_1 = d_2 = d_3 = 1$ they show that there is an element of rank 5. All ranks for border rank 3 partially symmetric tensors are described in [9] when $k = 3$, $d_1 = d_2 = d_3 = 1$ and $n_i = 1$ for at least one integer i. Therefore our Theorem 1.7 is the natural extension of the two extreme cases (tensors without any symmetry where $d_i = 1$ for all $i = 1, \ldots, k$ and totally symmetric case where $k = 1$).

In the special case of tensors without any symmetry, i.e. $T \in V_1 \otimes \cdots \otimes V_k$, we will be able to show, in Theorem 1.8, that all ranks among 3 and $k-1$ arise and if $\dim V_i \geq 3$ for all i's then also all the ranks between k and $2k-1$ arise, therefore this result is sharp (cf. Remark 3.11). In the proof of this theorem we will describe the structure of our solutions: they are all obtained from $(\mathbb{P}^2)^k$ taking as a border scheme a degree 3 connected curvilinear scheme. The two critical cases are rank $k-1$ on $(\mathbb{P}^1)^k$ and rank $2k-1$ on $(\mathbb{P}^2)^k$ and the other cases can be deduced from one of these two.

In [5] we defined the notion of curvilinear rank for symmetric tensors to be the minimum length of a curvilinear scheme whose span contains a given symmetric tensor. We can extend some of the ideas in [5] and some of those used in our proof of Theorem 1.7 to the case of partially symmetric tensors and prove that, if a partially symmetric tensor is contained in the span of a special degree c curvilinear scheme with α components, the rank of this tensor is bounded by $2\alpha+c\left(-1+\sum_{i=1}^{k}d_i\right)$ (cf. Theorem 1.11).

1. Notation, Definitions and Statements

In this section we introduce the basic geometric tools that we will use all along the paper.

Notation 1.1. We indicate with

$$\nu_{n_1^1,\dots,n_k^1}:\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}\to\mathbb{P}^M,\ \text{where } M=\left(\prod_{i=1}^{k}(n_i+1)\right)-1$$

the *Segre embedding* of the multi-projective space $\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}$, i.e. the embedding of $\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}$ by the complete linear system $|\mathcal{O}_{\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}}(1,\dots,1)|$.

For each $i\in\{1,\dots,k\}$ let

$$\pi_i:\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}\to\mathbb{P}^{n_i}$$

denote the projection onto the i-th factor.

Let

$$\tau_i:\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}\to\mathbb{P}^{n_1}\times\cdots\times\hat{\mathbb{P}}^{n_i}\times\cdots\times\mathbb{P}^{n_k}$$

denote the projection onto all the factors different from $\mathbb{P}^{n_i}$.

Let $\varepsilon_i\in\mathbb{N}^k$ be the k-tuple of integers $\varepsilon_i=(0,\dots,1,\dots,0)$ with 1 only in the i-th position. We say that *a curve $C\subset\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}$ has multi-degree* $(a_1,\dots,a_k)$ if for all $i=1,\dots,k$ the line bundle $\mathcal{O}_C(\varepsilon_i)$ has degree a_i.

We say that *a morphism $h:\mathbb{P}^1\to\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}$ has multi-degree* $(a_1,\dots,a_k)$ if, for all $i=1,\dots,k$:

$$h^*\left(\mathcal{O}_{\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}}(\varepsilon_i)\right)\cong\mathcal{O}_{\mathbb{P}^1}(a_i).$$

Let

$$\nu_{n_1^{d_1},\dots,n_k^{d_k}}:\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}\to\mathbb{P}^N,\ \text{where } N=\left(\prod_{i=1}^{k}\binom{d_i+n_i}{n_i}\right)-1$$

denote the Segre-Veronese embedding of $\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}$ of multi-degree $(d_1,\dots,d_k)$ and define

$$X:=\nu_{n_1^{d_1},\dots,n_k^{d_k}}(\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k})$$

to be the *Segre-Veronese variety.*

The name " Segre-Veronese " is classically due to the fact that when the d_i's are all equal to 1, then the variety X is called " Segre variety "; while when $k=1$ then X is known to be a " Veronese variety ".

Remark 1.2. An element of X is the projective class of a decomposable partially symmetric tensor $T\in S^{d_1}V_1\otimes\cdots\otimes S^{d_k}V_k$ where $\mathbb{P}(V_i)=\mathbb{P}^{n_i}$. More precisely $p\in X$ if there exists $T\in S^{d_1}V_1\otimes\cdots\otimes S^{d_k}V_k$ such that $p=[T]=[v_{1,i}^{\otimes d_1}\otimes\cdots\otimes v_{k,i}^{\otimes d_k}]$ with $[v_{j,i}]\in\mathbb{P}^{n_i}$.

Definition 1.3. The *s-th secant variety* $\sigma_s(X)$ of X is the Zariski closure of the union of all s-secant $\mathbb{P}^{s-1}$ to X. The *tangential variety* $\tau(X)$ is the Zariski closure of the union of all tangent lines to X.

Observe that

$$X=\sigma_1(X)\subset\tau(X)\subset\sigma_2(X)\subset\cdots\subset\sigma_{s-1}(X)\subset\sigma_s(X)\subset\cdots\subset\mathbb{P}^N. \tag{3}$$

Definition 1.4. The X-*rank* $r_X(p)$ of an element $p \in \mathbb{P}^N$ is the minimum integer s such that there exist a $\mathbb{P}^{s-1} \subset \mathbb{P}^N$ which is s-secant to X and containing p.

We indicate with $\mathcal{S}(p)$ the set of sets of points of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ " evincing " the X-rank of $p \in \mathbb{P}^N$, i.e.

$$\mathcal{S}(p) := \left\{ \{x_1, \ldots, x_s\} \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \mid r_X(p) = s \text{ and } p \in \langle \nu_{n_1^{d_1}, \ldots, n_k^{d_k}}(x_1), \ldots, \nu_{n_1^{d_1}, \ldots, n_k^{d_k}}(x_s) \rangle \right\}.$$

Definition 1.5. The X-*border rank* $br_X(p)$ of an element $p \in \mathbb{P}^N$ is the minimum integer s such that $p \in \sigma_s(X)$.

Remark 1.6. For any $p \in \mathbb{P}^N = \mathbb{P}(S^{d_1}V_1 \otimes \cdots \otimes S^{d_k}V_k)$ we obviously have that $br_X(p) \leq r_X(P)$. In fact $p \in \mathbb{P}^N$ of rank r is such that there exist a tensor $T \in S^{d_1}V_1 \otimes \cdots \otimes S^{d_k}V_k$ that can be minimally written as in (2); while an element $p \in \mathbb{P}^N$ has border rank s if and only if there exist a sequence of rank r tensors $T_i \in S^{d_1}V_1 \otimes \cdots \otimes S^{d_k}V_k$ such that $\lim_{i \to \infty} T_i = T$ and $p = [T]$.

The first result that we prove in Section 2 is an upper bound for the rank of points in $\sigma_3(X)$.

Theorem 1.7. *The rank of an element* $p \in \sigma_3(X)$ *is* $r_X(p) \leq -1 + \sum_{i=1}^k 2d_i$.

In the case $d_i = 1$ for all $i = 1, \ldots, k$, i.e. if X is the Segre variety, we fill in all low ranks with points of $\sigma_3(X) \setminus \sigma_2(X)$. In Section 3 we prove the following result.

Theorem 1.8. *Assume* $k \geq 3$ *and let* $X \subset \mathbb{P}^M$ *be the Segre variety of* k *factors as in Notation 1.1. Let* α *be the cardinality of the set* $\{i \in \{1, \ldots, k\} \mid n_i \geq 2\}$. *For each* $x \in \{3, \ldots, \alpha + k - 1\}$ *there is* $p \in \sigma_3(X) \setminus \sigma_2(X)$ *with* $r_X(p) = x$.

Remark 1.9. If $\alpha = k$, i.e. if $n_i \geq 2$ for all i's, Theorems 1.7 and 1.8 give the ranks of the points of $\sigma_3(X) \setminus \sigma_2(X)$. In Remark 3.11 we discuss the reason why we do not know the rank of a specific $p \in \sigma_3(X) \setminus \sigma_2(X)$.

Moreover, in the case of Segre varieties where factors have dimension $n_i \geq 2$, Theorem 1.8 says that all ranks from 3 to $2k - 1$ can be attained. Therefore the above result is sharp.

As remarked in the Introduction, we can extend some of the ideas of [5] on the notion of curvilinear rank to some tools used in our proof of Theorem 1.7 to the case of partially symmetric tensors.

Definition 1.10. A scheme $Z \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is *curvilinear* if it is a finite union of disjoint schemes of the form $\mathcal{O}_{C_i, P_i}/m_{p_i}^{e_i}$ for smooth points $p_i \in \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ on reduced curves $C_i \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. Equivalently Z is curvilinear if the tangent space at each of its connected component supported at the p_i's has Zariski dimension ≤ 1. We define the *curvilinear rank* $\mathrm{Cr}(p)$ of a point $p \in \mathbb{P}^N$ as:

$$\mathrm{Cr}(p) := \min \left\{ \deg(Z) \mid \nu_{n_1^{d_1}, \ldots, n_k^{d_k}}(Z) \subset X, \ Z \text{ curvilinear}, \ p \in \langle \nu_{n_1^{d_1}, \ldots, n_k^{d_k}}(Z) \rangle \right\}.$$

In Section 4 we prove the following result.

Theorem 1.11. *If there exists a degree* c *curvilinear scheme* $Z \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ *such that* $p \in \langle \nu_{n_1^{d_1}, \ldots, n_k^{d_k}}(Z) \rangle$ *and* Z *has* α *connected components, each of them mapped by* $\nu_{n_1^{d_1}, \ldots, n_k^{d_k}}$ *into a linearly independent zero-dimensional sub-scheme of* $\mathbb{P}^N$, *then* $r_X(p) \leq 2\alpha + c\left(-1 + \sum_{i=1}^k d_i\right)$.

2. Proof of Theorem 1.7

Remark 2.1. Fix a degree 3 connected curvilinear scheme $E \subset \mathbb{P}^2$ not contained in a line and a point $u \in \mathbb{P}^1$. The scheme E is contained in a smooth conic. Hence there is an embedding $f : \mathbb{P}^1 \to \mathbb{P}^2$ with $f(\mathbb{P}^1) = C$ and $f(3u) = E$.

Remark 2.2. For any couple of points $u, o \in \mathbb{P}^1$, there is an isomorphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ such that $f(u) = o$. For any such f we have $f(3u) = 3o$.

Remark 2.3. Fix two points $u, o \in \mathbb{P}^1$. There is a morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ such that $f(u) = o$, f is ramified at u and $\deg(f) = 2$, i.e. $f^*(\mathcal{O}_{\mathbb{P}^1}(1)) \cong \mathcal{O}_{\mathbb{P}^1}(2)$. Since $\deg(f) = 2$, f has only order 1 ramification at u. Thus $f(3u) = 2o$ (as schemes).

We recall the following lemma proved in [4, Lemma 3.3].

Lemma 2.4 (Autarky). *Let $p \in \langle X\rangle$ with X being the Segre-Veronese variety of $\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}$ embedded in multidegree $(d_1,\dots,d_k)$. If there exist $\mathbb{P}^{m_i}$, $i=1,\dots,k$, with $m_i\le n_i$, such that $p\in\langle \nu_{m_1^{d_1},\dots,m_k^{d_k}}(\mathbb{P}^{m_1}\times\cdots\times\mathbb{P}^{m_k})\rangle$, then the X-rank of p is the same as the Y-rank of p where Y is the Segre-Veronese $\nu_{m_1^{d_1},\dots,m_k^{d_k}}(\mathbb{P}^{m_1}\times\cdots\times\mathbb{P}^{m_k})$.*

Corollary 2.5. *Let $\Gamma\subset\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}$ be a 0-dimensional scheme of minimal degree such that $p\in\langle\nu_{n_1^{d_1},\dots,n_k^{d_k}}(\Gamma)\rangle$, then the X-rank of p is equal to its Y-rank where Y is the Segre-Veronese embedding of $\mathbb{P}^{m_1}\times\cdots\times\mathbb{P}^{m_k}$ where each $m_i=\dim\langle\pi_i(\Gamma)\rangle-1\le\deg(\pi_i(\Gamma))-1$ (π_i as in Notation 1.1). If there exists an index i such that $\deg(\pi_i(\Gamma))=1$, then we can take Y to be the Segre-Veronese embedding of $\mathbb{P}^{m_1}\times\cdots\times\hat{\mathbb{P}}^{m_i}\times\cdots\times\mathbb{P}^{m_k}$.*

Proof. Consider the projections $\pi_i:\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}$ onto the i-th factor $\mathbb{P}^{n_i}$ as in Notation 1.1. It may happen that $\deg(\pi_i(\Gamma))$ can be any value from 1 to $\deg(\Gamma)$.

By the just recalled Autarky Lemma (cf. Lemma 2.4), we may assume that each $\pi_i(\Gamma)$ spans the whole $\mathbb{P}^{n_i}$. Therefore if there is an index $i\in\{1,\dots,k\}$ such that $\deg(\pi_i(\Gamma))=1$ we can take $p\in\mathbb{P}^{n_1}\times\cdots\times\hat{\mathbb{P}}^{n_i}\times\cdots\times\mathbb{P}^{n_k}$. Moreover the autarkic fact that we can assume $\mathbb{P}^{n_i}$ to be $\langle\pi_i(\Gamma)\rangle$ implies that we can replace each $\mathbb{P}^{n_i}$ with $\mathbb{P}^{\dim\langle\pi_i(\Gamma)\rangle-1}$ and clearly $\dim\langle\pi_i(\Gamma)\rangle\le\deg(\pi_i(\Gamma))$. □

Proof of Theorem 1.7: Because of the filtration of secants varieties (3), for a given element $p\in\sigma_3(X)$, it may happen that either $p\in X$, or $p\in\sigma_2(X)\setminus X$ or $p\in\sigma_3(X)\setminus\sigma_2(X)$. We distinguish among these cases.

(1) If $p\in X$, then $r_X(p)=1$.

(2) If $p\in\sigma_2(X)\setminus X$ then either p lies on a honest bisecant line to X (and in this case obviously $r_X(p)=2$) or p belongs to certain tangent line to X. In this latter case, the minimum number $h\le k$ of factors containing such a tangent line is the minimum integer such that $p\in\langle\nu_{n_1^{d_1},\dots,n_h^{d_h}}(\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_h})\rangle$ (maybe reordering factors). In [4, Theorem 3.1] we proved that, if this is the case, then $r_X(p)=\sum_{i=1}^h d_i$.

(3) From now on we assume that $p\in\sigma_3(X)\setminus\sigma_2(X)$. By [10, Theorem 1.2] there is short list of zero-dimensional schemes $\Gamma\subset\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}$ such that $p\in\langle\nu_{n_1^{d_1},\dots,n_k^{d_k}}(\Gamma)\rangle$, therefore, in order to prove Theorem 1.7, it is sufficient to bound the rank of the points in $\langle\nu_{n_1^{d_1},\dots,n_k^{d_k}}(\Gamma)\rangle$ for each Γ in their list.

Since $p\in\sigma_3(X)\setminus\sigma_2(X)$, The possibilities for Γ are only the following: either Γ is a smooth degree 3 zero-dimensional scheme (case (3a) below), or it is the union of a degree 2 scheme supported at one point and a simple point (case (3b)), or it is a curvilinear degree 3 scheme (case (3c)) or, finally, a very particular degree 4 scheme with 2 connected components of degree 2 (case (3d)).

(3a) If Γ is a set of 3 distinct points, then obviously $r_X(p)=3$ ([10, Case (i), Theorem 1.2]).

(3b) If Γ is a disjoint union of a simple point a and a degree 2 connected scheme ([10, Case (ii), Theorem 1.2]), then there is a point q on a tangent line to X such that $p\in\langle\{\nu_{n_1^{d_1},\dots,n_k^{d_k}}(a),q\}\rangle$. Hence $r_X(p)\le 1+r_X(q)\le 1+\sum_{i=1}^k d_i$ (for the rank on the tangential variety of X see [2]). Since $d_i>0$ for all i's and $k\ge 2$, then $1+\sum_{i=1}^k d_i\le -1+\sum_{i=1}^k 2d_i$.

(3c) Assume $\deg(\Gamma)=3$ and that Γ is connected ([10, Case (iii), Theorem 1.2]) supported at a point $\{o\}:=\Gamma_{\mathrm{red}}$. Since the case $k=1$ is true by [7, Theorem 37], we can prove the theorem by using induction on k, with the case $k=1$ as the starting case.
Since $\deg(\Gamma)=3$, by Corollary 2.5, we can assume that p belongs to a Segre-Veronese variety of k factors all of them being either $\mathbb{P}^1$'s or $\mathbb{P}^2$'s, i.e., after having reordered the factors,

$$p\in\nu_{1^{d_1},\dots,1^{d_a},2^{d_{a+1}},\dots 2^{d_k}}(\mathbb{P}^1)^a\times(\mathbb{P}^2)^b.$$

The $\mathbb{P}^1$'s correspond to the cases in which either $\deg(\pi_i(\Gamma))=3$ and $\dim\langle\pi_i(\Gamma)\rangle=1$ (i.e. $\pi_i(\Gamma)$ is contained in a line of the original $\mathbb{P}^{n_i}$), or $\deg(\pi_i(\Gamma))=2$ (notice that in this case $\pi_{i|\Gamma}$ is not an embedding). The $\mathbb{P}^2$'s correspond to the cases in which

$\dim\langle\pi_i(\Gamma)\rangle = 2$, $\deg(\pi_i(\Gamma)) = 3$. Finally we can exclude all the cases in which $\deg(\pi_i(\Gamma)) = 1$ because, again by Corollary 2.5, we would have that p belongs to a Segre-Veronse variety of less factors and then this won't give the highest bound for the rank of p.
Now fix a point $u \in \mathbb{P}^1$. By Remarks 2.1, 2.2 and 2.3 there is

$$f_i : \mathbb{P}^1 \to \mathbb{P}^{n_i} \text{ with } f_i(3u) = \pi_i(\Gamma). \tag{4}$$

Consider the map

$$f = (f_1, \dots, f_k) : \mathbb{P}^1 \to \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}.$$

We have $f(u) = \{o\}$ and $\pi_i(f(3u)) = f_i(3u) = \pi_i(\Gamma)$. Since $\pi_i(f(3u)) = \pi_i(\Gamma)$ for all i's, the universal property of products gives $f(3u) = \Gamma$. The map f has multi-degree $(a_1, \dots, a_k)$ where $a_i = 1$ if $n_i = 1$ and $\deg(\pi_i(\Gamma)) = 3$, and $a_i = 2$ in all other cases. Notice that f_i is an embedding if $\deg(\pi_i(\Gamma)) \neq 2$. Since $\deg(\pi_i(\Gamma)) = 2$ if and only if $\pi_i^{-1}(o_i)$ contains the line spanned by the degree 2 sub-scheme of Γ, we have $\deg(\pi_i(\Gamma)) = 2$ for at most one index i. Since $k \geq 2$, f is an embedding. Set

$$C := \nu_{n_1^{d_1}, \dots, n_k^{d_k}}\left(f\left(\mathbb{P}^1\right)\right) \text{ and } Z := \nu_{n_1^{d_1}, \dots, n_k^{d_k}}(\Gamma).$$

The curve C is smooth and rational of degree $\delta := \sum_{i=1}^k a_i d_i$. Note that $\delta \leq \sum_{i=1}^k 2d_i$. Hence to prove Theorem 1.7 in this case it is sufficient to show that $r_C(p) \leq \delta - 1$ because clearly $r_C(p) \geq r_X(p)$ since $C \subset X$.
By assumption $p \in \langle Z\rangle$. Since $p \notin \sigma_2(X)$, $\langle Z\rangle$ is not a line of $\mathbb{P}^N$. Hence $\langle Z\rangle$ is a plane because $\deg(Z) = \deg(\Gamma) = 3$. Since C is a degree δ smooth rational curve, we have $\dim\langle C\rangle \leq \delta$. By [14, Proposition 5.1] we have $r_C(p) \leq \dim\langle C\rangle$. Hence it is sufficient to prove the case $\delta = \dim\langle C\rangle$, i.e. we may assume that C is a rational normal curve in its linear span.

If $\delta \geq 4$, since Z is connected and of degree 3, by Sylvester's theorem (cf. [11]) we have p has C-border rank 3 and $r_C(p) = \delta - 1$, concluding the proof in this case.

If $\delta \leq 3$, we have $\sigma_2(C) = \langle C\rangle$ and hence $p \in \sigma_2(X)$, contradicting $p \in \sigma_3(X) \setminus \sigma_2(X)$.

(3d) Assume that Γ has degree 4 ([10, Case (iv), Theorem 1.2]). J. Buczyński and J.M. Landsberg show that p belongs to the span of two tangent lines to X whose set theoretic intersections with X span a line which is contained in X. This means that $\Gamma = v \sqcup w$ with v, w being two degree 2 reduced zero-dimensional schemes with support contained in a line $L \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ and moreover that the multi-degree of L is ε_i for some $i = 1, \dots, k$ (cfr. Notation 1.1). This case occurs only when $d_i = 1$, i.e. when $\nu_{n_1^{d_1}, \dots, n_k^{d_k}}(L) = \nu_{n_1^1, \dots, n_k^1}(L) = \tilde{L}$ is a line.
Observe that $\tilde{v} := \nu_{n_1^{d_1}, \dots, n_k^{d_k}}(v)$ and $\tilde{w} := \nu_{n_1^{d_1}, \dots, n_k^{d_k}}(w)$ are two tangent vectors to X. In [2, Theorem 1] we prove that the X-rank of a point $p \in T_o(X)$ for a certain point $o = (o_1, \dots, o_k) \in X$, is the minimum number $\eta_X(p)$ for which there exist $E \subseteq \{1, \dots, k\}$ such that $\sharp(E) = \eta_X(p)$ and $T_o(X) \subseteq \langle \cup_{i\in E} Y_{o,i}\rangle$ where $Y_{o,i}$ is the n_i-dimensional linear subspace obtained by fixing all coordinates $j \in \{1, \dots, k\} \setminus \{i\}$ equal to $o_j \in \mathbb{P}^n_i$. Let I and J be the sets playing the role of E for $\langle\tilde{v}\rangle$ and $\langle\tilde{w}\rangle$ respectively and set $I' = I \setminus \{i\}$ (meaning that $I' = I$ if $i \notin I$ and $I' = I \setminus \{i\}$ otherwise) and $J' = J \setminus \{i\}$. Now take

$$\alpha := \sum_{j\in I'} d_j + \sum_{j\in J'} d_j + d_i$$

and note that $\alpha \leq -1 + \sum_{h=1}^k 2d_h$, therefore if we prove that $r_X(p) \leq \alpha$ we are done. Let $D_j \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, $j \in I'$, be the line of multi-degree ε_J containing $\pi_j(v)$, and let T_j, $j \in J'$, be the line of X of multi-degree ε_j containing $\pi_j(w)$. The curve $L \cup \left(\bigcup_{j\in I'} D_j\right)$ contains v and the curve $L \cup \left(\bigcup_{j\in J'} T_j\right)$ contains w. Hence the curve

$$T := L \cup \left(\bigcup_{j\in I'} D_j\right) \cup \left(\bigcup_{j\in J'} T_j\right)$$

is a reduced and connected curve containing Γ. Since $p \in \langle \nu_{n_1^{d_1},\dots,n_k^{d_k}}(\Gamma)\rangle$, we have that if we call $\tilde{T} := \nu_{n_1^{d_1},\dots,n_k^{d_k}}(T)$ then $p \in \langle \tilde{T}\rangle$ and $r_X(p) \le r_{\tilde{T}}(p)$. The curve $\tilde{T}$ is a connected curve whose irreducible components are smooth rational curves and with $\deg(\tilde{T}) = \alpha$. Hence $\dim\langle \tilde{T}\rangle \le \alpha$. Since $\tilde{T}$ is reduced and connected, as in [14, Proposition 4.1] and in [11], we get $r_{\tilde{T}}(p) \le \alpha$. Summing up $r_X(p) \le r_{\tilde{T}}(p) \le \alpha \le -1 + \sum_{h=1}^{k} 2d_h$.

□

3. Proof of Theorem 1.8

Autarky Lemma (proved in [4, Lemma 3.3] and recalled here in Lemma 2.4) is true also for the border rank ([9, Proposition 2.1]). This allows to formulate the analog of Corollary 2.5 for border rank. Therefore, in order to prove Theorem 1.8 and $x \le k-1$, we can limit ourselves to the study of the case $n_i = 1$ for all i's. This is the reason why in the first part of this section we will always work with the Segre variety of $\mathbb{P}^1$'s. Let

$$\nu_{1^{(k)}} : (\mathbb{P}^1)^k \to \mathbb{P}^r, \ r = 2^k - 1 \tag{5}$$

be the Segre embedding of k copies of $\mathbb{P}^1$'s and $X := \nu_{1^{(k)}}((\mathbb{P}^1)^k)$; and let

$$\nu_{1^{(k-1)}} : (\mathbb{P}^1)^{k-1} \to \mathbb{P}^{r'}, \ r' = 2^{k-1} - 1 \tag{6}$$

be the the Segre embedding of $k-1$ copies of $\mathbb{P}^1$'s and $X' := \nu_{1^{(k-1)}}((\mathbb{P}^1)^{k-1})$.

Proposition 3.1. *Assume $k \ge 3$. Let $\Gamma \subset (\mathbb{P}^1)^k$ be a degree 3 connected curvilinear scheme such that $\deg(\pi_i(\Gamma)) = 3$ for all i's, and let β be the only degree 2 sub-scheme of Γ. For all $p \in \langle \nu_{1^{(k)}}(\Gamma)\rangle \setminus \langle \nu_{1^{(k)}}(\beta)\rangle$ we have that*
(a) if $k = 3$, then $2 \le r_X(p) \le 3$ and $r_X(p) = 2$ if p is general in $\langle \nu_{1^{(k)}}(\Gamma)\rangle$;
(b) if $k \ge 4$, then $r_X(p) = k-1$.

Proof. Since $\Gamma \subset (\mathbb{P}^1)^k$ is connected, it has support at only one point; all along this proof we set

$$o := \mathrm{Supp}(\Gamma) \in (\mathbb{P}^1)^k. \tag{7}$$

First of all recall that in step (3a) of the proof of Theorem 1.7 we obtained an embedding $f = (f_1, \dots, f_k)$ with $f_i : \mathbb{P}^1 \to \mathbb{P}^1$ an isomorphism (see (4)); moreover we can fix a point $u \in \mathbb{P}^1$ such that $f(u) = o$ and $\Gamma = f(3u)$. We proved that

$$C := \nu_{1^{(k)}}(f(\mathbb{P}^1))$$

is a degree k rational normal curve in its linear span. Obviously

$$r_X(p) \le r_C(p).$$

If $k \ge 4$ Sylvester's theorem implies $r_C(p) = k-1$.

Now assume $k = 3$. Since a degree 3 rational plane curve has a unique singular point, for any $q \in \langle C\rangle$ there is a unique line $L \subset \langle C\rangle = \mathbb{P}^3$ with $\deg(L \cap C) = 2$. Thus $r_C(p) = 2$ (resp. $r_C(p) = 3$) if and only if $p \notin \tau(C)$ (resp. $p \in \tau(C)$, cfr. Definition 1.3). Since $\tau(C)$ is a degree 4 surface, by Riemann-Hurwitz, we see that both cases occur and that $r_C(p) = 2$ (and hence $r_X(p) = 2$ if p is general in $\langle \nu_{1^{(k)}}(\Gamma)\rangle$).

Claim 1. *Let the point $o \in (\mathbb{P}^1)^k$ be, as in (7), the support of Γ. Fix any $F \in |\mathcal{O}_{(\mathbb{P}^1)^k}(\varepsilon_k)|$ such that $o \notin F$. Then $\langle \nu_{1^{(k)}}(\Gamma)\rangle \cap \langle \nu_{1^{(k)}}(F)\rangle = \emptyset$.*

Proof of Claim 1. It is sufficient to show that $h^0(\mathcal{I}_{F\cup\Gamma}(1,\dots,1)) = h^0(\mathcal{I}_F(1,\dots,1)) - 3$, i.e. $h^0(\mathcal{I}_\Gamma(1,\dots,1,0)) = h^0(\mathcal{O}_{(\mathbb{P}^1)^k}(1,\dots,1,0)) - 3$. This is true because $f_1,\dots,f_{k-1}$ (recalled at the beginning of the proof this Proposition 3.1 and introduced in (4)) are isomorphisms. □

(a) Assume $k = 3$. Since $r_X(p) \le r_C(p) \le 3$ and $r_C(p) = 2$ for a general p in $\langle \nu_{1^{(3)}}(\Gamma)\rangle$, we only need to prove that $r_X(p) > 1$. The case $r_X(p) = 1$ corresponds to a completely decomposable tensor: $p = \nu_{1^{(3)}}(q)$ for some $q \in (\mathbb{P}^1)^3$. Clearly $r_X(\nu_{1^{(3)}}(o)) = 1$ but $o \in \langle \beta\rangle$ then, since we took $p \in \langle \nu_{1^{(3)}}(\Gamma)\rangle \setminus \langle \nu_{1^{(3)}}(\beta)\rangle$, we have $p \ne \nu_{1^{(3)}}(o)$ and in particular $q \ne o$. In this case we can add q to Γ and get that $h^1(\mathcal{I}_{q\cup\Gamma}(1,1,1)) > 0$ by [1, Lemma 1]. Since $\deg(f_i(\Gamma)) = 3$, for all i's, every point of $\langle \beta\rangle \setminus \{o\}$ has rank 2. Since $q := (q_1, q_2, q_3) \ne o := (o_1, o_2, o_3)$

we have $q_i \neq o_i$ for some i, say for $i = 3$. Take $F \in |\mathcal{O}_{(\mathbb{P}^1)^3}(\varepsilon_3)|$ such that $q \in F$ and $o \notin F$. Hence $F \cap (\Gamma \cup \{q\}) = \{q\}$. We have $h^1(F, \mathcal{I}_{q,F}(1,1,1)) = 0$, because $\mathcal{O}_{(\mathbb{P}^1)^3}(1,1,1)$ is spanned. Claim 1 gives $h^1(\mathcal{I}_\Gamma(1,1,0)) = 0$. The residual exact sequence of F in $(\mathbb{P}^1)^3$ gives $h^1(\mathcal{I}_{\Gamma\cup\{q\}}(1,1,1)) = 0$, a contradiction.

(b) From now on we assume $k \geq 4$ and that Proposition 3.1 is true for a smaller number of factors. Since $X \supset C$, we have $r_X(p) \leq k-1$ (in fact, as we already recalled above, $r_C(p) = k-1$ by Sylvester's theorem). We need to prove that we actually have an equality, so we assume $r_X(p) \leq k-2$ and we will get a contradiction.

Take a set of points $S \in \mathcal{S}(p)$ of $(\mathbb{P}^1)^k$ evincing the X-rank of p (see Definition 1.4) and consider $v = (v_1, \dots, v_k) \in S \subset (\mathbb{P}^1)^k$ to be a point appearing in a decomposition of p. We can always assume that, if $o = (o_1, \dots, o_k)$, then $v_k \neq o_k$: such a $v \in S \subset \mathcal{S}(p)$ exists because, by Autarky (here recalled in Lemma 2.4), no element of $\mathcal{S}(p)$ is contained in $(\mathbb{P}^1)^{k-1} \times \{o_k\}$.

Consider the pre-image

$$D := \pi_k^{-1}(v_k).$$

Clearly by construction $o \notin D$ hence for any $q \in (\mathbb{P}^1)^k \setminus D$ we have $h^1(\mathcal{I}_{q\cup D}(1,\dots,1)) = h^1(\mathcal{I}_q(1,\dots,1,0)) = 0$, because $\mathcal{O}_{(\mathbb{P}^1)^k}(1,\dots,1,0)$ is globally generated. This implies that $\langle \nu_{1^{(k)}}(D)\rangle$ intersects X only in $\nu_{1^{(k)}}(D)$.

Now consider

$$\ell : \mathbb{P}^{2^k-1} \setminus \langle \nu_{1^{(k)}}(D)\rangle \to \mathbb{P}^{2^{k-1}-1}$$

the linear projection from $\langle \nu_{1^{(k)}}(D)\rangle$. Since $p \notin \langle \nu_{1^{(k)}}(D)\rangle$ (Claim 1), ℓ is defined at p. Moreover the map ℓ induces a rational map $\nu_{1^{(k)}}((\mathbb{P}^1)^k \setminus D) \to \nu_{1^{(k-1)}}((\mathbb{P}^1)^{k-1})$ which is induced by the projection $\tau_k : (\mathbb{P}^1)^k \to (\mathbb{P}^1)^{k-1}$ defined in Notation 1.1. We have

$$\ell \circ \nu_{1^{(k)}} = \nu_{1^{(k-1)}} \circ \tau_k.$$

Since $o \notin D$, we have $\ell(\langle\Gamma\rangle) = \langle \nu_{1^{(k-1)}}(\Gamma')\rangle$, where $\Gamma' = \tau_k(\Gamma)$. Hence $p' := \ell(p) \in \langle \nu_{1^{(k-1)}}(\Gamma')\rangle$. By [2] every element of $\langle \nu_{1^{(k-1)}}(\beta)\rangle \setminus \nu_{1^{(k-1)}}(o')$, with $o' := \tau_k(o)$, has X'-rank $k-1$. Since $\deg(\pi_i(\Gamma)) = 3$ for all i's, we have $\deg(\pi_i(\beta)) = 2$ for $i = 1, \dots, k-1$. This implies that the minimal sub-scheme α of Γ' such that $p' \in \langle \nu_{1^{(k-1)}}(\alpha)\rangle$ is such that $\alpha \neq \beta$ where β is the degree 2 sub-scheme of Γ'. Now let $S' \subset (\mathbb{P}^1)^{k-1}$ be the projection by τ_k of the set of points of $S \subset \mathcal{S}(p)$ that are not in D, i.e. $S' := \tau_k(S \setminus S \cap D)$. Since $\sharp(S') \leq k-2$ and $p' \in \langle \nu_{1^{(k-1)}}(\Gamma')\rangle$, the inductive assumption gives $\alpha \neq \Gamma'$ (it works even when $k = 4$). Hence $\alpha = \{o'\}$. Thus $p \in \langle \nu_{1^{(k)}}(\{o\} \cup D)\rangle$. Hence $\dim(\langle \nu_{1^{(k)}}(\Gamma \cup D)\rangle) \leq \dim(\langle \nu_{1^{(k)}}(D)\rangle) + 2$, contradicting Claim 1.

□

We need the following lemma, which is the projective version of an obvious linear algebra exercise.

Lemma 3.2. *Fix two linear spaces $L_1 \subsetneq L_2 \subset \mathbb{P}^m$ and a finite set $E \subset L_2$ spanning L_2. Let $\ell : \mathbb{P}^m \setminus L_1 \to \mathbb{P}^z$, $z := m - 1 - \dim L_1$, be the linear projection from L_1. Then $\ell(L_2 \setminus L_1)$ is a linear space spanned by the set $\ell(E \setminus E \cap L_1)$.*

Notation 3.3. Fix $(a,b) \in \mathbb{N}^2 \setminus \{(0,0)\}$. Let $\Delta_{a,b}$ be the set of all pairs (f,o), where $o \in \mathbb{P}^1$, $f : \mathbb{P}^1 \to (\mathbb{P}^2)^a \times (\mathbb{P}^1)^b$, each $\pi_i \circ f$, $1 \leq i \leq a$, is a degree 2 embedding and, for $a+1 \leq i \leq b$, $\pi_i \circ f$ is an isomorphism.

Lemma 3.4. *Set $\tilde{G} = \mathrm{Aut}(\mathbb{P}^2)^a \times \mathrm{Aut}(\mathbb{P}^1)^b$, $G := \tilde{G} \times \mathrm{Aut}(\mathbb{P}^1)$. Let G acts on $\Delta_{a,b}$ via $(g,h)(f,o) = (g \circ f \circ h^{-1}, h(o))$. Then this action is transitive, i.e., for $(f,o), (f',o')$ we have $(g,h) \in G$ such that $h(o) = o'$ and $g \circ f \circ h^{-1} = f'$.*

Proof. Fix any $h \in \mathrm{Aut}(\mathbb{P}^1)$ such that $h(o) = o'$ and write $\tilde{f} := f \circ h^{-1}$.
Write $\tilde{f} = (\tilde{f}_1, \dots, \tilde{f}_a, \tilde{f}_{a+1}, \dots, \tilde{f}_{a+b})$ and $f' = (f'_1, \dots, f'_a, f'_{a+1}, \dots, f'_{a+b})$ with $\tilde{f}_i := \pi_i \circ \tilde{f}$ and $f'_i := \pi_i \circ f'$. We need to find $g = (g_1, \dots, g_a, g_{a+1}, \dots, g_{a+b}) \in \tilde{G}$ such that $g \circ \tilde{f} = f'$, i.e. by the universal property of maps to products, we need to find $g = (g_1, \dots, g_a, g_{a+1}, \dots, g_{a+b}) \in \tilde{G}$ such that $g_i \circ \tilde{f}_i = f'_i$ for all i.

If $a+1 \leq i \leq a+b$ take $g_i := f'_i \circ \tilde{f}_i^{-1}$.

Now we fix i such that $1 \leq i \leq a$. We have two degree 2 embeddings $f'_i : \mathbb{P}^1 \to \mathbb{P}^2$ and $\tilde{f}_i : \mathbb{P}^1 \to \mathbb{P}^2$. Any two such maps are equivalent, up to an automorphism of $\mathbb{P}^2$, because these embeddings are induced by the complete linear system of the anticanonical line bundle of $\mathbb{P}^1$. Thus there is $g_i \in \mathrm{Aut}(\mathbb{P}^2)$ such that $g_i \circ \tilde{f}_i = f_i'$. □

Notation 3.5. Take $Y = (\mathbb{P}^2)^a \times (\mathbb{P}^1)^b$ and let $\nu_{2^{(a)},1^{(b)}} : Y \to \mathbb{P}^N$, $N := 3^a 2^b - 1$, be the Segre embedding of Y. Let $\Gamma_{a,b}$ (resp. $\Gamma'_{a,b}$) be the set of all $p \in \mathbb{P}^N$, such there is $(f,o) \in \Delta_{a,b}$ with $p \in \langle \nu_{2^{(a)},1^{(b)}}(f(3o))\rangle$ (resp. and $p \notin \langle \nu_{2^{(a)},1^{(b)}}(f(2o))\rangle$).

Since the image of an algebraic set by a morphism is constructible, $\Gamma_{a,b}$ and $\Gamma'_{a,b}$ of Notation 3.5 are constructible sets. The closure of $\Gamma_{a,b}$ in $\mathbb{P}^N$ is irreducible. Therefore we are allowed to inquire about the rank of a general element of $\Gamma_{a,b}$. If either $a > 0$ or $b \geq 2$, then $\Gamma'_{a,b} \neq \emptyset$ and the closures in $\mathbb{P}^N$ of $\Gamma_{a,b}$ and $\Gamma'_{a,b}$ are the same.

Lemma 3.6. *For all $k \geq 3$ we have $r_X(p) = 2k-1$ for a general $p \in \Gamma_{k,0}$ as in Notation 3.5.*

Proof. We use induction on k, the case $k = 3$ being true by [10, Theorem 1.8].
Now assume $k \geq 4$. Call $\nu_{2^{(k)}} : (\mathbb{P}^2)^k \to \mathbb{P}^r$, $r := 3^k - 1$, the Segre embedding. Fix $a \in \mathbb{P}^1$. For each $1 \leq i \leq k$ let $f_i : \mathbb{P}^1 \to \mathbb{P}^2$ be a degree 2 embedding. Let $f = (f_1, \dots, f_k) : \mathbb{P}^1 \to (\mathbb{P}^2)^k$ be the embedding with $f_i = \pi_i \circ f$ for all i. As in step (3c) of the proof of Theorem 1.7 we see that the curve $C := \nu_{2^{(k)}}(f(\mathbb{P}^1))$ is a rational normal curve of degree $2k$ in its linear span. Fix $a \in \mathbb{P}^1$ and set $o := (o_1, \dots, o_k) := f(a)$ and $A := f(3a)$. The scheme $\nu_{2^{(k)}}(A)$ has degree 3 and it is curvilinear. Fix a general $p \in \langle \nu_{2^{(k)}}(A)\rangle \setminus \langle \nu_{2^{(k)}}(2o)\rangle$. Since p has border rank 3 with respect to the rational normal curve C, Sylvester's theorem gives $r_C(p) = 2k-1$. Hence $r_X(p) \leq 2k-1$. To prove the lemma for the integer k it is sufficient to prove that $r_X(p) \geq 2k-1$.

Assume $r_X(p) \leq 2k-2$ and fix $B \in \mathcal{S}(p)$.

(a) In this step we assume the existence of a line $L \subset \mathbb{P}^{n_k}$ such that $o_k \notin L$ and $\sharp(Y' \cap B) \geq 2$, where $Y' := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_{k-1}} \times L$. We have $Y' \in |\mathcal{O}_Y(\varepsilon_k)|$. Since $o_k \notin L$, we have $o \notin Y'$ and hence $A \cap Y' = \emptyset$. Set $B' := B \setminus B \cap Y'$. Set $A' := \tau_k(A)$ where τ_k is defined in Notation 1.1. Since $k \geq 3$ and $(f_1, f_2) : \mathbb{P}^1 \to \mathbb{P}^2 \times \mathbb{P}^2$ is an embedding, we have $\deg(A') = 3$. Let $\nu_{2^{(k-1)}} : (\mathbb{P}^2)^{k-1} \to \mathbb{P}^s$, $s = 3^{k-1} - 1$, be the Segre embedding of $(\mathbb{P}^2)^{k-1}$. Note that the linear projection from L of $\mathbb{P}^2 \setminus L$ sends $\mathbb{P}^2 \setminus L$ onto a point. Set $E := \langle \nu_{2^{(k)}}(Y')\rangle$. We have $\dim E = 2 \cdot 3^{k-1} - 1$. Let $\ell : \mathbb{P}^M \setminus E \to \mathbb{P}^s$ denote the linear projection from E. Since $A \cap Y' = \emptyset$, $\ell(\nu_{2^{(k)}}(A))$ is a well-defined zero-dimensional scheme. Note that $\nu_{2^1,2^1}(f_1, f_2)(\mathbb{P}^1)$ is not a line of the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$. Since $k \geq 3$, we get that $\nu_{2^{(k-1)}}(A')$ spans a plane. Hence $\ell(\nu_{2^{(k)}}(A)) = A'$ is linearly independent, i.e. $\langle \nu_{2^{(k)}}(A)\rangle \cap E = \emptyset$. Hence $p' := \ell(p)$ is well-defined and in particular it is well-defined its rank with respect to the Segre variety $X' := \nu_{2^{(k-1)}}((\mathbb{P}^2)^{k-1})$. Since $\dim\langle \nu_{2^{(k)}}(A)\rangle = \dim\langle \nu_{2^{(k)}}(A')\rangle$ and p is general in $\langle \nu_{2^{(k)}}(A)\rangle$, p' is general in $\langle \nu_{2^{(k-1)}}(A')\rangle$. By the inductive assumption (case $k \geq 5$) or by [10, Theorem 1.8] (case $k = 4$), we have $r_{X'}(p') = 2k-3$. Since $p \in \langle \nu_{2^{(k)}}(B)\rangle$, Lemma 3.2 applied to $E := \nu_{2^{(k)}}(B)$, $m = 3^k - 1$ and $L_1 = E$, gives $p' \in \langle \nu_{2^{(k-1)}}(B')\rangle$. Since $\sharp(B') \leq \sharp(B) - 2 < 3k-3$, we get a contradiction.

(b) Assume the non-existence of a line $L \subset \mathbb{P}^{n_k}$ such that $o_k \notin L$ and $\sharp(Y' \cap B) \geq 2$. By Autarky we have $B \nsubseteq (\mathbb{P}^2)^{k-1} \times \{o_k\}$. Hence the assumption of this step is equivalent to assuming the existence of $b \in B$ such that $\pi_k(b) \neq o_k$, but $\pi_k(B)$ is contained in the line $R \subset \mathbb{P}^{n_2}$ spanned by o_k and $\pi_k(b)$. Hence $B \subset (\mathbb{P}^2)^{k-1} \times R$, contradicting Autarky, because $n_k = 2$ and $f_k(3a)$ spans $\mathbb{P}^2$.

□

Lemma 3.7. *Let $\nu_{2^{(1)},1^{(1)}}(Y)$ be the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^1$. We have $\Gamma_{1,1} \nsubseteq \nu_{2^{(1)},1^{(1)}}(Y)$.*

Proof. We have $\tau(\nu_{2^{(1)},1^{(1)}}(Y)) \supsetneq \nu_{2^{(1)},1^{(1)}}(Y)$. Since a general tangent vector of Y is of the form $f(2o)$ with $(f,o) \in \Delta_{1,1}$, we get $\Gamma_{1,1} \nsubseteq \nu_{2^{(1)},1^{(1)}}(Y)$. □

Definition 3.8. Let $X \subset \mathbb{P}^N$ be any variety, Z a zero-dimensional scheme and H an effective Cartier divisor. We define the scheme $Res_H(Z) \subset \mathbb{P}^N$ to be the *residue scheme of Z with respect to H*, namely the subscheme of $\mathbb{P}^N$ whose ideal sheaf is $\mathcal{I}_Z : \mathcal{I}_H$.

Lemma 3.9. *Take $Y = (\mathbb{P}^2)^2$. For every $p \in \Gamma'_{2,0}$ we have $r_X(p) > 2$ (cf. Notation 3.5).*

Proof. Assume the existence of a set $B \subset Y$ such that $\sharp(B) \le 2$ and $p \in \langle \nu_{2^{(2)}}(B)\rangle$. Since $B \in \mathcal{S}(p)$, we have $p \notin \langle \nu_{2^{(2)}}(B')\rangle$ for any $B' \subsetneq B$. Take $(f,o) \in \Delta_{2,0}$ such that $p \in \langle \nu_{2^{(2)}}(f(3o))\rangle$ and $p \notin \langle f(2o)\rangle$. Set $A := f(3o)$. By assumption we have $p \notin \langle \nu_{2^{(2)}}(A')\rangle$ for any $A' \subsetneq A$. In particular $B \ne \{o\}$. By [1, Lemma 1] we have $h^1(\mathcal{I}_{A\cup B}(1,1)) > 0$. Since $\sharp(B) \le 2$, there is a line $R \subset \mathbb{P}^2$ such that $\pi_1(B) \subset R$. Set $H := R \times \mathbb{P}^2 \in |\mathcal{O}_Y(1,0)|$ and call $\nu' : H \to \mathbb{P}^5$ the Segre embedding of H. We have $\mathrm{Res}_H(A\cup B) \subseteq A$. Since $\pi_2(A)$ spans $\mathbb{P}^2$ by the definition of $\Gamma_{2,0}$, $\pi_{2|\mathrm{Res}_H(A\cup B)}$ is an embedding and $\pi_2(A\cup B)$ is linearly independent. The residual exact sequence of H in Y gives $h^1(H, \mathcal{I}_{H\cap(A\cup B),H}(1,1)) > 0$. Hence $\langle \nu'(H\cap A)\rangle \cap \langle \nu'(H\cap B)\rangle \ne \emptyset$. Since $\pi_1(A)$ spans $\mathbb{P}^2$, we have $A \nsubseteq H$. Thus $H \cap A \subsetneq A$. By the definition of $\Gamma'_{2,0}$ we have $p \notin \langle \nu_{2^{(2)}}(H\cap A)\rangle$. Set $J := \langle \nu_{2^{(2)}}(A)\rangle \cap \nu_{2^{(2)}}(Y)$. Since the only linear subspaces of $\nu_{2^{(2)}}(Y)$ are the ones contained in a ruling of Y and $(f,o) \in \Delta_{2,0}$, the plane $\langle \nu_{2^{(2)}}(A)\rangle$ is not contained in $\nu_{2^{(2)}}(Y)$. Hence $J \nsubseteq \langle \nu_{2^{(2)}}(A)\rangle$. Since $\nu_{2^{(2)}}(Y)$ is scheme-theoretically cut out by quadrics, J is cut out by plane conics. Write $J = \nu_{2^{(2)}}(I)$ with $I \subset Y$. J is not a reducible conic or a double line or a line, because $\pi_i(A)$ spans $\mathbb{P}^2$, $i = 1,2$, while all linear subspaces of $\nu_{2^{(2)}}(Y)$ are contained in a ruling of Y. If J were a smooth conic we would have that either $\pi_1(I)$ spans $\mathbb{P}^2$ and $\pi_2(I)$ is a point, or $\pi_2(I)$ spans $\mathbb{P}^2$ and $\pi_1(I)$ is a point or $\pi_1(I)$ and $\pi_2(I)$ are lines, contradicting the assumption that each $\pi_i(A)$ spans $\mathbb{P}^2$. Thus J is a zero-dimensional scheme of degree ≤ 4. Since $A\cup B \subseteq I$, we get that either $B = \{o\}$ (and we excluded this case) or $B = \{o,q\}$ for some $q \in A$ with $q \ne o$. Thus $\deg(A\cup B) = 4$. We have $h^1(\mathcal{I}_{A\cup B}(1,1)) \ne 0$ ([1, Lemma 1]). Since p has not rank 2 with respect to $\nu_{2^{(2)}}(C)$, we have $q \notin C$. Thus there is $M \in |\mathcal{O}_Y(1,1)|$ with $M \supset C$ and $q \notin M$. Thus $M \cap (A\cup B) = A$ and $\mathrm{Res}_M(A\cup B) = \{q\}$. Thus $h^1(\mathcal{I}_Q) = 0$. Since $h^1(\mathcal{I}_A(1,1)) = 0$, the residual exact sequence of M in Y gives a contradiction. □

Lemma 3.10. *Fix integers $a \ge 0$ and $b \ge 0$ with $a+b \ge 3$. We have $r_X(p) = 2a+b-1$ for a general $p \in \Gamma_{a,b}$ (cf. Notation 3.5).*

Proof. The case $a = 0$ is true by Proposition 3.1. The case $b = 0$ is true by Lemma 3.6. Thus we may assume that $a > 0$ and $b > 0$. Set $k := a+b$. Take $(f,o) \in \Delta_{a,b}$ such that p is a general element of $\langle \nu_{2^{(a)},1^{(b)}}(A)\rangle$ with $A := 3o$. Set $C := f(\mathbb{P}^1)$, $f_i := \pi_i \circ f$ and $o_i := \pi_i(f(o))$. Since $\nu_{2^{(a)},1^{(b)}}(C)$ is a degree $2a+b$ rational normal curve in its linear span and $2a+b \ge 4$, Sylvester's theorem gives $r_{\nu_{2^{(a)},1^{(b)}}(C)} = 2a+b-1$. Thus $r_X(p) \le 2a+b-1$. Assume $r_X(p) \le 2a+b-2$ and take $B \in \mathcal{S}(p)$. By Autarky we have $B \nsubseteq (\mathbb{P}^2)^a \times (\mathbb{P}^1)^{b-1} \times \{o_k\}$. Take $z \in B$ such that $b_k := \pi_k(z) \ne o_k$. Set $Y' := (\mathbb{P}^2)^a \times (\mathbb{P}^1)^{b-1} \times \{b_k\}$. Let $\nu_{2^{(a)},1^{(b-1)}} := (\mathbb{P}^2)^a \times (\mathbb{P}^1)^{b-1} \to \mathbb{P}^s$, $s := -1 + 3^a 2^{b-1}$, be the Segre embedding of $(\mathbb{P}^2)^a \times (\mathbb{P}^1)^{b-1}$. Set $E := \langle \nu_{2^{(a)},1^{(b)}}(Y')\rangle$. We have $\dim E + 1 = 2\cdot 3^2$. Let $\ell : \mathbb{P}^M \setminus E \to \mathbb{P}^s$ the linear projection from E. Set $A' := \tau_k(A)$ (as in Notation 1.1). As in the proof of Lemma 3.6 we get $E \cap \langle \nu_{2^{(a)},1^{(b)}}(A)\rangle = \emptyset$, $\nu_{2^{(a)},1^{(b-1)}}(A') = \ell(A)$, $p' := \ell(p)$ is a general element of $\langle \nu_{2^{(a)},1^{(b-1)}}(A')\rangle$.

(a) Assume $(a,b) = (1,2)$. Since $\nu_{2^1,1^2}(Y) \nsubseteq \Gamma_{1,2}$, p is general in $\Gamma_{1,2}$ and $\sharp(B) \le 2$, we have $\sharp(B) = 2$. Thus $\sharp(A') = 1$ and so $p' \in \nu_{2^{(1)},1^{(1)}}(\mathbb{P}^2 \times \mathbb{P}^1)$. Hence a general element of $\Gamma_{1,1}$ has rank 1, contradicting Lemma 3.7.
(b) Assume $(a,b) = (2,1)$. We use Lemma 3.9.
(c) By the previous steps we may assume $a+b \ge 4$, $a > 0$, $b > 0$ and use induction on the integer $a+b$. (and hence by the inductive assumption applied to $(a,b-1)$ it has rank $2a+b-2$), while $p' \in \langle \nu_{2^{(a)},1^{(b-1)}}(B \setminus B\cap Y')\rangle$ with $\sharp(B \setminus B \cap Y') \le x-2$ (because $b_k \in \pi_k(Y') \cap \pi_k(B)$), a contradiction.

□

Proof of Theorem 1.8: First assume $x \le k-1$. If $x = 3$, then we may take as p a general point of $\sigma_3(X)$. Now assume $x \ge 4$ and hence $k \ge 5$. Apply Proposition 3.1 to $(\mathbb{P}^1)^{x+1}$ and then use Autarky (Lemma 2.4). Now assume $k \le x \le 2k-1$. For $x = 2k-1$ use Lemma 3.6 and Autarky. For each $x \in \{4,\dots,2k-2\}$ use the case $a = x+1-k$ and $b = k-a$ of Lemma 3.10 and then apply Autarky. □

Remark 3.11. Take the set-up of Theorem 1.8. If $n_i \ge 2$ for all i, then Theorem 1.8 gives all ranks of points of $\sigma_3(X) \setminus \sigma_2(X)$, but it does not say the rank of each point of $\sigma_3(X) \setminus \sigma_2(X)$. One problem is that in Lemma 3.6 we do not check all ranks of points of $\Gamma'_{1,1}$. A bigger problem is that the inductive proof should be adapted and the induction must start. These problems may be not deal-breakers, but there is a class of points of $\sigma_3(X)\setminus\sigma_2(X)$ (occurring even if $n_i=1$ for some i) for which we do not have a good upper bound for the rank (except that $r_X(p)\le 2k-1$). These are the points $p\in\langle\nu_{n_1^1,\dots,n_k^1}(A)\rangle$ with $A\subset Y$ a connected curvilinear scheme of degree 3 and $\deg(\pi_i(A))=2$ for some i, because in this case $A\nsubseteq C$ with $C\subset Y$ and $\nu_{n_1^1,\dots,n_k^1}(C)$ a rational normal curve in its linear span. We have no idea about the rank of these points.

4. Proof of Theorem 1.11

Lemma 4.1. *Fix an integer $c>0$ and $u\in\mathbb{P}^1$. Let $E=cu\subset\mathbb{P}^1$ be the degree c effective divisor of $\mathbb{P}^1$ with u as its support. Let $g:E\to\mathbb{P}^n$ be any morphism. Then there is a non-negative integer $e\le c$ and a morphism $h:\mathbb{P}^1\to\mathbb{P}^n$ such that $h^*(\mathcal{O}_{\mathbb{P}^n}(1))\cong\mathcal{O}_{\mathbb{P}^1}(e)$ and $h_{|E}=g$.*

Proof. Every line bundle on E is trivial. We fix an isomorphism between $g^*(\mathcal{O}_{\mathbb{P}^n}(1))$ and $\mathcal{O}_E(c)$. After this identification, g is induced by $n+1$ sections $u_0,\dots,u_n$ of $\mathcal{O}_E(c)$ such that at least one of them has a non-zero restriction at $\{u\}$. The map $H^0(\mathcal{O}_{\mathbb{P}^1}(c))\to H^0(\mathcal{O}_E(c))$ is surjective and its kernel is the section associated to the divisor cu. Hence there are $v_0,\dots,v_n\in H^0(\mathcal{O}_{\mathbb{P}^1}(c))$ with $v_{i|E}=u_i$ for all i. Not all sections $v_0,\dots,v_n$ vanish at 0. If they have no common zero, then they define a morphism $\mathbb{P}^1\to\mathbb{P}^n$ extending g and we may take $e=c$. Now assume that they have a base locus and call F the scheme-theoretic base locus of the linear system associated to $v_0,\dots,v_m$. We have $\deg(F)\le c$. Set $e:=c-\deg(F)$ and $S:=F_{\mathrm{red}}$. The sections $v_0,\dots,v_n$ induce a morphism $f:\mathbb{P}^1\setminus S\to\mathbb{P}^n$ with $f_{|E}=g$. See $v_0,\dots,v_n$ as elements of $|\mathcal{O}_{\mathbb{P}^1}(c)|$ and set $u_i:=u-F\in|\mathcal{O}_{\mathbb{P}^1}(e)|$. By construction the linear system spanned by $u_0,\dots,u_n$ has no base points, hence it induces a morphism $h:\mathbb{P}^1\to\mathbb{P}^n$ such that $h^*(\mathcal{O}_{\mathbb{P}^n}(1))\cong\mathcal{O}_{\mathbb{P}^1}(e)$. We have $h_{|\mathbb{P}^1\setminus S}=f$ and hence $h_{|E}=g$. □

Proof of Theorem 1.11: Let $Z\subset\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}$ such that $p\in\langle\nu_{{n_1}^{d_1},\dots,{n_k}^{d_k}}(Z)\rangle$ and Z has Z_1, … , Z_α connected components, By assumption there is $p_i\in\langle\nu_{{n_1}^{d_1},\dots,{n_k}^{d_k}}(Z_i)\rangle$ such that $p\in\langle\{p_1,\dots,p_\alpha\}\rangle$. Note that if Theorem 1.11 is true for each (Z_i,p_i), then it is true for Z. Hence it is sufficient to prove Theorem 1.11 under the additional assumption that Z is connected, so from now on we assume

- Z connected.

Moreover, since $r_X(p)=1\le 2-1+\sum_i d_i$ if $c=1$, we may also assume that

- $\deg Z=c\ge 2$.

Finally, since the real-valued function $x\mapsto x\left(-1+\sum_{i=1}^k d_i\right)$ is increasing for $x\ge 1$, with no loss of generality we may assume that, for any $G\subsetneq Z$,

- $p\notin\langle\nu_{{n_1}^{d_1},\dots,{n_k}^{d_k}}(G)\rangle$.

Fix $u\in\mathbb{P}^1$ and let $E=cu\subset\mathbb{P}^1$ be the degree c effective divisor of $\mathbb{P}^1$ with u as its support. Since Z is curvilinear and $\deg(Z)=c$, we have $Z\cong E$ as abstract zero-dimensional schemes. Let $g:E\to\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}$ be the composition of an isomorphism $E\to Z$ with the inclusion $Z\hookrightarrow\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}$:

$$g:E\to Z\hookrightarrow\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}.$$

Set $g_i:=\pi_i\circ g$. If we apply Lemma 4.1 to each g_i, we get the existence of an integer $c_i\in\{0,\dots,c\}$ and of a morphism $h_i:\mathbb{P}^1\to\mathbb{P}^{n_i}$ such that $h_{i|Z}=g_i$ and ${h_i}^*(\mathcal{O}_{\mathbb{P}^{n_i}}(1))\cong\mathcal{O}_{\mathbb{P}^1}(c_i)$. The map

$$h=(h_1,\dots,h_k):\mathbb{P}^1\to\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}\tag{8}$$

has multi-degree $(c_1,\dots,c_k)$. The curve

$$D:=h(\mathbb{P}^1)$$

is an integral rational curve containing Z. Since $p\in\langle\nu_{{n_1}^{d_1},\dots,{n_k}^{d_k}}(Z)\rangle$, we have

$$r_X(p)\le r_{\nu_{{n_1}^{d_1},\dots,{n_k}^{d_k}}(D)}(p).$$

Thus it is sufficient to prove that, if we call $\tilde{D}:=\nu_{{n_1}^{d_1},\dots,{n_k}^{d_k}}(D)$, then $r_{\tilde{D}}(p)\le 2+c\left(-1+\sum_{i=1}^k d_i\right)$. Since $c_i\le c$ for all i, it is sufficient to prove that $r_{\tilde{D}}(p)\le 2-c+\sum_{i=1}^k c_id_i$.

Set $\tilde{Z} := \nu_{{n_1}^{d_1},\dots,{n_k}^{d_k}}(Z)$, $m := \dim(\langle \tilde{D}\rangle)$ and

$$f = \nu_{{n_1}^{d_1},\dots,{n_k}^{d_k}} \circ h : \mathbb{P}^1 \to \mathbb{P}^N.$$

By assumption $\tilde{Z}$ is linearly independent in $\langle \tilde{D}\rangle \cong \mathbb{P}^m$ and in particular $c \leq m+1$.

(a) Assume that the map h defined in (8) is birational onto its image. The curve $\tilde{D} \subset \mathbb{P}^N$ just defined is a rational curve of degree $a := \sum_{i=1}^k c_i d_i$ contained in the projective space $\mathbb{P}^m := \langle \tilde{D}\rangle$ and non-degenerate in $\mathbb{P}^m$. Note that $a \geq m$ and that $p \in \langle \tilde{Z}\rangle$.

(1) First assume that $a = m$. In this case $\tilde{D}$ is a rational normal curve of $\mathbb{P}^m$. If $c \leq \lceil (a+1)/2 \rceil$, then Sylvester's theorem implies that $r_{\tilde{D}}(p) = a + 2 - c = 2 - c + \sum_{i=1}^k c_i d_i$. Now assume $c > \lceil (a+1)/2 \rceil$. Since $\tilde{Z}$ is connected and curvilinear and $p \notin \langle G\rangle$ for any $G \subsetneq \tilde{Z}$, Sylvester's theorem implies $r_{\tilde{D}}(p) \leq c$.

(2) Now assume $m < a$. There is a rational normal curve $C \subset \mathbb{P}^a$ and a linear subspace $W \subset \mathbb{P}^a$ such that $\dim(W) = a - m - 1$, $C \cap W = \emptyset$ and h is the composition of the degree a complete embedding $j := \mathbb{P}^1 \hookrightarrow \mathbb{P}^a$ and the linear projection $\ell : \mathbb{P}^a \setminus W \to \mathbb{P}^m$ from W. The scheme $E' := j(E)$ is a degree c curvilinear scheme and ℓ maps E' isomorphically onto $\tilde{Z}$. Since $\tilde{Z}$ is linearly independent, then $\langle E'\rangle \cap W = \emptyset$ and ℓ maps isomorphically $\langle E'\rangle$ onto $\langle \tilde{Z}\rangle$. Thus there is a unique $q \in \langle E'\rangle$ such that $\ell(q) = p$. Take any finite set $S \subset j(\mathbb{P}^1)$ with $q \in \langle S\rangle$. Since $C \cap W = \emptyset$, $\ell(S)$ is a well-defined subset of $\tilde{D}$ with cardinality $\leq \sharp(S)$. Hence $r_{\tilde{D}}(p) \leq r_C(q)$. As in step (a1) we see that either $r_C(q) = a + 2 - c$ (case $c \leq \lceil (a+1)/2 \rceil$) or $r_C(q) \leq c$ (case $c > \lceil (a+1)/2 \rceil$).

(b) Now assume that h is not birational onto its image, but it has degree $k \geq 2$. Note that k divides c_i for all i. In this case we will prove that $r_{\tilde{D}}(p) \leq 2 - c + \sum_{i=1}^k c_i d_i / k$. Let $h' : \mathbb{P}^1 \to h(\mathbb{P}^1)$ denote the normalization map. There is a degree k map $h'' : \mathbb{P}^1 \to \mathbb{P}^1$ such that h is the composition of $h' \circ h''$ and the inclusion $h(\mathbb{P}^1) \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. We have $Z = h'(E')$, where $E' = cu'$ and $u' = h''(u)$. We use E' and h' instead of E and h and repeat verbatim step (a).

□

Acknowledgements

We want to thank the anonymous referee and the Handling Editor Jan Draisma for their careful jobs that improved the presentation of this paper. A special thank to the referee for her/his very interesting questions that encouraged us in giving a better version of Theorem 1.8.

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