

A Characterization of Symplectic Grassmannians

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Abstract We provide a characterization of Symplectic Grassmannians in terms of their Varieties of Minimal Rational Tangents.

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1 Introduction

Families of rational curves have been shown to be a powerful tool in birational geometry (see [6, 15]). Remarkably, the local analysis of these families via certain differential geometric techniques has been very useful in order to recover global properties of Fano manifolds of Picard number one (see [9, 11], and the references therein). In fact, many geometric features of these varieties can be read from the projective geometry of the set of tangent directions of a family of minimal rational curves at the general point of the variety –the so-called variety of minimal rational tangents, or VMRT, for short.

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An archetypal result of this type, due to Mok and to Hong-Hwang [17, 8], allows us to recognize a homogeneous Fano manifold X of Picard number one by looking at its VMRT at a general point x , and at its embedding into $\mathbb{P}(\Omega_{X,x})$. In a nutshell, their strategy consists of constructing a biholomorphism between analytic open subsets of X and the model manifold preserving the VMRT structure, and then using the so-called Cartan-Fubini extension principle to extend it along the rational curves of the family. This characterization works for all rational homogeneous manifolds of Picard number one whenever the VMRT is rational homogeneous, which is always the case except for the short root cases; namely for symplectic Grassmannians, and for two varieties of type F_4 . In the former cases, quoting Mok ([18]): “*An important feature for the primary examples of symplectic Grassmannians [...] is the existence of local differential-geometric invariants which cannot possibly be captured by the VMRT at a general point.*”

In this paper we show that, if we impose that the VMRT is the expected one at every point of the variety, we may still characterize symplectic Grassmannians (see Theorem 2.4 for the precise statement). In fact, we may use a family of minimal rational curves to construct a smooth projective variety \bar{X} , dominating X , supporting as many independent \mathbb{P}^1 -bundle structures as its Picard number. Then we may use [20] to claim that \bar{X} is a flag manifold (of type C), and X is, subsequently, a symplectic Grassmannian. The same argument had been successfully used in [21] to characterize rational homogeneous manifolds of Picard number one corresponding to long roots. In that case the existence of a family of rational curves with smooth evaluation and the appropriate fibers allowed to complete the proof, without making any assumption on its image into $\mathbb{P}(\Omega_X)$. In the case of symplectic Grassmannians, the existence of an adequate embedding of the family into $\mathbb{P}(\Omega_X)$ seems to be crucial in order to carry out the proof. We believe that a similar technique may lead also to the characterization of the last two remaining short root cases.

The structure of the paper is the following: Section 2 contains some preliminary material regarding rational curves, symplectic Grassmannians and flag bundles. In Section 3 we show how to reconstruct, upon the VMRT, certain vector bundles over the manifold X (corresponding, a posteriori, to universal bundles on the symplectic Grassmannian), and study some of their properties. They can be used (see Section 4) to prove that a certain subvariety $H\mathcal{U}$ of the family of rational curves is indeed a subfamily. Section 5 contains the last technical ingredient of the proof: the existence of a nondegenerate skew-symmetric form on one of those bundles. This will allow us in Section 6 to construct a flag bundle over $H\mathcal{U}$ of the appropriate type, which we prove to be a flag manifold.

2 Preliminaries

2.1 Families of rational curves

For the readers' convenience, we will introduce here some notation that we will use along the paper, regarding rational curves on algebraic varieties. We refer to [15] for a complete account on this topic.

Given a smooth complex projective variety Y , a *family of rational curves on Y* is, by definition, the normalization \mathcal{N} of an irreducible component of the scheme $\text{RatCurves}^n(Y)$. Each of these families comes equipped with a smooth \mathbb{P}^1 -fibration $p : \mathcal{V} \rightarrow \mathcal{N}$ and an evaluation morphism $q : \mathcal{V} \rightarrow Y$. The family \mathcal{N} is called *dominating* if q is dominant, and *unsplit* if \mathcal{N} is proper. Given a point $y \in Y$, we denote by \mathcal{N}_y the normalization of $q^{-1}(y)$, and by \mathcal{V}_y its fiber product with \mathcal{V} over \mathcal{N} . For a general point $y \in Y$, $\dim \mathcal{N}_y = -K_Y \cdot \ell - 2$, where ℓ is an element of the family. The composition $q^* \Omega_Y \rightarrow \Omega_{\mathcal{V}} \rightarrow \Omega_{\mathcal{V}/\mathcal{N}}$ gives rise to a rational map $\tau : \mathcal{V} \dashrightarrow \mathbb{P}(\Omega_Y)$, that we call the *tangent map*. A point $P \in \mathcal{V}$ for which this map is not defined is called a *cusps* of the family.

Remark 2.1 Let us denote by $Y^\circ \subset Y$ a nonempty subset such that the tangent map τ is defined at $\mathcal{V}^\circ := q^{-1}(Y^\circ)$. By definition, it satisfies

$$\tau^* \mathcal{O}_{\mathbb{P}(\Omega_{Y^\circ})}(1) = (\Omega_{\mathcal{V}/\mathcal{N}})_{|\mathcal{V}^\circ} = \mathcal{O}_{\mathcal{V}^\circ}(K_{\mathcal{V}/\mathcal{N}}).$$

We will be mostly interested in the case in which \mathcal{N} is a dominating, unsplit family such that $q : \mathcal{V} \rightarrow Y$ is a smooth morphism, and we will simply say that the family is *beautiful*. Note that in this case the varieties \mathcal{V} and \mathcal{N} are smooth as well. On the other hand, the first two assumptions allow us to say, by [13, Theorem 3.3], that the set of cusps of \mathcal{N} does not dominate Y , and τ is defined over \mathcal{N}_y , for $y \in Y$ general. We denote the restriction of τ to \mathcal{N}_y by τ_y . Its image is usually called the *VMRT* of \mathcal{N} at y ; it is known that, for a general y , τ_y is its normalization (see [12, Theorem 1], [13, Theorem 3.4]).

Remark 2.2 Beautiful families arise naturally, for instance, in the context of Campana-Peternell conjecture: any family of rational curves of minimal degree with respect to a fixed ample line bundle on a Fano manifold with nef tangent bundle is a beautiful family (see for instance [19, Proposition 2.10]).

Notation *Along the paper we will often consider vector bundles on the projective line \mathbb{P}^1 . For simplicity, we will denote by $E(a_1^{k_1}, \dots, a_r^{k_r})$ the vector bundle $\bigoplus_{j=1}^r \mathcal{O}(a_j)^{\oplus k_j}$ on \mathbb{P}^1 .*

Remark 2.3 Given a curve $C = p^{-1}(c)$, $c \in \mathcal{N}$, of a beautiful family \mathcal{N} of rational curves in Y , we have a surjective map $(T_{\mathcal{V}})_{|C} \rightarrow (q^* T_Y)_{|C}$. Since $(T_{\mathcal{V}})_{|C} \cong E(2, 0^{\dim(\mathcal{N})})$, it follows that $(q^* T_Y)_{|C}$ is nef, that is, $q|_C : C \rightarrow Y$ is *free* (cf. [15, Definition II.3.1]). In particular, the natural map from \mathcal{N} to $\text{Chow}(Y)$ is injective, that is two different elements of \mathcal{N} cannot correspond to the same curve in Y (see, for instance, [14, Proof of Lemma 5.2]).

2.2 Statement of the Main Theorem

The goal of this paper is to characterize rational homogeneous manifolds of the form $\mathrm{Sp}(2n)/P_r$, where P_r is a maximal parabolic subgroup, determined by a node $r \in \{2, \dots, n-1\}$ in the Dynkin diagram of $\mathrm{Sp}(2n)$ –where the nodes of the diagram are numbered as in (2.1). A variety of this kind is classically interpreted as the *Symplectic Grassmannian* parametrizing the $(r-1)$ -dimensional linear subspaces in \mathbb{P}^{2n-1} that are isotropic with respect to a certain nondegenerate skew-symmetric form.

$$\begin{array}{ccccccc} 1 & 2 & 3 & \dots & n-2 & n-1 & n \\ \circ & \circ & \circ & \cdots & \circ & \circ & \circ \\ & & & & & \longleftarrow & \\ & & & & & & C_n \end{array} \quad (2.1)$$

The Plücker embedding of $\mathrm{Sp}(2n)/P_r$ is covered by lines, and the family of these lines in $\mathrm{Sp}(2n)/P_r$ is beautiful. Furthermore, its evaluation morphism is isotrivial, with fibers isomorphic to the smooth projective variety

$$\mathcal{C} := \mathbb{P}(\mathcal{O}_{\mathbb{P}^{r-1}}(2) \oplus \mathcal{O}_{\mathbb{P}^{r-1}}(1)^{2n-2r}), \quad r \in \{2, \dots, n-1\},$$

and its tangent map is an embedding (see [9, 1.4.7]). The restriction of this embedding to any fiber is projectively equivalent to the embedding of \mathcal{C} into $\mathbb{P}(H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(1))) = \mathbb{P}(H^0(\mathbb{P}^{r-1}, \mathcal{O}_{\mathbb{P}^{r-1}}(2) \oplus \mathcal{O}_{\mathbb{P}^{r-1}}(1)^{2n-2r}))$.

Note that the homogeneous manifold $\mathrm{Sp}(2n)/P_{r-1, r+1}$ (where $P_{r-1, r+1}$ denotes the parabolic subgroup determined by the nodes $r-1, r+1$) can be identified with a subfamily of the family of lines in $\mathrm{Sp}(2n)/P_r$, which we call the family of *isotropic lines*. Its evaluation morphism has fibers isomorphic to

$$HC := \mathbb{P}(\mathcal{O}_{\mathbb{P}^{r-1}}(1)^{2n-2r}).$$

We can now give a precise statement of the main result in this paper.

Theorem 2.4 *Let X be a Fano manifold of Picard number one, endowed with a beautiful family of rational curves. Assume that its tangent map τ is a morphism, satisfying that τ_x is projectively equivalent to the embedding of \mathcal{C} into $\mathbb{P}(H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(1)))$, for every $x \in X$. Then X is isomorphic to a symplectic Grassmannian.*

2.3 Flag bundles

Let G be a semisimple group, $P \subset G$ a parabolic subgroup, and \mathcal{D} the Dynkin diagram of G . A G/P -bundle over a smooth projective variety Y is a smooth projective morphism $q : E \rightarrow Y$ such that all the fibers of q over closed points are isomorphic to G/P . If $P = B$ is a Borel subgroup of G we will also call E a flag bundle of type \mathcal{D} or a \mathcal{D} -bundle. We present here some general facts on these bundles, without proofs, and refer the reader to [21] for details.

Assume Y is simply connected (for instance, if it is rationally connected); whenever the identity component of the automorphism group of G/P is a

semisimple group with the same Lie algebra as G , a G/P -bundle $q : E \rightarrow Y$ determines a \mathcal{D} -bundle $\bar{q} : \bar{E} \rightarrow Y$ dominating it. Furthermore, in this case, given a parabolic subgroup $P' \subset G$, containing B , there exist a G/P' -bundle $q' : E' \rightarrow Y$, and a contraction $\bar{q}' : \bar{E} \rightarrow E'$ such that $\bar{q} = q' \circ \bar{q}'$, whose fibers are isomorphic to P'/B (which is a rational homogeneous space, quotient of a Levi part of the group P'); for $P' = P$, we get our original bundle q . In particular, when P' is a minimal parabolic subgroup properly containing B , then $\bar{q}' : \bar{E} \rightarrow E'$ is a smooth \mathbb{P}^1 -bundle. Given two parabolic subgroups $P' \subset P''$, containing B , the associated bundles and contractions fit in the following commutative diagram:

$$\begin{array}{ccccc}
 & & \bar{q}'' & & \\
 & & \curvearrowright & & \\
 \bar{E} & \xrightarrow{\quad} & E' & \xrightarrow{\quad} & E'' \\
 & \searrow \bar{q}' & \downarrow q' & \swarrow q'' & \\
 & & Y & &
 \end{array}$$

$\bar{q} : \bar{E} \rightarrow Y$ (diagonal arrow from \bar{E} to Y)
 $q' : E' \rightarrow Y$ (vertical arrow from E' to Y)
 $q'' : E'' \rightarrow Y$ (diagonal arrow from E'' to Y)
 $\bar{q}' : \bar{E} \rightarrow E'$ (diagonal arrow from \bar{E} to E')
 $q : \bar{E} \rightarrow E'$ (horizontal arrow from \bar{E} to E')
 $q'' : E' \rightarrow E''$ (horizontal arrow from E' to E'')

In the case $Y \simeq \mathbb{P}^1$, a \mathcal{D} -bundle is determined by the intersection numbers of a minimal section C of the bundle with the relative canonical bundles K_i of the \mathbb{P}^1 -bundle arising from the minimal parabolic subgroups $P_i \subset G$ properly containing B ([21, Proposition 3.17]). Each of these subgroups corresponds to a node i of the Dynkin diagram \mathcal{D} , therefore we may represent the \mathcal{D} -bundle by the *tagged Dynkin diagram* obtained by tagging the node i with the intersection number $K_i \cdot C$ (see [21, Remark 3.18]).

3 Projective geometry

3.1 Setup

Throughout the rest of the paper X will denote a Fano manifold of Picard number one, admitting a beautiful family of rational curves \mathcal{M} , satisfying the assumptions of Theorem 2.4. We denote by $p : \mathcal{U} \rightarrow \mathcal{M}$ the universal family, by $q : \mathcal{U} \rightarrow X$ the evaluation morphism and by $\tau : \mathcal{U} \rightarrow \mathbb{P}(\Omega_X)$ the tangent map; finally, given $x \in X$, we denote by \mathcal{M}_x the fiber $q^{-1}(x)$.

3.2 Projective geometry of the VMRT

For every $x \in X$, the variety \mathcal{M}_x is isomorphic to the blow-up $\pi_{2,x}$ of a $(2n - r - 1)$ -dimensional projective space $\mathbb{P}(\mathcal{F}_{2,x})$ along a $(2n - 2r - 1)$ -dimensional subspace $\mathbb{P}(\mathcal{Q}_x)$. Denoting by φ_x the associated linear map $\mathcal{F}_{2,x} \rightarrow \mathcal{Q}_x$, and setting $\mathcal{F}_{1,x} := \ker(\varphi_x)$, the exceptional divisor of the blow-up map $\mathcal{M}_x \rightarrow \mathbb{P}(\mathcal{F}_{2,x})$ is isomorphic to $\mathbb{P}(\mathcal{F}_{1,x}) \times \mathbb{P}(\mathcal{Q}_x)$. Moreover, the blow-up can be seen

as a resolution of indeterminacies $\pi_{1,x} : \mathcal{M}_x \rightarrow \mathbb{P}(\mathcal{F}_{1,x})$ of the linear projection from $\mathbb{P}(\mathcal{F}_{2,x})$ onto $\mathbb{P}(\mathcal{F}_{1,x})$. Summing up:

$$\begin{array}{ccc}
 & & \mathbb{P}(\mathcal{F}_{1,x}) \\
 & \text{nat. proj.} \nearrow & \uparrow \pi_{1,x} \\
 \mathbb{P}(\mathcal{F}_{1,x}) \times \mathbb{P}(\mathcal{Q}_x) & \hookrightarrow \mathcal{M}_x & \mathbb{P}^{2n-2r}\text{-bdl.} \\
 & \searrow \text{nat. proj.} & \downarrow \pi_{2,x} \\
 & & \mathbb{P}(\mathcal{Q}_x) \hookrightarrow \mathbb{P}(\mathcal{F}_{2,x})
 \end{array}
 \quad \begin{array}{l}
 (3.1) \\
 \text{lin. proj.}
 \end{array}$$

The variety \mathcal{M}_x is embedded into $\mathbb{P}(\Omega_{X,x})$ via the complete linear system of the line bundle $\pi_{1,x}^* \mathcal{O}_{\mathbb{P}(\mathcal{F}_{1,x})}(1) \otimes \pi_{2,x}^* \mathcal{O}_{\mathbb{P}(\mathcal{F}_{2,x})}(1)$, whose restriction to $\mathbb{P}(\mathcal{F}_{1,x}) \times \mathbb{P}(\mathcal{Q}_x)$ provides the Segre embedding of this variety into its linear span, which is isomorphic to $\mathbb{P}(\mathcal{F}_{1,x} \otimes \mathcal{Q}_x)$. We then have an epimorphism $\Omega_{X,x} \rightarrow \mathcal{F}_{1,x} \otimes \mathcal{Q}_x$ whose kernel is isomorphic to $S^2 \mathcal{F}_{1,x}$, since the linear projection of \mathcal{M}_x from $\mathbb{P}(\mathcal{F}_{1,x} \otimes \mathcal{Q}_x)$ onto $\mathbb{P}(S^2 \mathcal{F}_{1,x})$ has the second Veronese embedding of $\mathbb{P}(\mathcal{F}_{1,x})$ as its image.

3.3 Relative projective geometry

We will now see how the above constructions can be done relatively over X , under the assumptions of Setup 3.1.

We start by noting that the automorphism group of \mathcal{M}_x equals the parabolic subgroup of $\text{Aut}(\mathbb{P}(\mathcal{F}_{2,x}))$ defined as the stabilizer $\text{Aut}(\mathbb{P}(\mathcal{F}_{2,x}))_{\mathbb{P}(\mathcal{Q}_x)}$ of $\mathbb{P}(\mathcal{Q}_x)$ (that is, the subgroup of automorphisms that leave $\mathbb{P}(\mathcal{Q}_x)$ invariant). Since this holds for every $x \in X$, the theorem of Fischer and Grauert (cf. [3, Theorem I.10.1]) tells us that the variety \mathcal{U} is determined by a degree one cocycle θ on X (considered as a complex manifold with the analytic topology) with values in the group $\text{Aut}(\mathbb{P}(\mathcal{F}_{2,x}))_{\mathbb{P}(\mathcal{Q}_x)}$.

On the other hand, every element of $\text{Aut}(\mathbb{P}(\mathcal{F}_{2,x}))_{\mathbb{P}(\mathcal{Q}_x)}$ extends to a unique automorphism of $\mathbb{P}(\Omega_{X,x})$ (because \mathcal{M}_x is linearly normal in this space), and the image of θ via the natural map

$$H^1(X, \text{Aut}(\mathbb{P}(\mathcal{F}_{2,x}))_{\mathbb{P}(\mathcal{Q}_x)}) \rightarrow H^1(X, \text{Aut}(\mathbb{P}(\Omega_{X,x})))$$

defines the projective bundle $\mathbb{P}(\Omega_X)$ over X . Since this is the projectivization of a vector bundle, this cocycle is defined over the group $\text{GL}(T_{X,x})$ and, since the pullback of $\text{Aut}(\mathbb{P}(\Omega_{X,x}))$ into $\text{GL}(T_{X,x})$ is $\text{GL}(\mathcal{F}_{2,x}^\vee)_{\mathcal{Q}_x^\vee} \subset \text{GL}(\mathcal{F}_{2,x}^\vee)$, then the fiber bundle $\mathcal{U} \rightarrow X$ is defined by a cocycle in $H^1(X, \text{GL}(\mathcal{F}_{2,x}^\vee)_{\mathcal{Q}_x^\vee})$. Abusing notation, we denote this cocycle by θ .

Via the natural homomorphisms $\text{GL}(\mathcal{F}_{2,x}^\vee)_{\mathcal{Q}_x^\vee} \rightarrow \text{GL}(\mathcal{F}_{2,x}^\vee)$, $\text{GL}(\mathcal{F}_{2,x}^\vee)_{\mathcal{Q}_x^\vee} \rightarrow \text{GL}(\mathcal{Q}_x^\vee)$, $\text{GL}(\mathcal{F}_{2,x}^\vee)_{\mathcal{Q}_x^\vee} \rightarrow \text{GL}(\mathcal{F}_{1,x}^\vee)$, the cocycle θ defines three vector bundles over X , which we denote by \mathcal{F}_2^\vee , \mathcal{Q}^\vee , and \mathcal{F}_1^\vee , respectively; the dual bundles fit into a short exact sequence:

$$0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{Q} \rightarrow 0. \quad (3.2)$$

Moreover, we have an action of $\mathrm{GL}(\mathcal{F}_{2,x}^\vee)_{\mathcal{Q}_x^\vee}$ on $T_{X,x}$ which together with θ defines the tangent bundle T_X . It has a fixed subspace $\mathcal{F}_{1,x}^\vee \otimes \mathcal{Q}_x^\vee$, so that we have a sequence of $(\mathrm{GL}(\mathcal{F}_{2,x}^\vee)_{\mathcal{Q}_x^\vee})$ -modules

$$0 \rightarrow \mathcal{F}_{1,x}^\vee \otimes \mathcal{Q}_x^\vee \rightarrow T_{X,x} \rightarrow S^2\mathcal{F}_{1,x}^\vee \rightarrow 0,$$

which provides via θ a sequence of vector bundles:

$$0 \rightarrow \mathcal{F}_1^\vee \otimes \mathcal{Q}^\vee \rightarrow T_X \rightarrow S^2\mathcal{F}_1^\vee \rightarrow 0. \quad (3.3)$$

Furthermore, this sequence fits together with sequence (3.2), twisted with \mathcal{F}_1 , into the following commutative diagram, with short-exact rows and columns:

$$\begin{array}{ccccc} \wedge^2 \mathcal{F}_1 & \xlongequal{\quad} & \wedge^2 \mathcal{F}_1 & & \\ \downarrow & & \downarrow & & \\ \otimes^2 \mathcal{F}_1 & \longrightarrow & \mathcal{F}_1 \otimes \mathcal{F}_2 & \longrightarrow & \mathcal{F}_1 \otimes \mathcal{Q} \\ \downarrow & & \downarrow & & \parallel \\ S^2\mathcal{F}_1 & \longrightarrow & \Omega_X & \longrightarrow & \mathcal{F}_1 \otimes \mathcal{Q} \end{array} \quad (3.4)$$

At the point x the arrows of the diagram (3.1) are equivariant with respect to the action of $\mathrm{GL}(\mathcal{F}_{2,x}^\vee)_{\mathcal{Q}_x^\vee}$, hence the cocycle θ allows us to construct morphisms:

$$\begin{array}{ccccc} & & & \mathcal{U}_1 & \\ & & & \nearrow & \\ & & & \pi_1 & \\ \mathrm{HU} & \hookrightarrow & \mathcal{U} & \xrightarrow{q} & X \\ & & \searrow & & \nearrow \\ & & \mathbb{P}(\mathcal{Q}) & \hookrightarrow & \mathcal{U}_2 \end{array}$$

where $\mathcal{U}_1 := \mathbb{P}(\mathcal{F}_1)$, $\mathcal{U}_2 := \mathbb{P}(\mathcal{F}_2)$, and $\mathrm{HU} := \mathbb{P}(\mathcal{F}_1) \times_X \mathbb{P}(\mathcal{Q})$.

Moreover, denoting by $\mathcal{O}_{\mathcal{U}_i}(1)$ the tautological line bundles of \mathcal{U}_i , satisfying $q_{i*}(\mathcal{O}_{\mathcal{U}_i}(1)) = \mathcal{F}_i$, for $i = 1, 2$, and by $\mathcal{O}_{\mathcal{U}}(1)$ the pull-back $\tau^*\mathcal{O}_{\mathbb{P}(\Omega_X)}(1)$ we may write:

$$\mathcal{O}_{\mathcal{U}}(1) \cong \pi_1^*\mathcal{O}_{\mathcal{U}_1}(1) \otimes \pi_2^*\mathcal{O}_{\mathcal{U}_2}(1). \quad (3.5)$$

Note also that HU is the intersection of \mathcal{U} with $\mathbb{P}(\mathcal{F}_1 \otimes \mathcal{Q})$. This is a fiber bundle over X defined by a cocycle $\theta' \in H^1(X, \mathrm{Aut}(\mathbb{P}(\mathcal{F}_{1,x})) \times \mathrm{Aut}(\mathbb{P}(\mathcal{Q}_x)))$. Following [21], we would like to construct relative flags with respect to this fiber bundle, but, in order to obtain the complete flag of type C_n we need to reduce the cocycle θ' to a cocycle with values in $H^1(X, \mathrm{Aut}(\mathbb{P}(\mathcal{F}_{1,x})) \times \mathrm{Sp}(\mathbb{P}(\mathcal{Q}_x)))$. In other words, one of our goals will be to find a skew-symmetric isomorphism $\mathcal{Q} \cong \mathcal{Q}^\vee$.

Finally, we may also consider \mathcal{U} as a projective bundle $\mathbb{P}(\mathcal{G})$ over \mathcal{U}_1 , where $\mathcal{G} := \pi_{1*}\mathcal{O}_{\mathcal{U}}(1)$. Since the restriction map $\mathcal{O}_{\mathcal{U}}(1) \rightarrow \mathcal{O}_{\mathrm{HU}}(1)$ pushed-forward to X provides the quotient $\Omega_X \rightarrow \mathcal{F}_1 \otimes \mathcal{Q}$, whose kernel is $S^2\mathcal{F}_1$, it follows that we have an exact sequence on \mathcal{U}_1 :

$$0 \rightarrow \mathcal{O}_{\mathcal{U}_1}(2) \rightarrow \mathcal{G} \rightarrow q_1^*\mathcal{Q} \otimes \mathcal{O}_{\mathcal{U}_1}(1) \rightarrow 0. \quad (3.6)$$

In particular:

$$\mathcal{O}_{\mathcal{U}}(\mathrm{HU}) \cong \mathcal{O}_{\mathcal{U}}(1) \otimes \pi_1^*\mathcal{O}_{\mathcal{U}_1}(-2). \quad (3.7)$$

4 The family of isotropic lines in X

The main goal of this section is to show that the variety HU is a \mathbb{P}^1 -bundle over its image HM in \mathcal{M} , which will then be the natural candidate for the family of isotropic lines in X . This will be done by showing that the intersection number of $\mathcal{O}_{\mathcal{U}}(\mathrm{HU})$ with a fiber ℓ' of p is zero. By Formula 3.7 and Remark 2.1 we have

$$\mathcal{O}_{\mathcal{U}}(\mathrm{HU}) \cdot \ell' = -2 - 2\pi_1^*\mathcal{O}_{\mathcal{U}_1}(1) \cdot \ell', \quad (4.1)$$

so we need to show that $\pi_1^*\mathcal{O}_{\mathcal{U}_1}(1) \cdot \ell' = -1$.

Lemma 4.1 *Let ℓ' be a fiber of p and denote by ℓ_1 its image in \mathcal{U}_1 . Then ℓ_1 is a free rational curve, and so there exists a unique irreducible component \mathcal{M}_1 of $\mathrm{RatCurves}^n(\mathcal{U}_1)$ containing the class of ℓ_1 . Moreover the natural morphism $\mathcal{M} \rightarrow \mathcal{M}_1$ is injective, and $-K_{\mathcal{U}_1} \cdot \ell_1 \geq 2n - 2r + 2$, equality holding if and only if $\mathcal{M} \rightarrow \mathcal{M}_1$ is surjective.*

Proof The freeness of ℓ_1 can be proved as in Remark 2.3, and it implies the smoothness of the scheme $\mathrm{RatCurves}^n(\mathcal{U}_1)$ at the point corresponding to ℓ_1 ([15, Theorem. II.1.7, Theorem. II.2.15]). In particular, there is a unique component containing ℓ_1 , that we denote $\mathcal{M}_1 \subset \mathrm{RatCurves}^n(\mathcal{U}_1)$. Since fibers of p are mapped to different curves by q (see Remark 2.3), and therefore by π_1 , the natural morphism $\mathcal{M} \rightarrow \mathcal{M}_1$ is injective. By [15, Theorem. II.1.2]) we have $\dim \mathcal{M}_1 = -K_{\mathcal{U}_1} \cdot \ell_1 + \dim \mathcal{U}_1 - 3$. Finally, being $\dim \mathcal{M} = \dim \mathcal{U} - 1 = \dim \mathcal{U}_1 + 2n - 2r - 1$, the last assertion follows.

Lemma 4.2 *Let ℓ' be a fiber of p and let ℓ be its image in X . Then*

$$c_1(\mathcal{F}_1) \cdot \ell = -1 + r(\pi_1^*\mathcal{O}_{\mathcal{U}_1}(1) \cdot \ell' + 1) \left(1 + \frac{1-r}{2n-2r+1} \right), \text{ and}$$

$$c_1(\mathcal{F}_2) \cdot \ell = -1 - (\pi_1^*\mathcal{O}_{\mathcal{U}_1}(1) \cdot \ell' + 1)(2n - 3r + 2).$$

Proof We start by proving the second equality. Writing the relative canonical $\mathcal{O}_{\mathcal{U}}(K_{\mathcal{U}/X}) = \mathcal{O}_{\mathcal{U}}(K_{\mathcal{U}/\mathcal{U}_2}) \otimes \pi_2^*\mathcal{O}_{\mathcal{U}_2}(K_{\mathcal{U}_2/X})$ as

$$\mathcal{O}_{\mathcal{U}}(K_{\mathcal{U}/X}) = \mathcal{O}_{\mathcal{U}}((r-1)\mathrm{HU}) \otimes \pi_2^*\mathcal{O}_{\mathcal{U}_2}(-2n+r) \otimes q^*\det(\mathcal{F}_2)$$

and using (3.5) and (3.7), we get

$$\mathcal{O}_{\mathcal{U}}(K_{\mathcal{U}/X}) = \mathcal{O}_{\mathcal{U}}(-2n + 2r - 1) \otimes \pi_1^* \mathcal{O}_{\mathcal{U}_1}(2n - 3r + 2) \otimes q^* \det(\mathcal{F}_2);$$

now, intersecting with ℓ' and using the projection formula we obtain the formula for $c_1(\mathcal{F}_2) \cdot \ell$. Now, from the second column of diagram (3.4) we get

$$\det(\Omega_X) = \det(\mathcal{F}_2)^{\otimes r} \otimes \det(\mathcal{F}_1)^{\otimes 2n-2r+1},$$

and intersecting with ℓ we get the desired formula for $c_1(\mathcal{F}_1) \cdot \ell$.

Lemma 4.3 *Let ℓ' be a fiber of p . Then $\pi_1^* \mathcal{O}_{\mathcal{U}_1}(1) \cdot \ell' = -1$.*

Proof By Lemma 4.1 we have

$$1 - r \leq -\pi_1^* K_{\mathcal{U}_1/X} \cdot \ell' = \pi_1^* \mathcal{O}_{\mathcal{U}_1}(r) \cdot \ell' - q^* c_1(\mathcal{F}_1) \cdot \ell',$$

from which we get

$$c_1(\mathcal{F}_1) \cdot \ell \leq r(\pi_1^* \mathcal{O}_{\mathcal{U}_1}(1) \cdot \ell' + 1) - 1.$$

Combining this with the expression of $c_1(\mathcal{F}_1) \cdot \ell$ obtained in Lemma 4.2 we get

$$r(\pi_1^* \mathcal{O}_{\mathcal{U}_1}(1) \cdot \ell' + 1) \frac{1 - r}{2n - 2r + 1} \leq 0,$$

which gives $\pi_1^* \mathcal{O}_{\mathcal{U}_1}(1) \cdot \ell' \geq -1$.

On the other hand $\mathcal{O}_{\mathcal{U}}(\mathrm{H}\mathcal{U}) \cdot \ell' \geq 0$, since ℓ' belongs to a dominating family and $\mathrm{H}\mathcal{U}$ is an effective divisor; this implies, by formula (4.1), that $\pi_1^* \mathcal{O}_{\mathcal{U}_1}(1) \cdot \ell' \leq -1$, so equality holds.

Corollary 4.4 *With the same notation as above:*

1. $c_1(\mathcal{F}_1) \cdot \ell = -1$,
2. $(\mathcal{F}_1)_{|\ell} \simeq E(-1, 0^{r-1})$, and
3. $\mathcal{O}_{\mathcal{U}}(\mathrm{H}\mathcal{U}) \cdot \ell' = 0$.

In particular $\mathrm{HM} = p(\mathrm{H}\mathcal{U})$ parametrizes a subfamily of rational curves in X , which has codimension one in \mathcal{M} .

Proof The first assertion follows combining Lemmata 4.2 and 4.3. For the second, we notice that, by sequence (3.3), the vector bundle $S^2 \mathcal{F}_1^\vee$ is nef on ℓ , hence also \mathcal{F}_1^\vee is nef on ℓ . Since $c_1(\mathcal{F}_1^\vee) \cdot \ell = 1$, then (2) follows. The third assertion follows combining Formula (4.1) and Lemma 4.3.

Proposition 4.5 *Let \mathcal{M} and \mathcal{M}_1 be as in Lemma 4.1. Then the natural map $\mathcal{M} \rightarrow \mathcal{M}_1$ is an isomorphism. In particular \mathcal{M}_1 is a beautiful family of rational curves on \mathcal{U}_1 .*

Proof Using the canonical bundle formula for $p_1 : \mathcal{U}_1 \rightarrow X$, taking into account Lemma 4.3 and Corollary 4.4 (1) we get $-K_{\mathcal{U}_1} \cdot \ell_1 = 2n - 2r + 2$, so, by Lemma 4.1 the morphism $\mathcal{M} \rightarrow \mathcal{M}_1$ is surjective. Since \mathcal{M} is smooth and \mathcal{M}_1 is normal, then $\mathcal{M} \simeq \mathcal{M}_1$. In particular \mathcal{M}_1 is unsplit; it is, moreover, beautiful, since its evaluation morphism π_1 is surjective and smooth.

Remark 4.6 Note that the VMRT of the family \mathcal{M}_1 at every point P of \mathcal{U}_1 is a linear subspace of $\mathbb{P}(\Omega_{\mathcal{U}_1, P})$. In fact setting $x := q_1(P)$ and $L := q_1^{-1}(x)$, the tangent map $\tau_x : \mathcal{M}_x \rightarrow \mathbb{P}(\Omega_{X, x})$ factors via $\tau_1 : \mathcal{M}_x \rightarrow \mathbb{P}(\Omega_{\mathcal{U}_1|L})$:

$$\begin{array}{ccccc}
 & & \tau_x & & \\
 & \searrow & \curvearrowright & \searrow & \\
 \mathcal{M}_x & \xrightarrow{\tau_1} & \mathbb{P}(\Omega_{\mathcal{U}_1|L}) & \dashrightarrow & \mathbb{P}(\Omega_{X, x}) \times L & \xrightarrow{\quad} & \mathbb{P}(\Omega_{X, x}) \\
 & \searrow & \downarrow & \swarrow & & & \\
 & & L & & & &
 \end{array}$$

The tangent map of \mathcal{M}_1 at P is the restriction of τ_1 to the fiber $\mathcal{M}_{1, P}$ of $\mathcal{M}_x \rightarrow L$ over P . Since $\mathcal{M}_{1, P}$ goes one-to-one onto a linear subspace of $\mathbb{P}(\Omega_{X, x})$, and the second horizontal map is a linear projection (at every fiber over L), it follows that $\tau_1(\mathcal{M}_{1, P})$ is also linear.

5 Reduction of the defining group

As we have seen in Section 3.3, the variety $\text{H}\mathcal{U}$ is defined, as a bundle over X , by a cocycle $\theta' \in H^1(X, \text{Aut}(\mathbb{P}(\mathcal{F}_{1, x})) \times \text{Aut}(\mathbb{P}(\mathcal{Q}_x)))$. We will prove that there exists a skew-symmetric isomorphism $\omega_x : \mathcal{Q}_x^\vee \rightarrow \mathcal{Q}_x$, such that $\theta' \in H^1(X, \text{Aut}(\mathbb{P}(\mathcal{F}_{1, x})) \times \text{Sp}(\mathcal{Q}_x^\vee))$, where $\text{Sp}(\mathcal{Q}_x^\vee)$ denotes the subgroup of $\text{GL}(\mathcal{Q}_x^\vee)$ preserving ω_x . Equivalently, it is enough to prove that there exists a skew-symmetric isomorphism $\omega : \mathcal{Q}^\vee \rightarrow \mathcal{Q}$. We start by showing that \mathcal{U}_1 admits a second contraction, which is a projective bundle.

It is known ([1, Theorem 1.1]) that if a smooth complex projective variety Y has a family of minimal rational curves whose VMRT at a general point is a k -dimensional linear subspace, then there is a dense open subset $Y^\circ \subset Y$ and a \mathbb{P}^{k+1} -fibration $\rho^\circ : Y^\circ \rightarrow W^\circ$ such that the general element of the family is a line in a fiber of ρ° . The following Lemma shows that, under stronger assumptions, the fibration is defined on the whole Y .

Lemma 5.1 *Let Y be a smooth variety, and let $p : \mathcal{V} \rightarrow \mathcal{N}$ be a beautiful family of rational curves, satisfying that the tangent map is regular on \mathcal{V} , and the VMRT at every point $y \in Y$ is a linearly embedded $\mathbb{P}^k \subset \mathbb{P}(\Omega_{Y, y})$. Then there exists an equidimensional morphism $\rho : Y \rightarrow \mathcal{W}$, which contracts curves parametrized by \mathcal{N} , whose general fiber is a projective space of dimension $k+1$.*

Proof If the Picard number of Y is one, then Y is a projective space by [10, Proposition 5] (or by an easy application of [1, Theorem 1.1]), so we can assume that $\rho_Y \geq 2$. Let $q : \mathcal{V} \rightarrow Y$ be the evaluation morphism associated with \mathcal{N} . For a general $y \in Y$, since τ_y is the normalization of its image, we get that $\mathcal{N}_y \simeq \mathbb{P}^k$. Then all the fibers of q are isomorphic to \mathbb{P}^k by the smooth rigidity of the projective space (see [22]).

Given any point $y \in Y$, consider the \mathbb{P}^1 -bundle $p_y : \mathcal{V}_y \rightarrow \mathcal{N}_y$ defined as the fiber product of $p : \mathcal{V} \rightarrow \mathcal{N}$ and $p|_{\mathcal{N}_y} : \mathcal{N}_y \rightarrow \mathcal{N}$, whose image into Y via

the evaluation morphism is $\text{Locus}(\mathcal{N}_y) := q(p^{-1}(p(\mathcal{N}_y)))$; then $\text{Locus}(\mathcal{N}_y)$ is an irreducible closed subset of Y of dimension $k + 1$.

We claim first that, given two points $y, z \in Y$ then $z \in \text{Locus}(\mathcal{N}_y)$ if and only if $\text{Locus}(\mathcal{N}_y) = \text{Locus}(\mathcal{N}_z)$.

As in [2, Lemma 2.3] one can prove that \mathcal{V}_y is the blow-up of a \mathbb{P}^{k+1} at a point, and that there is a generically injective map $g : (\mathbb{P}^{k+1}, y') \rightarrow (\text{Locus}(\mathcal{N}_y), y)$ which maps lines by y' birationally to curves parametrized by \mathcal{N}_y . Let z' be a preimage of z via g and let ℓ be the line passing by y' and z' ; then lines by z' are mapped to curves in Y passing by z which are algebraically equivalent to $g(\ell)$. Therefore these curves belong to a component of $\text{RatCurves}^n(Y)$ containing $g(\ell)$. Since this component is unique by the freeness of $g(\ell)$, they belong to \mathcal{N} , hence $\text{Locus}(\mathcal{N}_y) \subset \text{Locus}(\mathcal{N}_z)$. Since both the loci are irreducible and of the same dimension the claim follows.

Consider now the rationally connected fibration $\rho : Y \dashrightarrow \mathcal{W}$ associated with \mathcal{N} ; by the claim every equivalence class has the same dimension, hence, by [4, Proposition 1] ρ is an (equidimensional) morphism.

By [15, Theorem II.1.2] we have $\dim \mathcal{N} = -K_Y \cdot g(\ell) + \dim Y - 3$, so $-K_Y \cdot g(\ell) = k + 2$. By adjunction and by [5], the general fiber is a projective space of dimension $k + 1$.

We now apply Lemma 5.1 to the case $Y = \mathcal{U}_1$ and $\mathcal{V} = \mathcal{M}_1$, obtaining:

Corollary 5.2 *There exist a smooth variety \mathcal{W} and smooth morphism $\rho : \mathcal{U}_1 \rightarrow \mathcal{W}$, which contracts curves parametrized by \mathcal{M}_1 , whose fibers are projective spaces of dimension $2n - 2r + 1$.*

Proof By Lemma 5.1 there exists an equidimensional morphism $\rho : \mathcal{U}_1 \rightarrow \mathcal{W}$, which contracts curves parametrized by \mathcal{M}_1 , whose general fiber is a projective space of dimension $2n - 2r + 1$. We then notice that, by Corollary 4.4 (1), $q_1^* \det(\mathcal{F}_1^\vee)$ restricts to $\mathcal{O}(1)$ on the general fiber, so ρ is a projective bundle by [7, Lemma 2.12] and \mathcal{W} is smooth.

Proposition 5.3 *There exists a skew-symmetric isomorphism $\omega : \mathcal{Q}^\vee \rightarrow \mathcal{Q}$. In particular, the bundle $\mathbb{P}(\mathcal{Q})$ is determined by a cocycle in $H^1(X, \text{Sp}(\mathcal{Q}_x^\vee))$.*

Proof By construction, since $\mathbb{P}(\mathcal{G}) = \mathcal{U}$ may be described as the universal family of lines in the fibers of $\rho : \mathcal{U}_1 \rightarrow \mathcal{W}$, it follows that $\mathbb{P}(\mathcal{G}) = \mathbb{P}(\Omega_{\mathcal{U}_1/\mathcal{W}})$. In particular there exists a line bundle \mathcal{L}_1 on \mathcal{U}_1 such that

$$\mathcal{O}_{\mathbb{P}(\mathcal{G})}(1) = \mathcal{O}_{\mathbb{P}(\Omega_{\mathcal{U}_1/\mathcal{W}})}(1) \otimes \pi_1^* \mathcal{L}_1.$$

Since both $\mathcal{O}_{\mathbb{P}(\mathcal{G})}(1)$ and $\mathcal{O}_{\mathbb{P}(\Omega_{\mathcal{U}_1/\mathcal{W}})}(1)$ have degree -2 on a fiber of p , which is mapped to a line in a fiber of ρ , there exists $\mathcal{L} \in \text{Pic}(\mathcal{W})$ such that $\mathcal{L}_1 = \rho^* \mathcal{L}$, so $\mathcal{G} = \Omega_{\mathcal{U}_1/\mathcal{W}} \otimes \rho^* \mathcal{L}$. In particular, the dual of the exact sequence (3.6) tensored with $\rho^* \mathcal{L}$ is

$$0 \rightarrow q_1^* \mathcal{Q}^\vee \otimes \mathcal{O}_{\mathcal{U}_1}(-1) \otimes \rho^* \mathcal{L} \rightarrow T_{\mathcal{U}_1/\mathcal{W}} \rightarrow \mathcal{O}_{\mathcal{U}_1}(-2) \otimes \rho^* \mathcal{L} \rightarrow 0,$$

which, restricted to a fiber $P \cong \mathbb{P}^{2n-2r+1}$ of ρ becomes, recalling Lemma 4.3:

$$0 \rightarrow (q_1^* \mathcal{Q}^\vee)|_P(1) \rightarrow T_P \rightarrow \mathcal{O}_P(2) \rightarrow 0.$$

Let us denote $\mathcal{D} := q_1^* \mathcal{Q}^\vee \otimes \mathcal{O}_{\mathcal{U}_1}(-1) \otimes \rho^* \mathcal{L}$. It defines a distribution in \mathcal{U}_1 , contained in the relative tangent bundle $T_{\mathcal{U}_1/\mathcal{W}}$. Hence, we may consider the O'Neill skew-symmetric tensor (induced by the Lie bracket of holomorphic sections of \mathcal{D}):

$$\omega : \bigwedge^2 \mathcal{D} \rightarrow \frac{T_{\mathcal{U}_1/\mathcal{W}}}{\mathcal{D}},$$

providing a homomorphism of sheaves (that we denote also by ω):

$$\omega : \mathcal{D} \rightarrow \mathcal{D}^\vee \otimes \mathcal{O}_{\mathcal{U}_1}(-2) \otimes \rho^* \mathcal{L}.$$

In order to prove that ω is an isomorphism, we may check it on the restriction to every fiber $P \cong \mathbb{P}^{2n-2r+1}$ of ρ . But on each of these fibers, $\mathcal{D}|_P$ is the kernel of a surjective map $T_P \rightarrow \mathcal{O}_P(2)$, hence it is a contact distribution (see [16, Example 2.1]), and then we know that the O'Neill tensor $\omega|_P : \mathcal{D}|_P \rightarrow \mathcal{D}|_P^\vee \otimes \mathcal{O}_P(2)$ is an isomorphism.

We conclude that there is an isomorphism:

$$\omega : q_1^* \mathcal{Q}^\vee \otimes \mathcal{O}_{\mathcal{U}_1}(-1) \otimes \rho^* \mathcal{L} \rightarrow q_1^* \mathcal{Q} \otimes \mathcal{O}_{\mathcal{U}_1}(-1).$$

Hence \mathcal{L} is trivial and, twisting ω with $\mathcal{O}_{\mathcal{U}_1}(1)$ and pushing it forward to X , it provides a skew-symmetric isomorphism from \mathcal{Q}^\vee to \mathcal{Q} .

6 Construction of relative flags and conclusion

As we have already remarked, we will consider the family of rational curves parametrized by HM , denoted $p : \text{HU} \rightarrow \text{HM}$, and the evaluation map $q : \text{HU} \rightarrow X$, which is the fiber product $\mathbb{P}(\mathcal{F}_1) \times_X \mathbb{P}(\mathcal{Q})$. We may consider, on one hand, the A_{r-1} -bundle $\overline{\mathcal{U}}_1 \rightarrow X$ associated to $\mathcal{U}_1 \rightarrow X$, whose fibers at every point $x \in X$ are the complete flags of $\mathbb{P}(\mathcal{F}_{1,x})$. On the other hand, if we denote $\mathcal{U}_3 := \mathbb{P}(\mathcal{Q})$, we may also construct, by Proposition 5.3, a C_{n-r} -bundle, $\overline{\mathcal{U}}_3 \rightarrow X$, which factors through the map $\mathcal{U}_3 \rightarrow X$, and whose fibers over every point $x \in X$ are the flags in $\mathbb{P}(\mathcal{Q}_x)$ that are isotropic with respect to ω_x . We finally define $\overline{\text{HU}} := \overline{\mathcal{U}}_1 \times_X \overline{\mathcal{U}}_3$, and denote by $\overline{q} : \overline{\text{HU}} \rightarrow X$ the projection map, that factors:

$$\begin{array}{ccc} & \overline{q} & \\ & \curvearrowright & \\ \overline{\text{HU}} & \longrightarrow & \text{HU} \xrightarrow{q} X \end{array}$$

The bundle $\overline{\text{HU}} \rightarrow X$ is an $(A_{r-1} \sqcup C_{n-r})$ -bundle over X . We number the nodes of this diagram over the set $D = \{1, 2, \dots, r-1, r+1, r+2, \dots, n\}$, as in Figure (6.1) below, so that the fibers of q are the homogeneous manifolds corresponding to the Dynkin diagram $A_{r-1} \sqcup C_{n-r}$ marked at the set $I := \{r-1, r+1\}$ (see for instance [19, Section 2.2]).

$$\begin{array}{ccccccccccc}
\circ & \text{---} & \circ & \text{---} & \cdots & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \cdots & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ \\
1 & & 2 & & & & r-2 & & r-1 & & r+1 & & r+2 & & & & n-1 & & n \\
& & & & & & A_{r-1} & & \underbrace{\hspace{2cm}}_I & & & & C_{n-r} & & & & & & & & &
\end{array} \quad (6.1)$$

Proposition 6.1 *The variety $\overline{\text{HU}}$ is the complete flag manifold of type C_n , and X is a symplectic Grassmannian.*

Proof It is enough to prove that $\overline{\text{HU}}$ is a complete flag manifold, since in this case, being a target of a contraction of $\overline{\text{HU}}$, X would be a rational homogeneous manifold; since the only rational homogeneous manifolds of Picard number one for which \mathcal{M}_x is as stated are the symplectic Grassmannians, the statement follows.

Now, in order to prove that $\overline{\text{HU}}$ is a complete flag, we use [21, Theorem A.1], for which we need to show that $\overline{\text{HU}}$ admits $\rho_{\overline{\text{HU}}}$ independent \mathbb{P}^1 -bundle structures. This is obvious in the case $n = 3$, $r = 2$, so we may assume that $I \subsetneq D$. Note that $\overline{\text{HU}}$ has already $\rho_{\overline{\text{HU}}} - 1$ different \mathbb{P}^1 -bundle structures, coming from the flag bundle construction described in Section 2.3. Hence we only need to find the remaining \mathbb{P}^1 -bundle structure, which will be constructed by means of minimal sections over isotropic lines.

Let $f : \mathbb{P}^1 \rightarrow X$ be the normalization of any curve Γ of the family HM and consider the pull-back $f^*\overline{\text{HU}}$ with the natural section $s : \mathbb{P}^1 \rightarrow s(\mathbb{P}^1) \subset f^*\overline{\text{HU}}$. Denote by $\bar{s} : \mathbb{P}^1 \rightarrow s^*f^*\overline{\text{HU}}$ a minimal section of the bundle $s^*f^*\overline{\text{HU}}$ over \mathbb{P}^1 and by $\bar{\Gamma}$ its image. We have then the following commutative diagram:

$$\begin{array}{ccccc}
& & \bar{s} & & \\
& & \curvearrowright & & \\
s^*f^*\overline{\text{HU}} & \xrightarrow{\pi} & \mathbb{P}^1 & & \\
\downarrow \lrcorner & & \downarrow s & \searrow & \\
f^*\overline{\text{HU}} & \xrightarrow{\pi} & f^*\overline{\text{HU}} & \xrightarrow{q} & \mathbb{P}^1 \\
\downarrow \lrcorner & & \downarrow \lrcorner & & \downarrow f \\
\overline{\text{HU}} & \xrightarrow{\pi} & \overline{\text{HU}} & \xrightarrow{q} & X \\
\leftarrow p \circ \pi & & \leftarrow p & & \\
\text{HM} & & & &
\end{array}$$

As in [21, Section 4] one can show that $\bar{\Gamma}$ is a minimal section of both the $(A_{r-1} \sqcup C_{n-r})$ -bundle $f^*\overline{\text{HU}}$ and the $(A_{r-2} \sqcup C_{n-r-1})$ -bundle $s^*f^*\overline{\text{HU}}$; moreover $s^*f^*\overline{\text{HU}}$ is determined by the tagged Dynkin diagram obtained by eliminating the nodes indexed by I from the tagged Dynkin diagram of $f^*\overline{\text{HU}}$.

Now we want to compute the tag $(d_1, \dots, d_{r-1}, d_{r+1}, \dots, d_n)$ of $f^*\overline{\text{HU}}$; to this aim we use [21, Formula (11)]:

$$\pi^* K_{\overline{\text{HU}}/X} \cdot \bar{\Gamma} = \sum_{i \in D} b_i d_i - \sum_{j \notin I} c_j d_j$$

where the b_i 's (resp. the c_j 's) can be read from [21, Table 1], for $\mathcal{D} = A_{r-1} \sqcup C_{n-r}$ (resp. $\mathcal{D} = A_{r-2} \sqcup C_{n-r-1}$), obtaining

$$2n - r - 1 = \pi^* K_{\overline{\text{HU}}/X} \cdot \overline{F} = (r-1)d_{r-1} + (2n-2r)d_{r+1} + \sum_{j \notin I} a_j d_j,$$

$$a_j = \begin{cases} j & \text{for } j = 1, \dots, r-2 \\ 2n-2r & \text{for } j = r+2, \dots, n-1 \\ n-r & \text{for } j = n \end{cases}$$

Now, since all the d_k are nonnegative, and d_{r-1}, d_{r+1} are strictly positive (see [21, Proof of (4.2.2)]) we have that $d_{r-1} = d_{r+1} = 1$ and $d_k = 0$ for every other k . In particular, by [21, Proposition 3.17], the flag bundle $s^* f^* \overline{\text{HU}}$ is trivial, and we may deduce, reasoning as in [21, Corollary 4.3], that there exists a smooth projective variety $\overline{\text{HM}}$ such that the morphism $\overline{\text{HU}} \rightarrow \text{HM}$ factors via a smooth \mathbb{P}^1 -bundle $\overline{p} : \overline{\text{HU}} \rightarrow \overline{\text{HM}}$. Now, as in [21, Proof of Theorem 1.1] we can show that in $\text{NE}(\overline{\text{HU}})$ there exists $n = \rho_{\overline{\text{HU}}}$ independent K_X -negative classes generating n extremal rays, whose associated elementary contractions $\pi_i : X \rightarrow X_i$ are smooth \mathbb{P}^1 -bundles, and we conclude by [21, Theorem A.1].

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