Infinite Dimensional Analysis, Quantum Probability and Related Topics

Existence and regularity of the density for solutions of stochastic differential equations with boundary noise

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We study the existence and regularity of densities for the solution of a nonlinear heat diffusion with stochastic perturbation of Brownian and fractional Brownian motion type: we use the Malliavin calculus in order to prove that, if the non linear term is suitably regular, then the law of the solution has a smooth density with respect to the Lebesgue measure.

Keywords: Density of the solution; fractional Brownian motion; stochastic boundary condition.

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1. Introduction

Stochastic reaction-diffusion equations with white-noise boundary conditions of Robin-Neumann type were introduced in the literature in the '90s of last century, see the seminal paper by Sowers.[1] One of the main issues is the regularity of the solution inside the domain as well as near the boundary. In a recent paper,[2] it was studied a class of nonlinear heat diffusions with stochastic perturbation of Brownian and fractional Brownian motion type. In this note, we proceed with the analysis of such equations, by considering in particular the problem of regularity in the Malliavin sense of the solution of such problem. An extension of these results to the case of stochastic reaction-diffusion equations with Dirichlet boundary conditions as proposed in[3,4] will be given in a subsequent paper.

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In particular, we consider in this paper the following nonlinear diffusion equation on the half-line

$$\begin{cases} \dfrac{\partial}{\partial t} u(t,x) = \dfrac{1}{2} \dfrac{\partial^2}{\partial x^2} u(t,x) + f(t,x,u(t,x)), \qquad x > 0, \quad t \in [0,T], \\ \\ \dfrac{\partial}{\partial x} u(t,0) + \displaystyle\int_S g(\sigma)\,\mathrm{d}B_{\sigma,t} = 0, \qquad t \in [0,T]. \end{cases} \tag{1.1}$$

for a suitable regular function f (see Hypothesis (4.1)–(4.2)–(4.3) for a precise definition). The stochastic process $B_{\sigma,t}$ is a Gaussian stochastic process on a measurable space $[0,T] \times S$; this process can be either a Brownian motion or a fractional Brownian motion with Hurst parameter $H > 1/2$.
We remark that, as opposite to the papers,[1,2] we have chosen to fix the one dimensional domain $\mathbb{R}_+$ in order to emphasize the results instead of the technicalities. We claim that most of the results still holds in the multidimensional case.

We interpret Eq. (1.1) in the sense of Walsh.[5] Let $g_N(t,x,y)$ and $p_N(t,x)$ be the Green kernel and the Poisson kernel for the heat equation in $[0,T] \times D$ with Neumann boundary conditions, see Definition 3.1. Then a solution of (1.1) is the process $u(t,x)$ that satisfies the evolution equation

$$u(t,x) = \int_0^t \int_S p_N(s,x) g(\sigma)\,\mathrm{d}B_{s,\sigma} + \int_0^t \int_{\mathbb{R}_+} g_N(t-s,x,y) f(s,y,u(s,y))\,\mathrm{d}y\,\mathrm{d}s. \tag{1.2}$$

Our main result provides the existence and regularity of the density (with respect to Lebesgue measure) of the random variable $u(t,x)$ with $t \in]0,T]$ and $x > 0$. Our approach is rather standard in this context, as we rely on techniques of the Malliavin calculus and a-priori estimates for the solution.

Stochastic heat equation in a bounded domain in one spatial dimension, with homogeneous boundary conditions, has been studied by many authors: see, e.g., Walsh[5] and the problem of existence and regularity of the density for equation with additive or multiplicative noise has been addressed by many authors: we quote in particular Pardoux and Zhang,[6] Bally and Pardoux,[7] Morien[8] and Mueller and Nualart.[9] For the multidimensional case, the literature is, in our knowledge, much scarcer: we quote, for instance, Márquez-Carreras *et al.*[10] and the recent paper by Marinelli *et al.*[11] Finally, as far as we know this is the first attempt to study a problem of existence and regularity of the density for solutions of stochastic evolution equations with boundary noise.

Notation. In the sequel, we shall indicate with C a constant that may varies from line to line. In certain cases, we write $C_{\alpha,\beta,\dots}$ to emphasize the dependence of the constant on the parameters $\alpha, \beta, \dots$.

2. Preliminaries on Malliavin calculus

Let us recall some basic facts about the Malliavin calculus with respect to (standard and fractional) Brownian motion; for full details, we refer to.[12]

Fix a measurable space $(S, \mathcal{S})$ with a finite measure μ on it, as well as a time interval $[0, T]$.

We are given a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a centered Gaussian family $B = \{B(h),\, h \in \mathcal{H}\}$ defined in Ω. The space $\mathcal{H}$ is constructed below.

Recall that a fractional Brownian motion $B^H = \{B^H(t),\; t \in [0,T]\}$ is a centered Gaussian process with covariance function

$$R_H(t,s) = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}), \qquad s,t \in [0,T].$$

Let $\mathcal{E}$ be the space of step functions on $[0,T] \times S$. We denote by $\mathcal{H}$ the closure of $\mathcal{E}$ with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]} \times \mathbf{1}_A, \mathbf{1}_{[0,s]} \times \mathbf{1}_B \rangle_{\mathcal{H}} = R_H(t,s)\, \mu(A \cap B);$$

notice that in case $H = 1/2$ then the first component in $\mathcal{H}$ is the standard L^2 space with respect to the Lebesgue measure on $[0,T]$, so that for $\phi, \psi \in \mathcal{H}$ we have

$$\langle \phi, \psi \rangle_{\mathcal{H}} = \int_0^T \int_S \phi(s,\sigma)\psi(s,\sigma)\, \mu(\mathrm{d}\sigma)\, \mathrm{d}s.$$

In case of a fractional Brownian motion with Hurst parameter $H > 1/2$ it holds

$$\langle \phi, \psi \rangle_{\mathcal{H}} = \int_S \int_0^T \int_0^T |s-t|^{2H-2} \phi(s,\sigma)\psi(t,\sigma)\, \mathrm{d}t\, \mathrm{d}s\, \mu(\mathrm{d}\sigma),$$

$\phi, \psi \in \mathcal{H}$.

Thus, in case $H = 1/2$, we say that the Gaussian family B is associated to a Brownian motion process $B_{s,\sigma}$ on $\mathcal{H}$ and in case $H > \frac{1}{2}$ it is associated to a fractional Brownian motion $B_{s,\sigma}$ via the identification

$$B(\phi) = \int_S \int_0^T \phi(s,\sigma)\, \mathrm{d}B_{s,\sigma}, \qquad \phi \in \mathcal{H}.$$

A $\mathcal{F}$-measurable real valued random variable F is said to be cylindrical if it can be written as

$$F = f\left(B(\phi^1), \dots, B(\phi^n)\right) ,$$

where $\phi^i \in \mathcal{H}$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a C^∞ bounded function. The set of cylindrical random variables is denoted by $\mathcal{S}$. The Malliavin derivative of $F \in \mathcal{S}$ is the stochastic process $DF = \{D_{s,\sigma}F,\; s \in [0,T],\; \sigma \in S\}$ given by

$$D_{s,\sigma}F = \sum_{i=1}^n \phi^i(s,\sigma) \frac{\partial f}{\partial x_i} \left(B(\phi^1), \dots, B(\phi^n)\right).$$

More generally, we can introduce iterated derivatives. If $F \in \mathcal{S}$, we set

$$D^k_{t_1,\dots,t_k,\sigma_1,\dots,\sigma_k} F = D_{t_1,\sigma_1} \dots D_{t_k,\sigma_k} F.$$

For any $p \geq 1$, the operator D^k is closable from $\mathcal{S}$ into $L^p\left(\mathcal{C}([0,T] \times S, \mathbb{R}), \mathcal{H}^{\otimes k}\right)$. We denote by $\mathbb{D}^{k,p}(\mathcal{H})$ the closure of the class of cylindrical random variables with respect to the norm

$$\|F\|_{k,p} = \left(\mathbb{E}\left(|F|^p\right) + \sum_{j=1}^{k} \mathbb{E}\left(\left\|D^j F\right\|^p_{\mathcal{H}^{\otimes j}}\right)\right)^{\frac{1}{p}},$$

and

$$\mathbb{D}^\infty(\mathcal{H}) = \bigcap_{p \geq 1} \bigcap_{k \geq 1} \mathbb{D}^{k,p}(\mathcal{H}).$$

We also introduce the localized spaces $\mathbb{D}^{k,p}_{\text{loc}}(\mathcal{H})$ by saying that a random variable F belongs to $\mathbb{D}^{k,p}_{\text{loc}}(\mathcal{H})$ if there exists a sequence of sets $\Omega_n \subset \Omega$ and random variables $F_n \in \mathbb{D}^{k,p}(\mathcal{H})$ such that $\Omega_n \uparrow \Omega$ almost surely and such that $F = F_n$ on Ω_n.

We then have the following key result which stems from Nualart[12] Theorem 2.1.2 and Corollary 2.1.2:

Theorem 2.1. *Let $F = (F_1, \dots, F_n)$ be a $\mathcal{F}$-measurable random vector such that:*

(1) For every $i = 1, \dots, n$, $F_i \in \mathbb{D}^{1,2}_{\text{loc}}(\mathcal{H})$;
(2) The Malliavin matrix of the random vector F: $\Gamma = \left(\langle DF^i, DF^j \rangle_{\mathcal{H}}\right)_{1 \leq i,j \leq n}$ is invertible almost surely.

Then the law of F has a density with respect to the Lebesgue measure on $\mathbb{R}^n$.
If moreover $F \in \mathbb{D}^\infty(\mathcal{H})$ and, for every $p > 1$,

$$\mathbb{E}\left(|\det \Gamma|^{-p}\right) < +\infty,$$

then this density is smooth.

The following result is useful in the proof of regularity for the solution of the stochastic differential equation (1.1). We recall here for the sake of completeness. A proof can be found, for instance, in Nualart[12] Lemma 1.5.3.

Proposition 2.1. *Let $\{F_n\}$ be a sequence of variables in $\mathbb{D}^{k,p}$ for some $p > 1$. Assume that the sequence F_n converges to F in $L^p(\Omega)$ and that*

$$\sup_n \|F_n\|_{k,p} < \infty. \tag{2.1}$$

Then F belongs to $\mathbb{D}^{k,p}$.

3. The stochastic convolution process

We are here concerned with the solution of the one dimensional heat diffusion problem with inhomogeneous white noise boundary condition

$$\begin{cases} \dfrac{\partial}{\partial t} z(t,x) = \dfrac{1}{2}\dfrac{\partial^2}{\partial x^2} z(t,x), & x>0, \quad t\in[0,T], \\[2ex] \dfrac{\partial}{\partial x} z(t,0) + \displaystyle\int_S g(\sigma)\,\mathrm{d}B_{\sigma,t} = 0, & t\in[0,T]. \end{cases} \tag{3.1}$$

With no loss of generality we assume that

$$\int_S |g(\sigma)|^2\,\mu(\mathrm{d}\sigma) = 1. \tag{3.2}$$

Then the solution to (3.1) is given by the *stochastic convolution process*

$$z(t,x) = \int_0^t \int_S p_N(s,x) g(\sigma)\,\mathrm{d}B_{s,\sigma}. \tag{3.3}$$

Definition 3.1 (The Green and Poisson kernel). The solution of the deterministic problem

$$\begin{cases} \dfrac{\partial}{\partial t} u(t,x) = \dfrac{1}{2}\dfrac{\partial^2}{\partial x^2} u(t,x) + f(t,x), & x>0, \quad t\in[0,T], \\[2ex] \dfrac{\partial}{\partial x} u(t,0) + g(t) = 0, & t\in[0,T]. \end{cases} \tag{3.4}$$

is given by the expression

$$u(t,x) = \int_0^t p_N(t-s,x) g(s)\,\mathrm{d}s + \int_0^t \int_{\mathbb{R}_+} g_N(t-s,x,y) f(s,y)\,\mathrm{d}y\,\mathrm{d}s \tag{3.5}$$

for sufficiently smooth data f and g. In this case, the Green g_N and Poisson p_N kernels with Neumann boundary conditions are explicitly given

$$\begin{aligned} g_N(t,x,y) &= \frac{1}{\sqrt{2\pi t}} \left(\exp(-\frac{(x-y)^2}{2t}) + \exp(-\frac{(x+y)^2}{2t}) \right), \\ p_N(t,x) &= \frac{1}{\sqrt{2\pi t}} \exp(-\frac{x^2}{2t}). \end{aligned}$$

Lemma 3.1. *The Poisson kernel $p_N(s,x)$ belongs to $\mathcal{H}$ for every $x>0$, and for $H>1/2$ $p_N(s,x)$ belongs to $\mathcal{H}$ for every $x\ge 0$.*
In particular we have the estimate

$$\|p_N(\cdot,x)\mathbf{1}_{(0,t)}(\cdot)\|^2_{\mathcal{H}} \le C \int_0^t s^{2H-2} e^{-x^2/s}\,\mathrm{d}s \tag{3.6}$$

valid for any $H \ge 1/2$.

Proof. We may directly compute the relevant integrals; notice that for the case $H > 1/2$, due to the monotonicity of $p_N(s, x)$ in x, it is sufficient to prove that

$$\int_0^T \int_0^T p_N(s,0) p_N(t,0) |t-s|^{2H-2} \, \mathrm{d}s \, \mathrm{d}t < +\infty,$$

and actually we have that:

$$\begin{aligned}\int_0^T \int_0^T p_N(s,0) p_N(t,0) |t-s|^{2H-2} \, \mathrm{d}s \, \mathrm{d}t &= \int_0^T \int_0^T \frac{1}{2\pi\sqrt{ts}} |t-s|^{2H-2} \, \mathrm{d}s \, \mathrm{d}t \\ &= \int_0^1 \int_0^1 \frac{1}{2\pi T \sqrt{\sigma\rho}} T^{2H-2} |\sigma-\rho|^{2H-2} T^2 \, \mathrm{d}\sigma \, \mathrm{d}\rho \\ &= \frac{T^{2H-1}}{2\pi} \int_0^1 \int_0^1 \frac{1}{\sqrt{\sigma\rho}} |\sigma-\rho|^{2H-2} \, \mathrm{d}\sigma \, \mathrm{d}\rho = C_H T^{2H-1}. \quad (3.7)\end{aligned}$$

In the Brownian case, we compute

$$\int_0^T p_N(s,x)^2 \, \mathrm{d}s = \frac{1}{2\pi} \int_0^T e^{-x^2/s} \frac{\mathrm{d}s}{s} = \frac{1}{2\pi} \int_{x^2/T}^\infty e^{-s} \frac{\mathrm{d}s}{s} \le C \, |\log(x)|$$

and this quantity is finite for any $x > 0$ while it degenerates as $x \to 0$. □

We first provide the global regularity of the solution of problem (3.1).

Theorem 3.1. *For any $p \ge 2$ the stochastic convolution process $z(t,x)$ belongs to $L^p_{\mathcal{F}}((0,T) \times \mathbb{R}_+) = L^p_{\mathcal{F}}(\Omega; L^p((0,T) \times \mathbb{R}_+)$.*

Proof. We start from the identity

$$\mathbb{E}|z(t,x)|^p = \|p_N(\cdot,x) \mathbf{1}_{(0,t)}(\cdot)\|^p_{\mathcal{H}}$$

valid for any $H \ge 1/2$ and $p \ge 2$. Recall (3.6). Then if $p = 2$ an application of Fubini's theorem leads to the estimate

$$\|z\|^2_{L^2_{\mathcal{F}}} = \int_0^T \int_{\mathbb{R}_+} (T-s) s^{2H-2} e^{-x^2/s} \, \mathrm{d}x \, \mathrm{d}s \le C \, T^{2H+1/2};$$

if $p > 2$ then we first apply Hölder's inequality for some $\varepsilon \in (0, \frac{1}{p})$ (although it is really necessary only for $H = 1/2$), then Fubini's theorem to get

$$\begin{aligned}\|z\|^p_{L^p_{\mathcal{F}}} &\le C \int_0^T \int_{\mathbb{R}_+} \left(\int_0^t \left| s^{-\frac{2}{p}-\varepsilon} e^{-x^2/s} \right|^{p/2} \mathrm{d}s \right) \left(\int_0^t \left| s^{-\frac{p-2}{p}+\varepsilon+(2H-1)} \right|^{\frac{p}{p-2}} \mathrm{d}s \right)^{\frac{p-2}{2}} \mathrm{d}x \, \mathrm{d}t \\ &\le C \int_0^T \int_{\mathbb{R}_+} t^{(\varepsilon+2H-1)\frac{p}{2}} \int_0^t s^{-1-\varepsilon\frac{p}{2}} e^{-\frac{p}{2}x^2/s} \, \mathrm{d}s \, \mathrm{d}x \, \mathrm{d}t \le C \int_0^T t^{(\varepsilon+2H-1)\frac{p}{2}} \int_0^t s^{-(1+\varepsilon p)/2} \, \mathrm{d}s \, \mathrm{d}t \\ &\le C \int_0^T t^{(\varepsilon+2H-1)\frac{p}{2}} \, t^{(1-\varepsilon p)/2} \, \mathrm{d}t \le C \, T^{(2H-1)\frac{p}{2}+\frac{3}{2}}. \end{aligned}$$ □

We next prove that the solution is suitably smooth in space. The following results are proved by appealing to the Kolmogorov's continuity criterium. Notice that there is a difference between the two cases: for the Brownian motion case there is a discontinuity on the boundary, while in the fractional Brownian motion case $H > 1/2$ the process is continuous up to the boundary.

Lemma 3.2. *Let $H = 1/2$. For any $\varepsilon > 0$ and $L < \infty$ there exists a version of the process $x \mapsto z(t,x)$ that is a.s. C^∞ on $[\varepsilon, L]$.*

Proof. We estimate first

$$\begin{aligned} |p_N(s,x) - p_N(s,y)| &= \left| \int_x^y \frac{\partial}{\partial z} p_N(s,z)\,\mathrm{d}z \right| = \left| \int_x^y \left(-\frac{z}{s}\right) p_N(s,z)\,\mathrm{d}z \right| \\ &\le \frac{1}{s} p_N(s,x) \frac{|x^2 - y^2|}{2} \le \frac{1}{s}\, L|x-y|\, p_N(s,x). \end{aligned}$$

Then we observe that

$$\begin{aligned} |p_N(s,x) - p_N(s,y)| &\le |p_N(s,x) - p_N(s,y)|^{1-\alpha} |p_N(s,x) - p_N(s,y)|^{\alpha} \\ &\le p_N(s,x)^{1-\alpha} \left| \frac{1}{s} p_N(s,x)|x-y|L \right|^{\alpha} \le p_N(s,x) \left(\frac{L|x-y|}{s} \right)^{\alpha} \end{aligned} \tag{3.8}$$

hence, recalling (3.2) and using the Burkholder-Davis-Gundy inequality, we get

$$\begin{aligned} \mathbb{E}|z(t,x) - z(t,y)|^p &\le \left[\int_0^t |p_N(s,x) - p_N(s,y)|^2\,\mathrm{d}s \right]^{p/2} \\ \le \left[\int_0^t \left| p_N(s,\varepsilon) L^{\alpha} \frac{1}{s^{\alpha}} \right|^2 \mathrm{d}s \right]^{\frac{p}{2}} |x-y|^{\alpha p} &\le C_L \left[\int_0^t \frac{e^{-\frac{\varepsilon^2}{s}}}{s^{1+2\alpha}} \mathrm{d}s \right]^{\frac{p}{2}} |x-y|^{\alpha p} \\ = C_L \left[\varepsilon^{-4\alpha} \int_{\frac{\varepsilon^2}{t}}^{\infty} e^{-z} z^{2\alpha - 1} \mathrm{d}z \right]^{\frac{p}{2}} |x-y|^{\alpha p} &\le C_L\ \Gamma(2\alpha)\ \varepsilon^{-2p\alpha} |x-y|^{\alpha p} \end{aligned}$$

where for the convergence of the Gamma function it is sufficient to require $\alpha > 0$. Then, Kolmogorov's continuity theorem guarantees the existence of a Hölder continuous version of $z(t,\cdot)$ with Hölder exponent $\alpha - 1/p$ and then, the thesis follows thanks to the arbitrariness of $\alpha > 0$ and $p \geq 2$. □

Following the idea in Sowers,[1] we can estimate the irregularity of the process near the boundary. It holds that even if the process is not regular up to the origin, if we moltiplicate the process by x^α we get a bounded (and continuous) process up to the origin. More formally we have the following.

Lemma 3.3. *Let $H = 1/2$. For any $\epsilon > 0$ the process $x \to x^\alpha z(t,x)$ is continuous in $[0, L]$ for any $\alpha > 0$.*

Proof. We aim to prove that

$$\lim_{x \to 0} \mathbb{E}|x^\alpha z(t,x)|^2 = 0.$$

We have

$$\lim_{x\to 0} \mathbb{E}|x^\alpha z(t,x)|^2 = \lim_{x\to 0} \mathbb{E}\left| x^\alpha \int_0^t \int_S p_N(s,x) g(\sigma)\,\mathrm{d}B_{s,\sigma}\right|^2$$
$$\le \lim_{x\to 0} x^{2\alpha} \int_0^t |p_N(s,x)|^2 \,\mathrm{d}s = \lim_{x\to 0} \int_0^t x^{2\alpha} \frac{e^{-\frac{x^2}{s}}}{s} \mathrm{d}s.$$

We perform a change of variables and we get

$$\lim_{x\to 0} \mathbb{E}|x^\alpha z(t,x)|^2 \le \lim_{x\to 0} x^{2\alpha} \Gamma[0, \tfrac{x^2}{t}] \approx \lim_{x\to 0} x^{2\alpha} \log\left(\tfrac{x^2}{t}\right) = 0$$

which guarantees the convergence for any $\alpha > 0$. □

Lemma 3.4. *Let $H > 1/2$. For any $\varepsilon > 0$ and $L < \infty$ there exists a version of the process $x \mapsto z(t,x)$ that is C^∞ on $[\varepsilon, L]$.*

Proof.

Using the Burkholder-Davis-Gundy inequality and the same estimates used in the proof of Lemma 3.2, we estimate

$$\mathbb{E}|z(t,x) - z(t,y)|^p = \mathbb{E}\left| \int_0^t \int_S [p_N(s,x) - p_N(s,y)]\, g(\sigma)\,\mathrm{d}B_{\sigma,s}\right|^p$$
$$\le \left[\int_0^t \int_0^t |r-s|^{2H-2} \left(p_N(s,\varepsilon) L^\alpha \frac{1}{s^\alpha}\right) \left(p_N(r,\varepsilon) L^\alpha \frac{1}{r^\alpha}\right) \mathrm{d}s\mathrm{d}r\right]^{\frac{p}{2}} |x-y|^{\alpha p}$$
$$\le C_L |x-y|^{\alpha p} \left[\int_0^t \int_0^t \frac{|r-s|^{2H-2}}{s^\alpha r^\alpha} p_N(s,\varepsilon) p_N(r,\varepsilon) \mathrm{d}s\mathrm{d}r\right]^{\frac{p}{2}}$$

Using a change of variables we estimate

$$\begin{aligned} &\int_0^t \int_0^t \frac{|r-s|^{2H-2}}{s^\alpha r^\alpha} p_N(s,\varepsilon) p_N(r,\varepsilon)\,\mathrm{d}s\,\mathrm{d}r \\ &\quad = (\varepsilon^2)^{2H-2\alpha-1} \int_{\frac{\varepsilon^2}{t}}^\infty \int_{\frac{\varepsilon^2}{t}}^\infty |z-w|^{2H-2} z^{\alpha+\frac12-2H} w^{\alpha+\frac12-2H} e^{-\frac{z}{2}} e^{-\frac{w}{2}} \mathrm{d}z\mathrm{d}w \\ &\quad = 2(\varepsilon^2)^{2H-2\alpha-1} \int_{\frac{\varepsilon^2}{t}}^\infty \int_{\frac{\varepsilon^2}{t}}^z (z-w)^{2H-2} z^{\alpha+\frac12-2H} w^{\alpha+\frac12-2H} e^{-\frac{z}{2}} e^{-\frac{w}{2}}\,\mathrm{d}z\,\mathrm{d}w \\ &\quad \le 2(\varepsilon^2)^{2H-2\alpha-1} \int_0^\infty \int_0^z (z-w)^{2H-2} z^{\alpha+\frac12-2H} w^{\alpha+\frac12-2H} e^{-\frac{z}{2}} e^{-\frac{w}{2}}\,\mathrm{d}z\,\mathrm{d}w \end{aligned} \tag{3.9}$$

We analyze first the inner integral to get

$$\int_0^z e^{-\frac{w}{2}} (z-w)^{2H-2} w^{\alpha+1/2-2H} \mathrm{d}w$$
$$= z^{2H-2+\alpha+1/2-2H+1} \int_0^1 e^{-\frac{\sigma z}{2}} (1-\sigma)^{2H-2} \sigma^{\alpha+1/2-2H} \mathrm{d}\sigma$$
$$\le z^{\alpha-1/2} \int_0^1 (1-\sigma)^{2H-2} \sigma^{\alpha+1/2-2H} \mathrm{d}\sigma = \frac{\Gamma(2H-1)\Gamma(\alpha-2H+3/2)}{\Gamma(\alpha+1/2)}\, z^{\alpha-1/2}$$

which converges for any $H > 1/2$ and $\alpha > 0$.
Then (3.9) is estimated by

$$2(\varepsilon^2)^{2H-2\alpha-1} C \int_0^\infty e^{-\frac{z}{2}} z^{2\alpha-2H} \mathrm{d}z = 2(\varepsilon^2)^{2H-2\alpha-1} C\, \Gamma(2\alpha - 2H + 1)$$

which converges provided $\alpha > H - 1/2$. So we get

$$\mathbb{E}|z(t,x) - z(t,y)|^p \leq 2(\varepsilon^2)^{2H-2\alpha-1} C_{L,H,\alpha} |x-y|^{\alpha p} \tag{3.10}$$

and we conclude the proof as in Lemma 3.2. □

Notice that previous lemma does not imply the smoothness of the solution up to the origin. In fact, the assumption $\alpha > H - 1/2$ implies that the right hand side of (3.10) blows up as $\varepsilon \to 0$. Thus we shall consider the regularity of the solution near the origin in the following lemma.

Lemma 3.5. *Let $H > 1/2$. There exists a version of the process $x \mapsto z(t,x)$ that is continuous up to zero. Moreover it is γ-Hölder continuous with $\gamma < 2H - 1$.*

Proof. Using as before the Burkholder-Davis-Gundy inequality, estimate (3.8) (with $L = x$), and the usual change of variables we have

$$\begin{aligned}
\mathbb{E}|z(t,0) - z(t,x)|^p &\leq \left[\int_0^t \int_0^t p_N(s,0) \frac{x^{2\alpha}}{s^\alpha} p_N(r,0) \frac{x^{2\alpha}}{r^\alpha} |r-s|^{2H-2} \mathrm{d}r\, \mathrm{d}s \right]^{\frac{p}{2}} \\
&\leq C x^{2\alpha p} \left[\int_0^t \int_0^t \frac{1}{s^{\alpha+\frac12}} \frac{1}{r^{\alpha+\frac12}} |r-s|^{2H-2} \mathrm{d}r\, \mathrm{d}s \right]^{\frac{p}{2}} \\
&= C x^{2\alpha p} \left(t^{2H-2\alpha-1} \right)^{\frac{p}{2}} \left[\int_0^1 \int_0^1 \frac{1}{\sigma^{\alpha+\frac12}} \frac{1}{\rho^{\alpha+\frac12}} |\sigma - \rho|^{2H-2} \mathrm{d}\sigma\, \mathrm{d}\rho \right]^{\frac{p}{2}} \\
&\leq C x^{2\alpha p} \left(t^{2H-2\alpha-1} \right)^{\frac{p}{2}} C_{H,\alpha}
\end{aligned}$$

where the term $C_{H,\alpha}$ is finite provided that $\alpha < H - 1/2$:

$$\begin{aligned}
\int_0^1 \int_0^1 \frac{1}{\sigma^{\alpha+\frac12}} \frac{1}{\rho^{\alpha+\frac12}} |\sigma-\rho|^{2H-2}\, \mathrm{d}\sigma\, \mathrm{d}\rho &= 2 \int_0^1 \int_0^\sigma \frac{1}{\sigma^{\alpha+\frac12}} \frac{1}{\rho^{\alpha+\frac12}} (\sigma-\rho)^{2H-2}\, \mathrm{d}\rho\, \mathrm{d}\sigma \\
&= 2 \int_0^1 \sigma^{2H-2\alpha-2} \left(\int_0^1 r^{-\alpha-\frac12} (1-r)^{2H-2}\, \mathrm{d}r \right) \mathrm{d}\sigma \\
&= \left(\frac{2}{2H-2\alpha-1} \right) \frac{\Gamma(\frac12 - \alpha)\Gamma(2H-1)}{\Gamma(2H-\alpha-\frac12)}.
\end{aligned}$$

By the Kolmogorov's continuity theorem we conclude that there exists a version of the process $z(t,x)$ which is a.s. γ-Hölder continuous in x with $\gamma < 2H - 1$ on $[0, L]$. □

Next, we prove the smoothness in Malliavin's sense of the process $z(t,x)$. This is a key step towards the prove of existence of the density.

Lemma 3.6. *The stochastic convolution process belongs to $\mathbb{D}^\infty$.*

Proof. Simply notice that $D\delta(u) = u$ for any deterministic function $u \in \mathcal{H}$, where we use the notation $\delta(u)$ for the Wiener integral. Therefore, higher order derivatives vanishes and the thesis follows. □

We finally prove the existence of the density of the random variable $z(t,x)$ with respect to the Lebesgue measure on $\mathbb{R}$. We shall use the criterion for absolute continuity stated in Theorem 2.1.

Lemma 3.7. *Assume* (3.2). *Then the random variable $z(t,x)$, $x > 0$ and $t \in]0,T]$, has a smooth density with respect to the Lebesgue measure on $\mathbb{R}$.*

Proof. The thesis follows from Theorem 2.1 and the estimate

$$\|Dz(t,x)\|_{\mathcal{H}}^2 \geq \delta > 0 \qquad \text{a.s..} \tag{3.11}$$

We may prove (3.11) by a direct computation. First, for $H = 1/2$, we have

$$\int_0^t \int_S |g(\sigma)p_N(s,x)|^2 \, \mu(\mathrm{d}\sigma) \, \mathrm{d}s \\ \geq \int_S |g(\sigma)|^2 \, \mu(\mathrm{d}\sigma) \int_0^t \frac{1}{2\pi s} \exp(-x^2/s) \, \mathrm{d}s = \frac{1}{2\pi} \Gamma[0, \tfrac{x^2}{t}],$$

where $\Gamma(0,z)$ is the incomplete Gamma function and $0 < \Gamma[0,z] < \infty$ for any $z > 0$. Next, in the fractional Brownian motion case $H > 1/2$, recalling (3.2), we have

$$\int_S \int_0^t \int_0^t |g(\sigma)|^2 p_N(r,x) p_N(s,x) |s-r|^{2H-2} \, \mathrm{d}s \, \mathrm{d}r \, \mu(\mathrm{d}\sigma) \\ \geq \left(\int_S |g(\sigma)|^2 \, \mu(\mathrm{d}\sigma) \right) \int_{t/2}^t \int_{t/2}^t \exp(-\tfrac{x^2}{2s}) \exp(-\tfrac{x^2}{2r}) \frac{1}{2\pi\sqrt{sr}} |s-r|^{2H-2} \, \mathrm{d}r \, \mathrm{d}s \\ \geq \exp(-\tfrac{2x^2}{t}) \int_{t/2}^t \int_{t/2}^t \frac{1}{2\pi\sqrt{sr}} |s-r|^{2H-2} \, \mathrm{d}r \, \mathrm{d}s = C_H \exp(-\tfrac{2x^2}{t}) t^{2H-1}.$$

□

Remark 3.1. The above proof shows, in particular, that in the fractional Brownian motion case the stochastic convolution term is regular (in the sense of Malliavin calculus) up to the boundary $x = 0$, as opposed to the Brownian motion case.

4. Existence of the solution for the nonlinear equation and its regularity

We consider in this section the nonlinear diffusion equation on the half-line (1.1). We assume that the function f has the following form

$$f(t,x,u) = f_0(t,x) + f_1(t,x)f(u),$$

where

$$f_0(t,x) \in L^\infty(0,T;L^p(\mathbb{R}_+)), \qquad p \geq 2, \tag{4.1}$$

$$f_1(t,x) \in L^\infty((0,T)\times\mathbb{R}_+), \text{ with } \|f_1(t,x)\|_\infty \leq L_1 \tag{4.2}$$

for some $L_1 < +\infty$, and

$$\begin{gathered} f(u) \text{ is Lipschitz continuous and belongs to the class } C^1(\mathbb{R}), \\ \text{with } f(0)=0 \text{ and } |\partial_u f(u)| \leq L \end{gathered} \tag{4.3}$$

for some $L < \infty$.
In the sequel, in order to improve the regularity of the solution, we shall introduce the following additional condition, which substitutes (4.3):

$$f(u) \text{ belongs to the class } C^\infty(\mathbb{R}), \text{ with } f(0)=0 \text{ and } |\partial_u^n f(u)| \leq L \tag{4.4}$$

for any $n \geq 1$ and some $L < \infty$.

Theorem 4.1. *Assume that Hypothesis* (4.1) *holds for some* $p \geq 2$*, and take also* (4.2) *and* (4.3)*. There exists a unique solution* $u(t,x)$ *for problem* (1.2) *that belongs to* $L^p_{\mathcal{F}}((0,T)\times\mathbb{R}_+)$ *for any* $p \geq 2$*.*
Moreover, let $p = 2$ *in* (4.1)*. Then* $u(t,x) \in \mathbb{D}^{1,2}$*, and the derivative* $D_{r,\sigma}u(t,x)$ *satisfies the following equation:*

$$\begin{aligned} D_{r,\sigma}u(t,x) = {} & p_N(r,x)g(\sigma)\,\mathbf{1}_{[0,t]}(r) \\ & + \int_r^t \int_{\mathbb{R}_+} g_N(t-s,x,y)\partial_u f(s,y,u(s,y))\, D_{r,\sigma}u(s,y)\,\mathrm{d}y\,\mathrm{d}s. \end{aligned} \tag{4.5}$$

Proof. [Existence and uniqueness of the solution]
Let us consider the Picard approximations of $u(t,x)$ defined as usual

$$u_{n+1}(t,x) = \int_0^t \int_S p_N(s,x)g(\sigma)\,\mathrm{d}B_{s,\sigma} + \int_0^t \int_{\mathbb{R}_+} g_N(t-s,x,y)f(s,y,u_n(s,y))\,\mathrm{d}y\,\mathrm{d}s. \tag{4.6}$$

First, we prove that $u_n(t,x)$ converges in $L^p(\Omega)$ to $u(t,x)$ as $n\to\infty$, for every $(t,x) \in [0,T]\times\mathbb{R}_+$. So let us estimate, for every $p \geq 2$,

$$\begin{aligned} & \mathbb{E}(|u_{n+1}(t,x) - u_n(t,x)|^p) \\ & \quad \leq \mathbb{E}\left| \int_0^t \int_{\mathbb{R}_+} g_N(t-s,x,y)\,[f(s,y,u_n(s,y)) - f(s,y,u_{n-1}(s,y))]\,\mathrm{d}y\,\mathrm{d}s \right|^p. \end{aligned}$$

Using Hölder's inequality, then hypothesis (4.2) and (4.3) on f and the trivial inequality $\int g_N(t,x,y)\,\mathrm{d}y \le 1$

$$\begin{aligned}
&\mathbb{E}(|u_{n+1}(t,x) - u_n(t,x)|^p) \\
&\quad \le C \left(\int_0^t \int_{\mathbb{R}_+} g_N(t-s,x,y)\,\mathrm{d}y\,\mathrm{d}s \right)^{p-1} \cdot \\
&\qquad \cdot \int_0^t \int_{\mathbb{R}_+} g_N(t-s,x,y)\,\mathrm{d}y \sup_{(r,y)\in[0,s]\times\mathbb{R}_+} \mathbb{E}|u_n(r,y) - u_{n-1}(r,y)|^p\,\mathrm{d}s \\
&\quad \le C\, t^{p-1} \int_0^t \sup_{(r,y)\in[0,s]\times\mathbb{R}_+} \mathbb{E}|u_n(r,y) - u_{n-1}(r,y)|^p\,\mathrm{d}s.
\end{aligned}$$

Defining now

$$\phi_{n+1}(t) := \sup_{(s,x)\in[0,t]\times\mathbb{R}_+} \mathbb{E}(|u_{n+1}(s,x) - u_n(s,x)|^p), \qquad t \in [0,T],$$

and recalling the above estimates we get

$$\phi_{n+1}(t) \le C_T \int_0^t \phi_n(s)\,\mathrm{d}s.$$

Iterating this inequality it follows that

$$\sum_n \phi_n(t) < \infty.$$

Hence $u_n(t,x)$ converges in $L^p(\Omega)$ as n tends to infinity, uniformly in $(t,x) \in [0,T] \times \mathbb{R}_+$. Let us denote the limit by $u(t,x)$. The process $u(t,x)$ is adapted, bounded in the p-mean and satisfies (1.2), hence it is a solution for (1.1). □

Proof. [Global regularity of the solution]

We aim to prove that the solution belongs to $L^p_{\mathcal{F}}((0,T)\times\mathbb{R}_+)$ for any $p \ge 2$. To this end, we recall from Theorem 3.1 that the stochastic convolution term indeed is in this space, so it is sufficient to estimate the convolution term. Setting for simplicity $f(s,y,u(s,y)) = \varphi(s,y)$, we estimate by the continuous version of Minkowski's inequality (see for instance Stroock[13] Theorem 6.2.14)

$$\begin{aligned}
&\mathbb{E}\left| \int_0^t \int_{\mathbb{R}_+} g_N(t-s,x,y)\varphi(s,y)\,\mathrm{d}y\,\mathrm{d}s \right|^p \\
&\qquad\qquad \le \left(\int_0^t \int_{\mathbb{R}_+} \left(\mathbb{E}|g_N(t-s,x,y)\varphi(s,y)|^p\right)^{1/p} \mathrm{d}y\,\mathrm{d}s \right)^p
\end{aligned}$$

then by Hölder's inequality and finally by the estimates on the Green kernel g_N

$$\begin{aligned}
&\mathbb{E}\left|\int_0^t \int_{\mathbb{R}_+} g_N(t-s,x,y)\varphi(s,y)\,\mathrm{d}y\,\mathrm{d}s\right|^p \\
&\le \left(\int_0^t \int_{\mathbb{R}_+} g_N(t-s,x,y)\,\mathrm{d}y\,\mathrm{d}s\right)^{p-1}\left(\int_0^t \int_{\mathbb{R}_+} g_N(t-s,x,y)\,\mathbb{E}|\varphi(s,y)|^p\,\mathrm{d}y\,\mathrm{d}s\right) \\
&\qquad\qquad \le t^{p-1}\left(\int_0^t \int_{\mathbb{R}_+} g_N(t-s,x,y)\,\mathbb{E}|\varphi(s,y)|^p\,\mathrm{d}y\,\mathrm{d}s\right).
\end{aligned}$$

Therefore, taking the integral in $\mathbb{R}_+$, we get by an application of Fubini's theorem

$$\begin{aligned}
&\int_{\mathbb{R}_+} \mathbb{E}\left|\int_0^t \int_{\mathbb{R}_+} g_N(t-s,x,y)\varphi(s,y)\,\mathrm{d}y\,\mathrm{d}s\right|^p \mathrm{d}x \\
&\qquad \le \int_{\mathbb{R}_+} t^{p-1}\int_0^t \int_{\mathbb{R}_+} g_N(t-s,x,y)\,\mathbb{E}|\varphi(s,y)|^p\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x \\
&\le t^{p-1}\int_0^t \int_{\mathbb{R}_+}\left(\int_{\mathbb{R}_+} g_N(t-s,x,y)\,\mathrm{d}x\right)\mathbb{E}|\varphi(s,y)|^p\,\mathrm{d}y\,\mathrm{d}s \le t^{p-1}\|\varphi\|^p_{L^p_{\mathcal{F}}((0,t)\times\mathbb{R}_+)}.
\end{aligned}$$

Let us return to (1.2); taking the p-th power and then integrating in $\mathrm{d}x$ on both sides we have

$$\|u(t,\cdot)\|^p_{L^p_{\mathcal{F}}(\mathbb{R}_+)} \le \|z(t,\cdot)\|^p_{L^p(\mathbb{R}_+)} + t^{p-1}\int_0^t \|\varphi(s,\cdot)\|^p_{L^p_{\mathcal{F}}(\mathbb{R}_+)}\,\mathrm{d}s \tag{4.7}$$

Recalling the definition of $\varphi(s,y)$ and using the hypothesis (4.1) – (4.3) we have

$$\begin{aligned}
\|\varphi(s,\cdot)\|^p_{L^p_{\mathcal{F}}(\mathbb{R}_+)} &= \int_{\mathbb{R}_+} \mathbb{E}|f(s,y,u(s,y))|^p\,\mathrm{d}y = \int_{\mathbb{R}_+} \mathbb{E}|f_0(s,y)+f_1(s,y)f(u)|^p \mathrm{d}y \\
&\le c_p\left(\int_{\mathbb{R}_+} |f_0(s,y)|^p \mathrm{d}y + \int_{\mathbb{R}^+} |f_1(s,y)|^p \mathbb{E}|f(u)|^p \mathrm{d}y\right) \\
&\le c\|f_0(s,\cdot)\|^p_{L^p(\mathbb{R}^+)} + c \sup_{(r,y)\in(0,s)\times\mathbb{R}_+} |f_1(r,y)|^p \|u(s,\cdot)\|^p_{L^p_{\mathcal{F}}(\mathbb{R}_+)} \\
&\le c\|f_0\|^p_{L^\infty(0,T;L^p(\mathbb{R}_+))} + c\|f_1\|^p_{L^\infty((0,T)\times\mathbb{R}_+)}\|u(s,\cdot)\|^p_{L^p_{\mathcal{F}}(\mathbb{R}_+)} \\
&\le c_0 + c_1\|u(s,\cdot)\|^p_{L^p_{\mathcal{F}}(\mathbb{R}_+)}
\end{aligned}$$

Now we use the above estimate and Gronwall's inequality to get

$$\|u(t,\cdot)\|^p_{L^p_{\mathcal{F}}(\mathbb{R}_+)} \le C_{0,T} + C_{1,T}\|z(t,\cdot)\|^p_{L^p_{\mathcal{F}}(\mathbb{R}_+)}$$

and the conclusion follows by taking one more integral on $(0,T)$. □

Proof. [Existence and uniqueness of the Malliavin derivative]

The proof of this part is based on the convergence result recalled in Proposition 2.1.
Let us consider the Picard approximations of $u(t,x)$ defined by (4.6). Then, in view of Proposition 2.1, it is sufficient to show that

$$\sup_n \mathbb{E}\|Du_n(t,x)\|^2_{\mathcal{H}} \le C < +\infty. \tag{4.8}$$

We proceed by induction starting from (4.6) to get

$$\begin{aligned} D_{r,\sigma}u_{n+1}(t,x) &= p_N(r,x)g(\sigma)\mathbf{1}_{[0,t]}(r) \\ &\quad + \int_r^t \int_{\mathbb{R}_+} g_N(t-s,x,y)\partial_u f(s,y,u_n(s,y))D_{r,\sigma}u_n(s,y)\,\mathrm{d}s\,\mathrm{d}y \end{aligned} \tag{4.9}$$

which leads to the following estimate

$$\mathbb{E}\|Du_{n+1}(t,x)\|^2_{\mathcal{H}} \le 2\|p_N(\cdot,x)g(\cdot)\mathbf{1}_{(0,t)}(\cdot)\|^2_{\mathcal{H}} + 2\mathbb{E}\,\|\phi_n(t,x;r,\sigma)\|^2_{\mathcal{H}}$$

where, for simplicity, we set

$$\phi_n(t,x;r,\sigma) := \int_r^t \int_{\mathbb{R}_+} g_N(t-s,x,y)\partial_u f(s,y,u_n(s,y))D_{r,\sigma}u_n(s,y)\,\mathrm{d}y\,\mathrm{d}s.$$

We separately estimate the various terms.
Recall from (3.6) the estimate, valid for any $H \ge 1/2$,

$$c(t,x) := \mathbb{E}\|p_N(r,x)g(\sigma)\mathbf{1}_{(0,t)}(r)\|^2_{\mathcal{H}} \le C\int_0^t s^{2H-2}e^{-x^2/s}\,\mathrm{d}s;$$

now, by the inequality

$$\left\|\int_{E_1} f(x,y)\,\mu(\mathrm{d}x)\right\|_{E_2} \le \int_{E_1} \|f(x,\cdot)\|_{E_2}\,\mu(\mathrm{d}x)$$

we get

$$\begin{aligned} \mathbb{E}\|\phi_n(t,x;r,\sigma)\|^2_{\mathcal{H}} =& \mathbb{E}\left\|\int_r^t \int_{\mathbb{R}_+} g_N(t-s,x,y)\partial_u f(s,y,u_n(s,y))Du_n(s,y)\,\mathrm{d}y\,\mathrm{d}s\right\|^2_{\mathcal{H}} \\ \le& \mathbb{E}\left|\int_r^t \int_{\mathbb{R}_+} \|g_N(t-s,x,y)\partial_u f(s,y,u_n(s,y))Du_n(s,y)\|_{\mathcal{H}}\,\mathrm{d}y\,\mathrm{d}s\right|^2 \\ \le& C_L\,\mathbb{E}\left|\int_r^t \int_{\mathbb{R}_+} g_N(t-s,x,y)\,\|Du_n(s,y)\|_{\mathcal{H}}\,\mathrm{d}y\,\mathrm{d}s\right|^2 \\ \le& C_L\left(\int_r^t \int_{\mathbb{R}^+} g_N(t-s,x,y)\,\mathrm{d}y\,\mathrm{d}s\right)\cdot \\ &\cdot\left(\int_r^t \int_{\mathbb{R}_+} g_N(t-s,x,y)\mathbb{E}\,\|Du_n(s,y)\|^2_{\mathcal{H}}\,\mathrm{d}y\,\mathrm{d}s\right) \\ \le& C_{L,T}\int_r^t \int_{\mathbb{R}_+} g_N(t-s,x,y)\mathbb{E}\,\|Du_n(s,y)\|^2_{\mathcal{H}}\,\mathrm{d}y\,\mathrm{d}s \end{aligned}$$

(where in the second to last inequality we have used Hölder's inequality and the properties of the Green kernel).

Let us now set

$$v_n(t,x) := \mathbb{E}\|Du_n(t,x)\|^2_{\mathcal{H}};$$

we have, from previous estimates, that

$$v_{n+1}(t,x) \le 2c(t,x) + \gamma \int_0^t \int_{\mathbb{R}^+} g_N(t-s,y,x) v_n(s,y)\,\mathrm{d}y\,\mathrm{d}s,$$

where $\gamma = 2L_1^2 L^2 T$ is a finite constant. Next, we take convolution in both sides with respect to the Green kernel $g_N(\theta - t, x, z)$ for some $\theta > t$ and $z \in \mathbb{R}_+$ and we get

$$\int_{\mathbb{R}^+} g_N(\theta - t, x, z) v_{n+1}(t,x)\mathrm{d}x \le 2 \int_{\mathbb{R}^+} c(t,x)\, g_N(\theta - t, x, z)\,\mathrm{d}x \\ + \gamma \int_0^t \int_{\mathbb{R}^+} \left(\int_{\mathbb{R}^+} g_N(\theta - t, x, z) g_N(t-s,y,x)\mathrm{d}x \right) v_n(s,y)\,\mathrm{d}y\,\mathrm{d}s.$$

For this computation, and later references, we state in a separate lemma the following result.

Lemma 4.1. *For any $p \ge 1$ set*

$$\tilde{c}_p(\theta,t,z) = \int_{\mathbb{R}^+} c(t,x)^p\, g_N(\theta - t, x, z)\,\mathrm{d}x.$$

Then $\tilde{c}_p(\theta,t,z) < +\infty$ for any $z > 0$, $t \le \theta$ and $H \ge 1/2$.

Proof.

We have, by definition

$$\tilde{c}_p(\theta,t,z) \le C \int_{\mathbb{R}^+} \left(\int_0^t e^{-x^2/s} s^{2H-2}\,\mathrm{d}s \right)^p g_N(\theta - t, x, z)\,\mathrm{d}x;$$

we first apply (in case $p > 1$) Minkowski's inequality

$$\tilde{c}_p(\theta,t,z) \le C \left(\int_0^t s^{2H-2} \left(\int_{\mathbb{R}_+} e^{-px^2/s} g_N(\theta - t, x, z)\,\mathrm{d}x \right)^{1/p} \mathrm{d}s \right)^p;$$

then we apply Fubini's theorem and the properties of Gaussian kernel and we obtain

$$\tilde{c}_p(\theta,t,z) \le C \left(\int_0^t \frac{s^{2H-2+1/2p}}{[2(\theta-t)p + s]^{1/2p}} \exp\!\Big(-\frac{z^2}{2p(\theta-t)+s}\Big)\,\mathrm{d}s \right)^p.$$

This quantity is clearly finite for any $t < \theta$; further, if $t = \theta$ then the integral function simplifies and we have

$$\tilde{c}_p(\theta,\theta,z) \le C\, z^{2p(2H-1)} \Gamma[1-2H, \frac{z^2}{\theta}]^p. \tag{4.10}$$

□

Setting for simplicity $\tilde{c}(\theta, t, z) := 2\tilde{c}_1(\theta, t, z)$ and using the Chapman-Kolmogorov relation we finally get

$$\int_{\mathbb{R}^+} g_N(\theta - t, x, z) v_{n+1}(t, x)\, \mathrm{d}x \le \tilde{c}(\theta, t, z) + \gamma \int_0^t \int_{\mathbb{R}^+} g_N(\theta - s, y, z) v_n(s, y)\, \mathrm{d}y\, \mathrm{d}s.$$

Setting now

$$\Phi_{n+1}(\theta, t, z) := \int_{\mathbb{R}^+} g_N(\theta - t, x, z) v_{n+1}(t, x)\, \mathrm{d}x,$$

where $\Phi_0 = 0$ and $\Phi_1 = \tilde{c}$, we get

$$\Phi_{n+1}(\theta, t, z) \le \tilde{c}(\theta, t, z) + \gamma \int_0^t \Phi_n(\theta, s, z)\, \mathrm{d}s.$$

Proceeding recursively we then obtain

$$\begin{aligned}
\Phi_{n+1}(\theta, t, z) &\le \tilde{c} + \gamma \int_0^t \Phi_n(\theta, s, z) \mathrm{d}s \le \tilde{c} + \gamma \int_0^t \left(\tilde{c} + \gamma \int_0^s \Phi_n(\theta, s_1, z) \mathrm{d}s_1 \right) \mathrm{d}s \\
&\le c + \gamma \int_0^t \left(\tilde{c} + \gamma \int_0^s \left(\tilde{c} + \gamma \int_0^{s_1} \Phi_{n-2}(\theta, s_2, z)\, \mathrm{d}s_2 \right) \mathrm{d}s_1 \right) \mathrm{d}s \\
&\le \tilde{c} + \int_0^t \tilde{c}\ \mathrm{d}s + \int_0^t \int_0^s \tilde{c}\ \mathrm{d}s_1\, \mathrm{d}s + \gamma^3 \int_0^t \int_0^s \int_0^{s_1} \Phi_{n-2}(\theta, s_2, z) \mathrm{d}s_2\, \mathrm{d}s_1\, \mathrm{d}s \\
&\le \sum_{k=0}^n \tilde{c}\, \frac{t^k}{k!} + \gamma^{n+2} \int_0^t \int_0^s \int_0^{s_1} \cdots \int_0^{s_n} \Phi_0(\theta, s_n, z) \mathrm{d}s_{n+1} \cdots \mathrm{d}s \\
&\le \tilde{c}\, e^t + \gamma^{n+2} \frac{t^{n-1}}{(n-1)!} \|\Phi_0(\theta, \cdot, z)\|^2_{L^2(0,t)}.
\end{aligned}$$

Since the second term in the expression above is zero, we have that

$$\Phi_{n+1}(\theta, t, z) \le \tilde{c}(\theta, t, z) e^t \le \tilde{c}(\theta, t, z)\, e^T < \infty$$

for any $0 < t \le \theta \le T$ and any $z > 0$; recalling the definition of Φ_n and taking the limit as $t \uparrow \theta$, we get $v_{n+1}(t, x) = \lim \Phi_{n+1}(t, t, x)$ hence the above inequality proves that, for every n,

$$\sup_n \mathbb{E} \|Du_n(t, x)\|^2_{\mathcal{H}} < \infty,$$

which concludes the proof of (4.8) and, therefore, of the theorem. □

In the following results we improve the regularity properties of the solution. First, we show that the Malliavin derivative is p-mean integrable, for any $p \ge 2$. Later, we show that under suitable assumptions on the nonlinear term, the solution is smooth in the Malliavin sense, i.e., it belongs to the space $\mathbb{D}^\infty$.

Theorem 4.2. *Assume* (4.1) *holds for* $p \ge 2$, *as well as* (4.2) *and* (4.3). *Then the random variable* $u(t, x)$, *for* $x > 0$ *and* $t \in (0, T]$, *belongs to the space* $\mathbb{D}^{1,p}$ *and*

$$\mathbb{E} \|Du(t, x)\|^p_{\mathcal{H}} \le C_p < \infty.$$

Proof. We know from previous results that the solution $u(t,x)$ is p-mean integrable. We therefore aim to prove that

$$\mathbb{E}\|Du(t,x)\|_{\mathcal{H}}^p < \infty.$$

Recalling that

$$u(t,x) = \int_0^t \int_S p_N(s,x) g(\sigma)\, \mathrm{d}B_{s,\sigma} + \int_0^t \int_{\mathbb{R}_+} g_N(t-s,x,y) f(s,y,u(s,y))\, \mathrm{d}y\, \mathrm{d}s \tag{4.11}$$

and setting

$$z(t,x) := \int_0^t \int_S p_N(s,x) g(\sigma)\, \mathrm{d}B_{s,\sigma},$$
$$F(t,x) := \int_0^t \int_{\mathbb{R}_+} g_N(t-s,x,y) f(s,y,u(s,y))\, \mathrm{d}y\, \mathrm{d}s$$

we have that

$$\mathbb{E}\|Du(t,x)\|_{\mathcal{H}}^p \le c_p \left(\mathbb{E}\|Dz(t,x)\|_{\mathcal{H}}^p + \mathbb{E}\|DF(t,x)\|_{\mathcal{H}}^p\right)$$

that is, using the notation of previous theorem,

$$\mathbb{E}\|Du(t,x)\|_{\mathcal{H}}^p \le c_p c(t,x)^{p/2} + c_p\, \mathbb{E}\|DF(t,x)\|_{\mathcal{H}}^p.$$

Now, using again Minkowski's inequality, we have that

$$\begin{aligned}\mathbb{E}\|DF(t,x)\|_{\mathcal{H}}^p &\le \left(\int_r^t \int_{\mathbb{R}+} g_N(t-s,x,y) \partial_u f(s,y,u(s,y)) \mathbb{E}\|Du(s,y)\|_{\mathcal{H}}\, \mathrm{d}y\, \mathrm{d}s\right)^p \\ &\le C_{L,p} \left(\int_r^t \int_{\mathbb{R}+} g_N(t-s,x,y) \left(\mathbb{E}\|Du(s,y)\|_{\mathcal{H}}^p\right)^{\frac{1}{p}}\, \mathrm{d}y\, \mathrm{d}s\right)^p \end{aligned} \tag{4.12}$$

Set now

$$v(t,x) := \mathbb{E}\|Du(t,x)\|_{\mathcal{H}}^p,$$

thanks to the above computations and using Hölder's inequality, we get

$$\begin{aligned} v(t,x) &\le c_p\, c(t,x)^{p/2} + C_{L,p} \left(\int_r^t \int_{\mathbb{R}+} g(t-s,x,y) v(s,y)^{\frac{1}{p}}\, \mathrm{d}y\, \mathrm{d}s\right)^p \\ &\le c_p\, c(t,x)^{p/2} \\ &\quad + C_{L,p} \left(\int_r^t \int_{\mathbb{R}+} g(t-s,x,y) \mathrm{d}y \mathrm{d}s\right)^{p-1} \left(\int_r^t \int_{\mathbb{R}+} g(t-s,x,y) v(s,y)\, \mathrm{d}y\, \mathrm{d}s\right) \\ &\le c_p\, c(t,x)^{p/2} + C_{L,p} t^{p-1} \left(\int_r^t \int_{\mathbb{R}+} g(t-s,x,y) v(s,y)\, \mathrm{d}y\, \mathrm{d}s\right) \end{aligned}$$

Following now the same ideas used in the proof of Theorem 4.1, we do a stochastic convolution on both sides, obtaining

$$\int_{\mathbb{R}+} g(\theta-t,x,z) v(t,x)\, \mathrm{d}x \le \tilde{c}_p(\theta,t,z) + C_{L,p,T} \int_0^t \int_{\mathbb{R}+} g(\theta-s,y,z) v(s,y)\, \mathrm{d}y\, \mathrm{d}s$$

where

$$\tilde{c}_p(\theta,t,z) = c_p \int_{\mathbb{R}^+} c(t,x)^{p/2}\, g_N(\theta - t,x,z)\,\mathrm{d}x$$

is studied in Lemma 4.1. Setting

$$\Phi(\theta,t,z) := \int_{\mathbb{R}^+} g(\theta - t,x,z) v(t,x)\,\mathrm{d}x,$$

from an application of Gronwall's lemma (in the t variable) we get

$$\Phi(\theta,t,z) \le \tilde{c}_p + C\int_0^t \Phi(\theta,s,z)\mathrm{d}s \qquad \Longrightarrow \qquad \Phi(\theta,\theta,z) \le \tilde{c}_p(\theta,\theta,z)e^{C\theta}$$

and since $\lim_{t\to\theta} \Phi(\theta,t,z) = v(\theta,z)$ it follows

$$v(\theta,z) \le \tilde{c}_p(\theta,\theta,z)e^{C\theta}$$

and so, recalling estimate (4.10), we conclude that

$$\mathbb{E}\|Du(t,x)\|^p_{\mathcal{H}} < \infty \tag{4.13}$$

for all $x > 0$ and $t > 0$, for any $H \ge 1/2$. □

Theorem 4.3. *Assume that* (4.1) *holds for any* $p \ge 2$ *and that* (4.2) *and* (4.4) *holds. Then the random variable* $u(t,x)$, $x > 0$ *and* $t \in (0,T]$, *belongs to the space* $\mathbb{D}^\infty$ *and for all* $p \ge 2$ *and* $M \ge 1$

$$\mathbb{E}\|D^M u(t,x)\|^p_{\mathcal{H}^{\otimes M}} \le C_{p,M} < \infty.$$

Proof.

The proof's idea is, again, to use the Picard approximation scheme and then prove that

$$\sup_n \mathbb{E}\|D^M u_n(t,x)\|^p_{\mathcal{H}^{\otimes M}} \le C_{p,M} < \infty, \tag{4.14}$$

for all $p > 1$ and $M \ge 1$. The conclusion follows from successive applications of Proposition 2.1 for each $M \ge 1$.

Estimate (4.14) is proved by induction on M, using the evolution equation for $D^M u(t,x)$ obtained by differentiating (4.9) M times.
For $M = 1$ it is easily seen that (4.14) holds true, simply by Theorem 4.2. Let us see explicitly show how estimate (4.14) holds in the case $M = 2$. Then the general induction step will be clear. Proceeding by induction we get that the Malliavin derivative $D^2 u_n(t,x)$ takes values in $\mathcal{H}^{\otimes 2}$ and satisfies

$$\begin{aligned} D^2 u_{n+1}(t,x) &= \int_0^t \int_{\mathbb{R}^+} g_N(t-s,x,y)\partial_u^2 f(s,y,u_n(s,y))(Du_n(s,y))^{\otimes 2}\,\mathrm{d}y\,\mathrm{d}s \\ &\quad + \int_0^t \int_{\mathbb{R}^+} g_N(t-s,x,y)\partial_u f(s,y,u_n(s,y))D^2 u_n(s,y)\,\mathrm{d}y\,\mathrm{d}s =: A_n + B_n \end{aligned}$$

hence we get

$$\mathbb{E}\|D^2u_{n+1}(t,x)\|^p_{\mathcal{H}^{\otimes 2}} \le c_p \left(\mathbb{E}\|A_n\|^p_{\mathcal{H}^{\otimes 2}} + \mathbb{E}\|B_n\|^p_{\mathcal{H}^{\otimes 2}}\right).$$

For the first term, using Minkowski's inequality, then Hölder inequality and then Tonelli's Theorem, we have that

$$\begin{aligned}\mathbb{E}\|A_n\|^p_{\mathcal{H}^{\otimes 2}} &= \left\|\int_0^t \int_{\mathbb{R}^+} g_N(t-s,x,y)\partial_u^2 f(s,y,u_n(s,y))(Du_n(s,y))^{\otimes 2}\,\mathrm{d}y\,\mathrm{d}s\right\|^p_{\mathcal{H}^{\otimes 2}}\\ &\le \left(\int_0^t \int_{\mathbb{R}^+} g_N(t-s,x,y)\mathbb{E}\|\partial_u^2 f(s,y,u_n(s,y))(Du_n(s,y))^{\otimes 2}\|_{\mathcal{H}^{\otimes 2}}\,\mathrm{d}y\,\mathrm{d}s\right)^p\\ &\le C_{t,p}\left(\int_0^t \int_{\mathbb{R}^+} g_N(t-s,x,y)\mathbb{E}\|\partial_u^2 f(s,y,u_n(s,y))(Du_n(s,y))^{\otimes 2}\|^2_{\mathcal{H}^{\otimes 2}}\,\mathrm{d}y\,\mathrm{d}s\right)^{\frac{p}{2}}\\ &\le C_{t,p}\int_0^t \int_{\mathbb{R}^+} g_N(t-s,x,y)\mathbb{E}\|\partial_u^2 f(s,y,u_n(s,y))(Du_n(s,y))^{\otimes 2}\|^{\frac{p}{2}}_{\mathcal{H}^{\otimes 2}}\,\mathrm{d}y\,\mathrm{d}s.\end{aligned}$$

By Chauchy-Schwartz inequality we have

$$\mathbb{E}\|\partial_u^2 f(s,y,u_n(s,y))(Du_n(s,y))^{\otimes 2}\|^{\frac{p}{2}}_{\mathcal{H}^{\otimes 2}} \le \left(\mathbb{E}|\partial_u^2 f(s,y,u_n(s,y))|^p\right)^{\frac{1}{2}}\left(\mathbb{E}\|Du_n(s,y)\|^{2p}_{\mathcal{H}}\right)^{\frac{1}{2}}$$

which is finite thanks to hypothesis (4.2) and (4.4) and Theorem 4.2.
So the first term is finite. As regards the second term, proceeding as above we get:

$$\begin{aligned}\mathbb{E}\|B_n\|^p_{\mathcal{H}^{\otimes 2}} &\le \left(\int_0^t \int_{\mathbb{R}^+} g_N(t-s,x,y)\partial_u f(s,y,u_n(s,y))\mathbb{E}\|D^2u_n(s,y)\|_{\mathcal{H}_{\otimes 2}}\,\mathrm{d}y\,\mathrm{d}s\right)^p\\ &\le C_L\left(\int_0^t \int_{\mathbb{R}^+} g_N(t-s,x,y)\left(\mathbb{E}\|D^2u_n(s,y)\|^p_{\mathcal{H}_{\otimes 2}}\right)^{\frac{1}{p}}\,\mathrm{d}y\,\mathrm{d}s\right)^p;\end{aligned}$$

setting now

$$v_n(t,x) := \mathbb{E}\|D^2_n u(t,x)\|^p_{\mathcal{H}_{\otimes 2}}$$

and using again Hölder's inequality, we get

$$v_{n+1}(t,x) \le C_{L,T,p} + c_{p,T,L}\int_0^t \int_{\mathbb{R}^+} g_N(t-s,x,y)v_n(s,y)\,\mathrm{d}y\,\mathrm{d}s$$

and the thesis follows, using the same idea of the proof of Theorem 4.1, by iteration.

□

5. Existence of the density for the solution of the nonlinear equation

In this section we prove the existence of the density of the random variable $u(t,x)$ with respect to the Lebesgue measure on $\mathbb{R}$. We shall use the criterion for absolute continuity stated in Theorem 2.1 and obtain the following result.

Theorem 5.1. *The random variable $u(t,x)$, $x>0$ and $t\in(0,T]$, has a density with respect to the Lebesgue measure on $\mathbb{R}$.*

In order to get some relevant estimate, it is useful to consider a smaller time interval than $(0,t)$ and consider the $\mathcal{H}$-norm of $u(t,x)$ on $(t-\delta,t)$, for some $\delta > 0$ small enough. Then, we define, for every $\phi \in \mathcal{H}$, the norm

$$\|\phi\|_{\mathcal{H}_\delta} := \|\mathbf{1}_{(t-\delta,t)}(\cdot)\phi\|_{\mathcal{H}}.$$

It is then straightforward to get

$$\|\phi\|_{\mathcal{H}} \geq \|\phi\|_{\mathcal{H}_\delta}.$$

Proof. [Proof of Theorem 5.1]

The existence of the density follows from Theorem 2.1 and the estimate

$$\mathbb{E}\|Du(t,x)\|^2_{\mathcal{H}} > 0 \qquad \text{a.s.} \tag{5.1}$$

In turn, to prove that (5.1) holds $\mathbb{P}$-a.s., the idea is to prove that

$$\mathbb{P}\left(\|Du(t,x)\|^2_{\mathcal{H}} < \varepsilon\right) \to 0$$

as $\varepsilon \to 0$.

Observe that, using the same notation as in the proof of Theorem 4.2,

$$\begin{aligned}\|Du(t,x)\|^2_{\mathcal{H}} \geq \|Du(t,x)\|^2_{\mathcal{H}_\delta} &= \|Dz(t,x) + DF(t,x)\|^2_{\mathcal{H}_\delta} \\ &\geq \frac{1}{2}\|Dz(t,x)\|^2_{\mathcal{H}_\delta} - \|DF(t,x)\|^2_{\mathcal{H}_\delta}.\end{aligned}$$

Using then Chebyshev's inequality we have

$$\begin{aligned}\mathbb{P}\left(\|Du(t,x)\|^2_{\mathcal{H}} < \varepsilon\right) &\leq \mathbb{P}\left(\|Du(t,x)\|^2_{\mathcal{H}_\delta} < \varepsilon\right) \\ &\leq \mathbb{P}\left(\|DF(t,x)\|^2_{\mathcal{H}_\delta} \geq \frac{1}{2}\|Dz(t,x)\|^2_{\mathcal{H}_\delta} - \varepsilon\right) \\ &\leq \frac{\mathbb{E}\|DF(t,x)\|^{2\tilde{p}}_{\mathcal{H}_\delta}}{\left(\frac{1}{2}\|Dz(t,x)\|^2_{\mathcal{H}_\delta} - \varepsilon\right)^{\tilde{p}}}\end{aligned}$$

Using the estimates obtained in Lemmas Appendix A.1 e Appendix A.2, and choosing δ such that

$$\frac{1}{2}C_{T,x,\delta_0}\delta^{2H} = 2\varepsilon,$$

for every $\varepsilon < \varepsilon_0 = \frac{1}{4}C_{T,x,\delta_0}\delta_0^{2H}$, we have

$$\mathbb{P}(\|Du(t,x)\|^2_{\mathcal{H}} \leq \varepsilon) \leq \frac{C_{L,\tilde{p},H,T}\delta^{2\tilde{p}}}{\varepsilon^{\tilde{p}}} = C_{L,\tilde{p},H,T,x,\delta_0}(\varepsilon^{\frac{1}{H}-1})^{\tilde{p}}, \tag{5.2}$$

and since $\frac{1}{H} - 1 > 0$ for any $H \in [\frac{1}{2}, 1)$, the above estimate allows to conclude the proof. □

6. Smoothness of the density for the solution of the nonlinear equation

In case the coefficients of the equation are more regular than just Lipschitz continuous, we can prove the smoothness of the density of the random variable $u(t,x)$ that solves Eq. (1.1) for $t>0$ and $x>0$, analogously with the result in Lemma 3.7.

Theorem 6.1. *Assume that the conditions in Theorem 4.3 holds, i.e.,* (4.1) *holds for any* $p \geq 2$*, and further* (4.2) *and* (4.4) *hold true. Then for each* $t \in (0,T)$ *and* $x>0$*, the random variable* $u(t,x)$ *has a density with respect to the Lebesgue measure that is infinitely differentiable.*

Proof. The idea is to apply Theorem 2.1. From Theorem 4.3 we have that $u(t,x) \in \mathbb{D}^\infty$. It remains to prove that $\mathbb{E}(\|Du(t,x)\|)^{-p} < +\infty$ for every $p \geq 1$. By Nualart[12] Lemma 2.3.1, it suffices to prove that, for any $q \geq 2$, there exits $\varepsilon_0(q) > 0$ such that, for all $\varepsilon < \varepsilon_0$,

$$\mathbb{P}(\|Du(t,x)\|^2_{\mathcal{H}} < \varepsilon) < \varepsilon^q$$

and this condition immediately follows from the estimate (5.2) above, choosing $\tilde{p} = \frac{qH}{1-H}$. □

Appendix A. Some supplementary lemmas

In this section we prove a couple of lemmas, which are necessary for the proof of Theorem 5.1; the first one suitably modifies the result in Theorem 4.2 to the space $\mathcal{H}_\delta$.

Lemma Appendix A.1. *For every* $x>0$ *and* $t \in [0,T]$

$$\mathbb{E}\|DF(t,x)\|^{2p}_{\mathcal{H}_\delta} < C_{L,p,H,T}\delta^{2p}.$$

Proof. Recall from Theorem 4.2 that

$$F(t,x) := \int_0^t \int_{\mathbb{R}_+} g_N(t-s,x,y) f(s,y,u(s,y))\,\mathrm{d}y\,\mathrm{d}s;$$

we have that

$$\mathbf{1}_{(t-\delta,t)}(s) D_{s,\sigma}F(t,x) = \mathbf{1}_{(t-\delta,t)}(s) \int_0^t \int_{\mathbb{R}_+} g_N(t-r,x,y)\partial_u f(r,y,u(r,y)) D_{s,\sigma}u(r,y)\,\mathrm{d}y\,\mathrm{d}r$$

and since $D_{s,\sigma}u(r,x) = 0$ for every $r<s$, we get

$$= \mathbf{1}_{(t-\delta,t)}(s) \int_{t-\delta}^t \int_{\mathbb{R}_+} g_N(t-r,x,y)\partial_u f(r,y,u(r,y)) D_{s,\sigma}u(r,y)\,\mathrm{d}y\,\mathrm{d}r.$$

Using again Minkowski's inequality, then Hölder's inequality and finally Theorem 4.2, we get

$$\mathbb{E}\|DF(t,x)\|_{\mathcal{H}_\delta}^{2p} \le C_L \left(\mathbb{E}\int_{t-\delta}^t \int_{\mathbb{R}^+} g_N(t-s,x,y)\|\mathbf{1}_{(t-\delta,t)}D(u(s,y))\|_{\mathcal{H}}\,\mathrm{d}y\,\mathrm{d}s\right)^{2p}$$
$$\le C_L \left(\int_{t-\delta}^t \int_{\mathbb{R}^+} g_N(t-s,x,y)\,\mathrm{d}y\mathrm{d}s\right)^{2p-1} \int_{t-\delta}^t \int_{\mathbb{R}^+} g_N(t-s,x,y)\mathbb{E}\|Du(s,y)\|_{\mathcal{H}}^{2p}\,\mathrm{d}y\,\mathrm{d}s$$

Recall, again from Theorem 4.2, the notation

$$\Phi(\theta,t,z) := \int_{\mathbb{R}^+} g(\theta-t,x,z)v(t,x)\,\mathrm{d}x,$$

and the bound in (4.13)

$$\mathbb{E}\|Du(t,x)\|_{\mathcal{H}}^{2p} \le C_{T,H,x} < \infty \qquad \text{for all } x>0 \text{ and } t>0, \text{ for any } H \ge 1/2,$$

we finally get

$$\mathbb{E}\|DF(t,x)\|_{\mathcal{H}_\delta}^{2p} \le C_L \delta^{2p-1}\int_{t-\delta}^t \Phi(t,s,x)\,\mathrm{d}s \le C_{L,p,H,T}\delta^{2p}$$

as required. □

Next lemma is concerned with the norm of the Malliavin derivative $Dz(t,x)$ of the stochastic convolution process $z(t,x)$ in the space $\mathcal{H}_\delta$ and is a refinement of the results in Lemma 3.7.

Lemma Appendix A.2. *Given $\delta_0 > 0$, for every $\delta < \delta_0$ and every $x > 0$ and $t \in]0,T]$,*

$$\|Dz(t,x)\|_{\mathcal{H}_\delta}^2 \ge C_{T,x,\delta_0}\delta^{2H}.$$

Proof. Recall from assumption (3.2) that

$$\int_S |g(\sigma)|^2\,\mu(\mathrm{d}\sigma) = 1.$$

We proceed separately in the cases $H > 1/2$ and $H = 1/2$.
In the first, fixed $\delta_0 > 0$, for every $\delta < \delta_0$, we get

$$\begin{aligned}\|Dz(t,x)\|_{\mathcal{H}_\delta}^2 &= \int_{t-\delta}^t \int_{t-\delta}^t |r-s|^{2H-2}p_N(s,x)p_N(r,x)\,\mathrm{d}r\,\mathrm{d}s \\ &= \frac{1}{2\pi}\int_{t-\delta}^t \int_{t-\delta}^t |r-s|^{2H-2}\frac{e^{-\frac{x^2}{s}}e^{-\frac{x^2}{r}}}{\sqrt{rs}}\,\mathrm{d}r\,\mathrm{d}s \\ &\ge \frac{1}{\pi}e^{-\frac{2x^2}{(t-\delta)}}\int_0^\delta \int_0^s \frac{|r-s|^{2H-2}}{\sqrt{(t-r)(t-s)}}\,\mathrm{d}r\,\mathrm{d}s\end{aligned}$$

since, trivially, $t - r < t$ implies $(t-r)^{-1/2} > t^{-1/2}$,

$$\begin{aligned} &\geq \frac{1}{\pi} e^{-\frac{2x^2}{(t-\delta)}} \int_0^\delta \int_0^s \frac{|r-s|^{2H-2}}{\sqrt{t(t-s)}} \, \mathrm{d}r \, \mathrm{d}s \\ &= \frac{1}{\pi} e^{-\frac{2x^2}{(t-\delta)}} \int_0^\delta \frac{s^{2H-1}}{(2H-1)\sqrt{t(t-s)}} \, \mathrm{d}s \\ &\geq \frac{\delta^{2H} \; e^{-\frac{2x^2}{(t-\delta)}}}{2\pi t H(2H-1)} \geq C_{T,x,\delta_0} \delta^{2H} \end{aligned}$$

as required.
Now we consider the case $H = 1/2$. It holds, proceeding as in Lemma 3.7,

$$\|\mathbf{1}_{(t-\delta,t)} Dz(t,x)\|_{\mathcal{H}}^2 = \int_{t-\delta}^t |p_N(s,x)|^2 \, \mathrm{d}s \geq C_x \, \delta$$

and the conclusion follows. □

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