SEQUENTIALLY COHEN-MACAULAY MIXED PRODUCT IDEALS

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Abstract. We classify the ideals of mixed products that are sequentially Cohen-Macaulay.

1. Introduction

The class of ideals of mixed products is a special class of square-free monomial ideals. They were first introduced by G. Restuccia and R. Villarreal (see [8] and [12]), who studied the normality of such ideals.

In [6], C. Ionescu and G. Rinaldo studied the Castelnuovo-Mumford regularity, the depth and dimension of mixed product ideals and characterize when they are Cohen-Macaulay. In [9] the author calculated the Betti numbers of their finite free resolutions. In [5], L. T. Hoa and N. D. Tam studied these ideals in a broader situation.

Let $S = K[\mathbf{x}, \mathbf{y}]$ be a polynomial ring over a field K in two disjoint sets of variables $\mathbf{x} = \{x_1, \ldots, x_n\}$, $\mathbf{y} = \{y_1, \ldots, y_m\}$. The *ideals of mixed products* are the proper ideals

$$\sum_{i=1}^{s} I_{q_i} J_{r_i} \quad q_i, r_i \in \mathbb{Z}_{\geq 0} \tag{1.1}$$

where I_{q_i} (resp. J_{r_i}) is the ideal of S generated by all the square-free monomials of degree q_i (resp. r_i) in the variables $\mathbf{x}$ (resp. $\mathbf{y}$). We set $I_0 = J_0 = S$ and $I_{q_i} = (0)$ (resp. $J_{r_i} = (0)$) if $q_i > n$ (resp. $r_i > m$). In the articles mentioned only two summands of 1.1 are allowed. In this article we classify the ideals of mixed product that are sequentially Cohen-Macaluay and Cohen-Macaulay for any $s \in \mathbb{N}$. Recently, a number of authors have been interested in classifying sequentially Cohen-Macaulay rings related to combinatorial structures (for example see [3], [4], [11]). This paper is inserted in this area and the tools used are essentially Stanley-Reisner rings and Alexander dual.

In section 2 we recall some preliminaries about simplicial complexes and questions related to commutative algebra. In section 3 we study the primary decomposition of mixed product ideals, we introduce the vectors $\bar{q}$, $\bar{r}$ that

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are uniquely determined by the values q_i, r_i for $i = 1, \dots, s$, m and n, and we classify the Cohen-Macaulay mixed product ideals in terms of the vectors $\bar{q}$ and $\bar{r}$. The vectors $\bar{q}$ and $\bar{r}$ are used also to classify the sequentially Cohen-Macaulay mixed product ideals in the last section.

2. Preliminaries

In this section we recall some concepts on simplicial complexes that we will use in the article (see [1], [7], [10]).

Set $V = \{x_1, \dots, x_n\}$. A *simplicial complex* Δ on the vertex set V is a collection of subsets of V such that (i) $\{x_i\} \in \Delta$ for all $x_i \in V$ and (ii) $F \in \Delta$ and $G \subseteq F$ imply $G \in \Delta$. An element $F \in \Delta$ is called a *face* of Δ. For $F \subset V$ we define the *dimension* of F by $\dim F = |F| - 1$, where $|F|$ is the cardinality of the set F. A maximal face of Δ with respect to inclusion is called a *facet* of Δ. If all facets of Δ have the same dimension, then Δ is called *pure*.

A simplicial complex Δ is called *shellable* if the facets of Δ can be given a linear order $F_1, \dots, F_t$ such that for all $1 \le i < j \le t$, there exist some $v \in F_j \setminus F_i$ and some $k \in \{1, \dots, j-1\}$ with $F_j \setminus F_k = \{v\}$.

Moreover, a pure simplicial complex Δ is *strongly connected* if for every two facets F and G of Δ there is a sequence of facets $F = F_0, F_1, \dots, F_t = G$ such that $\dim(F_i \cap F_{i+1}) = \dim \Delta - 1$ for each $i = 0, \dots, t-1$.

The *Stanley–Reisner ideal* of Δ, denoted by I_Δ, is the squarefree monomial ideal of $S = K[x_1, \dots, x_n]$ generated by

$$\{x_{i_1} x_{i_2} \cdots x_{i_p} : 1 \le i_1 < \cdots < i_p \le n, \ \{x_{i_1}, \dots, x_{i_p}\} \notin \Delta\},$$

and $K[\Delta] = K[x_1, \dots, x_n]/I_\Delta$ is called the *Stanley–Reisner ring* of Δ. It is known that

$$I_\Delta = \bigcap_{F \in \mathcal{F}(\Delta)} P_F \tag{2.1}$$

with $P_F = (\{x_1 \dots, x_n\} \setminus F)$.

Let $I = (\mathbf{x}^{\alpha_1}, \dots, \mathbf{x}^{\alpha_q}) \subset K[\mathbf{x}] = K[x_1, \dots, x_n]$ be a square-free monomial ideal, with $\alpha_i = (\alpha_{i_1}, \dots, \alpha_{i_n}) \in \{0,1\}^n$. The *Alexander dual* of I is the ideal

$$I^* = \bigcap_{i=1}^{q} \mathfrak{m}_{\alpha_i}, \tag{2.2}$$

where $\mathfrak{m}_{\alpha_i} = (x_j : \alpha_{i_j} = 1)$. It is known that $(I^*)^* = I$. We also have that if I, J are squarefree monomial ideals of $S = K[x_1, \dots, x_n]$ then

$$(I + J)^* = I^* \cap J^*. \tag{2.3}$$

3. Cohen-Macaulay mixed product ideals

Let $S = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$ be a polynomial ring over a field K and let

$$\sum_{i=1}^{s} I_{q_i} J_{r_i}, \quad q_i, r_i \in \mathbb{Z}_{\geq 0}, \ s \in \mathbb{N}, \tag{3.1}$$

be an ideal of mixed product as in 1.1. In this section we study the primary decomposition of the ideal 3.1 and give a criterion for its Cohen-Macaulayness. Under the assumption that no summands in 3.1 is a subset of another summand, we set

$$0 \leq q_1 < q_2 < \ldots < q_s \leq n. \tag{3.2}$$

Under this assumption and because of the ordering 3.2 we have

$$0 \leq r_s < r_{s-1} < \ldots < r_1 \leq m. \tag{3.3}$$

Throughout this paper we always assume 3.2 and 3.3.

Proposition 3.1. *Let $S = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$, then*

$$(\sum_{i=1}^{s} I_{q_i} J_{r_i})^* = I_{n-q_1+1} + \sum_{i=1}^{s-1} I_{n-q_{i+1}+1} J_{m-r_i+1} + J_{m-r_s+1}. \tag{3.4}$$

Proof. We prove the assertion by induction on s. If $s = 1$ we have that either $q_1 = 0$ (resp. $r_1 = 0$) and $r_1 \neq 0$ (resp. $q_1 \neq 0$) or $q_1 \neq 0$ and $r_1 \neq 0$. The assertion for the first and the second case follows respectively by Proposition 2.2 and Corollary 2.4 of [9]. Now suppose that

$$(\sum_{i=1}^{s-1} I_{q_i} J_{r_i})^* = I_{n-q_1+1} + (\sum_{i=1}^{s-2} I_{n-q_{i+1}} J_{m-r_i+1}) + J_{m-r_{s-1}+1}.$$

By equation 2.3 we have

$$(I_{q_s} J_{r_s} + \sum_{i=1}^{s-1} I_{q_i} J_{r_i})^* = (I_{q_s} J_{r_s})^* \cap (\sum_{i=1}^{s-1} I_{q_i} J_{r_i})^*$$

that is equal to, by Corollary 2.4 of [9] and induction hypothesis,

$$(I_{n-q_s+1} + J_{m-r_s+1}) \cap (I_{n-q_1+1} + \sum_{i=1}^{s-2} I_{n-q_{i+1}+1} J_{m-r_i+1} + J_{m-r_{s-1}+1}). \tag{3.5}$$

We observe, since $s > 1$ and $q_s > q_i$ for all i, that $q_s \neq 0$. Let $H = I_{n-q_1+1} + \sum_{i=1}^{s-2} I_{n-q_{i+1}+1} J_{m-r_i+1}$. If we apply the modular law to 3.5 we have

$$(I_{n-q_s+1} + J_{m-r_s+1}) \cap H + (I_{n-q_s+1} + J_{m-r_s+1}) \cap J_{m-r_{s-1}+1}.$$

Since by hypothesis $q_i < q_s \leq n$ we have $I_{n-q_s+1} \supset H$ and observing that $J_{m-r_s+1} \supset J_{m-r_{s-1}+1}$, the assertion follows easily. □

Remark 3.2. *By Proposition 3.1 we have that the class of mixed pruduct ideals with a finite set of summand is closed under Alexander duality (see also* [9]*, Remark 2.5).*

Corollary 3.3. *Let* $S = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$ *and let*

$$\mathcal{X}_i = \{X \subset \{x_1, \ldots, x_n\} : |X| = i\}, \mathcal{Y}_j = \{Y \subset \{y_1, \ldots, y_m\} : |Y| = j\},$$

with $\mathcal{X}_i = \emptyset$ *if* $i > n$ *and* $\mathcal{Y}_j = \emptyset$ *if* $j > m$*. Then*

$$\sum_{i=1}^{s} I_{q_i} J_{r_i} = \mathcal{P}_x \cap \mathcal{P}_{xy} \cap \mathcal{P}_y$$

where

$$\mathcal{P}_x = \bigcap_{X \in \mathcal{X}_{n-q_1+1}} (X), \qquad \mathcal{P}_y = \bigcap_{Y \in \mathcal{Y}_{m-r_s+1}} (Y),$$

$$\mathcal{P}_{xy} = \bigcap_{i=1}^{s-1} \left(\bigcap_{X,Y} ((X) + (Y)) \right), \ X \in \mathcal{X}_{n-q_{i+1}+1},\ Y \in \mathcal{Y}_{m-r_i+1}.$$

Proof. By Alexander duality and Proposition 3.1 we have that

$$\sum_{i=1}^{s} I_{q_i} J_{r_i} = (I_{n-q_1+1} + \sum_{i=1}^{s-1} I_{n-q_{i+1}+1} J_{m-r_i+1} + J_{m-r_s+1})^*.$$

By equation 2.2 the assertion follows. □

Corollary 3.4. *Let* $S = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$ *and let* $\sum_{i=1}^{s} I_{q_i} J_{r_i}$ *be the mixed product ideal on the ring* S*. Let* $h = \text{height} \sum_{i=1}^{s} I_{q_i} J_{r_i}$*, then the ideal is unmixed if and only if the following conditions are satisfied:*

1. $m + n - (q_{i+1} + r_i) + 2 = h$, $\forall i = 1, \ldots, s-1$*;*
2. *if* $q_1 > 0$ *then* $n - q_1 + 1 = h$*;*
3. *if* $r_s > 0$ *then* $m - r_s + 1 = h$*.*

Definition 3.5. *Let* $S = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$ *and let* $\sum_{i=1}^{s} I_{q_i} J_{r_i}$ *be the mixed product ideal on the ring* S*. We define* $s' \in \mathbb{N}$ *such that*

$$s' = \begin{cases} s+1 & \text{if } q_1 > 0 \text{ and } r_s > 0 \\ s & \text{if } q_1 > 0 \text{ and } r_s = 0 \text{ or } q_1 = 0 \text{ and } r_s > 0 \\ s-1 & \text{if } q_1 = r_s = 0 \end{cases}$$

and the two vectors $\bar{q} = (q(1), \ldots, q(s'))$, $\bar{r} = (r(1), \ldots, r(s')) \in \mathbb{Z}_{\geq 0}^{s'}$ *such that*

$$q(i) = \begin{cases} q_i - 1 & \text{if } q_1 > 0 \\ q_{i+1} - 1 & \text{if } q_1 = 0 \end{cases}$$

$$r(i) = \begin{cases} r_{i-1} - 1 & \text{if } q_1 > 0 \\ r_i - 1 & \text{if } q_1 = 0 \end{cases}$$

with $i = 1, \ldots, s'$ *and* $r_0 = m + 1$ *and* $q_{s+1} = n + 1$*.*

Proposition 3.6. *Let $S = K[x_1, \dots, x_n, y_1, \dots, y_m]$ and let $I_\Delta = \sum_{i=1}^{s} I_{q_i} J_{r_i}$ be a mixed product ideal. Using the notation of Definition 3.5 there exists a partition of $\mathcal{F}(\Delta)$, $\mathcal{F}(\Delta) = \mathcal{F}_1(\Delta) \cup \ldots \cup \mathcal{F}_{s'}(\Delta)$ such that*

$$\begin{aligned}\mathcal{F}_k(\Delta) = &\{\{x_{i_1}, \dots, x_{i_{q(k)}}, y_{j_1}, \dots, y_{j_{r(k)}}\} : \\ &1 \leq i_1 < \ldots < i_{q(k)} \leq n,\ 1 \leq j_1 < \ldots < j_{r(k)} \leq m\},\end{aligned} \tag{3.6}$$

with $k = 1, \dots, s'$.

Proof. By Corollary 3.3 and equation 2.1 the assertion follows. □

From now on we associate to a mixed product ideal $\sum_{i=1}^{s} I_{q_i} J_{r_i}$ the value $s' \in \mathbb{N}$ and the vectors $\bar{q} = (q(1), \dots, q(s'))$, $\bar{r} = (r(1), \dots, r(s')) \in \mathbb{Z}_{\geq 0}^{s'}$ defined in 3.5. We also give, for the sake of completeness, a way to compute the sequences $0 \leq q_1 < \ldots < q_s \leq n$, $0 \leq r_s < \ldots < r_1 \leq m$ by the vectors $\bar{q} = (q(1), \dots, q(s'))$ and $\bar{r} = (r(1), \dots, r(s'))$.

Definition 3.7. *Let $s' \in \mathbb{N}$, $\bar{q} = (q(1), \dots, q(s'))$, $\bar{r} = (r(1), \dots, r(s')) \in \mathbb{Z}_{\geq 0}^{s'}$, with $0 \leq q(1) < \ldots < q(s') \leq n$, $0 \leq r(s') < \ldots < r(1) \leq m$. We define $s \in \mathbb{N}$ such that*

$$s = \begin{cases} s' - 1 & \text{if } r(1) = m \text{ and } q(s') = n \\ s' & \text{if } r(1) = m \text{ and } q(s') < n \text{ or } r(1) < m \text{ and } q(s') = n \\ s' + 1 & \text{if } r(1) < m \text{ and } q(s') < n \end{cases}$$

and the two sequences $0 \leq q_1 < \ldots < q_s \leq n$, $0 \leq r_s < \ldots < r_1 \leq m$ such that

$$q_i = \begin{cases} q(i) + 1 & \text{if } r(1) = m \\ q(i-1) + 1 & \text{if } r(1) < m \end{cases}$$

$$r_i = \begin{cases} r(i+1) + 1 & \text{if } r(1) = m \\ r(i) + 1 & \text{if } r(1) < m \end{cases}$$

with $i = 1, \dots, s$ and $q(0) = r(s'+1) = -1$.

Lemma 3.8. *Let $S = K[x_1, \dots, x_n, y_1, \dots, y_m]$, $I_\Delta = \sum_{i=1}^{s} I_{q_i} J_{r_i}$ be a mixed product ideal and keep the notation of Proposition 3.6. Then for each $F \in \mathcal{F}_i(\Delta)$ and for each $G \in \mathcal{F}_j(\Delta)$ with $1 \leq i < j \leq s'$ we have*

$$\dim F \cap G \leq q(i) + r(j) - 1.$$

Proof. By Proposition 3.6 we have $|F| = q(i) + r(i)$ and $|G| = q(j) + r(j)$. By the ordering in 3.2 and 3.3 and the Definition 3.5 we have $q(i) < q(j)$, $r(i) > r(j)$ for all $1 \leq i < j \leq s'$ and the assertion follows. □

Lemma 3.9. *Let $S = K[x_1, \dots, x_n, y_1, \dots, y_m]$ and let $I_\Delta = \sum_{i=1}^{s} I_{q_i} J_{r_i}$ be a mixed product ideal and keep the notation of Proposition 3.6. Let I_Δ be unmixed and let $F \in \mathcal{F}_i(\Delta)$ and $G \in \mathcal{F}_j(\Delta)$ with $\dim F \cap G = \dim \Delta - 1$. If $i < j$ (resp. $i > j$) then*

(1) *$j = i + 1$ (resp. $j = i - 1$);*

(2) $q(i+1) = q(i) + 1$ *(resp.* $q(i-1) = q(i) - 1$*);*
(3) $r(i+1) = r(i) - 1$ *(resp.* $r(i-1) = r(i) + 1$*).*

Proof. (1) We assume $i < j$. By Lemma 3.8 and since Δ is pure we have the following inequality

$$\dim F \cap G = \dim \Delta - 1 = q(i) + r(i) - 2 \leq q(i) + r(j) - 1. \tag{3.7}$$

Since $r(j) > r(i)$ and by the inequality 3.7 we obtain $r(j) < r(i) \leq r(j)+1$ that is $r(i) = r(j) + 1$. Therefore $j = i + 1$ and $r(i+1) = r(i) - 1$. By similar arguments we easily complete the proof of the assertion. □

Lemma 3.10. *Let* $S = K[x_1, \dots, x_n, y_1, \dots, y_m]$ *and let* $I_\Delta = \sum_{i=1}^{s} I_{q_i} J_{r_i}$ *be a mixed product ideal. If* $q(i+1) = q(i) + 1$ *for* $i = 1, \dots, s' - 1$ *then* Δ *shellable.*

Proof. We consider the partition $\mathcal{F}(\Delta) = \mathcal{F}_1(\Delta) \cup \ldots \cup \mathcal{F}_{s'}(\Delta)$ defined in Proposition 3.6. We set a linear order $\prec$ on the facets $\mathcal{F}(\Delta)$ such that $F \prec G$ with $F \in \mathcal{F}_k(\Delta)$, $G \in \mathcal{F}_{k'}(\Delta)$ if either $k < k'$ or $k = k'$ with

$$F = \{x_{i_1}, \dots, x_{i_{q(k)}}, y_{j_1}, \dots, y_{j_{r(k)}}\},\ G = \{x_{i'_1}, \dots, x_{i'_{q(k)}}, y_{j'_1}, \dots, y_{j'_{r(k)}}\},$$

$1 \leq i_1 < \ldots < i_{q(k)} \leq n$, $1 \leq j_1 < \ldots < j_{r(k)} \leq m$, $1 \leq i'_1 < \ldots < i'_{q(k)} \leq n$, $1 \leq j'_1 < \ldots < j'_{r(k)} \leq m$ and there exists p, $1 \leq p \leq q(k)$, such that $i_k = i'_k$ for $k = 1, \dots, p-1$ but $i_p < i'_p$ or $i_k = i'_k$ for all $k = 1, \dots, q(k)$ and exists p', $1 \leq p' \leq r(k)$, such that $j_k = j'_k$ for $k = 1, \dots, p'-1$ but $j_{p'} < j'_{p'}$.

Suppose $F \prec G$ with $F \in \mathcal{F}_i(\Delta)$ and $G \in \mathcal{F}_j(\Delta)$ with $i < j$. Since $q(i) < q(j)$ there exists $x_k \in G \setminus F$. Now let $G_k = G \setminus \{x_k\}$. We observe that there exists $F_k \in \mathcal{F}_{j-1}(\Delta)$ such that $F_k \supset G_k$, in fact by hypothesis $q(j-1) = q(j) - 1$ and $r(j-1) > r(j)$. Hence $G \setminus F_k = \{x_k\}$.

Suppose $F \prec G$ with $F, G \in \mathcal{F}_i(\Delta)$. We may assume $x_k \in G \setminus F$, in fact if such x_k does not exist we can consider the case $y_k \in G \setminus F$ in an analogous way. Since $F \prec G$ there exists $x_{k'} \in F \setminus G$ such that $k' < k$. We set $F_k = (G \setminus \{x_k\}) \cup \{x_{k'}\}$. We observe that $F_k \in \mathcal{F}_i(\Delta)$, $F_k \prec G$ and $G \setminus F_k = \{x_k\}$. The assertion follows. □

By the same argument we have the following

Lemma 3.11. *Let* $S = K[x_1, \dots, x_n, y_1, \dots, y_m]$ *and let* $I_\Delta = \sum_{i=1}^{s} I_{q_i} J_{r_i}$ *be a mixed product ideal. If* $r(i+1) = r(i) - 1$ *for* $i = 1, \dots, s' - 1$ *then* Δ *is shellable.*

Theorem 3.12. *Let* $S = K[x_1, \dots, x_n, y_1, \dots, y_m]$, $I_\Delta = \sum_{i=1}^{s} I_{q_i} J_{r_i}$ *be a mixed product ideal,* $K[\Delta] = S/I_\Delta$. *The following conditions are equivalent:*

(1) $q(i+1) = q(i) + 1$ *and* $r(i+1) = r(i) - 1$ *for* $i = 1, \dots, s' - 1$*;*
(2) Δ *is pure shellable;*
(3) $K[\Delta]$ *is Cohen-Macaulay;*
(4) Δ *is strongly connected.*

Proof. (1)⇒(2). By Lemma 3.10 (or equivalently 3.11) we have that $K[\Delta]$ is shellable. We observe that $q(i+1)+r(i+1) = q(i)+r(i)$ for $i = 1, \dots, s'-1$. Hence Δ is pure.

(2)⇒(3). Always true.

(3)⇒(4). Always true.

(4)⇒(1). Let $i = 1, \dots, s'-1$ and let $F \in \mathcal{F}_i(\Delta)$ and $G \in \mathcal{F}_{i+1}(\Delta)$. Since Δ is strongly connected there exists a sequence of facets $F = F_0, F_1, \dots, F_t = G$ such that $\dim F_k \cap F_{k+1} = \dim \Delta - 1$ for $k = 0, \dots, t-1$. We observe that there exists $k \in \{0, \dots, t-1\}$ such that

$$F_k \in \bigcup_{j \le i} \mathcal{F}_j(\Delta), \qquad F_{k+1} \in \bigcup_{j \ge i+1} \mathcal{F}_j(\Delta).$$

Let $F_k \in \mathcal{F}_{i-d}(\Delta)$ and $F_{k+1} \in \mathcal{F}_{i+1+d'}(\Delta)$ with $0 \le d \le i-1$, $0 \le d' \le s'-i-1$. Since $q(i-d) \le q(i) - d$ and $r(i+1+d') \le r(i) - 1 - d'$, by Lemma 3.8 we obtain

$$\dim F_k \cap F_{k+1} \le q(i) + r(i) - (d + d') - 2.$$

On the other hand $\dim F_k \cap F_{k+1} = \dim \Delta - 1 = q(i) + r(i) - 2$. Hence $d = d' = 0$. The assertion follows by Lemma 3.9. □

4. Sequentially Cohen-Macaulay mixed product ideals

In this section we classify the sequentially Cohen-Macaulay mixed product ideals. We recall some definitions and results useful for our purpose and we continue to use the notation defined in section 3.

Definition 4.1. *Let K be a field, $S = K[x_1, \dots, x_n]$ be a polynomial ring. A graded S-module is called* sequentially Cohen-Macaulay *(over K), if there exists a finite filtration of graded S-modules*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_t = M$$

such that each M_i/M_{i-1} is Cohen-Macaulay, and the Krull dimensions of the quotients are increasing:

$$\dim(M_1/M_0) < \dim(M_2/M_1) < \cdots < \dim(M_t/M_{t-1}).$$

Definition 4.2. *Let Δ be a simplicial complex then we define the pure simplicial complexes $\Delta^{[l-1]}$ whose facets are*

$$\mathcal{F}(\Delta^{[l-1]}) = \{F \in \Delta : \dim(F) = l - 1\}, \qquad 0 \le l \le \dim(\Delta) + 1.$$

A fundamental result about sequentially Cohen-Macaulay Stanley-Reisner rings $K[\Delta]$ is the following

Theorem 4.3 ([2])**.** *$K[\Delta]$ is sequentially Cohen-Macaulay if and only if $K[\Delta^{[l-1]}]$ is Cohen-Macaulay for $0 \le l \le \dim(\Delta) + 1$.*

Remark 4.4. *Let $S = K[x_1, \dots, x_n, y_1, \dots, y_m]$, $I_\Delta = \sum_{i=1}^{s} I_{q_i} J_{r_i}$ be a mixed product ideal, $K[\Delta] = S/I_\Delta$ and let $\mathcal{F}(\Delta)$ be partitioned as shown in Proposition 3.6, that is $\mathcal{F}(\Delta) = \mathcal{F}_1(\Delta) \cup \ldots \cup \mathcal{F}_{s'}(\Delta)$ such that*

$$\begin{aligned}\mathcal{F}_k(\Delta) =& \{\{x_{i_1}, \dots, x_{i_{q(k)}}, y_{j_1}, \dots, y_{j_{r(k)}}\} : \\ & 1 \le i_1 < \ldots < i_{q(k)} \le n,\, 1 \le j_1 < \ldots < j_{r(k)} \le m\},\end{aligned}$$

with $k = 1, \dots, s'$. If we set an l with $0 \le l \le \dim(\Delta) + 1$ then for each $k \in \{1, \dots, s'\}$ we have that $\mathcal{F}_k(\Delta^{[l-1]}) = \mathcal{F}_{k1} \cup \ldots \cup \mathcal{F}_{kt_k}$ where

$$\begin{aligned}&\mathcal{F}_{kj} = \{\{x_{i_1}, \dots, x_{i_{q_k(j)}}, y_{j_1}, \dots, y_{j_{r_k(j)}}\} : \\ &\quad 1 \le i_1 < \ldots < i_{q_k(j)} \le n,\, 1 \le j_1 < \ldots < j_{r_k(j)} \le m\} \text{ with } j = 1, \dots, t_k,\end{aligned}$$

satisfies the following properties:

1. *$q_k(t_k) = \min\{q(k), l\}$,*
2. *$r_k(1) = \min\{r(k), l\}$,*
3. *$q_k(i) = q_k(i+1) - 1$, $r_k(i+1) = r_k(i) - 1$ for $i = 1, \dots, t_k - 1$.*

Definition 4.5. *Let $\bar{q} = (q(1), \dots, q(s'))$, $\bar{r} = (r(1), \dots, r(s')) \in \mathbb{Z}_{\ge 0}^{s'}$, we define the following function $\sigma : \{1, \dots, s'\} \to \mathbb{Z}_{\ge 0}$*

$$\sigma(i) = q(i) + r(i).$$

Lemma 4.6. *Let $S = K[x_1, \dots, x_n, y_1, \dots, y_m]$, $I_\Delta = \sum_{i=1}^{s} I_{q_i} J_{r_i}$ be a mixed product ideal and let $K[\Delta] = S/I_\Delta$. If $K[\Delta]$ is sequentially Cohen-Macaulay then*

1. *for all $i \in \{1, \dots, s'-1\}$ either $q(i) = q(i+1)-1$ or $r(i) = r(i+1)+1$;*
2. *there exists $k \in \{1, \dots, s'\}$ such that $\sigma(1) \le \sigma(2) \le \ldots \le \sigma(k) \ge \sigma(k+1) \ge \sigma(s')$.*

Proof. If $K[\Delta]$ is sequentially Cohen-Macaulay then $K[\Delta^{[l-1]}]$ is Cohen-Macaulay for $0 \le l \le \dim \Delta + 1$. Hence $\Delta^{[l-1]}$ is strongly connected for $0 \le l \le \dim \Delta + 1$. We observe that if we negate property (1) (resp. (2)) we find an l such that $\Delta^{[l-1]}$ is not strongly connected.

(1) We suppose that there exists k with $1 \le k \le s' - 1$ such that

$$q(k) < q(k+1) - 1 \text{ and } r(k) > r(k+1) + 1.$$

Let $l = \min\{\sigma(k), \sigma(k+1)\}$ and we assume that $l = \sigma(k)$. We observe that

$$\mathcal{F}(\Delta^{[l-1]}) = \bigcup_{i=1}^{s'} \mathcal{F}_i(\Delta^{[l-1]}) \tag{4.1}$$

where the union is not disjoint. Since $l = \sigma(k) \le \sigma(k+1)$ we have $\mathcal{F}_k(\Delta^{[l-1]}) = \mathcal{F}_k(\Delta)$ and $\mathcal{F}_{k+1}(\Delta^{[l-1]}) \ne \emptyset$.

We show that for all $F \in \mathcal{F}_i(\Delta)$ with $\mathcal{F}_i(\Delta^{[l-1]}) \ne \emptyset$ and for all $G \in \mathcal{F}_j(\Delta)$ with $\mathcal{F}_j(\Delta^{[l-1]}) \ne \emptyset$ with $1 \le i \le k < j \le s'$ we have that

$$\dim F \cap G < \dim \Delta^{[l-1]} - 1. \tag{4.2}$$

If the inequality 4.2 is satisfied also the facets $F' \subset F$ with $F' \in \mathcal{F}_i(\Delta^{[l-1]})$ and $G' \subset G$ with $G' \in \mathcal{F}_j(\Delta^{[l-1]})$ inherit this property.

Hence $\Delta^{[l-1]}$ is not strongly connected.

By Lemma 3.8 we have that $\dim F \cap G \leq q(i) + r(j) - 1 \leq q(k) + r(j) - 1$. Since $r(k) > r(k+1) + 1$ we have $q(k) + r(k) > q(k) + r(k+1) + 1$ that is

$$l - 2 = q(k) + r(k) - 2 > q(k) + r(j) - 1 \geq \dim F \cap G.$$

The case $l = \sigma(k+1)$ follows by similar arguments.

(2) We suppose that there exist k, k^-, k^+, with $1 \leq k^- < k < k^+ \leq s'$ such that

$$\sigma(k^-) > \sigma(k) < \sigma(k^+).$$

Let $l = \min\{\sigma(k^-), \sigma(k^+)\}$ and we assume that $l = \sigma(k^-)$.

Hence $\mathcal{F}_{k^-}(\Delta^{[l-1]}) = \mathcal{F}_{k^-}(\Delta)$ and, since $l \leq \sigma(k^+)$, $\mathcal{F}_{k^+}(\Delta^{[l-1]}) \neq \emptyset$. By Lemma 3.8 and using similar arguments of (1) it is easy to show that, for all $F \in \mathcal{F}_i(\Delta)$ with $\mathcal{F}_i(\Delta^{[l-1]}) \neq \emptyset$ and for all $G \in \mathcal{F}_j(\Delta)$ with $\mathcal{F}_j(\Delta^{[l-1]}) \neq \emptyset$ with $1 \leq i < k < j \leq s'$,

$$\dim F \cap G \leq q(i) + r(j) - 1 < q(k) + r(k) - 1 < l - 1.$$

The assertion follows since $\mathcal{F}_k(\Delta^{[l-1]}) = \emptyset$. □

We come to the main result of this section.

Theorem 4.7. *Let $S = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$, $I_\Delta = \sum_{i=1}^{s} I_{q_i} J_{r_i}$ be a mixed product ideal and let $K[\Delta] = S/I_\Delta$. The following conditions are equivalent:*

1. *$K[\Delta]$ is sequentially Cohen-Macaulay.*
2. *The following conditions hold:*
   - (a) *$q(i) = q(i+1) - 1$ or $r(i) = r(i+1) + 1$ with $i = 1, \ldots, s' - 1$;*
   - (b) *there exists $k \in \{1, \ldots, s'\}$ such that $\sigma(1) \leq \sigma(2) \leq \ldots \leq \sigma(k) \geq \sigma(k+1) \geq \sigma(s')$.*

Proof. (1)⇒(2). See Lemma 4.6.

(2)⇒(1). We need to show that for all l with $0 \leq l \leq \dim \Delta + 1$ we have $K[\Delta^{[l-1]}]$ is Cohen-Macaulay. Let $\Delta' = \Delta^{[l-1]}$, by Remark 4.4 we have that

$$\mathcal{F}(\Delta') = \bigcup \mathcal{F}_{kj} \text{ with } k = 1, \ldots, s', j = 1, \ldots, t_k, \tag{4.3}$$

where

$$\mathcal{F}_{kj} \cap \mathcal{F}_{k'j'} = \begin{cases} \mathcal{F}_{kj} = \mathcal{F}_{k'j'} & \text{if } q_k(j) = q_{k'}(j') \\ \emptyset & \text{if } q_k(j) \neq q_{k'}(j') \end{cases}$$

for all $k, k' \in \{1, \ldots, s'\}$, $j = 1, \ldots, t_k$, $j' = 1, \ldots, t_{k'}$. If we remove the redundant elements in 4.3 and sort the remaining ones in an increasing order by $q_k(j)$ with $k = 1, \ldots, s'$ and $j = 1, \ldots, t_k$, we obtain a partition, with $\bar{q}' = (q'(1), \ldots, q'(t'))$, $\bar{r}' = (r'(1), \ldots, r'(t'))$ and $q'(i) < q'(i+1)$ for $i = 1, \ldots, t' - 1$. Since Δ' is pure by definition, it is sufficient to show that $q'(i+1) = q'(i) + 1$ for $i = 1, \ldots, t' - 1$ by Theorem 3.12.

Let $q'(i)$ be an entry of the vector $\bar{q}'$ with $i = 1, \dots, t' - 1$, then $q'(i) < l$ and there exists $q_k(j)$ related to 4.3 with $q'(i) = q_k(j)$ with $k = 1, \dots, s'$ and $j = 1, \dots, t_k$.

If $j < t_k$ by property (3) of Remark 4.4 we are done. If $j = t_k$ this implies that in the partition induced by 4.3 there exists $k' > k$ such that $\mathcal{F}_{k'}(\Delta') \neq \emptyset$. Hence by the condition (1.b), $\sigma(k+1) \geq \min\{\sigma(k), \sigma(k')\} \geq l$, therefore $\mathcal{F}_{k+1}(\Delta') \neq \emptyset$. By condition (1.a), if $q(k+1) = q(k) + 1$ and by property (1) of Remark 4.4 we have $q_{k+1}(t_{k+1}) = q(k) + 1$. If $q(k+1) \neq q(k) + 1$ then $r(k+1) = r(k) - 1$ and this implies by property (2) of Remark 4.4 that $r_{k+1}(1) = r(k) - 1$, hence $q_{k+1}(1) = l - (r(k) - 1) = q(k) + 1$.

□

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