

ON POLYNOMIALS WITH GIVEN HILBERT FUNCTION AND APPLICATIONS

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ABSTRACT. Using Macaulay's correspondence we study the family of Artinian Gorenstein local algebras with fixed symmetric Hilbert function decomposition. As an application we give a new lower bound for the dimension of cactus varieties of the third Veronese embedding. We discuss the case of cubic surfaces, where interesting phenomena occur.

INTRODUCTION

Macaulay established a correspondence between polynomials and Artinian local Gorenstein algebras. In particular, any polynomial is a dual socle generator of an Artinian local Gorenstein algebra. In this paper we interpret the Hilbert function of the algebra as a Hilbert function of the corresponding polynomial, and give a description of the set of polynomials with given symmetric Hilbert function decomposition, in a fixed polynomial ring. We consider polynomials f in a divided power ring $S = K_{\text{dp}}[x_1, \dots, x_n]$, and a polynomial ring $T = K[y_1, \dots, y_n]$ acting on S by contraction (see Section 0.1). The Artinian local Gorenstein algebra A associated to $f \in S$ is the quotient T/f^\perp where f^\perp is the annihilator ideal of f . Thus $\text{Spec}(T/f^\perp) \subset \text{Spec}(T)$ is a local Gorenstein scheme supported at the origin of the affine space $\text{Spec}(T)$.

The application we have in mind is that of apolarity and the dimension of cactus varieties of cubic forms. Cactus varieties are generalizations of secant varieties.

Definition 1. Let $X \subset \mathbb{P}^N$ be a projective variety. The r -th cactus variety $\text{Cactus}_r(X)$ of X is the closure of the union of the linear spaces spanned by length r subschemes on X .

Here we abuse slightly the notation of variety, since the cactus variety is often a reducible algebraic set. We are interested in the case when $X \simeq \mathbb{P}^n$ is embedded into $\mathbb{P}^{\binom{n+3}{3}-1}$ by the third Veronese embedding.

Consider, like above, a divided power ring $\bar{S} = K_{\text{dp}}[x_0, \dots, x_n]$, and a polynomial ring $\bar{T} = K[y_0, \dots, y_n]$ acting on \bar{S} by contraction. A cubic form $F \in \bar{S}_3$ up to multiplication by scalars is a point in $\mathbb{P}(\bar{S}_3)$. The pure cubes form a n -dimensional subvariety $V_{3,n} \subset \mathbb{P}(\bar{S}_3)$. The least length of a subscheme $Z \subset V_{3,n}$ whose linear span contains F is called the *cactus rank* of F . This notion was introduced in [Iarrobino 1995, Definition 4D] under the name scheme length. The closure of the set of cubic forms with cactus rank r is the r -th cactus variety of $V_{3,n}$, denoted $\text{Cactus}_r(V_{3,n})$. Via the contraction action, \bar{T} is the natural homogeneous coordinate ring on $\mathbb{P}(\bar{S}_1)$, and a $Z \subset V_{3,n}$ contains F in its span, if and only if its homogeneous ideal $I_Z \subset \bar{T}$ is contained in F^\perp . This classical fact is called the *apolarity lemma* and is the motivation for defining a subscheme $Z \subset \mathbb{P}(\bar{S}_1)$ *apolar* to F if $I_Z \subset F^\perp$.

2010 *Mathematics Subject Classification.* Primary 13H10, Secondary 14Q15, 14C05.
Key words and phrases. cactus rank, Artinian Gorenstein local algebra.

We apply Macaulay's correspondence to investigate local Gorenstein schemes that are apolar to F . Our main result is the following lower bound on the dimension of cactus varieties of cubic forms.

Theorem 2 (Corollary 44). *Let $n \geq 8$ and $18 \leq r \leq 2n + 2$ and let $V_{3,n}$ be the third Veronese embedding of \mathbb{P}^n in \mathbb{P}^N , with $N = \binom{n+3}{3}$. Then*

$$\dim \text{Cactus}_r(V_{3,n}) \geq \begin{cases} (rn + r - 1) + \frac{r(r-2)(r-16)}{48} - 1 & \text{if } r \geq 18 \text{ even,} \\ (rn + r - 1) + \frac{(r-1)(r-3)(r-17)}{48} - 2 & \text{if } r \geq 19 \text{ odd.} \end{cases}$$

Hence, under these assumptions the r -th secant variety of $V_{3,n}$ is strictly contained in the r -th cactus variety of $V_{3,n}$.

The r -th secant variety of $V_{3,n}$ is $\sigma_r(V_{3,n}) = \overline{\bigcup_{P_1, \dots, P_r \in V_{3,n}} \langle P_1, \dots, P_r \rangle}$. The fact that the inclusion of $\sigma_r(V_{3,n})$ in $\text{Cactus}_r(V_{3,n})$ is strict is a consequence of the inequality

$$\dim \sigma_r(V_{3,n}) \leq rn + r - 1.$$

The right-hand side is the expected dimension of the secant variety, and it is an easy parameter count that gives an upper bound. The actual dimension of the secant variety is known thanks to the Alexander and Hirschowitz Theorem [Alexander, Hirschowitz 1995].

For $r \geq 2n+2$ the variety $\text{Cactus}_r(V_{3,n})$ fills the ambient space, see [Bernardi, Ranestad 2012]. Observe that $\sigma_r(V_{3,n}) = \text{Cactus}_r(V_{3,n})$, for $r \leq 13$, see [Casnati, Notari 2011] for the cases $r \leq 11$, and [Casnati, Jelisiejew, Notari 2015] for the remaining cases.

The link between the Artinian local Gorenstein algebras and apolar schemes to a homogeneous form $F \in \bar{S}$ is provided by the fact that if $f = F(1, x_1, \dots, x_n)$, then

$$\text{Spec}(K[y_1, \dots, y_n]/f^\perp) \subset \mathbb{P}(\bar{S}_1)$$

is a local scheme supported at $[1 : 0 : \dots : 0]$ that is apolar to F . We define

Definition 3. The minimal length of a local apolar scheme to F is called the *local cactus rank* of F .

That link is strengthened by the following result, where we call the sum of the homogeneous terms of a polynomial g of degree at most d the *degree- d tail* of g .

Proposition 4. *Let F be a homogeneous polynomial of degree d , and let $f = F(1, x_1, \dots, x_n)$. Let Γ be a zero-dimensional scheme of minimal length among local schemes supported at $[l] = [1 : 0 : \dots : 0]$ that are apolar to F . Then Γ is the affine apolar scheme of a polynomial g whose degree- d tail equals f .*

A particularly important problem is to find the cactus rank of a general form in \bar{S}_d , i.e. the minimal r such that $\text{Cactus}_r(V_{d,n}) = \mathbb{P}(\bar{S}_d)$. Our results do not improve previous known bounds, so this remains a major open problem in this theory. We refer the interested reader to [Iarrobino, Kanev 1999, Proposition 5.67], [Bernardi, Ranestad 2012, Theorem 3], [Buczyńska, Buczyński 2014], and [Bernardi, Brachat, Mourrain 2014].

Now, the first step in order to be able to compute the cactus rank, is to describe the structure of minimal apolar schemes. If we start by considering a minimal zero-dimensional scheme Γ apolar to a form F and decompose it as $\Gamma = \Gamma_1 \cup \dots \cup \Gamma_k$, where each Γ_i is a scheme supported on one point, then we take the corresponding decomposition $F = F_1 + \dots + F_k$, where Γ_i is a minimal local scheme apolar to F_i for $i = 1, \dots, k$. According to Proposition 4, one would like to find invariants for local apolar Gorenstein schemes, parameterizing the degree- d tails of all polynomials that define a scheme with given invariant. Iarrobino's analysis ([Iarrobino 1994]),

that we recall in the first section, provides one such discrete invariant: the symmetric Hilbert function decomposition. Then, if one wants to estimate the dimension of the sets of polynomials with the same local cactus rank, one needs to understand the structure of the polynomials with the same symmetric Hilbert function decomposition. This is what we do in Section 2 by the use of standard and exotic forms, that explain the unlucky behavior that the number of variables involved in each homogeneous summand of a given polynomial f may be larger than what is expected from the Hilbert function and its symmetric decomposition. Motivated by this, we describe the family \mathcal{F}_f^r of polynomials g whose linear partials and Hilbert function coincide with those of f , and such that $g - f$ is part of an exotic summand for g . In Proposition 21 we show that the polynomials in \mathcal{F}_f^r define isomorphic apolar algebras and compute in Corollary 23 its dimension. In Proposition 24 we give a decomposition of any polynomial as a sum of a polynomial in standard form and an exotic summand. In Section 3 we focus on the local cactus rank by proving Proposition 4 and computing the local cactus rank of a general cubic surface. Finally in Section 4 we use this description in order to estimate the dimension of the set of all the polynomials with the same symmetric Hilbert function decomposition, which allows us to estimate the dimension of the Cactus variety. In particular, we apply these results to the special cases of Hilbert functions $(1, m - 1, m - 1, 1)$ and $(1, m - 1, m - 1, 1, 1)$ to prove our lower bound for this dimension in Corollary 44.

0.1. Notations. For main applications we consider homogeneous forms in $\bar{S} = K_{\text{dp}}[x_0, \dots, x_n]$ and their dehomogenization in $S = K_{\text{dp}}[x_1, \dots, x_n]$:

$$\pi_{x_0} : \bar{S} \rightarrow S; \quad F(x_0, x_1, \dots, x_n) \mapsto F(1, x_1, \dots, x_n).$$

We consider the action of the polynomial ring $\bar{T} := K[y_0, \dots, y_n]$ on \bar{S} by contraction: if $\alpha = (\alpha_0, \dots, \alpha_n)$ and $\beta = (\beta_0, \dots, \beta_n)$ are multi-indices, then

$$y^\alpha(x^\beta) = \begin{cases} x^{\beta - \alpha} & \text{if } \beta \geq \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, we consider the action of the polynomial ring $T := K[y_1, \dots, y_n]$ on S :

$$T \times S \rightarrow S; \quad (\psi, f) \mapsto \psi(f)$$

defined by this contraction restricted to y^α, x^β , when $\alpha_0 = \beta_0 = 0$.

Note that we are using the same notation for ordinary powers in T and divided powers in S , unlike what is usually done in the literature (for properties of divided power rings see for instance [Iarrobino, Kanev 1999, Appendix A], where the divided power x^β would be written as $x^{[\beta]}$).

In characteristic 0 we could have used ordinary differentiation, and therefore, by abuse of language, we call $\psi(f) \in S$ a *partial* of a polynomial $f \in S$ for any $\psi \in T$.

1. PRELIMINARIES

We begin this section by presenting the Macaulay correspondence between polynomials and Artinian Gorenstein local rings, which is the starting point of our theory.

1.1. Macaulay correspondence. Let K be an algebraically closed field of characteristic other than 2, 3 and consider the divided power ring $S := K_{\text{dp}}[x_1, \dots, x_n]$. Consider the action of the polynomial ring $T := K[y_1, \dots, y_n]$ on S defined in Subsection 0.1.

Let S_1 and T_1 be the degree-one parts of S and T respectively. With respect to the action above (classically known as *apolarity*), S_1 and T_1 are natural dual spaces and $\langle x_1, \dots, x_n \rangle$ and $\langle y_1, \dots, y_n \rangle$ are dual bases. The annihilator of a polynomial f of degree d is an ideal in T

which we denote by $f^\perp \subset T$. The quotient $T_f := T/f^\perp$ is a local Artinian Gorenstein ring (see e.g. [Iarrobino, Kanev 1999, Lemma 2.14]). In fact, T_f is finitely generated as a K -module so it is *Artinian*. The image of T_1 in T_f generates the unique maximal ideal \mathfrak{m} , so T_f is local. Furthermore T_f has a 1-dimensional socle, the annihilator of the maximal ideal, namely \mathfrak{m}^d , so T_f is *Gorenstein*. In addition, if f is a form, f^\perp is a homogeneous ideal and therefore T_f is an Artinian Gorenstein *graded* local ring.

1.2. Symmetric decomposition of the Hilbert function of a polynomial. We consider now a polynomial $f \in S = K_{\text{dp}}[x_1, \dots, x_n]$ and let $f^\perp \subset T = K[y_1, \dots, y_n]$ be its annihilator with respect to contraction. We shall interpret a Hilbert function for the local Artinian Gorenstein quotient $T_f = T/f^\perp$ in terms of a filtration of the space of partials of the polynomial f . In particular we recall and interpret Iarrobino's analysis of Hilbert functions on associated graded algebras of T_f and their symmetric decomposition. We will apply this analysis in the next section to characterize the polynomials with a given Hilbert function.

The local Artinian Gorenstein quotient ring $T_f = T/f^\perp$ is naturally isomorphic to

$$\text{Diff}(f) = \{\psi(f) \mid \psi \in T\}$$

the space of all partials of f , as a T -module.

Following [Iarrobino 1994], we consider Hilbert functions on graded rings associated to two filtrations of T_f . Let \mathfrak{m} be the maximal ideal of T_f . The \mathfrak{m} -adic filtration

$$T_f = \mathfrak{m}^0 \supset \mathfrak{m} \supset \mathfrak{m}^2 \supset \dots \supset \mathfrak{m}^d \supset \mathfrak{m}^{d+1} = 0$$

where $d = \deg f$, defines an associated graded ring

$$T_f^* = \bigoplus_{i=0}^d \mathfrak{m}^i / \mathfrak{m}^{i+1}$$

whose Hilbert function we denote by H_f . The Löewy filtration

$$T_f = (0 : \mathfrak{m}^{d+1}) \supset (0 : \mathfrak{m}^d) \supset \dots \supset (0 : \mathfrak{m}^2) \supset (0 : \mathfrak{m}) \supset 0$$

induces the following sequence of ideals of T_f^* : For each $a = 0, 1, 2, \dots$ let

$$C_a = \bigoplus_{i=0}^{d-a} C_{a,i} = \bigoplus_{i=0}^{d-a} ((0 : \mathfrak{m}^{d+1-a-i}) \cap \mathfrak{m}^i) / ((0 : \mathfrak{m}^{d+1-a-i}) \cap \mathfrak{m}^{i+1}) \subset T_f^*,$$

and consider the T_f^* -modules

$$Q_a = C_a / C_{a+1}, \quad a = 0, 1, 2, \dots$$

and their respective Hilbert functions

$$\Delta_{Q,a} = H(Q_a).$$

Each Q_a decomposes as a sum $Q_a = \bigoplus_{i=0}^d Q_{a,i}$, where $Q_{a,i} = C_{a,i} / C_{a+1,i}$. We can check from the definitions that for $d \geq 2$ and any $a > d - 2$, we have $C_a = Q_a = 0$. The following is an important result on the structure of these modules.

Proposition 5. [Iarrobino 1994, Theorem 1.5] *The T_f^* -modules Q_a , $a = 0, 1, 2, \dots$, satisfy the following reflexivity condition*

$$Q_{a,i}^\vee \cong Q_{a,d-a-i}.$$

In particular, the Hilbert function $\Delta_{Q,a} = H(Q_a)$ is symmetric about $(d-a)/2$, and thus the Hilbert function $H(T_f^)$ of T_f^* has a symmetric decomposition*

$$H(T_f^*) = \sum_a \Delta_{Q,a}.$$

The possible symmetric decompositions of the Hilbert function is restricted by the fact that the partial sums of the symmetric decomposition are Hilbert functions of suitable quotients of T_f^* .

Corollary 6. [Iarrobino 1994, Section 5B, p. 69] *The Hilbert function of $T_f^*/C_{\alpha+1}$ satisfies*

$$H(T_f^*/C_{\alpha+1}) = \sum_{a=0}^{\alpha} \Delta_{Q,a}.$$

In particular every partial sum $\sum_{a=0}^{\alpha} \Delta_{Q,a}$ is the Hilbert function of a K -algebra generated in degree 1.

Iarrobino has listed all possible symmetric decompositions of Hilbert functions of rings T_f^* with $\dim T_f^* \leq 16$, see [Iarrobino 1994, Section 5].

We now interpret the ideal C_a and the module Q_a in terms of the space $\text{Diff}(f)$ of partials of f . This interpretation depends on the isomorphism

$$\tau : T/f^\perp \rightarrow \text{Diff}(f), \quad \psi \mapsto \psi(f),$$

of T -modules and thus K -vector spaces. Let $\text{Diff}(f)_i$ be the subspace of $\text{Diff}(f)$ of partials of degree at most i . The image of $(0 : \mathfrak{m}^i)$ under the map τ is precisely $\text{Diff}(f)_{i-1}$, so the L\"owey filtration

$$(0 : \mathfrak{m}) \subset (0 : \mathfrak{m}^2) \subset (0 : \mathfrak{m}^3) \subset \cdots \subset (0 : \mathfrak{m}^d) \subset (0 : \mathfrak{m}^{d+1}) = T_f$$

of T_f is mapped to the degree filtration

$$K = \text{Diff}(f)_0 \subset \text{Diff}(f)_1 \subset \text{Diff}(f)_2 \subset \cdots \subset \text{Diff}(f)_d = \text{Diff}(f)$$

of $\text{Diff}(f)$. Now

$$(0 : \mathfrak{m}^i)/(0 : \mathfrak{m}^{i-1}) \cong (\mathfrak{m}^{i-1}/\mathfrak{m}^i)^\vee,$$

so the integral function

$$H_f(0) = 1, \quad H_f(i) = \dim_K \text{Diff}(f)_i - \dim_K \text{Diff}(f)_{i-1}, \quad i = 1, \dots, d,$$

coincides with the Hilbert function of T_f^* :

$$H_f(i) = H(T_f^*)(i).$$

On the other hand, the \mathfrak{m} -adic filtration

$$T_f \supset \mathfrak{m} \supset \mathfrak{m}^2 \supset \cdots \supset \mathfrak{m}^d \supset \mathfrak{m}^{d+1} = 0$$

corresponds to an order filtration on $\text{Diff}(f)$. We call the *order* of $\psi \in T$ the smallest degree of a non-zero homogeneous term of ψ , and denote it by $\text{ord}(\psi)$. We call the *order* of a partial f' of f the largest order of a $\psi \in T$ such that $f' = \psi(f)$. Thus the image

$$\tau(\mathfrak{m}^i) \subseteq \text{Diff}(f)_{d-i}$$

is simply the space of partials of order at least i of f .

The isomorphism $Q_{a,i}^\vee \cong Q_{a,d-a-i}$ allows us to interpret the vector space $Q_{a,i}^\vee$ as parameterizing partials of f of degree i and order $d-a-i$, modulo partials of lower degree and larger order.

More precisely, let $\text{Diff}(f)_i^a \subset \text{Diff}(f)$ be the subspace of partials of degree at most i and order at least $d-i-a$, then

$$Q_{a,i}^\vee \cong \text{Diff}(f)_i^a / (\text{Diff}(f)_{i-1}^a + \text{Diff}(f)_i^{a-1}).$$

So

$$(1) \quad \Delta_{f,a}(i) = \dim_K (Q_{a,i}^\vee) = \dim_K (\text{Diff}(f)_i^a / (\text{Diff}(f)_{i-1}^a + \text{Diff}(f)_i^{a-1})).$$

Notation 7. We denote by Δ_f the symmetric decomposition $H_f = \sum_a \Delta_{f,a}$ of the Hilbert function H_f .

Consider the space of linear forms that are partials of f ,

$$\text{Lin}(f) := \text{Diff}(f)_1 \cap S_1,$$

and the linear subspaces

$$\text{Lin}(f)^a = \{l \in S_1 \mid l \text{ is a partial of } f \text{ of order at least } d - a - 1\} = \text{Diff}(f)_1^a \cap S_1.$$

We easily see that for each $a \geq 0$, we have an isomorphism $\text{Lin}(f)^a \simeq \text{Diff}(f)_1^a / \text{Diff}(f)_0$, so $\Delta_{f,a}(1) = \dim_K \text{Lin}(f)^a - \dim_K \text{Lin}(f)^{a-1}$. We obtain a canonical flag of subspaces of S_1 :

$$\text{Lin}(f)^0 \subseteq \text{Lin}(f)^1 \subseteq \dots \subseteq \text{Lin}(f)^{d-2} = \text{Lin}(f) \subseteq S_1.$$

Example 8. Let $f = x_1^3 x_2 + x_3^3 + x_4^2$. Its space of partials is generated by the elements in the following table, where the generators of each Q_a^\vee are arranged by degree; next to it, we have the symmetric decomposition of its Hilbert function:

Generators of the space of partials						Hilbert function decomposition						
degree	0	1	2	3	4	degree	0	1	2	3	4	
Q_0^\vee	1	x_1, x_2	$x_1^2, x_1 x_2$	$x_1^3, x_1^2 x_2$	f	$\Delta_{f,0}$	=	1	2	2	2	1
Q_1^\vee		x_3	x_3^2			$\Delta_{f,1}$	=	0	1	1	0	0
Q_2^\vee		x_4				$\Delta_{f,2}$	=	0	1	0	0	0
						H_f	=	1	4	3	2	1

For instance x_3^2 is a partial of order 1, since it is obtained as $y_3(f) = x_3^2$ and cannot be attained by a higher order element of T , so it is a generator of $Q_{1,2}^\vee$. Here we have $\text{Lin}(f)^0 = \langle x_1, x_2 \rangle$, $\text{Lin}(f)^1 = \langle x_1, x_2, x_3 \rangle$, and $\text{Lin}(f)^2 = \langle x_1, x_2, x_3, x_4 \rangle$.

In the next section we shall enumerate polynomials f with a given Hilbert function H_f using this symmetric decomposition Δ_f . For this purpose we denote a Hilbert function H by its values

$$H = (H(0), H(1), \dots, H(d))$$

and the decomposition Δ_f ,

$$H = \sum_a \Delta_a,$$

by its summands

$$\Delta = (\Delta_0, \dots, \Delta_{d-2}),$$

where each Δ_a is symmetric around $(d-a)/2$, i.e. $\Delta_a(i) = \Delta_a(d-a-i)$.

By Corollary 6, both H and each partial sum

$$\Delta_{\leq \alpha} = \sum_{a=0}^{\alpha} \Delta_a$$

are Hilbert functions of K -algebras generated in degree 1, so there are some immediate restrictions on these functions. First, Hilbert functions H and $\Delta_{\leq \alpha}$ have positive values and satisfy the Macaulay growth condition (cf. [Macaulay 1927]): If the i -binomial expansion of $\Delta_{\leq \alpha}(i)$ is

$$\Delta_{\leq \alpha}(i) = \binom{m_i}{i} + \binom{m_{i-1}}{i-1} + \dots + \binom{m_j}{j}; \quad m_i > m_{i-1} > \dots > m_j \geq j \geq 1,$$

then

$$(2) \quad \Delta_{\leq \alpha}(i+1) \leq \binom{m_i+1}{i+1} + \binom{m_{i-1}+1}{i} + \cdots + \binom{m_j+1}{j+1}.$$

Example 9. For $H(1) = 8$, $H(2) \geq 5$ and $1 + H(1) + \cdots + H(d-1) + 1 = 17$ the possible Hilbert functions H and their decompositions $H = \sum_i \Delta_i$ that satisfy the Macaulay growth conditions are the following:

$$\begin{array}{l} H = 1 \ 8 \ 7 \ 1 \\ \Delta_0 = 1 \ 7 \ 7 \ 1 \\ \Delta_1 = 0 \ 1 \ 0 \end{array}, \quad \begin{array}{l} H = 1 \ 8 \ 6 \ 1 \ 1 \\ \Delta_0 = 1 \ 1 \ 1 \ 1 \ 1 \\ \Delta_1 = 0 \ 5 \ 5 \ 0 \\ \Delta_2 = 0 \ 2 \ 0 \end{array}, \quad \begin{array}{l} H = 1 \ 8 \ 5 \ 1 \ 1 \ 1 \\ \Delta_0 = 1 \ 1 \ 1 \ 1 \ 1 \ 1 \\ \Delta_1 = 0 \ 0 \ 0 \ 0 \ 0 \\ \Delta_2 = 0 \ 4 \ 4 \ 0 \\ \Delta_3 = 0 \ 3 \ 0 \end{array}$$

$$\begin{array}{l} H = 1 \ 8 \ 5 \ 2 \ 1 \\ \Delta_0 = 1 \ 2 \ 3 \ 2 \ 1 \\ \Delta_1 = 0 \ 2 \ 2 \ 0 \\ \Delta_2 = 0 \ 4 \ 0 \end{array}, \quad \begin{array}{l} H = 1 \ 8 \ 5 \ 2 \ 1 \\ \Delta_0 = 1 \ 2 \ 2 \ 2 \ 1 \\ \Delta_1 = 0 \ 3 \ 3 \ 0 \\ \Delta_2 = 0 \ 3 \ 0 \end{array}.$$

2. STANDARD FORMS AND EXOTIC FORMS

At this point of the analysis we would like to find a precise description of all the polynomials having the same symmetric Hilbert function decomposition. To this purpose we have firstly to deal with the fact that the number of variables involved in each homogeneous summand of a given polynomial f may be larger than what is expected from the Hilbert function. This will be explained by the appearance of what we will call exotic summands of f . We will analyze their role in a description of all polynomials that have a given Hilbert function.

Let us start with some examples clarifying the kind of phenomena that we have to treat.

2.1. Standard and “Exotic” examples. Let A be a local Artinian Gorenstein algebra. As explained before it can be represented as a quotient $A \simeq T/I$ where $I = f^\perp$ for a polynomial $f \in S$. If the ideal I is fixed, then f is unique up to action by a unit of T , but clearly the choice of I such that $A \simeq T/I$ is not unique. In this section we wish to shed some light on how this choice can be made.

Example 10. Consider the ring $A = K[\varepsilon]/(\varepsilon^4)$ and polynomials $f = x_1^3$ and $h = x_1^3 + x_1x_2$ in $K_{\text{dp}}[x_1, x_2]$. Then

$$A \simeq \frac{K[y_1, y_2]}{(y_1^4, y_2)} = \frac{K[y_1, y_2]}{f^\perp},$$

but also

$$A \simeq \frac{K[y_1, y_2]}{(y_1^4, y_2 - y_1^2)} = \frac{K[y_1, y_2]}{h^\perp}.$$

Note that both x_1 and x_2 occur in h , but while x_1 is a partial of this polynomial, x_2 is not, since its space of partials is $\text{Diff}(h) = \langle h, x_1^2 + x_2, x_1, 1 \rangle$.

Now consider the ring $B = K[y_1, y_2]/(y_1^4 - y_2^2, y_1y_2)$ and polynomials $p = x_1^4 + x_2^2$ and $q = x_1^4 + x_1^2x_2$ in $K_{\text{dp}}[x_1, x_2]$. Then

$$B = \frac{K[y_1, y_2]}{p^\perp},$$

and

$$B \simeq \frac{K[y_1, y_2]}{(y_1^3 - y_1y_2, y_2^2)} = \frac{K[y_1, y_2]}{q^\perp}.$$

In this case, x_2 is an order-one partial of both p and q , since $y_2(p) = x_2$ and $(-y_2 + y_1^2)(q) = x_2$. However, x_2 occurs in q in degree 3, which may be surprising for a linear form that is a partial of order one.

As we will see in the remainder of this section, the most common behaviour is the one we can observe in Example 8: both x_1 and x_2 are partials of order 3 and they occur in f_4 ; x_3 and x_4 are partials of orders 2 and 1, and they occur in f_3 and f_2 , respectively.

2.2. Description of the standard and the “exotic” phenomena. Referring the notation of Example 10, we want to distinguish polynomials like g and p that have a “standard” behavior from the ones like h and q where either one finds a variable that does not occur in the partials or a partial whose order does not “match” with the degree of the corresponding variable. To this end, in Definition 11 we will define *standard* forms of polynomials. Intuitively, they correspond to minimal embeddings of algebras, in terms of variables appearing in the related polynomials f . Let $f \in S$ and $A = T/f^\perp$. Moreover let

$$(3) \quad f = f_d + f_{d-1} + \cdots + f_0$$

be the decomposition in homogeneous summands.

In Section 1 we defined the Hilbert function of A and its symmetric decomposition Δ . In particular, we saw that $\Delta_a(1) = \dim_K(\text{Lin}(f)^a / \text{Lin}(f)^{a-1})$, which is space of linear partials of f of order exactly $d - a - 1$. Let

$$n_i = \sum_{j=0}^i \Delta_j(1) = \dim_K \text{Lin}(f)^i,$$

i.e. the dimension of the space of linear partials of f of order at least $d - i - 1$. By degree reasons this space is contained in the space of linear partials of $f_{d-i} + f_{d-i-1} + \cdots + f_d$. But, as we have seen in Example 10, a linear form may occur in f_{d-i} and be a partial of f of order less than $d - i - 1$ or not be a partial of f at all.

First of all let us fix here a basis of linear forms x_1, \dots, x_n in S_1 that agrees with the filtration by $\text{Lin}(f)^i$:

$$(4) \quad \text{Lin}(f)^0 = \langle x_1, \dots, x_{n_0} \rangle \subseteq \text{Lin}(f)^1 = \langle x_1, \dots, x_{n_1} \rangle \subseteq \cdots \\ \cdots \subseteq \text{Lin}(f)^{d-2} = \langle x_1, \dots, x_{n_{d-2}} \rangle \subseteq S_1 = \langle x_1, \dots, x_n \rangle.$$

Definition 11. Let $f \in S$ be a polynomial with homogeneous decomposition $f = f_d + \cdots + f_0$. Let Δ be the symmetric decomposition of the Hilbert function of T/f^\perp . We say that $f \in S$ is in *standard form* if

$$f_{d-i} \in K_{\text{dp}}[\text{Lin}(f)^i] = K_{\text{dp}}[x_1, \dots, x_{n_i}], \quad \text{for all } i,$$

where x_1, \dots, x_n is any choice of basis for S_1 as in (4).

We define the linear space of standard forms.

$$\text{StandardForms} = \{f \in P_{\leq d} \mid \forall_i f_{d-i} \in K_{\text{dp}}[\text{Lin}(f)^i]\} = \{f \in P_{\leq d} \mid \forall_i f_i \in K_{\text{dp}}[\text{Lin}(f)^{d-i}]\}.$$

A first important property of standard forms is the following:

Proposition 12. *The leading summand of a partial of f of degree $d - i$ and order j lies in $K_{\text{dp}}[\text{Lin}(f)^{i-j}]$.*

Proof. Let g be the leading summand of a partial of f of degree $d - i$ and order j , then any partial x of degree one of g is a partial of order at least $d - i + j - 1$ of f and therefore lies in $\text{Lin}(f)^{i-j}$. Therefore $g \in K_{\text{dp}}[\text{Lin}(f)^{i-j}]$. \square

There may be variables appearing in f that do not show up in the leading summands of partials of f . It is tempting to call them exotic variables, but we reserve exotic for the non-standard part of f .

Definition 13. Let $f = f_d + \cdots + f_0 \in S$ be the homogeneous decomposition of f , and choose a basis for S_1 as in (4). The *exotic summand* of degree $d - i$ of f is the form

$$f_{d-i,\infty} \in \langle x_{n_i+1}, \dots, x_n \rangle K_{\text{dp}}[x_1, \dots, x_n]$$

such that the degree $d - i$ homogeneous summand of f can be written as

$$f_{d-i} = f_{d-i,i} + f_{d-i,\infty},$$

with $f_{d-i,i} \in K_{\text{dp}}[x_1, \dots, x_{n_i}]$.

Thus f is in standard form if and only if all its exotic summands are zero.

Example 14. Let us see how the above definitions work in the cases of Example 10. We have $H_A = (1, 1, 1, 1)$ and $\Delta_0 = (1, 1, 1, 1)$, $\Delta_i = \mathbf{0}$ for $i > 0$. Therefore $n_i = 1$ for all i .

Now we have $f = x_1^3 \in K_{\text{dp}}[x_1]$ so that x_1^3 is in standard form. On the other hand, $x_1 x_2 \notin K_{\text{dp}}[x_1]$ so that $h = x_1^3 + x_1 x_2$ is not in standard form. In fact $x_1 x_2$ is an exotic summand for h .

For the ring B , we get $H_B = (1, 2, 1, 1, 1)$, with symmetric decomposition $\Delta_0 = (1, 1, 1, 1, 1)$, $\Delta_1 = (0, 0, 0, 0, 0)$, $\Delta_2 = (0, 1, 0, 0, 0)$, $\Delta_i = \mathbf{0}$ for $i > 2$. Therefore $n_0 = n_1 = 1$, and $n_i = 2$ for all $i \geq 2$. As above we can check that p is in standard form, but q is not, having $x_1^2 x_2$ as an exotic summand.

2.3. Existence of standard forms and their presentation. Let $A = T/f^\perp$ be an Artinian Gorenstein algebra. One could ask:

- (1) if there exists a presentation $A \simeq T/g^\perp$ with g in standard form.
- (2) in case such a presentation exists, whether there are any relations between g and f .

Fortunately, these questions have quite satisfactory answers, as we explain below. We need some notation.

Notation 15. Let \hat{T} denote the power series ring obtained by completing T at the ideal of the origin. In coordinates, $\hat{T} := K[[y_1, \dots, y_n]]$. We may interpret S as subset of functionals on \hat{T} via the pairing defined by

$$\langle y^\alpha, x^\beta \rangle = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$$(5) \quad \langle \sigma, f \rangle = \langle 1, \sigma(f) \rangle,$$

as seen by decomposing σ and f into monomials. In particular if $\sigma(f) = 0$, then $\langle \sigma, f \rangle = 0$.

Let $\varphi : \hat{T} \rightarrow \hat{T}$ be an automorphism of \hat{T} . It induces a dual map $\varphi^\vee : S \rightarrow S$ defined by the condition

$$(6) \quad \langle \varphi(\sigma), f \rangle = \langle \sigma, \varphi^\vee(f) \rangle, \quad \text{for all } \sigma \in T, f \in P.$$

Let I be a finite colength ideal of T supported at the origin. Then $T/I = \hat{T}/I$. Clearly, the quotients \hat{T}/I and $\hat{T}/\varphi(I)$ are isomorphic. Moreover if $I = f^\perp$, then $\varphi(I) = ((\varphi^{-1})^\vee(f))^\perp$. The fundamental result is that for every f we may find φ so that $\varphi^\vee(f)$ is in standard form. In fact we prove that φ may be chosen “with no linear part”.

We now make this precise.

Definition 16. Let M be the unique maximal ideal of \hat{T} . Let $\varphi : \hat{T} \rightarrow \hat{T}$ be an automorphism. We say that φ is of order (at least) two if

$$\varphi(y_i) = y_i \pmod{(y_1, \dots, y_n)^2}$$

for all i .

Remark 17. Every automorphism of \hat{T} induces a linear action on M/M^2 . The order two automorphisms are precisely those which act trivially. Thus they form a normal subgroup. In particular an inverse of an order two automorphism is also of order two.

Theorem 18 (Existence of standard forms). *Let $f \in S$ be a polynomial with symmetric Hilbert function decomposition Δ . Then there is an automorphism $\varphi : \hat{T} \rightarrow \hat{T}$ such that $\varphi^\vee(f)$ is in standard form. Consequently, $f = \psi^\vee(g)$ for an element $g \in \text{StandardForms}$ and an automorphism $\psi = \varphi^{-1}$.*

Moreover one can choose φ and ψ of order two.

Proof. For the existence of φ see [Iarrobino 1994, Theorem 5.3AB]. Take one such φ . We will compose φ with a linear map to obtain the required order two automorphism.

We have $\varphi(y_i) = \sum \lambda_{ij}y_j + s_i$, where $s_i \in (y_1, \dots, y_n)^2$. Let τ be the linear automorphism of \hat{T} defined by $\tau(y_i) = \sum \lambda_{ij}y_j$. Then $(\varphi \circ \tau^{-1})(y_i) \equiv y_i \pmod{M^2}$ so $\varphi \circ \tau^{-1}$ is an automorphism of order two. Let $g = \varphi^\vee(f)$. By definition g is in standard form. Since τ is a linear automorphism of \hat{T} , the map $(\tau^{-1})^\vee$ is simply a linear transformation of S , i.e. a change of variables. The definition of being in standard form is coordinate free, so that $(\tau^{-1})^\vee$ preserves being in standard form. In particular $h = (\tau^{-1})^\vee(g)$ is in standard form.

But $h = (\varphi \circ \tau^{-1})^\vee(f)$, so that $\varphi \circ \tau^{-1}$ is a required automorphism of order two. \square

It is also important and interesting to see an explicit description of the action $\varphi^\vee : S \rightarrow S$ of an automorphism $\varphi : \hat{T} \rightarrow \hat{T}$. For this, recall that S is a *divided power ring*: $x^\alpha \cdot x^\beta = \binom{\alpha+\beta}{\alpha} x^{\alpha+\beta}$, where $\binom{\alpha+\beta}{\alpha} = \prod \binom{\alpha_i+\beta_i}{\alpha_i}$.

Proposition 19. *Let $\varphi : \hat{T} \rightarrow \hat{T}$ be an automorphism. Let $D_i = \varphi(y_i) - y_i \in \hat{T}$. For a multi-index α denote $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$. Let $f \in S$. Then*

$$\varphi^\vee(f) = \sum_{\alpha \in \mathbb{N}_0^n} x^\alpha \cdot (D^\alpha(f)) = f + \sum_{i=1}^n x_i \cdot (D_i(f)) + \cdots$$

Proof. See [Jelisiejew 2015, Proposition 1.8]. \square

Example 20. Let us illustrate Theorem 18 in the setup of Example 14. We should find a g in standard form and an automorphism ψ of $K[y_1, y_2]$ such that $\psi^\vee(g) = h = x_1^3 + x_1x_2$. We see that the linear partials of h are spanned by x_1 , so if we wish ψ to be of order at least two, we must have $g \in K_{\text{dp}}[x_1]$. According to Proposition 19, $\psi^\vee(g) = \sum_{\alpha \in \mathbb{N}_0^n} x^\alpha \cdot (D^\alpha(g))$, for some elements of order at least two $D_1, D_2 \in K[[y_1, y_2]]$. Since g must have degree three, $(D_i D_j)(g) = 0$, for any i and j , so

$$x_1^3 + x_1x_2 = g + x_1D_1(g) + x_2D_2(g).$$

This implies that $D_1(g) = 0$ and $D_2(g) = x_1$ (and consequently that $g = x_1^3$), so we must have $D_1 = 0 \pmod{(y_1, y_2)^4}$ and $D_2 = y_1^2 \pmod{(y_1, y_2)^4}$. Therefore we can choose $\psi : \hat{T} \rightarrow \hat{T}$ to be the automorphism defined by $\psi(y_1) = y_1$ and $\psi(y_2) = y_2 + y_1^2$.

2.4. Description of exotic summands. For parameterization purposes and dimension counts, it is interesting to consider families of polynomials yielding isomorphic algebras, or at least sharing the same Hilbert and symmetric decomposition. Given a polynomial $f \in K_{\text{dp}}[x_1, \dots, x_k]$ such that $\text{Lin}(f) = \langle x_1, \dots, x_k \rangle$, we consider the family

$$(7) \quad \mathcal{F}_f^r := \left\{ g \in K_{\text{dp}}[x_1, \dots, x_{k+r}] \mid g - f \in (x_{k+1}, \dots, x_{k+r}), \right. \\ \left. H_g = H_f, \text{Lin}(g) = \text{Lin}(f) = \langle x_1, \dots, x_k \rangle \right\}.$$

The next result gives a characterisation of the elements in this family. We will use the notation $\text{LS}(f)$ for the leading summand of a polynomial f , i.e. if $f = f_d + \dots + f_0$ is its decomposition into homogeneous summands, $\text{LS}(f) = f_d$.

Proposition 21. *Let $f \in K_{\text{dp}}[x_1, \dots, x_k]$ and assume*

$$\text{Lin}(f) = \langle x_1, \dots, x_k \rangle.$$

Let $g \in K_{\text{dp}}[x_1, \dots, x_{k+r}]$ be any polynomial. Then $g \in \mathcal{F}_f^r$ if and only if there are elements $\phi_1, \dots, \phi_r \in K[y_1, \dots, y_k]$ of order at least two such that

$$(8) \quad g = \sum_{i_1, \dots, i_r \geq 0} x_{k+1}^{i_1} \cdots x_{k+r}^{i_r} \cdot (\phi_1^{i_1} \cdots \phi_r^{i_r})(f).$$

In particular, for each $g \in \mathcal{F}_f^r$, the algebras $K[y_1, \dots, y_{k+r}]/g^\perp$ and $K[y_1, \dots, y_k]/f^\perp$ are isomorphic.

Proof. Let l be the dimension of $\text{Diff}(f)$ and choose a basis h_1, \dots, h_l for this vector space such that $\text{LS}(h_1), \dots, \text{LS}(h_l)$ are linearly independent. Choose elements $\psi_1, \dots, \psi_l \in K[y_1, \dots, y_k]$ such that for each i , $\psi_i(f) = h_i$. Let $g \in \mathcal{F}_f^r$ and write

$$g = f + x_{k+1}g_1 + \cdots + x_{k+r}g_r$$

in such a way that for each j , we have $g_j \in K_{\text{dp}}[x_1, \dots, x_{k+j}]$. Then for each i ,

$$\psi_i(g) = \psi_i(f) + x_{k+1}\psi_i(g_1) + \cdots + x_{k+r}\psi_i(g_r).$$

Now Proposition 12 tells us that $\text{LS}(\psi_i(g)) \in K_{\text{dp}}[x_1, \dots, x_k]$ and since $\psi_i(f)$ cannot be cancelled by the terms in $x_{k+1}\psi_i(g_1) + \cdots + x_{k+r}\psi_i(g_r)$, we must have

$$\text{LS}(\psi_i(g)) = \text{LS}(\psi_i(f)) = \text{LS}(h_i).$$

But this implies that $\psi_1(g), \dots, \psi_l(g)$ form a linearly independent set, and since the dimension of $\text{Diff}(g)$ is also l (since g and f yield algebras with the same Hilbert function), we get

$$\text{Diff}(g) = \langle \psi_1(g), \dots, \psi_l(g) \rangle.$$

In addition, we know that the variables x_{k+1}, \dots, x_{k+r} cannot occur in the leading summand of g , also by Proposition 12, so the polynomials g_1, \dots, g_r have degree at most $d-2$. Now $y_{k+r}(g) = g_r$, so g_r is a partial of g , which means that there is some $\phi_r \in \langle \psi_1, \dots, \psi_l \rangle$ such that $g_r = \phi_r(g)$. Moreover ϕ_r has order at least one, because $\deg g_r \leq d-2$, so $\phi_r^{d+1}(g) = 0$. Denote $\hat{g}_j = x_{k+1}g_1 + \cdots + x_{k+j}g_j$, and observe that $y_{k+s}(\hat{g}_j) = g_s$, if $s \leq j$, and $y_{k+s}(\hat{g}_j) = 0$, otherwise. So,

$$\begin{aligned} g &= f + \hat{g}_{r-1} + x_{k+r} \cdot \phi_r(g) \\ &= f + \hat{g}_{r-1} + x_{k+r} \cdot \phi_r(f + \hat{g}_{r-1} + x_{k+r} \cdot \phi_r(g)) \\ &= f + \hat{g}_{r-1} + x_{k+r} \cdot \phi_r(f + \hat{g}_{r-1}) + x_{k+r}^2 \cdot \phi_r^2(g) \end{aligned}$$

and iterating this further we get

$$g = \sum_{i \geq 0} x_{k+r}^i \cdot \phi_r^i(f + \hat{g}_{r-1}).$$

Applying y_{k+r-1} to both sides of this equality, we get

$$y_{k+r-1}(g) = \sum_{i \geq 0} x_{k+r}^i \cdot \phi_r^i(y_{k+r-1}(f + \hat{g}_{r-1})) = \sum_{i \geq 0} x_{k+r}^i \cdot \phi_r^i(g_{r-1}).$$

Again there must be some $\phi_{r-1} \in \langle \psi_1, \dots, \psi_l \rangle$ such that $\sum_{i \geq 0} x_{k+r}^i \cdot \phi_r^i(g_{r-1}) = \phi_{r-1}(g)$. So,

$$\begin{aligned} g &= \sum_{i \geq 0} x_{k+r}^i \cdot \phi_r^i(f + \hat{g}_{r-2} + x_{k+r-1}g_{r-1}) \\ &= \sum_{i \geq 0} x_{k+r}^i \cdot \phi_r^i(f + \hat{g}_{r-2}) + x_{k+r-1} \sum_{i \geq 0} x_{k+r}^i \cdot \phi_r^i(g_{r-1}) \\ &= \sum_{i \geq 0} x_{k+r}^i \cdot \phi_r^i(f + \hat{g}_{r-2}) + x_{k+r-1} \cdot \phi_{r-1}(g). \end{aligned}$$

Iterating we get

$$g = \sum_{i_{r-1}, i_r \geq 0} x_{k+r-1}^{i_{r-1}} x_{k+r}^{i_r} \cdot \phi_{r-1}^{i_{r-1}} \phi_r^{i_r}(f + \hat{g}_{r-2}).$$

Applying the remaining operators $y_{k+r-2}, \dots, y_{k+1}$ the same way, we obtain (8). It remains to show that ϕ_1, \dots, ϕ_r have order at least two. Without loss of generality, suppose that ϕ_1 has order one, and write $\phi_1 = \phi' + \phi''$, where $\phi' \in \langle y_1, \dots, y_k \rangle$ and $\text{ord } \phi'' \geq 2$. Let $t \in \langle x_1, \dots, x_k \rangle$ be such that $\phi'(t) = 1$. Since t is a partial of f , there exists $\eta \in K[y_1, \dots, y_k]$ such that $\eta(f) = t$. Note that $(\phi_u \eta)(f) = \phi_u(t)$ is a constant, for all $u > 1$. So

$$\eta(g) = \sum_{i_1, \dots, i_r \geq 0} x_{k+1}^{i_1} \cdots x_{k+r}^{i_r} \cdot \phi_1^{i_1} \cdots \phi_r^{i_r} \eta(f) = t + x_{k+1} + \sum_{u > 1} x_{k+u} \phi_u(t),$$

and therefore $\eta(g)$ is a partial of g of degree one that does not belong to $\langle 1, x_1, \dots, x_k \rangle$, a contradiction.

For the converse, suppose that g admits a presentation as in (8). Then clearly we have $g - f \in \langle x_{k+1}, \dots, x_{k+r} \rangle$. Let φ be the automorphism of $K[y_1, \dots, y_{k+r}]$ defined by $\varphi(y_i) = y_i$ for $1 \leq i \leq k$ and $\varphi(y_{k+i}) = y_{k+i} + \phi_i$, for $1 \leq i \leq r$. Then by Proposition 19, we see that $g = \varphi^\vee(f)$. So φ induces an isomorphism between $K[y_1, \dots, y_{k+r}]/g^\perp$ and $K[y_1, \dots, y_k]/f^\perp$, which proves the last statement, and shows that $H_g = H_f$. Finally, if $t \in \text{Lin}(f)$ and we take $\eta \in K[y_1, \dots, y_k]$ such that $\eta(f) = t$, we may apply η to both sides of (8) to get $\eta(g) = t$. So $\text{Lin}(g) \subseteq \text{Lin}(f)$ and since they have the same dimension, equality must hold, and $g \in \mathcal{F}_f^r$. \square

Remark 22. We can get an alternative proof of Proposition 21 if we take $g \in \mathcal{F}_f^r$ and use Proposition 12 to see that since the linear partials of g lie in $\langle x_1, \dots, x_k \rangle$, the leading terms of partials of g only involve these variables, like we did in the proof. Now if we denote $(g^\perp)^*$ the ideal generated by the initial forms of elements of g^\perp , we know that $(g^\perp)^*$ is the annihilator of the set of leading summands of all partials of g (see a discussion on this at the beginning of Section 2 in [Casnati, Notari 2013], but also Proposition 3 in [Emsalem 1978]). Therefore for each $j > k$, $y_j \in (g^\perp)^*$, which means that there is an element $D_j \in T$ of order at least two such that $y_j - D_j \in g^\perp$. If a variable y_{j_1} occurs in a D_{j_2} , with $j_1, j_2 > k$ then we may replace y_{j_1} by D_{j_1} in D_{j_2} . Each time we do this, the minimal degree of y_{j_1} in D_{j_2} grows. When this degree exceeds $\deg g$, we may discard the remaining part, so the process eventually ends

and we may assume that each $D_j \in k[y_1, \dots, y_k]$. We can then consider the automorphism $\varphi : \hat{T} \rightarrow \hat{T}$ which sends y_j to itself for $j \leq k$ and y_j to $y_j - D_j$ for $j > k$. We can now show that $\varphi^\vee(g) = f$ by showing that for any element $\phi \in T$, $\langle \phi, \varphi^\vee(g) \rangle = \langle \phi, f \rangle$.

Note that in the definition of the family \mathcal{F}_f^r and the hypotheses of Proposition 21, the polynomial f need not be in standard form. This result gives us a way of adding exotic summands to a polynomial without changing the Hilbert function of the algebra it yields. It also gives us the following result:

Corollary 23. *Let $f \in K_{\text{dp}}[x_1, \dots, x_k] \subset K_{\text{dp}}[x_1, \dots, x_{k+r}]$ be a polynomial of degree d and assume $\text{Lin}(f) = \langle x_1, \dots, x_k \rangle$. Then the family $\mathcal{F}_f^r \subset K_{\text{dp}}[x_1, \dots, x_{k+r}]_{\leq d}$ has dimension*

$$\dim \mathcal{F}_f^r = r \cdot \sum_{\substack{a \geq 0 \\ i \leq d-a-2}} \Delta_{f,a}(i)$$

The next result shows that all exotic summands may be obtained in a similar description.

Proposition 24. *Let $f \in S$ be a polynomial of degree d and choose a basis x_1, \dots, x_n for S_1 that agrees with the filtration in (4). Then we can write $f = f_{\text{st}} + f_{\text{ex}}$ such that f_{st} is in standard form and*

$$(9) \quad f_{\text{ex}} = \sum_{\substack{\alpha > 0 \\ \alpha_1 = \dots = \alpha_{n_1} = 0}} x^\alpha \cdot (D^\alpha(f_{\text{st}})).$$

where $D_{n_1+1}, \dots, D_n \in T$ have order at least two and

- (1) $d - a \leq \deg D_k(f_{\text{st}}) \leq d - 2$, if $D_k(f_{\text{st}}) \neq 0$ and $n_{a-1} < k \leq n_a$, for any $a \geq 2$;
- (2) $\deg D_k(f_{\text{st}}) \leq d - 2$, if $k > n_{d-2}$.

Proof. Applying Theorem 18 and Proposition 19 to f , we know that there exists a polynomial g in standard form such that

$$(10) \quad f = \varphi^\vee(g) = \sum_{\alpha \in \mathbb{N}_0^n} x^\alpha \cdot D^\alpha(g),$$

with $D_1, \dots, D_n \in T$ of order at least two.

Claim. The basis x_1, \dots, x_n also satisfies the filtration in (4) for the polynomial g . To see this, let $t \in \langle x_1, \dots, x_n \rangle$ be a linear partial of g of order j , and $\eta \in T$ an element of order j such that $\eta(g) = t$. If $\langle \cdot, \cdot \rangle : T \times S$ is the usual pairing, using the rule in (6), we have

$$\langle \phi, (\varphi^{-1}(\eta))(f) \rangle = \langle (\varphi^{-1}(\eta)) \cdot \phi, f \rangle = \langle \varphi(\phi), t \rangle,$$

for any $\phi \in T$. Since φ is of order at least two, this implies $(\varphi^{-1}(\eta))(f) = t$. We also know that $\text{ord } \varphi^{-1}(\eta) = \text{ord } \eta$, so t is also a linear partial of f of order j . So f and g have the same linear partials and those partials have the same order, which proves the claim.

Now we wish to show that g can be replaced by a polynomial g' , also in standard form, so that we do not need the first operators D_1, \dots, D_{n_1} . We consider expression (10), and we start by factoring the power of any variable, say $x_1^{\alpha_1}$, and the power of the corresponding operator, $D_1^{\alpha_1}$. Note that $D_1^{\alpha_1}$ obviously commutes with the remaining operators $D_2^{\alpha_2} \cdots D_n^{\alpha_n}$. In order to move also the term $x_1^{\alpha_1}$ to the right side of $D_2^{\alpha_2} \cdots D_n^{\alpha_n}$, we can apply the rule

$$x_i(\phi(h)) = \phi(x_i h) - \phi^{(i)}(h), \quad \text{for any } \phi \in T \text{ and } h \in S,$$

where $\phi^{(i)} = \frac{\partial}{\partial y_i} \phi$. We obtain

$$\begin{aligned} f &= \sum_{\alpha \in \mathbb{N}_0^n} (x_2^{\alpha_2} \cdots x_n^{\alpha_n}) x_1^{\alpha_1} \cdot ((D_2^{\alpha_2} \cdots D_n^{\alpha_n}) D_1^{\alpha_1}(g)), \\ &= \sum_{i \geq 0} \sum_{\alpha_2, \dots, \alpha_n} (x_2^{\alpha_2} \cdots x_n^{\alpha_n}) \cdot \left((D_2 - D_2^{(1)} D_1)^{\alpha_2} \cdots (D_n - D_n^{(1)} D_1)^{\alpha_n} \right) (x_1^i \cdot D_1^i(g)), \\ &= \sum_{\alpha_2, \dots, \alpha_n} (x_2^{\alpha_2} \cdots x_n^{\alpha_n}) \cdot ((D'_2)^{\alpha_2} \cdots (D'_n)^{\alpha_n})(g'), \end{aligned}$$

where $g' = \sum_{i \geq 0} x_1^i \cdot D_1^i(g)$ and $D'_i = D_i - D_i^{(1)} D_1$. The equality between the first and the second lines can be checked by hand in a straightforward, even if cumbersome, computation. Observe that in this way we get an extra piece $D_i^{(1)}$, but this does not change the fact that the new elements D'_1, \dots, D'_n have order at least two. Repeating this procedure, we can rewrite

$$f = \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ \alpha_1 = \dots = \alpha_{n_1} = 0}} x^\alpha \cdot (D''^\alpha(g'')), \quad \text{where} \quad g'' = \sum_{\alpha \in \mathbb{N}_0^{n_1}} x^\alpha \cdot ((D'')^\alpha(g)),$$

with some modified $D''_1, \dots, D''_n \in T$ of order at least two. Since any terms of f and any terms of g involving only the variables x_1, \dots, x_{n_1} are not exotic (because the choice of basis x_1, \dots, x_n agrees with the filtration in (4) for both f and g) we have that g'' is in standard form.

Now we know that for any $k > n_1$, $D''_k(g'')$ is a partial of degree at most $d - 2$, otherwise x_k would occur in the leading summand of f , and all such linear partials belong to $\langle x_1, \dots, x_{n_0} \rangle$. Finally, suppose that for some k we have $n_{a-1} < k \leq n_a$ but $\deg D''_k(g'') < d - a$. Then the term $x_k D''_k(g'')$ has degree at most $d - a$ and belongs to $K_{\text{dp}}[x_1, \dots, x_{n_a}]$, so it is not part of an exotic summand. The terms $x_k^2 D''_k{}^2(g''), \dots, x_k^{d-1} D''_k{}^{d-1}(g'')$ have lower degree so are also not part of exotic summands. Therefore we can perform another modification as above, with the variable x_k and the corresponding operator D''_k . Doing this for every such k , we may replace g'' by some g''' also in standard form, and f will be written as in (9), with $f_{\text{st}} = g'''$. \square

Example 25. Let f be a polynomial of degree five such that $H_f = (1, 1, 1, 1, 1)$; this is the minimal possible Hilbert function. Then H_f is symmetric, so that the only possible symmetric decomposition is $\Delta_{f,0} = H_f = (1, 1, 1, 1, 1)$ and all other $\Delta_{f,i}$ equal to zero vectors.

By Theorem 18 we see that $f = \varphi^\vee(g)$, where g is in standard form and φ is of order at least two. We have $\Delta_{g,i} = \Delta_{f,i}$ for all i , so $\dim_K \text{Lin}(g)^i = 1$ for all i . Choose a generator x for this space. Let $g = g_5 + \cdots + g_0$ be the decomposition into homogeneous summands. From the definition of standard form we see that $g_i \in K[x]$ for all i , i.e. $g \in K[x]$. So we may write $g = a_5 x^5 + a_4 x^4 + \cdots + a_0$ for constants a_i . Since g has degree five, we have $a_5 \neq 0$ and by changing x we may assume that $a_5 = 1$.

Now $f = \sum_{\alpha \in \mathbb{N}_0^n} x^\alpha \cdot (D^\alpha(g))$, where $D_i = \varphi(y_i) - y_i \in M^2$. Then $D_i D_j D_k(g) = 0$ for all i, j, k and the sum becomes shorter: $f = g + \sum_i x_i \cdot D_i(g) + \sum_{i,j} x_i x_j \cdot D_i D_j(g)$. We see that $\deg(D_i(g)) \leq 3$ so that

$$f = x^5 + \sum_{i=1}^n \lambda_i x_i \cdot x^3 + \sum_{i=1}^n \mu_i x_i x^2 + \sum_{i,j} \lambda_i \lambda_j x_i x_j x + Q,$$

where λ_i, μ_i are constants and Q is a polynomial of degree at most two, partially depending on λ_i and μ_i . What is the dimension of possible f obtained this way? Each D_i may be chosen as an element of the square of the maximal ideal of $T/(x^5)^\perp$, therefore we have $(5-2)$ -dimensional

choice. Together with the choice of x , we obtain at most a $4(\dim \text{Spec}(T)) = 4n$ -dimensional family.

3. APOLARITY AND LOCAL CACTUS RANK

We shall now apply our analysis of Hilbert functions of polynomials to apolar subschemes of a homogeneous form. Recall from Section 0.1 that we denote by $\bar{S} = K_{\text{dp}}[x_0, \dots, x_n]$ and $\bar{T} = K[y_0, \dots, y_n]$.

Definition 26. A subscheme $Z \subset \mathbb{P}(\bar{S}_1)$ is apolar to a form $F \in \bar{S}$ if its homogeneous ideal $I_Z \subset \bar{T}$ is contained in F^\perp .

Apolarity for a subscheme $Z \subset \mathbb{P}(\bar{S}_1)$ to a form F of degree d may be given by the following natural interpretation in terms of the d -uple embedding of Z , the image $\nu_d(Z) \subset \mathbb{P}(\bar{S}_d)$ where $\nu_d : \mathbb{P}(\bar{S}_1) \rightarrow \mathbb{P}(\bar{S}_d)$, $[l] \mapsto [l^d]$.

Lemma 27. (*Apolarity Lemma*). *A scheme $Z \subset \mathbb{P}(\bar{S}_1)$ is apolar to $F \in \bar{S}_d$ if and only if $[F] \in \langle \nu_d(Z) \rangle \subset \mathbb{P}(\bar{S}_d)$.*

Proof. If Z is apolar to F , then $(I_Z)_d \subseteq (F^\perp)_d$ and we get $[F] \in V((I_Z)_d) = \langle \nu_d(Z) \rangle \subset \mathbb{P}(\bar{S}_d)$, so the “only if” part follows. For the “if” part, $(I_Z)_e \subset (F^\perp)_e = \bar{T}_e$ when $e > d$, so it remains to consider $\Psi \in (I_Z)_e$, for some $e \leq d$. In this case, $\bar{T}_{d-e}\Psi \subset (I_Z)_d$. So if $[F] \in \langle \nu_d(Z) \rangle$, then $\bar{T}_{d-e}\Psi \subset (F^\perp)_d$. But $\bar{T}_{d-e}\Psi(F) = 0$ only if $\Psi(F) = 0$, so the “if” part follows also. \square

We are particularly interested in minimal apolar zero-dimensional subschemes to a form, their length is called the cactus rank of the form. The closure of the set of forms with a given cactus rank is called a cactus variety of forms, although it may be reducible. Minimal apolar zero-dimensional schemes are locally Gorenstein, so our first aim is to describe the forms that have a given minimal length local Gorenstein scheme. For any form F a minimal apolar zero-dimensional scheme decomposes into local Artinian Gorenstein schemes. This decomposition corresponds to an additive decomposition of the form F . In particular, the cactus variety is the join of varieties of forms whose minimal apolar scheme is local.

Let F be a form of degree d in \bar{S} and $[l] \in \mathbb{P}(\bar{S}_1)$. Consider the family $\mathcal{Z} \subset \text{Hilb}(\mathbb{P}(\bar{S}_1))$ of subschemes apolar to F and supported at $[l]$. We construct a particular subscheme $Z_{F,l} \in \mathcal{Z}$ that we will call the natural apolar subscheme of F at $[l]$. The element $[l] \in \mathbb{P}(\bar{S}_1)$ defines a hyperplane $V(l) \subset \mathbb{P}(\bar{T}_1)$. The complement U of this hyperplane is isomorphic to an affine space $\text{Spec}(\bar{S}_l)$. Moreover, we get a homomorphism $\pi = \pi_{[l]} : \bar{S} \rightarrow \bar{S}_l$ corresponding to passing from the homogeneous coordinate ring S of $\mathbb{P}(\bar{T}_1)$ to the coordinate ring of U .

Choose dual bases $\langle l, l_1, \dots, l_n \rangle$ and $\langle l', l'_1, \dots, l'_n \rangle$ for \bar{S}_1 and \bar{T}_1 , respectively. Let $\bar{S}_l = K[l_1, \dots, l_n]$ and let $\bar{T}_l = K[l'_1, \dots, l'_n]$. Then $(\bar{S}_l)_1$ and $(\bar{T}_l)_1$ are natural dual spaces like \bar{S}_1 and \bar{T}_1 above, and \bar{T}_l is the coordinate ring of the affine space U' that contains the point $[l]$ and is the complement of the hyperplane $V(l') \subset \mathbb{P}(\bar{S}_1)$. Given any polynomial $g \in \bar{S}_l$ we denote by Z_g the subscheme $V(g^\perp) \subseteq U' \subset \mathbb{P}(\bar{S}_1)$.

Definition 28. Let $F \in S$ be any form and $l \in \bar{S}_1$ a linear form. Take $f := \pi_{[l]}(F)$ and $f^\perp \subseteq \bar{T}_l$. We define $Z_{F,l}$ to be the subscheme $Z_f = V(f^\perp) \subseteq U' \subset \mathbb{P}(\bar{S}_1)$.

Since $Z_{F,l}$ is finite, it is a closed subset of $\mathbb{P}(\bar{S}_1)$. By construction, the support of $Z_{F,l}$ is $[l] \in \mathbb{P}(\bar{S}_1)$.

In the definition of $Z_{F,l}$ we have not used any particular coordinate system. However, to simplify the following proofs we fix coordinates. First, note that for every lifting $l \in \bar{S}_1$ of $[l]$ we have a canonical isomorphism $\bar{S}_l \simeq \bar{S}/(l-1)$. Changing x_0, \dots, x_n if necessary we

may assume that $l = x_0$. Then \bar{S}_l may be identified with $S = K_{\text{dp}}[x_1, \dots, x_n]$ and \bar{T}_l with $T = K[y_1, \dots, y_n]$. The homomorphism

$$\pi := \pi_{x_0} : \bar{S} \rightarrow S$$

sends x_0 to 1 and x_i to itself for $i > 0$. Note that π induces an isomorphism between the space \bar{S}_d of homogeneous polynomials of degree d and the space $(S)_{\leq d}$ of polynomials of degree d . Furthermore we fix a “dual” homomorphism

$$\pi^* : \bar{T} \rightarrow T$$

sending y_0 to 1 and y_i to itself for $i > 0$.

Let $G \in \bar{S}$ be a homogeneous polynomial, let $\Psi \in \bar{T}$ be a homogeneous differential operator, and denote $g = \pi(G)$ and $\psi = \pi^*(\Psi)$. In general, $\pi(\Psi(G))$ and $\psi(g)$ are different polynomials, but the following lemma gives the basic relation between them.

Lemma 29. *Let $G \in \bar{S}$ be a homogeneous polynomial, let $\Psi \in \bar{T}$ be a homogeneous operator, with $\deg G \geq \deg \Psi$, and let $d = \deg G - \deg \Psi$. Let $g = \pi(G)$ and $\psi = \pi^*(\Psi)$, then the degree- d tails of $\pi(\Psi(G))$ and $\psi(g)$ are equal. Moreover, if G is divisible by $x_0^{\deg \Psi}$, then $\pi(\Psi(G)) = \psi(g)$.*

Proof. The first statement is equivalent to saying that the images of $\pi(\Psi(G))$ and $\psi(g)$ are equal in the linear space $S/(S)_{>d}$. Therefore, both statements are linear with respect to Ψ and G and it is enough to prove them in the case when G and Ψ are monomials. Let $G = x_0^{\alpha_0} \dots x_n^{\alpha_n}$ and $\Psi = y_0^{\beta_0} \dots y_n^{\beta_n}$. By definition

$$\Psi(G) = \begin{cases} x_0^{\alpha_0 - \beta_0} x_1^{\alpha_1 - \beta_1} \dots x_n^{\alpha_n - \beta_n} & \text{if } \alpha_i \geq \beta_i \text{ for all } i \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$\psi(g) = \begin{cases} x_1^{\alpha_1 - \beta_1} \dots x_n^{\alpha_n - \beta_n} & \text{if } \alpha_i \geq \beta_i \text{ for all } i \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

We consider two cases. First, suppose $\alpha_0 \geq \beta_0$. Then the conditions $\forall i \geq 0, \alpha_i - \beta_i \geq 0$ and $\forall i \geq 1, \alpha_i - \beta_i \geq 0$ are equivalent. Thus $\pi(\Psi(G)) = \psi(g)$ and so their images in $S/(S)_{>d}$ agree.

Next, suppose that $\alpha_0 < \beta_0$. Then $\Psi(G) = 0$. Suppose $\psi(g) \neq 0$ in S . Then $\psi(g) = x_1^{\alpha_1 - \beta_1} \dots x_n^{\alpha_n - \beta_n}$ is a monomial of degree $\sum_{i \geq 1} \alpha_i - \beta_i = d - (\alpha_0 - \beta_0) > d$, thus its image in $S/(S)_{>d}$ is zero, i.e. equal to the image of $\Psi(G)$. This finishes the proof of the first claim.

For the second claim, note that by assumption $\alpha_0 \geq \deg \Psi \geq \beta_0$, thus the proof of the first case above applies, giving $\pi(\Psi(G)) = \psi(g)$. \square

Corollary 30. *Let $G \in \bar{S}$ and $\Psi \in \bar{T}$ be homogeneous polynomials. Let $g = \pi(G)$ and $\psi = \pi^*(\Psi)$. If $\psi(g) = 0$, then $\Psi(G) = 0$.*

Proof. Let $b = \deg(\Psi)$, $d = \deg G - \deg \Psi$, and $G' := x_0^b G$. Then $g = \pi(G')$ and G', Ψ, g, ψ satisfy assumptions of Lemma 29, therefore $\pi(\Psi(G')) = \psi(g) = 0$. Since π is an isomorphism between \bar{S}_d and $(S)_{\leq d}$ we have $\Psi(G') = 0$. But $G = y_0^b(G')$, so that

$$\Psi(G) = \Psi(y_0^b(G')) = y_0^b \Psi(G') = 0. \quad \square$$

Corollary 31. *Let $F \in S$ be a homogeneous polynomial and $l \in \bar{S}_1$ any linear form. Then the scheme $Z_{F,l}$ (see Definition 28) is apolar to F .*

Proof. Take any homogeneous form $\Psi \in I(Z_{F,l})$. Then $\psi = \pi^*(\Psi)$ is an element annihilating $f = \pi(F)$. By Corollary 30 we have $\Psi(F) = 0$, i.e. $\Psi \in F^\perp$. \square

Remark 32. It is easy to characterize the d -uple embedding $\nu_d(Z_{F,l})$ in a manner similar to the proof of the Apolarity Lemma 27.

For $l \in \bar{S}_1$, let $l^\perp \cap \bar{T}_1$ be the subspace of linear forms in \bar{T}_1 that annihilate l , and let $(l^\perp)^e$ be its e -th symmetric product. Then $(l^\perp)^e(F) \subset \bar{S}_{d-e}$ for $e \leq d$ is a subspace of partials of F of degree $d - e$, and the linear span of $\nu_d(Z_{F,l})$ is given by

$$\langle \nu_d(Z_{F,l}) \rangle = \mathbb{P}(l^d \oplus l^{d-1} \cdot (l^\perp)^{d-1}(F) \oplus \cdots \oplus l \cdot (l^\perp)^1(F) \oplus F).$$

Furthermore $\nu_d(Z_{F,l}) = \langle \nu_d(Z_{F,l}) \rangle \cap V_{d,n} \subset \mathbb{P}(\bar{S}_d)$.

The following lemma is a private communication from Jarosław Buczyński.

Lemma 33 (Buczyński). *If Z is any local scheme in $\mathbb{P}(\bar{S}_1)$ apolar to a homogeneous polynomial $F \in S$ and supported at $[l] = [1 : 0 : \cdots : 0]$, then there exists a closed subscheme $Z' \subseteq Z$, apolar to F , such that $Z' = Z_{G,l}$, for some $G \in \bar{S}$. Moreover, $F = \Psi(G)$ for some $\Psi \in \bar{T}$.*

Proof. By [Buczyńska, Buczyński 2014, Proposition 2.2, Lemma 2.3], the scheme Z contains a closed Gorenstein subscheme Z' apolar to F .

Let $g \in S$ be a polynomial such that $Z' = V(g^\perp)$ and let $G \in \bar{S}$ be a homogenization of g such that G is divisible by x_0^d , where $d = \deg(F)$. Then $g = \pi(G)$ and $Z' = Z_{G,l}$. Lemma 29 asserts that $(G^\perp)_s = (I_{Z_{G,l}})_s$ for any $s \leq d$, and therefore $(G^\perp)_s = (I_{Z_{G,l}})_s \subseteq (F^\perp)_s$. Since F is of degree d , it follows that $G^\perp \subseteq F^\perp$, so F is a partial of G . \square

We may now prove Proposition 4.

Proof (Proposition 4). By Lemma 33, we may assume that $\Gamma = Z_{G,l}$ and that $F = \Psi(G)$ for some homogeneous $\Psi \in K[y_0, \dots, y_n]$. Let $g = \pi(G)$ and $\psi = \pi^*(\Psi)$. By Lemma 29 the polynomial f is the degree- d tail of $\psi(g)$.

Clearly $\text{Diff}(\psi(g)) \subseteq \text{Diff}(g)$, thus $Z_{\psi(g)}$ is a closed subscheme of $Z_g = \Gamma$. By minimality of Γ it is enough to prove that $Z_{\psi(g)}$ is a scheme apolar to F . Let ψ' be such that $\psi'(\psi(g)) = 0$ and let $\Psi' \in \bar{T}$ be a homogeneous polynomial such that $\psi' = \pi^*(\Psi')$. Then $\pi^*(\Psi'\Psi) = \psi'\psi$. By Corollary 30 we have $\Psi'(\Psi(G)) = 0$. Since $F = \Psi(G)$ we get that $\Psi'(F) = 0$, thus $Z_{\psi(g)}$ is apolar to F . \square

3.1. The local cactus rank of a general cubic surface. In this subsection we restrict to characteristic 0. First we present an example of a quartic polynomial whose cubic tail has more partials than the polynomial itself. Similar examples play a role in our computation of the local cactus rank of a general cubic surface, the main issue in this section.

Example 34. Let $f = x_1^2 x_2 + x_2^2$ and $g = x_1^4 + f$. Then $\text{Diff}(f)$ has a basis

$$f, x_1^2 + x_2, x_1 x_2, x_1, x_2, 1,$$

thus $\dim_K \text{Diff}(f) = 6$. On the other hand, $\text{Diff}(g)$ is spanned by

$$g, x_1^3 + x_1 x_2, x_1^2 + x_2, x_1, 1,$$

so that $\dim_K \text{Diff}(g) = 5$. Notice that $y_2 - y_1^2 \in g^\perp$, so that x_2 does not appear in the leading summand of any partial of g (cf. Proposition 12).

For our computation of the local cactus rank of a general cubic surface $V(F)$, we need to translate the generality assumptions on F into properties of its partials. First note the following algebraic–geometric correspondences for $\Psi \in \bar{T}_1$:

- (1) $\Psi^3(F) = 0$ if and only if $[\Psi] \in V(F) \subseteq \mathbb{P}(\bar{T}_1)$,
- (2) $\Psi^2(F) = 0$ if and only if $[\Psi]$ is a singular point of the (hyper)surface $V(F)$,
- (3) $\Psi(F) = 0$ if and only if $[\Psi]$ is a cone point of $V(F)$.

Lemma 35. *Let $F \in S$ be a general cubic form in four variables. Then*

- (1) *The set of $[\Psi] \in \mathbb{P}(\bar{T}_1)$ such that the quadric $\Psi(F)$ has rank less than 4 is an irreducible surface of degree 4. Furthermore, the quadric $\Psi(F)$ has rank less than 4 if and only if there exist a $\Psi' \in \bar{T}_1$ such that $(\Psi\Psi')(F) = 0$.*
- (2) *There are no points $[\Psi] \in V(F) \subset \mathbb{P}(\bar{T}_1)$, such that $\Psi(F)$ is a quadric of rank less than 3 i.e. if $\Psi_3^3(F) = 0$ and $(\Psi_2\Psi_3)(F) = 0$, then $\Psi_2^3(F) \neq 0$.*
- (3) *The cubic surface $V(F)$ is smooth, i.e. $\Psi^2(F) \neq 0$ for every nonzero $\Psi \in \bar{T}_1$.*
- (4) *The cubic surface $V(F)$ has no Eckhardt points, i.e. no plane section is a cone.*

Proof. These facts are classical. For a good recent reference see [Dolgachev 2012, 9.4]. \square

Proposition 36. *Let $F \in \bar{S}_3$ be a general smooth cubic form in four variables. Then the local cactus rank of F is 7. For every linear form $l \in \bar{S}_1$, the apolar scheme of the dehomogenization $\pi_{[l]}(F)$ has length 8, while if $\{l = 0\}$ defines a singular curve section of $V(F) \subset \mathbb{P}(\bar{T}_1)$ whose tangent cone at the singular point is a square, then there is a length 7 scheme supported at $[l] \in \mathbb{P}(\bar{S}_1)$ that is apolar to F .*

Proof. Let $F \in \bar{S} = K_{\text{dp}}[x_0, x_1, x_2, x_3]$ be a general cubic form in the sense of Lemma 35.

We claim that for all non-zero $l \in \bar{S}_1$, the Hilbert function is

$$(11) \quad H_{F_l} = (1, 3, 3, 1).$$

Indeed, suppose it is not so, then $H_{F_l} \leq (1, 3, 2, 1)$ for some non-zero $l \in \bar{S}_1$ and there exists a non-zero linear form $\Psi \in l^\perp$, such that $\psi(F_l)$ has degree at most one, where $\psi = \pi^*(\Psi)$. Let $F_l = f_3 + f_2 + f_1 + f_0$ be the decomposition into homogeneous components, then $F = f_3 + lf_2 + l^2f_1 + l^3f_0$. Since $\deg \psi(F_l) \leq 1$ we have $\Psi(f_3) = 0$. But then $\Psi(F)$ is divisible by l , so it is a quadric of rank at most 2. On the other hand $\Psi(l) = 0$, so $\Psi^2(\Psi(F)) = \Psi^3(F) = 0$, so $[\Psi] \in V(F) \subset \mathbb{P}(\bar{T}_1)$. This contradicts the generality assumption Lemma 35.2 of F .

The cubic surface $V(F)$ has a one dimensional family of plane cuspidal cubic sections, and finitely many reducible plane sections that are unions of a smooth conic and a tangent line. In either case, the tangent cone at the singular point is a square. We pick one such plane section. After a linear change of coordinates, we may assume that this plane section is $V(F, x_0)$, that it is singular at $V(x_0, x_1, x_2)$ and that $V(x_0, x_1^2)$ is the tangent cone, so that the dehomogenization $\pi_{[x_0]}(F)$ has the form

$$\pi_{[x_0]}(F) = f + x_1^2x_3 + x_3^2 + x_3l$$

where f and l are polynomials in $K_{\text{dp}}[x_1, x_2]$ of degree three and one respectively. The plane section $V(F, x_0) = V(f_3 + x_1^2x_3, x_0)$ where f_3 is the cubic summand of f . By Lemma 35.4, the linear partials of $f_3 + x_1^2x_3$ fill $\langle x_1, x_2, x_3 \rangle$. For later use we note that y_1^2 is the only monomial quadric such that $y_1^2(f_3 + x_1^2x_3)$ contains x_3 . Hence we have $(y_2, y_3) \cdot (y_1, y_2, y_3)(f) = \langle 1, x_1, x_2 \rangle$. Since $l - y_1^2(f) \in K_{\text{dp}}[x_1, x_2]$ we can find $\sigma_0 \in (y_2, y_3)(y_1, y_2, y_3)$ such that $\sigma_0(f) = l - y_1^2(f) \pmod{\langle x_1 \rangle}$. Then there also exists $\sigma = \sigma_0 + ay_1^3 \in (y_1, y_2, y_3)^2$ such that $\sigma(x_1^4 + f) = l - y_1^2(f)$. Clearly $\sigma_0x_1^4 = 0$.

Let $G = x_1^4 + x_0F$. Then $y_0(G) = F$, hence $G^\perp \subseteq F^\perp$. By Lemma 31 we have $I(Z_{G, x_0}) \subseteq G^\perp$, so we conclude that $I(Z_{G, x_0}) \subseteq F^\perp$, i.e. that the local Gorenstein scheme Z_{G, x_0} is apolar to F . We claim that $\text{length}(Z_{G, x_0}) \leq 7$, and hence that the local cactus rank of F is at most 7.

We prove the claim, by showing that $x_1^2x_3 + x_3^2 + x_3l$ is an exotic summand for $G_{x_0} := \pi_{[x_0]}(G) = x_1^4 + f + x_1^2x_3 + x_3^2 + x_3l$. For this we consider the partials

$$y_3(G_{x_0}) = x_1^2 + x_3 + l, \quad y_1^2(G_{x_0}) = x_1^2 + x_3 + y_1^2(f).$$

We have $(y_3 - y_1^2)(G_{x_0}) = l - y_1^2(f) = \sigma(G_{x_0})$, so $y_3 - y_1^2 - \sigma$ annihilates G_{x_0} . If we take $\psi = y_1^2 + \sigma$, then $\psi^2 G_{x_0} = y_1^4(G_{x_0}) = 1$ and we may write

$$G_{x_0} = x_1^4 + f + x_3\psi(x_1^4 + f) + x_3^2\psi^2(x_1^4 + f)$$

which shows, by Proposition 19, that $x_1^2x_3 + x_3^2 + x_3l$ is an exotic summand for G_{x_0} . Thus G_{x_0} has the Hilbert function of the binary polynomial $x_1^4 + f$. The maximal values of this function is clearly $(1, 2, 2, 1, 1)$, so this proves the claim that $\text{length}(Z_{G, x_0}) \leq 7$.

Finally, suppose that there exists a local Gorenstein scheme of length at most 6 apolar to F . It must be defined by some polynomial g whose cubic tail coincides with F_l for some l . Thus g has degree at least four, and its Hilbert function is $H_g = (1, 2, 1, 1, 1)$, $H_g = (1, 1, 1, 1, 1)$ or $H_g = (1, 1, 1, 1, 1, 1)$. In the first two cases, g has degree 4 and the leading summand of g is a pure power l^4 . Each order-one partial of the cubic summand $F_{l,3}$ of F_l is therefore proportional to l^2 . In particular $\pi^*(\Psi)(F_{l,3}) = 0$ for some $\Psi \in l^\perp \subset \bar{T}_1$, so $\Psi(F)$ is divisible by l contradicting, as above, the generality assumption on F .

In the case $H_g = (1, 1, 1, 1, 1)$ we see that F_l is the degree-at-most-three part of g and the standard form of g is x^5 . By Example 25, we have a $4 \cdot 3$ -dimensional choice of F_l . Together with the choice of l , we obtain a 16-dimensional variety of possible F , thus such F is not general. \square

4. ON THE DIMENSION OF THE CACTUS VARIETIES OF CUBIC FORMS

In this section we consider polynomials with Hilbert function $(1, m-1, m-1, 1)$ and $(1, m-1, m-1, 1, 1)$ and derive lower bounds on the dimension of the Cactus variety of cubic forms $\text{Cactus}_{2m}(V_{3,n})$ and $\text{Cactus}_{2m+1}(V_{3,n})$, respectively.

The cactus variety $\text{Cactus}_r(V_{3,n})$ of the third Veronese embedding $V_{3,n} \subset \mathbb{P}(\bar{S}_3)$ is, according to Proposition 4, the closure of the family of cubic forms $[F]$ admitting a decomposition $F = F_1 + \dots + F_s$ and distinct linear forms $l_1, \dots, l_s \in \bar{S}_1$ and forms $G_1, \dots, G_s \in S$, such that $Z_{G_1, l_1} \cup \dots \cup Z_{G_s, l_s}$ has length at most r , and the dehomogenization $f_i = \pi_{[l_i]}(F)$ of F at l_i is the cubic tail of the dehomogenization of G_i at l_i (see Definition 28).

To get a lower bound on the dimension of the cactus variety, we consider the extreme opposite to the higher secants, namely linear spaces that intersect $V_{3,n}$ in a local scheme. In particular we consider the closure

$$W_{2m}(V_{3,n}) \subset \text{Cactus}_r(V_{3,n}) \subset \mathbb{P}(\bar{S}_3)$$

of the family of cubic forms $[F]$, for which there exist a linear form $l \in \bar{S}_1$ and a form $G \in S$, such that $f = \pi_{[l]}(F)$ is the cubic tail of $g = \pi_{[l]}(G)$ and the polynomial g has Hilbert function $(1, m-1, m-1, 1)$. We define $W_{2m+1}(V_{3,n})$ analogously, using the Hilbert function $(1, m-1, m-1, 1, 1)$. In the first case g is itself a cubic polynomial, i.e. its own cubic tail, while in the second case, g is a quartic polynomial.

To find the dimension of $W_r(V_{3,n})$ when $r = 2m$, we note that it is the union over $l \in \bar{S}_1$ of varieties isomorphic to the projectivisation of $V_r(n)$, the family of cubic polynomials $f \in K[x_1, \dots, x_n]$ with Hilbert function $(1, m-1, m-1, 1)$. When $r = 2m+1$, the variety $W_r(V_{3,n})$ is union over l of varieties isomorphic to the projectivisation of $V_r(n)$ of $\text{Tails}_r(3, n)$, the family of cubic polynomials $f \in K[x_1, \dots, x_n]$, that are tails of polynomials $g \in K[x_1, \dots, x_n]$ with Hilbert function $(1, m-1, m-1, 1, 1)$.

Example 37. If $f \in K_{\text{dp}}[x_1, \dots, x_n]$ has Hilbert function $H_f = (1, m-1, m-1, 1)$, its only possible symmetric decomposition is

$$H_f = \Delta = \Delta_0 = (1, m-1, m-1, 1),$$

and therefore $\deg f = 3$, $\Delta_0(1) = m - 1$ and

$$(n_0, n_1) = (m - 1, m - 1).$$

If f_α is a general cubic polynomial in $K_{\text{dp}}[x_1, \dots, x_{m-1}]$, and f_β is any quadratic polynomial in $\langle x_m, \dots, x_n \rangle \langle x_1, \dots, x_{m-1}, 1 \rangle$, then f_α is in standard form and f_β is an exotic summand for $f_\alpha + f_\beta$ (cf. Definition 13). Furthermore $H_{f_\alpha + f_\beta} = \Delta$ and so $f_\alpha + f_\beta \in V_{2m}(n)$. The subspace

$$\langle x_1, \dots, x_{m-1} \rangle = \text{Lin}(f)^0$$

is determined by f , and the variety of subspaces $\langle x_1, \dots, x_{m-1} \rangle \subset \langle x_1, \dots, x_n \rangle$ has dimension $(n - m + 1)(m - 1)$, so we get

$$\dim V_{2m}(n) = \binom{m+2}{3} + (2m-1)(n-m+1).$$

Notice furthermore that the affine variety $V_{2m}(n)$ is a cone, so that its projectivisation has dimension one less.

Example 38. If $g \in K_{\text{dp}}[x_1, \dots, x_n]$ has Hilbert function $H_g = (1, m - 1, m - 1, 1, 1)$, its only possible symmetric decomposition is

$$\Delta = (\Delta_0, \Delta_1) = ((1, 1, 1, 1, 1), (0, m - 2, m - 2, 0)),$$

and therefore $\deg g = 4$, $\Delta_0(1) = 1$, $\Delta_1(1) = m - 2$ and

$$(n_0, n_1, n_2) = (1, m - 1, m - 1).$$

If f_α is a general cubic polynomial in $K_{\text{dp}}[x_1, \dots, x_{m-1}]$, and f_β is a cubic polynomial of the form $l_0 x_1^2 + l_0^2 + f_\infty$, where $l_0 \in \langle x_m, \dots, x_n \rangle$ and $f_\infty \in \langle x_m, \dots, x_n \rangle \langle x_1, \dots, x_{m-1}, 1 \rangle$, then $x_1^4 + f_\alpha$ is in standard form and f_β is an exotic summand for

$$g = x_1^4 + f_\alpha + f_\beta = x_1^4 + f_\alpha + l_0 x_1^2 + l_0^2 + f_\infty.$$

Furthermore

$$H_g = \Delta_0 + \Delta_1 = (1, m - 1, m - 1, 1, 1),$$

and so $f_\alpha + f_\beta \in \text{Tails}_{2m+1}(3, n)$.

The flag of subspaces

$$\langle x_1 \rangle = \text{Lin}(g)^0 \subset \langle x_1, \dots, x_{m-1} \rangle = \text{Lin}(g)^1 \subset \langle x_1, \dots, x_n \rangle$$

is determined by g , and the variety of such flags has dimension $m - 2 + (n - m + 1)(m - 1)$, so we get

$$\dim \text{Tails}_{2m+1}(3, n) = \binom{m+2}{3} + (2m-1)(n-m+1) + n - 1.$$

Notice that, since the summand $l_0 x_1^2 + l_0^2$ is quadratic in the form l_0 , the affine variety $\text{Tails}_{2m+1}(3, n)$ is not a cone, so its projectivisation has the same dimension.

We use the Examples 37 and 38 to give a lower bound on the dimension of the union of linear spaces that intersect $V_{3,n}$ in a local subscheme.

Proposition 39. *Let $3 \leq m \leq n$. The union $\text{Cactus}_{2m}^L(V_{3,n})$ of linear spans in $\mathbb{P}^{\binom{n+3}{3}-1}$ of local subschemes in $V_{3,n}$ of length $2m$ has dimension*

$$\dim \text{Cactus}_{2m}^L(V_{3,n}) \geq \binom{m+2}{3} + 2m(n-m) + 3m - 2.$$

Let $4 \leq m \leq n$. The union $\text{Cactus}_{2m+1}^L(V_{3,n})$ of linear spans in $\mathbb{P}^{\binom{n+3}{3}-1}$ of local subschemes in $V_{3,n}$ of length $2m+1$ has dimension

$$\dim \text{Cactus}_{2m+1}^L(V_{3,n}) \geq \binom{m+2}{3} + 2m(n-m) + 3m + n - 2.$$

Proof. Clearly $W_{2m}(V_{3,n}) \subset \text{Cactus}_{2m}^L(V_{3,n})$ and $W_{2m+1}(V_{3,n}) \subset \text{Cactus}_{2m+1}^L(V_{3,n})$, so we get the inequality by computing the dimension of these subvarieties.

Let $m > 2$. $W_{2m}(V_{3,n})$ is the union as l varies, of projective varieties whose affine cones are isomorphic to $V_{2m}(n)$, so $W_{2m}(V_{3,n})$ has dimension

$$\dim W_{2m}(V_{3,n}) \leq \dim V_{2m}(n) - 1 + n$$

equal to the right hand side in the lemma. Similarly, $W_{2m+1}(V_{3,n})$ is the union, as l varies, of varieties isomorphic to $\text{Tails}_{2m+1}(3, n)$, so

$$\dim W_{2m+1}(V_{3,n}) \leq \dim \text{Tails}_{2m+1}(3, n) + n.$$

In both cases the right hand side is the dimension of the given parametrization of the variety $W_r(V_{3,n})$. To get equality, we show that the parameterization is generically one to one.

When r is even, we show that for a general $[F] \in W_r(V_{3,n})$ there is a unique l such that $Z_{F,l}$ has length r . When r is odd, we show that there is a unique l such that $f = \pi_l(F)$ is the tail of a quartic polynomial g_l whose apolar scheme Z_{g_l} has length r .

Let $r = 2m \leq 2n$, and assume that $[F] \in W_r(V_{3,n})$ is general. Let $l \in \bar{S}_1$ and $f = \pi_l(F)$ be the local polynomial of F at l such that f has Hilbert function $(1, m-1, m-1, 1)$. Then $F = F_3 + lF_2$ where F_3 depends on $m-1$ variables. Therefore $V(F, l)$ is a cone inside the hyperplane $V(l)$. Let $[y]$ be a point in the $n-m$ dimensional linear vertex of $V(F, l)$. Then $y(F_3) = y(l) = 0$. Furthermore, since all partials of F_l of degree 1 are partials of F_3 , we have $y^2(F_2) = 0$. In particular, $y^2(F) = 0$ and $y(F) = l \cdot l'$, so $V(F)$ is singular at $[y]$ with a tangent cone of rank 2. On the other hand, if $V(F)$ is singular at $[y]$ with a tangent cone of rank 2, then

$$F = F_3 + lF_2 + l \cdot l'x,$$

where $y(x) = 1$, $y(F_3) = y(F_2) = y(l) = y(l') = 0$ and F_3 has Hilbert function $(1, m-1, m-1, 1)$ for some $m \leq n$.

If $r = 2m+1$, and assume that $[F] \in W_r(V_{3,n})$ is general. Let $l \in \bar{S}_1$ and $f = \pi_l(F)$ be the local polynomial of F at l such that f has Hilbert function $(1, m, m, 1)$, but is the cubic tail of a quartic polynomial g with Hilbert function $(1, m-1, m-1, 1, 1)$. When $m \leq n$, then

$$g = \alpha_0 l_0^4 + g_3 + \alpha_1 l_0^2 l' + \alpha_2 (l')^2 + g_2$$

where g_3 and l_0 depends on m variables and l' is a hidden variable for g . Thus

$$f_3 = g_3 + \alpha_1 l_0^2 l', \quad F = g_3 + \alpha_1 l_0^2 l' + \alpha_2 l (l')^2 + l g_2$$

and $V(F, l)$ is a singular cubic hypersurface with a double point whose tangent cone is a square. In fact, if $m < n$, then $V(F, l)$ is a cone with linear vertex of dimension $n-m-1$ over such a singular hypersurface.

In both cases, if $m < n$, let l_1, \dots, l_{n-m} are general linear forms. If $r = 2m$, then the linear section $V(F, l_1, \dots, l_{n-m})$ still has a singular point whose tangent cone has rank 2. If $r = 2m+1$, then the linear section $V(F_3, l, l_1, \dots, l_{n-m})$ is still a singular cubic hypersurface inside a $(m-1)$ -dimensional linear subspace with a non reduced tangent cone at the singular point. The proof of uniqueness of l may therefore in both cases be reduced to the case, when $n = m$.

The following is a classical result.

Lemma 40. *The set of singular cubic hypersurfaces in \mathbb{P}^n , $n > 2$ whose tangent cone at the singular point has rank at most 2, form a subvariety of codimension $\binom{n-1}{2} + 1$ in $\mathbb{P}^{\binom{n+3}{3}-1}$, and the general member in the set has exactly one singular point.*

Proof. It suffices to note that the set of singular cubic hypersurfaces in \mathbb{P}^n form a hypersurface in $\mathbb{P}^{\binom{n+3}{3}-1}$. The general point in this hypersurface, the discriminant, corresponds to a cubic hypersurface with a quadratic singularity, i.e. the tangent cone is a quadric of rank n . In the space of quadrics of rank at most n , the quadrics of rank 2 form a subvariety of codimension $\binom{n-1}{2}$. The two codimensions add up to the codimension in the lemma. For uniqueness it suffices to fix a quadric q of rank 2 and a point p in its vertex, and notice, by Bertini's theorem, that the general cubic hypersurface through for which q is the tangent cone at p is smooth elsewhere. \square

Remark 41. Notice that the codimension in lemma is consistent with the dimensions of $W_{2n}(V_{3,n})$. When $n = m$ in the proposition, we get

$$\dim W_{2n,n}(V_{3,n}) = \binom{n+3}{3} - 1 - \binom{n-1}{2} - 1.$$

For the case r is odd, we show that

Lemma 42. *The set of cubic hypersurfaces in \mathbb{P}^n , $n > 3$ with a singular hyperplane section whose tangent cone at the singular point is a square, form a subvariety of codimension $\binom{n-2}{2} - 1$ in $\mathbb{P}^{\binom{n+3}{3}-1}$, and when $n > 4$, the general member in the set has exactly one such hyperplane section.*

Proof. Assume $V(F)$ is a general cubic of dimension $n - 1 > 2$ with a singular hyperplane section $V(F, l)$ whose tangent cone at the singular point is a square. Let $p \in V(F, l)$ be a singular point and $V(l, (l')^2)$ the tangent cone in $V(l)$. We may choose coordinates x_0, \dots, x_n , so that $l = x_0, l' = x_1$ and $p = [0 : \dots : 0 : 1]$. Then

$$F = F_3 + x_1^2 x_n + x_0 F_2,$$

where $F_3 \in K_{\text{dp}}[x_1, \dots, x_{n-1}]$ and $F_3 + x_1^2 x_n$ is the cubic form defining the singular hyperplane section $V(F, x_0)$ and $F_2 \in K_{\text{dp}}[x_0, \dots, x_n]$. Thus F depends on $\binom{n+1}{3} + 1 + \binom{n+2}{2}$ coefficients. Now, p, l and l' varies in a $(n + (n - 1) + (n - 2))$ dimensional variety, so we get that cubics with a singular hyperplane section whose tangent cone at the singular point is a square form a variety of codimension

$$\binom{n+3}{3} - \left(\binom{n+1}{3} + 1 + \binom{n+2}{2} \right) - 3(n-1) = \binom{n+1}{2} - 1 - 3(n-1) = \binom{n-2}{2} - 1$$

When $n > 4$, this codimension is positive.

The forms F , when F_3 and F_2 vary, define a linear system of cubic hypersurfaces with base locus supported at $p = [0 : \dots : 0 : 1]$. The general member is smooth, and the tangent hyperplane section at p is singular only at p , and the tangent cone at p is a square. If this hyperplane section is not unique with this property, there is another point q distinct from p such that the tangent hyperplane section at q also has this property. To count dimensions, we fix two flags $p \in L_p \subset H_p$ and $q \in L_q \subset H_q$, and consider the space of smooth cubic hypersurfaces through p and q , whose tangent hyperplanes are H_p and H_q and whose tangent cones at p and q are squares with support along L_p and L_q respectively. Notice that H_p and H_q are distinct, while $L_p \cap H_q$ may equal $L_q \cap H_p$. This gives two cases for the dimension count. These are both similar to the dimension count above and show that the variety of cubic hypersurfaces with two special points as above, has positive codimension in the variety

of cubics with only one such point when $n > 4$. Therefore the last statement of the lemma follows. \square

Remark 43. Notice that codimension in the lemma is consistent with the dimension of $W_{2n+1}(V_{3,n})$. When $n = m$ in the proposition, we get

$$\dim W_{2n+1,n}(V_{3,n}) = \binom{n+3}{3} - 1 - \binom{n-2}{2} + 1.$$

We conclude that the parameterization of $W_r(V_{3,n})$ is birational for any even r with $5 < r < 2n + 1$, and any odd r with $8 < r \leq 2n + 1$, and hence that the dimension formulas of Proposition 39 are the dimensions of $W_{2m}(V_{3,n})$ and $W_{2m+1}(V_{3,n})$ respectively. \square

We rewrite the formulas for the dimensions of $W_{2m}(V_{3,n})$ and $W_{2m+1}(V_{3,n})$ in terms of the lengths $r = 2m$ (resp. $r = 2m + 1$):

$$\dim W_r(V_{3,n}) = \begin{cases} (rn + r - 1) + \frac{r(r-2)(r-16)}{48} - 1 & \text{if } 5 < r < 2n + 1, r \text{ even,} \\ (rn + r - 1) + \frac{(r-1)(r-3)(r-17)}{48} - 2 & \text{if } 8 < r < 2n + 2, r \text{ odd.} \end{cases}$$

Corollary 44. *When $18 \leq r \leq 2n + 2$, then*

$$\dim \text{Cactus}_r(V_{3,n}) \geq \begin{cases} (rn + r - 1) + \frac{r(r-2)(r-16)}{48} - 1 & \text{if } r \geq 18 \text{ even,} \\ (rn + r - 1) + \frac{(r-1)(r-3)(r-17)}{48} - 2 & \text{if } r \geq 19 \text{ odd.} \end{cases}$$

For each possible Hilbert function for local schemes of length r , one may define a variety analogous to $W_r(V_{3,n})$. The dimensions of these varieties are in general not known, and this remains an obstacle to finding a precise dimension for the cactus variety $\text{Cactus}_r(V_{3,n})$.

Finally we leave an open question: we know that the cactus rank of a general cubic surface equals the rank, which is 5, while the local cactus rank is 7 (see Proposition 36), but we do not know whether for a larger number of variables the local cactus rank and the cactus rank agree.

Question 45. *Is the cactus rank of a cubic form in $K_{\text{dp}}[x_0, \dots, x_n]$ always computed locally, when $n \geq 8$ and the cactus rank is at least 18?*

ACKNOWLEDGEMENTS

JJ and KR thank Jarosław Buczyński for fruitful discussions and the Homing Plus programme of Foundation for Polish Science, co-financed from European Union, Regional Development Fund for partial support of their mutual visits. PMM thanks Anthony Iarrobino and Jerzy Weyman for the invitation and hospitality at Northeastern University, and is grateful to Anthony Iarrobino for having introduced him to this subject and for very fruitful discussions. He also thanks James Adler for help with language. AB was partially supported by Project Galaad of INRIA Sophia Antipolis Méditerranée, France, Marie Curie Intra-European Fellowships for Career Development (FP7-PEOPLE-2009-IEF): “DECONSTRUCT”, GNSAGA of INDAM, Mathematical Department Giuseppe Peano of Turin, Italy, and Politecnico of Turin, Italy. JJ is a doctoral fellow at the Warsaw Center of Mathematics and Computer Science financed by the Polish program KNOW and by Polish National Science Center, project 2014/13/N/ST1/02640 and a member of “Computational complexity, generalised Waring type problems and tensor decompositions” project within “Canaletto”, the executive program for scientific and technological cooperation between Italy and Poland, 2013-2015. PMM was partially supported by Fundação para a Ciência e Tecnologia, projects “Geometria Algébrica em Portugal”, PTDC/MAT/099275/2008, “Comunidade Portuguesa de Geometria Algébrica”,

PTDC/MAT-GEO/0675/2012, and sabbatical leave grant SFRH/BSAB/1392/2013, by CIMA – Centro de Investigação em Matemática e Aplicações, Universidade de Évora, projects PEst-OE/MAT/UI0117/2011 and PEst-OE/MAT/UI0117/2014, and by Fundação de Amparo à Pesquisa do Estado de São Paulo, grant 2014/12558–9. KR was supported by the RCN project no 239015 “Special Geometries”.

REFERENCES

- [Alexander, Hirschowitz 1995] Alexander, James, Hirschowitz, André: Polynomial interpolation in several variables. *J. of Alg. Geom.*, **4** (1995), no. 2, pp. 201–222.
- [Bernardi, Brachat, Mourrain 2014] Bernardi, Alessandra, Brachat, Jérôme, Mourrain, Bernard: A comparison of different notions of ranks of symmetric tensors, *Linear Algebra and its Applications*, **460**, 2014, pp. 205–230.
- [Bernardi, Ranestad 2012] Bernardi, Alessandra, Ranestad, Kristian: On the cactus rank of cubics forms, *J. Symbolic Comput.* **50** (2012), pp. 291–297.
- [Buczyńska, Buczyński 2014] Buczyńska, Weronika, Buczyński, Jaroslaw: Secant varieties to high degree Veronese reembeddings, catalecticant matrices and smoothable Gorenstein schemes, *J. Algebraic Geom.* **23** (2014), no. 1, pp. 63–90.
- [Casnati, Notari 2011] Casnati, Gianfranco, Notari, Roberto: On the irreducibility and the singularities of the Gorenstein locus of the punctual Hilbert scheme of degree 10, *J. Pure Appl. Algebra* **215** (2011), no. 6, pp. 1243–1254.
- [Casnati, Notari 2013] Casnati, Gianfranco, Notari, Roberto: A structure theorem for 2-stretched Gorenstein algebras, arXiv:1312.2191, to appear in *J. Commut. Algebra*.
- [Casnati, Jelisiejew, Notari 2015] Casnati, Gianfranco, Jelisiejew, Joachim, Notari, Roberto: Irreducibility of the Gorenstein loci of Hilbert schemes via ray families, *Algebra Number Theory* **9** (2015), no. 7, pp. 1525–1570.
- [Dolgachev 2012] Dolgachev, Igor V.: Classical algebraic geometry: a modern view. Cambridge University Press, (2012).
- [Emsalem 1978] Emsalem, Jacques: Géométrie des points épais, *Bull. Soc. Math. France* **106**, (1978), no. 4, pp. 399–416.
- [Iarrobino 1994] Iarrobino, Anthony: Associated graded algebra of a Gorenstein Artin Algebra, *Mem. Amer. Math. Soc.* **107**, (1994), no. 514, Amer. Math. Soc. Providence.
- [Iarrobino 1995] Iarrobino, Anthony: Inverse system of a symbolic power. II. The Waring problem for forms, *J. Algebra* **174**, (1995), no. 3, pp. 1091–1110.
- [Iarrobino, Kanev 1999] Iarrobino, Anthony, Kanev, Vassil: Power Sums, Gorenstein Algebras and Determinantal Loci, *Lecture Notes in Mathematics* **1721**, Springer-Verlag, Berlin Heidelberg New York (1999).
- [Jelisiejew 2015] Jelisiejew, Joachim: Classifying local Artinian Gorenstein algebras, arXiv:1511.08007.
- [Macaulay 1927] Macaulay, Francis Sowerby: Some properties of enumeration in the theory of modular systems, *Proc. London Math. Soc.* **S2-26** (1927), no. 1, pp. 531–555.

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