PRODUCT-QUOTIENT SURFACES: NEW INVARIANTS AND ALGORITHMS

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Abstract. In this article we suggest a new approach to the systematic, computer-aided construction and to the classification of product-quotient surfaces, introducing a new invariant, the integer γ, which depends only on the singularities of the quotient model $X = (C_1 \times C_2)/G$. It turns out that γ is related to the codimension of the subspace of $H^{1,1}$ generated by algebraic curves coming from the construction (i.e., the classes of the two fibers and the Hirzebruch-Jung strings arising from the minimal resolution of singularities of X).

Profiting from this new insight we developed and implemented an algorithm in the computer algebra program MAGMA which constructs all regular product-quotient surfaces with given values of γ and geometric genus. Being far better than the previous algorithms, we are able to construct a substantial number of new regular product-quotient surfaces of geometric genus zero. We prove that only two of these are of general type, raising the number of known families of product-quotient surfaces of general type with genus zero to 75. This gives evidence to the conjecture that there is an effective bound $\Gamma(p_g, q) \geq \gamma$ (cf. Conjecture 4.5).

Finally we introduce a duality among product-quotient surfaces and prove that the dual surface of a surface of geometric genus zero has maximal Picard number, thus providing several new examples of surfaces with maximal Picard number.

1. Introduction

Let G be a finite group acting on two compact Riemann surfaces C_1, C_2 of respective genera $g_1, g_2 \geq 2$. We shall consider the diagonal action of G on $C_1 \times C_2$ and in this situation we say for short: the action of G on $C_1 \times C_2$ is *unmixed*. By [Cat00] we may assume w.l.o.g. that G acts faithfully on both factors.

Definition 1.1. *The minimal resolution S of the singularities of $X = (C_1 \times C_2)/G$, where G is a finite group with an unmixed action on the product of two compact Riemann surfaces C_1, C_2 of respective genera at least two, is called a* product-quotient surface*.*

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X is called the quotient model *of the product-quotient surface.*

In the last years several people have been studying product-quotient surfaces and quite some literature is nowadays available (cf. *e.g* [Cat00, Zuc01, BC04, BCG08, MP10, Pol09, Pol10, Fra11, BCGP12, BP12, FP13, Pen12, GP13]...).

The authors (partially in collaboration with F. Catanese, D. Frapporti and F. Grunewald) have been focusing mainly on the systematic construction and classification of product-quotient surfaces of general type with geometric genus $p_g = 0$. Our previous results may be summarized as follows.

Theorem 1.2 ([BC04], [BCG08] [BCGP12],[BP12])**.**

1) Product-quotient surfaces isogenous to a product *(i.e., G acts freely) with $p_g(S) = q(S) = 0$ form 13 irreducible connected components of the Gieseker moduli space of surfaces of general type.*

2) Minimal *product-quotient surfaces with $p_g = 0$ of general type form 72 irreducible families, including the 13 families in point 1.*

3) There is exactly one product-quotient surface with $p_g = 0$, $K_S^2 > 0$ which is not minimal.

Even if quite some effort has been put and new techniques have been developed, the following problem remains open:

Problem 1.3. *Classify all product-quotient surfaces of general type with $p_g = 0$.*

By theorem 1.2 it remains to classify all *non-minimal* product-quotient surfaces of general type with geometric genus zero. In [BP12] he authors wrote a MAGMA script producing all regular product-quotient surfaces with $p_g = 0$ and fixed K_S^2.

As already noticed in *loc. cit*, one approach to solve the above problem is

1) prove that $K_S^2 \leq -C$ implies that S is not of general type for some explicit integer C;
2) use a suitable algorithm to construct all regular product-quotient surfaces with $p_g = 0$ and $-C < K^2 < 0$.

At the moment, not only an explicit bound is out of reach, but also the algorithm used in [BP12] is very slow for $K_S^2 < 0$, hence far from being good enough to make step 2 work even for small C.

In the present article we suggest a different approach to solve problem 1.3.

The key observation is the following: inspecting the list of surfaces in Theorem 1.2 (cf. [BP12], tables 1, 2), one notices that all minimal product-quotient surfaces with $p_g = 0$ have the property that $H^{1,1}(S)$ is generated by the fibres of the two fibrations and the irreducible components of the exceptional divisor of the minimal resolution of singularities σ, whereas for the single non-minimal product-quotient surface with $K_S^2 > 0$, this is not the case. Here the fibres and the exceptional curves generate a subspace of codimension 2.

This remark led us to study the subspace of $H^{1,1}(S)$ generated by the fibres of the two fibrations and the irreducible components of the exceptional divisor of the minimal resolution of singularities σ for a general product-quotient surface S. We shall prove in this article, that its codimension is even, and equal to $2(p_g(S)+\gamma)$ (cf. Proposition 4.2), where γ is an invariant depending only on some numerical data of the singularities of X.

Note that then in particular: $p_g = 0 \Rightarrow \gamma \geq 0$.

Remark 1.4. Looking at the program used in [BP12] for the case $p_g = 0$, one notices that almost half of the computations had to deal with the case $\gamma < 0$. This information could be used now to speed up the computations quite a bit.

Instead, we chose to write a different MAGMA script, substituting (as input) γ to K^2. The result is a much quicker program, producing dozens of new regular product-quotient surfaces with $p_g = 0$ (and several with $p_g > 0$, on which we do not report here).

Our computations suggest the following

Conjecture 1.5. *Let S be a product-quotient surface. Then S is minimal if and only if $p_g(S) + \gamma = 0$.*

We shall prove the conjecture for surfaces with vanishing geometric genus (cf. Theorem 6.2).

Running our program for $\gamma = 1, 2, 3$, produces three examples of surfaces of general type, two with $\gamma = 1$ (including the surface in Theorem 1.2, 4), and one with $\gamma = 2$: the two new examples, both Numerical Godeaux surfaces, are described in Section 7. Together with the results [BP12] we have 75 families of product-quotient surfaces of general type with $p_g = q = 0$ and we conjecture that this is a complete list.

What we can prove, is the following:

Proposition 1.6. *Let S be a product-quotient surface of general type with $p_g = 0$ not among the 75 families just mentioned. Then*

- *either $\gamma \geq 4$,*
- *or $\gamma = 3$ and X has a singular point of multiplicity at least 14,*
- *or $\gamma = 2$ and X has a singular point of multiplicity at least 45.*

On the way to prove the above we construct a substantial number of product-quotient surfaces not of general type, collected in the tables 1, 2, 3, 4 and 5.

Coming back to Problem 1.3, Our new approach allows to substitute part 1) of the proposed solution of Problem 1.3 by the following:

Conjecture (4.5). *There is an explicit function $\Gamma = \Gamma(p_g, q)$ such that, for the quotient model X of every product-quotient surface S of general type*

$$\gamma(X) \leq \Gamma(p_g(S), q(S)).$$

We give some motivation for this Conjecture in Section 8, proving the above conjecture under some additional hypotheses.

Finally in section 9 we construct a duality among regular product-quotient surfaces allowing, among other things, to give a new interpretation of the "half-codimension" $p_g + \gamma$, which in fact turns out to be equal to the geometric genus of the dual product-quotient surface.

An interesting result in this last section is Corollary 9.4, showing that the dual of every product-quotient surface of geometric genus zero has automatically maximal Picard number. Thus the dual surfaces of the surfaces in tables 2, 3, 4 and 5 provide more than 100 families of surfaces with $1 \leq p_g \leq 3$ and maximal Picard number.

2. Notation

In this chapter we fix the notation, which will be valid throughout the paper.

Let C be an algebraic curve, G a finite group acting faithfully on it, $C' = C/G$. We associate to the pair (C, G), after certain choices on C/G ([BCP12, Section 4] for details), an

- *appropriate orbifold homomorphism* $\varphi\colon \mathbb{T}(g(C/G); m_1, \dots, m_r) \to G$,

which allows (up to the above made choices) to reconstruct (C, G).

Equivalently, one can give

- a *generating vector* ([Pol10, Definition 1.1]) of G of *signature* (or *type*) $(g(C/G); m_1, \dots, m_r)$,

where $g(C/G)$ is the genus of the quotient curve.

We will say that the action of G on C has *signature* $(g(C/G); m_1, \dots, m_r)$.

We will also need the number

- $\Theta := \Theta(g(C/G); m_1, \dots, m_r) := 2g(C/G) - 2 + \sum \left(1 - \frac{1}{m_i}\right) > 0$,

which relates the genus of C and the order of G by the Hurwitz formula

$$2g(C) - 2 = |G|\Theta. \tag{1}$$

In the following C_1, C_2 will be two algebraic curves of respective genera $g_1, g_2 \geq 2$, G a finite group acting faithfully on both curves.

We consider the quotient surface $X := (C_1 \times C_2)/G$ by the diagonal action, and the minimal resolution of its singularities $\sigma\colon S \to X$. We will refer to S as

- a *product-quotient* surface and
- to X as its *quotient model*.

We will denote by $\overline{S}$ the minimal model of S.

As usual, $p_g(S)$ (or simply p_g) will be the geometric genus $h^2(\mathcal{O}_S)$, and $q(S)$ (or simply q) will be the irregularity $h^1(\mathcal{O}_S)$. We will also denote by χ or $\chi(S) = 1 - q + p_g$ the Euler characteristic of the structure sheaf $\mathcal{O}_S$ of S.

We will say that the quotient model X has *type*

$$((g(C_1/G); m_1, \dots, m_r), (g(C_2/G); n_1, \dots, n_s)),$$

if the action of G on C_1 has signature $(g(C_1/G); m_1, \dots, m_r)$ and the action of G on C_2 has signature $(g(C_2/G); n_1, \dots, n_s)$; we will write Θ_1 for $\Theta(g(C_1/G); m_1, \dots, m_r)$ and Θ_2 for $\Theta(g(C_2/G); n_1, \dots, n_s)$.

All singularities of X are cyclic quotient singularities, locally isomorphic to the quotient of $\mathbb{C}^2$ by the cyclic group generated by $(x, y) \mapsto (e^{\frac{2\pi i}{n}}x, e^{\frac{2q\pi i}{n}}y)$ for two relatively prime positive integers q, n with $q < n$. We will say that the singularity is *of type* $\frac{q}{n}$, instead of using the classical notation $\frac{1}{n}(1, q)$.

We denote by q' the integer between 1 and $n-1$ which is the multiplicative inverse of q modulo n, whence a singularity of type $\frac{q}{n}$ is also of type $\frac{q_1}{n_1}$ if and only if $n = n_1$ and q_1 is either q or q'.

We associate four numbers to each cyclic quotient singularity, depending only on its type.

Definition 2.1. *For each rational number* $0 < \frac{q}{n} < 1$ *we consider its continued fraction*

$$\frac{n}{q} = b_1 - \cfrac{1}{b_2 - \frac{1}{b_3 - \dots}} =: [b_1, \dots, b_l];$$

writing $\frac{q}{n} = [b_1, \dots, b_l]$, $b_i \in \mathbb{N}$, $b_i \geq 2$.

We define:

- $l\left(\frac{q}{n}\right)$ *is the length of the continued fraction;*
- $\gamma\left(\frac{q}{n}\right) := \frac{1}{6}\left[\frac{q+q'}{n} + \sum_{i=1}^{l(\frac{q}{n})}(b_i - 3)\right]$;
- $\mu\left(\frac{q}{n}\right) = 1 - \frac{1}{n}$.
- $I\left(\frac{q}{n}\right) = \frac{n}{\gcd(n, q+1)}$.

It is well known that if $\frac{q}{n} = [b_1, \dots, b_l]$, then $\frac{q'}{n} = [b_l, \dots, b_1]$. It follows immediately that l, γ, μ and I do not change when substituting q with q', and therefore the following definition is well posed.

Definition 2.2. *Let x be a singular point of X, of type $\frac{q}{n}$. Then we define* $l_x := l\left(\frac{q}{n}\right)$; $\gamma_x := \gamma\left(\frac{q}{n}\right)$; $\mu_x := \mu\left(\frac{q}{n}\right)$; $I_x := I\left(\frac{q}{n}\right)$.

A *representation of the basket of singularities* of the quotient model X is a multiset

$$\mathcal{B}(X) := \left\{\lambda \times \frac{a}{n} : X \text{ has exactly } \lambda \text{ singularities of type } \frac{a}{n}\right\}.$$

E.g., $\mathcal{B} = \{2 \times \frac{1}{3}, \frac{3}{4}\}$ means that the singular locus of X consists of two $\frac{1}{3}$-points and one $\frac{3}{4}$-point.

Consider the equivalence relation generated by "$\frac{a}{n}$ is equivalent to $\frac{a'}{n}$", where $a' = a^{-1}$ in $(\mathbb{Z}/n\mathbb{Z})^*$, on the multisets of the above form. A *basket of singularities* $\mathfrak{B}$ is then an equivalence class.

We globalize l, γ, μ and I as follows.

Definition 2.3 (**Invariants of the basket $\mathfrak{B}$**). *Let $\mathfrak{B}$ be the basket of singularities of the quotient model X of a product-quotient surface S. Then*

$$l(X) := \sum_{x \in \mathfrak{B}} l_x; \quad \gamma(X) := \sum_{x \in \mathfrak{B}} \gamma_x; \quad \mu(X) := \sum_{x \in \mathfrak{B}} \mu_x; \quad I(X) := \mathrm{lcm}_{x \in \mathfrak{B}} I_x.$$

Remark 2.4. I is the index of X, the minimal positive integer such that IK_X is a Cartier divisor. It is the only number, among the numbers defined in Definition 2.1, which was already considered in [BP12]. The numbers l ,γ and μ are convenient substitutes of the numbers e, k, and B considered in [BP12]. For the convenience of the reader, we recall the definition of e, k and B in terms of the new invariants:

$$e = l + \mu; \quad k = 6\gamma + l - 2\mu; \quad B = 3\,(2\gamma + l)\,. \tag{2}$$

3. Hodge theory of Product-Quotient Surfaces

We start with the following:

Proposition 3.1.

(1) *For all $k \neq 2$, $H^k(S, \mathbb{C}) \cong H^k(X, \mathbb{C})$.*
(2) *$H^2(S, \mathbb{C}) \cong H^2(X, \mathbb{C}) \oplus \mathbb{C}^{l(X)}$.*

Proof. 1) Let X° be the smooth locus of X. For each singular point x of X, choose a small neighbourhood U_x of x which may be retracted to the point x and set

- $U := \bigcup U_x$,
- $U_x^\circ := U_x \setminus \{x\} = U_x \cap X^\circ$,
- $U^\circ = U \cap X^\circ$.

We also consider

- $S^\circ := \sigma^{-1}(X^\circ)$,
- $V_x := \sigma^{-1}(U_x)$, $V_x^\circ := \sigma^{-1}(U_x^\circ)$,
- $V := \sigma^{-1}(U)$, $V^\circ := \sigma^{-1}(U^\circ)$.

The Mayer-Vietoris exact sequences corresponding to the decompositions

$$X = X^\circ \cup U, \quad S = S^\circ \cup V$$

give a commutative diagram
(3)

$$\begin{array}{ccccccccc}
H^{k-1}(X^\circ) \oplus H^{k-1}(U) & \longrightarrow & H^{k-1}(U^\circ) & \longrightarrow & H^k(X) & \longrightarrow & H^k(X^\circ) \oplus H^k(U) & \longrightarrow & H^k(U^\circ) \\
\downarrow{\scriptstyle b_{k-1} \oplus c_{k-1}} & & \downarrow{\scriptstyle d_{k-1}} & & \downarrow{\scriptstyle a_k} & & \downarrow{\scriptstyle b_k \oplus c_k} & & \downarrow{\scriptstyle d_k} \\
H^{k-1}(S^\circ) \oplus H^{k-1}(V) & \longrightarrow & H^{k-1}(V^\circ) & \longrightarrow & H^k(S) & \longrightarrow & H^k(S^\circ) \oplus H^k(V) & \longrightarrow & H^k(V^\circ)\,.
\end{array}$$

The vertical maps are induced by suitable restrictions of σ.

Since $\sigma_{|S^\circ}$ and $\sigma_{|V^\circ}$ are homeomorphisms, all the maps b_q and d_q are isomorphisms. Moreover, since U_x retracts to a point and V_x to a tree of l_x rational curves, c_k is an isomorphism for all $k \neq 2$, and c_2 is the (injective) map $0 \to \mathbb{C}^l$.

By the Five Lemma, it follows that all maps a_k with $k \neq 2, 3$ are isomorphisms, while the Four Lemma implies that a_2 is injective and a_3 is surjective.

Let $A_1, \dots, A_l$ be the exceptional divisors of σ. Since V retracts to the union of the A_i, the inclusions yield an isomorphism $H^2(V) \cong \oplus_1^l H^2(A_i)$, so $H^2(V) \cong \mathbb{C}^l$. Moreover, identifying by Poincaré duality $H^2(S)$ with $H_2(S)^*$, the map $H^2(S) \to H^2(A_i) \cong \mathbb{C}$ induced by inclusion sends each linear form ϕ to $\phi(A_i)$. Since the intersection form on the A_i is negative definite, it follows that the map $H^2(S) \to H^2(V) \cong \oplus H^2(A_i)$ is surjective.

Then standard diagram chasing shows that a_3 is injective, hence an isomorphism.

2) We have just shown that all maps a_k,b_k, c_k and d_k are isomorphisms with the exception of a_2 and c_2. Moreover, a_2 and c_2 are injective, and $\dim(\operatorname{coker} c_2) = l$. Since the alternating sum of the dimensions of the vector spaces in a finite exact sequence is zero, comparing the two long exact sequences in (3) we obtain $\dim H^2(S) = \dim H^2(X) + l$. □

For $H^2(X, \mathbb{C})$ we can prove the following:

Proposition 3.2.

- $\dim H^2(X, \mathbb{C}) \equiv 0 \mod 2$,
- $\dim H^2(X, \mathbb{C}) \geq 2$.

Proof. By the Hodge decomposition we know that

$$\begin{aligned} H^2(C_1 \times C_2, \mathbb{C}) \cong & H^0(\Omega^2_{C_1 \times C_2}) \oplus H^1(\Omega^1_{C_1 \times C_2}) \oplus H^2(\mathcal{O}_{C_1 \times C_2}) \\ \cong & H^0(\Omega^2_{C_1 \times C_2}) \oplus H^1(\Omega^1_{C_1 \times C_2}) \oplus H^0(\Omega^2_{C_1 \times C_2})^*. \end{aligned}$$

Therefore the G-invariant part of $H^2(C_1 \times C_2, \mathbb{C})$ decomposes as

$$\begin{aligned} H^2(X, \mathbb{C}) \cong & H^2(C_1 \times C_2, \mathbb{C})^G \\ \cong & H^0(\Omega^2_{C_1 \times C_2})^G \oplus H^1(\Omega^1_{C_1 \times C_2})^G \oplus (H^0(\Omega^2_{C_1 \times C_2})^*)^G \\ \cong & H^0(\Omega^2_{C_1 \times C_2})^G \oplus H^1(\Omega^1_{C_1 \times C_2})^G \oplus (H^0(\Omega^2_{C_1 \times C_2})^G)^*. \end{aligned}$$

Therefore, writing as usual h^q for the dimension of H^q, $h^2(X, \mathbb{C}) = 2 \cdot h^0(\Omega^2_{C_1 \times C_2})^G + h^1(\Omega^1_{C_1 \times C_2})^G$, whence the claim is proven once we show that $h^1(\Omega^1_{C_1 \times C_2})^G \equiv 0 \mod 2$.

By Künneth's formula (cf. e.g. [Ka67]) and Hodge theory we have:

$$\begin{aligned} H^1(\Omega^1_{C_1 \times C_2}) \cong & \left(H^1(\Omega^1_{C_1}) \otimes H^0(\mathcal{O}_{C_2})\right) \oplus \left(H^1(\Omega^1_{C_2}) \otimes H^0(\mathcal{O}_{C_1})\right) \\ & \oplus \left(H^0(\Omega^1_{C_1}) \otimes H^1(\mathcal{O}_{C_2})\right) \oplus \left(H^0(\Omega^1_{C_2}) \otimes H^1(\mathcal{O}_{C_1})\right) \\ \cong & \left(H^1(\Omega^1_{C_1}) \otimes H^0(\mathcal{O}_{C_2})\right) \oplus \left(H^1(\Omega^1_{C_2}) \otimes H^0(\mathcal{O}_{C_1})\right) \\ & \oplus \left(H^0(\Omega^1_{C_1}) \otimes \overline{H^0(\Omega^1_{C_2})}\right) \oplus \left(\overline{H^0(\Omega^1_{C_1})} \otimes H^0(\Omega^1_{C_2})\right). \end{aligned}$$

It is well known that if χ is the character of the G-module $H^0(\Omega^1_{C_1})$, then $\bar{\chi}$ is the character of the G-module $\overline{H^0(\Omega^1_{C_1})}$. From this fact it follows that

$$\left(H^0(\Omega^1_{C_1}) \otimes \overline{H^0(\Omega^1_{C_2})}\right)^G \oplus \left(\overline{H^0(\Omega^1_{C_1})} \otimes H^0(\Omega^1_{C_2})\right)^G \cong V \oplus \bar{V},$$

where

$$V := \left(H^0(\Omega^1_{C_1}) \otimes \overline{H^0(\Omega^1_{C_2})}\right)^G.$$

Since the fundamental class of C_i is G-invariant, we have

$$\left(H^1(\Omega^1_{C_1}) \otimes H^0(\mathcal{O}_{C_2})\right) \oplus \left(H^1(\Omega^1_{C_2}) \otimes H^0(\mathcal{O}_{C_1})\right) =$$

$$\left(H^1(\Omega^1_{C_1}) \otimes H^0(\mathcal{O}_{C_2})\right)^G \oplus \left(H^1(\Omega^1_{C_2}) \otimes H^0(\mathcal{O}_{C_1})\right)^G \cong \mathbb{C}^2.$$

This proves the claim. □

Consider the inclusion

$$j\colon X^\circ := X \setminus \mathrm{Sing}(X) \to X$$

and define $\tilde{\Omega}^p_X := j_*\Omega^p_{X^\circ}$.

Theorem 3.3 ([Ste77], (1.10),(1.11), (1.12))**.**

(1) $\tilde{\Omega}^p_X$ *is coherent for all* p*;*
(2) $\tilde{\Omega}^p_X = \sigma_*\Omega^p_S$*, for all* p*;*
(3) $\tilde{\Omega}^p_X = (\pi_*\Omega^p_{C_1\times C_2})^G$*;*
(4) *there is a morphism of spectral sequences*

$$\begin{array}{ccc} E_1^{pq} = H^q(X, \tilde{\Omega}^p_X) & \Longrightarrow & H^{p+q}(X,\mathbb{C}) \\ \downarrow \sigma^* & & \downarrow \sigma^* \\ E_1'^{pq} = H^q(S, \Omega^p_S) & \Longrightarrow & H^{p+q}(S,\mathbb{C}), \end{array}$$

which is injective at the E_1*-level.*

Proposition 3.4. *If* $p_g(S) = 0$*, then* $H^0(C_1 \times C_2, \Omega^2_{C_1\times C_2})^G = 0$*. In particular,* $H^2(X,\mathbb{C}) \cong H^1(C_1 \times C_2, \Omega^1_{C_1\times C_2})^G$*.*

Proof. By Theorem 3.3, $H^0(X, \tilde{\Omega}^2_X) \to H^0(S, \Omega^2_S) = 0$ is injective, and $H^0(X, \tilde{\Omega}^2_X) = H^0(C_1 \times C_2, \Omega^2_{C_1\times C_2})^G$. □

We recall the following version of Schur's lemma (cf. *e.g.* [Se77, Proposition 4]):

Lemma 3.5. *Let* G *be a finite group and let* W *be an irreducible* G*-representation. Then*

(1) $\dim(W \otimes W^*)^G = 1$*;*

(2) *if W' is an irreducible G-representation not isomorphic to W^*, then* $\dim(W \otimes W')^G = 0$.

Remark 3.6.

(1) Proposition 3.4 shows that the singularities of the quotient-model X give no conditions of adjunction for canonical forms, even if the singularities are not canonical. This is not true for bicanonical forms.

(2) The above results (especially the proof of prop. 3.2) make clear that the condition that S has vanishing geometric genus gives strong restrictions on the G-modules $H^0(C_i, \Omega^1_{C_i})$. For example, using Schur's lemma, we can list the following properties:
   (a) if χ is an irreducible character of G, then $H^0(\Omega^1_{C_1})^\chi = 0$ or $H^0(\Omega^1_{C_2})^{\bar{\chi}} = 0$;
   (b) $\dim H^2(X, \mathbb{C}) > 2$ if and only if there is an irreducible non selfdual character χ of G such that $H^0(\Omega^1_{C_1})^\chi \neq 0$ and $H^0(\Omega^1_{C_2})^\chi \neq 0$.

   Each time that such a situation occurs, the dimension of $\dim H^2(X, \mathbb{C})$ is raised by two.

An immediate consequence of the above considerations is the following:

Proposition 3.7. *Let $X = (C_1 \times C_2)/G$ be the quotient model of a regular product-quotient surface with $p_g = 0$. Assume moreover that all irreducible representations of G are selfdual (e.g. $G = \mathfrak{S}_n$). Then $h^2(X, \mathbb{C}) = 2$.*

4. The invariant γ

The formulas for K_S^2, χ and q in [BP12] translate, in the notation of the present paper, as follows.

Proposition 4.1 ([BP12], Prop. 1.6 and Cor. 1.7 and [Ser96])**.**

$$K_S^2 = 8\chi - 2\gamma - l, \qquad \chi = \frac{(g_1 - 1)(g_2 - 1)}{|G|} + \frac{\mu - 2\gamma}{4}, \qquad q = g_1 + g_2.$$

Observe that the new invariant γ is (as defined in 2.1 and 2.3) a priori a rational number.

But, in fact, we are going to show in the next proposition that γ is an integer, bounded from below by $-p_g(S)$.

Proposition 4.2.

$$\gamma(X) + p_g(S) \in \mathbb{N}.$$

Moreover, if $\gamma(X) + p_g(S) = 0$, then S has maximal Picard number.

Proof. The intersection form on $H^2(S, \mathbb{C})$ shows that the fibres of the two fibrations $S \to C_i/G$, and the l irreducible exceptional curves of σ form a set of $l + 2$ linearly independent classes in $H^1(S, \Omega^1_S)$. Therefore we have

$$h^{1,1}(S) - l - 2 \in \mathbb{N}.$$

By Proposition 3.1,we know that $\dim H^2(S,\mathbb{C}) = l + \dim H^2(X,\mathbb{C})$ and, by Proposition 3.2, we see that $h^{1,1}$ has the same parity as l. Therefore $h^{1,1} - l - 2 \in 2\mathbb{N}$.

The claim follows, using Noether's formula, Hodge theory and Proposition 4.2, since

$$\begin{aligned} 2(\gamma + p_g) =& - K_S^2 + 8\chi - l + 2p_g \\ =& c_2(S) - 4\chi - l + 2p_g \\ =& 2 - 2b_1 + b_2 - 4 + 4q - 4p_g - l + 2p_g \\ =& h^{1,1} - l - 2. \end{aligned}$$

In particular, if $\gamma(X) + p_g(S) = 0$, then $H^{1,1}(S)$ is generated by algebraic curves (the fibres of the two fibrations and the exceptional curves of σ) and therefore S has maximal Picard number. □

Remark 4.3. From Proposition 3.1 and the proof of Proposition 4.2, we get that $h^2(X,\mathbb{C}) = 2(\gamma + 2p_g + 1)$. In particular, by Proposition 3.7, if $X = (C_1 \times C_2)/G$ is the quotient model of a regular product-quotient surface with $p_g = 0$, and if all irreducible representations of G are selfdual, then $\gamma = 0$.

The next proposition implies that the possible values of γ distribute symmetrically around zero.

Proposition 4.4. $\gamma(\frac{q}{n}) = -\gamma(\frac{n-q}{n})$.

Proof. Write $\frac{n}{q} = [b_1, \dots, b_l]$, $\frac{n}{n-q} = [a_1, \dots, a_k]$. Then by [Rie74, Lemma 4]

$$\sum_1^k (a_i - 1) = \sum_1^l (b_i - 1) = k + l - 1.$$

Therefore

$$\begin{aligned} 6\left(\gamma\left(\frac{q}{n}\right) + \gamma\left(\frac{n-q}{n}\right)\right) &= \frac{q+q'}{n} + \frac{n-q+n-q'}{n} + \sum_{i=1}^l (b_i - 3) + \sum_{i=1}^k (a_i - 3) = \\ &= 2 + \sum_{i=1}^l (b_i - 1) - 2l + \sum_{i=1}^k (a_i - 1) - 2k = 0. \end{aligned}$$

□

What concerns an upper bound for γ in terms of the invariants of S, we have the following

Conjecture 4.5. *There is an explicit function $\Gamma = \Gamma(p_g, q)$ such that, for the quotient model X of every product-quotient surface S of general type*

$$\gamma(X) \leq \Gamma(p_g(S), q(S)).$$

5. A classification algorithm for surfaces of general type with given p_g, q and γ

In [BP12] we developed an algorithm producing all product-quotient surfaces with given values of K_S^2, and $\chi(\mathcal{O}_S)$ (as input).

In the following we shall show that we can substitute γ to K_S^2; in other words, fixing χ and $\gamma \in \mathbb{N}$, we also get a finite problem. In particular, answering in the affirmative Conjecture 4.5 we would have an algorithm constructing all product-quotient surfaces with fixed values of q and p_g.

To ease the forthcoming formulas, we also introduce the following:

Definition 5.1. $\xi := \xi(X) := 4\chi + 2\gamma - \mu \in \mathbb{Q}$.

Remark 5.2. Observe that ξ only depends on χ and on the basket $\mathfrak{B}$. Moreover,

$$\xi(X) = \frac{4(g_1 - 1)(g_2 - 1)}{|G|} = \frac{K_X^2}{2}.$$

We recall the following theorem due to Xiao Gang:

Theorem 5.3 ([Xia96])**.** *Let T be a minimal surface of general type and G a finite group of automorphisms of T, such that T/G is of general type. Let Y be the minimal model of a resolution of singularities of T/G. Then*

$$1 \leq K_Y^2 \leq \frac{K_T^2}{|G|}.$$

Using remark 5.2 we immediately get the following lower bound for ξ.

Corollary 5.4.

$$\xi(X) \geq \frac{1}{2} K_S^2 \geq \frac{1}{2}.$$

Proof. This follows immediately, since $\frac{K^2_{C_1 \times C_2}}{|G|} = K_X^2 = 2\xi$. □

We consider the two natural fibrations

$$f_1 \colon S \to C_1/G, \quad f_2 \colon S \to C_2/G,$$

and denote the generic fibre of f_i by F_i. Observe that F_1 is isomorphic to C_2 and F_2 is isomorphic to C_1.

These fibrations have been studied in detail in [Pol10]. If the type of X is $((g_1; m_1, \ldots, m_r), (g_2; n_1, \ldots, n_s))$, then f_1 has exactly r reducible fibers, all non reduced, of the form:

$$F_1 \equiv m_i F_1^{(i)} + \sum a_j A_j, \quad 1 \leq i \leq r,$$

where the A_j's are contracted by σ. Similarly the second fibration $f_2 \colon S \to C_2/G$ with general fibre F_2 isomorphic to C_1, has s reducible fibers of the form $n_i F_2^{(i)} + \sum b_j A_j$.

Remark 5.5. [cf. [Ser96], Theorem 2.1] Each singular point x of X lies on $\sigma(F_1^{(i)})$ for one i. Moreover, if x is of type $\frac{q}{n}$, then n divides m_i.

We will need the following result by F. Polizzi, computing the self intersection $(F_1^{(i)})^2$ from the types of the singularities of X along $\sigma(F_1^{(i)})$.

Proposition 5.6 ([Pol10], Proposition 2.8)**.**

$$\sum_{x \in Sing X \cap \sigma(F_1^{(i)})} \frac{q}{n}(x) = -(F_1^{(i)})^2 \in \mathbb{N},$$

where x is a singular point of type $\frac{q}{n}(x)$.

Moreover, if $x \in \sigma(F_1^{(i)}) \cap \sigma(F_2^{(j)})$ and the contribution of x to $(F_1^{(i)})^2$ is $\frac{q}{n}$, then its contribution to $(F_2^{(j)})^2$ is $\frac{q'}{n}$.

We shall show now (Proposition 5.11) that for fixed γ, p_g and q there is a finite list containing all possible signatures involved in the construction of product-quotient surfaces with those values of γ, p_g and q.

Before doing this, we need to recall further invariants, the integers α_i, which were already considered in our previous papers.

Definition 5.7.

$$\alpha_1 := \frac{4\chi + 2\gamma - \mu}{2\Theta_1} = \frac{\xi}{2\Theta_1}, \qquad \alpha_2 := \frac{4\chi + 2\gamma - \mu}{2\Theta_2} = \frac{\xi}{2\Theta_2}.$$

In fact, we have (cf. e.g. [BCGP12])

Proposition 5.8. $\alpha_i = g_{i+1} - 1 \in \mathbb{N}$.

Proof. W.l.o.g. we can assume $i = 1$. Then

$$\alpha_1 = \frac{\xi}{2\Theta_1} = \frac{2(g_1 - 1)(g_2 - 1)}{|G|\Theta_1} = g_2 - 1 \in \mathbb{N}.$$

□

The following inequality allows to bound the multiplicities in the signatures in terms of the genera of the involved curves.

Theorem 5.9 ([Wim95])**.** *Let H be a cyclic group of automorphisms of a compact Riemann surface C of genus $g \geq 2$. Then $|H| \leq 4g + 2$.*

In fact, an immediate consequence of Wiman's inequality is the following:

Corollary 5.10. *For all $1 \leq i \leq r$, $1 \leq j \leq s$ we have:*

$$m_i, n_j \leq 2 \min\left(\left(\frac{\xi}{\Theta_1} + 3\right), \left(\frac{\xi}{\Theta_2} + 3\right)\right).$$

The next proposition gives upper bounds for r, s, m_i and n_j in terms of ξ and $g(C_i/G)$.

Proposition 5.11. *The following inequalities hold:*

a) $r \leq \xi + 4 - 2g(C_1/G)$;

b) *if* $g(C_1/G) > 0$ *or* $r > 3$*, then for all* $1 \leq i \leq r$ *it holds*

$$m_i \leq 3 + \frac{2\xi + 1 + \sqrt{(3(4g(C_1/G) + r - 3) + 2\xi + 1)^2 - 12(4g(C_1/G) + r - 3)}}{4g(C_1/G) + r - 3}$$
$$< 6 + \frac{4\xi + 2}{4g(C_1/G) + r - 3}$$

c) *if* $g(C_1/G) = 0$ *and* $r \leq 3$*, then* $r = 3$ *and*

$$m_i \leq 6[\xi + 1 + \sqrt{\xi(\xi + 2)}] < 12(\xi + 1)$$

for $1 \leq i \leq 3$*.*

Analogous bounds hold for s, n_j*.*

Proof. a) By $1 \leq \alpha_1 = \frac{\xi}{2\Theta_1}$, $2\Theta_1 \leq \xi$. Since by definition $\Theta_1 \geq 2g(C_1/G) - 2 + \frac{r}{2}$,

$$r \leq 2\Theta_1 + 4 - 2g(C_1/G) \leq \xi + 4 - 2g(C_1/G)$$

b) If $r = 0$ there is nothing to prove, so we may assume $r \geq 1$. Let m_1 be the maximum of the m_i. Note that by definition

$$\Theta_1 \geq 2g(C_1/G) + \frac{r-3}{2} - \frac{1}{m_1} = \frac{m_1(4g(C_1/G) + r - 3) - 2}{2m_1}.$$

By assumption $m_1(4g(C_1/G) + r - 3) - 2 \geq 0$. Moreover $m_1(4g(C_i) + r - 3) - 2 = 0$ implies that the signature is $(0; 2, 2, 2)$, which implies $\Theta_1 = -\frac{1}{2} < 0$, a contradiction. So $m_1(4g(C_1/G) + r - 3) - 2 > 0$, whence, from corollary 5.10,

$$m_1 \leq 2\left(\frac{\xi}{\Theta_1} + 3\right) \leq 2\left(\frac{2m_1\xi}{m_1(4g(C_1/G) + r - 3) - 2} + 3\right),$$

so

$$m_1^2(4g(C_1/G) + r - 3) - 2m_1(3(4g(C_1/G) + r - 3) + 2\xi + 1) + 12 \leq 0.$$

This immediately implies the desired inequality.

c) By corollary 5.4 we have $\xi \geq \frac{1}{2} > 0$, and therefore the claimed upper bound for m_i is > 6. Therefore we can assume w.l.o.g that $m_1 > 6$.

Since $\Theta_1 > 0$ it follows $r \geq 3$ (so $r = 3$) and $\Theta_1 + \frac{1}{m_1} \geq \frac{1}{6}$ with equality if and only if the signature is $(0; 2, 3, m_1)$. So $\Theta_1 \geq \frac{m_1 - 6}{6m_1}$ and

$$m_1 \leq 2\left(\frac{\xi}{\Theta_1} + 3\right) \leq 2\left(\frac{6m_1\xi}{m_1 - 6} + 3\right)$$

which is equivalent to

$$m_1^2 - 12\,(\xi + 1)\,m_1 + 36 \leq 0$$

and we can conclude as before. □

We are now prepared to give the necessary bounds in order to show that given γ, p_g and q, there is a finite number of families of product-quotient surfaces with these invariants.

Recall that $g(C_i/G)$, $i = 1, 2$, is bounded by q (Proposition 4.1), whence it is enough to produce upper bounds for the remaining natural numbers involved, i.e., we need to bound r, s, m_i and n_j in terms of p_g and q.

Remark 5.12. If S is of general type then

$$\# \operatorname{Sing} X = \#\mathfrak{B}(X) \leq 8\chi + 4\gamma - 1.$$

Proof. The inequality follows by Corollary 5.4 since, by the definition of μ, $\#\mathfrak{B}(X) \leq 2\mu$. □

We now give an upper bound for the multiplicity of each singularity of X in terms of p_g, q and γ. This, together with remark 5.12, produces a finite list of possibilities for the basket of singularities of the quotient model of a product quotient surface with given values of p_g, q and γ.

Proposition 5.13. *Let S be of general type. Then:*
a) *if $\frac{q}{n} \in \mathfrak{B}$, then $n \leq 12(4\chi + 2\gamma - 1)$;*
b) *if moreover $\gamma \neq 0$, then $n \leq 12(4\chi + 2\gamma - \frac{3}{2})$.*

Proof. a) If the basket is empty, then the claim is empty. Otherwise assume that there is a singular point x of type $\frac{q}{n}$, and let m_i be the multiplicity of the central component of the fibre of f_1 containing it. Then by lemma 5.6 there is at least one further singular point on the same fibre, and, if there is only one, it is of type $\frac{n-q}{n}$. It follows $\mu \geq 2 - \frac{2}{n}$.

By proposition 5.11 and remark 5.5 we know that $n \leq m_i < 12(4\chi+2\gamma-\mu+1) \leq 12(4\chi + 2\gamma - 1 + \frac{2}{n})$. Therefore

$$n - \frac{24}{n} < 12(4\chi + 2\gamma - 1).$$

If $4\chi + 2\gamma - 1 \geq 2$, then the righthand side is bigger than 24, hence

$$n \leq 12(4\chi + 2\gamma - 1). \tag{4}$$

By Proposition 4.2 and Corollary 5.4, $4\chi + 2\gamma - 1$ is a positive integer, so it remains to consider only the case $4\chi+2\gamma-1 = 1$. In this case Corollary 5.4 yields $\mu \leq \frac{3}{2}$, and therefore either there are three points of multiplicity 2 or there are exactly two singular points, both of multiplicity $n \leq 4$. In all cases (4) hold.

b) If the basket contains exactly 2 elements, they are by Proposition 5.6 of respective type $\frac{q}{n}$ and $\frac{n-q}{n}$ and then by Proposition 4.4 $\gamma = 0$. Therefore $\gamma \neq 0$ implies that there are at least three singular points, and a straightforward computation gives $\mu \geq \frac{5}{2} - \frac{3}{n}$, whence

$$n - \frac{36}{n} < 12\left(4\chi + 2\gamma - \frac{3}{2}\right).$$

The claim follows by the same argument as in the previous case. □

Remark 5.14. We have shown that the classification problem is finite. In fact, we know that there are finitely many possibilities for the basket of singularities. If we fix a basket $\mathfrak{B}$, then we have to show that there are finitely many possibilities for

- the order of the group G, and for
- the two types $t_1 = (g(C_1/G); m_1, \ldots, m_r)$ and $t_2 = (g(C_2/G); n_1, \ldots, n_s)$.

Note that by proposition 5.15, a), $|G|$ is determined by t_1 and t_2. The length r (resp. s) of t_1 (resp. t_2) is bounded by proposition 5.11, a), whereas a bound for the m_i (resp. n_j) is given by loc.cit. b), c).

We are now ready to write an algorithm producing, for each fixed value of the triple (p_g, q, γ), all product quotient surfaces with those values of p_g, q and γ. Still, for implementing a reasonable (quick) algorithm it is convenient to use also the following additional informations which we have proved in [BP12].

Proposition 5.15.

a) $|G| = \frac{4\alpha_1\alpha_2}{\xi} = \frac{\xi}{\Theta_1\Theta_2}$;
b) *for each* i, $\frac{I\xi}{\Theta_1 m_i} \in \mathbb{N}$;
c) *there are at most* $\frac{|\mathfrak{B}|}{2}$ *indices such that* $\frac{I\xi}{2\Theta_1 m_i} \notin \mathbb{N}$;
d) $m_i \leq \frac{1+I\xi}{f}$, *where* $f := \max(\frac{1}{6}, \frac{r-3}{2})$;
e) *except for at most* $\frac{|\mathfrak{B}|}{2}$ *indices, it holds:* $m_i \leq \frac{2+I\xi}{2f}$

Similar statements as b), c), d) obviously hold for $(n_1, \ldots, n_s)$.

Proof. a) follows by Remark 5.2 and Proposition 5.8;
b-c) see [BP12], proposition 1.13;
d) let m_1 be the biggest of the m_i's; then $\Theta_1 + \frac{1}{m_1} \geq f$ whence:

$$m_i \leq m_1 \leq \frac{1 + \Theta_1 m_1}{f} \leq \frac{1 + I\xi}{f};$$

where $f := \max(\frac{1}{6}, \frac{r-3}{2})$;
e) similar. □

We describe now explicitly an algorithm producing all product quotient surfaces of general type with fixed p_g, q and γ.

Indeed, Corollary 5.12 and Proposition 5.13 produce, once fixed p_g, q and γ, a finite list of possible baskets. The basket determines also μ, l and ξ.

Moreover, $0 \leq g(C_1/G) \leq q$ varies also in a finite set (and determines $g(C_2/G) = q - g(C_1/G)$).

For each basket in the list, and for each choice of $g(C_1/G)$, Proposition 5.11 gives a finite list of possible signatures for the action of G on C_1 (and similarly on C_2). Most of the signature obtained can be excluded by using the other conditions we know:

- Remark 5.5 ensures that for each singularity of type $\frac{q}{n}$ there is an i such that $n|m_i$;
- $\alpha \in \mathbb{N}$;
- Proposition 5.15, b), c), d), e).

Finally, for each pair of signatures, we can run a search over all groups of the order predicted by Proposition 5.11, a), whether there is a pair of generating vectors of the prescribed signatures.

We have implemented this algorithm in MAGMA ([BCP97]) in the case $q = 0$. The interested reader may download the commented script from
`http://www.science.unitn.it/~pignatel/papers/RegP-QByPgGamma.magma`

The command *ExistingSurfaces(p_g,γ,M)* has two outputs: a list of regular product-quotient surfaces with the given values of p_g and γ, and quotient model whose singularity of maximal multiplicity has multiplicity M, and a list of *skipped* cases, pairs (group,signature) which the computer could not handle (for technical reasons): if there is a regular product-quotient surface with those values of p_g and γ which is not in the first output, group and signature are in the second output.

To get all product-quotient surfaces with given values of p_g and γ one should run it with M up to the maximum predicted in Proposition 5.13, and then check the second output for missing surfaces. In all cases we run we could show, by argument similar to those used in [BP12], that the first list is complete; in other words, that the computation skipped by the computer do not give rise to a product-quotient surface.

6. Does γ detect minimality?

In [BP12] the authors ran a computer program whose output lists all product-quotient surfaces with $p_g = 0$ and $K_S^2 \geq 1$. Inspecting the output it turned out that all surfaces are minimal (hence of general type) with the exception of one case. All minimal product-quotient surfaces satisfy $\gamma(S) = 0$, while the only non-minimal surface in the list has $\gamma = 1$. It seems therefore natural to conjecture that γ is related to the minimality of a product-quotient surface. Or, more ambitiously, that one can bound the number of exceptional (-1)-cycles on a product-quotient surface in terms of γ.

We make the following

Conjecture 6.1. *Let S be a regular product-quotient surface of general type. Then*
$$\gamma(S) + p_g(S) = 0 \iff S \text{ is minimal.}$$

In the sequel we shall give a proof of this conjecture in the special case $p_g = 0$. In fact, we have

Theorem 6.2. *Let S be a product-quotient surface of general type with $p_g = 0$. Then*
$$\gamma(S) = 0 \iff S \text{ is minimal.}$$

TABLE 1. Product-quotient surfaces with $\gamma = p_g = 0$ not of general type

	K_S^2	Sing X	t_1	t_2	G
1)	0	$\frac{1}{6}, \frac{5}{6}, 2 \times \frac{1}{2}$	$2,4,6$	$2,4,6$	SmallGroup(192,955)
2)	0	$\frac{1}{6}, \frac{5}{6}, 2 \times \frac{1}{2}$	$2,4,6$	$2,5,6$	SmallGroup(120,34)
3)	0	$\frac{1}{6}, \frac{5}{6}, 2 \times \frac{1}{2}$	$2,4,6$	$2,2,2,6$	SmallGroup(48,48)
4)	-2	$2 \times \frac{1}{5}, 2 \times \frac{4}{5}$	$2,5,5$	$2,5,5$	SmallGroup(80,49)
5)	0	$4 \times \frac{2}{5}$	$2,5,5$	$2,5,5$	SmallGroup(80,49)
6)	0	$2 \times \frac{1}{4}, 2 \times \frac{3}{4}$	$2,4,5$	$3,4,4$	SmallGroup(120,34)
7)	0	$2\times, \frac{1}{4}, 2 \times \frac{3}{4}$	$2,2,2,4$	$2,2,2,4$	SmallGroup(16,11)
8)	0	$2 \times \frac{1}{4}, 2 \times \frac{3}{4}$	$2,2,2,4$	$3,4,4$	SmallGroup(24,12)
9)	0	$2 \times \frac{1}{4}, 2 \times \frac{3}{4}$	$3,4,4$	$3,4,4$	SmallGroup(36,9)
10)	-1	$\frac{1}{5}, 2 \times \frac{2}{5}, \frac{4}{5}$	$2,5,5$	$3,3,5$	SmallGroup(60,5)

Remark 6.3. Unfortunately, we do not have a conceptional proof of the above theorem, which could shed some light on a possible connection between the number of exceptional cycles on a product-quotient surface and the invariant γ, or $\gamma + p_g$. The proof is just a case by case inspection of the output of the MAGMA script listing all product-quotient surfaces with $p_g = \gamma = 0$.

Proof. Running the MAGMA script "ExistingSurfaces(0,0,M)" for $M \leq 36$, we only have to take care of the surfaces S with $K_S^2 \leq 0$. In fact, if $K_S^2 > 0$, it has already been proven in [BP12] (cf. also Theorem 1.2 and the corresponding tables) that in these cases $\gamma = 0$.

Therefore the proof is finished once we show that the cases with $K_S \leq 0$ in the output of "ExistingSurfaces(0,0,M)" for $M \leq 36$ are not of general type.

This will be taken care of in the remaining part of the section. □

First of all we list the output of the surfaces with $K_S^2 \leq 0$ in table 1.

We need the following:

Proposition 6.4. *Let S be a product-quotient surface and let $A_1, \dots A_l$ be the exceptional curves of σ of respective selfintersection b_i. Assume that*

$$E \sim \frac{\mu_1}{|G|} F_1 + \frac{\mu_2}{|G|} F_2 - \sum_{i=1}^{l} a_i A_i \in H^2(S, \mathbb{Q}).$$

Then $\mu_i \in \mathbb{N}$. *Moreover, let* M *be the intersection matrix of the basket (i.e., of the* A_i*'s), and set*

$$b := \begin{pmatrix} K_S A_1 \\ K_S A_2 \\ \cdot \\ \cdot \\ \cdot \\ K_S A_l \end{pmatrix} = \begin{pmatrix} b_1 - 2 \\ b_2 - 2 \\ \cdot \\ \cdot \\ \cdot \\ b_l - 2 \end{pmatrix}, \quad e := \begin{pmatrix} EA_1 \\ EA_2 \\ \cdot \\ \cdot \\ \cdot \\ EA_l \end{pmatrix}.$$

Then

$$K_S E = \mu_1 \Theta_2 + \mu_2 \Theta_1 + e^T M^{-1} b; \tag{5}$$

$$E^2 = \frac{2\mu_1\mu_2}{|G|} + e^T M^{-1} e. \tag{6}$$

Proof. Note that $\mu_1 = EF_2$; in particular $\mu_1 \in \mathbb{N}$. Similarly $\mu_2 \in \mathbb{N}$.

Since

$$\sigma^* K_X \equiv \frac{1}{|G|} \left((2g_1 - 2)F_1 + (2g_2 - 2)F_2 \right) \equiv \Theta_1 F_1 + \Theta_2 F_2,$$

then $K_S \equiv \Theta_1 F_1 + \Theta_2 F_2 - \mathfrak{A}$, where $\mathfrak{A}$ is of the form $\sum_{i=1}^{l} \alpha_i A_i$ for some $\alpha_i \in \mathbb{Q}$. Set

$$a := \begin{pmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_l \end{pmatrix}, \quad \alpha := \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_l \end{pmatrix}.$$

Since $\forall i$ $A_i F_1 = A_i F_2 = 0$, then $\mathfrak{A} A_i = -K_S A_i = -(b_i - 2)$; in other words $M\alpha = -b$. Similarly $Ma = -e$. Since M is invertible, we can also write $b = -M^{-1}\alpha$, $e = -M^{-1}a$.

Then $K_S E = \mu_1 \Theta_2 + \mu_2 \Theta_1 + \sum a_i A_i \mathfrak{A} = \mu_1 \Theta_2 + \mu_2 \Theta_1 + a^T M \alpha = \mu_1 \Theta_2 + \mu_2 \Theta_1 + e^T M^{-1} b$. Similarly $E^2 = \frac{2\mu_1\mu_2}{|G|} - \left(\sum_{i=1}^{l} a_i A_i \right)^2 = \frac{2\mu_1\mu_2}{|G|} + a^T M a = \frac{2\mu_1\mu_2}{|G|} + e^T M^{-1} e$. □

Remark 6.5. By the proof of Proposition 4.2, if $p_g + \gamma = 0$, the set $\{A_i, F_j\}$ is a basis of $H^2(X, \mathbb{Q})$, so the assumption of Proposition 6.4 is automatically verified by every curve E.

To show that the surfaces in table 1 are not of general type we argue by cotradiction, assuming that they are of general type, and showing that the minimal model has $K_{\overline{S}}^2 < 0$. To do that, we look for rational curves E with selfintersection -1, and study their image $\sigma(E)$ in the quotient model X.

We recall that

Proposition 6.6. *Let* $\alpha\colon \mathbb{P}^1 \to X$ *be a generically injective map (i.e.,* $\alpha(\mathbb{P}^1) \subset X$ *is a rational curve). Then* $\alpha^{-1}(\mathrm{Sing}(X))$ *has cardinality at least three.*

Proof. This has been shown in the proof of [BP12, Proposition 4.7] □

Proposition 6.7. *Let* S *be a smooth surface of general type and let* $C \subset S$ *be an irreducible curve with* $K_S C \leq 0$. *Then* C *is smooth and rational.*

Proof. See [BP12, Remark 4.3] □

Corollary 6.8. *If* S *is a surface of general type,* E *a* $(-1)-$*curve on* S *and* C *a curve with* $C^2 = -b$. *Then* $CE \leq \max(1, b-3)$.

Proof. Else, contracting E, we obtain a surface of general type with a curve, the image of C, contradicting Proposition 6.7. □

We will also need the following

Lemma 6.9. *Let* S *be a product-quotient surface of general type. Suppose that the exceptional locus of* σ *consists of*

i) *curves of self intersection (-3) and (-2), or*
ii) *at most two smooth rational curves of self-intersection (-3) or (-4), and (-2)-curves.*

Then S *is minimal.*

Proof. i) This is [FP13, Corollary 4.8].

ii) Assume that S contains a (-1)-curve E. Note that E cannot intersect two different (-2)-curves or, contracting it, we would get two (-1)-curves intersecting transversally, impossible on a surface of general type. Then by Proposition 6.6 and Corollary 6.8 the exceptional locus contains two curves of self-intersection (-3) or (-4), say E_1 and E_2, $EE_1 = EE_2 = 1$ and moreover E intersects exactly one (-2)-curve, transversally. After contracting E, then the image of the (-2)-curve we get two rational curves of self intersections (-1) or (-2), intersecting each other with multiplicity bigger than one, which is impossible on a surface of general type. □

Remark 6.10. Observe that if we arrive, after contracting one or more exceptional curves, to a configuration as in the previous lemma with maybe singular (-4) resp-(-3)-curves, the same argument applies, showing that on a surface of general type there cannot be more (-1)-curves.

We can now prove that all surfaces in table 1 are not of general type.

Lemma 6.11. *The product-quotient surfaces 1), 2), 3) in table 1 are not of general type.*

Proof. In this case the basket is $\{\frac{1}{6}, \frac{5}{6}, 2 \times \frac{1}{2}\}$. We have 8 curves $A_, \ldots, A_8$, which we order in a natural way, such that $A_1^2 = -6$, and A_7, A_8 are the inverse images of the nodes.

Assume that there exist a (-1)-curve E. In the notation of Proposition 6.4

$$b = \begin{pmatrix} 4 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}.$$

We notice again that E cannot intersect two different (-2)-curves, so by Proposition 6.6 and corollary 6.8, $BA_1 \geq 2$. But then E cannot intersect $A_2, \ldots, A_6$ since else, after contracting it we could contract enough other curves intersecting the image of A_1 to contradict proposition 6.7. The possibilities left for e are thus

$$a) \begin{pmatrix} 3 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \quad b) \begin{pmatrix} 2 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \end{pmatrix}, \quad c) \begin{pmatrix} 2 \\ 0 \\ \cdot \\ \cdot \\ 1 \\ 0 \end{pmatrix}.$$

Note that the second and third case are symmetric, one obtained from the other exchanging the two nodes. Therefore it suffices to treat only the cases a) and b).

Applying proposition 6.4 and substituting $EK_S = E^2 = -1$ in equations 6, 5 we get in each of the three cases:

(1) here $\Theta_1 = \Theta_2 = \frac{1}{12}$ and
 a) $\mu_1 + \mu_2 = 12$, $\mu_1\mu_2 = 48$,
 b) $\mu_1 + \mu_2 = 4$, $\mu_1\mu_2 = 16$;
(2) here $\Theta_1 = \frac{1}{12}$, $\Theta_2 = \frac{2}{15}$ and
 a) $8\mu_1 + 5\mu_2 = 60$, $\mu_1\mu_2 = 30$,
 b) $8\mu_1 + 5\mu_2 = 20$, $\mu_1\mu_2 = 10$;
(3) here $\Theta_1 = \frac{1}{12}$, $\Theta_2 = \frac{1}{3}$ and
 a) $4\mu_1 + \mu_2 = 12$, $\mu_1\mu_2 = 12$,
 b) $4\mu_1 + \mu_2 = 4$, $\mu_1\mu_2 = 4$.

In all cases there are no integral solutions, a contradiction. □

Lemma 6.12. *The product-quotient surface 4) in table 1 is not of general type.*

Proof. Here the basket is $\{2 \times \frac{1}{5}, 2 \times \frac{4}{5}\}$. Assume that S is of general type. Since $K_S^2 = -1$ there must be a (-1)-curve E on S. E has to intersect at least one (-5)-curve, and cannot intersect any rational (-2)-curve (or, as in the previous proof, after contracting it, we could contract enough curves to contradict Proposition 6.7).

So E passes twice through one of the (-5)-curves and at least once through the other. After contracting E we get a surface S' with a configuration of rational curves as in Remark 6.10. Therefore S' is minimal, a contradiction, since $K_{S'}^2 = -1$. □

Lemma 6.13. *The product-quotient surfaces 5), 6), 7), 8), 9) in table 1 are not of general type.*

Proof. Here the basket is $\{2 \times \frac{1}{4}, 2 \times \frac{3}{4}\}$ or $\{4 \times \frac{2}{5}\}$. In all cases, if S was of general type, it would be minimal by Lemma 6.9. A contradiction, since in all cases$K_S^2 = 0$. □

Lemma 6.14. *The product-quotient surface 10) in table 1 is not of general type.*

Proof. Here the basket is $\{\frac{1}{5}, 2 \times \frac{2}{5}, \frac{4}{5}\}$. Assume that S is of general type. Then S contains a (-1)-curve E. After contracting E, which has to pass at least once through the (-5)-curve and at least once through a (-3)-curve, we get a surface S' with a configuration of rational curves as in Remark 6.10 and we get a contradiction since $K_{S'}^2 = 0$. □

This concludes the proof of Theorem 6.2.

7. Surfaces of general type with $p_g = 0$ and $\gamma > 0$

We shall give now a detailed description of the minimal models of the three product-quotient surfaces of general type with $p_g = 0$ and $\gamma > 0$ which we found running our computer program. In fact, we believe that there are no more non-minimal product-quotient surfaces of general type with $p_g = 0$ left.

7.1. **A numerical Godeaux surface with torsion of order** 4. The group G is the subgroup of order 96 of the permutation group S_8 generated by (123), (12)(34), (57) and (5678)(12).

Its action on $\{1, \dots, 8\}$ has two orbits, $\{1, \dots, 4\}$ and $\{5, \dots, 8\}$. Indeed G is an index 2 subgroup of $S_4 \times D_4$ where S_4 is the permutation group of $\{1, 2, 3, 4\}$, and D_4 is the isometry group of the square, embedded in S_8 by considering its action on the vertices of the square and labeling them counterclockwise as $5, 6, 7, 8$.

The curves C_1 and C_2 are very similar, they are both G-covers of $\mathbb{P}^1$ branched on $\{p_1 = 1, p_2 = 0, p_3 = \infty\}$ with respective generating vectors

- $\{g_1 := (123)(57), g_2 := (4321)(56)(78), g_3 := (g_1 g_2)^{-1})\}$;
- $\{g_1' := (123)(57), g_2' := (4321)(5678), g_3' := (g_1' g_2')^{-1})\}$.

Their respective signatures are $(0; 6, 4, 4)$ and $(0; 6, 4, 2)$.

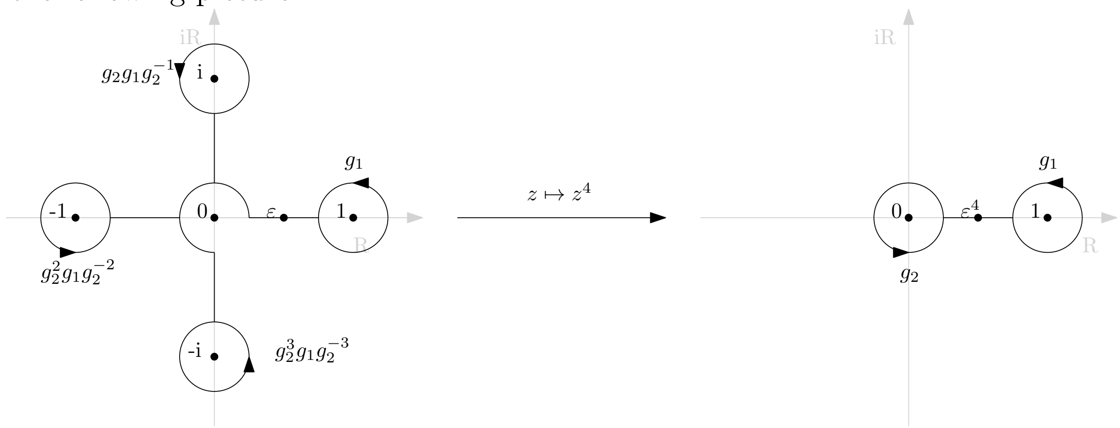
Our computer program shows that

Proposition 7.1. *The product-quotient surface S with quotient model $X = (C_1 \times C_2)/G$ above has $p_g = q = K_S^2 = 0$, $\pi_1(X) \cong \mathbb{Z}_4$ and $\gamma = 1$. The basket of singularities of X is $\{2 \times \frac{1}{6}, \frac{2}{3}, 2 \times \frac{1}{2}\}$. All singular points of X are mapped onto $(1, 1)$ by the natural map $X = (C_1 \times C_2)/G \to \mathbb{P}^1 \times \mathbb{P}^1 = C_1/G \times C_2/G$.*

We consider the map $\mathbb{P}^1 \xrightarrow{z\mapsto z^4} \mathbb{P}^1$, and the normalization of the fibre product as in the following commutative diagram (for $i = 1, 2$):

$$\begin{array}{ccc} C_i' & \xrightarrow{\zeta_i} & C_i \\ \downarrow{\scriptstyle \lambda_i} & & \downarrow \\ \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1 \end{array}$$

Then λ_i is a $G-$cover of $\mathbb{P}^1$ branched in the $4-$th roots of unity. Lifting loops as in the following picture



we see that λ_1 is the $G-$cover with generating vector $\{g_1, g_2g_1g_2^{-1}, g_2^2g_1g_2^{-2}, g_3g_1g_2^{-3}\}$ and λ_2 is the $G-$cover with the analogous generating vector obtained substituting g_i with g_i'.

Remark 7.2. It is worth mentioning that here the word "generating" is a slight abuse of notation, since the above elements do not generate the whole group G. This implies that C' is not connected, the number of connected components being the index of the subgroup generated by $\{g_1, g_2g_1g_2^{-1}, g_2^2g_1g_2^{-2}, g_3g_1g_2^{-3}\}$ in G; this does not affect in any way our argument.

The reader can easily check that the two generating vectors coincide, so λ_1 and λ_2 are isomorphic $G-$covers. In particular, we have a map $\zeta' : \Gamma \cong C_1' \cong C_2' \xrightarrow{(\zeta_1,\zeta_2)} C_1 \times C_2$ which is $G-$equivariant, hence induces a morphism on the quotient

$$\zeta \colon \Gamma/G \cong \mathbb{P}^1 \to X = (C_1 \times C_2)/G$$

and $E' := \zeta(\mathbb{P}^1)$ is a rational curve on X.

Denote by A_1, A_2 the inverse images of the singularities $\frac{1}{6}$ in S, and by E the strict transform of E'.

Proposition 7.3. *E is a smooth rational curve with $K_SE = E^2 = -1$. Moreover, $E(A_1 + A_2) = 4$, and $EA_i = 0$ for every further exceptional curve A_i of σ.*

Proof. First of all, let us show that ζ is generically injective. In fact, composing with the map $X \to \mathbb{P}^1 \times \mathbb{P}^1$ we get the map $\mathbb{P}^1 \xrightarrow{z \mapsto z^4} \mathbb{P}^1 \times \mathbb{P}^1$; this shows that ζ is d-to-1 for a positive integer d which is a divisor of 4.

On the other hand, since all singular points of X lie over $(1,1)$, this also shows that only the 4^{th} roots of unity may be mapped to singular points of X. So E' will pass at most $\frac{4}{d}$ times through singular points of X, and we get by Proposition 6.6 that $d = 1$.

The smoothness of E follows easily by a local computation. The only points of E' contained in $\operatorname{Sing} X$, the 4^{th} roots of unity, have stabilizer of order 6, so they are mapped to singular points of multiplicity 6. This implies $E(A_1 + A_2) = 4$ and $EA_i = 0$ for every further exceptional curve of σ.

Then $E = \sigma^* E' - \frac{a_1}{6} A_1 - \frac{a_2}{6} A_2$ with $a_1 + a_2 = 4$. Moreover $K_X E' = \frac{K_{C_1 \times C_2} \zeta'(\Gamma)}{|G|} = \frac{4|G|\Theta_1 + 4|G|\Theta_2}{|G|} = 4(\Theta_1 + \Theta_2) = \frac{5}{3}$. Therefore

$$K_S E = K_X E' - \frac{a_1}{6} K_S A_1 - \frac{a_2}{6} K_S A_2 = \frac{5}{3} - \frac{4}{6}(a_1 + a_2) = -1.$$

□

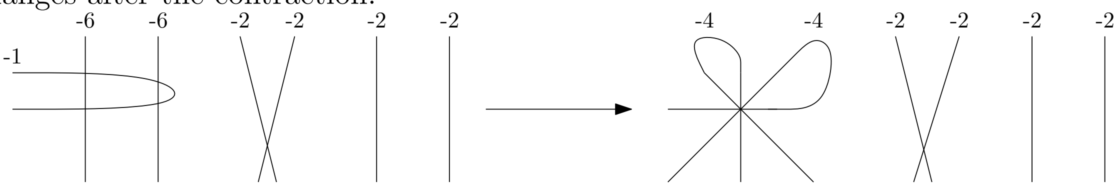
Finally we can prove

Theorem 7.4. *Contracting E we get a minimal surface. In particular the minimal model of S is a numerical Godeaux surface with torsion of order* 4.

Proof. Since $K^2_{\overline{S}} > 0$, $q = 0$, $\pi_1(\overline{S}) \neq 0$, by the Enriques-Kodaira classification $\overline{S}$ is of general type.

By corollary 6.8, $EA_1, EA_2 \leq 3$, so (EA_1, EA_2) equals either $(2,2)$, or $(3,1)$, or $(1,3)$.

In the first case, the following picture describes how the configuration of curves changes after the contraction.



The minimality follows then directly by remark 6.10. A similar argument gives the minimality in the other two cases. □

7.2. **A numerical Godeaux surface with torsion of order** 5. The group G is $\mathbb{Z}_5^2$. The curves C_1 and C_2 are two G-covers of $\mathbb{P}^1$ branched on $\{p_1 = 1, p_2 = 0, p_3 = \infty\}$ with respective generating vectors

- $\{g_1 := (1,0), g_2 := (0,1), g_3 := (g_1 g_2)^{-1})\}$;
- $\{g'_1 := (1.0), g'_2 := (1,1), g'_3 := (g'_1 g'_2)^{-1})\}$.

Both signatures are $(0; 5, 5, 5)$. Our computer program shows that

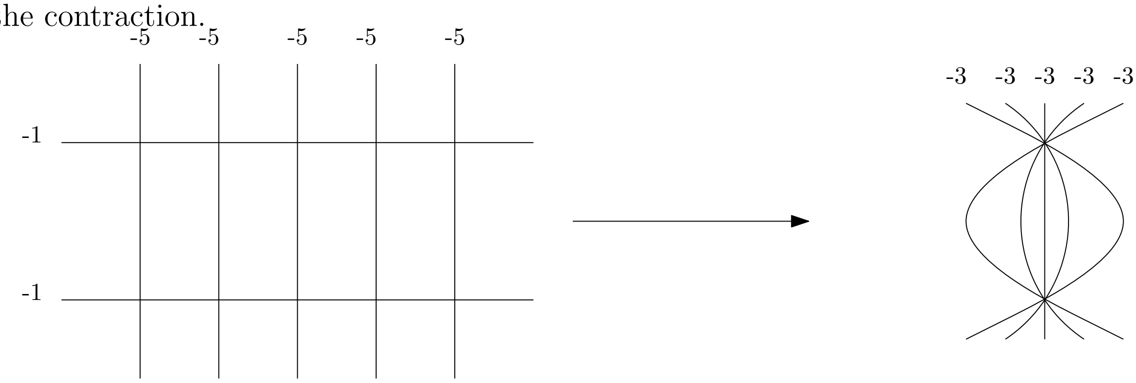
Proposition 7.5. *The product-quotient surface S with quotient model $X = (C_1 \times C_2)/G$ above has $p_g = q = K_S^2 = -1$, $\pi_1(X) \cong \mathbb{Z}_5$ and $\gamma = 2$. The basket of singularities of X is $\{5 \times \frac{1}{5}\}$. All singular points of X are mapped onto $(1,1)$ by the natural map $X = (C_1 \times C_2)/G \to \mathbb{P}^1 \times \mathbb{P}^1 = C_1/G \times C_2/G$.*

Since all singularities lie over $(1,1)$, they all lie in the same fibre of each of the two isotrivial fibrations, whose central components we denote by E_1 and E_2 rspectively. They are $\mathbb{Z}_5$-quotients of C_2 resp. C_1 with 5 branching points; Hurwitz' formula shows that both E_1 and E_2 are rational. By Proposition 5.6 $E_1^2 = E_2^2 = -5\frac{1}{5} = -1$. So both curves are exceptional divisors of the first kind.

Theorem 7.6. *Contracting E_1 and E_2 we get a minimal surface. In particular, the minimal model of S is a numerical Godeaux surface with torsion of order 5.*

Proof. Since $K_{\overline{S}}^2 > 0$, $q = 0$, $\pi_1(\overline{S}) \neq 0$, by the Enriques-Kodaira classification $\overline{S}$ is of general type.

The following picture describes how the configuration of curves changes after the contraction.



The minimality follows then directly by remark 6.10. □

7.3. **Are there more product-quotient surfaces of general type with $p_g = 0$?** By the results in[BP12] and Theorem 6.2 there are exactly 72 families of surfaces of general type with $p_g = \gamma = 0$. By Proposition 4.2 all missing product-quotient surfaces of general type have $\gamma > 0$. We know three examples of them, the *fake Godeaux* described in [BP12] (with $K_S^2 = \gamma = 1$, $K_{\overline{S}}^2 = 3$), and the two numerical Godeaux surfaces described in this section.

We can prove the following

Proposition 7.7. *Let S be a product-quotient surface of general type with $p_g = 0$ not among the 75 families just mentioned. Then*

- *either $\gamma \geq 4$,*
- *or $\gamma = 3$ and X has a singular point of multiplicity at least 14,*
- *or $\gamma = 2$ and X has a singular point of multiplicity at least 45.*

The proof is obtained by running our program for $\gamma = 1$ and multiplicity up to 54 (the maximal possible value by Proposition 5.13), $\gamma = 2$ and multiplicity up to

TABLE 2. Product-quotient surfaces not of general type with $p_g = q = 0$, $\gamma = 1$

γ	K_S^2	Sing X	t_1	t_2	G
1	-2	$4 \times \frac{1}{2}, 4 \times \frac{1}{4}$	$4,4,4$	$4,4,4$	(16,2)
1	-3	$2 \times \frac{1}{2}, \frac{1}{3}, 2 \times \frac{2}{3}, 2 \times \frac{1}{6}$	$2,6,6$	$2,6,6$	(48,49)
1	-3	$4 \times \frac{1}{2}, \frac{1}{7}, 2 \times \frac{2}{7}$	$2,3,7$	$4,4,7$	(168,42)
1	-3	$4 \times \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{5}{8}$	$2,4,8$	$4,4,8$	(32,11)
1	-4	$6 \times \frac{1}{2}, \frac{2}{3}, 2 \times \frac{1}{6}$	$2,4,6$	$2,2,2,6$	(24,8)
1	-4	$2 \times \frac{1}{3}, 3 \times \frac{2}{3}, 2 \times \frac{1}{6}$	$3,3,6$	$3,3,6$	(36,11)
1	-4	$2 \times \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{7}, 2 \times \frac{2}{7}$	$2,3,7$	$3,4,7$	(168,42)
1	-4	$7 \times \frac{1}{2}, \frac{1}{8}, \frac{3}{8}$	$2,3,8$	$2,2,2,8$	(48,29)
1	-4	$2 \times \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{1}{8}, \frac{5}{8}$	$2,3,8$	$3,4,8$	(96,64)
1	-5	$2 \times \frac{1}{3}, 2 \times \frac{2}{3}, \frac{1}{7}, 2 \times \frac{2}{7}$	$2,3,7$	$3,3,7$	(168,42)
1	-5	$2 \times \frac{1}{3}, 2 \times \frac{2}{3}, \frac{1}{7}, 2 \times \frac{2}{7}$	$3,3,7$	$3,3,7$	(21,1)
1	-5	$2 \times \frac{1}{2}, 2 \times \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{5}{8}$	$2,4,8$	$2,4,8$	(64,32)
1	-8	$2 \times \frac{3}{4}, 2 \times \frac{1}{8}, 2 \times \frac{5}{8}$	$2,8,8$	$2,8,8$	(16,5)
1	-8	$4 \times \frac{1}{2}, 2 \times \frac{3}{4}, \frac{1}{12}, \frac{5}{12}$	$2,4,12$	$2,4,12$	(24,5)

44 (here the maximal value by 5.13 is 78, $\gamma = 3$ and multiplicity up to 13, and then by showing case by case that the resulting surface is not of general type.

The full list of the cases to consider is the tables 2, 3, 4 and 5. Note that in the last column we list only the SmallGroup identifier of the MAGMA database of groups up to order 2000, i.e. (n,m) means the m-th group of order n.

We skip the details of the proof, which is rather long (since the cases are many) and most of the times straightforward, repeating arguments already used in this paper. Still in a few cases quite some effort is needed to show that the surface is not of general type. Unfortunately, we do not know a systematic way to prove that certain product-quotient surfaces cannot be of general type.

Remark 7.8. Note that the surfaces of general type we have found have singular points of multiplicity much smaller than the bounds in Proposition 7.7, giving some evidence to the conjecture that there are no other examples. Still, we can't prove it without proving first Conjecture 4.5 at least in the case $p_g = q = 0$, finding $\Gamma(0,0)$ explicitly.

TABLE 3. Product-quotient surfaces not of general type with $p_g = q = 0$, $K^2 \geq -8$, $\gamma = 2$ and singularities of multiplicity at most 44

γ	K_S^2	Sing X	t_1	t_2	G
2	-3	$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, 2 \times \frac{1}{8}$	$2,3,8$	$4,6,8$	(192,181)
2	-4	$5 \times \frac{1}{2}, \frac{1}{4}, 2 \times \frac{1}{8}$	$2,4,8$	$4,4,8$	(64,8)
2	-5	$3 \times \frac{1}{2}, 3 \times \frac{1}{3}, 3 \times \frac{1}{6}$	$3,6,6$	$3,6,6$	(18,5)
2	-6	$8 \times \frac{1}{3}, 2 \times \frac{1}{6}$	$3,3,6$	$3,3,6$	(36,11)
2	-6	$7 \times \frac{1}{2}, \frac{1}{4}, 2 \times \frac{1}{8}$	$2,4,8$	$2,4,8$	(128,75)
2	-6	$\frac{1}{4}, \frac{3}{4}, 2 \times \frac{1}{8}, 2 \times \frac{3}{8}$	$2,4,8$	$2,2,8,8$	(32,9)
2	-6	$\frac{1}{2}, 2 \times \frac{1}{3}, 2 \times \frac{2}{3}, \frac{1}{4}, 2 \times \frac{1}{8}$	$2,3,8$	$3,4,8$	(192,181)
2	-6	$3 \times \frac{1}{2}, 2 \times \frac{1}{4}, \frac{3}{4}, 2 \times \frac{1}{8}$	$2,4,8$	$2,2,4,8$	(32,9)
2	-6	$7 \times \frac{1}{2}, 2 \times \frac{1}{5}, \frac{1}{10}$	$2,5,10$	$2,5,10$	(50,3)
2	-6	$4 \times \frac{1}{2}, 2 \times \frac{1}{4}, \frac{1}{12}, \frac{5}{12}$	$2,4,12$	$2,2,4,12$	(24,5)
2	-6	$2 \times \frac{1}{2}, 4 \times \frac{1}{3}, \frac{1}{12}, \frac{7}{12}$	$2,3,12$	$3,6,12$	(72,42)
2	-7	$2 \times \frac{1}{3}, 2 \times \frac{2}{3}, 5 \times \frac{1}{5}$	$3,3,5$	$3,3,5$	(75,2)
2	-7	$5 \times \frac{1}{2}, 3 \times \frac{1}{3}, 3 \times \frac{1}{6}$	$2,6,6$	$2,6,6$	(36,12)
2	-7	$5 \times \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, 2 \times \frac{1}{8}$	$2,3,8$	$2,6,8$	192,181)
2	-7	$2 \times \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, 2 \times \frac{1}{4}, \frac{1}{12}, \frac{5}{12}$	$2,3,12$	$4,12,12$	(48,33)
2	-8	$4 \times \frac{1}{2}, 2 \times \frac{2}{3}, 4 \times \frac{1}{6}$	$2,4,6$	$2,2,6,6$	(24,8)
2	-8	$4 \times \frac{1}{2}, 2 \times \frac{2}{3}, 4 \times \frac{1}{6}$	$2,6,6$	$2,2,6,6$	(12,5)
2	-8	$6 \times \frac{1}{2}, 2 \times \frac{1}{8}, 2 \times \frac{3}{8}$	$2,4,8$	$2,2,8,8$	(16,8)
2	-8	$6 \times \frac{1}{2}, 2 \times \frac{1}{8}, 2 \times \frac{3}{8}$	$2,3,8$	$2,2,8,8$	(48,29)
2	-8	$9 \times \frac{1}{2}, \frac{1}{4}, 2 \times \frac{1}{8}$	$2,4,8$	$2,4,8$	(64,8)
2	-8	$2 \times \frac{1}{2}, 2 \times \frac{1}{4}, 2 \times \frac{1}{8}, 2 \times \frac{5}{8}$	$2,8,8$	$2,8,8$	(32,5)
2	-8	$2 \times \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, 2 \times \frac{1}{8}, 2 \times \frac{3}{8}$	$2,8,8$	$4,8,8$	(16,5)
2	-8	$6 \times \frac{1}{2}, \frac{4}{5}, 2 \times \frac{1}{10}$	$2,4,10$	$2,2,2,10$	(40,8)
2	-8	$4 \times \frac{1}{2}, 4 \times \frac{1}{3}, \frac{1}{12}, \frac{7}{12}$	$2,6,12$	$2,6,12$	(24,10)
2	-8	$2 \times \frac{1}{2}, 4 \times \frac{1}{3}, 2 \times \frac{2}{3}, \frac{1}{4}, \frac{1}{12}$	$2,3,12$	$2,3,12$	(192,194)
2	-8	$4 \times \frac{1}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{12}, \frac{7}{12}$	$2,3,12$	$3,4,12$	(72,42)
2	-8	$2 \times \frac{3}{20}, 2 \times \frac{1}{4}, 6 \times \frac{1}{2}$	$2,4,20$	$2,4,20$	(40,5)

TABLE 4. Product-quotient surfaces not of general type with $p_g = q = 0$, $K^2 \leq -9$, $\gamma = 2$ and singularities of multiplicity at most 44

γ	K_S^2	Sing X	t_1	t_2	G
2	-9	$5 \times \frac{1}{5}, 4 \times \frac{2}{5}$	$5,5,5$	$5,5,5$	(5,1)
2	-9	$\frac{1}{3}, \frac{2}{3}, 2 \times \frac{1}{7}, 4 \times \frac{2}{7}$	$2,3,7$	$3,7,7$	(168,42)
2	-9	$4 \times \frac{1}{2}, 5 \times \frac{1}{4}, \frac{1}{8}, \frac{5}{8}$	$2,4,8$	$2,4,8$	(32,11)
2	-9	$\frac{1}{3}, \frac{2}{3}, 2 \times \frac{1}{4}, 2 \times \frac{1}{8}, 2 \times \frac{5}{8}$	$2,3,8$	$3,8,8$	(96,64)
2	-9	$7 \times \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, 2 \times \frac{1}{5}, \frac{1}{10}$	$2,3,10$	$2,3,10$	(150,5)
2	-9	$4 \times \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, 2 \times \frac{1}{4}, \frac{1}{12}, \frac{5}{12}$	$2,12,12$	$3,4,12$	(12,2)
2	-9	$2 \times \frac{1}{3}, 2 \times \frac{2}{3}, \frac{1}{13}, 2 \times \frac{3}{13}$	$3,3,13$	$3,3,13$	(39,1)
2	-9	$\frac{1}{13}, 2 \times \frac{3}{13}, 2 \times \frac{1}{3}, 2 \times \frac{2}{3}$	$3,3,13$	$3,3,13$	(39,1)
2	-10	$2 \times \frac{1}{3}, 4 \times \frac{2}{3}, 4 \times \frac{1}{6}$	$3,6,6$	$3,6,6$	(6,2)
2	-10	$4 \times \frac{1}{2}, 2 \times \frac{1}{7}, 4 \times \frac{2}{7}$	$2,3,7$	$2,7,7$	(168,42)
2	-10	$2 \times \frac{1}{2}, 2 \times \frac{1}{3}, 2 \times \frac{2}{3}, 2 \times \frac{1}{8}, 2 \times \frac{3}{8}$	$2,3,8$	$3,8,8$	(48,29)
2	-10	$7 \times \frac{1}{2}, 2 \times \frac{1}{4}, \frac{3}{4}, 2 \times \frac{1}{8}$	$2,4,8$	$2,4,8$	(32,9)
2	-10	$8 \times \frac{1}{2}, 2 \times \frac{1}{8}, 2 \times \frac{3}{8}$	$2,8,8$	$2,8,8$	(8,1)
2	-10	$4 \times \frac{1}{2}, 2 \times \frac{1}{4}, 2 \times \frac{1}{8}, 2 \times \frac{5}{8}$	$2,8,8$	$2,8,8$	(16,6)
2	-10	$4 \times \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, 2 \times \frac{1}{8}, 2 \times \frac{3}{8}$	$2,8,8$	$4,8,8$	(8,1)
2	-10	$5 \times \frac{1}{2}, 2 \times \frac{1}{3}, 2 \times \frac{2}{3}, \frac{1}{4}, 2 \times \frac{1}{8}$	$2,3,8$	$2,3,8$	(192,181)
2	-10	$2 \times \frac{1}{2}, 2 \times \frac{1}{5}, 3 \times \frac{2}{5}, \frac{1}{10}, \frac{3}{10}$	$2,5,10$	$5,10,10$	(10,2)
2	-10	$6 \times \frac{1}{2}, 4 \times \frac{1}{3}, \frac{1}{12}, \frac{7}{12}$	$2,3,12$	$2,3,12$	(72,42)
2	-10	$5 \times \frac{1}{2}, 3 \times \frac{2}{3}, \frac{1}{4}, \frac{1}{6}, \frac{1}{12}$	$2,3,12$	$2,6,12$	(48,33)
2	-10	$8 \times \frac{1}{2}, 2 \times \frac{1}{4}, \frac{1}{12}, \frac{5}{12}$	$2,4,12$	$2,4,12$	(24,5)
2	-11	$4 \times \frac{1}{2}, \frac{1}{5}, 2 \times \frac{4}{5}, 2 \times \frac{1}{10}$	$2,10,10$	$2,10,10$	(20,5)
2	-12	$6 \times \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{4}{5}, 2 \times \frac{1}{10}$	$2,4,10$	$2,4,10$	(40,8)
2	-12	$2 \times \frac{1}{2}, 4 \times \frac{2}{5}, \frac{4}{5}, 2 \times \frac{1}{10}$	$2,5,10$	$5,10,10$	(10,2)
2	-12	$7 \times \frac{1}{2}, \frac{1}{16}, \frac{1}{4}, \frac{7}{16}, \frac{3}{4}$	$2,4,16$	$2,4,16$	(32,19)
2	-14	$2 \times \frac{5}{6}, 2 \times \frac{1}{12}, 2 \times \frac{7}{12}$	$2,12,12$	$2,12,12$	(24,9)
2	-14	$5 \times \frac{1}{2}, 2 \times \frac{3}{4}, \frac{5}{6}, 2 \times \frac{1}{12}$	$2,4,12$	$2,4,12$	(48,14)
2	-14	$6 \times \frac{1}{2}, 2 \times \frac{3}{4}, \frac{9}{20}, \frac{1}{20}$	$2,4,20$	$2,4,20$	(40,5)

TABLE 5. Product-quotient surfaces not of general type with $p_g = q = 0$, $\gamma = 3$ and singularities of multiplicity at most 13

γ	K_S^2	Sing X	t_1	t_2	G
3	-9	$7 \times \frac{1}{3}, 4 \times \frac{1}{6}$	$3, 6, 6$	$3, 6, 6$	(18,3)
3	-10	$12 \times \frac{1}{4}$	$4, 4, 4$	$4, 4, 4$	(16,2)
3	-10	$3 \times \frac{2}{3}, 6 \times \frac{1}{6}$	$3, 3, 6$	$3, 3, 6$	(108,22)
3	-12	$2 \times \frac{1}{2}, 3 \times \frac{2}{3}, 6 \times \frac{1}{6}$	$2, 6, 6, 6$	$3, 6, 6$	(6,2)
3	-12	$7 \times \frac{1}{2}, 2 \times \frac{1}{3}, 5 \times \frac{1}{6}$	$2, 6, 6$	$2, 6, 6$	(36,12)
3	-15	$5 \times \frac{1}{7}, 4 \times \frac{3}{7}$	$7, 7, 7$	$7, 7, 7$	(7,1)
3	-13	$3 \times \frac{1}{7}, 6 \times \frac{2}{7}$	$2, 3, 7$	$7, 7, 7$	(168,42)
3	-13	$3 \times \frac{1}{7}, 6 \times \frac{2}{7}$	$7, 7, 7$	$7, 7, 7$	(7,1)
3	-12	$3 \times \frac{1}{8}, 3 \times \frac{3}{8}, 5 \times \frac{1}{2}$	$2, 3, 8$	$2, 8, 8, 8$	(48,29)
3	-10	$2 \times \frac{1}{8}, 5 \times \frac{1}{4}, 5 \times \frac{1}{2}$	$2, 4, 8$	$2, 4, 8$	(128,75)
3	-16	$4 \times \frac{1}{8}, 2 \times \frac{3}{4}, 8 \times \frac{1}{2}$	$2, 8, 8$	$2, 8, 8$	(16,5)
3	-12	$2 \times \frac{1}{8}, 5 \times \frac{1}{4}, 7 \times \frac{1}{2}$	$2, 4, 8$	$2, 4, 8$	(64,8)
3	-6	$2 \times \frac{1}{8}, 5 \times \frac{1}{4}, \frac{1}{2}$	$2, 8, 8$	$2, 8, 8$	(64,6)
3	-10	$4 \times \frac{1}{8}, 2 \times \frac{3}{4}, 2 \times \frac{1}{2}$	$2, 4, 8, 8$	$2, 8, 8$	(16,5)
3	-12	$4 \times \frac{1}{8}, 2 \times \frac{3}{4}, 4 \times \frac{1}{2}$	$2, 4, 8$	$2, 2, 8, 8$	(32,9)
3	-12	$4 \times \frac{1}{8}, 2 \times \frac{3}{4}, 4 \times \frac{1}{2}$	$2, 8, 8$	$2, 2, 8, 8$	(16,5)
3	-8	$4 \times \frac{1}{8}, 2 \times \frac{3}{4}$	$2, 4, 8$	$2, 2, 2, 8, 8$	(32,9)
3	-16	$4 \times \frac{1}{8}, 2 \times \frac{1}{4}, 4 \times \frac{3}{4}$	$4, 8, 8$	$4, 8, 8$	(8,1)
3	-12	$2 \times \frac{1}{8}, 6 \times \frac{1}{4}, 2 \times \frac{5}{8}$	$4, 8, 8$	$4, 8, 8$	(8,1)
3	-13	$2 \times \frac{1}{9}, 2 \times \frac{2}{9}, 5 \times \frac{1}{3}, 2 \times \frac{2}{3}$	$3, 9, 9$	$3, 9, 9$	(9,1)
3	-9	$3 \times \frac{1}{9}, 2 \times \frac{1}{3}, 3 \times \frac{2}{3}$	$3, 3, 9$	$3, 3, 9$	(81,9)
3	-13	$2 \times \frac{1}{9}, 3 \times \frac{2}{9}, 3 \times \frac{1}{3}, \frac{4}{9}$	$3, 9, 9$	$9, 9, 9$	(9,1)
3	-6	$3 \times \frac{1}{9}, 2 \times \frac{2}{3}, \frac{1}{3}$	$3, 9, 9$	$3, 9, 9$	(27,2)
3	-12	$\frac{1}{12}, 5 \times \frac{1}{4}, 4 \times \frac{1}{3}, 2 \times \frac{2}{3}$	$3, 4, 12$	$3, 4, 12$	(12,2)
3	-11	$\frac{1}{12}, 2 \times \frac{1}{6}, \frac{1}{4}, \frac{3}{4}, 3 \times \frac{1}{3}, \frac{7}{12}$	$3, 12, 12$	$4, 6, 12$	(12,2)
3	-14	$2 \times \frac{1}{12}, 2 \times \frac{1}{3}, 2 \times \frac{7}{12}, 8 \times \frac{2}{3}$	$2, 3, 12$	$3, 12, 12$	(72,42)
3	-12	$2 \times \frac{1}{12}, 2 \times \frac{5}{12}, 6 \times \frac{1}{2}$	$2, 4, 12$	$2, 2, 12, 12$	(24,5)
3	-12	$\frac{1}{12}, \frac{1}{6}, \frac{1}{4}, 6 \times \frac{1}{3}, 5 \times \frac{1}{2}$	$2, 3, 12$	$2, 3, 12$	(144,27)
3	-16	$2 \times \frac{1}{12}, 2 \times \frac{1}{4}, 9 \times \frac{1}{2}, \frac{5}{6}$	$2, 4, 12$	$2, 4, 12$	(48,14)
3	-16	$2 \times \frac{1}{12}, 2 \times \frac{5}{12}, 10 \times \frac{1}{2}$	$2, 12, 12$	$2, 12, 12$	(12,2)
3	-13	$\frac{1}{12}, 2 \times \frac{1}{6}, 3 \times \frac{1}{3}, 6 \times \frac{1}{2}, \frac{7}{12}$	$2, 6, 12$	$2, 6, 12$	(24,10)
3	-13	$\frac{1}{12}, 2 \times \frac{1}{6}, 3 \times \frac{1}{3}, 6 \times \frac{1}{2}, \frac{7}{12}$	$2, 3, 12$	$2, 3, 12$	(216,92)
3	-15	$\frac{1}{12}, 2 \times \frac{1}{4}, \frac{1}{3}, 6 \times \frac{2}{3}$	$2, 3, 12$	$3, 12, 12$	(48,33)
3	-14	$2 \times \frac{1}{12}, 2 \times \frac{1}{4}, 2 \times \frac{1}{3}, \frac{1}{2}, 2 \times \frac{2}{3}, \frac{5}{6}$	$3, 4, 12$	$6, 12, 12$	(12,2)
3	-16	$2 \times \frac{1}{12}, \frac{1}{6}, 2 \times \frac{5}{12}, 4 \times \frac{1}{2}, \frac{5}{6}$	$2, 12, 12$	$6, 12, 12$	(12,2)
3	-8	$2 \times \frac{1}{12}, 2 \times \frac{1}{4}, \frac{1}{2}, \frac{5}{6}$	$2, 4, 12$	$2, 2, 4, 12$	(48,14)
3	-11	$2 \times \frac{1}{12}, 2 \times \frac{1}{4}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{5}{6}$	$2, 12, 12$	$4, 6, 12$	(24,9)

8. Upper bounds for γ under some additional hypotheses

In this section we will give some evidence to Conjecture 4.5, establishing an upper bound $\Gamma(p_g, q)$ for γ for product-quotient surfaces of general type under some additional hypotheses.

Write

$$K_S = P + N = \sigma^* K_X - \mathcal{A},$$

where $P + N$ is the Zariski decomposition of the canonical divisor of the product-quotient surface S.

Remark 8.1. By construction P, $\sigma^* K_X$ are nef, N, $\mathcal{A}$ are effective; and

$$PN = \sigma^* K_X \mathcal{A} = 0.$$

In particular, $K_S^2 = P^2 + N^2 = K_X^2 + \mathcal{A}^2$.

Recall that

- $K_S^2 = 8\chi - 2\gamma - l$;
- $1 \leq P^2 \in \mathbb{N}$;
- $\nu := -N^2$ is the number of (-1)-cycles on S;
- $-\mathcal{A}^2 = K_S \mathcal{A} = 6\gamma + l - 2\mu \geq 0$.

Lemma 8.2. *$\forall \delta \geq 0$ such that $\delta\nu \leq N\sigma^* K_X$ holds*

$$8\chi + \left(\frac{6}{1+\delta} - 2\right)\gamma - \frac{\delta}{1+\delta} l - \frac{2\mu}{1+\delta} \geq 1 \tag{7}$$

Proof. From the assumption $N\sigma^* K_X \geq \nu\delta$ follows $K_S \mathcal{A} \geq N\mathcal{A} = N(\sigma^* K_X - K_S) \geq \nu(1 + \delta)$ and therefore

$$\nu \leq \frac{6\gamma + l - 2\mu}{1 + \delta}$$

so

$$1 \leq P^2 = K_S^2 - N^2 \leq 8\chi + \left(\frac{6}{1+\delta} - 2\right)\gamma - \frac{\delta}{1+\delta} l - \frac{2\mu}{1+\delta}$$

□

Remark 8.3. Since K_X is nef, we can set $\delta = 0$ in (7) obtaining a further proof of Corollary 5.4.

If S is minimal, all $\delta > 0$ verify the assumptions, and the statement gives just, when $\delta \to \infty$, the well known $K_S^2 \geq 1$. In the more complicated case $N > 0$, the maximal possible δ is the average of the values of $E\sigma^* K_X$ where E varies over the exceptional divisors of the first kind.

Writing E' for the unique irreducible component with self intersection (-1) of an exceptional divisor of the first kind E, we note that, since $\sigma(E')$ is a curve, K_X is ample and IK_X is Cartier, $E\sigma^* K_X \geq E'\sigma^* K_X \geq \frac{1}{I}$. So equation (7) holds for $\delta = \frac{1}{I}$.

Remark 8.4. When $N\sigma^* K_X \geq (2+\varepsilon)\nu$, $\varepsilon > 0$, Lemma 8.2 implies Conjecture 4.5, since then (7) implies

$$\frac{2\varepsilon}{3+\varepsilon}\gamma \leq 8\chi - 1 - \frac{2+\varepsilon}{3+\varepsilon}l - \frac{2\mu}{3+\varepsilon} \leq 8\chi - 1$$

so

$$\gamma \leq \frac{3+\varepsilon}{2\varepsilon}(8\chi - 1) \tag{8}$$

Unfortunately, in the fake Godeaux case described in [BP12] we have $\nu = 2$, $E_1\sigma^* K_X = 1$, $E_2\sigma^* K_X = 11/7$, hence

$$N\sigma^* K_X = 9/7\nu.$$

Lemma 8.2 gives further evidence to Conjecture 4.5 under the assumption that the Θ_i are not too small. In fact, we have the following result.

Proposition 8.5. *If both Θ_1 and Θ_2 are not smaller than $1 + \frac{\varepsilon}{2}$, $\varepsilon > 0$ then (8) holds.*

Proof. Let again E be an exceptional divisor of the first kind, E' a component of E with self intersection -1. If E' is not contained in one of the fibres, then

$$\sigma^* K_X E \geq \sigma^* K_X E' = \frac{K_{C_1\times C_2}\pi^*\sigma(E')}{|G|} \geq 2(\alpha_1 + \alpha_2)\frac{\Theta_1\Theta_2}{\xi} \geq \Theta_1 + \Theta_2.$$

Else $\sigma(E')$ is the central component of a singular fibre $F_1^{(i)}$ with multiplicity m_i, then

$$\sigma^* K_X E \geq \sigma^* K_X E' = \frac{K_{C_1\times C_2} m_i C_2}{|G|} \geq \frac{\Theta_1\Theta_2}{\xi} 2m_i\alpha_1 \geq m_i\Theta_1 \geq 2\Theta_1.$$

We conclude $E\sigma^* K_X \geq 2+\varepsilon$ and therefore $N\sigma^* K_X \geq (2+\varepsilon)\nu$ and then (8) follows from Lemma 8.2. □

A third type of hypothesis under which Lemma 8.2 implies Conjecture 4.5 is the assumption that $E(F_1 + F_2)$ is big enough. For example, we can show the following.

Proposition 8.6. *Assume that for every exceptional divisor of the first kind E, $E(F_1 + F_2) \geq 42(2+\varepsilon)$. Then (8) holds.*

Proof. Arguing as in the previous proposition

$$\sigma^* K_X E = \frac{K_{C_1\times C_2}\pi^*\sigma(E')}{|G|} \geq \frac{\Theta_1\Theta_2}{\xi} 84(2+\varepsilon)\alpha_{min} \geq 42(2+\varepsilon)\Theta_{min} \geq 2+\varepsilon$$

and we conclude as in the previous case. □

9. The dual surface of a product-quotient surface

In this section we assume furthermore that S is *regular*, i.e., $q(S) = 0$.

Suppose that S is given by a pair of generating vectors: $(a_1, \dots, a_s)$, $(b_1, \dots, b_t)$ of G.

Definition 9.1. *The* dual surface S' of S *is the product-quotient surface given by the pair of generating vectors:* $(a_1, \dots, a_s)$, $(b_t^{-1}, \dots, b_1^{-1})$.

Similarly we will denote by X' *the quotient model of* S'.

Remark 9.2. It is easy to see that $\frac{1}{n}(1, q) \in \mathfrak{B}(X) \iff \frac{1}{n}(1, n-q) \in \mathfrak{B}(X')$.

The numbers of S' are then immediately computed by those of S as follows.

Proposition 9.3. *Let* S *be a regular product-quotient surface, and denote by* S' *its dual surface. Set* $\gamma := \gamma(X)$, $\mu := \mu(X)$, $l := l(X)$, $\gamma' := \gamma(X')$, $\mu' := \mu(X')$, $l' := l(X')$. *Then:*

(1) $\gamma = -\gamma'$;
(2) $\mu = \mu'$,
(3) $\xi = \xi'$;
(4) $p_g(S') = p_g(S) + \gamma$.

Proof. Remark 9.2 describes the basket of the singularities of X' in terms of the basket of X.

Directly by the definition, and proposition 4.4

$$\gamma = -\gamma', \quad \mu = \mu', \quad \xi = \xi'$$

Then

$$\chi(S') = \frac{(g_1 - 1)(g_2 - 1)}{|G|} + \frac{1}{4}(\mu - 2\gamma') = \frac{(g_1 - 1)(g_2 - 1)}{|G|} + \frac{1}{4}(\mu + 2\gamma) =$$
$$= \chi(S) + \gamma.$$

In particular, since we assumed $q(S) = 0$, then

$$p_g(S') = p_g(S) + \gamma.$$

□

Note that this gives an independent proof of Proposition 4.2. Moreover, using Proposition 4.2, we obtain the following:

Corollary 9.4. *The dual surface of a product-quotient surface with* $p_g = 0$ *has maximal Picard number.*

Thus the dual surfaces of the surfaces in table 2 are surfaces with $p_g = 1$ and maximal Picard number. Similarly the dual surfaces of the surfaces in table 3, 4 and 5 are surfaces with maximal Picard number and geometric genus 2 and 3. Summing up, we get more than 100 families of surfaces with maximal Picard number and low genus.

For the index of S resp. S' we have:

Proposition 9.5.

$$\tau(S) := \frac{1}{3}(K_S^2 - 2e(S)) = -\frac{1}{3}B(\mathfrak{B}(X)) = -2\gamma - l,$$

$$\tau(S') := \frac{1}{3}(K_{S'}^2 - 2e(S')) = 2\gamma - l'.$$

Proof. Note that

$$e(S) = 2 + 2p_g(S) + h^{1,1}(S) = 2 + 2p_g(S) + 2 + 2(\gamma + p_g(S)) + l = \\ = 4(1 + p_g(S)) + 2\gamma + l = 4\chi(S) + 2\gamma + l.$$

Therefore

$$\tau(S) := \frac{1}{3}(K_S^2 - 2e(S)) = \frac{1}{3}(8\chi(S) - 2\gamma - l - 2e(S)) = \\ = \frac{1}{3}(8\chi(S) - 2\gamma - l - 8\chi(S) - 4\gamma - 2l) = \\ = \frac{1}{3}(-6\gamma - 3l) = -2\gamma - l.$$

From the previous calculation it follows that $\tau(S') = -2\gamma' - l'$. Using $\gamma' = -\gamma$ we get the second equation. □

Remark 9.6. Let $\bar{S}$ be the minimal model of S, then $\tau(S) + (-N^2) = \tau(\bar{S})$. Moreover, by [Ser96, Proposition 5.1 or 5.3], we know that for the minimal model of a product-quotient surface, the inequality $\tau(\bar{S}) < 0$ holds.

In particular, we get that $l' > 2\gamma$.

It follows immediately from the above:

$$\frac{1}{3}(B(\mathfrak{B}) + B(\mathfrak{B}')) = l + l' = -(\tau(S) + \tau(S')).$$

And it is also easy to see that

$$\frac{1}{3}B(\mathfrak{B}) = l + l' + \tau(S'),$$
$$\frac{1}{3}B(\mathfrak{B}') = l + l' + \tau(S).$$

Remark 9.7. Observe that when we go from S to the dual surface S', we consider on C_1 the same action of G as for S, whereas for C_2 we replace the action $y \mapsto g(y)$ by $y \mapsto \overline{g(\overline{y})}$.

Similarly we can replace $y \mapsto g(y)$ by $y \mapsto g\alpha(y)$ for any (holomorphic) automorphism α of C_2, getting many new surfaces from this construction (depending on the representation theory of G).

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