THE SET OF REGULAR VALUES (IN THE SENSE OF CLARKE) OF A LIPSCHITZ MAP. A SUFFICIENT CONDITION FOR THE RECTIFIABILITY OF CLASS C^3.

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ABSTRACT. Let n, N be positive integers such that $n < N$. We prove a result about the rectifiability of class C^3 of the set of regular values (in the sense of Clarke) of a Lipschitz map $\varphi : \mathbb{R}^n \to \mathbb{R}^N$.

1. Introduction and statement of main result

In this paper we prove a result about the rectifiability of class C^3 of the set of regular values (in the sense of Clarke) of a Lipschitz map

$$\varphi : \mathbb{R}^n \to \mathbb{R}^N \qquad (n < N).$$

Before we state it, let us recall some basic definitions.

A Borel subset S of $\mathbb{R}^N$ is said to be an $(\mathcal{H}^n, n)$ rectifiable set of class C^3 (or simply: a rectifiable set of class C^3), if there exist countably many n-dimensional submanifolds M_j of $\mathbb{R}^N$ of class C^3 such that

$$\mathcal{H}^n\left(S \backslash \bigcup_j M_j\right) = 0.$$

Analogously one can define the $(\mathcal{H}^n, n)$ rectifiable sets of class C^k, for each positive integer k. In particular, for $k = 1$ this notion is equivalent to that of n-rectifiable set, e.g. by [S, Lemma 11.1].

For $\gamma \in I(n, N)$ and $s \in \mathbb{R}^n$, let $\partial\varphi^\gamma(s)$ denote the Clarke subdifferential of the map

$$\varphi^\gamma := (\varphi^{\gamma_1}, \dots, \varphi^{\gamma_n}) : \mathbb{R}^n \to \mathbb{R}^n$$

namely

$$\partial\varphi^\gamma(s) := \mathrm{co}\left\{ \lim_{i\to\infty} D\varphi^\gamma(s_i) \,\middle|\, D\varphi^\gamma(s_i) \text{ exists}, s_i \to s \right\}$$

compare [CLSW, p.133]. The set $\partial\varphi^\gamma(s)$ is said to be "nonsingular" if every matrix in $\partial\varphi^\gamma(s)$ is of rank n. Observe that $D\varphi^\gamma(s) \in \partial\varphi^\gamma(s)$ whenever φ^γ

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is differentiable at s. In particular, $D\varphi^\gamma(s)$ is nonsingular provided $\partial\varphi^\gamma(s)$ is nonsingular. Define

$$\mathcal{R} := \{s \in \mathbb{R}^n \,|\, \partial\varphi^\gamma(s) \text{ is nonsingular for some } \gamma\}.$$

We can now state our theorem.

Theorem 1.1. *Consider a Lipschitz map*

$$\varphi : \mathbb{R}^n \to \mathbb{R}^N \qquad (n < N).$$

Moreover let

$$c_{1,i}, c_{2,i} : \mathbb{R}^n \to \mathbb{R}\backslash\{0\} \qquad (i = 1, \dots, n)$$

be a family of locally bounded functions, let

$$G_{1,i}, G_{2,i} : \mathbb{R}^n \to \mathbb{R}^N \quad (i = 1, \dots, n), \quad H_{ij} : \mathbb{R}^n \to \mathbb{R}^N \quad (i, j = 1, \dots, n)$$

be a family of Lipschitz maps and denote by A the set of points $t \in \mathbb{R}^n$ satisfying the following conditions:

(i) *The map φ and all the maps $G_{1,i}$ are differentiable at t;*

(ii) *The equality*

$$D_i\varphi(t) = c_{1,i}(t)G_{1,i}(t) = c_{2,i}(t)G_{2,i}(t) \tag{1.1}$$

holds for all $i = 1, \dots, n$;

(iii) *Moreover one has*

$$D_jG_{1,i}(t) = c_{2,j}(t)H_{ij}(t) \tag{1.2}$$

for all $i, j = 1, \dots, n$.

Also assume that

(iv) *For almost every $a \in A$ there exists a non-trivial ball B centered at a and such that*

$$\mathcal{L}^n(B\backslash A) = 0.$$

Then $\varphi(A \cap \mathcal{R})$ is an $(\mathcal{H}^n, n)$ rectifiable set of class C^3.

Remark 1.2. As an immediate corollary of Theorem 1.1, we get this result. Let

$$\varphi : \mathbb{R}^n \to \mathbb{R}^N, \qquad G_{1,i}, G_{2,i}, H_{ij} : \mathbb{R}^n \to \mathbb{R}^N \ (i, j = 1, \dots, n)$$

be a family of Lipschitz maps and let

$$c_{1,i}, c_{2,i} : \mathbb{R}^n \to \mathbb{R}\backslash\{0\} \qquad (i = 1, \dots, n),$$

be a family of bounded functions such that the equalities (1.1) and (1.2) hold almost everywhere in $\mathbb{R}^n$. Then the image $\varphi(\mathcal{R})$ is an $(\mathcal{H}^n, n)$ rectifiable set of class C^3.

Remark 1.3. Let each component φ_i of $\varphi : \mathbb{R}^n \to \mathbb{R}^N$ belong to $C^3(\mathbb{R}^n)$ and have uniformly bounded gradient $\nabla\varphi_i$. Moreover let the differential $D\varphi$ have rank n at each point of $\mathbb{R}^n$. Then the assumptions of Theorem 1.1 are trivially satisfied by setting

$$c_{1,i} := 1, \quad c_{2,i} := 1, \quad G_{1,i} := D_i\varphi, \quad G_{2,i} := D_i\varphi \qquad (i = 1, \ldots, n)$$

and

$$H_{ij} := D^2_{ij}\varphi \qquad (i, j = 1, \ldots, n)$$

with $A = \mathbb{R}^n$.

Rectifiable sets of class C^k have been been introduced in [AS] and provide a natural setting for the description of singularities of convex functions and convex surfaces, [A, AO]. More generally, it can be used to study the singularities of surfaces with generalized curvatures, [AO]. Rectifiability of class C^2 is strictly related to the context of Legendrian rectifiable subsets of $\mathbb{R}^N \times \mathbf{S}^{N-1}$, [Fu1, Fu2, D2, D3]. The level sets of a $W^{k,p}_{\mathrm{loc}}$ mapping between manifolds are rectifiable sets of class C^k, [BHS]. Applications of rectifiable sets of class C^H (with $H \geq 2$) to geometric variational problems can be found in [D4].

Finally, we would like to explain the reasons of our interest in conditions (1.1) and (1.2). In the particular case when $n = 1$, such conditions arised naturally in the context of one-dimensional generalized Gauss graphs (see [AST, D1], for the basic definitions and results) and of two-storey towers of one-dimensional generalized Gauss graphs (see [D4]). After that it was natural to explore the question of how those assumptions could be generalized in order to get results about higher order of rectifiability, including the case when $n \geq 2$. Then a general theorem for curves was provided in [D3], while in [D5] we started studying the case of general dimension by proving a result about the rectifiability of class C^2. Further results in this direction can be found in [AS] and [Fu1, Fu2]. Roughly speaking, the very basic idea and the proof-strategy in the present paper are the same as in [D5] namely: to use the celebrated Whitney extension Theorem to show that the image of φ is captured, up to $\mathcal{H}^n$ measure 0, by countably many high regular images of $\mathbb{R}^n$. More precisely, the main objective of this work is to get the set of third order Whitney estimates which allows one to perform the (countably many) extensions of class C^3 necessary to show that the image of φ is C^3-rectifiable. Such a result is the product of our efforts to prove a general theorem about rectifiability of class C^H in any dimension, which will be the subject of our future further investigations.

2. Reduction to graphs

Remark 2.1. Under the hypotheses of Theorem 1.1, let A' denote the set of $a \in A$ such that there exists a non-trivial ball B centered at a satisfying

$$\mathcal{L}^n(B \backslash A) = 0.$$

One has

$$\mathcal{L}^n(A \backslash A') = 0 \tag{2.1}$$

by assumption (iv). Hence, it will be enough to prove that $\varphi(A' \cap \mathcal{R})$ is an $(\mathcal{H}^n, n)$ rectifiable set of class C^3.

Remark 2.2. By the main theorem in [D5], we already know that $\varphi(A \cap \mathcal{R})$ (hence also $\varphi(A' \cap \mathcal{R})$) is an $(\mathcal{H}^n, n)$ rectifiable set of class C^2.

Remark 2.3. Let E be any subset of $\mathcal{R}$ and define

$$E^\gamma := \{s \in E \,|\, \partial\varphi^\gamma(s) \text{ is nonsingular}\}, \qquad \gamma \in I(n, N).$$

Then one obviously has

$$\bigcup_{\gamma \in I(n,N)} E^\gamma = E.$$

Remark 2.4. If $s \in \mathcal{R}^\gamma$, by the Lipschitz inverse function Theorem (e.g. [CLSW, Theorem 3.12]), there exist a neighborhood U (in $\mathbb{R}^n$) of s and a neighborhood V (in $\mathbb{R}^n$) of $\varphi^\gamma(s)$ such that

- $V = \varphi^\gamma(U)$ and $\varphi^\gamma|U : U \to V$ is invertible;
- $(\varphi^\gamma|U)^{-1}$ is Lipschitz.

Let $\overline{\gamma}$ denote the multi-index in $I(N-n, N)$ which complements γ in $\{1, 2, \dots, N\}$ in the natural increasing order and set (for $x \in \mathbb{R}^N$)

$$x^\gamma := (x^{\gamma_1}, \dots, x^{\gamma_n}), \qquad x^{\overline{\gamma}} := (x^{\overline{\gamma}_1}, \dots, x^{\overline{\gamma}_{N-n}}).$$

Then the map

$$f := \varphi^{\overline{\gamma}} \circ (\varphi^\gamma|U)^{-1} : V \to \mathbb{R}^{N-n}$$

is Lipschitz and its graph

$$\mathcal{G}_f^\gamma := \{x \in \mathbb{R}^N \,|\, x^\gamma \in V \text{ and } x^{\overline{\gamma}} = f(x^\gamma)\}$$

coincides with $\varphi(U)$.

By virtue of previous remarks, it will be enough to prove the following claim.

Theorem 2.5. *Under the assumptions of Theorem 1.1, let $\gamma \in I(n,N)$ and consider a map*

$$g : \mathbb{R}^n \to \mathbb{R}^{N-n}$$

of class C^2. Then $\varphi((A' \cap \mathcal{R})^\gamma) \cap \mathcal{G}_g^\gamma$ is an $(\mathcal{H}^n, n)$ rectifiable set of class C^3.

Remark 2.6. The remainder of our paper is devoted to proving Theorem 2.5. With no loss of generality, we can restrict our attention to the particular case when $\gamma = \{1, \dots, n\}$.

3. Preliminaries

3.1. **Further reduction of the claim.** From now on, for simplicity,

$$\mathcal{G}_g^{\{1,\dots,n\}}, \quad (A' \cap \mathcal{R})^{\{1,\dots,n\}}, \quad \varphi^{\{1,\dots,n\}}$$

will be denoted by $\mathcal{G}_g$, F and λ, respectively.

Define

$$L := \varphi^{-1}(\mathcal{G}_g) \cap F.$$

Without loss of generality, we can assume that $\mathcal{L}^n(L) < \infty$. Then, by a well-known regularity property of $\mathcal{L}^n$, for any given real number $\varepsilon > 0$ there exists a closed subset L_ε of $\mathbb{R}^n$ with

$$L_\varepsilon \subset L, \qquad \mathcal{L}^n(L \backslash L_\varepsilon) \le \varepsilon, \tag{3.1}$$

compare e.g. [M, Theorem 1.10]. Moreover, since L_ε is closed, one has

$$L_\varepsilon^* \subset L_\varepsilon \tag{3.2}$$

where L_ε^* is the set of density points of L_ε. Recall that

$$\mathcal{L}^n(L_\varepsilon \backslash L_\varepsilon^*) = 0 \tag{3.3}$$

by a well-known result of Lebesgue. In the special case that L has measure zero, we define $L_\varepsilon := \emptyset$, hence $L_\varepsilon^* := \emptyset$.

Observe that

$$\mathcal{G}_g \cap \varphi(F) \backslash \varphi(L_\varepsilon^*) \subset \varphi\left(\varphi^{-1}(\mathcal{G}_g) \cap F \backslash L_\varepsilon^*\right) = \varphi(L \backslash L_\varepsilon^*)$$

hence

$$\begin{aligned}
\mathcal{H}^n\left(\mathcal{G}_g \cap \varphi(F) \backslash \varphi(L_\varepsilon^*)\right) &\le \mathcal{H}^n\left(\varphi\left(L \backslash L_\varepsilon^*\right)\right) \\
&\le \int_{L \backslash L_\varepsilon^*} J_n \varphi \, d\mathcal{L}^n \\
&\le (\operatorname{Lip} \varphi)^n \mathcal{L}(L \backslash L_\varepsilon^*) \\
&\le \varepsilon \, (\operatorname{Lip} \varphi)^n
\end{aligned}$$

by the area formula (compare [F, §3.2.], [S, §8]), (3.1), (3.2) and (3.3). It follows that

$$\mathcal{H}^n\left(\mathcal{G}_g \cap \varphi(F)\setminus \bigcup_{j=1}^{\infty} \varphi(L^*_{1/j})\right) = 0.$$

Thus, to prove Theorem 2.5, it suffices to show that

$$\varphi(L^*_\varepsilon) \text{ is an } (\mathcal{H}^n, n) \text{ rectifiable set of class } C^3$$

for all $\varepsilon > 0$.

3.2. **Further notation.** Let us consider the projection

$$\Pi : \mathbb{R}^N \to \mathbb{R}^{N-n}, \qquad (x_1, \ldots, x_N) \mapsto (x_{n+1}, \ldots, x_N).$$

Moreover set

$$\mathcal{R}^{(0)}_s(\sigma) := g(\lambda(\sigma)) - g(\lambda(s)) - \sum_{i=1}^n D_i g(\lambda(s)) \left[\varphi^i(\sigma) - \varphi^i(s)\right] + \\ - \frac{1}{2} \sum_{i,j=1}^n D^2_{ij} g(\lambda(s)) \left[\varphi^i(\sigma) - \varphi^i(s)\right] \left[\varphi^j(\sigma) - \varphi^j(s)\right],$$

$$\mathcal{R}^{(1)}_{i;s}(\sigma) := D_i g(\lambda(\sigma)) - D_i g(\lambda(s)) - \sum_{j=1}^n D^2_{ij} g(\lambda(s)) \left[\varphi^j(\sigma) - \varphi^j(s)\right],$$

$$\mathcal{R}^{(2)}_{ij;s}(\sigma) := D^2_{ij} g(\lambda(\sigma)) - D^2_{ij} g(\lambda(s)).$$

For $h = 1, 2$, let G_h denote the $n \times n$ matrix field such that

$$[G_h(t)]^j_i := G^j_{h,i}(t), \qquad t \in \mathbb{R}^n \qquad (i, j = 1, \ldots, n).$$

Also let H be the $n^2 \times n$ matrix field defined by

$$[H(t)]^k_{ij} := H^k_{ij}(t), \qquad t \in \mathbb{R}^n \qquad (i, j, k = 1, \ldots, n)$$

where the couples ij (indexing the rows) are ordered lexicographically.

Then consider the $(n + n^2) \times (n + n^2)$ matrix field

$$M := \begin{bmatrix} G_1 & 0 \\ H & G_1 \otimes G_2 \end{bmatrix}$$

where the symbol $\otimes$ denotes the Kronecker product of matrices, [HJ, Sect. 4.2].

For $l = 1, \ldots, N - n$, let $D^2 g^l$ denote the $\mathbb{R}^{n^2}$-valued field such that

$$\left[D^2 g^l(t)\right]^{ij} := D^2_{ij} g^l(t), \qquad t \in \mathbb{R}^n \qquad (i, j = 1, \ldots, n)$$

where the lexicographical order is assumed.

Finally, given a matrix X and a index k, denote by

$$R_k(X), C_k(X)$$

the k-th row of X and k-th column of X, respectively.

4. The derivatives of g in terms of $\{G_1, G_2, H\}$ (under the assumptions of Theorem 2.5, with $\gamma = \{1, \ldots, n\}$)

Proposition 4.1. *Let $l \in \{1, \ldots, N-n\}$ and $s \in L_\varepsilon^*$. Then*

$$M(s)\ \left(Dg^l(\lambda(s)), D^2 g^l(\lambda(s))\right)^T = \left(G_1^{n+l}(s), H^{n+l}(s)\right)^T \tag{4.1}$$

where G_1^{n+l} and H^{n+l} are the the vector fields defined as follows:

$$G_1^{n+l} := (G_{1,1}^{n+l}, \ldots, G_{1,n}^{n+l})$$

and

$$H^{n+l} := [H_{ij}^{n+l}]_{i,j=1}^n \qquad (\textit{in lexicographical order}).$$

Proof. First of all, observe that

$$g(\lambda(t)) = \Pi\varphi(t)$$

for all $t \in \varphi^{-1}(\mathcal{G}_g)$. Since $L_\varepsilon^* \subset A$ the two members of this equality are both differentiable at s. Moreover s is a limit point of $L_\varepsilon \subset \varphi^{-1}(\mathcal{G}_g)$. It follows that

$$\sum_{j=1}^n D_j g(\lambda(s)) D_i \varphi^j(s) = \Pi D_i \varphi(s) \qquad (i = 1, \ldots, n)$$

namely

$$\sum_{j=1}^n D_j g(\lambda(s)) c_{1,i}(s) G_{1,i}^j(s) = c_{1,i}(s) \Pi G_{1,i}(s) \qquad (i = 1, \ldots, n)$$

by (1.1). Since $c_{1,i}(s) \neq 0$ $(i = 1, \ldots, n)$, we get

$$\sum_{j=1}^n D_j g^l(\lambda(s)) G_{1,i}^j(s) = G_{1,i}^{n+l}(s) \qquad (i = 1, \ldots, n) \tag{4.2}$$

i.e.

$$G_1(s) Dg^l(\lambda(s)) = G_1^{n+l}(s). \tag{4.3}$$

By the same argument as above, we can differentiate (4.2) and obtain

$$\sum_{j,k=1}^n D_{jk}^2 g^l(\lambda(s)) D_m \varphi^k(s)\, G_{1,i}^j(s) + \sum_{j=1}^n D_j g^l(\lambda(s)) D_m G_{1,i}^j(s) = D_m G_{1,i}^{n+l}(s)$$

for all $i, m = 1, \ldots, n$. By (1.2)

$$\sum_{j,k=1}^n D_{jk}^2 g^l(\lambda(s)) c_{2,m}(s) G_{2,m}^k(s) G_{1,i}^j(s) + \\ + \sum_{j=1}^n D_j g^l(\lambda(s)) c_{2,m}(s) H_{im}^j(s) = c_{2,m}(s) H_{im}^{n+l}(s)$$

for all $i, m = 1, \ldots, n$, namely

$$[G_1(s) \otimes G_2(s)]\, D^2 g^l(\lambda(s)) + H(s) Dg^l(\lambda(s)) = H^{n+l}(s). \tag{4.4}$$

We conclude by observing that the system of equalities (4.3) and (4.4) is equivalent to (4.1). □

In this result we investigate the properties of the matrix field $t \mapsto M(t)^{-1}$.

Proposition 4.2. *Let $s \in A$ be such that $D\lambda(s)$ is nonsingular (e.g. $s \in F$). Then there exists a nontrivial ball B, centered at s, such that*

- *For all $t \in B$, the matrices $G_1(t)$, $G_2(t)$ and $M(t)$ are invertible and*

$$M(t)^{-1} = \begin{bmatrix} G_1(t)^{-1} & 0 \\ -[G_1(t)^{-1} \otimes G_2(t)^{-1}]H(t)G_1(t)^{-1} & G_1(t)^{-1} \otimes G_2(t)^{-1} \end{bmatrix} \tag{4.5}$$

- *The map*
$$t \mapsto M(t)^{-1}, \qquad t \in B$$
is Lipschitz.

Proof. One has

$$D\lambda(s) = \left[\prod_{i=1}^{n} c_{1,i}(s)\right] G_1(s)^T = \left[\prod_{i=1}^{n} c_{2,i}(s)\right] G_2(s)^T$$

by (1.1), hence $G_1(s)$ and $G_2(s)$ are nonsingular. Moreover one has

$$\det M = \det G_1 \det(G_1 \otimes G_2) = (\det G_1)^{n+1}(\det G_2)^n \tag{4.6}$$

by [HJ, Sect. 4.2, Problem 1]. Thus

$$\det M(s) \neq 0.$$

Since the function $t \mapsto \det M(t)$ is continuous, there exists a nontrivial ball B centered at s and such that

$$|\det M(t)| \geq \frac{|\det M(s)|}{2} > 0$$

for all $t \in B$. As a consequence, $M(t)$ is invertible at every $t \in B$. The formula (4.5) follows at once observing that, for $t \in B$, the matrix $M(t)^{-1}$ has to be of the form (recall (4.6))

$$\begin{bmatrix} G_1(t)^{-1} & 0 \\ X(t) & [G_1(t) \otimes G_2(t)]^{-1} \end{bmatrix}$$

with $X(t)$ satisfying

$$H(t)G_1(t)^{-1} + [G_1(t) \otimes G_2(t)]X(t) = 0$$

and finally recalling that

$$[G_1(t) \otimes G_2(t)]^{-1} = G_1(t)^{-1} \otimes G_2(t)^{-1}$$

compare [HJ, Corollary 4.2.11]. This concludes the proof of the first claim. The second one follows by observing that the entries of M are Lipschitz. □

5. Whitney-type estimates (under the assumptions of Theorem 2.5, with $\gamma = \{1, \dots, n\}$)

Proposition 5.1. *Let $s \in L^*_\varepsilon$ and $t \in A \cap \varphi^{-1}(\mathcal{G}_g)$ be such that*

$$\mathcal{H}^1([s;t] \backslash A) = 0 \tag{5.1}$$

where $[s;t]$ denotes the segment joining s and t. Then the following estimate holds

$$\|\mathcal{R}^{(0)}_s(t)\| \le \left(\sup_{[s;t]} \|c_1\|\right)\left(\sup_{[s;t]} \|c_2\|\right)\Lambda_s \, \|t-s\|^3$$

where

$$c_1 := (c_{1,1}, \dots, c_{1,n}), \qquad c_2 := (c_{2,1}, \dots, c_{2,n})$$

and Λ_s is a constant not depending on t.

Proof. First of all, observe that:

- Since $s, t \in \varphi^{-1}(\mathcal{G}_g)$ one has $g(\lambda(s)) = \Pi\varphi(s)$ and $g(\lambda(t)) = \Pi\varphi(t)$;
- Consider the following parametrization of $[s;t]$

  $$\sigma : [0,1] \to \mathbb{R}^n, \qquad \rho \mapsto s + \rho(t-s).$$

  Then the function $\rho \mapsto \varphi(\sigma(\rho))$ is Lipschitz, hence it is differentiable almost everywhere in $[0,1]$. Moreover the assumption (5.1) implies that

  $$(\varphi \circ \sigma)'(\rho) = \sum_{i_1=1}^n (t^{i_1} - s^{i_1}) D_{i_1}\varphi(\sigma(\rho))$$

  at a.e. $\rho \in [0,1]$.

Recalling also (1.1), we obtain

$$\begin{aligned}
\mathcal{R}^{(0)}_s(t) &= \Pi\varphi(t) - \Pi\varphi(s) - \sum_{i=1}^n D_i g(\lambda(s)) \left[\varphi^i(t) - \varphi^i(s)\right] + \\
&\quad - \frac{1}{2}\sum_{i,j=1}^n D^2_{ij} g(\lambda(s)) \left[\varphi^i(t) - \varphi^i(s)\right]\left[\varphi^j(t) - \varphi^j(s)\right] \\
&= \sum_{h=1}^n (t^h - s^h) \int_0^1 \Bigg\{ \Pi D_h\varphi(\sigma(\rho)) - \sum_{i=1}^n D_i g(\lambda(s)) D_h\varphi^i(\sigma(\rho)) + \\
&\quad - \sum_{i,j=1}^n D^2_{ij} g(\lambda(s)) \left[\varphi^i(\sigma(\rho)) - \varphi^i(s)\right] D_h\varphi^j(\sigma(\rho)) \Bigg\} \, d\rho
\end{aligned}$$

that is

$$\mathcal{R}^{(0)}_s(t) = \sum_{h=1}^n (t^h - s^h) \int_0^1 c_{1,h}(\sigma(\rho)) \Phi_{s,h}(\sigma(\rho)) \, d\rho \tag{5.2}$$

where $\Phi_{s,h}$ denotes the Lipschitz map defined as follows

$$\Phi_{s,h} := \Pi G_{1,h} - \sum_{i=1}^{n} D_i g(\lambda(s)) G_{1,h}^i - \sum_{i,j=1}^{n} D_{ij}^2 g(\lambda(s)) \left[\varphi^i - \varphi^i(s)\right] G_{1,h}^j. \tag{5.3}$$

Now, since $\Phi_{s,h} \circ \sigma$ is Lipschitz, it is differentiable almost everywhere in $[0,1]$ and

$$(\Phi_{s,h} \circ \sigma)' = \sum_{k=1}^{n} (t^k - s^k)(D_k \Phi_{s,h}) \circ \sigma.$$

Moreover $\Phi_{s,h}(s) = 0$, by (4.2). By (5.3) and recalling (1.2), we get

$$\begin{aligned} \Phi_{s,h}(\sigma(\rho)) = \Phi_{s,h}(\sigma(\rho)) - \Phi_{s,h}(s) &= \int_0^\rho (\Phi_{s,h} \circ \sigma)' \\ &= \sum_{k=1}^{n} (t^k - s^k) \int_0^\rho (D_k \Phi_{s,h}) \circ \sigma \\ &= \sum_{k=1}^{n} (t^k - s^k) \int_0^\rho (c_{2,k} \circ \sigma)(\Psi_{s,hk} \circ \sigma) \end{aligned} \tag{5.4}$$

where $\Psi_{s,hk}$ is the Lipschitz map defined by

$$\begin{aligned} \Psi_{s,hk} := \Pi H_{hk} - \sum_{i=1}^{n} D_i g(\lambda(s)) H_{hk}^i + \\ - \sum_{i,j=1}^{n} D_{ij}^2 g(\lambda(s)) \left\{ G_{2,k}^i G_{1,h}^j + [\varphi^i - \varphi^i(s)] H_{hk}^j \right\}. \end{aligned}$$

Observe that

$$\begin{aligned} \Psi_{s,hk}(s) &= \Pi H_{hk}(s) - \sum_{i=1}^{n} D_i g(\lambda(s)) H_{hk}^i(s) - \sum_{i,j=1}^{n} D_{ij}^2 g(\lambda(s)) G_{2,k}^i(s) G_{1,h}^j(s) \\ &= 0 \end{aligned}$$

by (4.4). Hence (for all $r \in [0,1]$)

$$\begin{aligned} \|\Psi_{s,hk}(\sigma(r))\| = \|\Psi_{s,hk}(\sigma(r)) - \Psi_{s,hk}(s)\| &\le \|\sigma(r) - s\| \operatorname{Lip} \Psi_{s,hk} \\ &= r\|t - s\| \operatorname{Lip} \Psi_{s,hk} \\ &\le \|t - s\| \, \Lambda_s \end{aligned}$$

with

$$\Lambda_s := \max_{h,k=1,\dots,n} \left(\operatorname{Lip} \Psi_{s,hk} \right).$$

Recalling (5.4), we obtain

$$\|\Phi_{s,h}(\sigma(\rho))\| \le \left(\sup_{[s;t]} \|c_2\| \right) \Lambda_s \, \|t - s\|^2.$$

The conclusion follows at once from (5.2). ☐

Proposition 5.2. *Let $s \in L^*_\varepsilon$. Then there exists a nontrivial ball B, centered at s, such that*

$$\left\|\mathcal{R}^{(1)}_{i;s}(t)\right\| \leq \left(\sup_{[s;t]} \|c_2\|\right) \Sigma_s \|t-s\|^2 \qquad (i=1,\dots,n)$$

for all $t \in L^*_\varepsilon \cap B$ such that (5.1) is satisfied, where c_2 is defined as in Proposition 5.1 while Σ_s is a constant not depending on t and i.

Proof. Since $s \in L^*_\varepsilon \subset F$, there exists a ball B as in Proposition 4.2. Consider

$$t \in L^*_\varepsilon \cap B$$

such that (5.1) is satisfied. Then (for $l = 1,\dots,N-n$)

$$\begin{aligned}\left[\mathcal{R}^{(1)}_{i;s}(t)\right]^l &= D_i g^l(\lambda(t)) - D_i g^l(\lambda(s)) - \sum_{j=1}^n D^2_{ij} g^l(\lambda(s))\,[\varphi^j(t)-\varphi^j(s)] \\ &= R_i(G_1(t)^{-1}) \bullet G_1^{n+l}(t) - R_i(G_1(s)^{-1}) \bullet G_1^{n+l}(s) + \\ &\quad - \sum_{j=1}^n D^2_{ij} g^l(\lambda(s))\,[\varphi^j(t)-\varphi^j(s)]\end{aligned}$$

by Proposition 4.1 and Proposition 4.2. Moreover, if σ is the parametrization of $[s;t]$ defined above, the function

$$\Pi : \rho \mapsto R_i(G_1(\sigma(\rho))^{-1}) \bullet G_1^{n+l}(\sigma(\rho)), \qquad \rho \in [0,1]$$

is Lipschitz, hence it is differentiable almost everywhere in $[0,1]$. Recalling (5.1) and denoting with G_1^{-1} the map $r \mapsto G_1(r)^{-1}$ (by a convenient abuse of notation), we obtain

$$\Pi'(\rho) = \sum_{q=1}^n (t^q - s^q)\left\{R_i(D_q G_1^{-1}) \bullet G_1^{n+l} + R_i(G_1^{-1}) \bullet D_q G_1^{n+l}\right\}(\sigma(\rho))$$

for a.e. $\rho \in [0,1]$. By the well-known formula for the derivative of the inverse matrix field, compare [HJ, (6.5.7)], it follows that

$$\begin{aligned}\Pi'(\rho) &= \sum_{q=1}^n (t^q - s^q)\big\{R_i(G_1^{-1}) \bullet D_q G_1^{n+l} + \\ &\quad - R_i[G_1^{-1}(D_q G_1)G_1^{-1}] \bullet G_1^{n+l}\big\}(\sigma(\rho)) \\ &= \sum_{m,q=1}^n (t^q - s^q)\left\{[G_1^{-1}]_i^m D_q G_{1,m}^{n+l} - [G_1^{-1}(D_q G_1)G_1^{-1}]_i^m G_{1,m}^{n+l}\right\}(\sigma(\rho)) \\ &= \sum_{m,q=1}^n (t^q - s^q)\Bigg\{[G_1^{-1}]_i^m D_q G_{1,m}^{n+l} \\ &\quad - \sum_{h,k=1}^n [G_1^{-1}]_i^h (D_q G_{1,h}^k)[G_1^{-1}]_k^m G_{1,m}^{n+l}\Bigg\}(\sigma(\rho))\end{aligned}$$

for a.e. $\rho \in [0,1]$. Recalling (1.2), we get

$$\Pi'(\rho) = \sum_{m,q=1}^{n} c_{2,q}(\sigma(\rho))(t^q - s^q)\Big\{[G_1^{-1}]_i^m H_{mq}^{n+l} + - \sum_{h,k=1}^{n} [G_1^{-1}]_i^h H_{hq}^k [G_1^{-1}]_k^m G_{1,m}^{n+l}\Big\}(\sigma(\rho))$$

for a.e. $\rho \in [0,1]$. It follows that

$$\left[\mathcal{R}_{i;s}^{(1)}(t)\right]^l = \sum_{q=1}^{n}(t^q - s^q)\int_0^1 c_{2,q}(\sigma(\rho))\Theta_{q;s}^l(\sigma(\rho))\,d\rho \tag{5.5}$$

where $\Theta_{q;s}^l : B \to \mathbb{R}$ is the function defined as

$$\Theta_{q;s}^l := \sum_{m=1}^{n}\left\{[G_1^{-1}]_i^m H_{mq}^{n+l} - \sum_{h,k=1}^{n}[G_1^{-1}]_i^h H_{hq}^k [G_1^{-1}]_k^m G_{1,m}^{n+l} - D_{im}^2 g^l(\lambda(s)) G_{2,q}^m\right\}.$$

One has

$$\begin{aligned} D_{im}^2 g^l(\lambda(s)) &= \sum_{c,d=1}^{n}\left[G_1(s)^{-1}\otimes G_2(s)^{-1}\right]_{im}^{cd} H_{cd}^{n+l}(s) + \\ &\quad - \sum_{b,c,d,e=1}^{n}\left[G_1(s)^{-1}\otimes G_2(s)^{-1}\right]_{im}^{cd} H_{cd}^b(s)\left[G_1(s)^{-1}\right]_b^e G_{1,e}^{n+l}(s) \\ &= \sum_{c,d=1}^{n}\left[G_1(s)^{-1}\right]_i^c\left[G_2(s)^{-1}\right]_m^d H_{cd}^{n+l}(s) + \\ &\quad - \sum_{b,c,d,e=1}^{n}\left[G_1(s)^{-1}\right]_i^c\left[G_2(s)^{-1}\right]_m^d H_{cd}^b(s)\left[G_1(s)^{-1}\right]_b^e G_{1,e}^{n+l}(s) \end{aligned}$$

by Proposition 4.1 and Proposition 4.2. Hence the following equality holds

$$\begin{aligned} \sum_{m=1}^{n} D_{im}^2 g^l(\lambda(s)) G_{2,q}^m(s) &= \sum_{c,d=1}^{n}\left[G_1(s)^{-1}\right]_i^c H_{cd}^{n+l}(s)\,\delta_{dq} + \\ &\quad - \sum_{b,c,d,e=1}^{n}\left[G_1(s)^{-1}\right]_i^c H_{cd}^b(s)\left[G_1(s)^{-1}\right]_b^e G_{1,e}^{n+l}(s)\,\delta_{dq} \\ &= \sum_{c=1}^{n}\left[G_1(s)^{-1}\right]_i^c H_{cq}^{n+l}(s) + \\ &\quad - \sum_{b,c,e=1}^{n}\left[G_1(s)^{-1}\right]_i^c H_{cq}^b(s)\left[G_1(s)^{-1}\right]_b^e G_{1,e}^{n+l}(s) \end{aligned}$$

namely

$$\Theta_{q;s}^l(s) = 0.$$

Moreover $\Theta^l_{q;s}$ is Lipschitz, by Proposition 4.2. Then, if define

$$\Sigma_s := (N-n) \max_{\substack{q=1,\dots,n \\ l=1,\dots,N-n}} \left(\mathrm{Lip}\,\Theta^l_{q;s}\right),$$

we get

$$\left|\Theta^l_{q;s}(\sigma(\rho))\right| = \left|\Theta^l_{q;s}(\sigma(\rho)) - \Theta^l_{q;s}(s)\right| \le \frac{\Sigma_s}{N-n}\rho\|t-s\| \le \frac{\Sigma_s}{N-n}\|t-s\|$$

for all $q = 1, \dots, n$, for all $l = 1, \dots, N-n$ and for all $\rho \in [0,1]$. From (5.5) it finally follows that

$$\left\|\mathcal{R}^{(1)}_{i;s}(t)\right\| \le \sum_{l=1}^{N-n} \left|\left[\mathcal{R}^{(1)}_{i;s}(t)\right]^l\right| \le \left(\sup_{[s;t]}\|c_2\|\right)\Sigma_s\|t-s\|^2.$$

□

The estimate of the second order remainder term is established in the following result, which is an immediate conseguence of Proposition 4.2 and (4.1).

Proposition 5.3. *Let $s \in L^*_\varepsilon$. Then there exists a nontrivial ball B, centered at s, such that*

$$\left\|\mathcal{R}^{(2)}_{ij;s}(t)\right\| = \left\|D^2_{ij}g(\lambda(t)) - D^2_{ij}g(\lambda(s))\right\| \le \Gamma_s\,\|t-s\| \qquad (i,j = 1,\dots,n)$$

for all $t \in L^*_\varepsilon \cap B$, where Γ_s is a constant not depending on t and i, j.

6. Proof of Theorem 2.5

As we pointed out in Section 3.1, we are reduced to prove that $\varphi(L^*_\varepsilon)$ is an $(\mathcal{H}^n, n)$ rectifiable set of class C^3 (for all $\varepsilon > 0$).

For each positive integer h, define $\Gamma_{\varepsilon,h}$ as the set of $s \in L^*_\varepsilon$ such that

$$\|\mathcal{R}^{(0)}_s(t)\| \le h\|\lambda(t)-\lambda(s)\|^3 \tag{6.1}$$

and

$$\|\mathcal{R}^{(1)}_{i;s}(t)\| \le h\|\lambda(t)-\lambda(s)\|^2, \quad \|\mathcal{R}^{(2)}_{ij;s}(t)\| \le h\|\lambda(t)-\lambda(s)\| \tag{6.2}$$

for all $i, j = 1, \dots, n$ and for all $t \in L^*_\varepsilon$ satisfying

$$\|t-s\| \le \frac{1}{h}.$$

Proposition 6.1. *One has*

$$\bigcup_h \Gamma_{\varepsilon,h} = L^*_\varepsilon.$$

Proof. The inclusion

$$\bigcup_h \Gamma_{\varepsilon,h} \subset L^*_\varepsilon$$

is obvious. In order to prove the opposite inclusion, consider $s \in L^*_\varepsilon$ and let U and V be as in Remark 2.4. Observe that

$$\begin{aligned} \|t-s\| &= \left\|(\lambda|U)^{-1}(\lambda(t)) - (\lambda|U)^{-1}(\lambda(s))\right\| \\ &\le \mathrm{Lip}(\lambda|U)^{-1}\,\|\lambda(t)-\lambda(s)\| \end{aligned} \tag{6.3}$$

for all $t \in U$.

Since $s \in A'$, there exists a non-trivial ball B centered at s such that

$$B \subset U, \qquad \mathcal{L}^n(B\backslash A) = 0.$$

By shrinking, if need be, we may also assume that B is as in the claims of Proposition 5.2 and Proposition 5.3.

We now recall the following fact, proved in [D5]: given a null-measure subset Z of $\mathbb{R}^n$ and $s \in \mathbb{R}^n$, one has

$$\mathcal{H}^1(Z \cap [s;t]) = 0$$

for a.e. $t \in \mathbb{R}^n$.

For $Z := B\backslash A$, we get

$$\mathcal{H}^1([s;t]\backslash A) = \mathcal{H}^1(Z \cap [s;t]) = 0$$

for a.e. $t \in B$. Then Proposition 5.1 yields

$$\|\mathcal{R}^{(0)}_s(t)\| \le C\,\|t-s\|^3$$

for a.e. $t \in B \cap \varphi^{-1}(\mathcal{G}_g)$, where C is a suitable number which does not depend on t. By continuity we get

$$\|\mathcal{R}^{(0)}_s(t)\| \le C\,\|t-s\|^3$$

for all $t \in B \cap \varphi^{-1}(\mathcal{G}_g)$. Recalling (6.3) we conclude that

$$\|\mathcal{R}^{(0)}_s(t)\| \le C_0\,\|\lambda(t)-\lambda(s)\|^3, \qquad C_0 := C\,\left[\mathrm{Lip}(\lambda|U)^{-1}\right]^3$$

for all $t \in B\cap\varphi^{-1}(\mathcal{G}_g)$. Analogously, we can use Proposition 5.2, Proposition 5.3 and (6.3) to deduce the existence of two numbers C_1 and C_2 which do not depend on t and are such that

$$\|\mathcal{R}^{(1)}_{i;s}(t)\| \le C_1\,\|\lambda(t)-\lambda(s)\|^2 \qquad (i=1,\dots,n)$$

and

$$\|\mathcal{R}^{(2)}_{ij;s}(t)\| \le C_2\,\|\lambda(t)-\lambda(s)\| \qquad (i,j=1,\dots,n)$$

for all $t \in L^*_\varepsilon \cap B$.

Hence

$$s \in \Gamma_{\varepsilon,h}$$

provided h is big enough. $\square$

From Proposition 6.1 it follows that

$$\varphi(L_\varepsilon^*) = \bigcup_h \varphi(\Gamma_{\varepsilon,h})$$

hence it will be enough to verify that

$$\varphi(\Gamma_{\varepsilon,h}) \text{ is an } (\mathcal{H}^n, n) \text{ rectifiable set of class } C^3 \tag{6.4}$$

for all ε and h.

To prove this claim, we first consider a countable measurable covering $\{Q_l\}_{l=1}^\infty$ of $\Gamma_{\varepsilon,h}$ such that

$$\operatorname{diam} Q_l \ \leq \ \frac{1}{h}$$

for all l, and define

$$F_l := \overline{\lambda(\Gamma_{\varepsilon,h} \cap Q_l)}.$$

If $\xi, \eta \in F_l$, then there exist two sequences

$$\{s_k\}, \{t_k\} \subset \Gamma_{\varepsilon,h} \cap Q_l$$

such that

$$\lim_k \lambda(s_k) = \xi, \qquad \lim_k \lambda(t_k) = \eta.$$

By (6.1) and (6.2), for all k, one has

$$\|\mathcal{R}_{s_k}^{(0)}(t_k)\| \leq h\|\lambda(t_k) - \lambda(s_k)\|^3$$

and

$$\|\mathcal{R}_{i,s_k}^{(1)}(t_k)\| \leq h\|\lambda(t_k) - \lambda(s_k)\|^2, \quad \|\mathcal{R}_{ij,s_k}^{(2)}(t_k)\| \leq h\|\lambda(t_k) - \lambda(s_k)\|$$

for all $i, j = 1, \dots, n$. Letting $k \to \infty$, we obtain

$$\left\| g(\eta) - g(\xi) - \sum_{i=1}^n D_i g(\xi)(\eta^i - \xi^i) - \frac{1}{2} \sum_{i,j=1}^n D_{ij}^2 g(\xi)(\eta^i - \xi^i)(\eta^j - \xi^j) \right\| \leq h\|\eta - \xi\|^3,$$

$$\left\| D_i g(\eta) - D_i g(\xi) - \sum_{j=1}^n D_{ij}^2 g(\xi)(\eta^j - \xi^j) \right\| \leq h\|\eta - \xi\|^2 \qquad (i = 1, \dots, n)$$

and

$$\left\| D_{ij}^2 g(\eta) - D_{ij}^2 g(\xi) \right\| \leq h\|\eta - \xi\| \qquad (i, j = 1, \dots, n)$$

for all $\xi, \eta \in F_l$. By the Whitney extension Theorem [St, Ch. VI, §2.3] it follows that each $g|F_l$ can be extended to a map in $C^{2,1}(\mathbb{R}^n, \mathbb{R}^{N-n})$. Then the Lusin type result [F, §3.1.15] implies that $\varphi(\Gamma_{\varepsilon,h} \cap Q_l)$ is an $(\mathcal{H}^n, n)$ rectifiable set of class C^3. Finally, claim (6.4) follows observing that

$$\varphi(\Gamma_{\varepsilon,h}) = \bigcup_l \varphi(\Gamma_{\varepsilon,h} \cap Q_l).$$

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