

**Fano manifolds whose elementary contractions
are smooth \mathbb{P}^1 -fibrations
A geometric characterization of flag varieties**

Gianluca Occhetta, Luis E. Solá Conde, Kiwamu Watanabe,
and Jarosław A. Wiśniewski

ABSTRACT. The present paper provides a geometric characterization of complete flag varieties for semisimple algebraic groups. Namely, if X is a Fano manifold whose all elementary contractions are \mathbb{P}^1 -fibrations then X is isomorphic to the complete flag manifold G/B where G is a semi-simple Lie algebraic group and B is a Borel subgroup of G .

1. Introduction

Rational homogeneous manifolds constitute one of the most important class of examples of Fano manifolds. Whilst the representation theory of semisimple Lie groups provides a powerful tool to describe many different aspects of the geometry of this type of manifolds, they lack good intrinsic characterization within the class of Fano manifolds.

One of the most natural conjectures in this direction is the one proposed by F. Campana and T. Peternell in 1991 (cf. [5, Special case 11.2, Conjecture 11.1]):

CONJECTURE 1.1 (Campana–Peternell). *Any Fano manifold whose tangent bundle is nef is rational homogeneous.*

We note that the above conjecture is in the spirit of the celebrated Hartshorne–Frankel conjecture proved by Mori in [29], which constitutes one of the cornerstones of the Minimal Model Program in algebraic geometry.

The conjecture is known to be true in some special situations ([6, 26, 12, 31, 33, 30]). In the quoted references the proofs depend on detailed classifications of the manifolds satisfying the required properties, hence the general question of

2010 *Mathematics Subject Classification*. Primary 14J45; Secondary 14E30, 14M17.

First and second author were partially supported by PRIN project “Geometria delle varietà algebriche” and the Department of Mathematics of the University of Trento.

Second author partially supported by the Spanish government grant MTM2009-06964, and by the Korean National Researcher Program 2010-0020413 of NRF..

The third author was partially supported by JSPS KAKENHI Grant Number 24840008.

The fourth author was supported by Polish National Science Center grant 2013/08/A/ST1/00804. A part of this work was done when he visited CIRM at the University of Trento.

showing how the homogeneity follows from the nefness of the tangent bundle is still an open problem.

Recently, a possible strategy towards the solution of this problem has been proposed in [28]. The paper showed how the classification of manifolds of Picard number two with two smooth \mathbb{P}^1 -fibrations could be used recursively to associate a finite Dynkin diagram with any manifold with nef tangent bundle supporting as many smooth \mathbb{P}^1 -fibrations as its Picard number, identifying in this way a homogeneous model of X , to which it is conjecturally isomorphic. In [28] it was then proposed a method to prove homogeneity by reconstructing the manifold using families of rational curves, and this method was successfully applied to manifolds with associated Dynkin diagram of type A_n .

In this paper we generalize (using completely different techniques) the results in [28], associating a finite Dynkin diagram \mathcal{D} and a homogeneous model G/B with any Fano manifold X whose elementary contractions are smooth \mathbb{P}^1 -fibrations. We then prove the following

THEOREM 1.2. *Let X be a Fano manifold whose elementary contractions are smooth \mathbb{P}^1 -fibrations. Then X is isomorphic to a complete flag manifold G/B , where G is a semisimple algebraic group and B a Borel subgroup.*

Our proof of this statement is based on the following ideas: every smooth \mathbb{P}^1 -fibration in X provides an involution of the vector space $N^1(X)$ of classes of \mathbb{R} -divisors in X . We show in Section 2 that the set of these involutions generates a finite reflection group, which is the Weyl group W of a semisimple Lie algebra \mathfrak{g} , defining a Dynkin diagram \mathcal{D} and a homogeneous model G/B for X . We can then consider an isomorphism $\psi : N^1(X) \rightarrow N^1(G/B)$ preserving the relative canonical classes of the \mathbb{P}^1 -fibrations of X and G/B , using it to prove that they have the same dimension and the same cohomology of line bundles (see Corollary 2.25).

In Section 3 we use the \mathbb{P}^1 -fibrations in X to define a set of auxiliary manifolds called Bott-Samelson varieties, which are analogues of the Bott-Samelson varieties that appear classically in the study of Schubert cycles of homogeneous manifolds. We then show in Section 4 that the recursive construction of appropriately chosen Bott-Samelson varieties depends only on the combinatorics of the Weyl group W (cf. Propositions 4.6 and 4.8), and we infer from this the isomorphism between X and its homogeneous model.

This proof works with an exception, which occurs when a connected component of \mathcal{D} is F_4 , so, to conclude the proof of Theorem 1.2, we need to treat this case with different arguments, presented in Section 5.

Finally, in Section 6, we briefly discuss how our results merge into the general problem of Campana–Peternell Conjecture.

NOTATION 1.3. Along this paper, unless otherwise stated, X will denote a Fano manifold of Picard number $\rho_X = n$ whose elementary contractions $\pi_i : X \rightarrow X_i$ are smooth \mathbb{P}^1 -fibrations. The vector space of \mathbb{R} -divisors on X modulo numerical equivalence will be denoted by $N^1(X)$, and its dual, that is the vector space of real 1-cycles modulo numerical equivalence, by $N_1(X)$. The closure of the cone generated by effective 1-cycles, the so called Mori cone of X , will be denoted by $\overline{\text{NE}}(X) \subset N_1(X)$. For every i we will denote by Γ_i a fiber of π_i and by R_i the extremal ray of $\overline{\text{NE}}(X)$ generated by the class of Γ_i . The relative canonical divisor of π_i will be denoted by K_i .

2. Homogeneous models

In this section we will show how to associate a flag manifold G/B with a Fano manifold X whose elementary contractions π_i , $i = 1, \dots, n$, are smooth \mathbb{P}^1 -fibrations.

NOTATION 2.1. Since X is a Fano manifold, two divisors on X are numerically equivalent if and only if they are linearly equivalent; in particular we can identify the space $N^1(X)_{\mathbb{Z}}$ of divisors modulo numerical equivalence with $\text{Pic}(X)$. For this reason, with abuse of notation we will denote the numerical class of a divisor and the corresponding line bundle with the same letter, using additive notation.

As a first observation, we note the following:

LEMMA 2.2. *With the same notation as in 1.3, the classes $[\Gamma_i]$, $i = 1, \dots, n$ form a basis of the real vector space $N_1(X)$, and generate the Mori cone of X . In particular, X has Picard number n .*

PROOF. Since every Γ_i provides an unsplit dominating family of rational curves and $\mathbb{R}_{\geq 0}[\Gamma_i]$ are extremal rays of $\overline{\text{NE}}(X)$, the first part of the statement follows from a direct application of [1, Lemma 5.2]. Since X is a Fano manifold, its Mori cone is generated by the rays associated with its elementary contractions: this completes the proof. \blacksquare

We will discuss now some general facts about cohomology of divisors on smooth \mathbb{P}^1 -fibrations, that we will need later on.

2.1. Relative duality on \mathbb{P}^1 -fibrations. The following result may be understood as an extension to the case of smooth \mathbb{P}^1 -fibrations of the Borel-Weil-Bott Theorem on \mathbb{P}^1 . Using similar ideas, a short proof of the full Borel-Weil-Bott Theorem was given in [25].

LEMMA 2.3. *Let $\pi : M \rightarrow Y$ be a smooth \mathbb{P}^1 -fibration over a smooth manifold Y , denote by Γ one of its fibers and by K its relative canonical divisor. Then for every Cartier divisor D on M , setting $l := D \cdot \Gamma$ and $\text{sgn}(\alpha) := \alpha/|\alpha|$ for $\alpha \neq 0$, $\text{sgn}(0) := 1$, one has*

$$(1) \quad H^i(M, \mathcal{O}_M(D)) \cong H^{i+\text{sgn}(l+1)}(M, \mathcal{O}_M(D + (l+1)K)), \text{ for every } i \in \mathbb{Z}.$$

In particular:

- $H^i(M, \mathcal{O}_M(D)) = 0$ if $l = -1$, and
- $\chi(M, \mathcal{O}_M(D)) = -\chi(M, \mathcal{O}_M(D + (l+1)K))$ for every divisor D .

PROOF. Note first that $l+1 = D \cdot \Gamma + 1 = -((D + (l+1)K) \cdot \Gamma + 1)$, so we may assume, without loss of generality, that $l \leq -1$. In this case, moreover, equation (1) holds trivially for $i \leq -1$, so we may also assume that $i \geq 0$.

First of all we claim that there is a natural isomorphism:

$$R^1 \pi_* \mathcal{O}_M(D) \cong \pi_* \mathcal{O}_M(D + (l+1)K).$$

In fact, relative Serre duality provides $R^1 \pi_* \mathcal{O}_M(D) \cong (\pi_* \mathcal{O}_M(K - D))^\vee$, and the fact that the vector bundle on Y appearing in the right hand side coincides with $\pi_* \mathcal{O}_M(D + (l+1)K)$ follows from Lemma 2.4 below.

Then, applying the Leray spectral sequence with respect to the morphism π to both $\mathcal{O}_M(D)$ and $\mathcal{O}_M(D + (l+1)K)$, we obtain the required isomorphism: in the

left hand side, since $l \leq -1$, the only nonzero elements of the sequence are those of the form $H^i(Y, R^1 \pi_* \mathcal{O}_M(D))$, hence there is an isomorphism $H^{i+1}(M, \mathcal{O}_M(D)) \cong H^i(Y, R^1 \pi_* \mathcal{O}_M(D))$; on the right hand side, since $(D + (l+1)K) \cdot \Gamma = -l-2 \geq -1$, the only nonzero elements of the sequence are those of the form $H^i(Y, \pi_* \mathcal{O}_M(D + (l+1)K))$, so that $H^i(Y, \pi_* \mathcal{O}_M(D + (l+1)K)) \cong H^i(M, \mathcal{O}_M(D + (l+1)K))$. Summing up we have:

$$\begin{aligned} H^{i+1}(M, \mathcal{O}_M(D)) &\cong H^i(Y, R^1 \pi_* \mathcal{O}_M(D)) \cong H^i(Y, \pi_* \mathcal{O}_M(D + (l+1)K)) \\ &\cong H^i(M, \mathcal{O}_M(D + (l+1)K)). \end{aligned}$$

■

■

LEMMA 2.4. *Let $\pi : M \rightarrow Y$ be a smooth \mathbb{P}^1 -fibration over a smooth manifold Y , denote by Γ one of its fibers and by K its relative canonical divisor. Then for every divisor D we have*

$$(\pi_* \mathcal{O}_M(K - D))^\vee \cong \pi_* \mathcal{O}_M(D + (D \cdot \Gamma + 1)K).$$

PROOF. The statement is clearly true when $(K - D) \cdot \Gamma < 0$, so we may assume $l := D \cdot \Gamma \leq -2$.

If $\pi : M \rightarrow Y$ is a \mathbb{P}^1 -bundle or, equivalently, if M admits a divisor H satisfying $H \cdot \Gamma = 1$, then denoting by \mathcal{F} the rank two vector bundle $\pi_* \mathcal{O}_M(H)$, and writing $\mathcal{O}_M(D) \cong \mathcal{O}_M(lH) \otimes \pi^*(B)$ for some $B \in \text{Pic}(Y)$, we have:

$$\begin{aligned} \pi_* \mathcal{O}_M(K - D) &= S^{-(l+2)} \mathcal{F} \otimes B^\vee \otimes \det(\mathcal{F}) \\ \pi_* \mathcal{O}_M(D + (l+1)K) &= S^{-(l+2)} \mathcal{F} \otimes B \otimes \det(\mathcal{F})^{l+1} \end{aligned}$$

Since $\mathcal{F}^\vee = \mathcal{F} \otimes \det(\mathcal{F})^\vee$, it follows that

$$(S^{-(l+2)}(\mathcal{F}))^\vee = (S^{-(l+2)}(\mathcal{F})) \otimes \det(\mathcal{F})^{l+2}$$

which, together with our previous two equalities, concludes the proof.

We may then assume that $\pi : M \rightarrow Y$ is not a \mathbb{P}^1 -bundle and, in particular, that $l = D \cdot \Gamma$ is even. Let us denote by \mathcal{E} the rank three vector bundle $\pi_* \mathcal{O}_M(-K)$; the natural map $\pi^*(\mathcal{E}) \rightarrow \mathcal{O}_M(K)^\vee$ defines an inclusion $\iota : M \hookrightarrow P := \mathbb{P}(\mathcal{E})$ which makes the following diagram, where $p : P \rightarrow Y$ is the natural projection, commutative:

$$\begin{array}{ccc} M & & \\ \downarrow \iota & \searrow \pi & \\ P & \xrightarrow{p} & Y \end{array}$$

Note that, by construction, $M \subset P$ is a divisor whose associated line bundle is $\mathcal{O}_P(2)$, so that M defines an everywhere nondegenerate (symmetric) isomorphism $\mathcal{E} \cong \mathcal{E}^\vee$.

We may write $\mathcal{O}_M(D) \cong \mathcal{O}_M(-(l/2)K) \otimes \pi^* B$ for some $B \in \text{Pic}(Y)$, so that:

$$\begin{aligned} \pi_* \mathcal{O}_M(K - D) &= \pi_* \mathcal{O}_M((l/2 + 1)K) \otimes B^\vee = S^{-(l/2+1)} \mathcal{E} \otimes B^\vee \\ \pi_* \mathcal{O}_M(D + (l+1)K) &= S^{-(l/2+1)} \mathcal{E} \otimes B \end{aligned}$$

The proof is finished by using the isomorphism $\mathcal{E} \cong \mathcal{E}^\vee$. ■

2.2. Reflection groups. We will now apply results of the previous subsection to the case of a Fano manifold X satisfying Notation 1.3.

For every elementary contraction $\pi_i : X \rightarrow X$ we will consider the affine involution r'_i of $N^1(X)$ given by:

$$r'_i(D) := D + (D \cdot \Gamma_i + 1)K_i.$$

Denoting by T the translation by $K_X/2$ in $N^1(X)$, that is $T(D) := D + K_X/2$, every composition $r_i := T^{-1} \circ r'_i \circ T$ is a linear involution of $N^1(X)$ given by

$$r_i(D) = D + (D \cdot \Gamma_i)K_i,$$

which is a *reflection*, i.e. it is an order two automorphism that fixes a hyperplane in $N^1(X)$. In our case, $r_i(K_i) = -K_i$ and the fixed hyperplane is

$$(2) \quad \Gamma_i^\perp := \{D \mid D \cdot \Gamma_i = 0\} \subset N^1(X).$$

We will denote by $W \subset \text{GL}(N^1(X))$ the group generated by the reflections r_i , $i = 1, \dots, n$. Via conjugation with T , the group of affinities W' generated by the reflections r'_i may be identified with W . The following lemma shows how cohomology behaves with respect to the action of W' .

LEMMA 2.5. *If $D \in \text{Pic}(X)$ such that $w'(D) \in T(\text{Nef}(X))$ for some $w' \in W'$ then there exists $i \in \mathbb{Z}$, such that*

$$H^j(X, D) = 0, \text{ for } j \neq i.$$

PROOF. Note first that for $D \in T(\text{Nef}(X))$, one has

$$H^j(X, D) = H^j(X, K_X + D - K_X/2 - K_X/2) = 0, \text{ for } j \neq 0,$$

where the last equality follows, being $D - K_X/2$ nef and $-K_X/2$ ample, by Kodaira Vanishing Theorem. Since every $w' \in W'$ is a composition of reflections r'_i , $i = 1, \dots, n$, the result follows from recursive use of Lemma 2.3. \blacksquare

DEFINITION 2.6. Let $\chi_X : \text{Pic}(X) \rightarrow \mathbb{Z}$ be the function which associates with a line bundle its Euler characteristic, i.e. $\chi_X(L) = \chi(X, L)$; this function, as proved by Snapper (Cf. [17, Section 1, Theorem]), has the property that, given $L_1, \dots, L_t \in \text{Pic}(X)$, $\chi_X(m_1, \dots, m_t) = \chi(X, m_1L_1 + \dots + m_tL_t)$ is a numerical polynomial in m_1, \dots, m_t of degree $\leq \dim X$. Via the identification of the Picard group with $N^1(X)_\mathbb{Z}$ we can thus extend this function to a polynomial function $\chi_X : N^1(X) \rightarrow \mathbb{R}$. We also set $\chi^T := \chi_X \circ T$.

The following lemma describes the behaviour of χ^T with respect to W :

LEMMA 2.7. *For every \mathbb{R} -divisor D on X and every reflection r_i , the following equality is fulfilled*

$$(3) \quad \chi^T(D) = -\chi^T(r_i(D))$$

PROOF. It is enough to notice that the property holds in the lattice $\text{Pic}(X) \subset N^1(X)$, by Lemma 2.3. \blacksquare

COROLLARY 2.8. *For every \mathbb{R} -divisor D on X and every $w \in W$,*

$$(4) \quad \chi^T(D) = \pm \chi^T(w(D)).$$

We will also consider the dual action of W on the dual vector space $N_1(X)$, by considering, for every $w \in W$:

$$N_1(X) = \text{Hom}(N^1(X), \mathbb{R}) \ni C \rightarrow w^\vee(C) := C \circ w \in \text{Hom}(N^1(X), \mathbb{R}).$$

In other words, the action is chosen so that

$$(5) \quad w(D) \cdot C = D \cdot w^\vee(C), \text{ for all } D \in N^1(X), C \in N_1(X).$$

Moreover, the involutions r_i^\vee are reflections as well, given by

$$r_i^\vee(C) = C + (K_i \cdot C)\Gamma_i.$$

Note also that the action of W on $N_1(X)$ is clearly faithful, i.e. the morphism $W \rightarrow \text{GL}(N_1(X))$ defined by $w \mapsto w^\vee$ is injective. This observation will help us to prove the following:

PROPOSITION 2.9. *With the same notation as above, the function χ^T vanishes on all the hyperplanes determined by the cycles $w^\vee(\Gamma_i)$, and the group W is finite.*

PROOF. Since, for every i , the reflection r_i fixes the hyperplane Γ_i^\perp , it follows by Lemma 2.7 that the polynomial function χ^T vanishes on Γ_i^\perp , for all i . But then, by Corollary 2.8, χ^T vanishes on all the hyperplanes of the form $w(\Gamma_i^\perp)$, $w \in W$. In particular, it follows that the cardinality of this set of hyperplanes is smaller than or equal to the degree of χ^T , i.e. the dimension of X .

Every hyperplane $w(\Gamma_i^\perp)$ is uniquely determined by the class of the 1-cycle $(w^{-1})^\vee(\Gamma_i)$ in the quotient of $N_1(X)$ modulo homotheties, that is, the Grothendieck projectivization $\mathbb{P}(N^1(X))$. Denote by $Z \subset \mathbb{P}(N^1(X))$ the set of elements of the form $w(\Gamma_i^\perp) = [(w^{-1})^\vee(\Gamma_i)]$, $w \in W$ and $i = 1, \dots, n$, and consider the diagonal action of W on the (finite) product set Z^n .

In order to see that W is finite, it is enough to show that the isotropy subgroup $W^0 \subset W$ of elements of W fixing $([\Gamma_1], \dots, [\Gamma_n])$ is finite. By construction, the matrix of every element $w^\vee \in \text{GL}(N_1(X))$ with respect to the basis $\{\Gamma_1, \dots, \Gamma_n\}$ has integral coefficients and determinant ± 1 (since the properties hold for every generator r_i^\vee). On the other hand, if $w \in W^0$, then the matrix of w^\vee with respect to this base must be diagonal, hence its diagonal coefficients are all equal to ± 1 . In particular the image of W^0 in $\text{GL}(N_1(X))$ is finite and, since the action of W on $N_1(X)$ is faithful, W^0 is finite as well. \blacksquare

Let us denote by W^\vee the image of W into $\text{GL}(N_1(X))$. As a consequence of Proposition 2.9 we may state the following

COROLLARY 2.10. *With the same notation as above there exists a scalar product $\langle \cdot, \cdot \rangle$ in $N_1(X)$ invariant under the action of W^\vee . In particular, the reflections r_i^\vee are orthogonal with respect to the scalar product $\langle \cdot, \cdot \rangle$, and we may write:*

$$(6) \quad -K_i \cdot C = 2 \frac{\langle \Gamma_i, C \rangle}{\langle \Gamma_i, \Gamma_i \rangle}, \text{ for all } i = 1, \dots, n.$$

In particular $\{-K_i, i = 1, \dots, n\}$ is a basis of $N^1(X)$ as a vector space over \mathbb{R} .

PROOF. Consider any inner product (\cdot, \cdot) in $N_1(X)$. A new inner product $\langle \cdot, \cdot \rangle$ defined by $\langle x, y \rangle := \sum_{w \in W} (w(x), w(y))$, for all $x, y \in N_1(X)$, is W^\vee -invariant. \blacksquare

REMARK 2.11. The inner product $\langle \cdot, \cdot \rangle$ provided by the above corollary induces an isomorphism $\varphi : N_1(X) \rightarrow N^1(X)$ sending C to $\langle C, - \rangle$. The scalar product in $N^1(X)$ induced by $\langle \cdot, \cdot \rangle$ via this isomorphism, that we denote by $\langle \cdot, \cdot \rangle$ as well, is invariant by the action of W , since

$$(7) \quad w(\varphi(C)) = \varphi((w^\vee)^{-1}C), \text{ for all } C \in N_1(X), \text{ and } w \in W.$$

In particular, the reflections r_i are orthogonal with respect to $\langle \cdot, \cdot \rangle$ and we may write:

$$(8) \quad -\Gamma_i \cdot D = 2 \frac{\langle D, K_i \rangle}{\langle K_i, K_i \rangle}, \text{ for all } i.$$

2.3. Root systems and Dynkin diagrams. We will now show how to use the group W to construct a (reduced) root system in $N^1(X)$; this will allow us to obtain a finite Dynkin diagram \mathcal{D} which encodes the intersection matrix $-K_i \cdot \Gamma_j$. In turn this will allow us to associate with X a semisimple Lie group G , whose Weyl group is W , and a homogeneous model G/B .

NOTATION 2.12. Let us define

$$\Phi := \{w(-K_i) \mid w \in W, i = 1, \dots, n\} \subset N^1(X),$$

which is a finite set by Proposition 2.9.

The next step will be showing that Φ is a *root system* in $N^1(X)$. This means, by definition (cf. [4, VI, §1, Definition 1]), that:

- (R1) $\Phi \subset N^1(X)$ is a finite set of nonzero elements generating $N^1(X)$.
- (R2) For every $D \in \Phi$ there exists $\Gamma_D \in N_1(X)$ such that $D \cdot \Gamma_D = 2$ and such that Φ is invariant by the reflection r_D defined by $r_D(E) = E - (\Gamma_D \cdot E)D$, for all $E \in N^1(X)$.
- (R3) For every $D, D' \in \Phi$, $\Gamma_D \cdot D' \in \mathbb{Z}$.

A root system is said to be *reduced* if $D, kD \in \Phi$ implies $k = \pm 1$. The *Weyl group* of Φ is the group generated by the reflections r_D .

PROPOSITION 2.13. *With the same notation as above, the set Φ is a reduced root system in $N^1(X)$, whose Weyl group is W .*

PROOF. Property (R1) follows from Proposition 2.9 and Corollary 2.10. For (R2), given $D = w(-K_i) \in \Phi$, we take $\Gamma_D := (w^{-1})^\vee(\Gamma_i)$. Then we have $\Gamma_D \cdot D = -K_i \cdot \Gamma_i = 2$ and the reflection r_D leaves Φ invariant because with this choice of Γ_D the corresponding reflection r_D is equal to $w \circ r_i \circ w^{-1} \in W$, as a direct computation shows. Since, by definition, Φ is invariant by W , this shows that Property (R2) holds and that the W is the Weyl group of Φ .

For the third property, note that the lattice $\text{Pic}(X)$ is invariant by the action of W , hence, given $D = w(-K_i)$, $D' = w'(-K_j)$, the element $w^{-1} \circ w'(-K_j)$ is the numerical class of a Cartier divisor, therefore

$$\Gamma_D \cdot D' = (w^{-1})^\vee(\Gamma_i) \cdot w'(-K_j) = (\Gamma_i) \cdot w^{-1}(w'(-K_j)) \in \mathbb{Z}.$$

Finally, if Φ was not reduced, there would exist two indices i and j , an element $w \in W$ and a real number k with $|k| < 1$ such that $w(-K_i) = k(-K_j)$, contradicting the fact that the matrix of every element of W with respect to the basis $\{-K_1, \dots, -K_n\}$ has integral coefficients (every r_i satisfies this property, by definition). ■

We will call *Cartan matrix* of X the $n \times n$ matrix \mathcal{A} whose entries are given by $\mathcal{A}_{ij} := -K_i \cdot \Gamma_j \in \mathbb{Z}$. Note that, by definition, $\mathcal{A}_{ii} = 2$ for all i . The following lemma lists the possible values of the non-diagonal entries of \mathcal{A} .

LEMMA 2.14. *With the same notation as above, for every two distinct indices $i, j \in \{1, \dots, n\}$, the possible values of $(-K_i \cdot \Gamma_j, -K_j \cdot \Gamma_i)$ are:*

$$(0, 0), \quad (-1, -1), \quad (-1, -2), \quad (-2, -1), \quad (-1, -3), \quad (-3, -1).$$

PROOF. Note first that $\mathcal{A}_{ij} = 2\langle -K_i, -K_j \rangle / \langle -K_j, -K_j \rangle$, by equation (8). Then, being Φ a root system, the possible values of \mathcal{A}_{ij} are listed in [10, 9.4]. It is then enough to prove that $\mathcal{A}_{ij} \leq 0$ for $i \neq j$. In order to see this we consider the fiber product S of $\pi_i : X \rightarrow X_i$ and its restriction to a curve Γ_j :

$$\begin{array}{ccc} S & \longrightarrow & X \\ \uparrow s_j & \nearrow & \downarrow \pi_i \\ \Gamma_j & \xrightarrow{\pi_i|_{\Gamma_j}} & X_i \end{array}$$

The inclusion of Γ_j into X factors via S , providing a morphism: $s_j : \Gamma_j \rightarrow S$, whose image satisfies: $-K_{S|\Gamma_j} \cdot s_j(\Gamma_j) = -K_i \cdot \Gamma_j$. Now, if this intersection number were positive, one could easily prove that the curve $s_j(\Gamma_j)$ would be algebraic equivalent to a reducible 1-cycle. From this we would obtain that the family of Γ_j 's in X would not be unsplit, a contradiction. ■

NOTATION 2.15. The matrix \mathcal{A} is determined by its *Dynkin diagram* \mathcal{D} , which consists of a graph with n ordered nodes, where the nodes i and j are joined by $\mathcal{A}_{ij}\mathcal{A}_{ji}$ edges. When two nodes i and j are joined by a double or triple edge, we add to it an arrow, pointing towards i if $\mathcal{A}_{ij} > \mathcal{A}_{ji}$.

PROPOSITION 2.16. *With the same notation as above, every connected component of the Dynkin diagram \mathcal{D} is one of the following.*

$$A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, \text{ or } G_2.$$

PROOF. Without loss of generality, we may assume that \mathcal{D} is connected. Note that, by Lemma 2.14 above, in the euclidean space $(N^1(X), \langle \cdot, \cdot \rangle)$, the set

$$\{\epsilon_i := -K_i / \langle -K_i, -K_i \rangle^{\frac{1}{2}} \mid i = 1, \dots, n\}$$

is *admissible*, that is, its elements are n linearly independent unit vectors satisfying $\langle \epsilon_i, \epsilon_j \rangle \leq 0$ for $i \neq j$, and $4\langle \epsilon_i, \epsilon_j \rangle^2 = 0, 1, 2$ or 3 for $i \neq j$. Then the result follows from the proof of [10, Theorem 11.4]. ■

In particular, we may associate with X a semisimple Lie group G , and its semisimple Lie algebra \mathfrak{g} , determined by the Dynkin diagram \mathcal{D} . We choose a Cartan subgroup H , with Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and denote by $\overline{\Phi} \subset \mathfrak{h}^\vee$ the corresponding root system (cf. [10, 8]). Moreover, we denote by $\mathfrak{h}_{\mathbb{R}}^\vee$ the real vector subspace of \mathfrak{h}^\vee generated by $\overline{\Phi}$. Given a base of simple roots $\overline{\Delta} = \{\alpha_i \mid i = 1, \dots, n\}$ (see [10, 10.1] for the definition), we may assume that its elements have been ordered so that the corresponding Dynkin diagram is equal to \mathcal{D} . We will consider the morphism

$$\psi : N^1(X) \rightarrow \mathfrak{h}_{\mathbb{R}}^\vee, \text{ defined by } \psi(-K_i) = \alpha_i.$$

DEFINITION 2.17. Consider the Borel subgroup $B \subset G$ containing the Cartan subgroup H . The complete flag manifold G/B is a rational homogeneous space, which we will call the *rational homogeneous model of X* . We will use for G/B a similar notation as for X , adding an overline to distinguish the two cases (so we will use $\overline{\pi}_i, \overline{\Gamma}_i, -\overline{K}_i, \dots$).

COROLLARY 2.18. *With the same notation as above, the set $\Delta := \{-K_i \mid i = 1, \dots, n\}$ is a base of simple roots of Φ .*

PROOF. The Dynkin diagram \mathcal{D} contains the necessary information to recover $\overline{\Phi}$ up on $\overline{\Delta}$ and the corresponding reflections. Then, by construction, the isomorphism ψ defined above maps $\overline{\Delta}$ to Δ and $\overline{\Phi}$ to Φ , so the claim follows. ■

COROLLARY 2.19. *With the same notation as above, $\text{Amp}(X)$ is a Weyl chamber of W , and $\text{Nef}(X)$ is a fundamental domain of W .*

PROOF. The fact that Δ is a base of simple roots is equivalent to the set

$$U = \{D \in N^1(X) \mid \langle D, -K_i \rangle > 0, i = 1, \dots, n\}$$

to be a Weyl chamber of Φ (see [10, 10.1] for details). By equation (8), U is the set of classes having positive intersection with every Γ_i . Hence, by Lemma 2.2, U is the ample cone of X . The second part of the statement follows then from [10, 10.2]. ■

REMARK 2.20. Recall that, being Δ a base of simple roots, the root system Φ decomposes as a union $\Phi = \Phi^+ \cup \Phi^-$, where Φ^+ (respectively Φ^-) is the set of roots that are integer linear combination of elements of Δ with nonnegative (resp. nonpositive) coefficients. The elements in Φ^+ and Φ^- are called, respectively, positive and negative roots of Φ with respect to Δ . Moreover, by Corollary 2.19, taking any ample divisor A , the set Φ^+ (resp. Φ^-) can be described as the set of roots D satisfying $\langle A, D \rangle > 0$ (resp. $\langle A, D \rangle < 0$). See [10, 10.1] for details.

2.4. Length of divisors and cohomology. We will now show that the cohomology of divisors on X equals the one on the homogeneous model, i.e. $h^i(X, D) = h^i(G/B, \psi(D))$ for every $D \in \text{Pic}(X)$, $i \in \mathbb{Z}$; in particular the dimension of X is equal to the dimension of its homogeneous model.

We will call *translated Weyl chambers* the images by T of the Weyl chambers of Φ . In other words, they are the connected components of $N^1(X) \setminus \bigcup_{D \in \Phi} T(D^\perp)$.

LEMMA 2.21. *With the same notation as above, for every $D \in N^1(X)$, there exist $w' \in W'$ and a unique $D' \in T(\text{Nef}(X))$ such that $w'(D) = D'$. If moreover D lies in a translated Weyl chamber of X , w' is uniquely determined by D .*

PROOF. Clearly, W' acts on the set of translated Weyl chambers of Φ . Then the result follows from Corollary 2.19 and [10, 10.3]. ■

Following [10, 10.3], given an element $w \in W$, we will define its *length* $\lambda(w)$ as the minimal t such that w can be written as $w = r_{i_1} \circ \dots \circ r_{i_t}$; such an expression for w will be called *reduced*; the length of the identity is 0, by definition. Then, for every element $w' \in W'$, we define its *length* as $\lambda(w') := \lambda(T^{-1} \circ w' \circ T)$. In view of the previous lemma, for an element D lying in a translated Weyl chamber, we define its *length* $\lambda(D)$ as the length of the element w' sending it into $T(\text{Amp}(X))$. The following result relates the length $\lambda(D)$ with the cohomology of D :

PROPOSITION 2.22. *With the same notation as above, given $D \in \text{Pic}(X)$, we consider an element $w' \in W'$ such that $w'(D) \in T(\text{Nef}(X))$. Then:*

- *either $w'(D) \in T(\text{Nef}(X) \setminus \text{Amp}(X))$ and $H^j(X, D) = 0$ for all j ,*
- *or $w'(D) \in T(\text{Amp}(X))$ and $H^j(X, D) = 0$ for $j \neq \lambda(D)$.*

PROOF. We already know that, for any $D \in \text{Pic}(X)$ there exists an index i_0 for which $H^j(X, D) = 0$ for $j \neq i_0$ (see Lemma 2.5). If $D \in w'(T(\text{Nef}(X) \setminus \text{Amp}(X)))$ for some $w' \in W'$ then, there exists an index i such that $T^{-1}(w'^{-1}(D)) \cdot \Gamma_i = 0$. Hence, if $w = T^{-1} \circ w' \circ T$, we may assert that $T^{-1}(D) \cdot (w^{-1})^\vee(\Gamma_i) = 0$ and by Proposition 2.9, $\chi(D) = \chi^T(T^{-1}(D)) = 0$. From this we conclude that $H^j(X, D) = 0$ also for $j = i_0$.

If $D \in w'(T(\text{Amp}(X)))$, then w' is uniquely determined by D . Write $w' = r'_{i_1} \circ \dots \circ r'_{i_t}$, with $t = \lambda(D)$, and denote by A the only ample class such that $D = w'(T(A))$. Following the idea of the proof of [10, 10.3, Lemma A], we will prove our statement by induction on t . The statement is clear for $t = 0$, and we will assume by induction that $H^j(X, r'_{i_2} \circ \dots \circ r'_{i_t}(T(A))) = 0$ for $j \neq t - 1$. Then, by Lemma 2.3, it suffices to show that $r'_{i_2} \circ \dots \circ r'_{i_t}(T(A)) \cdot \Gamma_{i_1} > -1$. Assume by contradiction that $r_{i_2} \circ \dots \circ r_{i_t}(A) \cdot \Gamma_{i_1} < 0$.

Setting $w_1 := r_{i_2} \circ \dots \circ r_{i_t}$, this inequality tells us that $\langle w_1(A), -K_{i_1} \rangle < 0$, that is $\langle A, w_1^{-1}(-K_{i_1}) \rangle < 0$. This implies (see Remark 2.20) that $w_1^{-1}(-K_{i_1})$ is a negative root, which, by [10, 10.2 Lemma C], tells us that

$$w_1^{-1} \circ r_{i_1} = r_{i_t} \circ \dots \circ r_{i_2} \circ r_{i_1} = r_{i_t} \circ \dots \circ r_{i_{s-1}} \circ r_{i_{s+1}} \circ \dots \circ r_{i_1},$$

for some $s < t$. In particular this implies that $\lambda(r_{i_t} \circ \dots \circ r_{i_2} \circ r_{i_1}) < t$, a contradiction. ■

Let G/B be the homogenous model of X . Every line bundle on G/B is homogeneous, hence determined by a *weight*, that is, an element of \mathfrak{h}^\vee . In particular, $-\bar{K}_i$ is the homogeneous line bundle on G/B associated to the simple root α_i , and $N^1(G/B)$ may be identified with real vector space $\mathfrak{h}_\mathbb{R}^\vee$. In the sequel, we will interpret the morphism ψ defined above as an isomorphism from $N^1(X)$ to $N^1(G/B)$ sending $-K_i$ to $-\bar{K}_i$ for every i . Moreover, this isomorphism sends $-K_X$ to $-K_{G/B}$. In fact, since the Cartan matrix \mathcal{A} is nonsingular, the coefficients of $-K_X$ with respect to the basis $\{-K_1, \dots, -K_n\}$ are determined by the intersection numbers $-K_X \cdot \Gamma_i = 2$. The same holds for G/B so, in particular, denoting by \bar{T} the translation by $\frac{1}{2}K_{G/B}$ in $N^1(G/B)$, we have $\psi \circ T = \bar{T} \circ \psi$. Note finally that, by construction, ψ preserves the length of divisors. As a consequence, we get:

PROPOSITION 2.23. *With the same notation as above, let G/B be the homogenous model of X . Then $\dim X = \dim G/B$.*

PROOF. Since, by Serre duality, $H^{\dim X}(X, K_X) \neq 0$, the dimension of X equals, by Proposition 2.22, the length of K_X . But, by our previous observations on the map ψ , this equals the length of $K_{G/B}$, which, by the same argument, is equal to $\dim(G/B)$. ■

For every $D \in \Phi$, denote by $F_D : N^1(X) \rightarrow \mathbb{R}$ the linear operator defined by:

$$F_D(L) := -2 \frac{\langle D, L \rangle}{\langle D, K_X \rangle}.$$

The following result states that χ_X depends only on the root system Φ .

THEOREM 2.24. *Let $\Phi^+ \subset \Phi$ be the set of positive roots with respect to Δ . Then*

$$\chi_X = \prod_{D \in \Phi^+} (1 + F_D).$$

PROOF. We will show that $\chi^T = \prod_{D \in \Phi^+} F_D$. By Proposition 2.23 $\deg(\chi^T) = \deg(\chi) \leq \dim X = \dim G/B$. Moreover, since B is the subgroup associated to a Borel subalgebra \mathfrak{b} which is isomorphic to the direct sum of the root spaces \mathfrak{g}_α , $\alpha \in \overline{\Phi}^+$ (cf. [10, II.8]), it follows that $\dim G/B = \sharp(\overline{\Phi}^+)$. Since Φ is a reduced root system, two different elements in Φ^+ are not proportional, hence $\prod_{D \in \Phi^+} F_D$ vanishes on the $|\Phi^+|$ distinct hyperplanes $\{F_D = 0\}$. Since $\chi^T(-K_X/2) = 1$ and $F_D(-K_X/2) = 1$ for all $D \in \Phi^+$, it is enough to show that χ^T vanishes on the hyperplane $\{F_D = 0\}$ for every $D \in \Phi^+$. Given any $D \in \Phi^+$ there exists $w \in W$ such that $D = w(-K_i)$ for some i , and the hyperplane $\{F_D = 0\}$ is $\{D' \in N_1(X) \mid D' \cdot w^\vee(\Gamma_i) = 0\} = w(\Gamma_i^\perp)$, with Γ_i^\perp as in (2); we already noticed in the proof of Proposition 2.9 that χ^T vanishes on these hyperplanes. ■

As a consequence of Proposition 2.22 and Theorem 2.24, we clearly obtain the following result:

COROLLARY 2.25. *With the same notation as above, for every line bundle belonging to the subgroup of $\text{Pic}(X)$ generated by K_1, \dots, K_n*

$$h^i(X, L) = h^i(G/B, \psi(L)) \quad i \in \mathbb{Z}.$$

3. Bott-Samelson varieties

Let $I = \{1, 2, \dots, n\}$ be the set of indices parametrizing the nodes of the Dynkin diagram \mathcal{D} of X , and let G/B be the homogeneous model of X . Following [24], when dealing with finite sequences of indices in I we will use this notation:

NOTATION 3.1. Given a sequence $\ell = (l_1, \dots, l_r)$, $l_i \in I$, we will set

$$\left. \begin{aligned} \ell[s] &:= (l_1, \dots, l_{r-s}) \\ \ell(i) &:= (l_1, \dots, \widehat{l}_i, \dots, l_r) \\ \ell[s](i) &:= (l_1, \dots, \widehat{l}_i, \dots, l_{r-s}) \end{aligned} \right\} \text{ for } i \leq r-s \leq r$$

In particular $\ell[s][s'] = \ell[s+s']$ and $\ell[s](r-s) = \ell[s+1]$. If $\ell = (l_1, \dots, l_r)$ and $\ell' = (l'_1, \dots, l'_t)$ are two sequences as above we will denote by $\ell\ell'$ the sequence $(l_1, \dots, l_r, l'_1, \dots, l'_t)$. Moreover we will denote by ℓ^k the sequence obtained juxtaposing k times the indices of ℓ . Finally we will denote by $w(\ell)$ the element of the Weil group W of \mathcal{D} given by $w(\ell) := r_{l_1} \circ r_{l_2} \circ \dots \circ r_{l_r}$.

Fix a point $x \in X$. To every sequence $\ell = (l_1, \dots, l_r)$ in I we will associate a sequence of manifolds $Z_{\ell[s]}$, $s = 0, \dots, r$, called the *Bott-Samelson varieties* associated to the subsequences $\ell[s]$, together with morphisms

$$f_{\ell[s]} : Z_{\ell[s]} \rightarrow X, \quad p_{\ell[s+1]} : Z_{\ell[s]} \rightarrow Z_{\ell[s+1]}, \quad \sigma_{\ell[s+1]} : Z_{\ell[s+1]} \rightarrow Z_{\ell[s]}.$$

They are constructed in the following way: for $s = r$ we set $Z_{\ell[r]} := \{x\}$ and let $f_{\ell[r]} : \{x\} \rightarrow X$ be the inclusion. Then for $s < r$ we define $Z_{\ell[s]}$ recursively upon $Z_{\ell[s+1]}$ by considering the composition $g_{\ell[s+1]} := \pi_{l_{r-s}} \circ f_{\ell[s+1]} : Z_{\ell[s+1]} \rightarrow X_{l_{r-s}}$

and taking its fiber product with $\pi_{l_{r-s}}$:

$$\begin{array}{ccc}
 Z_{\ell[s]} & \xrightarrow{f_{\ell[s]}} & X \\
 \sigma_{\ell[s+1]} \uparrow & \nearrow p_{\ell[s+1]} & \downarrow \pi_{l_{r-s}} \\
 Z_{\ell[s+1]} & \xrightarrow{g_{\ell[s+1]}} & X_{l_{r-s}}
 \end{array}$$

Note that the map $f_{\ell[s+1]}$ factors naturally via $Z_{\ell[s]}$, providing a section $\sigma_{\ell[s+1]} : Z_{\ell[s+1]} \rightarrow Z_{\ell[s]}$ of $p_{\ell[s+1]}$. This shows in particular that $p_{\ell[s+1]}$ is a \mathbb{P}^1 -bundle. Pulling back to $Z_{\ell[s+1]}$ via $\sigma_{\ell[s+1]}$ the relative Euler sequence over $Z_{\ell[s]}$ we see that this \mathbb{P}^1 -bundle is the projectivization of an extension $\mathcal{F}_{\ell[s]}$ of $\mathcal{O}_{Z_{\ell[s+1]}}$ by $f_{\ell[s+1]}^* K_{l_{r-s}}$:

$$(9) \quad 0 \longrightarrow f_{\ell[s+1]}^* K_{l_{r-s}} \longrightarrow \mathcal{F}_{\ell[s]} \longrightarrow \mathcal{O}_{Z_{\ell[s+1]}} \longrightarrow 0.$$

The cohomology class determining it will be denoted by

$$(10) \quad \zeta_{\ell[s]} \in H^1(Z_{\ell[s+1]}, f_{\ell[s+1]}^* K_{l_{r-s}}).$$

The next lemma will be used to prove that if X has disconnected Dynkin diagram, then it is a product (see Corollary 3.20).

LEMMA 3.2. *Assume that $\mathcal{D} = \mathcal{D}_1 \sqcup \mathcal{D}_2$, and denote by $I_1, I_2 \subset I$ the sets of indices parametrizing nodes in \mathcal{D}_1 and \mathcal{D}_2 , respectively. Let $\ell_1 = (l_1, \dots, l_k), \ell_2 = (l'_1, \dots, l'_s)$ be sequences of indices in I_1 and I_2 , respectively. Then $Z_{\ell_1 \ell_2} = Z_{\ell_1} \times Z_{\ell_2}$.*

PROOF. We will use induction on the number of elements of ℓ_2 . If ℓ_2 consists only of the element l'_1 then $Z_{\ell_1 \ell_2}$ is obtained as the projectivization of the rank two vector bundle

$$(11) \quad 0 \longrightarrow f_{\ell_1}^* K_{l'_1} \longrightarrow \mathcal{F}_{\ell_1} \longrightarrow \mathcal{O}_{Z_{\ell_1}} \longrightarrow 0.$$

Since $K_{l'_1}$ is trivial on Z_{ℓ_1} also \mathcal{F}_{ℓ_1} is trivial.

For the general case, let us consider the following commutative diagram

$$\begin{array}{ccccc}
 Z_{\ell_1 \ell_2} & \longrightarrow & Z_{\ell_2} & \xrightarrow{f_{\ell_2}} & X \\
 p_{\ell_1 \ell_2[1]} \downarrow & & (1) \quad p_{\ell_2[1]} \downarrow & & (2) \quad \downarrow \pi_{l'_s} \\
 Z_{\ell_1 \ell_2[1]} & \longrightarrow & Z_{\ell_2[1]} & \xrightarrow{g_{\ell_2[1]}} & X_{l'_s} \\
 \downarrow & & (3) \quad \downarrow & & \\
 Z_{\ell_1} & \longrightarrow & \text{Spec}(\mathbb{C}) & &
 \end{array}$$

Squares (2) and (1)–(2) are Cartesian, by construction of the Bott-Samelson varieties Z_{ℓ_2} and $Z_{\ell_1 \ell_2}$, hence also square (1) is Cartesian. Square (3) is Cartesian by the induction assumption, hence also square (1)–(3) is Cartesian. \blacksquare

3.1. Divisors and 1-cycles on Bott-Samelson varieties. Let Z_ℓ be the Bott-Samelson variety corresponding to $\ell = (l_1, \dots, l_r)$. In this section we will describe two pairs of dual basis for the vector spaces $N^1(Z_\ell)$ and $N_1(Z_\ell)$.

NOTATION 3.3. Note that, by construction, linear and numerical equivalence of divisors on Z_ℓ are the same. In particular, as we did for X , we will abuse of notation, denoting the numerical class of a divisor and the corresponding line bundle with the same letter, using additive notation.

Let us denote by $\beta_{i(r-i)}$ the class in $N_1(Z_{\ell[r-i]})$ of the fibers of $p_{\ell[r-i+1]} : Z_{\ell[r-i]} \rightarrow Z_{\ell[r-i+1]}$. We will denote by $\beta_{i(s)}$ the image of this class into $N_1(Z_{\ell[s]})$, via push forward with the sections $\sigma_{\ell[r-j]}$, $j = i, \dots, r-s-1$. If $s = 0$ we will write β_i instead of $\beta_{i(0)}$. Note that, by construction, $f_{\ell[s]*}\beta_{i(s)} = [\Gamma_{l_i}]$. The next remark describes the behaviour of the cohomology class $\zeta_{\ell[s]}$ with respect to curves in the classes $\beta_{i(s+1)}$:

REMARK 3.4. Consider the projection $p_{\ell[s+1]} : Z_{\ell[s]} \rightarrow Z_{\ell[s+1]}$, whose fibers lie in the class $\beta_{r-s(s)}$, for some $s \geq 0$. Assume that there exists $i < r-s$ satisfying that $l_i = l_{r-s}$. Then, over a rational curve $C \cong \mathbb{P}^1$ in $Z_{\ell[s+1]}$ whose class is $\beta_{i(s+1)}$ the sequence (9) restricts to a sequence of the form

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^1}(-2) \rightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^1} \rightarrow 0$$

and the section $\sigma_{\ell[s+1]}$ restricted to C is the diagonal morphism. In particular $\beta_{i(s)} = \sigma_{\ell[s+1]*}\beta_{i(s+1)} = \beta_{r-s(s)} + \beta'_{i(s)}$, where $\beta'_{i(s)}$ denotes the class of a minimal section of $p_{\ell[s+1]}$ over C .

If else $i < r-s$ is such that $l_i \neq l_{r-s}$ then the restriction of the sequence (9) to a curve $C \cong \mathbb{P}^1$ in the class $\beta_{i(s+1)}$ splits (by Lemma 2.14), and $\beta_{i(s)} = \sigma_{\ell[s+1]*}\beta_{i(s+1)}$ is the class of a minimal section $p_{\ell[s+1]}$ over C . ■

COROLLARY 3.5. *Let $\ell = (l_1, \dots, l_r)$ be a sequence in I . Assume that there exists $i < r-s$ satisfying that $l_i = l_{r-s}$. Then the class $\zeta_{\ell[s]}$ defined in (10) is not trivial. In particular $h^1(Z_{\ell[s+1]}, f_{\ell[s+1]}^* K_{l_{r-s}}) \neq 0$.*

For every $i \leq r-s$ we will denote by $Z_{\ell[s](i)}$ the divisor in $Z_{\ell[s]}$ which is the image of the natural inclusion of Bott-Samelson varieties $Z_{\ell[s](i)} \hookrightarrow Z_{\ell[s]}$. It may also be described as the pull-back to $Z_{\ell[s]}$, via the projections $p_{\ell[r-j]}$, $j = i, \dots, r-s-1$ of the divisor $\sigma_{\ell[r-i+1]}(Z_{\ell[r-i+1]}) \subset Z_{\ell[r-i]}$. As usual, for $s = 0$ we write $Z_{\ell(i)}$ instead of $Z_{\ell[0](i)}$. Abusing notation we will denote by $Z_{\ell[s](i)}$ also the numerical class of $Z_{\ell[s](i)}$ in $N^1(Z_\ell)$. The following lemma is immediate:

LEMMA 3.6. (Cf. [24, Section 2 and Subsection 3.2]) *The divisor $Z = \sum Z_{\ell(i)}$ is simple normal crossing. Moreover the $Z_{\ell(i)}$'s (resp. the β_i 's), $i = 1, \dots, r$, form a basis of $N^1(Z_\ell)$ (resp. of $N_1(Z_\ell)$).*

Within $N^1(Z_\ell)$ we may also consider the dual basis of $\{\beta_i, i = 1, \dots, r\}$, that we denote by

$$\{H_i, i = 1, \dots, r\}.$$

By our previous description of the Bott-Samelson varieties as \mathbb{P}^1 -bundles – equation (9) –, every H_i is the tautological bundle of $Z_{\ell[r-i]} \cong \mathbb{P}(\mathcal{F}_{\ell[r-i]})$, that is, $p_{\ell[r-i]*}H_i = \mathcal{F}_{\ell[r-i]}$. In particular, these line bundles provide a \mathbb{Z} -basis of $\text{Pic}(Z_\ell)$.

The following lemma provides an adjunction formula for the evaluation morphisms $f_\ell : Z_\ell \rightarrow X$.

LEMMA 3.7. (Cf. [24, Lemma 5.1]) *With the same notation as above:*

$$(12) \quad f_\ell^*(-K_X/2) = \sum_{i=1}^r H_i \quad \text{and} \quad -K_{Z_\ell} = \sum_{i=1}^r Z_{\ell(i)} + f_\ell^*(-K_X/2).$$

PROOF. Since $f_\ell^*(-K_X/2) \cdot \beta_i = 1$ for all i , it follows that $f_\ell^*(-K_X/2) = \sum_{i=1}^r H_i$. On the other hand, note that, for every $s \leq r$ one has

$$f_{\ell[s+1]}^*(-K_{l_{r-s}}) = (Z_{\ell[s](r-s)})_{|Z_{\ell[s+1]}}.$$

Then using adjunction formula recursively, one gets:

$$-K_{Z_\ell} = \sum_{i=1}^r Z_{\ell(i)} + \sum_{i=1}^r H_i = \sum_{i=1}^r Z_{\ell(i)} + f_\ell^*(-K_X/2). \quad \blacksquare$$

Let us also define for every $t \leq r$, the following line bundles on Z_ℓ :

$$(13) \quad N_t = \sum_{i \leq t} H_i.$$

and the classes $\gamma_i \in N_1(Z_\ell)$, $i = 1, \dots, r$, defined by $N_t \cdot \gamma_i = \delta_i^t$.

Since, by definition, $N_t \cdot \beta_i = 1$ if $l_t = l_i$, $i \leq t$, and it is zero otherwise, the intersection matrix $(N_t \cdot \beta_i)$ is lower triangular and its diagonal entries are ones.

The classes γ_i might also be defined as follows: for every index $i \leq r$, satisfying that the set $\{k \in \mathbb{Z} \mid i < k \leq r, l_i = l_k\} \neq \emptyset$ denote by i^* its maximum. Then, according to Remark 3.4, the class $\beta_{i(r-i^*)}$ decomposes in $N_1(Z_{\ell[r-i^*]})$ as $\beta_{i^*(r-i^*)} + \beta'_{i(r-i^*)}$. Defining β'_i as the image of $\beta'_{i(r-i^*)}$ into $N_1(Z_\ell)$, via push forward with the sections $\sigma_{\ell[r-j]}$, $j = i^*, \dots, 1$, we may then write:

$$\gamma_i = \begin{cases} \beta_i & \text{if } \{k \in \mathbb{Z} \mid i < k \leq r, l_i = l_k\} = \emptyset \\ \beta'_i & \text{otherwise} \end{cases}$$

3.2. The Mori cone of the Bott-Samelson varieties. In this section we will show that the Nef and Mori cones of the Bott-Samelson varieties Z_ℓ are simplicial, describing their generators. Moreover we will describe the relative cone of the maps $f_\ell : Z_\ell \rightarrow X$.

LEMMA 3.8. *The divisor N_t is nef on Z_ℓ , for every $t \leq r$.*

PROOF. Set $k = l_t$ and consider the contraction $\pi^k : X \rightarrow X^k$ associated with the $(n-1)$ -dimensional face of $\overline{\text{NE}}(X)$ generated by the rays R_i , $i \neq k$, whose image X^k has Picard number one. Denote by $p : Z_\ell \rightarrow Z_{\ell[r-t]}$ the composition of the projections $p_{\ell[1]}, \dots, p_{\ell[r-t]}$.

By construction, $\varphi_t := \pi^k \circ f_{\ell[r-t]} \circ p$ contracts every curve whose class is β_j such that $l_j \neq k$ and every curve whose class is β_j with $l_j = k$ and $j > t$. Moreover, it contracts every curve whose class is β'_j , since either these curves are contracted by p if $j > t$ or they are contracted by $f_{\ell[r-t]} \circ p$. On the other hand, the image of β_t is $\pi^k(\Gamma_k)$, which is not a point.

We conclude that the kernel of induced linear map $\varphi_{t*} : N_1(Z_\ell) \rightarrow N_1(X^k)$ contains the linear space generated by the classes γ_i , $i \neq t$, which is a hyperplane. This hyperplane meets $\overline{\text{NE}}(Z_\ell)$ along the extremal face defined by φ_t . Since N_t vanishes on this hyperplane and it has positive intersection with $\gamma_t = \beta_t$, it follows that it is a supporting divisor of φ_t and, in particular, it is nef. \blacksquare

COROLLARY 3.9. *The Mori cone (respectively, the nef cone) of Z_ℓ is the simplicial cone generated by the classes γ_t (resp. N_t), $t \leq r$.*

PROOF. Let \mathcal{C} (respectively, \mathcal{K}) be the polyhedral cone generated by the γ_t 's (resp. by the N_t 's). Note that, by definition of the γ_t 's, these two cones are dual. Moreover, the cone \mathcal{C} is contained in $\overline{\text{NE}}(Z_\ell)$ and, by Lemma 3.8, \mathcal{K} is contained in $\text{Nef}(Z_\ell)$. Summing up we have:

$$\mathcal{K} \subseteq \text{Nef}(Z_\ell) = (\overline{\text{NE}}(Z_\ell))^\vee \subseteq \mathcal{C}^\vee = \mathcal{K}.$$

This concludes the proof. \blacksquare

COROLLARY 3.10. *Set $J = \{i \mid l_i = l_k \text{ for some } k > i\}$; then the Stein factorization of the map $f_\ell : Z_\ell \rightarrow X$ is the contraction of Z_ℓ associated to the extremal face of $\overline{\text{NE}}(Z_\ell)$ generated by $\{\gamma_i \mid i \in J\}$.*

PROOF. By Lemma 3.7 and equation (13) $f_\ell^*(-K_X/2) = \sum_{i \notin J} N_i$, and this nef divisor vanishes on the face spanned by $\{\gamma_i \mid i \in J\}$. \blacksquare

We will denote by $\overline{Z}_\ell, \overline{f}_\ell$, etc., the Bott-Samelson varieties and the corresponding morphisms, associated with the homogeneous model G/B , for every list ℓ . We will consider also the linear isomorphism

$$\psi_\ell : N^1(Z_\ell) \rightarrow N^1(\overline{Z}_\ell)$$

defined by $\psi_\ell(Z_{\ell(i)}) = \overline{Z}_{\ell(i)}$ for all i . By construction, its transposed map $\psi_\ell^t : N_1(\overline{Z}_\ell) \rightarrow N_1(Z_\ell)$, defined by the equality $\psi_\ell^t(C) \cdot D = C \cdot \psi_\ell(D)$ for all $C \in N_1(\overline{Z}_\ell)$, $D \in N^1(Z_\ell)$, satisfies $\psi_\ell^t(\overline{\beta}_i) = \beta_i$, hence we have the following

COROLLARY 3.11. *With the same notation as above, for every list ℓ , the map ψ_ℓ sends $\text{Nef}(Z_\ell)$ to $\text{Nef}(\overline{Z}_\ell)$, K_{Z_ℓ} to $K_{\overline{Z}_\ell}$, and $f_\ell^*(K_X)$ to $\overline{f}_\ell^*(K_{G/B})$.*

PROOF. The first part follows from Corollary 3.9. The second follows from the fact that the numerical classes of K_{Z_ℓ} and $f_\ell^*(K_X)$ are determined by their intersections with the classes β_i , and these intersection numbers are the same for X and its homogeneous model. \blacksquare

3.3. Cohomology of line bundles on Bott-Samelson varieties. In this section we will describe some properties of cohomology of line bundles on Bott-Samelson varieties, that will be the key tool to prove the uniqueness of these varieties in Section 4. Moreover, we will use them here to prove that the evaluation morphisms of Bott-Samelson varieties corresponding to a reduced expression of the longest element in W are birational.

REMARK 3.12. Note that, for any sequence $\ell = (l_1, \dots, l_r)$ and every line bundle L on X , setting $s = L \cdot \Gamma_{l_r}$, the following equalities are obvious:

$$(14) \quad f_\ell^* L = sH_r + p_{\ell[1]}^* f_{\ell[1]}^* L,$$

$$(15) \quad p_{\ell[1]*} f_\ell^* L = p_{\ell[1]*} sH_r \otimes f_{\ell[1]}^* L,$$

LEMMA 3.13. *Let Z_ℓ be the Bott-Samelson variety of X defined by the sequence $\ell = (l_1, \dots, l_r)$, and let L be a line bundle on X of degree s with respect to Γ_{l_r} . Then:*

- (1) *If $s \geq 0$ then $H^i(Z_\ell, f_\ell^* L) = H^i(Z_{\ell[1]}, p_{\ell[1]*} f_\ell^* L)$ for every i and the rank $s + 1$ bundle $p_{\ell[1]*} f_\ell^* L$ has a filtration on $\overline{Z}_{\ell[1]}$ with factors*

$$f_{\ell[1]}^*(L + tK_{l_r}), \quad t = 0, \dots, s.$$

- (2) If $s = -1$ then $H^i(Z_\ell, f_\ell^* L) = 0$ for every i ;
(3) If $s \leq -2$ then $H^i(Z_\ell, f_\ell^* L) = H^{i-1}(Z_{\ell[1]}, p_{\ell[1]*} f_\ell^*(L + (s+1)K_{l_r}))$ for every i and the rank $-(s+1)$ bundle $p_{\ell[1]*} f_\ell^*(L + (s+1)K_{l_r})$ has a filtration on $Z_{\ell[1]}$ with factors

$$f_{\ell[1]}^*(L + tK_{l_r}), \quad t = s+1, \dots, -1.$$

PROOF. The first part of (1) follows from the fact that the hypothesis $s \geq 0$ implies that $R^i p_{\ell[1]*} f_\ell^* L = 0$ for all $i > 0$. For the second part, note that the statement is clear for $s = 0$. In the case $s \geq 1$ we consider first the exact sequence on $Z_{\ell[1]}$:

$$0 \longrightarrow f_{\ell[1]}^* K_{l_r} \longrightarrow p_{\ell[1]*} H_r \longrightarrow \mathcal{O}_{Z_{\ell[1]}} \longrightarrow 0.$$

By [8, II.5.16.(c)], there exists a filtration of the rank $(s+1)$ vector bundle $p_{\ell[1]*} sH_r$ whose factors are multiples of $f_{\ell[1]}^* K_{l_r}$. Twisting this filtration with $f_{\ell[1]}^* L$ we obtain, by Remark 3.12, a filtration of $p_{\ell[1]*} f_\ell^* L$, whose factors are as stated.

Statement (2) is a direct consequence of Lemma 2.3, while (3) follows from the same Lemma and case (1). \blacksquare

For nef line bundles on X we may state the following vanishing theorem, consequence of the Kodaira-Kawamata-Viehweg Theorem:

LEMMA 3.14. *With the same notation as above, let D be a nef divisor on Z_ℓ . Then $H^i(Z_\ell, f_\ell^*(K_X/2) + D) = 0$ for $i > 0$. In particular $H^i(Z_\ell, L) = 0$ for every nef line bundle L on Z_ℓ and every $i > 0$.*

PROOF. Following [24, Lemma 6.1], one can show that there exist rational numbers $0 < \varepsilon_j \leq 1$ such that the \mathbb{Q} -divisor $\sum_{j=1}^r \varepsilon_j Z_{\ell(j)}$ is ample on Z_ℓ , hence the \mathbb{Q} -divisor $M = D + \sum_{j=1}^r \varepsilon_j Z_{\ell(j)}$ is ample, too. Set $a_j := 1 - \varepsilon_j$. Then, by Lemma 3.7 we may write $f_\ell^*(K_X/2) + D \equiv K_{Z_\ell} + M + \sum a_j Z_{\ell(j)}$ and the vanishing follows from [19, Theorem 2.64]. \blacksquare

Later on we would use the following stronger vanishing result, which in the rational homogeneous case is a special version of Kumar's vanishing [20, Theorem 8.1.8]:

LEMMA 3.15. *With the same notation as above, being $\ell = (l_1, \dots, l_r)$, then $H^i(Z_\ell, L - Z_{\ell(r)}) = 0$ for every nef line bundle L on Z_ℓ and every $i > 0$.*

PROOF. This proof has been taken from [21, Theorem 16]. Note first that there exist positive rational numbers $\varepsilon_1, \dots, \varepsilon_{r-1}$ such that $\sum_{j=1}^{r-1} \varepsilon_j Z_{\ell(j)}$ is the pull-back of an ample \mathbb{Q} -divisor in $Z_{\ell[1]}$. Hence, since $f_\ell^*(K_X/2)$ is nef and has degree one on γ_r , it follows that $A := \sum_{j=1}^{r-1} \varepsilon_j Z_{\ell(j)} - f_\ell^*(K_X/2)$ is an ample divisor in Z_ℓ .

Then, by Lemma 3.7, we may write:

$$L - Z_{\ell(r)} = K_{Z_\ell} + \sum_{j=1}^{r-1} Z_{\ell(j)} - f_\ell^*(K_X/2) + L,$$

and, setting $E := 1 + \sum_{j=1}^{r-1} \varepsilon_j \in \mathbb{Q}$, and $\Delta := \sum_{j=1}^{r-1} (1 - \frac{\varepsilon_j}{E}) Z_{\ell(j)}$, we have:

$$L - Z_{\ell(r)} = K_{Z_\ell} + \Delta + L + \sum_{j=1}^{r-1} \left(\frac{\varepsilon_j}{E} \right) Z_{\ell(j)} - f_\ell^*(K_X/2).$$

Since $L + \sum_{j=1}^{r-1} \left(\frac{\varepsilon_j}{E}\right) Z_{\ell(j)} - f_\ell^*(K_X/2) = L + \frac{1}{E}(A - (E-1)f_\ell^*(K_X/2))$ is ample and $0 < 1 - \frac{\varepsilon_j}{E} \leq 1$ for every j , the proof is concluded by [19, Theorem 2.64]. ■

Let $\mathbb{Z}[\text{Pic}(X)]$ be the group algebra of $\text{Pic}(X)$, in which, following [7] we will denote by e^L the element of $\mathbb{Z}[\text{Pic}(X)]$ corresponding to L under the canonical inclusion. For any reflection $r_i \in W$ let us define the *Demazure operator*

$$D_i : \mathbb{Z}[\text{Pic}(X)] \rightarrow \mathbb{Z}[\text{Pic}(X)]$$

by setting, for any L in $\text{Pic}(X)$

$$D_i(e^L) = \begin{cases} e^L + e^{L+K_i} + \dots + e^{r_i(L)} & \text{if } L \cdot \Gamma_i \geq 0 \\ 0 & \text{if } L \cdot \Gamma_i = -1 \\ -e^{L-K_i} - e^{L-2K_i} - \dots - e^{r_i(L)+K_i} & \text{if } L \cdot \Gamma_i \leq -2 \end{cases}$$

and extending it linearly. Note that, by definition, $D_i(e^L) = -D_i(e^{r_i(L)+K_i})$ for all $L \in \text{Pic}(X)$. Given a list $\ell = (l_1, \dots, l_r)$, $l_s \in I$ we define $D_\ell(e^L) = D_{l_1}(\dots(D_{l_r}(e^L)))$.

NOTATION 3.16. Let us set, for every element $\sum_j n_j e^{L_j} \in \mathbb{Z}[\text{Pic}(X)]$:

$$\begin{aligned} H^i\left(Z_\ell, \sum_j n_j e^{L_j}\right) &:= \bigoplus_j n_j H^i(Z_\ell, f_\ell^* L_j), \quad i \in \mathbb{Z}, \\ h^i\left(Z_\ell, \sum_j n_j e^{L_j}\right) &:= \dim H^i\left(Z_\ell, \sum_j n_j e^{L_j}\right), \quad i \in \mathbb{Z}, \\ \chi\left(Z_\ell, \sum_j n_j e^{L_j}\right) &:= \sum_j n_j \chi(Z_\ell, f_\ell^* L_j), \\ \deg\left(\sum_j n_j e^{L_j}\right) &:= \sum_j a_j n_j. \end{aligned}$$

We then have that Lemma 3.13 yields

PROPOSITION 3.17. *Let L be a line bundle on X . Then*

$$\chi(Z_\ell, e^L) = \deg(D_\ell(e^L)) = \chi(\bar{Z}_\ell, \bar{e}^{\psi(L)}).$$

If moreover L is nef, then

$$h^0(Z_\ell, e^L) = h^0(\bar{Z}_\ell, \bar{e}^{\psi(L)});$$

in particular $f_\ell^ L$ is nef and big if and only if $\bar{f}_\ell^* \psi(L)$ is nef and big.*

PROOF. By Lemma 3.13, for any line bundle $L \in \text{Pic}(X)$, our previous definitions allow us to write:

$$\chi(Z_\ell, e^L) = \chi(Z_\ell, f_\ell^* L) = \chi(Z_{\ell[1]}, D_{l_r}(e^L)),$$

so that, recursively, $\chi(Z_\ell, e^L) = \chi(Z_{\ell[r]}, D_\ell(e^L))$ and, since $\chi(Z_{\ell[r]}, f_{\ell[r]}^* L') = 1$ for every $L' \in \text{Pic}(X)$, we obtain the first equality. Since $\deg(D_\ell(e^L))$ depends only on intersection numbers with the curves Γ_i , the second equality follows. If L is nef the higher cohomology of $f_\ell^* L$ and $\bar{f}_\ell^* \psi(L)$ vanishes by Lemma 3.14. Finally the last statement follows from [23, Lemma 2.2.3]. ■

COROLLARY 3.18. *Let $\ell = (l_1, \dots, l_r)$ be a sequence. Then $\dim f_\ell(Z_\ell) = \dim Z_\ell$ if and only if $w(\ell)$ is reduced.*

PROOF. Let us consider the homogeneous model G/B of X and the corresponding Bott-Samelson variety \overline{Z}_ℓ , with evaluation $\overline{f}_\ell : \overline{Z}_\ell \rightarrow G/B$. It is known that the property holds for \overline{f}_ℓ , since $\overline{f}_\ell(\overline{Z}_\ell)$ is the Schubert variety $\overline{Bw(\ell)B}/B$ of G/B , whose dimension is $\lambda(w(\ell))$ so that, in particular, the divisor $\overline{f}_\ell^*(-K_{G/B})$ is big if and only if $w(\ell)$ is reduced. Since $-K_{G/B} = \psi(-K_X)$ the result follows from Proposition 3.17. \blacksquare

Consider a list $\ell = (l_1, \dots, l_r)$ satisfying that $r_{l_1} \circ \dots \circ r_{l_r} = w(\ell)$ is a reduced expression for the *longest element* of W . This means, by definition, that $w(\ell)$ is the unique element w_0 of maximal length in W (see [11, Sect. 1.8]). The next proposition shows that in this case the corresponding Bott-Samelson variety Z_ℓ is birational to X .

COROLLARY 3.19. *Let $\ell = (l_1, \dots, l_r)$ be a sequence such that $w(\ell)$ is a reduced expression of w_0 . Then the morphism $f_\ell : Z_\ell \rightarrow X$ is surjective and birational.*

PROOF. The surjectivity of f_ℓ follows from Proposition 2.23 and Corollary 3.18. Let L be an ample line bundle on X . By [3, Lemma 3.3.3 (b)] it is enough to show that, for $s \gg 0$, the restriction map $H^0(X, sL) \rightarrow H^0(Z_\ell, e^{sL})$, which is an injection by the surjectivity of f_ℓ , is an isomorphism.

On one hand, by Corollary 2.25 we have $h^0(X, sL) = h^0(G/B, \psi(sL))$. On the other, since \overline{f}_ℓ is birational (Cf. [7, Proposition 2]), it follows that $h^0(G/B, \psi(sL)) = h^0(\overline{Z}_\ell, \overline{e}^{\psi(sL)})$. The proof is then finished by Proposition 3.17. \blacksquare

We finish this section by noting that the birationality of a dominant evaluation map f_ℓ allows us to reduce Theorem 1.2 to the case in which the diagram \mathcal{D} is connected.

COROLLARY 3.20. *Assume that $\mathcal{D} = \mathcal{D}_1 \sqcup \mathcal{D}_2$. Then $X \simeq X_1 \times X_2$, where X_1 and X_2 are Fano manifolds whose elementary contractions are smooth \mathbb{P}^1 -fibrations and whose Dynkin diagrams are \mathcal{D}_1 and \mathcal{D}_2 , respectively.*

PROOF. As in Lemma 3.2 we consider two sequences ℓ_1 and ℓ_2 on the indices $I_1, I_2 \subset I$ determined by \mathcal{D}_1 and \mathcal{D}_2 , so that, $Z_{\ell_1 \ell_2} \simeq Z_{\ell_1} \times Z_{\ell_2}$. We will assume that they provide reduced expressions for the longest elements of the Weyl groups of \mathcal{D}_1 and \mathcal{D}_2 , respectively, so that the evaluation $f_{\ell_1 \ell_2} : Z_{\ell_1 \ell_2} \rightarrow X$ is birational.

In other words, $f_{\ell_1 \ell_2}$ is the contraction determined by the extremal face of $\overline{\text{NE}}(Z_{\ell_1 \ell_2})$ generated by $R := \{\gamma_i \mid l_i = l_k \text{ for some } k > i\}$ (cf. Corollary 3.10). This face is the convex hull of the two extremal faces generated by the sets $\{\gamma_i \in R \mid i \in I_1\}$, $\{\gamma_i \in R \mid i \in I_2\}$, that provide contractions of $h_{\ell_1} : Z_{\ell_1} \rightarrow X_1$ and $h_{\ell_2} : Z_{\ell_2} \rightarrow X_2$, satisfying that $X \simeq X_1 \times X_2$ and that, up to this isomorphism $f_{\ell_1 \ell_2} = h_{\ell_1} \times h_{\ell_2}$. In particular, X_1 and X_2 are smooth Fano manifolds.

Finally, for every elementary contraction $f : X_1 \rightarrow X'_1$, its pullback $X \simeq X_1 \times X_2 \rightarrow X'_1 \times X_2$ is an elementary contraction of X , hence it follows that f is a smooth \mathbb{P}^1 -fibration. The same holds for X_2 . \blacksquare

4. Homogeneity - General case

With the same notation as in the previous sections, we will further assume here (cf. Corollary 3.20) that the Dynkin diagram \mathcal{D} of X is connected, hence of type $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4$ or G_2 . We start by noting that the case G_2 follows from previous results:

LEMMA 4.1. *Let X be a Fano manifold whose elementary contractions are smooth \mathbb{P}^1 -fibrations. If its Dynkin diagram is of type G_2 , then X is isomorphic to its homogeneous model G/B .*

PROOF. Since the Cartan matrix of G_2 has determinant one, there exist two Cartier divisors L_1 and L_2 on X satisfying that $L_i \cdot \Gamma_j = \delta_i^j$. In particular the elementary contractions of X are \mathbb{P}^1 -bundles, and then the statement follows from [32, Theorem 1.1]. ■

We may then assume that \mathcal{D} is different from G_2 . Our strategy will be to compare first the Bott-Samelson varieties Z_ℓ of X with the Bott-Samelson varieties \overline{Z}_ℓ of its homogeneous model G/B . In fact, we will show (cf. Propositions 4.6 and 4.8) that, for a suitable choice of a sequence $\ell = (l_1, \dots, l_r)$ corresponding to a reduced expression of the longest element in W , we have

$$Z_{\ell[s]} \simeq \overline{Z}_{\ell[s]} \text{ for every } s = 0, \dots, r-1.$$

In order to describe explicitly the sequences ℓ we are going to use, let us introduce the following notation:

$$u_i = (1, 2, \dots, i), \quad d_i = (i, i-1, \dots, 1), \text{ for every } i = 1, \dots, n.$$

Ordering the nodes of \mathcal{D} as in [10, page 58], the table below lists the sequences we are going to use in each case. Whereas in the case of a Dynkin diagram without multiple edges we show our uniqueness result for any sequence ℓ , in the cases B_n and C_n , ℓ was chosen conveniently so that the uniqueness property holds. In the case of F_4 we have checked (using the software system Sage) that our proof, based on cohomological computations, does not work for any of the 2144892 possible sequences providing a reduced expression of the longest word. This is the reason why the proof of Theorem 1.2 in case $\mathcal{D} = F_4$ is different; we will present it in Section 5.

\mathcal{D}	A_n, D_n, E_n	B_n	C_n	F_4
ℓ	any	$(u_n)^n$	$(d_n)^n$	none

4.1. Uniqueness of Bott-Samelson varieties. Throughout the section we will use the notation for cohomology of line bundles introduced in 3.16. Let us start by stating a corollary of Lemma 3.14, which easily follows from the fact that $K_X/2 \cdot \Gamma_j = -1$ for every $j = 1, \dots, n$.

COROLLARY 4.2. *Let Z_ℓ be the Bott-Samelson variety of X defined by the sequence $\ell = (l_1, \dots, l_r)$, and L be a line bundle on X satisfying that $L \cdot \Gamma_{l_j} \geq -1$ for every $j = 1, \dots, r$. Then $H^i(Z_\ell, e^L) = 0$ for $i > 0$. In particular, if the node j and the node j' are connected then $H^i(Z_\ell, e^{K_j + K_{j'}}) = 0$ for $i > 0$.*

PROOF. It is enough to note that if L satisfies the required property, then, by Corollary 3.9, $L = f_\ell^*(L - K_X/2)$ is nef on Z_ℓ . Then the result follows by Lemma 3.14. ■

The following result gives a set of descent rules for cohomology of line bundles on Bott-Samelson varieties, which are consequences of Lemma 3.13:

LEMMA 4.3. *Let $\ell = (l_1, \dots, l_r)$ be the sequence defining the Bott-Samelson variety Z_ℓ , and let L be a line bundle on X , of degree s with respect to Γ_{l_r} . Then:*

- (DR1) If $s = 0$, then $H^i(Z_\ell, e^L) = H^i(Z_{\ell[1]}, e^L)$ for all i .
- (DR2) If $s = -1$, then $H^i(Z_\ell, e^L) = 0$ for all i .
- (DR3) If $s = -2$, then $H^i(Z_\ell, e^L) = H^{i-1}(Z_{\ell[1]}, e^{L-K_{l_r}})$, for all i .
- (DR4) If $s \geq 1$, then $h^i(Z_\ell, e^L) \leq h^i(Z_{\ell[1]}, D_{l_r}(e^L))$ for all i .
- (DR5) If $s = 1$ and $(L + K_{l_r}) \cdot \Gamma_{l_j} \geq -1$ for every $j = 1, \dots, r-1$, then $H^i(Z_\ell, e^L) = H^i(Z_{\ell[1]}, e^L)$ for all $i > 0$. In particular, this holds for $L = K_j$, whenever $K_j \cdot \Gamma_{l_r} = 1$.

PROOF. Items (DR1) to (DR4) follow directly from Lemma 3.13 and, eventually, the corresponding long exact sequences of cohomology associated to the filtrations provided there. For (DR5) one needs to use also Corollary 4.2. ■

In particular, we get the following:

PROPOSITION 4.4. *Let $\ell = (l_1, \dots, l_r)$ be a sequence in I , and $j \in I$, $k > 0$ be two integers such that $K_j \cdot \Gamma_{l_i} = 0, 1$ for every $i > r - k$; then $h^1(Z_\ell, e^{K_j}) = h^1(Z_{\ell[k]}, e^{K_j})$.*

PROOF. If $k = 1$, then the result follows by (DR1) if $K_j \cdot \Gamma_{l_r} = 0$ and by (DR5) if $K_j \cdot \Gamma_{l_r} = 1$. The general statement follows by recursion. ■

COROLLARY 4.5. *Let $\ell = (l_1, \dots, l_r)$ be a list in I satisfying that $K_{l_r} \cdot \Gamma_{l_i} \in \{0, 1, -2\}$ for every $i < r$, and set $J := \{i < r \mid l_i = l_r\}$. Then $H^1(Z_{\ell[1]}, e^{K_{l_r}})$ is generated by ζ_ℓ , and this cocycle is zero if and only if $J = \emptyset$.*

PROOF. If $J = \emptyset$, then, by Proposition 4.4, $H^1(Z_{\ell[1]}, e^{K_{l_r}})$ is isomorphic to $H^1(Z_{\ell[r-1]}, e^{K_j}) = H^1(\Gamma_{l_1}, K_j|_{\Gamma_{l_1}}) = 0$.

If $J \neq \emptyset$, denote by $r - s$ its maximum. Then Proposition 4.4 tells us that $H^1(Z_\ell, e^{K_j}) \cong H^1(Z_{\ell[s]}, e^{K_{l_r}})$. On the other hand (DR3) provides $h^1(Z_{\ell[s]}, e^{K_{l_r}}) = h^0(Z_{\ell[s+1]}, e^{K_{l_r} - K_{l_r}}) = 1$. Since, by Corollary 3.5, $\zeta_\ell \in H^1(Z_{\ell[1]}, e^{K_{l_r}})$ is nonzero, the claim follows. ■

We conclude from this the following uniqueness result for Bott-Samelson varieties in the case in which \mathcal{D} has no multiple edges:

PROPOSITION 4.6. *Assume that $\mathcal{D} = A_n, D_n$ or E_n . Then, for every sequence $\ell = (l_1, \dots, l_r)$, the manifold Z_ℓ is isomorphic to \bar{Z}_ℓ .*

PROOF. We will use induction on r , noting that the result is obvious for $r = 1$. Then, given ℓ , we may assume that $Z_{\ell[1]} \simeq \bar{Z}_{\ell[1]}$. It is then enough to show that the cocycles $\zeta_\ell, \bar{\zeta}_\ell \in H^1(Z_{\ell[1]}, e^{K_{l_r}})$ defining Z_ℓ and \bar{Z}_ℓ , respectively, are proportional, up to this isomorphism. Since $K_{l_r} \cdot \Gamma_{l_i} \in \{0, 1, -2\}$ for every $i < r$, by the assumptions on \mathcal{D} , this follows by Corollary 4.5. ■

The case in which $\mathcal{D} = B_n$ or C_n is more involved. Let us recall that in this situation, with the standard ordering (cf. [10, page 58]), $K_i \cdot \Gamma_j \in \{0, 1, -2\}$ unless $(i, j) = (n-1, n)$, in the case B_n , or $(i, j) = (n, n-1)$, in the case C_n . Hence, in order to obtain a uniqueness result, we will need to control cohomology groups of the type $H^1(Z_\ell, e^{K_{n-1}})$ and $H^1(Z_\ell, e^{K_n})$, for B_n and C_n , respectively, and some particular choices of ℓ . The precise statement we are going to prove is the following:

LEMMA 4.7. *Assume that $\mathcal{D} = B_n$ (resp. C_n) and, given $k \in \{1, \dots, n-1\}$, consider the sequence $\ell = (u_n)^k$ (resp. $\ell = (d_n)^k$). Then $H^1(Z_\ell, e^{K_{n-1}})$ (resp. $H^1(Z_\ell, e^{K_n})$) is 1-dimensional.*

PROOF. Let us consider first the case $\mathcal{D} = B_n$. By Corollary 3.5, it is enough to prove $h^1(Z_\ell, e^{K_{n-1}}) \leq 1$. We start by noting that $h^1(Z_{\ell[1]}, e^{K_{n-1}}) = 1$ by (DR3), and that $h^1(Z_{\ell[1]}, e^{K_{n-1}+K_n}) = 0$ by Corollary 4.2. Then (DR4) provides:

$$h^1(Z_\ell, e^{K_{n-1}}) \leq 1 + h^1(Z_{\ell[1]}, e^{K_{n-1}+2K_n}) = 1 + h^1(Z_{\ell[2]}, e^{K_{n-1}+2K_n}),$$

where the last equality follows by (DR1), since $(K_{n-1} + 2K_n) \cdot \Gamma_{n-1} = 0$.

Set $D_i := \sum_{j=i}^{n-1} K_j + 2K_n$ for every $i = 1, \dots, n-1$. The proof will be finished by showing that $h^1(Z_{\ell[2]}, e^{D_{n-1}}) = 0$.

Note that, since $D_i \cdot \Gamma_{i-1} = 1$ and $D_i \cdot \Gamma_j = 0$ for any $i = 2, \dots, n-1$ and any $j < i-1$, (DR4) and (DR1) provide

$$\begin{aligned} h^1(Z_{\ell[n-i+1]}, e^{D_i}) &\leq h^1(Z_{\ell[n-i+2]}, e^{D_i} + e^{D_{i-1}}) \\ &\leq h^1(Z_{\ell[n]}, e^{D_i}) + h^1(Z_{\ell[n-i+2]}, e^{D_{i-1}}), \end{aligned}$$

for every $i = 2, \dots, n-1$. Hence, recursively we obtain:

$$h^1(Z_{\ell[2]}, e^{D_{n-1}}) \leq \sum_{i=1}^{n-1} h^1(Z_{\ell[n]}, e^{D_i}).$$

Note that $\ell[n] = (u_n)^{k-1}$, hence the right hand side of this inequality is obviously zero if $k = 1$. In the case $k > 1$, since $D_i \cdot \Gamma_n = -2$ for every i , (DR3) provides:

$$h^1(Z_{\ell[n]}, e^{D_i}) = h^0(Z_{\ell[n+1]}, e^{D_i - K_n}).$$

Since $(D_i - K_n) \cdot \Gamma_j = 0$ for $j = i+1, \dots, n-1$, by (DR1) we have

$$h^0(Z_{\ell[n+1]}, e^{D_i - K_n}) = h^0(Z_{\ell[2n-i]}, e^{D_i - K_n}),$$

and this group is zero by (DR2).

The case $\mathcal{D} = C_n$ follows by a similar argument, in which we will need to consider the line bundles $D'_i := \sum_{j=i}^{n-1} 2K_j + K_n$, for $i = 1, \dots, n-1$. Note that $D'_i \cdot \Gamma_j$ equals -2 if $j = i$, 2 if $j = i-1$ and zero otherwise. In this situation (DR1), (DR4) and (DR3), together with Corollary 4.2, provide:

$$\begin{aligned} h^1(Z_\ell, e^{K_n}) &= h^1(Z_{\ell[n-2]}, e^{K_n}) \leq h^1(Z_{\ell[n-1]}, e^{K_n} + e^{K_{n-1}+K_n} + e^{D'_{n-1}}) \\ &\leq 1 + h^1(Z_{\ell[n-1]}, e^{D'_{n-1}}), \end{aligned}$$

and we may conclude by showing that $h^1(Z_{\ell[n-1]}, e^{D'_{n-1}}) = 0$. We claim first that

$$(16) \quad h^1(Z_{\ell[n-1]}, e^{D'_{n-1}}) \leq h^1(Z_{\ell[i(n-1)]}, e^{D'_{n-i}}), \text{ for all } i = 2, \dots, k.$$

We may write, by (DR1), (DR4) and Corollary 4.2:

$$\begin{aligned} h^1(Z_{\ell[(i-1)(n-1)]}, e^{D'_{n-i+1}}) &= h^1(Z_{\ell[i(n-1)-1]}, e^{D'_{n-i+1}}) \\ &\leq h^1(Z_{\ell[i(n-1)]}, e^{D'_{n-i+1}} + e^{D'_{n-i+1}+2K_{n-1}}) \end{aligned}$$

Since, by (DR3), $h^1(Z_{\ell[i(n-1)]}, e^{D'_{n-i+1}}) = h^0(Z_{\ell[i(n-1)+1]}, e^{D'_{n-i+1}-K_{n-i+1}})$, and this is zero by (DR2), the claim follows.

Now, using the inequality (16), together with (DR1) we may finally write:

$$h^1(Z_{\ell[n-1]}, e^{D'_{n-1}}) \leq h^1(Z_{\ell[k(n-1)]}, e^{D'_{n-k}}) = h^1(Z_{\ell[kn]}, e^{D'_{n-k}}),$$

which is trivially equal to zero. This finishes the proof. \blacksquare

We may now state the following uniqueness result:

PROPOSITION 4.8. *Assume that $\mathcal{D} = B_n$ (resp. C_n) and consider the sequence $\ell = (u_n)^n$ (resp. $(d_n)^n$). Then, for every i , the group $H^1(Z_{\ell[i]}, e^{K_n})$ is generated by $\zeta_{\ell[i-1]}$, and this cocycle is zero unless $i < n(n-1)$. In particular, $Z_{\ell[i]} \simeq \overline{Z}_{\ell[i]}$ for every i .*

PROOF. We will show the proof for $\mathcal{D} = B_n$, being the case $\mathcal{D} = C_n$ analogous. We proceed by induction on i . Let us assume that $Z_{\ell[i+1]} \simeq \overline{Z}_{\ell[i+1]}$ and consider the cocycles $\zeta_{\ell[i]}, \overline{\zeta}_{\ell[i]} \in H^1(Z_{\ell[i+1]}, e^{K_{l_{n^2-i}}})$. If $l_{n^2-i} \neq n-1$, then the result follows from Corollary 4.5. If $l_{n^2-i} = n-1$, then let k denote the integral part of $(n^2-i)/n$ and $\ell' = (u_n)^k$. If $k = 0$, then $H^1(Z_{\ell[i+1]}, e^{K_{n-1}}) = 0$ by Corollary 4.5. If $k > 0$, Proposition 4.4 and Lemma 4.7 provide:

$$H^1(Z_{\ell[i+1]}, e^{K_{n-1}}) \simeq H^1(Z_{(u_n)^k}, e^{K_{n-1}}) \simeq \mathbb{C}.$$

Since $\zeta_{\ell[i]}$ and $\overline{\zeta}_{\ell[i]}$ are nonzero by Corollary 3.5, they are necessarily proportional, from which we get $Z_{\ell[i]} \simeq \overline{Z}_{\ell[i]}$. \blacksquare

OF THEOREM 1.2 FOR $\mathcal{D} \neq F_4$. The case $\mathcal{D} = G_2$ has been considered in Lemma 4.1. Note also that $\dim X$ is equal to the dimension of its homogeneous model G/B , by Proposition 2.23. In particular, the assumption $\dim(X) \neq 24$ implies that \mathcal{D} is different from F_4 .

For $\mathcal{D} \neq G_2, F_4$ we know, from Propositions 4.6 and 4.8, that there exists a sequence $\ell = (l_1, \dots, l_r)$ satisfying that $w(\ell) = r_{l_1} \circ \dots \circ r_{l_r}$ is a reduced expression of the longest element of W , and such that $Z_\ell = \overline{Z}_\ell$. The two maps $f_\ell : Z_\ell \rightarrow X$ and $\overline{f}_\ell : Z_\ell \rightarrow G/B$ are both birational (cf. Corollary 3.19), and they both have the same Stein factorization, by Corollary 3.10. It follows that $X \simeq G/B$. \blacksquare

5. Manifolds whose Dynkin diagram is F_4

In this section we will prove Theorem 1.2 for the case $\mathcal{D} = F_4$, by following the line of argumentation of [28, Section 6]. More concretely, numbering the nodes of \mathcal{D} as in [10, p. 58], we will consider the contraction $\pi^1 : X \rightarrow X^1$ associated with the three-dimensional face of $\text{NE}(X)$ generated by the rays R_i $i \neq 1$; we will first show that X^1 is isomorphic to the corresponding rational homogeneous space by comparing their varieties of minimal rational tangents (see definition below), and then we will reconstruct inductively the variety X upon X^1 .

Let us briefly recall first some well-known facts on varieties of minimal rational tangents (cf. [13]), and introduce some notation.

NOTATION 5.1. Given a smooth projective variety Y , a family of rational curves on Y is an irreducible component of the scheme $\text{RatCurves}^n(Y)$ (see [18, II.2]). We say that a family \mathcal{H} is *unsplit* if \mathcal{H} is a proper \mathbb{C} -scheme.

For a general point $y \in Y$, let \mathcal{H}_y be the normalization of the family of \mathcal{H} -curves passing through y , and consider the corresponding universal family:

$$\begin{array}{ccc} \mathcal{U}_y & \xrightarrow{\iota_y} & Y \\ \sigma_y \uparrow & & \downarrow \pi_y \\ & & \mathcal{H}_y \end{array}$$

where σ_y is the unique section of π_y such that $\iota_y(\sigma_y(\mathcal{H}_y)) = \{y\}$. Setting $\mathcal{E}_y = (\pi_y)_* \mathcal{O}_{\mathcal{U}_y}(\sigma_y)$ we have that $\mathcal{U}_y \simeq \mathbb{P}(\mathcal{E}_y)$ and $\mathcal{O}_{\mathcal{U}_y}(\sigma_y)$ is the tautological line bundle.

A member of \mathcal{H}_y is called an \mathcal{H}_y -curve. We define the tangent map $\tau_y : \mathcal{H}_y \rightarrow \mathbb{P}(T_y Y^\vee)$ by assigning the tangent vector at y to each \mathcal{H}_y -curve which is smooth at y . We denote by $\mathcal{C}_y \subset \mathbb{P}(T_y Y^\vee)$ the closure of the image of τ_y , which is called the *VMRT (Variety of Minimal Rational Tangents)* at y . By [2, Sequence 2.6], $\tau_y^* \mathcal{O}_{\mathcal{C}_y}(1) = c_1(\mathcal{E}_y)$.

The VMRT's of a Fano manifold are not necessarily smooth. Later on we will need to use the following smoothness criterion.

LEMMA 5.2. *Let Y be a Fano manifold of Picard number one and denote by $\mathcal{O}_Y(1)$ the ample generator of $\text{Pic}(Y)$. Assume that $\mathcal{O}(1)$ is spanned and that the rational curves of a family \mathcal{H} have degree one with respect to $\mathcal{O}_Y(1)$. Then, at a general point $y \in Y$ the variety of minimal rational tangents \mathcal{C}_y of \mathcal{H} is smooth.*

PROOF. Note first that the hypotheses imply that the family \mathcal{H} is unsplit and that all its elements are smooth rational curves in Y . Hence we may apply [16, Theorem 1.3], to get that the tangent map τ_y is injective. According to [13, Proposition 1.4], it is then enough to show that every \mathcal{H}_y -curve C is standard, that is, $T_Y|_C \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus a} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus b}$, for some a, b .

To prove this, let $\psi : Y \rightarrow \mathbb{P}^d$ be a surjective morphism defined by a linear system in $|\mathcal{O}_Y(1)|$, where $d := \dim Y$. The image via ψ of C is a line in \mathbb{P}^d and, by the generality of $y \in Y$, C is not contained in the ramification locus of ψ . In particular, the natural morphism of normal bundles $N_{C/Y} \rightarrow N_{\psi(C)/\mathbb{P}^d} \cong \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d-1}$ is generically surjective. Since moreover, the generality of C also implies that $N_{C/Y}$ is globally generated, it follows that C is standard. ■

NOTATION 5.3. From now on X will be a Fano manifold whose elementary contractions are smooth \mathbb{P}^1 -fibrations, with Dynkin diagram F_4 . Besides the notation introduced in the previous sections of the paper, we will denote by R_J , $J \subset I := \{1, 2, 3, 4\}$, the extremal face of $\text{NE}(X)$ spanned by the rays R_j , $j \in J$. Its associated contraction will be denoted either by $\pi_J : X \rightarrow X_J$, or by $\pi^{J^c} : X \rightarrow X^{J^c}$, where J^c is the complement of J in I . Moreover, we will denote by \mathcal{D}_J the Dynkin diagram obtained from \mathcal{D} by deleting the nodes which are not in J and by W_J the Weyl group of \mathcal{D}_J . Finally, we will denote by \bar{X} the homogeneous model G/B of X , and we will add an overline to the usual notation $(\bar{R}_J, \bar{\pi}_J, \dots)$ to denote the corresponding extremal faces, contractions, etc.

5.1. Smoothness of the contractions of X . In this section we will show that, for every $J \subset I$, the contraction $\pi_J : X \rightarrow X_J$ is smooth, and any fiber of π_J is the complete flag manifold with Dynkin diagram \mathcal{D}_J .

We will follow here the notation and ideas from [18, IV. Section 4]. For each $j = 1, \dots, 4$ the proper prerelation given by $X \xleftarrow{s_j} U_j \xrightarrow{w_j} X$, and $X \xrightarrow{\sigma_j} U_j$, fitting in the diagram:

$$\begin{array}{ccc}
 U_j := X \times_{X_j} X & \xrightarrow{w_j} & X \\
 \sigma_j \uparrow & \nearrow \text{id} & \downarrow \pi_j \\
 X & \xrightarrow{\pi_j} & X_j
 \end{array}$$

The scheme U_j , together with the maps w_j, s_j , is a proper algebraic relation on X .

Moreover, any list $\ell = (l_1, \dots, l_r)$ in I defines an algebraic relation $U_\ell := U_{l_1} * \dots * U_{l_r}$ (see [18, IV. Definition 4.3]) which is proper ([18, IV. Lemma 4.3.1]) and satisfies, by construction, that $(x, y) \in \widetilde{U}_\ell$ if and only if $y \in f_\ell(Z_\ell)$, being Z_ℓ the Bott-Samelson variety determined by the list ℓ and by $Z_{\ell[r]} = \{x\}$.

Finally, for every subset $J \subset I$ we may consider the proalgebraic relation $\text{Chains}(J)$ defined as the countable union of the relations U_ℓ constructed with lists ℓ of indices in J . By [18, IV. Theorem 4.16] there exists a proper fibration $\pi : X^0 \rightarrow Y^0$, defined on an open subset $X^0 \subset X$, whose fibers are $\text{Chains}(J)$ -equivalence classes.

PROPOSITION 5.4. *Let J be any nonempty subset of I . Let $x \in X$ be any point, let $\ell = (l_1, \dots, l_r)$ be a reduced list such that $l_i \in J$ for all i and $w(\ell)$ is the longest word in W_J , and let Z_ℓ be the Bott-Samelson variety associated to ℓ such that $Z_{\ell[r]} = \{x\}$. Then $f_\ell(Z_\ell)$ is set-theoretically equal to $\pi_J^{-1}(\pi_J(x))$.*

PROOF. Since any $\text{Chains}(J)$ -equivalence class is contained in a fiber of π_J , clearly $f_\ell(Z_\ell) \subset \pi_J^{-1}(\pi_J(x))$, hence it is enough to show the opposite inclusion.

With the same notation as above, let H_Y be a general very ample effective divisor on Y such that $H_Y \cap Y_0 \neq \emptyset$. Set $H^0 = \pi^{-1}(H_Y)$ and denote by H the closure of H^0 in X . This divisor is trivial on curves $\Gamma_{j_1}, \dots, \Gamma_{j_k}$, hence on the face R_J . It follows that a general fiber of π_J is a $\text{Chains}(J)$ -class.

But then for some list ℓ' , the morphism $U_{\ell'} \rightarrow X \times_{X_J} X$ is dominant. Since the relation $U_{\ell'}$ is proper, the morphism is surjective and, in particular, $\pi_J^{-1}(\pi_J(x))$ is $\text{Chains}(J)$ -connected.

Finally, let us show that $f_\ell(Z_\ell) = \pi_J^{-1}(\pi_J(x))$. If this is not the case, since $\pi_J^{-1}(\pi_J(x))$ is $\text{Chains}(J)$ -connected, there exists $l_{r+1} \in J$ such that, being $\bar{\ell} = (l_1, \dots, l_r, l_{r+1})$, we have $f_\ell(Z_\ell) \subsetneq f_{\bar{\ell}}(Z_{\bar{\ell}})$. Since these two varieties are irreducible, $\dim(f_\ell(Z_\ell)) < \dim(f_{\bar{\ell}}(Z_{\bar{\ell}}))$, contradicting the reducedness of ℓ by Corollary 3.18. ■

PROPOSITION 5.5. *Let $\ell = (l_1, \dots, l_r)$ be a reduced list in I , and Z_ℓ an associated Bott-Samelson variety. Then the map $H^0(f_\ell(Z_\ell), L|_{f_\ell(Z_\ell)}) \rightarrow H^0(Z_\ell, f_\ell^*L)$ is an isomorphism for any nef line bundle L on X . In particular $f_\ell(Z_\ell)$ is normal and f_ℓ is birational.*

PROOF. Complete the list ℓ to a reduced expression $\bar{\ell}$ of the longest word $w(\bar{\ell}) \in W$. Consider the commutative diagram

$$\begin{array}{ccc} H^0(X, L) & \xrightarrow{\sim} & H^0(Z_{\bar{\ell}}, e^L) \\ \downarrow & & \downarrow \\ H^0(f_{\bar{\ell}[1]}(Z_{\bar{\ell}[1]}), L) & \hookrightarrow & H^0(Z_{\bar{\ell}[1]}, e^L) \\ \downarrow & & \downarrow \\ H^0(f_{\bar{\ell}[2]}(Z_{\bar{\ell}[2]}), L) & \hookrightarrow & H^0(Z_{\bar{\ell}[2]}, e^L) \\ \downarrow & & \downarrow \\ \dots & & \dots \end{array}$$

where the top horizontal map is an isomorphism by the birationality of $f_{\bar{\ell}}$ (Corollary 3.19) and the other horizontal maps are injective by the surjectivity of $f_{\bar{\ell}[k]} : Z_{\bar{\ell}[k]} \rightarrow X_{\bar{\ell}[k]}$. By Lemma 3.15, for any list ℓ' and any nef line bundle L on X we

have the vanishing $h^1(Z_{\ell'}, f_{\ell'}^* L - Z_{\ell'[1]}) = 0$, so the vertical maps on the right are surjective. Then, recursively, the horizontal maps are isomorphisms, and we obtain the normality of $f_{\ell}(Z_{\ell})$ from [3, Lemma 3.3.3]. \blacksquare

PROPOSITION 5.6. *Let $J \subset I$ be a proper nonempty subset. Then every fiber of π_J is the complete flag manifold with Dynkin diagram \mathcal{D}_J . Moreover π_J and X_J are smooth.*

PROOF. Let $x \in X$ be any point. By Proposition 5.4 the fiber of $\pi_J^{-1}(\pi_J(x))$ passing through x equals (set-theoretically) $f_{\ell}(Z_{\ell})$, where Z_{ℓ} is the Bott-Samelson variety constructed starting from x and corresponding to a list $\ell = (l_1, \dots, l_r)$ such that $w(\ell)$ is the longest word in W_J . By Propositions 4.6 and 4.8, ℓ can be chosen so that $Z_{\ell} \simeq \overline{Z}_{\ell}$, where \overline{Z}_{ℓ} is the corresponding Bott-Samelson variety of the homogeneous model $\overline{X} = G/B$ of X . Since $f_{\ell}(Z_{\ell})$ is normal and f_{ℓ} is birational by Proposition 5.5, the map f_{ℓ} is the contraction of Z_{ℓ} corresponding to the extremal face generated by cycles $\{\gamma_i \mid l_i = l_k \text{ for some } k > i\}$ (see Corollary 3.10), hence $f_{\ell}(Z_{\ell}) \simeq f_{\ell}(\overline{Z}_{\ell})$ is the complete flag manifold with Dynkin diagram \mathcal{D}_J .

For the second part, by [31, Lemma 4.13], it is enough to prove that the normal bundle $N_{f_{\ell}(Z_{\ell})/X}$ of $f_{\ell}(Z_{\ell})$ in X is trivial, for any $f_{\ell}(Z_{\ell})$.

In order to see this, we consider its restriction to any curve $\Gamma_j \subset f_{\ell}(Z_{\ell})$, $j \in J$, that is the cokernel of the natural inclusion $N_{\Gamma_j/f_{\ell}(Z_{\ell})} \rightarrow N_{\Gamma_j/X}$. Since these two bundles are trivial, $(N_{f_{\ell}(Z_{\ell})/X})|_{\Gamma_j}$ is trivial as well.

Denote $Z := f_{\ell}(Z_{\ell})$, $Y := \mathbb{P}(N_{f_{\ell}(Z_{\ell})/X})$, and $\pi : Y \rightarrow Z$ the natural projection. Let $\tilde{\Gamma}_j$ denote a minimal section of Y over Γ_j , for all j . By the previous argument, there is precisely one of these sections passing by every point of Y . For each $j \in J$, the family of deformations of $\tilde{\Gamma}_j$ is unsplit and dominates Y , hence we may consider the proper fibration $\varphi : Y^0 \rightarrow P^0$, defined on an open set $Y^0 \subset Y$, whose fibers are equivalence classes with respect to the relation determined by the chains of deformations of the $\tilde{\Gamma}_j$'s, $j \in J$. Let $y \in Y^0$ be a general point, and consider the fiber $\varphi^{-1}(\varphi(y)) \subset Y$, which is a proper closed subset because Y is not rationally connected by curves $\tilde{\Gamma}_j$, since its Picard number is $|J| + 1$.

Let V be the irreducible component of $\varphi^{-1}(\varphi(y))$ passing by y . It is a smooth projective variety satisfying that any curve $\tilde{\Gamma}_j$ meeting it is strictly contained in it. Then the restriction $\pi|_V : V \rightarrow X$ is surjective, and the proof may be finished by showing that it is an isomorphism.

Suppose this is not the case. Since Z is simply connected, then the branch locus $B \subset X$ of $\pi|_V$ is nonempty. Since X is rationally connected by the Γ_j 's, there exists an index $k \in J$ and a curve Γ_k meeting B in a point $b \in B$ and not contained in it. But then the inverse image of a general point $x \in \Gamma_k$ has $\deg(\pi|_V)$ inverse images, providing $\deg(\pi|_V)$ sections of the form $\tilde{\Gamma}_k$, over the curve Γ_k , contained in V . We conclude that b has $\deg(\pi|_V)$ inverse images as well, a contradiction. \blacksquare

5.2. Smoothness of the VMRT's. Let us consider, for any $i = 1, \dots, 4$, the contraction $\pi^i : X \rightarrow X^i$ and the unsplit family of rational curves \mathcal{H}_i in X^i containing all the curves of the form $\pi^i(\Gamma_i)$. In this section we will show the following:

PROPOSITION 5.7. *With the same notation as above, the VMRT of \mathcal{H}_i at a general point of X^i is smooth, for any $i = 1, \dots, 4$.*

By the smoothness of π^i (Proposition 5.6), X^i is a Fano manifold of Picard number one. Then the statement reduces to check that the ample generator of $\text{Pic}(X^i)$, that we denote by L_i , satisfies the hypotheses of Lemma 5.2. We will show this in Lemma 5.8 and Proposition 5.10 below.

Note first that the determinant of the Cartan matrix of F_4 is 1, hence there exist line bundles $\Lambda_i \in \text{Pic}(X)$ such that $\Lambda_i \cdot \Gamma_j = \delta_{ij}$, which allows to state the following:

LEMMA 5.8. *With the same notation as above, $\Lambda_i = (\pi^i)^*L_i$ and, in particular $L_i \cdot \pi_*^i \Gamma_i = 1$, for all i .*

PROOF. Since π^i is smooth by Proposition 5.6, $\pi_*^i \Lambda_i$ is a line bundle, satisfying $(\pi^i)^* \pi_*^i \Lambda_i = \Lambda_i$. Therefore $\pi_*^i \Lambda_i \cdot \pi_*^i \Gamma_i = \Lambda_i \cdot \Gamma_i = 1$, and so $\pi_*^i \Lambda_i$ is necessarily the ample generator of $\text{Pic}(X_i)$. ■

In order to check, finally, the spannedness of L_i we will first show how to construct global sections of Λ_i by means of Bott-Samelson varieties.

LEMMA 5.9. *With the same notation as above, there exists a list $\ell = (l_1, \dots, l_{23})$ in I such that $w(\ell)$ is reduced and $f_\ell(Z_\ell)$ is a divisor in the linear system $|\Lambda_i|$.*

PROOF. Let w_0 be the longest element in the Weyl group of F_4 , and let $\ell = (l_1, \dots, l_{23})$ be a list such that $w(\ell)$ is a reduced expression of $w_0 \circ r_i$. Since, for every $j \neq i$, $\lambda(w_0 \circ r_i \circ r_j) = \lambda(w_0) - 2$ (see [10, 10.3 Lemma A]), then $\ell_j = (l_1, \dots, l_{23}, j)$ satisfies that $w(\ell_j)$ is not reduced and, by Corollary 3.18, $\dim f_{\ell_j}(Z_{\ell_j}) < \dim Z_{\ell_j}$. Hence $f_{\ell_j}(Z_{\ell_j}) = f_\ell(Z_\ell)$ and, in particular, $f_\ell(Z_\ell) \cdot \Gamma_j = 0$.

On the other hand $\ell_i = (l_1, \dots, l_{23}, i)$ gives a reduced expression $w(\ell_i)$ of w_0 , hence f_{ℓ_i} is birational by Corollary 3.19 and $f_\ell(Z_\ell)$ is a unisecant divisor for π_i . ■

PROPOSITION 5.10. *With the same notation as above, the line bundle L_i is spanned by global sections.*

PROOF. Let ℓ be as in Lemma 5.9, and let $\ell' := (l_{23}, \dots, l_1)$. Given any $y \in X^i$, and taking $x \in X$, $y = \pi^i(x)$, we may consider the Bott-Samelson variety $Z_{\ell'}$ starting at $Z_{\ell'[23]} = \{x\}$. Take any point $x' \in X \setminus f_{\ell'}(Z_{\ell'})$ and consider the Bott-Samelson variety Z_ℓ with $Z_{\ell[23]} = \{x'\}$. Then, by construction, $x \notin f_\ell(Z_\ell)$, so $\pi^i(f_\ell(Z_\ell))$ is a divisor in the linear system $|L_i|$ not passing by x . This concludes the proof. ■

5.3. Determination of X^1 . We will prove now that X^1 is isomorphic to its homogeneous model, by showing first that its VMRT at a general point is the appropriate one. We will start by computing some numerical invariants of X^1 :

LEMMA 5.11. *With the same notation as above, X^1 is a Fano manifold of dimension 15 and index 8.*

PROOF. That X^1 is smooth and 15-dimensional follows from Propositions 2.23 and 5.6. The index of X^1 can then be seen as the minimum integer k such that $H^{15}(X^1, -kL_1) \neq 0$. Since

$$H^{15}(X^1, -kL_1) = H^{15}(X, -k\Lambda_1) = H^{15}(\overline{X}, -k\overline{\Lambda}_1) = H^{15}(\overline{X}^1, -k\overline{L}_1),$$

where the equality in the middle follows from Corollary 2.25, we get that the index of X^1 is equal to the index of \overline{X}^1 , which is 8. ■

PROPOSITION 5.12. *With the same notation as above, $X^1 \simeq \overline{X}^1$.*

PROOF. Let us denote by \mathbb{G} the Lagrangian Grassmannian of 3-dimensional subspaces in \mathbb{C}^6 which are isotropic with respect to a fixed symplectic form, i.e. the rational homogeneous space corresponding to the Dynkin diagram C_3 marked on the third node.

By Proposition 5.7, \mathcal{C}_x is smooth, for a general $x \in X^1$. Let $\bar{x} \in X$ be such that $\pi^1(\bar{x}) = x$, and consider the Bott-Samelson variety Z_ℓ , corresponding to the list $(2, 3, 4, 2, 3, 4, 2, 3, 4, 1)$, constructed starting from the point \bar{x} . By Corollary 4.5 and Proposition 4.8, we have $Z_\ell \simeq \mathbb{P}(f_{\ell[1]}^*(K_1) \oplus \mathcal{O}_{Z_{\ell[1]}}) \simeq \mathbb{P}(\bar{f}_{\ell[1]}^*(\bar{K}_1) \oplus \mathcal{O}_{\bar{Z}_{\ell[1]}}) \simeq \bar{Z}_\ell$.

By the universal property of \mathcal{H}_1 , there exists a morphism $h_x : Z_{\ell[1]} \rightarrow (\mathcal{H}_1)_x$ such that $p_{\ell[1]} : Z_\ell \rightarrow Z_{\ell[1]}$ is the pull-back of the universal family $\pi_x : \mathbb{P}(\mathcal{E}_x) \rightarrow (\mathcal{H}_1)_x$. Since h_x contracts curves on which $\mathcal{O}_{Z_{\ell[1]}} \oplus f_{\ell[1]}^*K_1$ is trivial, it factors through $\psi_1 := \pi^2 \circ f_{\ell[1]}$, and we have a commutative diagram:

$$\begin{array}{ccccccc}
 Z_\ell = \mathbb{P}(\mathcal{O}_{Z_{\ell[1]}} \oplus f_{\ell[1]}^*K_1) & \longrightarrow & \mathbb{P}(\mathcal{O}_{\mathbb{G}} \oplus \mathcal{O}_{\mathbb{G}}(1)) & \longrightarrow & \mathbb{P}(\mathcal{E}_x) & \xrightarrow{\iota} & X^1 \\
 \downarrow p_{\ell[1]} & & \downarrow & & \downarrow \pi_x & & \\
 Z_{\ell[1]} & \xrightarrow{\psi_1} & \mathbb{G} & \xrightarrow{\psi_2} & (\mathcal{H}_1)_x & &
 \end{array}$$

Since \mathbb{G} has dimension six and Picard number one, and $\dim \mathcal{H}_x = -K_{X^1} \cdot \pi_*^{-1}\Gamma_1 - 2 = 6$, the morphism ψ_2 , which is non constant –otherwise $f_{\ell[1]}^*K_1$ would be trivial–, is surjective, hence the composition $\varphi := \tau_x \circ \psi_2 : \mathbb{G} \rightarrow \mathcal{C}_x$ is a surjective morphism such that $\varphi^*\mathcal{O}_{\mathcal{C}_x}(1) = c_1(\psi_2^*\mathcal{E}_x) = \mathcal{O}_{\mathbb{G}}(1)$.

By [22, Main Theorem], φ is an isomorphism unless \mathcal{C}_x is a projective space. In the latter case, since $\varphi^*\mathcal{O}_{\mathcal{C}_x}(1) = \mathcal{O}_{\mathbb{G}}(1)$, \mathcal{C}_x would be a linear space, so, by [14, Proposition 5], X^1 would be the projective space, contradicting that X^1 has dimension 15 and index 8.

So we have proved that $\varphi : \mathbb{G} \rightarrow \mathcal{C}_x$ is an isomorphism, given by a linear projection of $\phi(\mathbb{G})$, where ϕ is the embedding given by the complete linear system $|\mathcal{O}_{\mathbb{G}}(1)|$. On the other hand, by [34, III. 1.4], $\phi(\mathbb{G})$ cannot be projected isomorphically. This implies that \mathcal{C}_x is projectively equivalent to \mathbb{G} embedded by the complete linear system $|\mathcal{O}_{\mathbb{G}}(1)|$. We can thus apply [9, Main Theorem] (which is a generalization of [27, Main Theorem]) to get that $X^1 \simeq \overline{X}^1$. \blacksquare

5.4. Reconstructing the complete flag. In order to prove that $X \simeq \overline{X}$ we will use the arguments in [28, Section 6]. Let us note that in the quoted reference X is assumed to be an FT-manifold, but this assumption is used only to show that the contractions of X have the properties described in [28, Proposition 4], which we have shown to hold (see section 5.1) in the case we are considering.

OF THEOREM 1.2, CASE F_4 . Arguing as in [28, Proposition 11] we can show that there is a commutative diagram

$$\begin{array}{ccccc}
 & & \pi^{\{1,2\},1} & & \\
 & & \curvearrowright & & \\
 X^{1,2} & \xrightarrow{\tilde{h}} & \bar{X}^{1,2} & \xrightarrow{\bar{\pi}^{\{1,2\},1}} & \bar{X}^1 \\
 \downarrow \pi^{\{1,2\},2} & & \downarrow \bar{\pi}^{\{1,2\},2} & & \\
 X^2 & \xrightarrow{h} & \bar{X}^2 & &
 \end{array}$$

Since $\dim X^2 = \dim \bar{X}^2$ and h is not constant, h is a finite surjective map. By [22, Main Theorem] \tilde{h} restricted to the fibers of $\pi^{\{1,2\},1}$ is an isomorphism. It follows that h is bijective, hence an isomorphism. We conclude by applying [28, Proposition 12] with $I_1 = \{1, 2\}$. \blacksquare

6. Consequences for the Campana-Peternell Conjecture

In this section we will show the implications of our methods to the Campana-Peternell Conjecture (see 1.1). Along this section X will denote a complex projective Fano manifold with nef tangent bundle T_X , and we will say that X is a CP-manifold. We define the width of X as a measure of how far is X from being a Fano manifold whose elementary contractions are smooth \mathbb{P}^1 -fibrations.

DEFINITION 6.1. Given a CP-manifold X , we define:

$$\tau(X) := \sum_C (-K_X \cdot C - 2) \in \mathbb{Z}_{\geq 0},$$

where the sum is taken over all the classes C of rational curves of minimal $(-K_X)$ -degree belonging to extremal rays of $\overline{\text{NE}}(X)$.

In particular $\tau(X) = 0$ if and only if every elementary contraction of X is a smooth \mathbb{P}^1 -fibration, i.e. Theorem 1.2 can be read as:

COROLLARY 6.2. *Any CP-manifold X with $\tau(X) = 0$ is isomorphic to the quotient of a semisimple group G by its Borel subgroup B .*

PROOF. Assume that $\tau(X) = 0$, and let $\pi : X \rightarrow X_i$ be an elementary contraction, associated to an extremal ray R_i generated by the class of a minimal rational curve Γ_i . Let $p : \mathcal{U} \rightarrow \mathcal{M}$ be the family of deformations of Γ_i , with evaluation morphism q . Since by hypothesis $-K_X \cdot \Gamma_i = 2$, then $q : \mathcal{U} \rightarrow X$ is finite. Moreover, following the proof of [15, Proposition 4], q is a holomorphic submersion. Now, since X is simply connected, it follows that q is an isomorphism, and that the contraction π is a \mathbb{P}^1 -fibration. \blacksquare

Then Conjecture 1.1 boils down to proving the following:

CONJECTURE 6.3. *Given a CP-manifold satisfying $\tau(X) > 0$, there exists a contraction $f : X' \rightarrow X$ from a CP-manifold X' satisfying $\tau(X') < \tau(X)$.*

References

- [1] C. Araujo, *Rational curves of minimal degree and characterizations of projective spaces*, Math. Ann. **335** (2006), 937–951.
- [2] C. Araujo and A.-M. Castravet, *Polarized minimal families of rational curves and higher Fano manifolds*, Amer. J. Math. **134** no. 1 (2012), 87–107.
- [3] M. Brion, and S. Kumar, “Frobenius Splitting Methods in Geometry and Representation Theory”, Progress in Mathematics **231**, Birkhäuser Boston Inc., Boston, MA, 2005.
- [4] N. Bourbaki, “Groupes et algèbres de Lie” (Chapitres IV, V et VI), Hermann, Paris, 1968.
- [5] F. Campana, and T. Peternell, *Projective manifolds whose tangent bundles are numerically effective*, Math. Ann. **289** no.1 (1991), 169–187.
- [6] F. Campana, and T. Peternell, *4-folds with numerically effective tangent bundles and second Betti numbers greater than one*, Manuscripta Math. **79** no. 3–4 (1993), 225–238.
- [7] M. Demazure, *Désingularisation des variétés de Schubert généralisées*, Ann. Sci. Éc. Norm. Supér. **7** no. 4 (1974), 53–88.
- [8] R. Hartshorne, “Algebraic Geometry”, Grad. Texts in Math. **52**, Springer–Verlag, New York, 1977.
- [9] J. Hong, J.-M. and Hwang, *Characterization of the rational homogeneous space associated to a long simple root by its variety of minimal rational tangents*, In: “Algebraic geometry in East Asia-Hanoi 2005”, Adv. Stud. Pure Math., 50, Math. Soc. Japan, Tokyo, 2008, 217–236.
- [10] J.E. Humphreys, “Introduction to Lie Algebras and Representation Theory”, Grad. Texts in Math. **9**, Springer-Verlag, New York, 1978.
- [11] J.E. Humphreys, “Reflection groups and Coxeter groups”, Cambridge Stud. Adv. Math. **29**, Cambridge University Press, Cambridge, 1990.
- [12] J.-M. Hwang, *Rigidity of rational homogeneous spaces*, In: “International Congress of Mathematicians. Vol. II”, Eur. Math. Soc., Zürich, 2006, 613–626.
- [13] J.-M. Hwang, *Geometry of minimal rational curves on Fano manifolds*, In: “School on Vanishing Theorems and Effective Results in Algebraic Geometry (Trieste, 2000)”, ICTP Lect. Notes **6**, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2001, 335–393.
- [14] J.-M. Hwang, *On the degrees of Fano four-folds of Picard number 1*. J. Reine Angew. Math. **556** (2003), 225–235.
- [15] J.-M. Hwang and N. Mok, *Rigidity of irreducible Hermitian symmetric spaces of the compact type under Kähler deformation*. Invent. Math. **131** no. 2 (1998), 393–418.
- [16] S. Kebekus, and S.J. Kovács, *Are rational curves determined by tangent vectors?*, Ann. Inst. Fourier (Grenoble) **54** no. 1 (2004), 53–79.
- [17] S. Kleiman, *Toward a numerical theory of ampleness*, Ann. of Math. (2) **84** no. 2 (1966), 293–344.
- [18] J. Kollár, “Rational curves on algebraic varieties”, Ergeb. Math. Grenzgeb. (3) **32**, Springer-Verlag, Berlin, 1996.
- [19] J. Kollár and S. Mori, “Birational geometry of algebraic varieties”, Cambridge Tracts in Math. **134**, Cambridge University Press, Cambridge, 1998.
- [20] S. Kumar, “Kac-Moody groups, their flag varieties and representation theory”, Progr. Math. **204**, Birkhuser Boston Inc., Boston, MA, 2002.
- [21] S. Kumar, *Borel-Weil-Bott theorem and geometry of Schubert varieties*. Notes of the lectures imparted at the MRC Meeting, Snowbird, Utah, 2012. Retrieved from the author’s webpage.
- [22] C.-H. Lau, *Holomorphic maps from rational homogeneous spaces onto projective manifolds*, J. Algebraic Geom. **18** no. 2 (2009), 223–256.
- [23] R. Lazarsfeld, “Positivity in Algebraic Geometry I”, Ergeb. Math. Grenzgeb. (3) **48**, Springer-Verlag, Berlin, 2004.
- [24] N. Lauritzen, and J.F. Thomsen, *Line bundles on Bott-Samelson varieties*, J. Algebraic Geom. **13** (2004), 461–473.
- [25] J. Lurie, *A proof of the Borel–Weil–Bott Theorem*, unpublished, available at the author’s webpage: <http://www.math.harvard.edu/~lurie/bwb.pdf>.
- [26] N. Mok, *On Fano manifolds with nef tangent bundles admitting 1-dimensional varieties of minimal rational tangents*, Trans. Amer. Math. Soc. **354** no.7 (2002), 2639–2658.
- [27] N. Mok, *Recognizing certain rational homogeneous manifolds of Picard number 1 from their varieties of minimal rational tangents*, AMS/IP Stud. Adv. Math. **42** (2008), 41–61.

- [28] R. Muñoz, G. Occhetta, L.E. Solá Conde, and K. Watanabe, *Rational curves, Dynkin diagrams and Fano manifolds with nef tangent bundles*, in Math. Ann. **361** no. 3 (2015), 583–609.
- [29] S. Mori, *Projective manifolds with ample tangent bundles*, Ann. of Math. (2) **110** no. 3 (1979), 593–606.
- [30] R. Pandharipande, *Convex rationally connected varieties*, Proc. Amer. Math. Soc. **141** no. 5 (2013), 1539–1543.
- [31] L.E. Solá Conde and J.A. Wiśniewski, *On manifolds whose tangent bundle is big and 1-ample*, Proc. London Math. Soc. (3) **89** no. 2 (2004), 273–290.
- [32] K. Watanabe, *\mathbb{P}^1 -bundles admitting another smooth morphism of relative dimension one*, J. Algebra **414** no. 15 (2014), 105–119.
- [33] K. Watanabe, *Fano 5-folds with nef tangent bundles and Picard numbers greater than one*, Math. Z. **276** no.1 (2014), 39–49.
- [34] F.L. Zak, “Tangents and secants of algebraic varieties”, Translations of Mathematical Monographs **127**, American Mathematical Society, Providence, RI, 1993.

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI TRENTO, VIA SOMMARIVE 14 I-38123 POVO DI TRENTO (TN), ITALY

E-mail address: gianluca.occhetta@unitn.it

KOREA INSTITUTE FOR ADVANCED STUDY, 85 HOEGIRO, DONGDAEMUN-GU, SEOUL, 130-722, REPUBLIC OF KOREA

E-mail address: lesolac@gmail.com

COURSE OF MATHEMATICS, PROGRAMS IN MATHEMATICS, ELECTRONICS AND INFORMATICS, GRADUATE SCHOOL OF SCIENCE AND ENGINEERING, SAITAMA UNIVERSITY. SHIMO-OKUBO 255, SAKURA-KU SAITAMA-SHI, 338-8570 JAPAN

E-mail address: kwatanab@rimath.saitama-u.ac.jp

INSTYTUT MATEMATYKI UW, BANACHA 2, 02-097 WARSZAWA, POLAND

E-mail address: J.Wisniewski@uw.edu.pl