# Fuzzy Maximal Eigenvalues of Fuzzy Pairwise Comparison Matrices

Jana Krejčí[1]

**Abstract.** In the Analytic Hierarchy Process (AHP), the maximal eigenvalue of a pairwise comparison matrix is used to verify its consistency using the Consistency index and to obtain the weights of objects from the pairwise comparison matrix. There have been many attempts to fuzzy the AHP, among which also the fuzzification of the maximal eigenvalue of a fuzzy pairwise comparison matrix was approached. Csutora & Buckley (2001) and Wang & Chin (2006) proposed formulas to obtain the fuzzy maximal eigenvalues of fuzzy pairwise comparison matrices whose elements were triangular and trapezoidal fuzzy numbers. However, the fuzzification was not done properly. The reciprocity of fuzzy pairwise comparison matrices was not taken into account, even though it is strictly required in the AHP. We propose new formulas for computing the fuzzy maximal eigenvalue of a fuzzy pairwise comparison matrix. For the sake of simplicity, only triangular fuzzy numbers are used. Afterwards, fuzzy maximal eigenvalues obtained using the new formulas are confronted with those obtained by the means of the formulas formerly proposed in the literature. Properties of the fuzzy maximal eigenvalues are described and an illustrative example is given.

**Keywords:** fuzzy maximal eigenvalue, fuzzy pairwise comparison matrix, reciprocity.

**JEL classification:** C44
**AMS classification:** 90C15

## 1 Introduction

In the Analytic Hierarchy Process, in the original method, Saaty proposed the Consistency Index CI to verify the consistency of pairwise comparison matrices and the eigenvalue method to obtain the weights of objects from pairwise comparison matrices. For the computation of the CI as well as the weights of the objects, the maximal eigenvalue of a pairwise comparison matrix is essential.

In the last few decades, the fuzzification of the AHP has become very popular among researchers since fuzzy elements can handle the vagueness of meaning of linguistic terms expressing intensities of decision makers' preferences. There have been many attempts to fuzzify the AHP, among which also the fuzzification of the eigenvalue method was proposed. In 2001, Csutora & Buckley [1] proposed formulas for obtaining the fuzzy maximal eigenvalue of a fuzzy pairwise comparison matrix whose elements where either triangular or trapezoidal fuzzy numbers. After that, using the fuzzy maximal eigenvalue, they proposed an algorithm for obtaining fuzzy weights of objects from the fuzzy pairwise comparison matrix. Five years later, the fuzzification of the eigenvalue method proposed by Csutora & Buckley was revised by Wang & Chin [4]. They adopted the formulas for computing the fuzzy maximal eigenvalue of a fuzzy pairwise comparison matrix with triangular fuzzy numbers and did a little modification in the formulas for obtaining the fuzzy maximal eigenvalue of a fuzzy pairwise comparison matrix with trapezoidal fuzzy numbers. Afterward, they proposed a new method for obtaining the fuzzy weights of objects using the fuzzy maximal eigenvalue of the fuzzy pairwise comparison matrix. However, neither Custora & Buckley [1] nor Wang & Chin [4] took into account the requirement of the reciprocity of pairwise comparison matrices during the construction of the formulas for obtaining the fuzzy maximal eigenvalue of a fuzzy pairwise comparison matrix. This is a serious flaw since the reciprocity is regarded as an essential property of pairwise comparison matrices in the AHP.

[1]University of Trento, Department of Industrial Engineering, Via Sommarive 9, 38123 Trento, Italy, jana.krejci@unitn.it

This paper focuses on the proper fuzzification of the formulas for obtaining the maximal eigenvalue of a pairwise comparison matrix. The formulas for computing the fuzzy maximal eigenvalue of a fuzzy pairwise comparison matrix proposed by Csutora & Buckley [1] and by Wang & Chin [4] are reviewed, and the deficiency of the formulas regarding the violation of the reciprocity condition is pointed out. Then, improvements to the formulas are proposed in order to obtain the actual maximal eigenvalue of a fuzzy pairwise comparison matrix.

The paper is organized as follows: In Section 2, triangular fuzzy numbers and arithmetic operations with them are defined. In Section 3, a pairwise comparison matrix and a fuzzy pairwise comparison matrix are defined, and their properties are reviewed. In Section 4, formulas for obtaining the fuzzy maximal eigenvalue of a fuzzy pairwise comparison matrix are dealt with. In Section 5, illustrative examples are given, and finally, in Section 6, a conclusion is formed.

## 2 Triangular fuzzy numbers

In this section, triangular fuzzy numbers and arithmetic operations with them are defined.

A triangular fuzzy number $\tilde{c}$ is a fuzzy number on $\mathbb{R}$ whose membership function is given as

$$\tilde{c}(x) = \begin{cases} \frac{x-c_l}{c_2-c_1}, & c_1 \leq x \leq c_2, \\ \frac{c_3-x}{c_3-c_2}, & c_2 < x \leq c_3, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where $c_1, c_2$ and $c_3$ are called the lower, the middle and the upper significant values of the triangular fuzzy number $\tilde{c}$. Every triangular fuzzy number can be uniquely described by a triplet of its significant values; the notation $\tilde{c} = (c_1, c_2, c_3)$ is used in the paper hereafter. A triangular fuzzy number $\tilde{c} = (c_1, c_2, c_3)$ is said to be positive if $c_1 > 0$.

Since in the fuzzification of the AHP only positive fuzzy numbers are used, we restrict definitions of arithmetic operations only to positive fuzzy numbers. According to the simplified standard arithmetic, the sum, the product and the quotient of two positive triangular fuzzy numbers $\tilde{c} = (c_1, c_2, c_3)$ and $\tilde{d} = (d_1, d_2, d_3)$ are triangular fuzzy numbers in the form $\tilde{c} + \tilde{d} = (c_1 + d_1, c_2 + d_2, c_3 + d_3)$, $\tilde{c} * \tilde{d} = (c_1 * d_1, c_2 * d_2, c_3 * d_3)$ and $\tilde{c}/\tilde{d} = (c_1/d_3, c_2/d_2, c_3/d_1)$. The reciprocal of a triangular fuzzy number $\tilde{c}$ is a triangular fuzzy number given in the form $\frac{1}{\tilde{c}} = \left(\frac{1}{c_3}, \frac{1}{c_2}, \frac{1}{c_1}\right)$.

However, the concept of the standard fuzzy arithmetic (in this paper the simplified standard arithmetic defined above) can be applied only if there are no interactions between the fuzzy numbers. In case of any interaction between the fuzzy numbers, the constrained fuzzy arithmetic by Klir & Pan [2] should be applied on the arithmetic operations.

Let $f$ be a continuous function, $f : \mathbb{R}^n \to \mathbb{R}$, let $\tilde{c}_i = (c_{i1}, c_{i2}, c_{i3})$, $i = 1, \ldots, n$, be triangular fuzzy numbers, and let $D$ be a relation in $\mathbb{R}^n$ describing interactions among the variables. Then, according to the simplified constrained fuzzy arithmetic, $\tilde{c} = (c_1, c_2, c_3)$ is a triangular fuzzy number whose significant values are computed as

$$\begin{aligned} c_1 &= \min \left\{ f(x_1, \ldots, x_n); (x_1, \ldots, x_n) \in D \cap [c_{11}, c_{13}] \times \cdots \times [c_{n1}, c_{n3}] \right\}, \\ c_2 &= f(c_{12}, \ldots, c_{n2}), \\ c_3 &= \max \left\{ f(x_1, \ldots, x_n); (x_1, \ldots, x_n) \in D \cap [c_{11}, c_{13}] \times \cdots \times [c_{n1}, c_{n3}] \right\}. \end{aligned} \tag{2}$$

## 3 Pairwise comparison matrices and fuzzy pairwise comparison matrices

In this Section, a pairwise comparison matrix and a fuzzy pairwise comparison matrix are defined, and their properties are reviewed.

*A pairwise comparison matrix* of $n$ objects is a square matrix $A = \{a_{ij}\}_{i,j=1}^n$ whose elements $a_{ij}$ are numbers from Saaty's scale [5] when the object in the $i$-th row is more important than the object in the $j$-th column or their reciprocals when the object in the $j$-th column is more important than the object in the $i$-th row. It follows that a pairwise comparison matrix has to be reciprocal, i.e. $a_{ji} = \frac{1}{a_{ij}}$ for $i,j = 1,\ldots,n, i \neq j$, and $a_{ii} = 1$ for $i = 1,\ldots,n$. A pairwise comparison matrix $A = \{a_{ij}\}_{i,j=1}^{n}$ is said to be *consistent* when $a_{ik} = a_{ij}a_{jk}$ holds for $i,j,k = 1,\ldots,n$.

*The maximal eigenvalue* $\lambda_{MAX}$ of a pairwise comparison matrix $A$ is the maximal solution to the equation $|A - \lambda I| = 0$ where $|.|$ denotes the determinant of a given matrix. For the sake of simplicity, the maximal eigenvalue of a given matrix is denoted only by $\lambda$ hereafter. According to Perron-Frobenius theorem for nonnegative matrices follows that, for any nonnegative matrix $A = \{a_{ij}\}_{i,j=1}^{n}$, its maximal eigenvalue $\lambda$ is nonnegative, i.e. $\lambda \geq 0$. Moreover, for any two matrices $A = \{a_{ij}\}_{i,j=1}^{n}, B = \{b_{ij}\}_{i,j=1}^{n}$ such that $a_{ij} \leq b_{ij}$ for $i,j = 1,\ldots,n$, and $a_{kl} < b_{kl}$ for $k,l \in \{1,\ldots,n\}$, the maximal eigenvalue $\lambda_A$ of $A$ is lower than the maximal eigenvalue $\lambda_B$ of $B$, i.e. $\lambda_A < \lambda_B$. Further, as was shown by Saaty in [6], for any positive reciprocal matrix $A = \{a_{ij}\}_{i,j=1}^{n}$, its maximal eigenvalue $\lambda$ is greater or equal to $n$, i.e. $\lambda \geq n$. The equality $\lambda = n$ holds if and only if the matrix $A = \{a_{ij}\}_{i,j=1}^{n}$ is consistent, i.e. $a_{ij}a_{jk} = a_{ik}$ for $i,j,k = 1,\ldots,n$.

*A fuzzy pairwise comparison matrix* of $n$ objects is a square matrix $\widetilde{A} = \{\tilde{a}_{ij}\}_{i,j=1}^{n}$ whose elements $\tilde{a}_{ij}$ are fuzzy numbers from a given scale when the object in the $i$-th row is more important than the object in the $j$-th column or their reciprocals when the object in the $j$-th column is more important than the object in the $i$-th row. In this paper, fuzzified Saaty's scales of triangular fuzzy numbers defined in [3] are applied. Analogously as the pairwise comparison matrix of real numbers, also the fuzzy pairwise comparison matrix is reciprocal, i.e. $\tilde{a}_{ij} = \frac{1}{\tilde{a}_{ji}}$ for $i,j = 1,\ldots,n$. On the main diagonal of a fuzzy pairwise comparison matrix we compare always one object with itself. Naturally, the objects are always equally important as themselves and there is no fuzziness in these comparisons. Therefore, we require $\tilde{a}_{ii} = 1, i = 1,\ldots,n$, for all fuzzy pairwise comparison matrices.

## 4 Fuzzy maximal eigenvalue

In this section, the formulas for obtaining the fuzzy maximal eigenvalue of a fuzzy pairwise comparison matrix formerly proposed in the literature are reviewed, deficiencies of the formulas are pointed out, and then new formulas are proposed. Subsequently, properties of the fuzzy maximal eigenvalue are discussed.

Csutora & Buckley [1] proposed formulas for obtaining $\alpha-$cuts, $\alpha \in [0,1]$, of the fuzzy maximal eigenvalue of a fuzzy pairwise comparison matrix. For the sake of simplicity, we focus here only on obtaining the significant values of the triangular fuzzy maximal eigenvalue. According to Csutora & Buckley [1], the significant values of the fuzzy maximal eigenvalue $\tilde{\lambda}^S = \left(\lambda_1^S, \lambda_2^S, \lambda_3^S\right)$ (the upper index $S$ stands for the standard arithmetic which is applied to obtain the fuzzy maximal eigenvalue) of a fuzzy pairwise comparison matrix $\widetilde{A} = \{\tilde{a}_{ij}\}_{i,j=1}^{n}, \tilde{a}_{ij} = (a_{ij1}, a_{ij2}, a_{ij3})$, are computed as

$$\begin{aligned} \lambda_1^S = &\quad max\left\{\lambda; |A_1 - \lambda I| = 0, A_1 = \{a_{ij1}\}_{i,j=1}^{n}\right\}, \\ \lambda_2^S = &\quad max\left\{\lambda; |A_2 - \lambda I| = 0, A_2 = \{a_{ij2}\}_{i,j=1}^{n}\right\}, \\ \lambda_3^S = &\quad max\left\{\lambda; |A_3 - \lambda I| = 0, A_3 = \{a_{ij3}\}_{i,j=1}^{n}\right\}. \end{aligned} \tag{3}$$

It means, the lower significant value $\lambda_1^S$ of the fuzzy maximal eigenvalue $\tilde{\lambda}^S$ of a fuzzy pairwise comparison matrix $\widetilde{A}$ is computed as the maximal eigenvalue of the matrix $A_1 = \{a_{ij1}\}_{i,j=1}^{n}$, whose elements are the lower significant values of the triangular fuzzy numbers from the fuzzy pairwise comparison matrix $\widetilde{A}$. Analogously, the middle significant value $\lambda_2^S$ is computed as the maximal eigenvalue of the matrix of the middle significant values of the triangular fuzzy numbers from the matrix $\widetilde{A}$, and the upper significant value $\lambda_3^S$ is computed as the maximal eigenvalue of the matrix of the upper significant values of the triangular fuzzy numbers from the matrix $\widetilde{A}$.

Obviously, the matrices $A_1$ and $A_3$ are not reciprocal. Even Csutora & Buckley [1] themselves observed this fact. Nevertheless, they did not consider it to be a flaw. Not even Wang & Chin [4], who adopted the formulas (3) to compute fuzzy weights of objects from fuzzy pairwise comparison matrices, realized the flaw. However, as was discussed in [3], the reciprocity is a key property of fuzzy pairwise comparison matrices, which has to be preserved. It means, there are interactions between the elements of a fuzzy pairwise comparison matrix which have to be taken into account during the computation. Therefore, the constrained fuzzy arithmetic (2) has to be applied on the formulas for obtaining the maximal eigenvalue of a pairwise comparison matrix.

By applying the simplified constrained fuzzy arithmetic, the formulas for obtaining the significant values of the fuzzy maximal eigenvalue of a fuzzy pairwise comparison matrix become considerably more complicated; an optimization tool is necessary to solve the problem. Let us analyze the problem. To obtain the fuzzy maximal eigenvalue of a fuzzy pairwise comparison matrix, we have to search through all reciprocal pairwise comparison matrices which can be generated by combining the elements lying in the closures of the supports of the fuzzy numbers in the fuzzy pairwise comparison matrix. We compute the maximal eigenvalue of each of these matrices, and then the lowest obtained maximal eigenvalue corresponds to the lower significant value of the resulting fuzzy maximal eigenvalue, and the greatest obtained maximal eigenvalue corresponds to the upper significant value of the resulting fuzzy maximal eigenvalue. The middle significant value of the resulting fuzzy maximal eigenvalue is computed as the maximal eigenvalue of the matrix of the middle significant values of the fuzzy numbers from the fuzzy pairwise comparison matrix since the middle significant values of the fuzzy numbers are the most possible ones and the matrix of these values is reciprocal. It means that the formula for obtaining the middle significant value of the fuzzy maximal eigenvalue remains the same as the formula proposed by Csutora & Buckley [1]. Formally, the formulas for obtaining the significant values $\lambda_1^R, \lambda_2^R$ and $\lambda_3^R$ of the maximal eigenvalue $\tilde{\lambda}^R$ (the upper index R stands for involving the reciprocity into the formulas for obtaining the fuzzy maximal eigenvalue) of a fuzzy pairwise comparison matrix $\widetilde{A} = \{\tilde{a}_{ij}\}_{i,j=1}^n, \tilde{a}_{ij} = (a_{ij1}, a_{ij2}, a_{ij3})$, are given in this form:

$$
\begin{aligned}
\lambda_1^R = &\quad \min\left\{\max\left\{\lambda; |A-\lambda I| = 0, A = \{a_{ij}\}_{i,j=1}^n\right\}; a_{ij} \in [a_{ij1}, a_{ij3}], a_{ij} = \tfrac{1}{a_{ji}}, a_{ii} = 1, i,j = 1,\ldots,n\right\}, \\
\lambda_2^R = &\quad \max\left\{\lambda; |A_2-\lambda I| = 0, A_2 = \{a_{ij2}\}_{i,j=1}^n\right\}, \\
\lambda_3^R = &\quad \max\left\{\max\left\{\lambda; |A-\lambda I| = 0, A = \{a_{ij}\}_{i,j=1}^n\right\}; a_{ij} \in [a_{ij1}, a_{ij3}], a_{ij} = \tfrac{1}{a_{ji}}, a_{ii} = 1, i,j = 1,\ldots,n\right\}.
\end{aligned}
\tag{4}
$$

Using the properties of the maximal eigenvalues reviewed in Section 3, we can derive some properties of the fuzzy maximal eigenvalues of fuzzy pairwise comparison matrices obtained both by the formulas (3) proposed by Csutora & Buckey [1] and by the new formulas (4).

First, the properties of the fuzzy maximal eigenvalue obtained by formulas (3) are summarized. Since a fuzzy pairwise comparison matrix $\widetilde{A} = \{\tilde{a}_{ij}\}_{i,j=1}^n, \tilde{a}_{ij} = (a_{ij1}, a_{ij2}, a_{ij3})$, is positive, i.e. $a_{ij1} > 0$ for $i,j = 1,\ldots,n$, the maximal eigenvalue of any matrix constructed from the elements from the closures of the supports of its fuzzy numbers is positive too. Further, because the inequalities $a_{ij1} < a_{ij2} < a_{ij3}$ hold for $i,j = 1,\ldots,n$, $i \neq j$, and $a_{ii1} = a_{ii2} = a_{ii3}$ for $i = 1,\ldots,n$, than clearly, for the maximal eigenvalues $\lambda_1^S, \lambda_2^S$ and $\lambda_3^S$ of the matrices $A_1 = \{a_{ij1}\}_{i,j=1}^n, A_2 = \{a_{ij2}\}_{i,j=1}^n$ and $A_3 = \{a_{ij3}\}_{i,j=1}^n$, the inequalities $\lambda_1^S < \lambda_2^S < \lambda_3^S$ hold. Since $A_2 = \{a_{ij2}\}_{i,j=1}^n$ is reciprocal, the inequality $\lambda_2^S \geq n$ holds for its maximal eigenvalue $\lambda_2^S$. The equality $\lambda_2^S = n$ occurs only if the matrix $A_2 = \{a_{ij2}\}_{i,j=1}^n$ is consistent, i.e. $a_{ij2}a_{jk2} = a_{ik2}$ for $i,j,k = 1,\ldots,n$.

The fuzzy maximal eigenvalue of a fuzzy pairwise comparison matrix obtained by the new formulas (4) has the following properties. Since all significant values of the fuzzy maximal eigenvalue $\tilde{\lambda}^R = (\lambda_1^R, \lambda_2^R, \lambda_3^R)$ are obtained from reciprocal matrices (see formulas (4)), the inequalities $\lambda_1^R \geq n, \lambda_2^R \geq n$ and $\lambda_3^R \geq n$ clearly hold. Further, because the lower significant value $\lambda_1^R$ of the fuzzy maximal eigenvalue $\tilde{\lambda}^R$ is obtained as the minimum of a given function, the upper significant value $\lambda_3^R$ is obtained as the maximum, and $\lambda_2^R$ is the maximal eigenvalue of a particular matrix, the inequalities $\lambda_1^R \leq \lambda_2^R \leq \lambda_3^R$ hold. In total, for the fuzzy maximal eigenvalue of a fuzzy pairwise comparison matrix obtained by formulas (4), the inequalities $n \leq \lambda_1^R \leq \lambda_2^R \leq \lambda_3^R$ hold. In special case, when the matrix $A_2$ of the middle significant values of the fuzzy numbers from the fuzzy pairwise comparison matrix $\widetilde{A}$ is consistent, the significant values of the fuzzy maximal eigenvalue $\tilde{\lambda}^R = (\lambda_1^R, \lambda_2^R, \lambda_3^R)$ of the fuzzy pairwise comparison matrix are in the form $n = \lambda_1^R = \lambda_2^R < \lambda_3^R$. In case that the matrix $A_2$ is not consistent but there exist elements in the closures of the supports of the fuzzy numbers from the fuzzy pairwise comparison matrix such that they form a consistent matrix, the significant values of the fuzzy maximal eigenvalue of the fuzzy pairwise comparison matrix are in the form $n = \lambda_1^R < \lambda_2^R < \lambda_3^R$.

Further, since $\lambda_1^S$ is the maximal eigenvalue of the matrix $A = \{a_{ij1}\}_{i,j=1}^n$, $\lambda_1^R$ is the maximal eigenvalue of a reciprocal matrix $A^* = \left\{a_{ij}^*\right\}_{i,j=1}^n, a_{ij}^* \in [a_{ij1}, a_{ij3}]$, and $a_{ij1} \le a_{ij}^*$ for $i,j = 1,\ldots,n$, then obviously the inequality $\lambda_1^S < \lambda_1^R$ holds. Analogously, since $\lambda_3^S$ is the maximal eigenvalue of the matrix $A = \{a_{ij3}\}_{i,j=1}^n$, $\lambda_1^R$ is the maximal eigenvalue of a reciprocal matrix $A^* = \left\{a_{ij}^*\right\}_{i,j=1}^n, a_{ij}^* \in [a_{ij1}, a_{ij3}]$, and $a_{ij3} \ge a_{ij}^*$ for $i,j = 1,\ldots,n$, then also the inequality $\lambda_3^R < \lambda_3^S$ holds. Therefore, the support of the fuzzy maximal eigenvalue $\tilde{\lambda}^R$ of a given fuzzy pairwise comparison matrix is a proper subset of the support of the fuzzy maximal eigenvalue $\tilde{\lambda}^S$ of the fuzzy matrix, i.e. $\left(\lambda_1^R, \lambda_3^R\right) \subset \left(\lambda_1^S, \lambda_3^S\right)$.

By applying the reciprocity condition on the formulas for obtaining the fuzzy maximal eigenvalue of a fuzzy pairwise comparison matrix, we eliminated all unfeasible combinations of elements from the supports of the fuzzy numbers in the fuzzy pairwise comparison matrix. Consequently, we obtained the fuzzy maximal eigenvalue $\tilde{\lambda}^R$ which is less vague than the fuzzy maximal eigenvalue $\tilde{\lambda}^S$ obtained by the formulas proposed by Csutora & Buckley. Moreover, the support of the fuzzy maximal eigenvalue $\tilde{\lambda}^R$ is a proper subset of the support of the fuzzy maximal eigenvalue $\tilde{\lambda}^S$ and represents the actual maximal eigenvalue of the fuzzy pairwise comparison matrix.

## 5 Illustrative example

In this section, three fuzzy pairwise comparison matrices are examined. The maximal eigenvalue of each of the matrices is computed both by the formulas (3) and by the new formulas (4), where the matrix reciprocity is taken into account, and the results are compared. For the construction of the fuzzy pairwise comparison matrices, fuzzified Saaty's scales defined in [3] are used.

First, let us assume the fuzzy pairwise comparison matrix $\widetilde{A}$ given in Table 1. The middle significant values of the triangular fuzzy numbers in the fuzzy pairwise comparison matrix form a consistent matrix, and therefore, as was mentioned in Section 4, its maximal eigenvalue is 3. The fuzzy maximal eigenvalue computed according to the formulas (3) is then $\tilde{\lambda}^S = (2.4983, 3, 3.7919)$, and the fuzzy maximal eigenvalue computed according to the formulas (4) is $\tilde{\lambda}^R = (3, 3, 3.0649)$. For better illustration, the fuzzy maximal eigenvalues are represented in Figure 1.

Now, let us assume the fuzzy pairwise comparison matrix $\widetilde{B}$ given in Table 1. The middle significant values of the triangular fuzzy numbers in the fuzzy matrix obviously do not form a consistent matrix. However, we can obtain a consistent matrix in the form

$$\begin{pmatrix} 1 & 3 & 9 \\ \frac{1}{3} & 1 & 3 \\ \frac{1}{9} & \frac{1}{3} & 1 \end{pmatrix} \tag{5}$$

by combining different significant values of the fuzzy numbers from the fuzzy pairwise comparison matrix $\widetilde{B}$. According to this observation, we know that the lower significant value of the fuzzy maximal eigenvalue of the fuzzy pairwise comparison matrix $\widetilde{B}$ obtained by the formulas (4) is 3. The fuzzy maximal eigenvalue computed according to the formulas (3) is then $\tilde{\lambda}^S = (2.3429, 3.0291, 4.3209)$, and the fuzzy maximal eigenvalue computed according to the formulas (4) is $\tilde{\lambda}^R = (3, 3.0291, 3.2948)$. Both the fuzzy maximal eigenvalues are represented in Figure 2.

Finally, let us assume the fuzzy pairwise comparison matrix $\widetilde{C}$ given in Table 1. Obviously, we can not obtain a consistent matrix by any combination of the elements from the closures of the supports of the fuzzy numbers from the fuzzy pairwise comparison matrix. Therefore, the lower significant value of the fuzzy maximal eigenvalue of the fuzzy pairwise comparison matrix obtained by the formulas (4) is greater than 3. The fuzzy maximal eigenvalue computed according to the formulas (3) is then $\tilde{\lambda}^S = (2.6794, 3.2085, 3.9458)$, and the fuzzy maximal eigenvalue computed according to the formulas (4) is $\tilde{\lambda}^R = (3.0291, 3.2085, 3.5608)$. Both the fuzzy maximal eigenvalues are represented in Figure 3.

Obviously, the fuzzy maximal eigenvalues of the fuzzy pairwise comparison matrices $\widetilde{A}, \widetilde{B}$ and $\widetilde{C}$ obtained by the formulas (4), which preserve the reciprocity of the matrices, are significantly less vague than the fuzzy maximal eigenvalues obtained by the formulas (3), which violate the reciprocity condition. Moreover, the closures of the supports of the fuzzy maximal eigenvalues obtained by the means of the formulas (4) are the proper subsets of the closures of the supports of the fuzzy maximal eigenvalues obtained by the means of the formulas (3).

| $\widetilde{A}$ | | | $\widetilde{B}$ | | | $\widetilde{C}$ | | |
|---|---|---|---|---|---|---|---|---|
| $1$ | $(1,2,3)$ | $(7,8,9)$ | $1$ | $(1,3,5)$ | $(7,9,9)$ | $1$ | $(3,5,7)$ | $(7,9,9)$ |
| $\left(\frac{1}{3},\frac{1}{2},1\right)$ | $1$ | $(3,4,5)$ | $\left(\frac{1}{5},\frac{1}{3},1\right)$ | $1$ | $(3,5,7)$ | $\left(\frac{1}{7},\frac{1}{5},\frac{1}{3}\right)$ | $1$ | $(5,7,9)$ |
| $\left(\frac{1}{9},\frac{1}{8},\frac{1}{7}\right)$ | $\left(\frac{1}{5},\frac{1}{4},\frac{1}{3}\right)$ | $1$ | $\left(\frac{1}{9},\frac{1}{9},\frac{1}{7}\right)$ | $\left(\frac{1}{7},\frac{1}{5},\frac{1}{3}\right)$ | $1$ | $\left(\frac{1}{9},\frac{1}{9},\frac{1}{7}\right)$ | $\left(\frac{1}{9},\frac{1}{7},\frac{1}{5}\right)$ | $1$ |
| $\tilde{\lambda}^S = (2.4983, 3, 3.7919)$ | | | $\tilde{\lambda}^S = (2.3429, 3.0291, 4.3209)$ | | | $\tilde{\lambda}^S = (2.6794, 3.2085, 3.9458)$ | | |
| $\tilde{\lambda}^R = (3, 3, 3.0649)$ | | | $\tilde{\lambda}^R = (3, 3.0291, 3.2948)$ | | | $\tilde{\lambda}^R = (3.0291, 3.2085, 3.5608)$ | | |

**Table 1** Fuzzy maximal eigenvalues of fuzzy pairwise comparison matrices

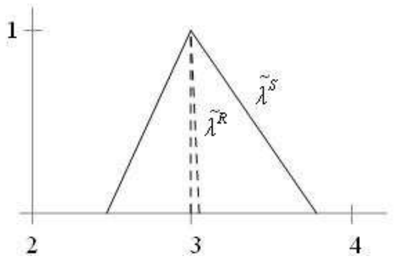


Figure 1 Fuzzy maximal eigenvalues of the fuzzy pairwise comparison matrix $\widetilde{A}$.

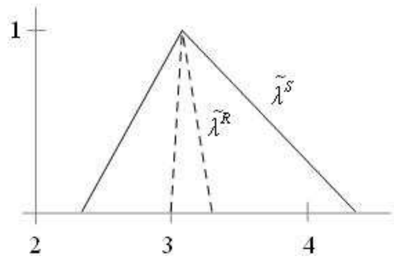


Figure 2 Fuzzy maximal eigenvalues of the fuzzy pairwise comparison matrix $\widetilde{B}$.

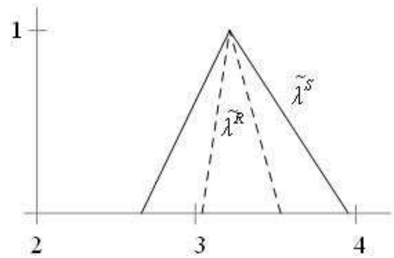


Figure 3 Fuzzy maximal eigenvalues of the fuzzy pairwise comparison matrix $\widetilde{C}$.

## 6 Conclusion

The proper fuzzification of the maximal eigenvalue of a pairwise comparison matrix was dealt with in this paper. The formulas for obtaining the fuzzy maximal eigenvalue of a fuzzy pairwise comparison matrix of triangular fuzzy numbers proposed by Csutora & Buckley in [1] and adopted by Wang & Chin in [4] were reviewed. Deficiences of their formulas were pointed out and new formulas were introduced afterwards. The new formulas are based on the constrained fuzzy arithmetic taking into account the reciprocity condition, which is required for all pairwise comparison matrices in the AHP. Subsequently, we derived some properties of the fuzzy maximal eigenvalue of a fuzzy pairwise comparison matrix obtained both by the formulas (3) and by the new formulas (4), where the reciprocity condition is taken into account. The fuzzy maximal eigenvalue of a fuzzy pairwise comparison matrix obtained by the new formulas (4) is less vague than the fuzzy maximal eigenvalue obtained by the formulas (3) proposed by Csutora & Buckley. Moreover, it represents the actual maximal eigenvalue of the fuzzy pairwise comparison matrix since all constraints derived from the AHP and from the simplified constrained fuzzy arithmetic are taken into account in the formulas.

## References

[1] Csutora, R., Buckley, J.J.: Fuzzy hierarchical analysis: the Lambda-Max method, *Fuzzy Sets and Systems* **120** (2001), 181–195.

[2] Klir, G. J., Pan, Y.: Constrained fuzzy arithmetic: Basic questions and some answers, *Soft Computing* **2** (1998), 100–108.

[3] Krejčí, J., Talašová, J.: A proper fuzzification of Saaty's scale and an improved method for computing fuzzy weights in fuzzy AHP, *Proceedings of the 31th International Conference on Mathematical Methods in Economics 2013, Jihlava* (2013), ISBN: 978-80-87035-76-4.

[4] Wang, Y.-M., Chin, K.-S.: An eigenvector method for generating normalized interval and fuzzy weights, *Applied Mathematics and Computation* **181** (2006), 1257–1275.

[5] Saaty, T. L.: *The Analytic Hierarchy Process*. McGraw Hill, New York, 1980.

[6] Saaty, T.L.: Decision-making with the AHP: Why is the principal eigenvector necessary, *European Journal of Operational Research* **145** (2003), 85–91