Deformation of piecewise differentiable curves in constrained variational calculus.

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Abstract

A survey of the geometric tools involved in the study of constrained variational calculus is presented. The central issue is the characterization of the admissible deformations of piecewise differentiable sections of a fibre bundle $\mathcal{V}_{n+1} \to \mathbb{R}$, in the presence of arbitrary non-holonomic constraints. Asynchronous displacements of the corners are explicitly considered. The coordinate-independent representation of the variational equation and the associated concepts of *infinitesimal control* and *absolute time derivative* are reviewed. In the resulting algebraic environment, every admissible section is assigned a corresponding *abnormality index*, identified with the co–rank of a suitable linear map. Sections with vanishing index are called *normal*. A section is called *ordinary* if every solution of the variational equation vanishing at the endpoints is tangent to some finite deformation with fixed endpoints. The interplay between abnormality index and ordinariness — in particular the fact that every normal evolution is automatically an ordinary one — is discussed.

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Introduction

Calculus of variations has a very old origin, dating back to the pioneering works of Euler, Lagrange and Weierstrass [1]. More recent contributions [11–22] have significantly improved the differential geometric approach to the subject.

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[1] For a modern exposition of the classical theory, see e.g. [1–10].

In this paper we review some foundational aspects of constrained variational calculus. The discussion deals with *parameterized curves*, namely with sections of a fiber bundle $\mathcal{V}_{n+1} \xrightarrow{t} \mathbb{R}$, called the *event space*, the projection t possibly identified with the *absolute time* of Classical Mechanics. The constraints are accounted for by a submanifold $\mathcal{A}$ of the first jet bundle of $j_1(\mathcal{V}_{n+1})$. Every continuous, piecewise differentiable section $\gamma : \mathbb{R} \to \mathcal{V}_{n+1}$ whose first jet extension $j_1(\gamma)$ factors through $\mathcal{A}$ is called an *admissible evolution*. The points of discontinuity of $j_1(\gamma)$ are called the *corners* of γ.

A basic task of constrained variational calculus is characterizing the extremals of a given action functional I among the class of admissible evolutions.

In this connection, the presence of kinetic constraints and the possible existence of corners raise some relevant questions: among others, the covariant characterization of the infinitesimal deformations, the geometrical interpretation of the concept of *normality* of a section, the relation between normality and deformability. A thorough analysis of these aspects may be found in [20].

The main results are reported in the following Sections. After a few preliminary remarks, the infinitesimal deformations of an admissible evolution γ are discussed via a revisitation of the *variational equation*. The central idea is the introduction of the concept of *infinitesimal control*, yielding a covariant characterization of the (infinite dimensional) vector space $\mathfrak{W}$ formed by the totality of admissible infinitesimal deformations.

In Section 2 the admissible evolutions are classified into *ordinary*, if every element of $\mathfrak{W}$ vanishing at the endpoints of γ is tangent to some finite deformations with fixed endpoints, and *exceptional* in the opposite case. Along the same guidelines, every admissible evolution is assigned a corresponding *abnormality index*, extending and expressing in geometrical terms the traditional attributes of normality and abnormality commonly found in the literature.

The interplay between abnormality index and ordinariness is eventually discussed in Subsection 2.2. The fact that all normal evolutions are automatically ordinary, proved by Hsu [12] in a linear context, is established in the case of arbitrary non–linear constraints and piecewise differentiable sections.

Section 3 provides arguments and examples that clarify some aspects of the concept of normality discussed in Sec. 2.1.

1. Overview of foregoing results

All definitions, conventions and results described in [20] will be freely used throughout. For convenience of the reader, a few basic aspects, especially relevant to the present discussion, are reported below.

1.1. Preliminaries

(i) Let $\mathcal{V}_{n+1} \xrightarrow{t} \mathbb{R}$ denote a $(n+1)$–dimensional fiber bundle, locally referred to fibred coordinates $t, q^1, \ldots, q^n$ and called the *event space*.

Every section $\gamma \colon \mathbb{R} \to \mathcal{V}_{n+1}$ is interpreted as the evolution of an abstract system with a finite number of degrees of freedom.

The first jet bundle $j_1(\mathcal{V}_{n+1}) \xrightarrow{\pi} \mathcal{V}_{n+1}$ (with π denoting the natural projection), referred to jet coordinates $t, q^i, \dot{q}^i$, is called the *velocity space*. The jet–extension of a section $\gamma : \mathbb{R} \to \mathcal{V}_{n+1}$ is indicated by $j_1(\gamma) \colon \mathbb{R} \to j_1(\mathcal{V}_{n+1})$.

The presence of non–holonomic constraints is accounted for by a submanifold $\mathcal{A}$ of $j_1(\mathcal{V}_{n+1})$, fibred over $\mathcal{V}_{n+1}$ and referred to local fibred coordinates $t, q^1, \ldots, q^n, z^1, \ldots, z^r$. The imbedding $\mathcal{A} \xrightarrow{i} j_1(\mathcal{V}_{n+1})$ is locally expressed as

$$\dot{q}^i = \psi^i(t, q^1, \ldots, q^n, z^1, \ldots, z^r) \qquad i = 1, \ldots, n.$$

A section $\gamma : \mathbb{R} \to \mathcal{V}_{n+1}$ is called *admissible* if and only there exists a section $\hat{\gamma} : \mathbb{R} \to \mathcal{A}$, called the *lift* of γ, locally described as $q^i = q^i(t), z^A = z^A(t)$ and satisfying $i \cdot \hat{\gamma} = j_1(\gamma)$. In the stated circumstance, the section $\hat{\gamma}$ too is called admissible. In coordinates, the admissibility condition reads

$$\frac{dq^i}{dt} = \psi^i(t, q^1(t), \ldots, q^n(t), z^1(t), \ldots, z^r(t)).$$

Every section $\sigma : \mathcal{V}_{n+1} \to \mathcal{A}$ is called a *control* for the system. A section γ is said to *belong* to a control σ if and only if its lift $\hat{\gamma}$ factors into $\hat{\gamma} = \sigma \cdot \gamma$, i.e. if and only if the jet extension $j_1(\gamma)$ coincides with the composite map $i \cdot \sigma \cdot \gamma : \mathbb{R} \to j_1(\mathcal{V}_{n+1})$. In local coordinates we have the representations

$$\sigma \quad : \quad z^A = z^A(t, q^1, \ldots, q^n), \tag{1a}$$

$$i \cdot \sigma : \quad \dot{q}^i = \psi^i(t, q^1, \ldots, q^n, z^A(t, q^1, \ldots, q^n)), \tag{1b}$$

showing that assigning σ determines every section $\gamma \in \sigma$ through the solution of a well posed Cauchy problem.

(ii) Let $V(\mathcal{V}_{n+1}) \xrightarrow{\pi} \mathcal{V}_{n+1}$ and $V^*(\mathcal{V}_{n+1}) \xrightarrow{\pi} \mathcal{V}_{n+1}$ respectively denote the *vertical bundle* relative to the fibration $\mathcal{V}_{n+1} \to \mathbb{R}$ and the associated dual bundle, π indicating, in both cases, the natural bundle projection.

The space $V^*(\mathcal{V}_{n+1})$ is naturally identified with the quotient of the cotangent bundle $T^*(\mathcal{V}_{n+1})$ by the equivalence relation

$$\sigma \sim \sigma' \iff \begin{cases} \pi(\sigma) = \pi(\sigma') \\ \sigma - \sigma' \propto dt_{\,|\pi(\sigma)}\,; \end{cases}$$

its elements of are called the *virtual* 1*–forms* over $\mathcal{V}_{n+1}$.

Every local coordinate system t, q^i in $\mathcal{V}_{n+1}$ induces fibred coordinates t, q^i, p_i in $V^*(\mathcal{V}_{n+1})$, with $p_i(\hat{\lambda}) := \left\langle \hat{\lambda}, \left(\frac{\partial}{\partial q^i}\right)_{\pi(\hat{\lambda})} \right\rangle \ \forall \hat{\lambda} \in V^*(\mathcal{V}_{n+1})$.

For any $g \in \mathscr{F}(\mathcal{V}_{n+1})$, the linear functional on $V(\mathcal{V}_{n+1})$ determined by the differential dg is denoted by δg and is called the *virtual differential* of g.

The vector bundles $V(\mathcal{V}_{n+1})$ and $V^*(\mathcal{V}_{n+1})$ generate a tensor algebra, known as the *virtual algebra* over $\mathcal{V}_{n+1}$. The fibred product $\mathcal{C}(\mathcal{A}) := \mathcal{A} \times_{\mathcal{V}_{n+1}} V^*(\mathcal{V}_{n+1})$, referred to local coordinates t, q^i, z^A, p_i, is called the *contact bundle*.

By construction, $\mathcal{C}(\mathcal{A})$ is a vector bundle over $\mathcal{A}$, canonically isomorphic to the subbundle of the cotangent space $T^*(\mathcal{A})$ locally spanned by the 1-forms $\omega^i = dq^i - \psi^i\, dt$.

(iii) Given any admissible section γ, we denote by $V(\gamma) \xrightarrow{t} \mathbb{R}$ the bundle of vertical vectors along γ, by $A(\hat{\gamma}) \xrightarrow{t} \mathbb{R}$ the totality of vectors along $\hat{\gamma}$ annihilating the 1–form dt and by $V(\hat{\gamma}) \subset A(\hat{\gamma})$ the totality of vertical vectors relative to the fibration $A(\hat{\gamma}) \xrightarrow{\pi} V(\gamma)$. All spaces are referred to fibred coordinates, respectively denoted by t, u^i, t, u^i, v^A and t, v^A, defined according to the identifications $u^i =< dq^i, \cdot >$, $v^A =< dz^A, \cdot >$.

The restriction of $V^*(\mathcal{V}_{n+1})$ to the curve γ determines a vector bundle $V^*(\gamma) \xrightarrow{t} \mathbb{R}$, dual to the vertical bundle $V(\gamma)$. The elements of $V^*(\mathcal{V}_{n+1})$ are called the *virtual* 1*–forms* along γ. The tensor algebra generated by $V(\gamma)$ and $V^*(\gamma)$ is called the *virtual algebra* along γ.

Preserving the notation δ for the virtual differential, every virtual tensor field w along γ is locally represented as $w = w^i{}_{j\cdots}(t) \left(\frac{\partial}{\partial q^i}\right)_\gamma \otimes \delta q^j{}_{|\gamma} \otimes \cdots$.

(iv) In the forthcoming discussion, we shall not deal with ordinary (differentiable) sections, but with *piecewise differentiable* ones, defined on *closed* intervals. In this connection, we recall the following definitions [20]:

- an admissible closed arc $\big(\gamma, [m, n]\big)$ in $\mathcal{V}_{n+1}$ is the restriction to a closed interval $[m, n]$ of an admissible section $\gamma : (m', n') \to \mathcal{V}_{n+1}$ defined on an open interval $(m', n') \supset [m, n]$;

- an evolution of the system in the interval $[t_0, t_1]$ is a finite collection

  $$\big(\gamma, [t_0, t_1]\big) := \big\{\big(\gamma^{(s)}, [a_{s-1}, a_s]\big),\ s = 1, \ldots, N,\ t_0 = a_0 < \cdots < a_N = t_1\big\}$$

  of admissible closed arcs satisfying the matching conditions

  $$\gamma^{(s)}(a_s) \,=\, \gamma^{(s+1)}(a_s)\ , \qquad \forall\, s = 1, \ldots, N-1. \tag{2}$$

On account of Eq. (2), the correspondence $t \to \gamma(t)$ is well–defined and continuous for all $t_0 \leqslant t \leqslant t_1$. The points $x_s := \gamma(a_s)$, $s = 1, \ldots, N-1$, are called the *corners* of γ. The tangent vector to the arc $\gamma^{(s)}$ is denoted by $\dot{\gamma}^{(s)}$.

Along the same guidelines, the lift of an admissible closed arc $\big(\gamma, [m, n]\big)$ is the restriction to $[m, n]$ of the lift $\hat{\gamma} : (m', n') \to \mathcal{A}$, while the lift $\hat{\gamma}$ of an evolution $\big\{\big(\gamma^{(s)}, [a_{s-1}, a_s]\big)\big\}$ is the family of lifts $\hat{\gamma}^{(s)}$, $s = 1, \ldots, N$, each restricted to the corresponding closed interval $[a_{s-1}, a_s]$.

The image $\hat{\gamma}(t)$ is well–defined for $t \neq a_1, \ldots, a_{N-1}$: the map $\hat{\gamma} : [t_0, t_1] \to \mathcal{A}$ may therefore be regarded as a (generally discontinuous) section of the velocity space.

1.2. Deformations

The representation of deformations in the presence of constraints is regarded as known [20]. A few technical aspects are reported below.

(i) An admissible deformation of an admissible closed arc $\big(\gamma, [m, n]\big)$ is a 1–parameter family $\big(\gamma_\xi, [m(\xi), n(\xi)]\big)$, $|\xi| < \varepsilon$ of admissible closed arcs, depending differentiably on ξ and satisfying the condition

$$\big(\gamma_0, [m(0), n(0)]\big) = \big(\gamma, [m, n]\big)\,.$$

An admissible deformation of an evolution $\big(\gamma,[t_0,t_1]\big)$ is likewise a collection $\big\{\big(\gamma_\xi^{(s)},[a_{s-1}(\xi),a_s(\xi)]\big)\big\}$ of admissible deformations of each single arc, satisfying the matching conditions

$$\gamma_\xi^{(s)}(a_s(\xi)) \,=\, \gamma_\xi^{(s+1)}(a_s(\xi)) \qquad \forall\,|\xi|<\varepsilon,\; s=1,\ldots,N-1. \tag{3}$$

For each s, the family of lifts $\hat\gamma_\xi^{(s)}$, restricted to the interval $[a_{s-1}(\xi),a_s(\xi)\,]$, is easily recognized to provide a deformation for the lift $\hat\gamma^{(s)}:[a_{s-1},a_s]\to\mathcal{A}$.

In what follows, we shall only consider deformations leaving the interval $[t_0,t_1]$ fixed, namely those satisfying the conditions $a_0(\xi)=t_0$, $a_N(\xi)=t_1$; no restriction is posed on the functions $a_s(\xi)$, $s=1,\ldots,N-1$, i.e. on the temporal placement of the corners.

For any $s=1,\ldots,N-1$, the curve $c_s(\xi):=\gamma_\xi^{(s)}(a_s(\xi))=\gamma_\xi^{(s+1)}(a_s(\xi))$ is called the *orbit* of the corner x_s under the given deformation.

In local coordinates, setting $q^i(\gamma_\xi^{(s)}(t))=\varphi^i_{(s)}(\xi,t)$, the matching conditions (3) read

$$\varphi^i_{(s)}(\xi,a_s(\xi)) \,=\, \varphi^i_{(s+1)}(\xi,a_s(\xi)), \tag{4}$$

while the representation of the orbit $c_s(\xi)$ takes the form

$$c_s(\xi): \quad t \,=\, a_s(\xi)\,, \quad q^i \,=\, \varphi^i_{(s)}(\xi,a_s(\xi)). \tag{5}$$

(ii) Given an admissible closed arc $\big(\gamma,[m,n]\big)$, an admissible infinitesimal deformation of $\big(\gamma,[m,n]\big)$ tangent to a finite deformation $\big(\gamma_\xi,[m(\xi),n(\xi)]\big)$ is a triple (α,X,β), where X is the restriction to $\big(\gamma,[m,n]\big)$ of the vector field tangent to the orbits of γ_ξ, while α,β are the derivatives

$$\alpha \,=\, \left.\frac{dm}{d\xi}\right|_{\xi=0}, \qquad \beta \,=\, \left.\frac{dn}{d\xi}\right|_{\xi=0}.$$

Likewise, an admissible infinitesimal deformation of an evolution $\big(\gamma,[t_0,t_1]\big)$ is a collection $\big\{\cdots\alpha_{s-1}\,,X_{(s)}\,,\alpha_s\cdots\big\}$ of admissible infinitesimal deformations of each single closed arc, with $\alpha_s=\frac{da_s}{d\xi}{}_{\xi=0}$ (in particular, with $\alpha_0=\alpha_N=0$ whenever the interval $[t_0,t_1]$ is held fixed).

The admissibility of each $X_{(s)}$ requires the existence of a corresponding lift $\hat X_{(s)}=X^i_{(s)}\big(\frac{\partial}{\partial q^i}\big)_{\hat\gamma^{(s)}}+X^A_{(s)}\big(\frac{\partial}{\partial z^A}\big)_{\hat\gamma^{(s)}}$ satisfying the *variational equation*

$$\frac{dX^i_{(s)}}{dt} \,=\, \left(\frac{\partial\psi^i}{\partial q^k}\right)_{\hat\gamma^{(s)}} X^k_{(s)} \,+\, \left(\frac{\partial\psi^i}{\partial z^A}\right)_{\hat\gamma^{(s)}} X^A_{(s)}\,. \tag{6}$$

Moreover, at each corner x_s, the matching conditions (4) entail the relations

$$X^i_{(s+1)}(a_s)-X^i_{(s)}(a_s) \,:=\, \big[X^i\big]_{x_s} \,=\, -\alpha_s\big[\psi^i\big]_{x_s}\,, \tag{7}$$

the symbol [...] denoting the *jump* of its argument.

1.3. Infinitesimal controls

(i) Given an admissible differentiable section $\gamma : \mathbb{R} \to \mathcal{V}_{n+1}$, an *infinitesimal control* along γ is a linear section $h : V(\gamma) \to A(\hat{\gamma})$, described in fibred coordinates as $v^A = h_i{}^A(t)\, u^i$.

The image $h(V(\gamma))$ defines a distribution along $\hat{\gamma}$, called the *horizontal distribution* associated with h, locally spanned by the vector fields

$$\tilde{\partial}_i := h\left[\left(\frac{\partial}{\partial q^i}\right)_{\gamma}\right] = \left(\frac{\partial}{\partial q^i}\right)_{\hat{\gamma}} + h_i{}^A \left(\frac{\partial}{\partial z^A}\right)_{\hat{\gamma}}. \tag{8}$$

Every $\hat{X} = X^i(t)\left(\frac{\partial}{\partial q^i}\right)_{\hat{\gamma}} + X^A(t)\left(\frac{\partial}{\partial z^A}\right)_{\hat{\gamma}} \in A(\hat{\gamma})$ may be uniquely decomposed into the sum of a horizontal vector $\mathcal{P}_H(\hat{X})$ and a vertical vector $\mathcal{P}_V(\hat{X})$, respectively defined by the equations

$$\mathcal{P}_H(\hat{X}) := h\big(\pi_*(\hat{X})\big) = X^i\, \tilde{\partial}_i\,,$$

$$\mathcal{P}_V(\hat{X}) := \hat{X} - \mathcal{P}_H(\hat{X}) = \big(X^A - X^i h_i{}^A\big)\left(\frac{\partial}{\partial z^A}\right)_{\hat{\gamma}} =: U^A \left(\frac{\partial}{\partial z^A}\right)_{\hat{\gamma}}.$$

A section $X : \mathbb{R} \to V(\gamma)$ is said to be h*–transported* along γ if the composite map $h \cdot X : \mathbb{R} \to A(\hat{\gamma})$ is an admissible infinitesimal deformation of $\hat{\gamma}$.

In view of Eqs. (6), (8), the h–transported sections form an n–dimensional vector space V_h, isomorphic to the standard fibre of $V(\gamma)$. Every infinitesimal control provides therefore a *trivialization* of the vector bundle $V(\gamma) \to \mathbb{R}$, summarized into the identification $V(\gamma) \simeq \mathbb{R} \times V_h$. By duality, this entails the analogous identification $V^*(\gamma) \simeq \mathbb{R} \times V_h{}^*$.

(ii) The notion of h–transport induces a derivation $\frac{D}{Dt}$ of the virtual tensor algebra along γ, called the *absolute time derivative*. Introducing the *temporal connection coefficients*

$$\tau_k{}^i := -\tilde{\partial}_k\big(\psi^i\big) = -\left(\frac{\partial \psi^i}{\partial q^k}\right)_{\hat{\gamma}} - h_k{}^A \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat{\gamma}}, \tag{9a}$$

we have the representation

$$\frac{D}{Dt}\left[Z^i{}_{j\,\dots}(t)\left(\frac{\partial}{\partial q^i}\right)_{\gamma} \otimes \delta q^j{}_{|\gamma} \otimes \cdots\right] := \frac{D Z^i{}_{j\,\dots}}{Dt}\left(\frac{\partial}{\partial q^i}\right)_{\gamma} \otimes \delta q^j{}_{|\gamma} \otimes \cdots \quad ,$$

with

$$\frac{D Z^i{}_{j\,\dots}}{Dt} = \frac{d Z^i{}_{j\,\dots}}{dt} + \tau_k{}^i\, Z^k{}_{j\,\dots} - \tau_j{}^k\, Z^i{}_{k\,\dots} + \cdots . \tag{9b}$$

Matters get simplified referring both bundles $V(\gamma), V^*(\gamma)$ to h–transported dual bases $e_{(a)} = e^{\,i}_{(a)}\left(\frac{\partial}{\partial q^i}\right)_{\gamma}$, $e^{(a)} = e^{(a)}_{\,i}\, \delta q^i{}_{|\gamma}$.

Setting $Z = \widetilde{Z}^a{}_{b\,\dots} e_{(a)} \otimes e^{(b)} \otimes \cdots$, Eq. (9a) takes then the form

$$\frac{DZ}{Dt} = \frac{d\widetilde{Z}^a{}_{b\,\cdots}}{dt}\, e_{(a)} \otimes e^{(b)} \otimes \cdots ,$$

i.e. it reduces to the ordinary derivative of the components.

Given any admissible infinitesimal deformation $X = X^i \big(\frac{\partial}{\partial x^i}\big)_\gamma$ of γ, lifting to a deformation $\hat{X}$ of $\hat{\gamma}$, and denoting by $U = U^A \big(\frac{\partial}{\partial z^A}\big)_{\hat{\gamma}}$ the vertical projection $\mathcal{P}_V(\hat{X})$, the variational equation (6) and the lift process may be cast into the form

$$\frac{DX^i}{Dt} = U^A \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat{\gamma}} , \tag{10a}$$

$$\hat{X} \,=\, h(X) + U\,. \tag{10b}$$

(iii) Assigning an infinitesimal control $h^{(s)}$ along each arc $\gamma^{(s)}$ of an evolution $\big(\gamma, [t_0, t_1]\big)$ and arguing as above, we conclude that every admissible infinitesimal deformation $\big\{\cdots \alpha_{s-1}, X_{(s)}, \alpha_s \cdots\big\}$ of γ is determined, up to initial data, by the coefficients $\alpha_1, \ldots, \alpha_{N-1}$ and by N vertical vector fields $U_{(s)} = U^A_{(s)} \big(\frac{\partial}{\partial z^A}\big)_{\hat{\gamma}^{(s)}}$ through the covariant variational equation

$$\frac{DX^i_{(s)}}{Dt} = U^A_{(s)} \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat{\gamma}^{(s)}} \qquad s = 1, \ldots, N, \tag{11a}$$

completed by the jump relations (7). Each lift $\hat{X}_{(s)}$ is then expressed as

$$\hat{X}_{(s)} \,=\, h^{(s)}\big(X_{(s)}\big) + U_{(s)}\,. \tag{11b}$$

(iv) Given an evolution $\big(\gamma, [t_0, t_1]\big)$ and a family $h = \{h^{(s)}\}$ of infinitesimal controls, we glue $h^{(s)}$–transport along each arc $\big(\gamma^{(s)}, [a_{s-1}, a_s]\big)$ and continuity at the corners into a global h*–transport law* along γ.

Once again, this provides a trivialization of the vector bundle $V(\gamma)$ into the cartesian product $[t_0, t_1] \times V_h$, with $V_h \simeq V(\gamma)_{|t}\ \ \forall\, t \in [t_0, t_1]$.

Given any infinitesimal deformation $\big\{\cdots \alpha_{s-1}, X_{(s)}, \alpha_s \cdots\big\}$, we merge all sections $X_{(s)}$ into a piecewise differentiable map $X : [t_0, t_1] \to V_h$, with jump discontinuities at $t = a_s$ expressed by Eq. (7).

The vertical projections $U_{(s)} = \mathcal{P}_V(\hat{X}_{(s)})$ are similarly merged into a single object U, henceforth (improperly) called a vertical vector field along $\hat{\gamma}$.

In this way Eqs. (11) becomes formally identical to Eqs. (10). In particular, in h–transported bases, the determination of the components $\tilde{X}^a$ in terms of U^A and of the scalars α_s relies on the equations

$$\frac{d\tilde{X}^a}{dt} = U^A\, e^{(a)}_i \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat{\gamma}} \qquad \forall\, t \neq a_s, \tag{12a}$$

completed by the jump conditions

$$[\tilde{X}^a]_{x_s} = -\alpha_s\, e^{(a)}_i(a_s)\, [\psi^i]_{x_s} \qquad s = 1, \ldots, N-1\,. \tag{12b}$$

2. The variational setup

2.1. Basic concepts

(i) Given an admissible piecewise differentiable section $\gamma : [t_0, t_1] \to \mathcal{V}_{n+1}$, let $\mathfrak{V}$ and $\mathfrak{W}$ respectively denote the infinite–dimensional vector space formed by the totality of vertical vector fields $U = \big\{U_{(s)}\,,\, s = 1,\dots,N\big\}$ along $\hat\gamma$ and the direct sum $\mathfrak{V} \oplus \mathbb{R}^{N-1}$. Also, let $h = \big(h^{(1)},\dots,h^{(N)}\big)$ denote a collection of (arbitrarily chosen) infinitesimal controls along the arcs of γ.

By Eqs. (12), every solution X of the variational equation is determined, up to initial data, by an element $(U, \underset{\sim}{\alpha}) := (U, \alpha_1, \dots, \alpha_{N-1}) \in \mathfrak{W}$.

In h–transported bases, for any $t \in (a_{r-1}, a_r]\,,\; r = 1,\dots,N$, the resulting expression reads

$$X(t) = \left(\tilde X^a(t_0) + \int_{t_0}^{t} U^A\, e^{(a)}_{\;i} \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat\gamma} dt \,-\, \sum_{s=1}^{r-1} \alpha_s\; e^{(a)}_{\;i}(a_s)\,[\psi^i(\hat\gamma)]_{x_s}\right) e_{(a)}(t)\,.$$

In particular, denoting by $\Upsilon : \mathfrak{W} \to V_h$ linear map defined by the equation

$$\Upsilon(U, \underset{\sim}{\alpha}) := \left(\int_{t_0}^{t_1} U^A\, e^{(a)}_{\;i} \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat\gamma} dt \,-\, \sum_{s=1}^{N-1} \alpha_s\; e^{(a)}_{\;i}(a_s)[\psi^i(\hat\gamma)]_{x_s}\right) e_{(a)}\,, \tag{13}$$

every admissible infinitesimal deformation vanishing at the endpoints of γ is uniquely determined by a vector $(U, \underset{\sim}{\alpha}) \in \ker(\Upsilon) \subset \mathfrak{W}$.

Actually, what really matters in a variational context is not the space $\ker(\Upsilon)$ itself, but the possibly smaller subspace $\Delta(\gamma) \subset \ker(\Upsilon)$ formed by the infinitesimal deformations tangent to admissible finite deformations with fixed endpoints. An evolution γ is called *ordinary* when $\Delta(\gamma) = \ker(\Upsilon)$, *exceptional* when $\Delta(\gamma) \subsetneq \ker(\Upsilon)$.

(ii) A further important information comes from the nature of the inclusion $\Upsilon(\mathfrak{W}) \subset V_h$: an evolution $\big(\gamma, [t_0, t_1]\big)$ is called *normal* if $\Upsilon(\mathfrak{W}) = V_h$, *abnormal* in the opposite case. It is called *locally normal* if its restriction to any closed subinterval $[t_0', t_1'] \subseteq [t_0, t_1]$ is normal. The co-dimension of the image space $\Upsilon(\mathfrak{W})$, henceforth denoted by p, is called the *abnormality index* of γ.

In connection with the stated definitions, a useful result is provided by the following

Proposition 1. *The annihilator* $\big(\Upsilon(\mathfrak{W})\big)^0 \subset V_h^*$ *coincides with the totality of h–transported virtual 1–forms* $\hat\rho = \rho_i\; \delta q^i{}_{|\gamma}$ *satisfying the conditions*

$$\rho_i \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat\gamma} = 0, \qquad A = 1,\dots,r, \tag{14a}$$

$$\rho_i(a_s)\big[\psi^i(\hat\gamma)\big]_{a_s} = 0, \qquad s = 1,\dots,N-1. \tag{14b}$$

Proof. On account of Eq. (13), the subspace $\big(\Upsilon(\mathfrak{W})\big)^0 \subset V^*_h$ consists of the totality of elements $\hat\rho = \rho_a\, e^{(a)} = \rho_a\, \overset{(a)}{e}_i\, \delta q^i{}_{|\gamma}$ satisfying the relation

$$\rho_a \left(\int_{t_0}^{t_1} U^A\, \overset{(a)}{e}_i \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat\gamma} dt \,-\, \sum_{s=1}^{N-1} \alpha_s\, \overset{(a)}{e}_i(a_s) \Big[\psi^i(\hat\gamma)\Big]_{a_s} \right) = 0 \quad \forall\, (U, \underset{\sim}{\alpha}) \in \mathfrak{W},$$

clearly equivalent to Eqs. (14). □

In view of Eqs. (9), (14a), the requirement of h–transport of $\hat\rho$ along each arc $\gamma^{(s)}$ is expressed in coordinates as

$$\frac{d\rho_i}{dt} \,+\, \rho_k \left(\frac{\partial \psi^k}{\partial q^i}\right)_{\hat\gamma} \,+\, \cancel{h_i{}^A\, \rho_k \left(\frac{\partial \psi^k}{\partial z^A}\right)_{\hat\gamma}} \,=\, 0.$$

The content of Proposition 1 is therefore independent of the choice of the infinitesimal control along γ.

On account of Proposition 1, a direct evaluation of the abnormality index may be based on the following

Corollary 1. *Given an infinitesimal control* $h\,:\,V(\gamma) \to A(\hat\gamma)$*, arbitrarily choose a* h*–transported basis* $e^{(a)} = \overset{(a)}{e}_i\, \delta q^i{}_{|\gamma}$ *and a symmetric, positive definite tensor field* G *along* γ*, meant as a collection* $G_{(s)} = G^{AB}_{(s)}(t)\big(\frac{\partial}{\partial z^A}\big)_{\hat\gamma} \otimes \big(\frac{\partial}{\partial z^B}\big)_{\hat\gamma}$*,* $s = 1, \ldots, N$*. Then, the matrix*

$$S^{ab} = \sum_{s=1}^{N} \int_{a_{s-1}}^{a_s} G^{AB}_{(s)} \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat\gamma^{(s)}} \left(\frac{\partial \psi^j}{\partial z^B}\right)_{\hat\gamma^{(s)}} \overset{(a)}{e}_i\, \overset{(b)}{e}_j\; dt \,+ \\ +\, \sum_{s=1}^{N-1} \big[\psi^i(\hat\gamma)\big]_{a_s} \big[\psi^k(\hat\gamma)\big]_{a_s} \overset{(a)}{e}_i(a_s)\, \overset{(b)}{e}_j(a_s) \qquad (15)$$

has rank $n - p$*.*

Proof. By construction, the matrix (15) determines a positive semidefinite quadratic form $S^{ab}\rho_a\,\rho_b$ on V^*_h. The kernel of S^{ab} is therefore identical to the totality of h–transported 1–forms $\rho = \rho_a\, e^{(a)} = \rho_i\, \delta q^i{}_{|\gamma}$ satisfying the relation

$$\sum_{s=1}^{N} \int_{a_{s-1}}^{a_s} G^{AB}_{(s)}\; \rho_i \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat\gamma^{(s)}} \rho_j \left(\frac{\partial \psi^k}{\partial z^B}\right)_{\hat\gamma^{(s)}} dt \,+\, \sum_{s=1}^{N-1} \Big(\rho_i \big[\psi^i(\hat\gamma)\big]_{a_s}\Big)^2 = 0.$$

Due to the positive definiteness of $G^{AB}_{(s)}$, the latter is equivalent to the pair of conditions (14). This proves $\dim[\ker(S^{ab})] = p \implies \operatorname{rank}(S^{ab}) = n - p$. □

2.2. Finite deformations with fixed endpoints: an existence theorem

(i) We shall now discuss the interplay between normality and ordinariness [2]. The following preliminaries help simplifying the subsequent discussion.

[2]For an alternative approach, valid in a linear context, see [12].

Proposition 2. *Let $\hat{\gamma} : (c,d) \to \mathcal{A}$ be the lift of an admissible differentiable section $\gamma : (c,d) \to \mathcal{V}_{n+1}$. Then, for any closed interval $[a,b] \subset (c,d)$ there exists a fibred local chart (U,k), $k = (t, q^1, \ldots, q^n, z^1, \ldots, z^r)$ satisfying the properties*

- $\hat{\gamma}(t) \in U \quad \forall\, t \in [a,b]$; (16a)
- *the intersection* $\hat{\gamma}\big((c,d)\big) \cap U$ *coincides with the curve* $q^i = z^A = 0$; (16b)
- $\psi^i\big(\hat{\gamma}(t)\big) = \left(\frac{\partial \psi^i}{\partial q^k}\right)_{\hat{\gamma}(t)} = 0 \qquad \forall\, \hat{\gamma}(t) \in U.$ (16c)

Proof. The existence of fibred local charts $(U, \hat{k})$, $\hat{k} = (t, \hat{q}^1, \cdots \hat{q}^n, \hat{z}^1, \cdots \hat{z}^r)$ satisfying Eqs. (16a), (16b) and the first condition (16c) is entirely straightforward. Choose any such chart, and denote by $\hat{\dot{q}}^i = \hat{\psi}^i(t, \hat{q}^i, \hat{z}^A)$ the corresponding representation of the imbedding $\mathcal{A} \to j_1(\mathcal{V}_{n+1})$. Under an arbitrary transformation $q^i = \alpha^i{}_j(t)\, \hat{q}^j$, $z^A = \hat{z}^A$ we have then the relations

$$\psi^i = \frac{d\alpha^i{}_j}{dt}\, \hat{q}^j + \alpha^i{}_j\, \hat{\psi}^j, \qquad \frac{\partial \psi^i}{\partial q^k} = \left(\frac{d\alpha^i{}_j}{dt} + \alpha^i{}_r\, \frac{\partial \hat{\psi}^r}{\partial \hat{q}^j} \right) (\alpha^{-1})^j{}_k\,.$$

Therefore, if the matrix $\alpha^i{}_j(t)$ obeys the transport law

$$\frac{d\alpha^i{}_j}{dt} + \alpha^i{}_r \left(\frac{\partial \hat{\psi}^r}{\partial \hat{q}^j} \right)_{\hat{\gamma}(t)} = 0,$$

the coordinates t, q^i, z^A fulfil all stated requirements. □

Every local chart (U,k) consistent with Eqs. (16) is said to be *adapted* to the arc $\big(\hat{\gamma}, [a,b]\big)$. The adaptedness property is stable under arbitrary transformations of the form

$$\bar{q}^i = \bar{q}^i(q^1, \ldots, q^n)\,, \qquad \bar{z}^A = \bar{z}^A(t, q^1, \ldots, q^n, z^1, \ldots, z^r), \tag{17}$$

with $\bar{q}^i(0, \ldots, 0) = \bar{z}^A(t, 0, \ldots, 0, \ldots, 0) = 0$.

Corollary 2. *Let $\hat{\gamma} = \big\{\big(\hat{\gamma}^{(s)}, [a_{s-1}, a_s]\big),\ s = 1, \ldots, N\big\}$ be the lift of an admissible piecewise differentiable section $\big(\gamma, [t_0, t_1]\big)$. Then, there exist fibred local charts (U_s, k_s), $k_s = \big(t, q^{\,1}_{(s)}, \ldots, q^{\,n}_{(s)}, z^{\,1}_{(s)}, \ldots, z^{\,r}_{(s)}\big)$ adapted to the arcs $\hat{\gamma}^{(s)}$ such that, in each intersection $\pi(U_s) \cap \pi(U_{s+1})$, the coordinate transformations $q^{\,i}_{(s+1)} = q^{\,i}_{(s+1)}(t, q^{\,1}_{(s)}, \ldots, q^{\,n}_{(s)})$ satisfy the conditions*

$$\left(\frac{\partial q^{\,i}_{(s+1)}}{\partial q^{\,j}_{(s)}} \right)_{\gamma(a_s)} = \delta^i_j\,; \qquad \left(\frac{\partial q^{\,i}_{(s+1)}}{\partial t} \right)_{\gamma(a_s)} = -\Big[\psi^i(\hat{\gamma})\Big]_{a_s}. \tag{18}$$

Proof. The conclusion follows at once from Proposition 2, observing that the freedom in the choice of the adapted coordinates, summarized into Eqs. (17), leaves full control on the values of the Jacobians $\left(\frac{\partial q^{\,i}_{(s+1)}}{\partial q^{\,j}_{(s)}} \right)_{\gamma(a_s)}$.

In particular, in the intersection $\pi(U_s)\cap\pi(U_{s+1})$, the arcs $\gamma^{(s)}, \gamma^{(s+1)}$ are respectively described by the equations

$$\begin{cases} q^{\,i}_{(s+1)}(\gamma^{(s)}) = q^{\,i}_{(s+1)}\big(t, q^{\,1}_{(s)}(\gamma^{(s)}),\dots,q^{\,n}_{(s)}(\gamma^{(s)})\big) = q^{\,i}_{(s+1)}(t,0,\dots,0), \\ q^{\,i}_{(s+1)}(\gamma^{(s+1)}) = 0. \end{cases}$$

In the coordinate system $t, q^{\,i}_{(s+1)}$, the jump of the tangent vector at the corner $\gamma(a_s)$ is therefore given by

$$\Big[\psi^i(\hat\gamma)\Big]_{a_s} = \left(\frac{dq^{\,i}_{(s+1)}(\gamma^{(s+1)})}{dt} - \frac{dq^{\,i}_{(s+1)}(\gamma^{(s)})}{dt}\right)_{\gamma(a_s)} = -\left(\frac{\partial q^{\,i}_{(s+1)}}{\partial t}\right)_{\gamma(a_s)}.$$

□

Every family of local charts $\big\{(U_s,k_s),\ s=1,\dots,N\big\}$ satisfying the requirements of Corollary 2 is said to be *adapted* to the section γ.

(ii) Assigning an adapted family of local charts singles out a distinguished class of controls $\sigma^{(s)}\colon \pi(U_s)\to U_s$, described in coordinates as

$$\sigma^{(s)}\colon \quad z^{\,A}_{(s)}(t,q^{\,1}_{(s)},\dots,q^{\,n}_{(s)}) \,=\, 0. \tag{19}$$

For each $s=1,\dots,N$, the restriction of the tangent map $\sigma^{(s)}_*$ to the vertical bundle $V(\gamma^{(s)})$ determines an infinitesimal control $h^{(s)}: V(\gamma^{(s)})\to\mathcal{A}(\gamma^{(s)})$, expressed in coordinates as

$$h^{(s)}\left(\frac{\partial}{\partial q^{\,i}_{(s)}}\right)_{\gamma^{(s)}(t)} = \sigma^{(s)}_*\left(\frac{\partial}{\partial q^{\,i}_{(s)}}\right)_{\gamma^{(s)}(t)} = \left(\frac{\partial}{\partial q^{\,i}_{(s)}}\right)_{\hat\gamma^{(s)}(t)} \iff \quad h_i{}^A(t) = 0.$$

In view of Eqs. (9), (16c), the absolute time derivative associated with $h^{(s)}$ coincides with the ordinary derivative.

Moreover, since, according to Eq. (18), the fields $\big(\frac{\partial}{\partial q^{\,i}_{(s)}}\big)_{\gamma^{(s)}}$ — and therefore also the virtual 1–forms $\delta q^i{}_{|\gamma^{(s)}}$ — match continuously at the corners, the sections $e_{(i)}:[t_0,t_1]\to V(\gamma)$, $e^{(i)}:[t_0,t_1]\to V^*(\gamma)$ respectively defined by

$$e_{(i)}(t) = \left(\frac{\partial}{\partial q^{\,i}_{(s)}}\right)_{\gamma^{(s)}(t)}, \quad e^{(i)}(t) \,=\, \delta q^i{}_{|\gamma^{(s)}(t)}, \quad a_{s-1}\le t\le a_s \tag{20}$$

form a dual bases for the vector spaces V_h, V^*_h of h–transported fields along γ.

(iii) Let us now come to the main question: given an admissible, piecewise differentiable section $\gamma := \big\{\big(\gamma^{(s)},[a_{s-1},a_s]\big)\big\}$, we single out a family of adapted local charts $\{(U_s,k_s)\}$, and denote by $h=\big\{h^{(s)}: V(\gamma^{(s)})\to\mathcal{A}(\gamma^{(s)})\big\}$ the corresponding infinitesimal control.

As pointed out in Sec. 2.1, every infinitesimal deformation X of γ vanishing at $t=t_0$ is determined by an element $(U,\underset{\sim}{\alpha})\in\mathfrak{W}$, namely by a vertical vector field U along $\hat\gamma$ and by a collection of real numbers $\underset{\sim}{\alpha}=(\alpha_1,\dots,\alpha_{N-1})$.

In particular, a necessary and sufficient condition for X to vanish at both endpoints of γ is expressed by the requirement $\Upsilon(U, \underset{\sim}{\alpha}) = 0$ which, in adapted coordinates, reads

$$\int_{t_0}^{t_1} U^A \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat{\gamma}} dt \,-\, \sum_{s=1}^{N-1} \alpha_s \left[\psi^i(\hat{\gamma})\right]_{a_s} \,=\, 0. \tag{21}$$

This, of course, does not ensure that *any* pair $(U, \underset{\sim}{\alpha})$ satisfying Eq. (21) determines an infinitesimal deformation tangent to a finite deformation with fixed endpoints. To analyse this aspect, we introduce an auxiliary tensor field $G^{AB} \left(\frac{\partial}{\partial z^A}\right)_{\hat{\gamma}} \otimes \left(\frac{\partial}{\partial z^B}\right)_{\hat{\gamma}}$ along $\hat{\gamma}$, with G^{AB} symmetric and positive definite.

We then set up a procedure assigning to each $(U, \underset{\sim}{\alpha}) \in \mathfrak{W}$ a finite deformation of γ, depending parametrically on a h–transported virtual 1–form $\nu \in V_h^*$: to this end, we introduce

- a family of functions

$$a_s(\xi, \nu) \,:=\, a_s \,+\, \alpha_s\, \xi \,-\, \tfrac{1}{2}\, \nu_i \left[\psi^i(\hat{\gamma})\right]_{a_s} \xi^{\,2}\,, \qquad s = 0, \ldots, N, \tag{22a}$$

  expressing the deformation of the temporal placement of the corners. For notational convenience, Eq. (22a) includes the values $s = 0$ and $s = N$, with the understanding $\alpha_0 = \alpha_N = 0$;

- a family of controls $\sigma^{(s)}_{(\xi,\nu)} \,:\, \pi(U_s) \,\to\, U_s$, $s = 1, \ldots, N$, described in adapted coordinates as [3]

$$z^{\,A}_{(s)} \;=\; U^A_{(s)}(t)\,\xi + \tfrac{1}{2}\,\nu_i \left(G^{AB}\,\frac{\partial \psi^i}{\partial z^B}\right)_{\hat{\gamma}^{(s)}} \xi^2 := z^{\,A}_{(s)}(\xi, \nu, t). \tag{22b}$$

We have then the following

Theorem 1. *For any open bounded subset $\Delta \subset V_h^*$ there exist an $\varepsilon > 0$ and a family $\gamma_{(\xi,\nu)} = \left\{\left(\gamma^{(s)}_{(\xi,\nu)}, \left[a_{s-1}(\xi,\nu), a_s(\xi,\nu)\right]\right)\right\}$ of piecewise differentiable admissible sections defined for $|\xi| < \varepsilon$, $\nu \in \Delta$ and satisfying the properties:*

a) $\gamma_{(0,\nu)}(t) = \gamma(t) \quad \forall\, \nu$;

b) $\gamma_{(\xi,\nu)}(t_0) = \gamma(t_0) \quad \forall\, \xi, \nu$;

c) $\gamma^{(s)}_{(\xi,\nu)}(a_s(\xi,\nu)) = \gamma^{(s+1)}_{(\xi,\nu)}(a_s(\xi,\nu)) \quad \forall\, \xi, \nu\,,\ \forall\, s = 1, \ldots, N-1$

d) *each arc $\gamma^{(s)}_{(\xi,\nu)}$ satisfies the condition $\sigma^{(s)}_{(\xi,\nu)} \cdot \gamma^{(s)}_{(\xi,\nu)} = \hat{\gamma}^{(s)}_{(\xi,\nu)}$, i.e. it* belongs *to the control $\sigma^{(s)}_{(\xi,\nu)}$ in the sense of Eq.* (1b).

[3] Eq. (22b) is strictly coordinate–dependent. As such, it has no invariant geometrical meaning, but is merely a tool for the subsequent construction of a family of finite deformations.

Proof. On account of Eq. (22b), for any bounded open subset $\Delta \subset V_h^*$, there exists $m > 0$ small enough as to ensure $\sigma^{(s)}_{(\xi,\nu)}(\pi(U_s)) \subset U_s \;\; \forall \nu \in \Delta,\; |\xi| < m$, $s = 1, \ldots, N$. Recall that, for each s, the control (22b) determines a flow on $\pi(U_s)$, depending parametrically on ξ and ν, and described by the equations

$$\frac{dq^i_{(s)}}{dt} = \psi^i_{(s)}(t, q^i_{(s)}, z^A_{(s)}(\xi,\nu,t)) := Z^i_{(s)}(\xi,\nu,t,q^i_{(s)}) \tag{23}$$

Regard the latter as a vector field $Z_{(s)} = \frac{\partial}{\partial t} + Z^i_{(s)} \frac{\partial}{\partial q^i_{(s)}}$ in the product manifold $(-m,m)\times\Delta\times\pi(U_s)$, denote by $\zeta^{(s)}_\tau$ the associated local 1–parameter group of diffeomorphisms, and restore the notation x_s for the corner $\gamma(a_s)$.

Then, on account of Eq. (16c), for any $\nu^* \in \Delta$, the orbit of $\zeta^{(s)}_\tau$ through the point $(0,\nu^*,x_{s-1})$ coincides with the coordinate line $\xi = 0$, $\nu = \nu^*$, $q^i_{(s)} = 0$ and, as such, it is defined for all τ in the interval $[0, b_s - a_{s-1}) \supset [0, a_s - a_{s-1}]$.

Taking the compactness of $\bar\Delta$ and Eq. (22a) into account, a standard result [23, 24] ensures the existence of an $\varepsilon > 0$ and of a family of open neighborhoods $W_{s-1} \ni x_{s-1}$, $s = 1, \ldots, N$, such that

- the map $\zeta^{(s)}_\tau$ is well defined on $(-\varepsilon,\varepsilon)\times\Delta\times W_{s-1}$ for all $\tau \in [0, b_s - a_{s-1})$;
- $a_s(\xi,\nu) - a_{s-1}(\xi,\nu) < b_s - a_{s-1}$.

From this, denoting by Σ_η the slice $t = \eta$ in $\mathcal{V}_{n+1}$, we conclude that, for each $|\xi| < \varepsilon$, $\nu \in \Delta$, the flow (23) determines a diffeomorphism of $W_{s-1} \cap \Sigma_{a_{s-1}(\xi,\nu)}$ onto a submanifold of $\Sigma_{a_s(\xi,\nu)}$. Without loss of generality we may arrange that the image of each $W_{s-1}\cap\Sigma_{a_{s-1}(\xi,\nu)}$ is contained in $W_s \cap \Sigma_{a_s(\xi,\nu)}$, $s = 1, \ldots, N$.

The rest is straightforward: for each $|\xi| < \varepsilon$, $\nu \in \Delta$, consider the sequence of closed arcs $\gamma^{(s)}_{(\xi,\nu)} : [a_{s-1}(\xi,\nu), a_s(\xi,\nu)] \to \pi(U_s)$ defined inductively by

$$\begin{aligned}
\gamma^{(1)}_{(\xi,\nu)}(t) &= \zeta^{(1)}_t(\xi,\nu,\gamma(t_0)) && t \in [t_0, a_1(\xi,\nu)]\,, \\
\gamma^{(s+1)}_{(\xi,\nu)}(t) &= \zeta^{(s+1)}_t\big(\xi,\nu,\gamma^{(s)}_{(\xi,\nu)}(a_s(\xi,\nu))\big) && t \in [a_s(\xi), a_{s+1}(\xi)]\,.
\end{aligned}$$

The collection $\gamma_{(\xi,\nu)} := \big\{\big(\gamma^{(s)}_{(\xi,\nu)}, [a_{s-1}(\xi,\nu), a_s(\xi,\nu)]\big),\; s = 1, \ldots, N\big\}$ is then easily recognized to fulfil all stated requirements. □

In adapted coordinates, each arc $\gamma^{(s)}_{(\xi,\nu)}$ is represented in the form

$$q^i_{(s)} = \varphi^i_{(s)}(\xi,\nu,t)\,, \qquad a_{s-1}(\xi,\nu) \le t \le a_s(\xi,\nu),$$

with the functions $\varphi^i_{(s)}$ satisfying the differential equations

$$\frac{\partial \varphi^i_{(s)}}{\partial t} = \psi^i\bigg(t\,,\, \varphi^i_{(s)}\,,\, \xi\, U^A_{(s)}(t) + \tfrac{1}{2}\,\nu_k\,\xi^2 \bigg(G^{AB}\,\frac{\partial \psi^k}{\partial z^B}\bigg)_{\hat\gamma^{(s)}(t)}\bigg) \tag{24a}$$

as well as the matching conditions

$$\varphi^{\,i}_{(s+1)}\big(\xi,\nu,a_s(\xi,\nu)\big) = q^{\,i}_{(s+1)}\big(a_s(\xi,\nu),\varphi^{\,i}_{(s)}\big(\xi,\nu,a_s(\xi,\nu)\big)\big)\,, \tag{24b}$$

$q^{\,i}_{(s+1)} = q^{\,i}_{(s+1)}(t,q^{\,1}_{(s)},\ldots,q^{\,n}_{(s)})$ denoting the transformation between adapted coordinates in the intersection $\pi(U_s)\cap\pi(U_{s+1})$.

From this, taking Eqs. (16c), (18) into account, it is easily seen that, for any $\nu\in\Delta$, the 1–parameter family of sections $\gamma_{(\xi,\nu)}$ is a deformation of γ, tangent to the infinitesimal deformation X determined by the vector $(U,\underset{\sim}{\alpha})\in\mathfrak{W}$. We have therefore the identification $\Big(\frac{\partial\varphi^{\,i}_{(s)}(\xi,\nu,t)}{\partial\xi}\Big)_{\xi=0} = X^{\,i}_{(s)}(t)$, completed by the jump relations (7).

After these preliminaries, let us now focus on two facts:

- on account of Theorem 1, given any bounded open subset $\Delta\subset V^*_h$, the correspondence $(\xi,\nu)\to\gamma_{(\xi,\nu)}(t_1)$ determines a differentiable map χ of the cartesian product $(-\varepsilon,\varepsilon)\times\Delta$ into the slice $t=t_1$ in $\mathcal{V}_{n+1}$, with values in a neighborhood of the point $\gamma(t_1)$;

- given any differentiable curve $\nu=\nu(\xi)$ in Δ, the 1–parameter family of sections $\gamma_{(\xi,\nu(\xi))}(t)$, $|\xi|<\varepsilon,\ t\in[t_0,t_1]$ is a deformation of γ, leaving the first endpoint $\gamma(t_0)$ fixed.

Both assertions are entirely obvious; in adapted coordinates, the map χ is represented in the form

$$q^{\,i}_{(N)}(\chi(\xi,\nu)) = \varphi^{\,i}_{(N)}(\xi,\nu,t_1) := \chi^i(\xi,\nu). \tag{25}$$

Exactly as above, it may be seen that, for any choice of the function $\nu(\xi)$, the deformation $\gamma_{(\xi,\nu(\xi))}$ is automatically tangent to the infinitesimal deformation X determined by the vector $(U,\underset{\sim}{\alpha})\in\mathfrak{W}$.

From this, taking the relations $\chi^i(0,\nu)=0$, $\big(\frac{\partial\chi^i}{\partial\xi}\big)_{\xi=0} = X^i(t_1)$ into account and recalling Taylor's theorem, we conclude that, whenever the condition $X(t_1)=0$ holds true, i.e. whenever the vector $(U,\underset{\sim}{\alpha})$ belongs to $\ker(\Upsilon)$, the functions χ^i admit the factorization

$$\chi^i = \tfrac{1}{2}\,\xi^2\,\theta^i(\xi,\nu), \tag{26}$$

with $\theta^i(\xi,\nu)$ regular at $\xi=0$.

In this way, the original problem is reduced to establishing under what circumstances the validity of Eqs. (21) is sufficient to ensure the existence of an $\varepsilon'>0$ and of a curve $\nu(\xi)$ satisfying $\chi\big(\xi,\nu(\xi)\big)=\gamma(t_1)\ \forall\,|\xi|<\varepsilon'$.

In adapted coordinates, on account of Eq. (26), the answer relies on the solvability of the equations

$$\theta^i(\xi,\nu_1,\ldots,\nu_n) = 0 \qquad i=1,\ldots,n \tag{27}$$

for the unknowns ν_i in a neighborhood of $\xi=0$.

To examine this aspect we notice that, in each adapted chart, Eqs. (16c), (24a) imply the transport law

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 \varphi^{\,i}_{(s)}}{\partial \xi^2}\right)_{\xi=0} = \left(\frac{\partial^2 \psi^i}{\partial q^k \partial q^r}\right)_{\hat\gamma(s)} X^k X^r + 2\left(\frac{\partial^2 \psi^i}{\partial q^k \partial z^A}\right)_{\hat\gamma(s)} X^k U^A + \\ + \left(\frac{\partial^2 \psi^i}{\partial z^A \partial z^B}\right)_{\hat\gamma(s)} U^A U^B + \left(\frac{\partial \psi^i}{\partial z^A}\right)_{\hat\gamma(s)} \nu_k \left(G^{AB}\,\frac{\partial \psi^k}{\partial z^B}\right)_{\hat\gamma(s)} .$$

In a similar way, from the matching conditions (24b), recalling Eqs. (18), (22a) and evaluating everything at $\xi = 0$, we get the jump relations

$$\left[\left(\frac{\partial^2 \varphi^{\,i}_{(s+1)}}{\partial \xi^2}\right)_{\xi=0} - \left(\frac{\partial^2 \varphi^{\,i}_{(s)}}{\partial \xi^2}\right)_{\xi=0}\right]_{x_s} = \alpha_s^{\,2}\,\frac{\partial^2 q^{\,i}_{(s+1)}}{\partial t^2} + 2\alpha_s\,\frac{\partial^2 q^{\,i}_{(s+1)}}{\partial t\,\partial q^{\,k}_{(s)}}\,X^{\,k}_{(s)} + \\ + \frac{\partial^2 q^{\,i}_{(s+1)}}{\partial q^{\,h}_{(s)}\,\partial q^{\,k}_{(s)}}\,X^{\,h}_{(s)} X^{\,k}_{(s)} - 2\alpha_s\left[\frac{dX^{\,i}_{(s+1)}}{dt} - \frac{dX^{\,i}_{(s)}}{dt}\right]_{x_s} + \left[\psi^i(\hat\gamma)\right]_{x_s}\left[\psi^k(\hat\gamma)\right]_{x_s}\nu_k$$

Summing up and recalling Eqs. (25), (26) we obtain the expression

$$\theta^i\big|_{\xi=0} = \left(\frac{\partial^2 \chi^i}{\partial \xi^2}\right)_{\xi=0} = \\ = \left(\sum_{s=1}^{N}\int_{a_{s-1}}^{a_s}\left(G^{AB}\,\frac{\partial \psi^i}{\partial z^A}\,\frac{\partial \psi^k}{\partial z^B}\right)_{\hat\gamma(s)} dt + \sum_{s=1}^{N-1}\left[\psi^i(\hat\gamma)\right]_{a_s}\left[\psi^k(\hat\gamma)\right]_{a_s}\right)\nu_k + b^i \tag{28}$$

with the terms $b^i \in \mathbb{R}$ summarizing all contributions independent of ν. We can therefore state

Proposition 3. *Let $\gamma : [t_0, t_1] \to \mathcal{V}_{n+1}$ be a continuous, piecewise differentiable, admissible section. Then, if the matrix*

$$S^{ik} := \int_{t_0}^{t_1}\left(G^{AB}\,\frac{\partial \psi^i}{\partial z^A}\,\frac{\partial \psi^k}{\partial z^B}\right)_{\hat\gamma} dt + \sum_{s=1}^{N-1}\left[\psi^i(\hat\gamma)\right]_{a_s}\left[\psi^k(\hat\gamma)\right]_{a_s} \tag{29}$$

is non–singular, every infinitesimal deformation of γ vanishing at the endpoints is tangent to a finite deformation with fixed endpoints.

Proof. On account of Eq. (28), the non–singularity of the matrix (29) ensures the solvability of Eq. (27) for the unknowns ν_i in a neighborhood of $\xi = 0$. □

Recalling Corollary 1, the definition of ordinariness and the fact that, in adapted coordinates, the matrices (15), (29) coincide, we conclude:

Corollary 3. *The normal evolutions form a subset of the ordinary ones.*

More generally, we have the following

Corollary 4. *Let $p\ (\geq 0)$ denote the abnormality index of γ. Then a sufficient condition for the ordinariness γ is the existence of an $(n-p)$–dimensional submanifold $S_{t_1} \subset \Sigma_{t_1}$ such that, given any deformation γ_ξ leaving $\gamma(t_0)$ fixed, the inclusion $\gamma_\xi(t_1) \in S_{t_1}$ is fulfilled for all ξ in a neighborhood of $\xi = 0$.*

Proof. Using the freedom expressed by Eq. (17), we choose the adapted coordinates in such a way that, in a neighborhood of $\gamma(t_1)$, the submanifold S_{t_1} has local equation $q^{p+1}_{(N)} = \cdots = q^n_{(N)} = 0$.

Given an infinitesimal deformation X generated by an element $(U, \underset{\sim}{\alpha}) \in \mathfrak{W}$, we then proceed exactly as above, ending up with a finite deformation $\gamma_{(\xi,\nu)}$, described in coordinates as $q^{\,i}_{(s)} = \varphi^{\,i}_{(s)}(\xi,\nu,t)$, $a_{s-1}(\xi,\nu) \leq t \leq a_s(\xi,\nu)$, with the functions $\varphi^{\,i}_{(s)}(\xi,\nu,t)$ satisfying all conditions of Theorem 1.

Once again, we focus on the "end–point map" $\chi(\xi,\nu) = \gamma_{(\xi,\nu)}(t_1)$ and on the fact that, under the assumption $X(t_1) = 0$, the functions $\chi^i(\xi,\nu)$ factorize into the product (26),

$$\chi^i = \varphi^{\,i}_{(N)}(\xi,\nu,t_1) = \tfrac{1}{2}\,\xi^2\,\theta^i(\xi,\nu),$$

with

$$\theta^i\big|_{\xi=0} = \left(\frac{\partial^2 \chi^i}{\partial \xi^2}\right)_{\xi=0} = b^i + S^{ij}\nu_j\,.$$

and

$$S^{ik} := \int_{t_0}^{t_1} \left(G^{AB}\,\frac{\partial \psi^i}{\partial z^A}\,\frac{\partial \psi^k}{\partial z^B}\right)_{\hat\gamma} dt \,+\, \sum_{s=1}^{N-1} \Big[\psi^i(\hat\gamma)\Big]_{a_s} \Big[\psi^k(\hat\gamma)\Big]_{a_s}$$

We then observe that the assumption $\chi^\alpha(\xi,\nu) = 0\ \forall\,|\xi| < \varepsilon$, $\alpha = p{+}1,\ldots,n$ implies $\theta^\alpha(\xi,\nu) = 0$, whence, in particular, $b^\alpha = S^{\alpha j} = 0$.

At the same time, Corollary 1 ensures the equality between the co-rank of S^{ij} and the abnormality index p of γ.

The matrix S^{ij} is therefore necessarily of the form

$$S^{ij} = \begin{pmatrix} S^{AB} & 0 \\ 0 & 0 \end{pmatrix}, \qquad A,B = 1,\ldots,p\,,$$

with $\det S^{AB} \neq 0$.

The rest is now straightforward: in order to establish the existence of a finite deformation with fixed endpoints tangent to X, we have to verify that the equations $\theta^i(\xi,\nu) = 0$ admit at least one solution $\nu = \nu(\xi)$ in a neighborhood of $\xi = 0$. And indeed, no matter how we choose the functions $\nu_{p+1}(\xi),\ldots,\nu_n(\xi)$, the equations $\theta^{p+1}(\xi,\nu) = \theta^n(\xi,\nu) = 0$ are identically satisfied, while the remaining ones form a system of p equations for the unknowns $\nu_1,\ldots,\nu_p$, whose solvability is ensured by the non singularity of the Jacobian $\frac{\partial\theta^A}{\partial\nu_B}\big|_{\xi=0} = S^{AB}$. $\square$

3. Examples

The following arguments help clarifying some aspects of the concept of normality discussed in Sec. 2.1.

Let $\gamma = \{(\gamma^{(s)}, [a_{s-1}, a_s])\}$ be a piecewise differentiable evolution. According to Proposition 1, if at least one arc $\gamma^{(s)}$ is normal, γ is necessarily normal.

More generally, an evolution may turn out to be normal even when *all* of its arcs $\gamma^{(s)}$ are abnormal. Examples in this sense are:

• $\mathcal{V}_{n+1} = \mathbb{R} \times E_2$, referred to coordinates t, x, y. Constraint $\dot{x}^2 + \dot{y}^2 = v^2$; imbedding $\mathcal{A} \to j_1(\mathcal{V}_{n+1})$ expressed in coordinates as $\dot{x} = v \cos z$, $\dot{y} = v \sin z$.

Consider a piecewise differentiable evolution γ consisting of two arcs:

$$\begin{array}{llll} \gamma^{(1)}: & x = 0, & y = vt & t_0 \leq t \leq 0 \\ \gamma^{(2)}: & x = vt, & y = 0 & 0 \leq t \leq t_1 \end{array}$$

Then, Eq. (14a) admits h–transported solutions $\rho^{(1)} = \alpha \delta y_{|\gamma}$, $\rho^{(2)} = \beta \delta x_{|\gamma}$ ($\alpha, \beta \in \mathbb{R}$) respectively along $\gamma^{(1)}$ and along $\gamma^{(2)}$. Both arcs are therefore abnormal. Nevertheless γ is normal, since no pair $\rho^{(1)}, \rho^{(2)}$ matches into a continuous non–zero virtual 1–form along γ.

• $\mathcal{V}_{n+1} = \mathbb{R} \times E_2$, referred to coordinates t, x, y. Constraint: $v^3 \dot{x} = (\dot{y}^2 - a^2 t^2)^2$; imbedding $\mathcal{A} \to j_1(\mathcal{V}_{n+1})$ expressed in coordinates as $\dot{x} = v^{-3}(z^2 - a^2 t^2)^2$, $\dot{y} = z$. Consider a piecewise differentiable evolution γ consisting of two arcs:

$$\begin{array}{llll} \gamma^{(1)}: & x = 0, & y = \dfrac{1}{2}\, a (t^2 - t^{*2}) & t_0 \leq t \leq t^* \\ \gamma^{(2)}: & x = \dfrac{a^4}{5 v^3}\, (t^5 - t^{*5}), & y = 0 & t^* \leq t \leq t_1 \end{array}$$

($t^* \neq 0$). Then, Eq. (14a) admits h–transported solutions of the form $\rho = \alpha \delta x_{|\gamma}$ along the whole of γ. Both arcs $\gamma^{(1)}$, $\gamma^{(2)}$ are therefore abnormal. In spite of this fact, γ is normal, since no solution satisfies condition (14b).

• By definition, local normality implies normality. The converse is generally untrue, as shown by the previous examples. A further example, not relying on the presence of corners, is the following: let the imbedding $\mathcal{A} \xrightarrow{i} j_1(\mathcal{V}_{n+1})$ be locally described by the equations

$$\begin{cases} \dot{q}^A = z^A & A = 1, \ldots, n-1 \\ \dot{q}^n = f(t)\, z^1 \end{cases}$$

with $f(t) = \exp(-1/t^2)$ for $t < 0$ and $f(t) = 0$ for $t \geq 0$.

Along any admissible section $\gamma \colon [t_0, t_1] \to \mathcal{V}_{n+1}$, the condition of h–transport and Eqs. (14a) are summarized into the requirements

$$\frac{d\lambda_i}{dt} = 0\,, \qquad \lambda_1 + \lambda_n f(t) = 0\,, \qquad \lambda_2 = \cdots = \lambda_{n-1} = 0\,. \tag{30}$$

In particular, if $t_0 < 0 < t_1$, we conclude that:

◇ γ is normal, since Eqs. (30) do not admit any non-zero solution in $[t_0, t_1]$;

◇ γ is not locally normal: Eqs. (30) do in fact admit solutions of the form $\lambda_1 = \cdots = \lambda_{n-1} = 0,\ \lambda_n = \text{const.}$ in any subinterval $[a, b] \subseteq [0, t_1]$.

References

[1] O. Bolza, *Lectures on the Calculus of Variations*, The University of Chicago Press, Chicago, 1904.

[2] L. Tonelli, *Fondamenti di Calcolo delle Variazioni I e II*, Nicola Zanichelli, Bologna, 1921.

[3] C. Caratheodory, *Calculus of Variations and Partial Differential Equations of First Order*, Chesea Publishing Company, New York, (1982), (originally published as *Variationsrechnung und Partielle Differentialgleichungen erster Ordnung* by B. G. Teubner, Berlin, 1935).

[4] G.A. Bliss, *Lectures on the calculus of the variations*, The University of Chicago Press, Chicago, 1946.

[5] M. Morse, *Variational Analysis*, John Wiley & Sons, New York, London, Sydney, Toronto, 1973.

[6] I.M. Gelfand, S.V. Fomin, *Calculus of variations*, Prentice Hall, Englewood Cliffs, New Jersey, 1963.

[7] H. Rund, *The Hamilton-Jacobi theory in the calculus of variations*, Van Nostrand, London, 1966.

[8] M. Giaquinta, S. Hildebrandt, *Calculus of variations I, II*, Springer-Verlag, Berlin Heidelberg New York, 1996.

[9] H. Sagan, *Introduction to the calculus of variations*, McGraw–Hill Book Company, New York, 1969.

[10] M.R. Hestenes, *Calculus of variations and optimal control theory*, Wiley, New York London Sydney, 1966.

[11] P. Griffiths, *Exterior differential systems and the calculus of variations*, Birkhauser, Boston, 1983.

[12] L. Hsu, Calculus of Variations via the Griffiths Formalism, *J. Differential Geometry* **36** (1992), 551-589.

[13] R. Montgomery, *A Tour of Subriemannian Geometries, Their Geodesics and Applications*, AMS, Math. Surveys and Monographs, Vol. 91, 2000.

[14] O. Krupková, *The Geometry of Ordinary Variational Equations*, Springer-Verlag, Berlin, Heidelberg, 1997.

[15] A.A. Agrachev, Yu.L. Sachov, *Control Theory from the Geometric Viewpoint*, Springer-Verlag, Berlin Heidelberg New York, 2004.

[16] X. Gracia, J. Marin-Solano, M.C̃. Munoz–Lecanda, Some geometric aspects of variational calculus in constrained systems, *Reports in Mathematical Physics* **51** (2003), 127-147.

[17] M. Barbero–Linan, A. Echeverria–Enriquez, D. Martin de Diego, M.C̃. Munoz–Lecanda, N. Roman–Roy, Skinner-Rusk unified formalism for optimal control systems and applications, *J. Phys. A: Math. Theor.* **40** (2007), 12071-12093.

[18] M. Barbero–Linan, M.C̃. Munoz–Lecanda, Constraint Algorithm for Extremals in Optimal Control Problems, *Int. J. Geom. Methods Mod. Phys.* **06** (2009), 1221.

[19] M. de Leon, F. Jimenez, D. Martin de Diego, Hamiltonian dynamics and constrained variational calculus: continuous and discrete settings, *J. Phys. A: Math. Theor.* **45** (2012), 1-29.

[20] E. Massa, D. Bruno, G. Luria, E. Pagani, Geometric constrained variational calculus. I: Piecewise smooth extremals, *Int. J. Geom. Methods Mod. Phys.* **12** (2015), 1550061.

[21] E. Massa, D. Bruno, G. Luria, E. Pagani, Geometric constrained variational calculus. II: The second variation (Part I), *Int. J. Geom. Methods Mod. Phys.* **13** (2016), 1550132.

[22] E. Massa, G. Luria, E. Pagani, Geometric constrained variational calculus. III: The second variation (Part II), *Int. J. Geom. Methods Mod. Phys.* **13** (2016), 1650038.

[23] W. Hurewicz, *Lectures on ordinary differential equations* John Wiley & Sons, Inc., and MIT Press, New York and Cambridge, Mass., (1958).

[24] F.W. Warner, *Foundations of Differential Manifolds and Lie Groups*, Springer–Verlag, New York, 1983.