THE MULTIPLE HOLOMORPH OF A FINITELY GENERATED ABELIAN GROUP

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ABSTRACT. W.H. Mills has determined, for a finitely generated abelian group G, the regular subgroups $N \cong G$ of $S(G)$, the group of permutations on the set G, which have the same holomorph as G, that is, such that $N_{S(G)}(N) = N_{S(G)}(\rho(G))$, where ρ is the (right) regular representation.

We give an alternative approach to Mills' result, which relies on a characterization of the regular subgroups of $N_{S(G)}(\rho(G))$ in terms of commutative ring structures on G.

We are led to solve, for the case of a finitely generated abelian group G, the following problem: given an abelian group $(G, +)$, what are the commutative ring structures $(G, +, \cdot)$ such that all automorphism of G as a group are also automorphisms of G as a ring?

1. INTRODUCTION

Let G be a group, and $\rho : G \to S(G)$ its right regular representation, where $S(G)$ is the group of permutations on the set G. The normalizer

$$\mathrm{Hol}(G) = N_{S(G)}(\rho(G))$$

of the image of ρ is the *holomorph* of G, and it is isomorphic to the natural extension of G by its automorphism group $\mathrm{Aut}(G)$. It is well-known that $\mathrm{Hol}(G) = N_{S(G)}(\lambda(G))$, where $\lambda : G \to S(G)$ is the left regular representation, since $[\rho(G), \lambda(G)] = 1$.

The *multiple holomorph* of G has been defined in G.A. Miller [Mil08] as

$$N_{S(G)}(\mathrm{Hol}(G)) = N_{S(G)}(N_{S(G)}(\rho(G))).$$

Miller has shown that the quotient group

$$T(G) = N_{S(G)}(\mathrm{Hol}(G)) / \mathrm{Hol}(G)$$

acts regularly by conjugation on the set of the regular subgroups N of $S(G)$ which are isomorphic to G and have the same holomorph as G,

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that is, the regular subgroups $N \cong G$ of $S(G)$ such that

$$N_{S(G)}(N) = N_{S(G)}(\rho(G)).$$

There has been some attention in the recent literature [Koh15] to the problem of determining, for G in a given class of groups, the set

$$\mathcal{H}(G) = \left\{ N \le S(G) : N \text{ is regular, } N \cong G \text{ and } N_{S(G)}(N) = \operatorname{Hol}(G) \right\}$$

and the group $T(G)$.

In 1951 W.H. Mills [Mil51] determined these data for a finitely-generated abelian group G, extending the results of [Mil08] for finite abelian groups. Miller enumerated the regular groups N such that $N \cong G$ and $N_{S(G)}(N) = \operatorname{Hol}(G)$. Mills noted that for N abelian and regular, the condition $N_{S(G)}(N) = \operatorname{Hol}(G)$ implies $N \cong G$. Later Mills showed in [Mil53] that if G abelian, N is a regular subgroup of $S(G)$, and $N_{S(G)}(N) = \operatorname{Hol}(G)$, then N is abelian as well.

In this paper, we redo Mills' work using the approach of [CDVS06], which allows us to translate the problem in terms of commutative rings. In particular, we are led to solve the following question, which might be of independent interest, for the case when G is a finitely-generated abelian group.

Question. *Let $(G, +)$ be an abelian group.*

What are the commutative ring structures $(G, +, \cdot)$ such that the automorphisms of G as a group are also automorphisms of G as a ring?

Theorem 5.3 states that if G is a finitely-generated abelian group, then $T(G)$ is an elementary abelian 2-group, of order 1, 2, or 4. In other words, for a given G, there are either 1, 2, or 4 regular subgroups N of $S(G)$ that are isomorphic to G, and such that $N_{S(G)}(N) = N_{S(G)}(\rho(G))$.

These regular subgroups are described, via the just mentioned ring connection, in Theorem 4.17. In 4.18 we also determine explicitly all the group structures G for which $|T(G)| > 1$.

The plan of the paper is the following. In Section 2 we define the various holomorphs, and set up the problem. In Section 3 we rephrase the problem in terms of rings. The classification of the rings is worked out in Section 4. The group $T(G)$ is discussed in Section 5.

2. Groups with the same holomorph

In this section, G is an additively written group.

The *holomorph* of a group G is the natural semidirect product

$$\operatorname{Aut}(G)\, G$$

of G by its automorphism group $\operatorname{Aut}(G)$. Let $S(G)$ be the group of permutations on the set G. Consider the (right) regular representation

$$\begin{aligned} \rho : G &\to S(G) \\ g &\mapsto (x \mapsto x + g). \end{aligned}$$

The following is well-known.

Proposition 2.1. *$N_{S(G)}(\rho(G)) = \operatorname{Aut}(G)\,\rho(G)$ is isomorphic to the holomorph $\operatorname{Aut}(G)\,G$ of G.*

Definition 2.2. *We write $\operatorname{Hol}(G) = N_{S(G)}(\rho(G))$. We will refer to either of the isomorphic groups $N_{S(G)}(\rho(G))$ and $\operatorname{Aut}(G)\,G$ as the* holomorph *of G.*

One may inquire, what are the regular subgroups $N \leq S(G)$ which have the same holomorph as G, that is, for which

$$\operatorname{Hol}(N) \cong N_{S(G)}(N) = N_{S(G)}(\rho(G)) = \operatorname{Hol}(G). \tag{2.1}$$

W.H. Mills has noted in [Mil51] that if (2.1) holds, then G and N need not be isomorphic.

When we restrict our attention to the regular subgroups N of $S(G)$ for which $N_{S(G)}(N) = \operatorname{Hol}(G)$ and $N \cong G$, we can appeal to a result of G.A. Miller [Mil08]. Miller found a characterization of these subgroups in terms of the *multiple holomorph* of G

$$N_{S(G)}(\operatorname{Hol}(G)) = N_{S(G)}(N_{S(G)}(\rho(G))).$$

Consider the set

$$\mathcal{H}(G) = \left\{ N \leq S(G) : N \text{ is regular, } N \cong G \text{ and } N_{S(G)}(N) = \operatorname{Hol}(G) \right\}.$$

Using the well-known fact that two regular subgroups of $S(G)$ are isomorphic if and only if they are conjugate in $S(G)$, Miller showed that the group $N_{S(G)}(\operatorname{Hol}(G))$ acts transitively on $\mathcal{H}(G)$ by conjugation. Clearly the stabilizer in $N_{S(G)}(\operatorname{Hol}(G))$ of any element $N \in \mathcal{H}(G)$ is $N_{S(G)}(N) = \operatorname{Hol}(G)$. We obtain

Theorem 2.3. *The group*

$$T(G) = N_{S(G)}(\operatorname{Hol}(G)) / \operatorname{Hol}(G)$$

acts regularly on $\mathcal{H}(G)$ by conjugation.

3. Regular normal subgroups of the holomorph

Given an abelian group G, we aim first at giving a description of the set

$$\mathcal{K}(G) = \{ N \leq S(G) : N \text{ is regular, } N \trianglelefteq \operatorname{Hol}(G) \} \supseteq \mathcal{H}(G).$$

It was noted in [FCC12] that the results of [CDVS06] on affine groups admit a straightforward extension to the case of holomorphs of abelian groups. We recall this here in our context.

Let $N \leq \operatorname{Hol}(G)$ be a regular subgroup. Write $\nu(g)$, with $g \in G$, for the unique element of N such that $0^{\nu(g)} = g$. (We write group actions as exponents.) Then there is a map $\gamma : G \to \operatorname{Aut}(G)$ such that for $g \in G$ we can write uniquely

$$\nu(g) = \gamma(g)\rho(g). \tag{3.1}$$

For $g, h \in G$ we have

$$\nu(g)\nu(h) = \gamma(g)\rho(g)\gamma(h)\rho(h) = \gamma(g)\gamma(h)\rho(g^{\gamma(h)} + h). \tag{3.2}$$

Since N is a subgroup of $S(G)$, and the expression (3.1) is unique, we obtain, for $g, h \in G$,

$$\gamma(g)\gamma(h) = \gamma(g^{\gamma(h)} + h). \tag{3.3}$$

Note, for later usage, that (3.3) can be rephrased, setting $k = g^{\gamma(h)}$, as

$$\gamma(k + h) = \gamma(k^{\gamma(h)^{-1}})\gamma(h), \tag{3.4}$$

fo $h, k \in G$.

To enforce $N \trianglelefteq \mathrm{Hol}(G)$, it is now enough to make sure that N is normalized by $\mathrm{Aut}(G)$. In fact, N is a transitive subgroup of $\mathrm{Hol}(G)$ acting on G. Since $\rho(G)$ is regular, the stabilizer of 0 in $\mathrm{Hol}(G) = \mathrm{Aut}(G)\rho(G)$ is $\mathrm{Aut}(G)$. Hence $\mathrm{Hol}(G)$ is the product of $\mathrm{Aut}(G)$ and N (and then it is a semidirect product, as N is regular).

In order for $\mathrm{Aut}(G)$ to normalize N, we must have that for all $\beta \in \mathrm{Aut}(G)$ and $g \in G$, the conjugate $\nu(g)^\beta$ of $\nu(g)$ by β in $S(G)$ lies in N. Since

$$\nu(g)^\beta = (\gamma(g)\rho(g))^\beta = \gamma(g)^\beta\rho(g)^\beta = \gamma(g)^\beta\rho(g^\beta),$$

uniqueness of (3.1) implies that this is equivalent to

$$\gamma(g^\beta) = \gamma(g)^\beta \tag{3.5}$$

for $g \in G$ and $\beta \in \mathrm{Aut}(G)$. Applying this to (3.4), we obtain

$$\gamma(k + h) = \gamma(k^{\gamma(h)^{-1}})\gamma(h) = \gamma(k)^{\gamma(h)^{-1}}\gamma(h) = \gamma(h)\gamma(k), \tag{3.6}$$

that is, $\gamma : G \to \mathrm{Aut}(G)$ is a homomorphism, as G is abelian.

Note that (3.3) follows from (3.5) and (3.6), as

$$\gamma(g^{\gamma(h)} + h) = \gamma(g)^{\gamma(h)}\gamma(h) = \gamma(g)\gamma(h).$$

We now state the characterization we will be exploiting in the rest of the paper.

Theorem 3.1. *Let G be an abelian group. The following data are equivalent.*

(1) An abelian regular subgroup $N \trianglelefteq \mathrm{Hol}(G)$, that is, an element of $\mathcal{K}(G)$.

(2) A homomorphism

$$\gamma : G \to \mathrm{Aut}(G)$$

such that for $g \in G$ and $\beta \in \mathrm{Aut}(G)$

$$\gamma(g^\beta) = \gamma(g)^\beta. \tag{3.7}$$

(3) A commutative rings structure $(G, +, \cdot)$ such that

(a) the operation $g \circ h = g + h + gh$ defines a group structure $(G, \circ)$,

(b) $ghk = 0$ for all $g, h, k \in G$, and

(c) *each automorphism of the group* $(G,+)$ *is also an automorphism of the ring* $(G,+,\cdot)$.

Moreover, under these assumptions

(i) *in terms of* (2)*, the operations of* (3) *are given by*

$$g \cdot h = -g + g^{\gamma(h)}, \qquad \textit{and} \qquad g \circ h = g^{\gamma(h)} + h.$$

for $g, h \in G$.

(ii) *The function* ν *of* (3.2) *defines an isomorphism* $(G,\circ) \to N$.

(iii) *Every automorphism of* G *is also an automorphism of* $(G,\circ)$.

Note that (3b) implies (3a). Note also that (ii) tells us that each ring structure as in (3) yields a distinct $N \in \mathcal{K}(G)$.

Proof. We have already seen that (1) and (2) are equivalent.

We now recall from [CDVS06, FCC12] that if N is a regular abelian subgroup of $\operatorname{Hol}(G)$, and γ is the associated function as in (2), then, setting, for $g, h \in G$

$$g \cdot h = -g + g^{\gamma(h)},$$

we obtain a ring structure $(G,+.\cdot)$ on G such that

$$g \circ h = g + h + gh = g^{\gamma(h)} + h$$

defines a group structure $(G,\circ)$.

To show that (2) implies (3b), we have to prove that for all $h, k \in G$ we have $\gamma(hk) = 1$. (This was already observed in a comment after Lemma 3 of [CDVS06].) In fact

$$\gamma(hk) = \gamma(h^{\gamma(k)} - h) = \gamma(h)^{\gamma(k)} \gamma(h)^{-1} = [\gamma(k), \gamma(-h)] = 1,$$

as $\gamma : G \to \operatorname{Aut}(G)$ is a homomorphism, and G is abelian.

To show that (2) implies (3c), let $h, k \in G$, and $\beta \in \operatorname{Aut}(G)$. We have

$$\begin{aligned} h^{\beta} \cdot k^{\beta} = -h^{\beta} + h^{\beta\gamma(k^{\beta})} = -h^{\beta} + h^{\beta\gamma(k)^{\beta}} = \\ = -h^{\beta} + h^{\gamma(k)\beta} = (-h + h^{\gamma(k)})^{\beta} = (h \cdot k)^{\beta}, \end{aligned}$$

where we have used (3.7).

The bijection ν introduced above is a homomorphism $(G,\circ) \to N$ by (3.2) and (3.3).

Finally, (iii) follows from

$$(g \circ h)^{\beta} = (g + h + gh)^{\beta} = g^{\beta} + h^{\beta} + g^{\beta} h^{\beta} = g^{\beta} \circ h^{\beta}$$

for $g, h \in G$ and $\beta \in \operatorname{Aut}(G)$.

Conversely, given a ring as in (3), the following calculations show that the function $\gamma : G \to S(G)$ given by $\gamma(g) : h \mapsto h + hg$ satisfies the conditions of (2). (Here $\gamma(g) \in S(G)$ because $\gamma(g)\gamma(-g) : h \mapsto (h + hg) + (h + hg)(-g) = h + h(g - g) = h$, where we have used (3b).)

$$(h+k)^{\gamma(g)} = h + k + (h+k)g = h + hg + k + kg = h^{\gamma(g)} + k^{\gamma(g)},$$

for all $g, h, k \in G$, shows that γ maps G into $\operatorname{Aut}(G)$.

$$g^{\gamma(h)\gamma(k)} = (g + gh) + (g + gh)k = g + g(h+k) = g^{\gamma(h+k)},$$

for all $g, h, k \in G$, where we have used (3b), shows that $\gamma : G \to \operatorname{Aut}(G)$ is a homomorphism.

$$h^{\gamma(g^{\beta})} = h + hg^{\beta} = (h^{\beta^{-1}} + h^{\beta^{-1}} g)^{\beta} = h^{\beta^{-1}\gamma(g)\beta} = h^{\gamma(g)^{\beta}},$$

for all $g, h \in G$, and $\beta \in \operatorname{Aut}(G)$, shows that γ satisfies (3.7). ☐

Suppose the finitely generated abelian group $(G, +)$ admits a ring structure $(G, +, \cdot)$ as in Theorem 3.1.(3). Taking $\beta \in \operatorname{Aut}(G, +)$ to be inversion $g \mapsto -g$, we get that for all $g, h \in G$ one has

$$-gh = (-g)(-h) = gh,$$

that is, all products satisfy

$$2 \cdot gh = 0. \tag{3.8}$$

We have obtained

Lemma 3.2. *In the commutative ring* $(G, +, \cdot)$ *as in Theorem 3.1.*(3) *we have*

(1) $2 \cdot gh = 0$*, for all* $g, h \in G$*, so that*
(2) if $(G, +)$ *has no elements of order* 2*, ring multiplication is trivial, and*
(3) $(g + h)^2 = g^2 + h^2$ *, for all* $g, h \in G$*.*

4. The classification

From now on, let G be a finitely generated abelian group.

For such G, Mills [Mil51] has determined the set

$$\mathcal{H}(G) = \left\{ N \le S(G) : N \text{ is regular, } N \cong G \text{ and } N_{S(G)}(N) = \operatorname{Hol}(G) \right\}.$$

In the following we will first determine the set

$$\mathcal{K}(G) = \{ N \le S(G) : N \text{ is regular, } N \trianglelefteq \operatorname{Hol}(G) \} \supseteq \mathcal{H}(G),$$

weeding out in the process the N for which $N_{S(G)}(N) > N_{S(G)}(\rho(G))$. In Section 5 we will show that the remaining groups are precisely the elements of $\mathcal{H}(G)$, and we will also determine the group $T(G)$.

According to Theorem 3.1, we proceed to find all ring structures $(G, +, \cdot)$ such that all automorphisms of G as a group are also automorphisms of G as a ring. We will usually tacitly ignore the trivial case when $G^2 = \{xy : x, y \in G\} = \{0\}$.

Write

$$G = F \times H \times K,$$

where F is free abelian of finite rank, H is a finite 2-group, and K is a finite group of odd order.

If $a \in K$ has odd order d, then according to Lemma 3.2(1), for all $b \in G$ we have $ab = d(ab) = (da)b = 0$. Therefore the odd part K lies in the annihilator. For the ring structure on G to be non-trivial, it has thus to be non-trivial on $F \times H$. Because of this, from now on we will assume

$$G = F \times H,$$

where F is free abelian of finite rank, and H, the torsion part, is a finite 2-group. We write

$$\Omega(H) = \{t \in H : 2t = 0\}.$$

By Lemma 3.2.(1), all products in the ring $(G, +, \cdot)$ lie in $\Omega(H)$. We regard $\Omega(H)$ as a vector space over the field $\mathbf{E} = \{0, 1\}$ with 2 elements.

4.1. **The case $F = 0$.**

We first discuss the structure of the 2-torsion part H in the case when the torsion-free part F is zero.

Write

$$H = \prod_{i=1}^{m} \langle x_i \rangle,$$

where $|x_i| = 2^{e_i}$, with $e_i > 0$, and $|x_i| \geq |x_j|$ for $i \leq j$. Write $t_i = 2^{e_i - 1} x_i$ for the involution in $\langle x_i \rangle$.

A *homogeneous component* of H will be a subgroup $\prod_{i=a}^{b} \langle x_i \rangle$, for some $a \leq b$, such that $|x_{a-1}| > |x_a| = |x_{a+1}| = \cdots = |x_b| > |x_{b+1}|$, where the first inequality does not occur if $a = 1$, and the last one does not occur if $b = m$.

Consider the following automorphisms of H.

(1) ξ_{ij}, for x_i, x_j in the same homogeneous component, exchanges x_i with x_j, and leaves all the other x_k fixed.
(2) γ_{ij}, for $i < j$, maps x_i to $x_i + x_j$, and leaves all the other x_k fixed.
(3) β_{ij}, for $i > j$, maps x_i to $x_i + 2^{e_j - e_i} x_j$, and leaves all the other x_k fixed. Note that β_{ij} maps t_i to $t_i + t_j$.

Proposition 4.1. *If $m = 1$, then we have the following possibilities.*

(1) $x_1^2 = 0$, and then multiplication is trivial. This is always the case when $|x_1| = 2$.
(2) $x_1^2 = t_1$, and then
  (a) if $|x_1| = 4$, then $N \notin \mathcal{H}(G)$;
  (b) if $|x_1| > 4$, then $N \in \mathcal{H}(G)$.

Proof. If ring multiplication is non-trivial, then $x_1^2 = t_1 \neq 0$.

If $|x_1| = 2$, we obtain $t_1^2 = t_1$, and thus $t_1^3 = t_1$, contradicting Theorem 3.1(3b).

If $|x_1| = 4$, we obtain $x_1 \circ x_1 = 2x_1 + t_1 = 0$, so $(H, \circ)$ is elementary abelian of order 4, so that its automorphism group is larger than that of H, and $N_{S(G)}(N) > N_{S(G)}(\rho(G))$. Therefore $N \notin \mathcal{H}(G)$.

If $|x_1| > 4$, then one sees immediately that x_1 retains its order in $(H, \circ)$. □

In this case we have thus two rings.

$$\begin{cases} n = 0, m = 1 \\ |x_1| > 4 \\ x_1^2 \in \{0, t_1\} \end{cases} \tag{4.1}$$

We now turn to the case $m > 2$. We will see that in most cases the number of ring structures depends only on the orders of x_1, x_2 (and possibly x_3), and their relationships. An exception is case 4.4, where the orders of $x_1, \dots, x_k$ matter, for an arbitrary $k \leq m$.

Lemma 4.2. *If $m \geq 2$, then*

$$x_1 x_2 = \eta_1 t_1 + \eta_2 t_2 \neq t_2, \quad \textit{for some } \eta_i \in \mathbf{E}.$$

Proof. We have $x_1 x_2 = \sum_{k=1}^{m} \eta_k t_k$ for some $\eta_k \in \mathbf{E}$. Applying β_{i1} to this, for $i > 2$, we see that

$$x_1 x_2 = (x_1 x_2)\beta_{i1} = (\sum_{k=1}^{m} \eta_k t_k)\beta_{i1} = (\sum_{k=1}^{m} \eta_k t_k) + \eta_i t_1,$$

whence $\eta_i = 0$ for $i > 2$.

It remains to show that $x_1 x_2 \neq t_2$. If $x_1 x_2 = t_2$, in the case when $|x_1| = |x_2|$ we have $x_1 x_2 = (x_1 x_2)\xi_{12} = t_1$, a contradiction; when $|x_1| > |x_2|$, that is, $e_1 > e_2$ we have $t_1 + t_2 = (t_2)\beta_{21} = (x_1 x_2)\beta_{21} = x_1(x_2 + 2^{e_1 - e_2} x_1) = x_1 x_2$, a contradiction, using the fact that Lemma 3.2 implies $2^{e_1 - e_2} x_1^2 = 0$. □

Lemma 4.3. *Suppose $m > 2$, $|x_1| > |x_3|$, and Suppose $m > 2$, $|x_1| > |x_3|$, and either $|x_2| > |x_3|$, or $\eta_2 = 0$ in Fact 4.2.*

Then $x_1 x_j = x_2 x_k = x_k x_j = 0$ for $k, j > 2$.

Proof. Let $k, j > 2$.

Note that $|x_1| > |x_3| \geq |x_k|$ implies

$$(t_1)\gamma_{1k} = (2^{e_1 - 1} x_1)\gamma_{1k} = 2^{e_1 - 1}(x_1 + x_k) = 2^{e_1 - 1} x_1 = t_1.$$

Similarly, if $|x_2| > |x_3|$ we have $(t_2)\gamma_{2j} = t_2$. Thus under the given hypotheses we have, for $\eta_1, \eta_2 \in \mathbf{E}$,

$$(\eta_1 t_1 + \eta_2 t_2)\gamma_{1i} = \eta_1 t_1 + \eta_2 t_2 = (\eta_1 t_1 + \eta_2 t_2)\gamma_{2j},$$

where the last equality depends on the fact that by assumption either $|x_2| > |x_3|$, and thus $(t_2)\gamma_{2j} = t_2$, or $\eta_2 = 0$.

Apply γ_{1k} to x_1x_2, and use Lemma 4.2, to get

$$x_1x_2 = (x_1x_2)\gamma_{1k} = (x_1 + x_k)x_2 = x_1x_2 + x_kx_2,$$

whence $x_kx_2 = 0$.

Apply γ_{2j} to x_1x_2 to get

$$x_1x_2 = (x_1x_2)\gamma_{2j} = x_1(x_2 + x_j) = x_1x_2 + x_1x_j,$$

whence $x_1x_j = 0$.

Finally, apply $\gamma_{1k}\gamma_{2j}$ to x_1x_2 to get

$$x_1x_2 = (x_1x_2)\gamma_{1k}\gamma_{2j} = (x_1 + x_k)(x_2 + x_j) = x_1x_2 + x_kx_j,$$

whence $x_kx_j = 0$. □

4.1.1. *Torsion case, $m \geq 2$, $x_1x_2 = t_1$.*

If $|x_1| = |x_2|$, applying ξ_{12} to $x_1x_2 = t_1$ we get $x_1x_2 = t_2$, a contradiction.

If $|x_1| > |x_2|$, we have $2^{e_1-1}x_2 = 0$, so that $t_1 = 2^{e_1-1}x_1 = 2^{e_1-1}(x_1 + x_2) = 2^{e_1-1}(x_1)\gamma_{12} = (t_1)\gamma_{12}$, which implies $x_1x_2 = t_1 = t_1\gamma_{12} = (x_1x_2)\gamma_{12} = (x_1 + x_2)x_2 = x_1x_2 + x_2^2$, so that $x_2^2 = 0$. Applying γ_{2i} to the last identity, we obtain $x_i^2 = 0$ for $i > 2$.

Using Lemma 4.3 we obtain $x_1x_j = x_2x_i = x_ix_j = 0$ for $i, j > 2$.

If $m > 2$ and $|x_2| = |x_3|$, we have $t_1 = (t_1)\xi_{23} = (x_1x_2)\xi_{23} = x_1x_3$, a contradiction.

We have obtained the following result.

Proposition 4.4. *The following rings share the same group structure, and give rise to groups $(G, \circ) \cong G$.*

$$\begin{cases} n = 0, m \geq 2 & \\ |x_1| > |x_2| & \\ |x_1| > 4 & \text{if } x_1^2 \neq 0 \\ |x_2| > |x_3| & \text{if } m > 2 \text{ and } x_1x_2 \neq 0 \\ x_1^2 \in \{0, t_1\} & \\ x_i^2 = 0, & \text{for } i > 1 \\ x_1x_2 \in \{0, t_1\} & \\ x_1x_i = x_2x_j = x_ix_j = 0, & \text{for } i, j > 2 \end{cases} \tag{4.2}$$

Here there are either four rings, two rings, or just the trivial ring:

- *if $|x_1| > 4$, and either $m = 2$, or $m > 2$ and $|x_2| > |x_3|$, there are four rings;*
- *if $|x_1| = 4$ and $m > 2$ there is just the trivial ring;*
- *there are two rings in the remaining cases.*

Note that we have required $|x_1| > 4$ if $x_1^2 = t_1 \neq 0$, as in Proposition 4.1, to make sure that x_1 retains its order in $(H, \circ)$.

Remark 4.5. *Clearly the elements x_i retain their orders in $(H, \circ)$. Moreover it is easy to see that $(H, \circ)$ is still the direct product of the subgroups spanned by the x_i, so that $(H, \circ)$ is isomorphic to H.*

4.1.2. *Torsion case, $m \geq 2$, $x_1 x_2 = t_1 + t_2$.*

If $|x_1| > |x_2|$, then $t_1 + t_2 = x_1 x_2 = (x_1 x_2)\beta_{21} = (t_1 + t_2)\beta_{21} = t_2$, a contradiction.

Therefore $|x_1| = |x_2|$. Applying γ_{12} and β_{12} to $x_1 x_2 = t_1 + t_2$ we obtain $x_1^2 = t_1$ and $x_2^2 = t_2$.

When $m > 2$, if $|x_2| = |x_3|$, applying ξ_{23} to $x_1 x_2 = t_1 + t_2$ we obtain $x_1 x_3 = t_1 + t_3$. But then applying γ_{23} to $x_1 x_2 = t_1 + t_2$ we obtain $t_2 + t_3 = x_1 x_2 + x_1 x_3 = t_1 + t_2 + t_3$, a contradiction.

Therefore if $m > 2$ we have $|x_2| > |x_3|$. Using Lemma 4.3 we obtain $x_1 x_j = x_2 x_i = x_i x_j = 0$ for $i, j > 2$. We have obtained the following result.

Proposition 4.6. *The following ring gives rise to a group $(G, \circ) \cong G$.*

$$\begin{cases} n = 0, m \geq 2 \\ |x_1| = |x_2| > 4 \\ |x_2| > |x_3| \ \text{ if } m > 2 \\ x_1^2 = t_1, x_2^2 = t_2 \\ x_1 x_2 = t_1 + t_2 \\ x_1 x_i = x_2 x_j = x_i x_j = 0, \quad \text{for } i, j > 2 \end{cases} \tag{4.3}$$

The same group obviously allows also trivial ring multiplication.

Note that we have to take $|x_1| > 4$ here, for the same argument of Proposition 4.1. And $(H, \circ)$ is isomorphic to H, as per Remark 4.5.

4.1.3. *Torsion case, $m \geq 2$, $x_1 x_2 = 0$.*

Applying γ_{12} to $x_1 x_2 = 0$ we obtain $x_2^2 = 0$.

The arguments of the proof of Lemma 4.3 yield $x_1 x_j = x_2 x_i = x_i x_j = 0$ for $i, j > 2$.

If $|x_1| = |x_2|$, then applying ξ_{12} to $x_2^2 = 0$ we obtain $x_1^2 = 0$, and the ring has trivial multiplication.

Therefore $|x_1| > |x_2|$, and the ring is described in (4.2).

4.2. **The case $F \neq 0$.**

Write

$$F = \prod_{i=1}^{n} \langle z_i \rangle ,$$

with all $z_i \neq 0$.

Consider the following automorphisms of $F \times H$, which are trivial on H.

(1) Ξ_{ij}, for $i \neq j$ exchanges z_i with z_j, and leaves H and all the other z_k fixed.

(2) Γ_{ij}, for $i \neq j$, maps z_i to $z_i + z_j$, and leaves H and all the other z_k fixed.
(3) ζ_{ig}, for $g \in H$, maps z_i to $z_i + g$, and leaves H and all the other z_k fixed.

Recall that $\Omega(H) = \{t \in G : 2t = 0\}$. By Lemma 3.2, the ring product on G yields a bilinear map

$$G/2G \times G/2G \to \Omega(H).$$

Since the automorphisms $\Xi_{ij}, \Gamma_{ij}, \zeta_{ig}, \xi_{ij}, \gamma_{ij}, \beta_{ij}$ generate all the automorphisms of $F/2F \times H/2H$, it will be easy to see that all rings $(G, +, \cdot)$ constructed in the following have the property that all the automorphisms of the group $(G, +)$ are also automorphisms of the ring.

Since by Lemma 3.2 the square map $z \to z^2$ is a group homomorphism $F \to \Omega(H)$, we may make the following

Assumption 4.7. *The indexing of the z_i is chosen so that if some square of the z_i is non-zero, then $z_1^2 \neq 0$.*

We first record the following well-know fact, which in our context can be seen using the β_{i1}.

Lemma 4.8. *Let $P \neq 1$ be a finite, abelian p-group. The following are equivalent:*

(1) P has a characteristic minimal subgroup*, that is, a characteristic subgroup of order p,*
(2) P has a unique *characteristic minimal subgroup, and*
(3) P is the direct product of a cyclic group of order p^e, for some $e \geq 1$, by a group of exponent less than p^e.

If these conditions are verified, the unique characteristic subgroup of order p is $P^{p^{e-1}}$.

Write

$$F^2 = \{ab : a, b \in F\} \subseteq \Omega(H)$$

for the set of products of elements of F.

Lemma 4.9.

(1) $\mathrm{Aut}(H)$ *acts trivially on the set F^2.*
(2) The set $F^2 \subseteq \Omega(H)$ is either zero or $\langle t_1 \rangle$. If it is non-zero, then it is the unique minimal characteristic subgroup of H.
(3) The set $\{z^2 : z \in F\} \subseteq \Omega(H)$ is either zero or $\langle t_1 \rangle$. If it is non-zero, then it is the unique minimal characteristic subgroup of H, and $n = 1$.

Note that when $F^2 \neq 0$ in 2 and $\{z^2 : z \in F\} \neq 0$ in 3, then by Lemma 4.8 either $m = 1$ or $|x_1| > |x_2|$.

Proof. To see (1), take an arbitrary automorphism of H, and extend it trivially to F.

Let u be an arbitrary non-zero element of F^2. Then $\langle u \rangle$ is a characteristic minimal subgroup of H, so that by Lemma 4.8 it is the unique characteristic minimal subgroup of H. This shows (2).

A similar argument yields the first part of (3). If $\{z^2 : z \in F\} \neq \{0\}$, then by Assumption 4.7 we have $z_1^2 \neq 0$, and thus $z_1^2 = t_1$. If $n > 1$, applying Ξ_{12} we see that $z_2^2 = t_1$, but then applying Γ_{12} to $z_1^2 = t_1$ we obtain

$$t_1 = z_1^2 = (z_1 + z_2)^2 = z_1^2 + z_2^2 = 2t_1 = 0,$$

a contradiction. □

Lemma 4.10. *If $F \neq 0$, then $H^2 = 0$.*

Proof. Consider arbitrary $i, j \leq m$. Since $z_1 x_i \in H$, it is fixed by ζ_{1x_j}, so that

$$z_1 x_i = (z_1 x_i)\zeta_{1x_j} = (z_1 + x_j)x_i = z_1 x_i + x_j x_i,$$

and thus $x_i x_j = 0$. □

Lemma 4.11. *If $n > 2$, then $F^2 = 0$.*

Proof. Let i, j, k be distinct indices. Applying Γ_{kj} to $z_i z_k$, we get

$$z_i z_k = z_i(z_k + z_j) = z_i z_k + z_i z_j,$$

whence $z_i z_j = 0$ for all i, j. □

If $z_1 x_1 = 0$, then applying the Ξ_{1i} and the γ_{1j} we see that $z_i x_j = 0$ for all i, j, that is, $FH = 0$.

Let us first consider the case when $FH \neq 0$, so that

$$z_1 x_1 = \sum_{k=1}^{n} \varepsilon_k t_k \neq 0.$$

Applying β_{i1} to this, for $i > 1$, we obtain $z_1 x_1 = \sum_{k=1}^{n} \varepsilon_k t_k + \varepsilon_i t_1$, so that $\varepsilon_i = 0$, and thus $z_1 x_1 = t_1$. Note that this implies $0 = z_1^2 x_1 = z_1 t_1$, so that $|x_1| \geq 4$. If $n > 1$, applying Ξ_{12} to $z_1 x_1 = t_1$ we get $z_2 x_1 = t_1$, and applying Γ_{12} we get $t_1 = (t_1)\Gamma_{12} = (z_1 x_1)\Gamma_{12} = (z_1 + z_2)x_1 = t_1 + t_1 = 0$, a contradiction. Therefore $n = 1$. We have obtained

Lemma 4.12. *If $FH \neq 0$, then $n = 1$ and $z_1 x_1 = t_1$.*

Applying the ξ_{1j}, we obtain that if

$$|x_1| = |x_2| = \cdots = |x_k| > |x_{k+1}|$$

(where we might have $k = m$, so that the final inequality does not occur), then

$$z_1 x_i = t_i, \text{ for } i \leq k, \qquad z_1 x_i = 0, \text{ for } i > k.$$

If $k > 1$, this implies $F^2 = 0$, by Lemma 4.9 and Lemma 4.8. Also, if $n > 2$, then $F^2 = FH = H^2 = 0$.

We are now able to discuss the possibilities for the products on F.

4.2.1. $F \neq 0$, $F^2 = 0$.
In this case we have

Proposition 4.13. *The following ring gives rise to a group* $(G, \circ) \cong G$.

$$
\begin{cases}
n = 1, m \geq 1 & \\
|x_1| = |x_2| = \cdots = |x_k| \geq 4 & \text{for some } k \leq m \\
|x_k| > |x_{k+1}|, & \text{if } k < m \\
z_1^2 = 0 & \\
z_1 x_i = t_i, & \text{for } i \leq k \\
z_1 x_i = 0, & \text{for } i > k \\
x_i x_j = 0, & \text{for all } i, j
\end{cases}
\tag{4.4}
$$

The same group obviously allows also trivial ring multiplication.

4.2.2. $F^2 \neq 0$, $z_1^2 \neq 0$.
By Lemma 4.9(3), we have $n = 1$ here, and

$$z_1^2 = t_1,$$

with $\langle\, t_1 \,\rangle$ characteristic in H. We have obtained the following.

Proposition 4.14. *The following rings give rise to groups* $(G, \circ) \cong G$.

$$
\begin{cases}
n = 1, m \geq 1 & \\
m = 1, \text{ or } m > 1 \text{ and } |x_1| > |x_2| & \\
|x_1| \geq 4 & \text{if } z_1 x_1 \neq 0 \\
z_1^2 \in \{0, t_1\} & \\
z_1 x_1 \in \{0, t_1\} & \\
x_i x_j = 0 & \text{for all } i, j
\end{cases}
\tag{4.5}
$$

These are two rings if $|x_1| = 2$ *(and then* $z_1 x_1 = 0$*, with* $x_1 = t_1$*), four rings if* $|x_1| \geq 4$.

Remark 4.15. *Note that this case comprises* (4.4) *when* $k = 1$ *in* (4.4).

4.2.3. $F^2 \neq 0$, $z_1^2 = 0$.
By Lemma 4.11, we have $n \leq 2$.
The case $n = 1$ does not occur, as it means $F^2 = 0$ here.
If $n = 2$, we have

$$z_1 z_2 = t_1$$

by Lemma 4.9(2). We have obtained the following.

Proposition 4.16. *The following two rings give rise to groups* $(G, \circ) \cong G$.

$$\begin{cases} n = 2, m \geq 1 & \\ |x_1| > |x_2| & \text{if } m > 1 \\ z_1^2 = z_2^2 = 0 & \\ z_1 z_2 \in \{0, t_1\} & \\ z_i x_j = 0, & \text{for all } i, j \\ x_i x_j = 0, & \text{for all } i, j \end{cases} \tag{4.6}$$

In all of these cases, it is easy to see that H is isomorphic to $(H, \circ)$, as per Remark 4.5.

We can sum up the results of this section in the following theorems which represents our main results.

Theorem 4.17. *Let* $(G, +)$ *be a finitely generated abelian group,*

$$G = F \times H,$$

where

$$F = \prod_{i=1}^{n} \langle z_i \rangle$$

is torsion-free, of rank n,

$$H = \prod_{i=1}^{m} \langle x_i \rangle$$

is a 2-group, with $|x_1| \geq |x_2| \geq \cdots \geq |x_m| > 1$.

The possible ring structures with non-trivial multiplication $(G, +, \cdot)$ *on* $(G, +)$, *such that*

(1) $(G, +) \cong (G, \circ)$, *and*
(2) all automorphisms of $(G, +)$ *are also automorphisms of* $(G, +, \cdot)$

are those listed under

$$(4.1), (4.2), (4.3), (4.4), (4.5), (4.6).$$

The groups from the different cases are pairwise non-isomorphic, except for (4.4) *and* (4.5), *as noted in Remark 4.15.*

In the cases (4.2) *and* (4.5) *we have two or four rings (including the ring with trivial multiplication) for the same group structure, in the other cases we have two.*

All of these G *can be enlarged to* $G \times D$, *where* D *is an abelian group of odd order which lies in the annihilator of the ring.*

Theorem 4.18. *In the notation of Theorem 4.17,* *(1) the groups G such that $|\mathcal{H}(G)| = |T(G)| = 4$ are the following.*

$$\begin{cases} n = 0, m \geq 2 \\ |x_1| > 4 \\ |x_1| > |x_2| \\ |x_2| > |x_3| \qquad \text{if } m > 2 \end{cases}$$

$$\begin{cases} n = 1, m \geq 1 \\ |x_1| \geq 4 \\ m = 1, \text{ or } m > 1 \text{ and } |x_1| > |x_2| \end{cases}$$

(2) the groups G such that $|\mathcal{H}(G)| = |T(G)| = 2$ are the following.

$$\begin{cases} n = 0, m = 1 \\ |x_1| > 4 \end{cases}$$

$$\begin{cases} n = 0, m = 2 \\ |x_1| = 4 \\ |x_2| = 2 \end{cases}$$

$$\begin{cases} n = 0, m > 2 \\ |x_1| > 4 \\ |x_1| > |x_2| \\ |x_2| = |x_3| \end{cases}$$

$$\begin{cases} n = 0, m \geq 2 \\ |x_1| = |x_2| > 4 \\ |x_2| > |x_3| \qquad \text{if } m > 2 \end{cases}$$

$$\begin{cases} n = 1, m \geq 1 \\ |x_1| = |x_2| = \cdots = |x_k| \geq 4, \quad \text{for some } k \leq m \\ |x_k| > |x_{k+1}| \qquad \text{if } k < m \end{cases}$$

$$\begin{cases} n = 1, m = 1 \\ |x_1| = 2 \end{cases}$$

$$\begin{cases} n = 2, m \geq 1 \\ |x_1| > |x_2| \qquad \text{if } m > 1 \end{cases}$$

(3) for all other groups G we have $|\mathcal{H}(G)| = |T(G)| = 1$.

5. The group $T(G)$

We first record the following

Lemma 5.1. *In the notation of Section 3, suppose $\vartheta \in S(G)$ is an isomorphism $\vartheta : G \to (G, \circ)$.*

Then ϑ conjugates $\rho(G)$ to N.

Proof. For $g, h \in G$ we have

$$g^{\rho(h)^{\vartheta}} = g^{\vartheta^{-1}\rho(h)\vartheta} = (g^{\vartheta^{-1}} + h)^{\vartheta} = g \circ h^{\vartheta} = g^{\nu(h^{\vartheta})},$$

whence $\rho(h)^{\vartheta} = \nu(h^{\vartheta})$. □

In the previous section we have determined, for a given finitely generated abelian group G, all regular subgroups N of $S(G)$ which are normal in $\operatorname{Hol}(G)$, that is, the elements of the set

$$\mathcal{K}(G) = \{N \le S(G) : N \text{ is regular, } N \trianglelefteq \operatorname{Hol}(G)\}.$$

We weeded out those $N \in \mathcal{K}(G)$ for which $N_{S(G)}(N) > N_{S(G)}(\rho(G))$, and seen that the remaining groups are isomorphic to G. Now if $N \in \mathcal{K}(G)$ and $\vartheta : G \to (G, \circ) \cong N$ is an isomorphism, by Lemma 5.1 we have

$$N_{S(G)}(\rho(G))^{\vartheta} = N_{S(G)}(\rho(G)^{\vartheta}) = N_{S(G)}(N) \ge N_{S(G)}(\rho(G)). \tag{5.1}$$

In this section we will prove the following Lemma.

Lemma 5.2. *For each of the regular subgroups $N \cong G$ of the previous section, there is $\vartheta \in S(G)$* of order two *which is an isomorphism $\vartheta : G \to (G, \circ)$.*

We will then have from (5.1)

$$N_{S(G)}(\rho(G)) = N_{S(G)}(\rho(G))^{\vartheta^2} \ge N_{S(G)}(\rho(G))^{\vartheta},$$

so that

$$N_{S(G)}(N) = N_{S(G)}(\rho(G)).$$

Therefore the regular subgroups $N \cong G$ of the previous section will turn out to be exactly the elements of the set

$$\mathcal{H}(G) = \left\{N \le S(G) : N \text{ is regular, } N \cong G \text{ and } N_{S(G)}(N) = \operatorname{Hol}(G)\right\}.$$

In the previous section we have shown that for each group structure $(G, +)$ there are $1, 2$, or 4 rings $(G, +, \cdot)$. Therefore we will have obtained

Theorem 5.3. *For each finitely generated abelian group G, the group $T(G)$ is elementary abelian, of order $1, 2$, or 4*

Proof of Lemma 5.2. We now describe, for each of the regular subgroups $N \cong G$ of the previous section, an element of $\vartheta \in S(G)$ of order two which yields an isomorphism $\vartheta : G \to (G, \circ)$.

In the previous section we have noted that in all the cases of Theorem 4.17, the generators z_i, x_j are still generators of $(G, \circ)$, they retain their orders in $(G, \circ)$, and $(G, \circ)$ is still a direct product of the cyclic subgroups generated by the z_i, x_j (see Remark 4.5). Therefore there is an isomorphism $\vartheta : G \mapsto (G, \circ)$ such that $z_i^\vartheta = z_i$ and $x_j^\vartheta = x_j$ for all j. This can be extended to the whole of G via

$$(x+y)^\vartheta = x^\vartheta \circ y^\vartheta = x^\vartheta + y^\vartheta + x^\vartheta y^\vartheta \tag{5.2}$$

for all $x, y \in G$.

Define a function

$$\begin{aligned} f : G &\to G \\ u &\mapsto u^\vartheta - u. \end{aligned}$$

We will be using several times the following simple observation

$$f(G) \subseteq G^2. \tag{5.3}$$

Recall from Lemma 3.2 that $2G^2 = 0$, and from Theorem 3.1(3b) that G^2 lies in the annihilator of the ring.

To prove (5.3), we proceed by induction on the length of u as a sum of the generators z_i, x_j. We have from (5.2), if y is one of these generators,

$$\begin{aligned} f(u+y) &= (u+y)^\vartheta - (u+y) \\ &= u^\vartheta - u + y^\vartheta - y + u^\vartheta y^\vartheta \\ &= f(u) + u^\vartheta y^\vartheta \in G^2, \end{aligned}$$

as $y^\vartheta = y$, for y a generator.

We have thus proved (5.3).

Note that for all $u, v \in G$ we have

$$\begin{aligned} (u+v)^\vartheta &= u^\vartheta + v^\vartheta + u^\vartheta v^\vartheta \\ &= u + f(u) + v + f(v) + (u + f(u))(v + f(v)) \\ &= u + v + f(u) + f(v) + uv, \end{aligned}$$

so that

$$f(u+v) = f(u) + f(v) + uv.$$

Therefore (5.3) yields $f(2u) = 2f(u) + u^2 = u^2$, so that

$$f(4u) = f(2u + 2u) = 2f(2u) + 4u^2 = 0. \tag{5.4}$$

Therefore

$$\begin{aligned} u^{\vartheta^2} &= (u + f(u))^\vartheta \\ &= u + f(u) + f(u + f(u)) \\ &= u + f(u) + f(u) + f(f(u)) + uf(u) \\ &= u + f(f(u)), \end{aligned} \tag{5.5}$$

by (5.3).

In the cases of Theorem 4.17 when $|x_1| > 4$, we have $f(G) \subseteq G^2 \leq 4H$. Now (5.5) and (5.4) yield $u^{\vartheta^2} = u$.

In the cases when $|x_1| = 4$, we have $G^2 = \langle t_1, \dots, t_k \rangle$ for some k, and $x_i^2 = 0$ for all i. Thus we have, for $i \le k$,

$$f(t_i) = f(2x_i) = 2f(x_i) + x_i^2 = 0, \tag{5.6}$$

so that (5.5) implies $u^{\vartheta^2} = u$.

Finally, when $|x_1| = 2$ in (4.5), we have $f(t_1) = f(x_1) = x_1^{\vartheta} - x_1 = 0$.

Therefore $\vartheta \in S(G)$ is in all cases an involution, as claimed. □

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