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Rationally cubic connected manifolds I: manifolds covered by lines

By Gianluca OCCHETTA and Valentina PATERNO

Abstract. In this paper we study smooth complex projective polarized varieties (X, H) of dimension $n \geq 2$ which admit a covering family V of rational curves of degree 3 with respect to H such that two general points of X may be joined by a curve parametrized by V , and such that there is a covering family of rational curves of H -degree one.

We prove that the Picard number of these manifolds is at most three, and that, if equality holds, (X, H) has an adjunction theoretic scroll structure over a smooth variety.

1. Introduction

At the end of the last century, the concepts of uniruled and rationally connected varieties were introduced as suitable higher dimensional analogues of ruled and rational surfaces. Uniruled varieties are algebraic varieties that are covered by rational curves, i.e., varieties that contain a rational curve through a general point. Among uniruled varieties, those that contain a rational curve through two general points are especially important. Varieties satisfying this property are called rationally connected and were introduced by Campana in [7] and by Kollár, Miyaoka and Mori in [19].

A natural problem about rationally connected varieties is to characterize them by means of bounding the degree of the rational curves connecting pairs of general points; it is easy to see that projective spaces are the only projective manifolds for which two general points can be connected by a rational curve of degree one with respect to a fixed ample line bundle; Ionescu and Russo have recently studied *conic-connected* manifolds embedded in projective space, i.e., projective manifolds such that two general points may be joined by a rational curve of degree 2 with respect to a fixed very ample line bundle. In [13] they proved that conic-connected manifolds $X \subset \mathbb{P}^N$ are Fano and have Picard number ρ_X less than or equal to 2 and

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classified the manifolds with Picard number two. A special case of conic-connected manifolds was previously studied by Kachi and Sato in [14].

In this paper we will consider *rationally cubic connected* manifolds (RCC-manifolds, for short), i.e., smooth complex projective polarized varieties (X, H) of dimension $n \geq 2$ which are rationally connected by irreducible rational curves of degree 3 with respect to a fixed ample line bundle H , or equivalently which admit a dominating family V of rational curves of degree 3 with respect to H such that two general points of X may be joined by a curve parametrized by V .

Unlike in the conic-connected case, there is no constant bounding the Picard number of RCC-manifolds, as shown in Example (3.1); the same example shows also that there are RCC-manifolds which are not Fano manifolds and which do not carry a dominating family of lines (i.e., curves of degree one with respect to H), this last property holding for all conic-connected manifolds of Picard number greater than one.

These considerations lead us to divide our analysis of RCC-manifolds in two parts: in the present paper we will deal with the ones which are covered by lines, while the remaining ones, which present very different geometric features, are treated in [21].

Rationally cubic connected manifolds covered by lines present more similarities with conic-connected manifolds; the first one is the presence of a bound on the Picard number, with a description of the border case.

THEOREM 1.1. *Let (X, H) be RCC with respect to a family V , and assume that (X, H) is covered by lines. Then $\rho_X \leq 3$, and if equality holds then there exist three families of lines $\mathcal{L}^1, \mathcal{L}^2, \mathcal{L}^3$ with $[V] = [\mathcal{L}^1] + [\mathcal{L}^2] + [\mathcal{L}^3]$ ($[]$ denotes the numerical class), such that X is $rc(\mathcal{L}^1, \mathcal{L}^2, \mathcal{L}^3)$ -connected, i.e., two general points of X can be connected by a connected chain of lines parametrized by the families \mathcal{L}^j (see Subsection (2.2) for more details).*

In the case of maximal Picard number we also prove a structure theorem, which shows that RCC-manifolds covered by lines have always a special adjunction theoretic scroll structure over a smooth variety:

THEOREM 1.2. *Let (X, H) be RCC with respect to a family V , assume that (X, H) is covered by lines and that $\rho_X = 3$. Then there is a covering family of lines whose numerical class spans an extremal ray of $NE(X)$ such that the associated extremal contraction $\varphi : X \rightarrow Y$ makes X into a special Bănică scroll (see Definition (2.13)) over a smooth variety Y .*

For conic-connected manifolds a stronger result holds, namely conic-connected

manifolds with maximal Picard number have a classical scroll structure; as Example (3.3) shows, this is not true for RCC-manifolds, i.e., there are RCC-manifolds with a scroll structure which has jumping fibers.

As for the question if a RCC-manifold covered by lines is a Fano manifold, we are not able to provide an answer. The big difference with the conic-connected case, which makes the problem definitely harder, is that the structure of the cone of curves of the manifold, which now lives in a three-dimensional vector space is not known: a priori, many different shapes and an unknown number of extremal rays are possible.

2. Background material

2.1. Families of rational curves and of rational 1-cycles

DEFINITION 2.1. A *family of rational curves* V on X is an irreducible component of the scheme $\text{Ratcurves}^n(X)$ (see [18, Definition II.2.11]).

Given a rational curve we will call a *family of deformations* of that curve any irreducible component of $\text{Ratcurves}^n(X)$ containing a point parametrizing that curve. We define $\text{Locus}(V)$ to be the set of points of X through which there is a curve among those parametrized by V ; we say that V is a *covering family* if $\text{Locus}(V) = X$ and that V is a *dominating family* if $\overline{\text{Locus}(V)} = X$.

By abuse of notation, given a line bundle $L \in \text{Pic}(X)$, we will denote by $L \cdot V$ the intersection number $L \cdot C$, with C any curve among those parametrized by V .

We will say that V is *unsplit* if it is proper; clearly, an unsplit dominating family is covering.

We denote by V_x the subscheme of V parametrizing rational curves passing through a point x and by $\text{Locus}(V_x)$ the set of points of X through which there is a curve among those parametrized by V_x . If, for a general point $x \in \text{Locus}(V)$, V_x is proper, then we will say that the family is *locally unsplit*. Moreover, we say that V is *generically unsplit* if, through a general $x \in \text{Locus}(V)$ and a general $y \in \text{Locus}(V_x)$ there is a finite number of curves parametrized by V .

DEFINITION 2.2. Let U be an open dense subset of X and $\pi: U \rightarrow Z$ a proper surjective morphism to a quasi-projective variety; we say that a family of rational curves V is a *horizontal dominating family with respect to π* if $\text{Locus}(V)$ dominates Z and curves parametrized by V are not contracted by π . Notice that V does not need to be a dominating family.

DEFINITION 2.3. We define a *Chow family of rational 1-cycles* \mathcal{W} to be an irreducible component of $\text{Chow}(X)$ parametrizing rational and connected 1-cycles. We define $\text{Locus}(\mathcal{W})$ to be the set of points of X through which there is a cycle

among those parametrized by \mathcal{W} ; notice that $\text{Locus}(\mathcal{W})$ is a closed subset of X ([18, II.2.3]). We say that \mathcal{W} is a *covering family* if $\text{Locus}(\mathcal{W}) = X$.

If V is a family of rational curves, the closure of the image of V in $\text{Chow}(X)$, denoted by \mathcal{V} , is called the *Chow family associated to V* . If V is proper, i.e., if the family is unsplit, then V is the normalization of the associated Chow family \mathcal{V} .

DEFINITION 2.4. Let V be a family of rational curves and let \mathcal{V} be the associated Chow family. We say that V (and also \mathcal{V}) is *quasi-unsplit* if every component of any reducible cycle parametrized by \mathcal{V} has numerical class proportional to the numerical class of a curve parametrized by V .

DEFINITION 2.5. Let V^1, \dots, V^k be families of rational curves on X and $Y \subset X$. We define $\text{Locus}(V^1)_Y$ to be the set of points $x \in X$ such that there exists a curve C among those parametrized by V^1 with $C \cap Y \neq \emptyset$ and $x \in C$. We inductively define $\text{Locus}(V^1, \dots, V^k)_Y := \text{Locus}(V^k)_{\text{Locus}(V^1, \dots, V^{k-1})_Y}$. Notice that, by this definition, we have $\text{Locus}(V)_x = \text{Locus}(V_x)$. Analogously we define $\text{Locus}(\mathcal{W}^1, \dots, \mathcal{W}^k)_Y$ for Chow families $\mathcal{W}^1, \dots, \mathcal{W}^k$ of rational 1-cycles.

Notation: If Γ is a 1-cycle, then we will denote by $[\Gamma]$ its numerical equivalence class in $N_1(X)$; if V is a family of rational curves, we will denote by $[V]$ the numerical equivalence class of any curve among those parametrized by V . A proper family will always be denoted by a calligraphic letter.

If $Y \subset X$, we will denote by $N_1(Y, X) \subseteq N_1(X)$ the vector subspace generated by numerical classes of curves of X contained in Y ; moreover, we will denote by $\text{NE}(Y, X) \subseteq \text{NE}(X)$ the subcone generated by numerical classes of curves of X contained in Y . The notation $\langle \dots \rangle$ will denote a linear subspace, while the notation $\langle \dots \rangle_c$ will denote a subcone.

DEFINITION 2.6. We say that k quasi-unsplit families V^1, \dots, V^k are numerically independent if in $N_1(X)$ we have $\dim \langle [V^1], \dots, [V^k] \rangle = k$.

For special families of rational curves we have useful dimensional estimates. The basic one is the following:

PROPOSITION 2.7. [18, IV.2.6] *Let V be a family of rational curves on X and $x \in \text{Locus}(V)$ a point such that every component of V_x is proper. Then*

$$(a) \dim \text{Locus}(V) + \dim \text{Locus}(V_x) \geq \dim X - K_X \cdot V - 1;$$

$$(b) \text{ every irreducible component of } \text{Locus}(V_x) \text{ has dimension } \geq -K_X \cdot V - 1.$$

REMARK 2.8. If V is a generically unsplit dominating family then, for a general $x \in X$ the inequalities in Proposition (2.7) are equalities, by [18, Proposition

II.3.10].

The following generalization of Proposition (2.7) will be often used:

LEMMA 2.9. (Cf. [1, Lemma 5.4]) *Let $Y \subset X$ be an irreducible closed subset and V^1, \dots, V^k numerically independent unsplit families of rational curves such that $\langle [V^1], \dots, [V^k] \rangle \cap \text{NE}(Y, X) = \underline{0}$. Then either $\text{Locus}(V^1, \dots, V^k)_Y = \emptyset$ or*

$$\dim \text{Locus}(V^1, \dots, V^k)_Y \geq \dim Y + \sum -K_X \cdot V^i - k.$$

A key fact underlying our strategy to obtain bounds on the Picard number, based on [18, Proposition II.4.19], is the following:

LEMMA 2.10. [1, Lemma 4.1] *Let $Y \subset X$ be a closed subset and \mathcal{V} a Chow family of rational 1-cycles. Then every curve contained in $\text{Locus}(\mathcal{V})_Y$ is numerically equivalent to a linear combination with rational coefficients of a curve contained in Y and of irreducible components of cycles Γ parametrized by \mathcal{V} such that $\Gamma \cap Y \neq \emptyset$.*

The following Corollary encompasses the most frequent usages of Lemma (2.10) in the paper:

COROLLARY 2.11. *Let V^1 be a locally unsplit family of rational curves, and V^2, \dots, V^k unsplit families of rational curves. Then, for a general $x \in \text{Locus}(V^1)$,*

(a) $N_1(\text{Locus}(V^1)_x, X) = \langle [V^1] \rangle$;

(b) $\text{Locus}(V^1, \dots, V^k)_x = \emptyset$ or $N_1(\text{Locus}(V^1, \dots, V^k)_x, X) \subseteq \langle [V^1], \dots, [V^k] \rangle$.

2.2. Contractions and fibrations

DEFINITION 2.12. Let X be a manifold such that K_X is not nef.

Denote by $\overline{\text{NE}}(X) \subset N_1(X)$ the closure of the cone of effective 1-cycles into the \mathbb{R} -vector space of 1-cycles modulo numerical equivalence, and by $\overline{\text{NE}}(X)_{K_X < 0}$ the set $\{z \in N_1(X) : K_X \cdot z < 0\}$. An *extremal face* is a face σ of $\overline{\text{NE}}(X)$, associated to some nef line bundle L , contained in the negative part of the cone with respect to K_X , i.e., $\sigma = \overline{\text{NE}}(X) \cap L^\perp \subset \overline{\text{NE}}(X)_{K_X < 0}$; an extremal face of dimension one is called an *extremal ray*.

To an extremal face σ is associated a morphism with connected fibers $\varphi_\sigma : X \rightarrow Z$ onto a normal variety, morphism which contracts the curves whose numerical class is in σ ; φ_σ is called an *extremal contraction* or a *Fano-Mori contraction*, while a Cartier divisor H such that $H = \varphi_\sigma^* A$ for an ample divisor A on Z is called a *supporting divisor* of the map φ_σ (or of the face σ). We usually denote with $\text{Exc}(\varphi_\sigma) := \{x \in X \mid \dim \varphi_\sigma^{-1}(\varphi_\sigma(x)) > 0\}$ the *exceptional locus* of φ_σ ; if φ_σ is

of fiber type then clearly $\text{Exc}(\varphi_\sigma) = X$. An extremal contraction associated to an extremal ray is called an *elementary contraction*.

DEFINITION 2.13. An elementary fiber type extremal contraction $\varphi : X \rightarrow Z$ onto a smooth variety Z is called a \mathbb{P} -*bundle* or a *classical scroll* if there exists a vector bundle \mathcal{E} of rank $\dim X - \dim Z + 1$ on Z such that $X \simeq \mathbb{P}(\mathcal{E})$.

An elementary fiber type extremal contraction $\varphi : X \rightarrow Z$ is called an *adjunction scroll* if there exists a φ -ample line bundle $H \in \text{Pic}(X)$ such that $K_X + (\dim X - \dim Z + 1)H$ is a supporting divisor of φ . A general fiber of such a contraction is, by the adjunction formula and Kobayashi–Ochiai Theorem [17], a projective space $\mathbb{P}^{\dim X - \dim Z}$.

Some special adjunction scroll contractions arise from projectivization of Bănică sheaves (Cf. [3]); in particular, if $\varphi : X \rightarrow Z$ is an adjunction scroll such that every fiber has dimension $\leq \dim X - \dim Z + 1$, then Z is smooth and X is the projectivization of a Bănică sheaf on Z (cfr. [3, Proposition 2.5]); we will call such a contraction a *special Bănică scroll*.

Proposition (2.7), in case V is the unsplit family of deformations of a rational curve of minimal anticanonical degree contained in a fiber of an extremal contraction, gives the *fiber locus inequality*:

PROPOSITION 2.14. [12, Theorem 0.4], [23, Theorem 1.1] *Let φ be a Fano–Mori contraction of X and let $E = \text{Exc}(\varphi)$ be its exceptional locus; let S be an irreducible component of a (non trivial) fiber of φ . Then*

$$\dim E + \dim S \geq \dim X + l - 1,$$

where $l = \min\{-K_X \cdot C \mid C \text{ is a rational curve in } S\}$. If φ is the contraction of a ray R , then $l(R) := l$ is called the *length* of the ray.

If X admits a fiber type extremal contraction, then it is uniruled; for the converse, we have that a covering family of rational curves determines a rational fibration, defined on an open set of X . We recall briefly this construction.

DEFINITION 2.15. Let $Y \subset X$ be a closed subset, and let $\mathcal{V}^1, \dots, \mathcal{V}^k$ Chow families of rational 1-cycles; define $\text{ChLocus}_m(\mathcal{V}^1, \dots, \mathcal{V}^k)_Y$ to be the set of points $x \in X$ such that there exist cycles $\Gamma_1, \dots, \Gamma_m$ with the following properties:

- Γ_i belongs to a family \mathcal{V}^j ;
- $\Gamma_i \cap \Gamma_{i+1} \neq \emptyset$;
- $\Gamma_1 \cap Y \neq \emptyset$ and $x \in \Gamma_m$,

i.e., $\text{ChLocus}_m(\mathcal{V}^1, \dots, \mathcal{V}^k)_Y$, is the set of points that can be joined to Y by a connected chain of at most m cycles belonging to the families \mathcal{V}^j .

Define a relation of *rational connectedness with respect to* $\mathcal{V}^1, \dots, \mathcal{V}^k$ on X in the following way: two points x and y of X are in $\text{rc}(\mathcal{V}^1, \dots, \mathcal{V}^k)$ -relation if there exists a chain of cycles in $\mathcal{V}^1, \dots, \mathcal{V}^k$ which joins x and y , i.e., if $y \in \text{ChLocus}_m(\mathcal{V}^1, \dots, \mathcal{V}^k)_x$ for some m . In particular, X is $\text{rc}(\mathcal{V}^1, \dots, \mathcal{V}^k)$ -connected if for some m we have $X = \text{ChLocus}_m(\mathcal{V}^1, \dots, \mathcal{V}^k)_x$.

To the proper prerelation defined by $\mathcal{V}^1, \dots, \mathcal{V}^k$ it is associated a fibration, which we will call the $\text{rc}(\mathcal{V}^1, \dots, \mathcal{V}^k)$ -fibration:

THEOREM 2.16. ([18, IV.4.16], Cf. [6]) *Let X be a normal and proper variety and $\mathcal{V}^1, \dots, \mathcal{V}^k$ Chow families of rational 1-cycles; then there exists an open subvariety $X^0 \subset X$ and a proper morphism with connected fibers $\pi: X^0 \rightarrow Z^0$ such that*

- *the $\text{rc}(\mathcal{V}^1, \dots, \mathcal{V}^k)$ -relation restricts to an equivalence relation on X^0 ;*
- *$\pi^{-1}(z)$ is a $\text{rc}(\mathcal{V}^1, \dots, \mathcal{V}^k)$ -equivalence class for every $z \in Z^0$;*
- *$\forall z \in Z^0$ and $\forall x, y \in \pi^{-1}(z)$, $x \in \text{ChLocus}_m(\mathcal{V}^1, \dots, \mathcal{V}^k)_y$ with $m \leq 2^{\dim X - \dim Z} - 1$.*

If \mathcal{V} is a covering Chow family of rational 1-cycles, associated to a quasi-unsplit dominating family V , and $\pi: X \dashrightarrow Z$ is the $\text{rc}(\mathcal{V})$ -fibration, then by [5, Proposition 1, (ii)] its indeterminacy locus B is the union of all $\text{rc}(\mathcal{V})$ -equivalence classes of dimension greater than $\dim X - \dim Z$.

Combining Theorem (2.16) with Lemma (2.10), we get the following:

PROPOSITION 2.17. (Cf. [1, Corollary 4.4]) *If X is rationally connected with respect to some Chow families of rational 1-cycles $\mathcal{V}^1, \dots, \mathcal{V}^k$, then $N_1(X)$ is generated by the classes of irreducible components of cycles parametrized by $\mathcal{V}^1, \dots, \mathcal{V}^k$. In particular, if $\mathcal{V}^1, \dots, \mathcal{V}^k$ are quasi-unsplit families, then $\rho_X \leq k$ and equality holds if and only if $\mathcal{V}^1, \dots, \mathcal{V}^k$ are numerically independent.*

2.3. Extremality of families of rational curves

The key observation for proving the extremality of the numerical class of a family of curves is a variation of an argument of Mori, contained in [4, Proof of Lemma 1.4.5]. We state it as follows:

LEMMA 2.18. *Let $Z \subset X$ be a closed subset and let V be a quasi-unsplit family of rational curves. Then, for every integer m , every curve contained in*

$\text{ChLocus}_m(\mathcal{V})_Z$ is numerically equivalent to a linear combination with rational coefficients

$$\lambda C_Z + \mu C_V,$$

where C_Z is a curve in Z , C_V is a curve among those parametrized by V and $\lambda \geq 0$.

We build on Lemma (2.18), to analyze particular situations which will appear in the proof of Theorem (1.2).

LEMMA 2.19. *Let $\mathcal{L}^1, \mathcal{L}^2$ and \mathcal{L}^3 be numerically independent unsplit families on X . Assume that for some point $x \in X$ and some integers m_1, m_2 we have $X = \text{ChLocus}_{m_1}(\mathcal{L}^1, \mathcal{L}^2)_{\text{ChLocus}_{m_2}(\mathcal{L}^3)_x}$. Then the numerical classes $[\mathcal{L}^1], [\mathcal{L}^2]$ lie in a (two-dimensional) extremal face of $\text{NE}(X)$.*

PROOF. First of all notice that, by Proposition (2.17), we have that $\rho_X = 3$.

By repeated applications of Lemma (2.18), starting with $Z := \text{ChLocus}_{m_2}(\mathcal{L}^3)_x$, every curve in X is numerically equivalent to $\sum_1^3 a_j [\mathcal{L}^j]$, with $a_3 \geq 0$.

Let $\Pi \subset N_1(X)$ be the plane defined by $[\mathcal{L}^1]$ and $[\mathcal{L}^2]$ and let C^1 and C^2 be two curves such that $[C^1] + [C^2] \in \Pi$; write $[C^i] = \sum c_j^i [\mathcal{L}^j]$, with $c_3^i \geq 0$.

Asking for $[C^1] + [C^2]$ to be in Π amounts to impose $c_3^1 + c_3^2 = 0$, hence $c_3^1 = c_3^2 = 0$ and both $[C^1]$ and $[C^2]$ belong to Π . \square

LEMMA 2.20. *Let $\mathcal{L}^1, \mathcal{L}^2$ and \mathcal{L}^3 be numerically independent unsplit families on X . Assume that for some point $x \in X$ and some positive integer m we have $X = \text{Locus}(\mathcal{L}^2, \mathcal{L}^1)_{\text{ChLocus}_m(\mathcal{L}^3)_x}$. Then $[\mathcal{L}^1]$ is extremal in $\text{NE}(X)$.*

PROOF. By Lemma (2.19) the numerical classes of \mathcal{L}^1 and \mathcal{L}^2 lie in an extremal face σ . Let $C \subset X$ be a curve whose numerical class is contained in σ .

Since $X = \text{Locus}(\mathcal{L}^1)_Z$ with $Z = \text{Locus}(\mathcal{L}^2)_{\text{ChLocus}_m(\mathcal{L}^3)_x}$, by Lemma (2.18) there is an effective curve $C_Z \subset \text{Locus}(\mathcal{L}^2)_{\text{ChLocus}_m(\mathcal{L}^3)_x}$ such that

$$C \equiv \alpha C_Z + \beta C_1$$

with C_1 parametrized by \mathcal{L}^1 and $\alpha \geq 0$.

Since $[C] \in \sigma$, then also $[C_Z] \in \sigma$; on the other hand, by Lemma (2.18) applied to Z we have that $[C_Z] \in \langle [\mathcal{L}^2], [\mathcal{L}^3] \rangle$, so $[C_Z] = \lambda [\mathcal{L}^2]$.

We have thus shown that every curve whose numerical class belongs to σ is equivalent to $\alpha' [\mathcal{L}^2] + \beta [\mathcal{L}^1]$ with $\alpha' \geq 0$. Let now B^1 and B^2 be two curves such that $[B^1] + [B^2] \in \mathbb{R}_+ [\mathcal{L}^1]$; by the extremality of σ , both $[B^1]$ and $[B^2]$ are contained in σ . Write $[B^i] = \alpha'_i [\mathcal{L}^2] + \beta_i [\mathcal{L}^1]$ with $\alpha'_i \geq 0$. Then it is clear that $[B^1] + [B^2] \in \mathbb{R}_+ [\mathcal{L}^1]$ if and only if $\alpha'_i = 0$ for $i = 1, 2$. \square

3. Examples

EXAMPLE 3.1. (RCC manifolds with large Picard number)

Let P_1, \dots, P_k be general points of \mathbb{P}^n and let $\varphi : X \rightarrow \mathbb{P}^n$ be the blow-up of \mathbb{P}^n at P_1, \dots, P_k , with

$$k \leq \binom{n+3}{3} - (2n+2).$$

Denote by E_i with $i = 1, \dots, k$ the exceptional divisors. Let V be the family of deformations of the strict transform of a general line in \mathbb{P}^n and define H to be

$$H := \varphi^* \mathcal{O}_{\mathbb{P}^n}(3) - \left(\sum_{i=1}^k E_i \right).$$

By [9, Main theorem] the line bundle H is very ample, thus the pair (X, H) is RCC with respect to V and $\rho_X = k + 1$.

Notice that, if $k \geq 2$ and $n > 2$ then X is not a Fano manifold. In fact, if we consider the strict transform ℓ of the line in \mathbb{P}^n passing through P_1 and P_2 , then, by the canonical bundle formula of the blow up, we have $-K_X \cdot \ell \leq 0$.

EXAMPLE 3.2. (Products)

Let Y be a conic-connected manifold with Picard number two (see Example (3.4) below for the classification), and denote by $\mathcal{O}_Y(1)$ the hyperplane line bundle of Y . Trivial examples can be obtained by taking the product $X := Y \times \mathbb{P}^r$, with projections p_1 and p_2 and setting H to be $p_1^* \mathcal{O}_Y(1) \otimes p_2^* \mathcal{O}_{\mathbb{P}^r}(1)$.

EXAMPLE 3.3. (Adjunction scrolls)

Let Y be $\mathbb{P}^{r_1} \times \mathbb{P}^{r_2} \times \mathbb{P}^{r_3}$ with $r_i \geq 2$, let X be a general member of the linear system $|\mathcal{O}(1, 1, 1)|$ and let H be the restriction to X of $\mathcal{O}(1, 1, 1)$. Then (X, H) is an RCC-manifold which has three extremal contractions which are adjunction scrolls. If $r_i < r_j + r_k$ then the contraction onto $\mathbb{P}^{r_j} \times \mathbb{P}^{r_k}$ has a $(r_j + r_k - r_i - 1)$ -dimensional family of jumping fibers, hence it is not a classical scroll. Notice that the condition is always fulfilled by at least two indexes, and, taking $r_1 = r_2 = r_3$, it is fulfilled by all, hence in this case X has no classical scroll contractions.

EXAMPLE 3.4. (Projective bundles)

Let Y be a conic-connected manifold of Picard number two; by [13, Theorem 2.2] Y is one of the following:

- (a) $\mathbb{P}^a \times \mathbb{P}^b \subset \mathbb{P}^{ab+a+b}$ Segre embedded.
- (b) A hyperplane section of the Segre embedding $\mathbb{P}^a \times \mathbb{P}^b \subset \mathbb{P}^{ab+a+b}$.

(c) $Y \simeq \mathbb{P}_{\mathbb{P}^a}(\mathcal{E})$ with $\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^r}(1)^{\oplus n-a} \oplus \mathcal{O}_{\mathbb{P}^a}(2)$, $a = 1, 2, \dots, n-1$, embedded by $|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$.

All these manifolds have two extremal contractions onto projective spaces, one of which is a classical scroll. Denote this contraction by φ_1 and the other contraction by φ_2 ; denote by \mathcal{H}_1 and \mathcal{H}_2 respectively the line bundles $\varphi_1^*\mathcal{O}_{\mathbb{P}}(1)$ and $\varphi_2^*\mathcal{O}_{\mathbb{P}}(1)$, where $\mathcal{O}_{\mathbb{P}}(1)$ is the hyperplane line bundle on the target projective space.

For every integer $r \geq 1$ consider the vector bundles $\mathcal{E}_i := (\mathcal{O}_Y)^{\oplus r} \oplus \mathcal{H}_i$ on Y and their projectivizations $X_i = \mathbb{P}(\mathcal{E}_i)$, with natural projections $\pi_i : X_i \rightarrow Y$.

Let ξ_i be the tautological line bundle of $\mathbb{P}(\mathcal{E}_i)$, and set $H := \xi_i + \pi_i^*\mathcal{H}_1 + \pi_i^*\mathcal{H}_2$; H is the sum of three nef line bundles which do not all vanish on the same class in $\overline{\text{NE}}(X_i) \setminus \{0\}$, hence it is ample.

The restriction of \mathcal{E}_i to smooth conics γ in Y is $\mathcal{O}_{\gamma}^{\oplus r} \oplus \mathcal{O}_{\gamma}(1)$; let V be the family of sections over smooth conics in Y corresponding to the surjections $(\mathcal{E}_i)|_{\gamma} \rightarrow \mathcal{O}_{\gamma}(1)$. We claim that (X_i, H) is RCC with respect to V ; first of all it's clear that

$$\xi_i \cdot V = \pi_i^*\mathcal{H}_1 \cdot V = \pi_i^*\mathcal{H}_2 \cdot V = 1,$$

hence $H \cdot V = 3$. Let now x and x' be general points in X_i ; let y and y' be the images of these points in Y and let γ be a conic in Y passing through y and y' . By the generality of x and x' we can assume that γ is smooth. Let Γ be the projectivization of the restriction of \mathcal{E}_i to γ . The variety Γ is isomorphic to the blow-up of \mathbb{P}^r in a linear subspace Λ of codimension two, and a general curve in V contained in Γ is the strict transform of a line in \mathbb{P}^r not meeting Λ . By the generality of x and x' there is a line in \mathbb{P}^r not meeting Λ whose strict transform contains x and x' .

It is straightforward to check that all the manifolds constructed in this way are Fano manifolds with three elementary contractions; notice that, depending on the choice of Y and \mathcal{H}_i , the other contractions of X_i can be of different kind, namely:

Y	\mathcal{H}_i	Contractions
$\mathbb{P}^r \times \mathbb{P}^s$	1 - 2	Fiber type - Divisorial
Hyperplane section of $\mathbb{P}^r \times \mathbb{P}^s$	1 - 2	Fiber type - Divisorial
Blow-up of \mathbb{P}^n along a linear subspace	1	Fiber type - Small
Blow-up of \mathbb{P}^n along a linear subspace	2	Divisorial - Divisorial

EXAMPLE 3.5. (More projective bundles)

In Example (3.4) we considered bundles on all possible conic-connected manifolds with Picard number two. Other examples can be constructed taking as base the product of two projective spaces; this is possible because, in this case, through a pair of general points there is not just one conic, but a one-parameter family of conics.

Let Y be $\mathbb{P}^a \times \mathbb{P}^b$ and let \mathcal{H}_1 and \mathcal{H}_2 be as in Example (3.4). For every integer $r \geq 1$ consider the vector bundles $\mathcal{E}_{ij} := (\mathcal{O}_Y)^{\oplus r} \oplus \mathcal{H}_i \oplus \mathcal{H}_j$ on Y and their projectivizations $X_{ij} = \mathbb{P}(\mathcal{E}_{ij})$, with natural projection $\pi_{ij} : X_{ij} \rightarrow Y$.

Let ξ_{ij} be the tautological line bundle of $\mathbb{P}(\mathcal{E}_{ij})$, and set $H := \xi_{ij} + \pi_{ij}^* \mathcal{H}_1 + \pi_{ij}^* \mathcal{H}_2$; H is the sum of three nef line bundles which do not all vanish on the same class in $\overline{\text{NE}}(X_{ij}) \setminus \{0\}$, hence it is ample.

The restriction of \mathcal{E}_{ij} to smooth conics γ in Y is $\mathcal{O}_\gamma^{\oplus r} \oplus \mathcal{O}_\gamma(1)^{\oplus 2}$; let V be the family of sections over smooth conics γ in Y corresponding to the surjections $(\mathcal{E}_{ij})|_\gamma \rightarrow \mathcal{O}_\gamma(1)$. We will show that (X_{ij}, H) is RCC with respect to V ; first of all it is clear that $H \cdot V = 3$, since

$$\xi_{ij} \cdot V = \pi_{ij}^* \mathcal{H}_1 \cdot V = \pi_{ij}^* \mathcal{H}_2 \cdot V = 1.$$

Let now x and x' be general points in X_{ij} : we claim that there is at most a finite number of curves in V passing through x and x' . If this were not the case, through x and x' there would be a reducible cycle Γ parametrized by V .

Since there is only one dominating family of lines – the lines in the fibers of π_i – and x and x' are general, the cycle Γ consists of a line in a fiber of π_i and a curve γ' such that $H \cdot \gamma' = 2$. By the generality of x and x' we have that there is no line in Y passing through y and y' , hence $\pi_{ij}^* \mathcal{H}_1 \cdot \gamma'$ and $\pi_{ij}^* \mathcal{H}_2 \cdot \gamma'$ are positive. Therefore $\xi_{ij} \cdot \gamma' = 0$ and γ' is a section over a conic γ corresponding to a surjection $(\mathcal{E}_{ij})|_\gamma \rightarrow \mathcal{O}_\gamma$, but, through a general point of X_{ij} there is no such a curve, and the claim is proved.

For a general conic γ passing through $y = \pi_{ij}(x)$ and $y' = \pi_{ij}(x')$ we can compute the dimension of the space of curves parametrized by V contained in $\pi_{ij}^{-1}(\gamma) \simeq \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}^{\oplus r} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}) := \mathbb{P}$. By [18, Theorem 1.2], this dimension is $-K_{\mathbb{P}} \cdot V + \dim \mathbb{P} - 3 = (r+2) + (r+2) - 3 = 2r+1$ (the minus three appears since we are working with $\text{Ratcurves}^n(X)$). Since there is a one parameter family of conics γ passing through y and y' , the dimension of the space of curves $T \subset V$ parametrizing curves meeting $F_y := \pi_{ij}^{-1}(y)$ and $F_{y'} := \pi_{ij}^{-1}(y')$ is $2r+2$.

Since F_y and $F_{y'}$ have both dimension $r+1$ and we have proved above that through two general points there is at most a finite number of curves parametrized by T , we can conclude that through two general points in F_y and $F_{y'}$ there is a curve parametrized by V .

It is straightforward to check that all the manifolds constructed in this way are Fano manifolds with three elementary contractions; notice that, depending on the

choice of \mathcal{H}_i and \mathcal{H}_j , the other contractions of X_{ij} can be of different kind, namely:

\mathcal{H}_i	\mathcal{H}_j	Contractions
1	1	Fiber type - Small
1	2	Divisorial - Divisorial

4. Preliminaries

Let (X, H) be a RCC manifold with respect to a family V . Notice that we are asking that a general cubic through two general points is irreducible. Examples in which (X, H) is connected by reducible cycles of degree three can be constructed by taking any projective bundle over the projective space which has a section.

Our assumptions on V can be rephrased by saying that for a general point $x \in X$ the subset $\text{Locus}(V)_x$ is dense in X ; by [10, Proposition 4.9] a general curve $f : \mathbb{P}^1 \rightarrow X$ parametrized by V is a 1-free curve, i.e.

$$f^*TX \simeq \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n)$$

with $a_1 \geq a_2 \geq \dots \geq a_n$ and $a_1 \geq 2, a_n \geq 1$. This implies that

$$-K_X \cdot V = -K_X \cdot f_*\mathbb{P}^1 = \sum_1^n a_i \geq n + 1. \quad (1)$$

Since the locus of the corresponding family of rational 1-cycles \mathcal{V} is closed and $\text{Locus}(V)_x \subset \text{Locus}(\mathcal{V})_x$, we have that $\text{Locus}(\mathcal{V})_x = X$ for a general $x \in X$. By Lemma (2.10) it follows that $N_1(X)$ is generated by the numerical classes of irreducible components of cycles parametrized by \mathcal{V} passing through x . In particular the Picard number of X is one if and only if, for some $x \in X$, the subfamily V_x is quasi-unsplit. More precisely we have the following:

PROPOSITION 4.1. *Let (X, H) be RCC with respect to a family V ; then*

1. *There exists $x \in X$ such that V_x is proper if and only if $(X, H) \simeq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(3))$;*
2. *There exists $x \in X$ such that V_x is quasi-unsplit but not proper if and only if X is a Fano manifold of Picard number one and index $n + 1 > r(X) \geq \frac{n+1}{3}$ with fundamental divisor H .*

PROOF. In the first case X is the projective space and V is the family of lines by [8, Main Theorem] or [16, Proof of Theorem 1.1]. In the second case the Picard number of X is one by Lemma (2.10), hence $-K_X \geq \frac{n+1}{3}H$ by taking intersection numbers with V . The existence of a reducible cycle in \mathcal{V} provides a curve with intersection number one with H . \square

5. RCC-manifolds with plenty of reducible cubics

The results in the previous section show that, if the Picard number of X is greater than one, through a general point there is at least one reducible cycle in \mathcal{V} whose components are not all numerically proportional to V . Since $H \cdot V = 3$, a cycle in \mathcal{V} can split into two or three irreducible rational components. From now on we will call a component of H -degree one a *line* and a component of H -degree two a *conic*.

Families of lines are easier to handle, since they cannot degenerate further, i.e., they are unsplit families; for this reason the first possibility that we consider is the following: through a general point of X there is a reducible cycle consisting of three lines.

DEFINITION 5.1. We will say that a manifold (X, H) which is RCC with respect to V is *covered by V -triplets of lines* if through a general point of X there is a connected rational 1-cycle $\ell^1 + \ell^2 + \ell^3$ such that $[\ell^1] + [\ell^2] + [\ell^3] = [V]$.

5.1. RCC-manifolds covered by triplets of lines

We start by considering the following more general situation:

DEFINITION 5.2. We will say that a manifold (X, H) which is RCC with respect to V is *connected by V -triplets of lines* if there exist three families of lines $\mathcal{L}^1, \mathcal{L}^2, \mathcal{L}^3$ with $[V] = [\mathcal{L}^1] + [\mathcal{L}^2] + [\mathcal{L}^3]$ such that X is $\text{rc}(\mathcal{L}^1, \mathcal{L}^2, \mathcal{L}^3)$ -connected.

PROPOSITION 5.3. *If (X, H) is connected by V -triplets of lines then $\rho_X \leq 3$. If equality holds then, up to reordering, \mathcal{L}^1 is a covering family, \mathcal{L}^2 is horizontal and dominating with respect to the $\text{rc}(\mathcal{L}^1)$ -fibration and \mathcal{L}^3 is horizontal and dominating with respect to the $\text{rc}(\mathcal{L}^1, \mathcal{L}^2)$ -fibration.*

PROOF. The first assertion follows from Proposition (2.17).

Assume now that $\rho_X = 3$; since X is $\text{rc}(\mathcal{L}^1, \mathcal{L}^2, \mathcal{L}^3)$ -connected at least one of the families, say \mathcal{L}^1 , is covering.

Let $\pi^1: X \dashrightarrow Z^1$ be the $\text{rc}(\mathcal{L}^1)$ -fibration; since $\rho_X = 3$, by Proposition (2.17), we have $\dim Z^1 > 0$. Two general fibers of π^1 are connected by chains of curves in \mathcal{L}^2 and \mathcal{L}^3 , so one of the families, say \mathcal{L}^2 , is horizontal and dominating with respect to π^1 . Let $\pi^2: X \dashrightarrow Z^2$ be the $\text{rc}(\mathcal{L}^1, \mathcal{L}^2)$ -fibration; since $\rho_X = 3$, by Proposition (2.17), we have $\dim Z^2 > 0$. Two general fibers of π^2 are connected by chains of curves parametrized by \mathcal{L}^3 , so \mathcal{L}^3 is horizontal and dominating with respect to π^2 . \square

We will now explore the relation between the property of being covered by V -triplets of lines and the property of being connected by V -triplets of lines:

PROPOSITION 5.4. *Assume that (X, H) is RCC-connected by a family V . If (X, H) is covered by V -triplets of lines then (X, H) is connected by V -triplets of lines. The converse holds if $\rho_X = 3$.*

PROOF. Consider the set of triplets of families of lines whose numerical classes add up to $[V]$: $\mathcal{S} = \{(\mathcal{L}_i^1, \mathcal{L}_i^2, \mathcal{L}_i^3) \mid [\mathcal{L}_i^1] + [\mathcal{L}_i^2] + [\mathcal{L}_i^3] = [V]\}_{i=1, \dots, k}$. For every $i = 1, \dots, k$ denote by B_i the set of points which are contained in a connected chain $\ell^1 \cup \ell^2 \cup \ell^3$, with ℓ^j parametrized by \mathcal{L}_i^j and $\ell^j \cap \ell^{j+1} \neq \emptyset$ for $j = 1, 2$. The set B_i can be written as the union of three closed subsets:

1. $B_i^1 := \text{Locus}(\mathcal{L}_i^2, \mathcal{L}_i^1)_{\text{Locus}(\mathcal{L}_i^3)}$,
2. $B_i^2 := e_2(p_2^{-1}(p_2(e_2^{-1}(\text{Locus}(\mathcal{L}_i^1)))) \cap p_2(e_2^{-1}(\text{Locus}(\mathcal{L}_i^3))))$,
3. $B_i^3 := \text{Locus}(\mathcal{L}_i^2, \mathcal{L}_i^3)_{\text{Locus}(\mathcal{L}_i^1)}$.

where e_2 and p_2 are the (proper) morphisms defined on the universal family over \mathcal{L}_i^2 appearing in the fundamental diagram

$$\begin{array}{ccc} \mathcal{U}_i^2 & \xrightarrow{e_2} & X \\ p_2 \downarrow & & \\ \mathcal{L}_i^2 & & \end{array}$$

Notice that the (closed) set B_i^j is exactly the set of points on curves parametrized by \mathcal{L}_i^j belonging to the chains.

If X is covered by V -triplets of lines, then X is contained in the union of the B_i^j ; since the B_i^j are a finite number and each of them is closed there is a pair of indexes (i_0, j_0) such that X is contained in $B_0 := B_{i_0}^{j_0}$. By construction the set B_i^j is contained in $\text{Locus}(\mathcal{L}_i^j)$, therefore the family $\mathcal{L}_{i_0}^{j_0}$ is covering.

To simplify notation we denote from now on by $\mathcal{L}^1, \mathcal{L}^2$ and \mathcal{L}^3 the families corresponding to the index i_0 . We also assume that $j_0 = 1$ – we don't lose in generality, even if the sets B_i^j have not the same definition for different j 's. Let us consider the rationally connected fibration $\pi^1 : X \dashrightarrow Z^1$ with respect to the family \mathcal{L}^1 . If $\dim Z^1 = 0$ then X is $\text{rc}(\mathcal{L}^1)$ -connected, so also $\text{rc}(\mathcal{L}^1, \mathcal{L}^2, \mathcal{L}^3)$ -connected, otherwise we claim that either \mathcal{L}^2 or \mathcal{L}^3 is horizontal and dominating with respect to π^1 .

To prove the claim recall that X is covered by connected cycles $\ell^1 + \ell^2 + \ell^3$, and observe that these cycles are not contracted by π^1 , otherwise also curves parametrized by V would be contracted, and Z^1 should be a point. Therefore a general fiber of π^1 meets a cycle $\ell^2 + \ell^3$ and does not contain it, so the claim follows.

Assume that \mathcal{L}^2 is horizontal and dominating with respect to π^1 and consider the $\text{rc}(\mathcal{L}^1, \mathcal{L}^2)$ -fibration $\pi^2 : X \dashrightarrow Z^2$. If $\dim Z^2 = 0$ then X is $\text{rc}(\mathcal{L}^1, \mathcal{L}^2)$ -connected, so also $\text{rc}(\mathcal{L}^1, \mathcal{L}^2, \mathcal{L}^3)$ -connected, otherwise we can prove, arguing as above, that \mathcal{L}^3 is horizontal and dominating with respect to π^2 .

We can thus consider the $\text{rc}(\mathcal{L}^1, \mathcal{L}^2, \mathcal{L}^3)$ -fibration $\pi^3 : X \dashrightarrow Z^3$; this fibration contracts the cycles $\ell^1 + \ell^2 + \ell^3$, hence contracts curves parametrized by $[V]$: it follows that $\dim Z^3 = 0$, i.e., X is $\text{rc}(\mathcal{L}^1, \mathcal{L}^2, \mathcal{L}^3)$ -connected.

Now we suppose that $\rho_X = 3$ and that (X, H) is connected by V -triplets of lines, i.e., that there exist three families of lines $\mathcal{L}^1, \mathcal{L}^2, \mathcal{L}^3$ with $[V] = [\mathcal{L}^1] + [\mathcal{L}^2] + [\mathcal{L}^3]$ such that X is $\text{rc}(\mathcal{L}^1, \mathcal{L}^2, \mathcal{L}^3)$ -connected; we want to prove that (X, H) is covered by V -triplets of lines.

Notice that the assumption on the Picard number implies that the families $\mathcal{L}^1, \mathcal{L}^2, \mathcal{L}^3$ are numerically independent. If all the families are covering, the statement is clear, so we can assume that \mathcal{L}^3 is not. Let $\pi^2 : X \dashrightarrow Z^2$ be the $\text{rc}(\mathcal{L}^1, \mathcal{L}^2)$ -fibration; since X is $\text{rc}(\mathcal{L}^1, \mathcal{L}^2, \mathcal{L}^3)$ -connected then \mathcal{L}^3 is horizontal and dominating with respect to π^2 . By Proposition (2.7) and Lemma (2.9) a general fiber F of π^2 has dimension

$$\dim F \geq -K_X \cdot \mathcal{L}^1 + \dim \text{Locus}(\mathcal{L}^2)_x - 1,$$

for x general in $\text{Locus}(\mathcal{L}^2)$. It follows that, for y general in $\text{Locus}(\mathcal{L}^3)$ that

$$\dim \text{Locus}(\mathcal{L}^3)_y + \dim \text{Locus}(\mathcal{L}^2)_x \leq n + K_X \cdot \mathcal{L}^1 + 1 \leq -K_X \cdot (\mathcal{L}^2 + \mathcal{L}^3); \quad (2)$$

on the other hand, by Proposition (2.7) we have

$$\dim \text{Locus}(\mathcal{L}^3)_y + \dim \text{Locus}(\mathcal{L}^2)_x \geq -K_X \cdot (\mathcal{L}^2 + \mathcal{L}^3) - 2. \quad (3)$$

Therefore, recalling that \mathcal{L}^3 is not covering we have that either $\dim \text{Locus}(\mathcal{L}^3)_y = -K_X \cdot \mathcal{L}^3 + 1$ or $-K_X \cdot \mathcal{L}^3$. In the former case $\text{Locus}(\mathcal{L}^3)_y$ dominates Z , hence $\text{Locus}(\mathcal{L}^3, \mathcal{L}^1, \mathcal{L}^2)_y$ is not empty and, by Lemma (2.9), its dimension is n .

Otherwise $\dim \text{Locus}(\mathcal{L}^3)_x = -K_X \cdot \mathcal{L}^3$ and, by Proposition (2.7), $\text{Locus}(\mathcal{L}^3)$ is a divisor D_3 ; since \mathcal{L}^3 is horizontal with respect to π^2 , then D_3 is not trivial on the fibers of π^2 .

By formulas (2) and (3) and Proposition (2.7) $\text{Locus}(\mathcal{L}^2)$ is either X or a divisor D_2 (and in this case clearly $D_2 \cdot \mathcal{L}^1 > 0$). In both cases through every point of X there is a connected cycle consisting of a line in \mathcal{L}^1 and a line in \mathcal{L}^2 . The divisor D_3 is positive on this cycle hence we get the required V -triplet of lines. \square

5.2. RCC-manifolds connected by reducible cubics

The next situation we are going to consider is again related to the presence of many reducible cycles, i.e., we will consider RCC-manifolds of Picard number greater than one, such that through two general points there is a reducible cycle parametrized by the closure of the connecting family. It turns out that the only such manifolds which are not covered by V -triplets of lines are products of two projective spaces polarized by $\mathcal{O}(1, 2)$.

PROPOSITION 5.5. *Assume that (X, H) is RCC with respect to a family V and that, given two general points $x, x' \in X$, there exists a reducible cycle parametrized by V passing through x and x' . Assume moreover that $\rho_X > 1$ and that X is not connected by V -triplets of lines. Then $(X, H) \simeq (\mathbb{P}^t \times \mathbb{P}^{n-t}, \mathcal{O}(1, 2))$.*

PROOF. By assumption there is no reducible cycle parametrized by V consisting of three lines passing through two general points, hence through two general points there exists a reducible cycle $\ell + \gamma$ parametrized by V consisting of a line and a conic.

Consider the pairs $\{(\mathcal{L}^j, C^j)\}_{j=1, \dots, k}$, where \mathcal{L}^j is a family of lines and C^j is a family of conics and $[\mathcal{L}^j] + [C^j] = [V]$; let \mathcal{C}^j be the Chow family associated to C^j , with universal family $\mathcal{U}_{\mathcal{C}^j}$.

Define, as in [18, IV.4], $\text{Chain}_1(\mathcal{L}^j) = \mathcal{U}_{\mathcal{L}^j} \times_{\mathcal{L}^j} \mathcal{U}_{\mathcal{L}^j}$, with projections $p_j^{\mathcal{L}}$ and $u_j^{\mathcal{L}}$ onto X and $\text{Chain}_1(\mathcal{C}^j) = \mathcal{U}_{\mathcal{C}^j} \times_{\mathcal{C}^j} \mathcal{U}_{\mathcal{C}^j}$, with projections $p_j^{\mathcal{C}}$ and $u_j^{\mathcal{C}}$ onto X . Let Y_j^1 and Y_j^2 be defined by the fiber squares:

$$\begin{array}{ccc} Y_j^1 & \longrightarrow & \text{Chain}_1(\mathcal{L}^j) \\ \downarrow & & \downarrow p_j^{\mathcal{L}} \\ \text{Chain}_1(\mathcal{C}^j) & \xrightarrow{u_j^{\mathcal{C}}} & X \end{array} \qquad \begin{array}{ccc} Y_j^2 & \longrightarrow & \text{Chain}_1(\mathcal{C}^j) \\ \downarrow & & \downarrow p_j^{\mathcal{C}} \\ \text{Chain}_1(\mathcal{L}^j) & \xrightarrow{u_j^{\mathcal{L}}} & X \end{array}$$

Our assumptions can be restated by saying that the natural morphism

$$\text{ev}: \bigcup_{j=1}^k (Y_j^1 \cup Y_j^2) \longrightarrow X \times X$$

is dominant. Since for every j the image of $Y_j = Y_j^1 \cup Y_j^2$ in $X \times X$ is closed, there exists an index \bar{j} such that $\text{ev}|_{Y_{\bar{j}}}: Y_{\bar{j}} \rightarrow X \times X$ is surjective. From now on we consider all objects corresponding to this index \bar{j} and we omit it.

Denote by ev^1 and ev^2 the restrictions of ev to Y^1 and Y^2 . The morphism ev^1 is the composition of the natural isomorphism $Y^1 \rightarrow Y^2$ with ev^2 and the involution

exchanging the factors of $X \times X$, hence both ev^1 and ev^2 are surjective. For (x, x') to be in the image of ev^1 (respectively ev^2) means that there is a cycle $\ell + \gamma$ with ℓ and γ parametrized by \mathcal{L} and \mathcal{C} such that $x \in \ell$ and $x' \in \gamma$ (respectively $x \in \gamma$ and $x' \in \ell$). So, by the surjectivity of ev^1 and ev^2 , for every $x \in X$

$$X = \text{Locus}(\mathcal{L}, \mathcal{C})_x = \text{Locus}(\mathcal{C}, \mathcal{L})_x.$$

It follows that both \mathcal{L} and \mathcal{C} are covering families.

For a general $x \in X$ we have that C_y is proper for any point $y \in \text{Locus}(\mathcal{L}_x)$; in fact, if this were not the case, then through x there would be a reducible cycle with numerical class $[V]$, consisting of three lines, contradicting our assumptions.

It follows, applying Lemma (2.10) with $Y = \text{Locus}(\mathcal{L})_x$ and $\mathcal{V} = \mathcal{C}$, that every curve in X is numerically equivalent to a linear combination of $[\mathcal{L}]$ and $[\mathcal{C}]$; in particular we have that $[\mathcal{L}]$ and $[\mathcal{C}]$ are not proportional, since we are assuming $\rho_X > 1$. Applying twice Lemma (2.18) we get that $\text{NE}(X) = \langle [\mathcal{L}], [\mathcal{C}] \rangle_{\mathcal{C}}$; in particular \mathcal{C} is a quasi-unsplit family.

Let $\varphi_{\mathcal{L}} : X \rightarrow Z$ be the contraction of the ray $\mathbb{R}_+[\mathcal{L}]$; since for a general x we have $X = \text{Locus}(\mathcal{C}, \mathcal{L})_x$ it follows that $\text{Locus}(\mathcal{C})_x$ dominates Z ; in the same way, denoting by $\varphi_{\mathcal{C}} : X \rightarrow W$ the contraction of the ray $\mathbb{R}_+[\mathcal{C}]$ we deduce that $\text{Locus}(\mathcal{L})_x$ dominates W . Fibers of two different extremal contractions can meet only in points, therefore $\dim Z + \dim W \geq n$, hence

$$\dim \text{Locus}(\mathcal{L})_x + \dim \text{Locus}(\mathcal{C})_x \geq n;$$

Equality holds, since $\dim(\text{Locus}(\mathcal{L})_x \cap \text{Locus}(\mathcal{C})_x) = 0$, so, by Remark (2.8)

$$-K_X \cdot (\mathcal{L} + \mathcal{C}) = \dim \text{Locus}(\mathcal{L})_x + \dim \text{Locus}(\mathcal{C})_x + 2 = n + 2;$$

it follows that both the extremal contractions of X are equidimensional.

Since $H \cdot \mathcal{L} = 1$ we can apply [11, Lemma 2.12] to get that $\varphi_{\mathcal{L}}$ gives to X a structure of \mathbb{P} -bundle over Z and Z is smooth: more precisely $X = \mathbb{P}_Z(\mathcal{E} := \varphi_{\mathcal{L}*}H)$.

A general fiber F of the contraction $\varphi_{\mathcal{C}}$ is $\text{Locus}(\mathcal{C})_x$ for some $x \in F$; we can apply [15, Theorem 3.6] to get that F is a projective space; therefore by [20, Theorem 4.1] also Z is a projective space. Let l be any line in Z ; consider $X_l := \varphi_{\mathcal{L}}^{-1}(l) = \mathbb{P}_l(\mathcal{E}_l)$; the image of X_l via $\varphi_{\mathcal{C}}$ has dimension smaller than X_l ; the only vector bundle on \mathbb{P}^1 such that its projectivization has a map (different by the projection onto \mathbb{P}^1) to a smaller dimensional variety, is the trivial one (and its twists). Therefore \mathcal{E} is uniform of splitting type (a, a, \dots, a) , hence \mathcal{E} splits by [2, Proposition 1.2].

It follows that X is a product of projective spaces, that \mathcal{C} is the family of lines in one of the factors and that $H = \mathcal{O}(1, 2)$. \square

COROLLARY 5.6. *Let (X, H) be RCC with respect to a family V and assume that $\rho_X > 1$. If V is not generically unsplit then either X is connected by V -triplets of lines or $(X, H) \simeq (\mathbb{P}^t \times \mathbb{P}^{n-t}, \mathcal{O}(1, 2))$.*

PROOF. The assertion follows from Mori Bend and Break Lemma; in fact, if V is not generically unsplit then through two general points $x, x' \in X$, there is a reducible cycle in \mathcal{V} . \square

6. RCC manifolds covered by lines: Picard number

In this section we are going to prove Theorem (1.1):

THEOREM. *Let (X, H) be RCC with respect to a family V , and assume that (X, H) is covered by lines. Then $\rho_X \leq 3$, and if equality holds then there exist three families of rational curves of H -degree one, $\mathcal{L}^1, \mathcal{L}^2, \mathcal{L}^3$ with $[V] = [\mathcal{L}^1] + [\mathcal{L}^2] + [\mathcal{L}^3]$, such that X is $rc(\mathcal{L}^1, \mathcal{L}^2, \mathcal{L}^3)$ -connected.*

In view of Proposition (5.3) and Corollary (5.6) we can confine to the following situation:

6.1. (X, H) is a RCC-manifold with respect to a generically unsplit family V , covered by lines and not connected by V -triplets of lines.

We will show that in this setting we have $\rho_X \leq 2$, so we assume, by contradiction, that $\rho_X \geq 3$.

Since V is generically unsplit, by [18, Corollary IV.2.9], we have that

$$-K_X \cdot V = n + 1. \quad (4)$$

Consider the set $\mathcal{B}' = \{(\mathcal{L}^i, C^i)\}$ of pairs of families (\mathcal{L}^i, C^i) – where \mathcal{L}^i is a family of lines and C^i is a family of conics and $[\mathcal{L}^j] + [C^j] = [V]$ – such that through a general point $x \in X$ there is a reducible cycle $\ell + \gamma$, with ℓ and γ parametrized respectively by \mathcal{L}^i and C^i and such that $[\mathcal{L}^i]$ is independent from $[V]$.

Since through every point of x there is a reducible cycle as above – otherwise, by Proposition (4.1), we will have $\rho_X = 1$ – and the families of lines and conics on X are a finite number, \mathcal{B}' is not empty.

Let $\mathcal{B} = \{(\mathcal{L}^i, C^i)\}_{i=1}^k$ be a maximal set of pairs in \mathcal{B}' such that $[V], [\mathcal{L}^1], \dots, [\mathcal{L}^k]$ are numerically independent; if one of the family of lines in the pairs belonging to \mathcal{B}' is covering, we choose it to be \mathcal{L}^1 . Denote by Π_i the two-dimensional vector subspace of $N_1(X)$ spanned by $[V]$ and $[\mathcal{L}^i]$. By Lemma (2.10) we have

$$N_1(X) = \langle [V], [\mathcal{L}^1], [C^1], \dots, [\mathcal{L}^k], [C^k] \rangle = \langle [V], [\mathcal{L}^1], [\mathcal{L}^2], \dots, [\mathcal{L}^k] \rangle,$$

hence the Picard number of X is $k + 1$.

CLAIM 6.2. Let (\mathcal{L}, C) be a pair in \mathcal{B} . If C is a dominating family then it is locally unsplit.

PROOF. Assume by contradiction that C is not locally unsplit. Arguing as in Proposition (5.4) we can show that there are two families of lines \mathcal{L}' and \mathcal{L}'' such that $[\mathcal{L}'] + [\mathcal{L}''] = [C]$, \mathcal{L}' is covering and \mathcal{L}'' is horizontal and dominating with respect to the $\text{rc}(\mathcal{L}')$ -fibration.

Since through a general point there is a reducible cycle $\gamma + \ell$, with γ and ℓ parametrized by C and \mathcal{L} , respectively, then either curves of \mathcal{L} are contracted by the $\text{rc}(\mathcal{L}', \mathcal{L}'')$ -fibration or \mathcal{L} is horizontal and dominating with respect to this fibration.

In both cases the $\text{rc}(\mathcal{L}', \mathcal{L}'', \mathcal{L})$ -fibration contracts both curves parametrized by C and curves parametrized by \mathcal{L} , hence also curves parametrized by V are contracted and X is connected by V -triplets of lines, a contradiction. \square

Case 1: \mathcal{L}^1 is not a covering family.

Denote by \mathcal{L} the covering family of lines.

Since no family of lines in \mathcal{B} is covering, then the families of conics are dominating. Moreover they are locally unsplit, in view of Claim (6.2). Therefore $\dim \text{Locus}(C^i)_x \cap \text{Locus}(C^j)_x = 0$ for every $i \neq j$; it follows that

$$-K_X \cdot (C^i + C^j) \leq \dim \text{Locus}(C^i)_x + \text{Locus}(C^j)_x + 2 \leq n + 2, \quad (5)$$

so, recalling that $-K_X \cdot (C^i + \mathcal{L}^i) = -K_X \cdot V = n + 1$ we also have

$$-K_X \cdot (\mathcal{L}^i + \mathcal{L}^j) \geq n. \quad (6)$$

For every $i = 1, \dots, k$ denote by E_i the set $\text{Locus}(C^i, \mathcal{L}^i)_x$; by Lemma (2.9) it has dimension $\dim E_i \geq n - 1$; since $E_i \subset \text{Locus}(\mathcal{L}^i)$, the inclusion is an equality and E_i is an irreducible divisor. Moreover, by Corollary (2.11) we have $N_1(E_i, X) = \langle [C^i], [\mathcal{L}^i] \rangle$.

We can assume that \mathcal{L} is not numerically proportional to V , otherwise the $\text{rc}(\mathcal{L})$ -fibration would take X to a point, and $\rho_X = 1$ by Proposition (2.17).

This implies that there is at least a divisor, say E_1 , which is not trivial (hence positive) on \mathcal{L} ; therefore the family \mathcal{L}^1 is horizontal and dominating with respect to the $\text{rc}(\mathcal{L})$ -fibration. We can assume that $[\mathcal{L}] \notin \Pi_1$, otherwise the $\text{rc}(\mathcal{L}, \mathcal{L}^1)$ -fibration $\pi : X \dashrightarrow Z$ would go to a point and $\rho_X = 2$.

Let F be a fiber of π and $x \in F \cap \text{Locus}(\mathcal{L}^1)$; then, by Proposition (2.7),

$$\dim F \geq \text{Locus}(\mathcal{L}^1, \mathcal{L})_x \geq \dim \text{Locus}(\mathcal{L}^1)_x + 1 \geq -K_X \cdot \mathcal{L}^1 + 1,$$

hence, recalling that $-K_X \cdot (\mathcal{L}^1 + C^1) = n + 1$,

$$\dim Z \leq n + K_X \cdot \mathcal{L}^1 - 1 = (n - 1) - (n + 1) - K_X \cdot C^1 = -K_X \cdot C^1 - 2.$$

On the other hand, since $[\mathcal{L}] \notin \Pi_1$ curves of C^1 are not contracted by π and, by Claim (6.2) C^1 is locally unsplit, we have, by Proposition (2.7), $\dim Z \geq \dim \text{Locus}(C^1)_x \geq -K_X \cdot C^1 - 1$, a contradiction.

Case 2 \mathcal{L}^1 is a covering family.

We will denote from now on the pair (\mathcal{L}^1, C^1) by (\mathcal{L}, C) . If C is quasi-unsplit, then the $\text{rc}(\mathcal{L}, \mathcal{C})$ -fibration (which contracts the curves parametrized by V) goes to a point and $\rho_X \leq 2$ by Proposition (2.17).

Therefore we can assume, from now on, that C is not quasi-unsplit. Let $x \in X$ be general; then C_y is proper for any point $y \in \text{Locus}(\mathcal{L}_x)$; in fact, if this were not the case, then through x there would be a reducible cycle with numerical class $[V]$, consisting of three lines, and, this, in view of Proposition (5.4), would contradict our assumptions.

By Lemma (2.10) $N_1(\text{Locus}(\mathcal{L}, \mathcal{C})_x, X) = \langle [\mathcal{L}], [C] \rangle$ and, by Lemma (2.9), $\dim \text{Locus}(\mathcal{L}, \mathcal{C})_x \geq n - 1$; if the inequality is strict, then we get the contradiction $\rho_X = 2$ by Lemma (2.10). (Notice that this is always the case if $n = 2$, so from now on we can assume $n \geq 3$).

If equality holds, then an irreducible component of $\text{Locus}(\mathcal{L}, \mathcal{C})_x$ is a divisor, that we will call D_x . If the intersection number $D_x \cdot \mathcal{L}$, which is nonnegative since \mathcal{L} is a covering family, is positive, we have $X = \text{Locus}(\mathcal{L})_{D_x}$ and again the contradiction $\rho_X = 2$ by Lemma (2.10).

If else $D_x \cdot \mathcal{L} = 0$, then every curve of \mathcal{L} which meets D_x is contained in it; in particular this implies that $x \in D_x$; this has two important consequences: the first one is that $D_x \cdot V > 0$; in fact being general, x can be joined to another general point $x' \notin D_x$ by a curve parametrized by V . The second one is that, since $x \in D_x \subset \text{Locus}(C)$ and x is general, then C is a dominating family, and so it is locally unsplit by Claim (6.2).

Let $(\overline{\mathcal{L}}, \overline{C}) \in \mathcal{B}$ be a pair different from (\mathcal{L}, C) . If $\overline{\mathcal{L}}$ is not covering then \overline{C} is dominating (and locally unsplit, by Claim (6.2)).

Then, since x is general, D_x meets a general curve of \overline{C} ; since $[\overline{C}] \notin N_1(D_x, X)$ we have $D_x \cdot \overline{C} > 0$ and hence, by the same reason, $\dim \text{Locus}(\overline{C})_x = 1$, forcing $-K_X \cdot \overline{C} = 2$.

Recalling that $-K_X \cdot (\overline{\mathcal{L}} + \overline{C}) = -K_X \cdot V = n + 1$, we have $-K_X \cdot \overline{\mathcal{L}} = n - 1$, hence,

by Proposition (2.7) and Lemma (2.9) we have $\dim \text{Locus}(\overline{\mathcal{L}}, \mathcal{L})_x = n$, and $\rho_X = 2$ by Corollary (2.11), a contradiction.

If also $\overline{\mathcal{L}}$ is covering we can repeat all the above arguments for the pair $(\overline{\mathcal{L}}, \overline{\mathcal{C}})$. For a general x we thus have two divisors D_x and \overline{D}_x , which clearly have non empty intersection. In particular D_x meets both $\text{Locus}(\overline{\mathcal{L}})_x$ and $\text{Locus}(\overline{\mathcal{C}})_x$. Since D_x cannot contain curves proportional either to $[\overline{\mathcal{L}}]$ or to $[\overline{\mathcal{C}}]$ we have $\dim \text{Locus}(\overline{\mathcal{L}})_x = \dim \text{Locus}(\overline{\mathcal{C}})_x = 1$, hence

$$n + 1 = -K_X \cdot (\overline{\mathcal{L}} + \overline{\mathcal{C}}) \leq \dim \text{Locus}(\overline{\mathcal{L}})_x + 1 + \dim \text{Locus}(\overline{\mathcal{C}})_x + 1 = 4.$$

So we are left with the case $n = 3$. Let $\ell' \cup \ell''$ be a reducible cycle parametrized by \mathcal{C} , such that $[\ell']$ and $[\ell'']$ are not contained in the two dimensional vector subspace $\langle [\mathcal{L}], [\mathcal{C}] \rangle$. If such a cycle does not exist, then $\rho_X = 2$ by Proposition (2.17) since the $\text{rc}(\mathcal{L}, \mathcal{C})$ -fibration goes to a point.

The divisor $K_X + H$ is trivial on C , hence, up to exchange the cycles, we have $-K_X \cdot \ell' \geq 1$, therefore, denoting by \mathcal{L}' a family of deformations of ℓ' , by Proposition (2.7), we have $\dim \text{Locus}(\mathcal{L}') \geq 2$. In particular \mathcal{L}' is horizontal and dominating with respect to the $\text{rc}(\mathcal{L})$ -fibration.

The $\text{rc}(\mathcal{L}, \mathcal{L}')$ -fibration π' has fibers of dimension two, since every fiber contains $\text{Locus}(\mathcal{L}', \mathcal{L})_x$ for some $x \in \text{Locus}(\mathcal{L}')$ and if π' goes to a point, then $\rho_X = 2$.

It follows that, denoting by \mathcal{L}'' a family of deformations of ℓ'' , \mathcal{L}'' is horizontal and dominating with respect to π' , and X is $\text{rc}(\mathcal{L}, \mathcal{L}', \mathcal{L}'')$ connected, against the assumptions. \square

7. RCC manifold covered by lines: scroll structure

In this section we are going to prove Theorem (1.2):

THEOREM. *Let (X, H) be RCC with respect to a family V , assume that (X, H) is covered by lines and that $\rho_X = 3$. Then there is a covering family of lines whose numerical class spans an extremal ray of $\text{NE}(X)$; the associated extremal contraction $\varphi : X \rightarrow Y$ makes X into a special Bănică scroll over a smooth Y .*

PROOF. By Theorem (1.1) and Proposition (5.3) we know that there exist three families of lines $\mathcal{L}^1, \mathcal{L}^2, \mathcal{L}^3$ such that $[\mathcal{L}^1] + [\mathcal{L}^2] + [\mathcal{L}^3] = [V]$. Moreover \mathcal{L}^1 is covering, \mathcal{L}^2 is horizontal and dominating with respect to the $\text{rc}(\mathcal{L}^1)$ -fibration $\pi^1 : X \dashrightarrow Z^1$, and \mathcal{L}^3 is horizontal and dominating with respect to the $\text{rc}(\mathcal{L}^1, \mathcal{L}^2)$ -fibration $\pi^2 : X \dashrightarrow Z^2$.

We will first show that among the families \mathcal{L}^i which are covering there is (at least) one whose numerical class generates an extremal ray of $\text{NE}(X)$. To this end, we divide the proof into cases, according to the number of families among the \mathcal{L}^i which

are covering; notice that, as shown by the examples in Section 3, all cases do really occur.

Case 1: The families \mathcal{L}^i , $i = 1, \dots, 3$, are all covering.

Assume that $[\mathcal{L}^3]$ does not span an extremal ray; then, by [5, Proposition 1, (ii)] there is a $\text{rc}(\mathcal{L}^3)$ -equivalence class of dimension greater than the general one, hence an irreducible component G of this class of dimension $\dim G \geq -K_X \cdot \mathcal{L}^3$. Consider $\text{Locus}(\mathcal{L}^1, \mathcal{L}^2)_G$; by Lemma (2.9) its dimension is at least $n - 1$; if this dimension is n then $[\mathcal{L}^1], [\mathcal{L}^2]$ lie in a two-dimensional extremal face of $\text{NE}(X)$ by Lemma (2.19).

We can draw the same conclusion if an irreducible component of $\text{Locus}(\mathcal{L}^1, \mathcal{L}^2)_G$ is a divisor D . In fact, if D is positive either on \mathcal{L}^1 or on \mathcal{L}^2 we have $X = \text{ChLocus}_{m_1}(\mathcal{L}^1, \mathcal{L}^2)_G$ and we apply again Lemma (2.19). If else $D \cdot \mathcal{L}^1 = D \cdot \mathcal{L}^2 = 0$, recalling that the numerical class in X of every curve in D can be written as $\sum a_i [\mathcal{L}^i]$ with $a_3 \geq 0$ by Lemma (2.18), we get that $D|_D$ is nef, hence D is nef and is a supporting divisor of a face which contains $[\mathcal{L}^1]$ and $[\mathcal{L}^2]$.

We can repeat the same argument starting from another family, say \mathcal{L}^2 ; therefore we prove that, if neither $[\mathcal{L}^3]$ nor $[\mathcal{L}^2]$ span an extremal ray, then $[\mathcal{L}^1]$ belongs to two different extremal faces of $\text{NE}(X)$, hence to an extremal ray.

Case 2: Two families among the \mathcal{L}^i , $i = 1, \dots, 3$, are covering.

If the second covering family is \mathcal{L}^3 , then it is horizontal and dominating with respect to π^1 ; moreover, since X is $\text{rc}(\mathcal{L}^1, \mathcal{L}^2, \mathcal{L}^3)$ -connected, \mathcal{L}^2 will be horizontal and dominating with respect to the $\text{rc}(\mathcal{L}^1, \mathcal{L}^3)$ -fibration, so, without loss of generality we can assume that \mathcal{L}^2 is covering and \mathcal{L}^3 is not.

Assume that $\text{codim Locus}(\mathcal{L}^3) \geq 2$; by Proposition (2.7) we have, for any $x \in \text{Locus}(\mathcal{L}^3)$, that $\dim \text{Locus}(\mathcal{L}^3)_x \geq -K_X \cdot \mathcal{L}^3 + 1$. By Lemma (2.9) for such an x we have $X = \text{Locus}(\mathcal{L}^3, \mathcal{L}^2, \mathcal{L}^1)_x$, hence $[\mathcal{L}^1]$ is extremal by Lemma (2.20).

Assume now that for a general $x \in \text{Locus}(\mathcal{L}^3)$ we have $\dim \text{Locus}(\mathcal{L}^3)_x = -K_X \cdot \mathcal{L}^3$; by Proposition (2.7) $\text{Locus}(\mathcal{L}^3)$ is a divisor, that we denote by D_3 ; since \mathcal{L}^3 is horizontal and dominating with respect to π^2 , then, for some $i = 1, 2$ we have $D_3 \cdot \mathcal{L}^i > 0$.

Consider $\text{Locus}(\mathcal{L}^2, \mathcal{L}^1)_{\text{Locus}(\mathcal{L}^3)_x}$; if it is X , then $[\mathcal{L}^1]$ is extremal by Lemma (2.20). If else $\text{Locus}(\mathcal{L}^2, \mathcal{L}^1)_{\text{Locus}(\mathcal{L}^3)_x}$ has dimension $n - 1$, we take D to be an irreducible component. If D is positive on \mathcal{L}^1 or \mathcal{L}^2 then $X = \text{ChLocus}_{m_1}(\mathcal{L}^1, \mathcal{L}^2)_{\text{Locus}(\mathcal{L}^3)_x}$ and $[\mathcal{L}^1]$ and $[\mathcal{L}^2]$ are in an extremal face by Lemma (2.19).

So assume that $D \cdot \mathcal{L}^1 = D \cdot \mathcal{L}^2 = 0$; as in Case 1 we can prove that D is nef.

Since D is trivial on \mathcal{L}^1 and \mathcal{L}^2 then $[\mathcal{L}^1]$ and $[\mathcal{L}^2]$ are contained in an extremal face σ .

If neither $[\mathcal{L}^1]$ nor $[\mathcal{L}^2]$ are extremal, then, being $-K_X$ positive on \mathcal{L}^1 and \mathcal{L}^2 , there is a Mori extremal ray R in σ . Without loss of generality we can assume that $[\mathcal{L}^1]$ is in the interior of the cone spanned by $[\mathcal{L}^2]$ and R . Let B be the indeterminacy locus of π^1 , let D_Z be a very ample divisor on $\pi^1(X \setminus B)$ and let $\widehat{D}_1 := \overline{(\pi^1)^{-1}D_Z}$. The divisor \widehat{D}_1 is trivial on \mathcal{L}^1 and positive on \mathcal{L}^2 , since this family is covering, hence \widehat{D}_1 is negative on R , so the exceptional locus of R is contained in the indeterminacy locus of π^1 and so has codimension at least two in X .

Let F be a fiber of the contraction associated to R ; by Proposition (2.14) F has dimension $\dim F \geq 2$. Then $\dim \text{Locus}(\mathcal{L}^1, \mathcal{L}^2)_F \geq -K_X \cdot (\mathcal{L}^1 + \mathcal{L}^2)$ by Lemma (2.9).

Since $D_3 \cdot \mathcal{L}^i > 0$ for some i the intersection $\text{Locus}(\mathcal{L}^3)_x \cap \text{Locus}(\mathcal{L}^1, \mathcal{L}^2)_F$ is not empty for some x , therefore

$$\dim \text{Locus}(\mathcal{L}^3)_x \cap \text{Locus}(\mathcal{L}^1, \mathcal{L}^2)_F \geq -K_X \cdot (\mathcal{L}^1 + \mathcal{L}^2 + \mathcal{L}^3) - n \geq 1,$$

a contradiction, since the numerical class of every curve in $\text{Locus}(\mathcal{L}^1, \mathcal{L}^2)_F$ belongs to σ and $[\mathcal{L}^3]$ does not.

Case 3: Only \mathcal{L}^1 is covering.

Let F be a general fiber of π^2 ; it contains $\text{Locus}(\mathcal{L}^2, \mathcal{L}^1)_x$ for some x , hence, by Proposition (2.7) and Lemma (2.9), it has dimension

$$\dim F \geq -K_X \cdot (\mathcal{L}_1 + \mathcal{L}_2) - 1.$$

It follows that

$$\dim Z^2 = \dim X - \dim F \leq -K_X \cdot \mathcal{L}^3$$

Since $\text{Locus}(\mathcal{L}^3)_x$ has dimension $\dim \text{Locus}(\mathcal{L}^3)_x \geq -K_X \cdot \mathcal{L}^3$ by Proposition (2.7), it dominates the target; moreover π^2 does not contract curves in $\text{Locus}(\mathcal{L}^3)_x$, so $\dim \text{Locus}(\mathcal{L}^3)_x = -K_X \cdot \mathcal{L}^3 = \dim Z^2$; by the first equality and Proposition (2.7) we get that $\dim \text{Locus}(\mathcal{L}^3) = n - 1$, while from the second we infer that, for some m , $X = \text{ChLocus}_m(\mathcal{L}^1, \mathcal{L}^2)_{\text{Locus}(\mathcal{L}^3)_x}$ and we apply Lemma (2.19) to get that $[\mathcal{L}^1]$ and $[\mathcal{L}^2]$ are contained in an extremal face σ .

Let D_3 be an irreducible component of $\text{Locus}(\mathcal{L}^3)$. If $D_3 \cdot \mathcal{L}^1 > 0$ then we can exchange the role of \mathcal{L}^2 and \mathcal{L}^3 and obtain that $[\mathcal{L}^1]$ and $[\mathcal{L}^3]$ belong to a face σ' . Therefore, belonging to two different extremal faces, $[\mathcal{L}^1]$ belongs to an extremal ray.

If else $D_3 \cdot \mathcal{L}^1 = 0$ then $D_3 \cdot \mathcal{L}^2 > 0$, since \mathcal{L}^3 is horizontal and dominating with respect to π^2 . Therefore, if $[\mathcal{L}^1]$ were not extremal in σ , then there would be a

curve C , with $[C] \in \sigma$ such that $D_3 \cdot C < 0$. Counting dimensions we get that D_3 is an irreducible component of $\text{Locus}(\mathcal{L}^2, \mathcal{L}^1, \mathcal{L}^3)_x = \text{Locus}(\mathcal{L}^1, \mathcal{L}^3)_{\text{Locus}(\mathcal{L}^2)_x}$ for a suitable x , hence every curve in D_3 is numerically equivalent to $\alpha C_1 + \beta C_2 + \gamma C_3$ with $\beta \geq 0$. In particular every curve in D_3 whose class is in σ is numerically equivalent to $\alpha \mathcal{L}^1 + \beta \mathcal{L}^2$ with $\beta \geq 0$, so D_3 is nef on σ , a contradiction.

We have thus shown the first part of the statement; we can assume that the covering family which spans an extremal ray R of $\text{NE}(X)$ is \mathcal{L}^1 . We will now show that there is an extremal contraction of X which is a special Bănică scroll.

Let $\varphi : X \rightarrow Y$ be the contraction associated to R . If the general fiber of φ has dimension $-K_X \cdot \mathcal{L}^1 - 1$ and any fiber has dimension $\leq -K_X \cdot \mathcal{L}^1$, then φ is a special Bănică scroll by [3, Proposition 2.5].

We will prove that, if this is not the case, then there is another contraction of X which is a projective bundle.

Assume first φ has a fiber F of dimension $\geq -K_X \cdot \mathcal{L}^1 + 1$.

On an open subset φ coincides with the $\text{rc}(\mathcal{L}^1)$ -fibration π^1 ; since $\text{Locus}(\mathcal{L}^2)$ is closed and \mathcal{L}^2 is horizontal and dominating with respect to π^1 , we deduce that $\text{Locus}(\mathcal{L}^2)$ meets every fiber of φ . Let F' be an irreducible component of $\text{Locus}(\mathcal{L}^2)_F$; then, applying Lemma (2.9), we get $\dim F' \geq -K_X \cdot (\mathcal{L}^1 + \mathcal{L}^2)$.

CLAIM 7.1. There exists $x \in X$ and a component F' of $\text{Locus}(\mathcal{L}^2)_F$ such that $\text{Locus}(\mathcal{L}^3)_x \cap F' \neq \emptyset$.

This is clear if $\mathcal{L}^1, \mathcal{L}^2$ and \mathcal{L}^3 are all covering families. If both \mathcal{L}^1 and \mathcal{L}^2 are covering, but \mathcal{L}^3 is not, we showed at the beginning of Case 2 that either for some x we have $X = \text{Locus}(\mathcal{L}^3, \mathcal{L}^2, \mathcal{L}^1)_x$ – and the claim follows – or $\text{Locus}(\mathcal{L}^3)$ is a divisor. The latter happens (Cf. Case 3) also if \mathcal{L}^1 is the only covering family.

So assume that $\text{Locus}(\mathcal{L}^3)$ is a divisor; since \mathcal{L}^3 is horizontal and dominating with respect to the $\text{rc}(\mathcal{L}^1, \mathcal{L}^2)$ -fibration π^2 , there exists an irreducible component D_3 of $\text{Locus}(\mathcal{L}^3)$ which has positive intersection number with at least one among the families \mathcal{L}^1 and \mathcal{L}^2 , hence $D_3 \cap F' \neq \emptyset$ and the claim follows.

Pick x such that $\text{Locus}(\mathcal{L}^3)_x$ meets F' ; for such a point we have

$$\dim \text{Locus}(\mathcal{L}^3)_x + \dim F' \leq n,$$

from which we get that $\dim \text{Locus}(\mathcal{L}^3)_x = -K_X \cdot \mathcal{L}^3 - 1$, and \mathcal{L}^3 is covering by Proposition (2.7).

Being \mathcal{L}^3 covering, we can swap the roles of \mathcal{L}^2 and \mathcal{L}^3 and, by the same argument, we get that also \mathcal{L}^2 is covering.

It follows that both $\text{Locus}(\mathcal{L}^2, \mathcal{L}^3)_F$ and $\text{Locus}(\mathcal{L}^3, \mathcal{L}^2)_F$ are nonempty, hence, by Lemma (2.9), we have $X = \text{Locus}(\mathcal{L}^2, \mathcal{L}^3)_F = \text{Locus}(\mathcal{L}^3, \mathcal{L}^2)_F$; by Lemma (2.20)

both $[\mathcal{L}^2]$ and $[\mathcal{L}^3]$ span an extremal ray.

Let $\psi : X \rightarrow Y'$ be the contraction of $\mathbb{R}_+[\mathcal{L}^2]$; we have

$$\dim Y' \leq n + K_X \cdot \mathcal{L}^2 + 1 \leq -K_X \cdot (\mathcal{L}^1 + \mathcal{L}^3);$$

on the other hand, since no curve in $\text{Locus}(\mathcal{L}^3)_F$ is contracted by ψ we have

$$\dim Y' \geq \dim \text{Locus}(\mathcal{L}^3)_F \geq \dim F - K_X \cdot \mathcal{L}^3 - 1 \geq -K_X \cdot (\mathcal{L}^1 + \mathcal{L}^3).$$

It follows that all inequalities are equalities; in particular a general fiber of ψ has dimension $-K_X \cdot \mathcal{L}^2 - 1$ and $\text{Locus}(\mathcal{L}^3)_F$ meets every fiber, hence ψ is equidimensional. Recalling that $H \cdot \mathcal{L}^2 = 1$ we get that ψ is a projective bundle on a smooth variety by [11, Lemma 2.12].

Assume now that every fiber of φ has dimension $-K_X \cdot \mathcal{L}^1$. If \mathcal{L}^2 is not a covering family then fibers of π^2 have dimension $\geq -K_X \cdot (\mathcal{L}^1 + \mathcal{L}^2)$, therefore \mathcal{L}^3 is covering. Therefore at least one among \mathcal{L}^2 and \mathcal{L}^3 is a covering family: say it is \mathcal{L}^2 . (Notice that the same argument will work if the covering family is \mathcal{L}^3 even if the situation is not symmetric in \mathcal{L}^2 and \mathcal{L}^3).

Assume that $[\mathcal{L}^2]$ does not span an extremal ray; then, by [5, Proposition 1, (ii)] there is a $\text{rc}(\mathcal{L}^2)$ -equivalence class of dimension greater than the general one, hence an irreducible component G of this class of dimension $\dim G \geq -K_X \cdot \mathcal{L}^2$.

Let $\tilde{G} = \varphi^{-1}(\varphi(G))$; it has dimension $\dim \tilde{G} \geq -K_X \cdot (\mathcal{L}^1 + \mathcal{L}^2)$. For some x we have $\text{Locus}(\mathcal{L}^3)_x \cap \tilde{G} \neq \emptyset$, hence $\dim \text{Locus}(\mathcal{L}^3)_x \leq n - \dim \tilde{G} \leq -K_X \cdot \mathcal{L}^3 - 1$, hence \mathcal{L}^3 is a covering family.

By Lemma (2.9) we have $X = \text{Locus}(\mathcal{L}^3)_{\tilde{G}}$, hence \tilde{G} meets all the $\text{rc}(\mathcal{L}^3)$ -classes, which are thus equidimensional, and \mathcal{L}^3 is extremal. As above we can show that the associated contraction is a projective bundle over a smooth base. \square

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Gianluca OCCHETTA

Università di Trento - Dipartimento di Matematica
 via Sommarive 14, I-38050 Povo (TN)
 gianluca.occhetta@unitn.it

Valentina PATERNO

Università di Trento - Dipartimento di Matematica
via Sommarive 14, I-38050 Povo (TN)
paterno@science.unitn.it