A Perron-Frobenius theory for block matrices associated to a multiplex network

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Abstract

The uniqueness of the Perron vector of a nonnegative block matrix associated to a multiplex network is discussed. The conclusions come from the relationships between the irreducibility of some nonnegative block matrix associated to a multiplex network and the irreducibility of the corresponding matrices to each layer as well as the irreducibility of the adjacency matrix of the projection network. In addition the computation of that Perron vector in terms of the Perron vectors of the blocks is also addressed. Finally we present the precise relations that allow to express the Perron eigenvector of the multiplex network in terms of the Perron eigenvectors of its layers.

1 Introduction and notation

Throughout the last years, the development of the theory of complex networks has been providing radically new ways of understanding many different processes and mechanisms from engineering and physical, information, social and biological sciences [1, 2, 3, 6, 8, 24].

One of its fundamental tools, the Perron-Frobenius theorem, has shown how a 100 years old mathematical result, of essentially theoretical content, may have applications in many different areas of science and technology, and, specifically, in the analysis of social and complex networks (see, for example [5, 21]).

A lot of work has been done during the last years to understand the structure and dynamics of complex systems composed by multiple systems and/or layers of connectivity, leading, recently, to the concepts of multilayer and multiplex networks [1, 6, 8].

Following [1], a multiplex network $\mathcal{M}$ with M layers is a set of layers $\{G_\alpha;\ \alpha \in \{1, \cdots, M\}\}$, where each layer is a (directed or undirected, weighted or unweighted) graph $G_\alpha = (X_\alpha, E_\alpha)$, with $X_\alpha = \{x_1, \cdots, x_N\}$. In other words, multiplex networks consist of a fixed set of nodes connected by different types of links. A very simple and clarifying type of examples of multiplex networks appear in the modelling of social systems, since each of them can be seen as a superposition of many complex social networks, where nodes represent individuals and links in different layers represent different types of social relations.

Very recently, some relevant aspects in the theory of multiplex networks have been considered with the help of adequate block matrix representations of the networks, and of the analysis and interpretation of their eigenvalues and eigenvectors [6, 7, 8, 27, 30, 31].

This analysis typically includes the study of the existence and uniqueness of a positive and normalized eigenvector (Perron vector). The existence is guaranteed if the corresponding matrix is irreducible (by using the classical Perron-Frobenius theorem). As for the spectral properties, it is possible to relate the irreducibility of such a matrix with the irreducibility in each layer and the irreducibility in the corresponding matrix of the projection network [7, 30]. Some of these considerations are properly addressed with the help of the Perron vector of the block matrix which represents the multiplex structure.

The main goal of this paper is twofold. Firstly we show the uniqueness of the Perron eigenvector of the nonnegative block matrix associated to a multiplex network when the matrices of the layers and the matrix of connections between layers (or influence matrix) have some properties. Secondly we show how the Perron vector of the multiplex network relates to the lower-dimension Perron vectors of the layers and the Perron vector of the influence matrix in a precise way. Remarkably this relationship is shown to be non linear; thus it becomes evident that the information framed in a multiplex network goes beyond a simple linear combination of the information provided by the layers.

This paper is divided in four sections. The first and second sections contain the notation employed and some background as well as a detailed description of the matrix products used along. The third section is entirely devoted to justifying the existence and uniqueness of the Perron eigenvector of the multiplex structure while the fourth section presents the precise (non linear) relations that allow to express the Perron eigenvector of a multiplex network in terms of the Perron eigenvectors of its layers. The computations of this section are collected in a final appendix.

In the rest of the paper a *multiplex network* is a set $\mathcal{M} = \{S_1, \cdots, S_m\}$ ($m \in \mathbb{N}$) of (directed or undirected, weighted or unweighted) complex networks $S_\ell = (X, E_\ell)$ (each of them called a *layer* or *state* of the multiplex network) where $X = \{1, \cdots, n\}$ is the same set of nodes for all ℓ and E_ℓ is the set of edges in each S_ℓ, that is

$$E_\ell = \{(i,j) \in X \times X \mid i \text{ is linked to } j \text{ in the layer } S_\ell\}\,.$$

The adjacency matrix of each layer S_ℓ will be denoted by $A_\ell = (a_{ij}(\ell)) \in \mathbb{R}^{n\times n}$.

In many situations, if we consider a multiplex network $\mathcal{M}$ of $m \in \mathbb{N}$ layers $\{S_1, \cdots, S_m\}$, we also take a nonnegative (meaning that its entries are all nonnegative) *influence matrix* $0 \le W = (w_{ij}) \in \mathbb{R}^{m\times m}$, where w_{ij} measures the *influence* of the layer S_i in the layer S_j. Note that if we consider a random walker in a multiplex network, then each w_{ij} can be understood as the probability of the walker jumping from layer S_i to layer S_j (i.e. W is the *transition matrix* between the states of the multiplex network in the stochastic process given by a multiplex random walker) and therefore W is a row stochastic matrix. Hence in the rest of the paper, we will always assume that the influence matrices W are row stochastic.

A multiplex network with only one layer, that is, a network in the standard sense, will be usually called here a monolayer (or monoplex) network. Given a multiplex network $\mathcal{M}$ several (monolayer) networks that give valuable information about $\mathcal{M}$ can be associated to it. A first example of these (monolayer) networks is the unweighted *projection network* $proj(\mathcal{M}) = (X, E)$, where X is the same set of nodes of the layers of $\mathcal{M}$ and

$$E = \left(\bigcup_{\ell=1}^{m} E_\ell\right).$$

It is clear that if $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is the adjacency matrix of $proy(\mathcal{M})$, then

$$a_{ij} = \begin{cases} 1 & \text{if } a_{ij}(\ell) = 1 \text{ for some } 1 \le \ell \le m \\ 0 & \text{otherwise.} \end{cases}$$

A first approach to the concept of multiplex networks could suggest that these new objects are actually (monolayer) networks with some (modular) structure in the mesoscale. It is clear that a (monolayer) network $\tilde{\mathcal{M}}$ can be associated to $\mathcal{M}$ as follows: $\tilde{\mathcal{M}} = (\tilde{X}, \tilde{E})$, where $\tilde{X}$ is the disjoint union of all the nodes of $S_1, \cdots, S_m$, i.e.

$$\tilde{X} = \bigcup_{1\le i\le m} X_i = \{(i,k)|\ i = 1, \dots, n,\ k = 1, \dots, m\}$$

(the node $(i,k) \in \tilde{X}$ shoud be understood as the i-node of X in k-state). Now $\tilde{E}$ is given by

$$\tilde{E} = \{((i,k),(j,k)) \mid (i,j) \in E_k,\ 1 \le k \le m\} \bigcup \{((i,k),(i,l)) \mid i \in X,\ 1 \le k \ne l \le m\}.$$

Notice that an edge $((i,k),(j,k))$ belonging to the first set in this union reflects that nodes i and j in X are linked in layer k (horizontal link) while and edge $((i,k)(i,l))$ belonging to the second set in the union means that the node i considered in layer k is connected to itself considered in layer l (vertical link).

Notice that $\tilde{\mathcal{M}}$ is a (monolayer) network with $n{\cdot}m$ nodes whose adjacency matrix can be written as the block matrix

$$\tilde{A} = \left(\begin{array}{c|c|c|c} A_1 & I_n & \cdots & I_n \\ \hline I_n & A_2 & \cdots & I_n \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline I_n & I_n & \cdots & A_m \end{array}\right) \in \mathbb{R}^{nm\times nm}.$$

It is important to remark that the behaviours of $\mathcal{M}$ and $\tilde{\mathcal{M}}$ are related but they are different since a single node of $\mathcal{M}$ belonging to several layers corresponds to m different nodes in $\tilde{\mathcal{M}}$. Hence the properties and behaviours of corresponding (monolayer) network $\tilde{\mathcal{M}}$ could be understood as a kind of non-linear *quotient* of the properties of the a multilayer $\mathcal{M}$.

Other examples of (monolayer) networks associated to a multiplex network $\mathcal{M}$ that give valuable information about the properties of $\mathcal{M}$ come from the study of several structural and dynamical properties of $\mathcal{M}$. In this paper we will consider the associated monolayer networks coming from the study of the eigenvector centrality of multiplex networks [30] and from random walkers in multiplex networks [7].

If we want to extend the concept of eigenvector centrality to multiplex network, in [30] the concept of global heterogeneous centrality of a multiplex network $\mathcal{M}$ with influence matrix W is introduced from the Perron vector of the block matrix

$$\mathbb{B}_0 = \left(\begin{array}{c|c|c|c} w_{11}A_1^t & w_{21}A_2^t & \cdots & w_{m1}A_m^t \\ \hline w_{12}A_1^t & w_{22}A_2^t & \cdots & w_{m2}A_m^t \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline w_{1m}A_1^t & w_{2m}A_2^t & \cdots & w_{mm}A_m^t \end{array}\right) \in \mathbb{R}^{nm\times nm},$$

where A_ℓ^t is the transpose of the adjacency matrix of layer S_ℓ. Note that this kind of block matrix also appears if we consider some random walkers in multiplex networks. In this case, the distribution of the stationary state of the random walker is given from the Perron vector of the block matrix

$$\mathbb{B}_1 = \left(\begin{array}{c|c|c|c} w_{11}L_1^t & w_{21}L_2^t & \cdots & w_{m1}L_m^t \\ \hline w_{12}L_1^t & w_{22}L_2^t & \cdots & w_{m2}L_m^t \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline w_{1m}L_1^t & w_{2m}L_2^t & \cdots & w_{mm}L_m^t \end{array}\right) \in \mathbb{R}^{nm\times nm},$$

where L_ℓ^t is the transpose of the row normalization of the adjacency matrix of layer S_ℓ, i.e. if $L_\ell = (L_{ij}(\ell))$, then for each $1 \le i, j \le n$

$$L_{ij}(\ell) = \frac{a_{ij}(\ell)}{\sum_k a_{ik}(\ell)}.$$

Note that each L_ℓ is row stochastic and therefore L_ℓ^t is colum stochastic.

Similarly, in [7] a general framework for random walkers in multiplex networks is introduced and the distribution of the stationary states of these random walkers are given from the Perron vector of some block matrices. In particular, if we consider random walkers with no cost in the transition between states, the distribution of the stationary state is given in terms of the Perron vector of

$$\mathbb{B}_2 = \left(\begin{array}{c|c|c|c} w_{11}L_1^t & w_{21}L_1^t & \cdots & w_{m1}L_1^t \\ \hline w_{12}L_2^t & w_{22}L_2^t & \cdots & w_{m2}L_2^t \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline w_{1m}L_m^t & w_{2m}L_m^t & \cdots & w_{mm}L_m^t \end{array}\right) \in \mathbb{R}^{nm\times nm},$$

while if we consider random walkers with cost in the transition between states, the distribution of the stationary state is given in terms of the Perron vector of

$$\mathbb{B}_3 = \left(\begin{array}{c|c|c|c} w_{11}L_1^t & w_{21}I_n & \cdots & w_{m1}I_n \\ \hline w_{12}I_n & w_{22}L_2^t & \cdots & w_{m2}I_n \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline w_{1m}I_n & w_{2m}I_n & \cdots & w_{mm}L_m^t \end{array}\right) \in \mathbb{R}^{nm\times nm}.$$

As we will see in section 3, it can be proven that, under some hypotheses, if the adjacency matrix of the projection network is irreducible, then these matrices are also irreducible and hence the corresponding random walkers have a unique stationary state.

This kind of arguments can be also applied to the supra-Laplacian $\mathcal{L}$ of a multiplex ([10] and [31]) since we have the splitting

$$\mathcal{L} = \mathcal{L}^m + \mathcal{L}^I,$$

where $\mathcal{L}^m$ stands for the supra-Laplacian of the independent layers and $\mathcal{L}^I$ for the interlayer supra-Laplacian. The first one is just the direct sum of the intralayer Laplacians,

$$\mathcal{L}^L = \left(\begin{array}{c|c|c|c} L_1 & 0 & \cdots & 0 \\ \hline 0 & L_2 & \cdots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \cdots & L_m \end{array}\right),$$

while the interlayer supra-Laplacian may be expressed as the Kronecker (or tensorial) product (see section 2) of the interlayer Laplacian and the $n \times n$ identity matrix I,

$$\mathcal{L}^I = L^I \otimes I\,.$$

2 Block Hadamard and Block Khatri-Rao Products

In addition to the conventional matrix product, there are some other matrix products which will be used throughout this paper.

Note that, for example,

$$\mathbb{B}_1 = \left(\begin{array}{c|c|c} w_{11}L_1^t & \cdots & w_{m1}L_m^t \\ \hline \vdots & \ddots & \vdots \\ \hline w_{1m}L_1^t & \cdots & w_{mm}L_m^t \end{array}\right),$$

is the Hadamard product of

$$\left(\begin{array}{c|c|c} w_{11}1_n & \cdots & w_{m1}1_n \\ \hline \vdots & \ddots & \vdots \\ \hline w_{1m}1_n & \cdots & w_{mm}1_n \end{array}\right) \text{ and } \left(\begin{array}{c|c|c} L_1^t & \cdots & L_m^t \\ \hline \vdots & \ddots & \vdots \\ \hline L_1^t & \cdots & L_m^t \end{array}\right),$$

where 1_n the matrix $n \times n$ whose components are all equal to one, or the generalized Khatri-Rao product of

$$\left(\begin{array}{c|c|c} w_{11} & \cdots & w_{m1} \\ \hline \vdots & \ddots & \vdots \\ \hline w_{1m} & \cdots & w_{mm} \end{array}\right) \text{ and } \left(\begin{array}{c|c|c} L_1^t & \cdots & L_m^t \\ \hline \vdots & \ddots & \vdots \\ \hline L_1^t & \cdots & L_m^t \end{array}\right).$$

This section provides a brief survey on such definitions and basic properties without proofs. Throughout this section we refer to some standard references of matrix theory for details.

Let us consider two matrices A and B of $m \times n$ and $p \times q$ orders respectively. Let us suppose that $A = [A_{ij}]$ is partitioned with A_{ij} of order $m_i \times n_j$ (A_{ij} is the $(i,j)^{th}$ block submatrix of A) and let $B = [B_{kl}]$ be partitioned with B_{kl} of order $p_k \times q_l$ (B_{kl} is the $(k,l)^{th}$ block submatrix of B). Denote by $m = \sum_{i=1}^t m_i$, $n = \sum_{j=1}^d n_j, p = \sum_{k=1}^u p_k$, and $q = \sum_{l=1}^v q_l$. For simplicity, we say that A and B are *compatible partitioned* if $A = [A_{ij}]_{i,j=1}^t$ and $B = [B_{ij}]_{i,j=1}^t$ are square matrices of order $m \times m$ and partitioned, respectively, with A_{ij} and B_{ij} of order $m_i \times m_j$ ($m = \sum_{i=1}^t m_i = \sum_{j=1}^t m_j$).

Let $A \otimes B$, $A \circ B$, $A \Theta B$, and $A * B$ be the Kronecker, Hadamard, Tracy-Singh, and Khatri-Rao products, respectively, of A and B. All the definitions of the mentioned four matrix products can be found in [17], [18] as follows:

(i) *Kronecker product*

The Kronecker product of matrices is also called the tensor product, or direct product of matrices. This product is applicable to any two matrices. We refer to [12] for a complete discussion.

Let $A = (a_{ij}) \in \mathbb{R}^{m \times n}$ and $B = (b_{ij}) \in \mathbb{R}^{p \times q}$. The Kronecker product of A and B is defined as

$$A \otimes B = (a_{ij}B)_{ij} = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix} \in \mathbb{R}^{mp \times nq}.$$

(ii) *Hadamard product*

The Hadamard product (elementwise multiplication), also referred to as the *Schur product*, arises in a wide variety of mathematical applications such as covariance matrices for independent zero mean random vectors and characteristic functions in probability theory. The reader is referred to [12], [33], [28] for more details about it.

Let $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{m \times n}$. The Hadamard product of A and B is defined as

$$A \circ B = (a_{ij}b_{ij})_{ij} = \begin{pmatrix} a_{11}b_{11} & a_{12}b_{12} & \cdots & a_{1n}b_{1n} \\ a_{21}b_{21} & a_{22}b_{22} & \cdots & a_{2n}b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}b_{m1} & a_{m2}b_{m2} & \cdots & a_{mn}b_{mn} \end{pmatrix} \in \mathbb{R}^{m \times n}.$$

(iii) *Tracy-Singh product*

$$A \Theta B = [A_{ij} \Theta B]_{ij} = [[A_{ij} \otimes B_{kl}]_{kl}]_{ij},$$

where $A = [A_{ij}]$, $B = [B_{kl}]$ are partitioned matrices of order $m \times n$ and $p \times q$, respectively, A_{ij} is of order $m_i \times n_j$, B_{kl} of order $p_k \times q_l$, $A_{ij} \otimes B_{kl}$ of order

$m_ip_k \times n_jq_l$, $A_{ij}\Theta B$ of order $m_ip \times n_jq$ ($m = \sum_{i=1}^{t} m_i$, $n = \sum_{j=1}^{d} n_j$, $p = \sum_{k=1}^{u} p_k$, $q = \sum_{l=1}^{v} q_l$), and $A\Theta B$ of order $mp \times nq$;

In order to avoid confusion we use parentheses for ordinary matrices, whose entries are numbers, multiplied as usual, and square brackets for cores (core matrices), whose entries are blocks.

(iv) *Generalized Khatri-Rao product*

$$A * B = [A_{ij} \otimes B_{ij}]_{ij}$$

where $A = [A_{ij}]$, $B = [B_{ij}]$ are partitioned matrices of order $m \times n$ and $p \times q$, respectively, A_{ij} is of order $m_i \times n_j$, B_{kl} of order $p_i \times q_j$, $A_{ij} \otimes B_{ij}$ of order $m_ip_i \times n_jq_j$ ($m = \sum_{i=1}^{t} m_i$, $n = \sum_{j=1}^{d} n_j$, $p = \sum_{i=1}^{t} p_i$, $q = \sum_{j=1}^{d} q_j$), and $A * B$ of order $M \times N$ ($M = \sum_{i=1}^{t} m_ip_i$, $N = \sum_{j=1}^{d} n_jq_j$).

Note that the generalized Khatri-Rao product is defined based on a particular matrix partitioning, i.e., different matrix partitionings will lead to different results. Note also that the Kronecker product, the Hadamard product and the Khatri-Rao product [14], [26] are all special cases of the generalized Khatri-Rao product based on different matrix partitionings.

Recall that given two matrices A and B with the same number of columns, m, and denoting their columns by a_i and b_i, respectively, the (column-wise) *Khatri-Rao* product is defined as $A * B = [a_1 \otimes b_1, a_2 \otimes b_2, \cdots, a_m \otimes b_m]$ (we refer to [20], [33] or [19] for details). Note that the Khatri-Rao product can be constructed by selecting columns from the Kronecker product. To show this, define the Kronecker selection matrix $S_m = I_m * I_m$ and verify $A * B = (A \otimes B)S_m$, where I_m is the identity matrix in $\mathbb{R}^{m \times m}$.

Additionally, [17] shows that the generalized Khatri-Rao product can be viewed as a generalized Hadamard product and the Tracy-Singh product as a generalized Kronecker product, as follows:

(1) for a nonpartitioned matrix A, their $A\Theta B$ is $A \otimes B$;

(2) for nonpartitioned matrices A and B of order $m \times n$, their $A * B$ is $A \circ B$.

The Khatri-Rao and Tracy-Singh products are related by the following relation [17], [18] :

$$A * B = Z_1^T (A\Theta B) Z_2,$$

where $A = [A_{ij}]$ is partitioned with A_{ij} of order $m_i \times n_j$ and $B = [B_{kl}]$ is partitioned with B_{kl} of order $p_k \times q_l$ ($m = \sum_{i=1}^{t} m_i$, $n = \sum_{j=1}^{d} n_j$, $p =$

$\sum_{k=1}^{u} p_k$, $q = \sum_{l=1}^{v} q_l$), Z_1 is an $mp \times r$ ($r = \sum_{i=1}^{t} m_i p_i$) matrix of zeros and ones, and Z_2 is an $nq \times s$ ($s = \sum_{j=1}^{d} n_j q_j$) matrix of zeros and ones such that $Z_1^T Z_1 = I_r, Z_2^T Z_2 = I_s$ (I_r and I_s are $r \times r$ and $s \times s$ identity matrices, resp.).

In particular, if $m = n$ and $p = q$, then there exists a $mp \times r$ ($r = \sum_{i=1}^{t} m_i p_i$) matrix Z such that $Z^T Z = I_r$ (I_r is an $r \times r$ identity matrix) and $A * B = Z^T (A \Theta B) Z$. Here

$$Z = \begin{bmatrix} Z_1 & & \\ & \ddots & \\ & & Z_t \end{bmatrix},$$

where each $Z_i = [0_{i1} \ \cdots \ 0_{ii-1} \ I_{m_i p_i} \ 0_{ii+1} \ \cdots \ 0_{it}]^T$ is a real matrix of zeros and ones, and 0_{ik} is a $m_i p_i \times m_i p_k$ zero matrix for any $k \neq i$. Note also that $Z_i^T Z_i = I$ and

$$Z_i^T (A_{ij} \Theta B) Z_j = Z_i^T (A_{ij} \otimes B_{kl})_{kl} Z_j = A_{ij} \otimes B_{ij}, \ i, j = 1, 2, \cdots, t.$$

The generalized Khatri-Rao product was also used, e.g., in [32].

Let A and B be matrices respectively expressed as $r \times t$ and $t \times u$ block matrices

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1t} \\ A_{21} & A_{22} & \cdots & A_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ A_{r1} & A_{r2} & \cdots & A_{rt} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1u} \\ B_{21} & B_{22} & \cdots & B_{2u} \\ \vdots & \vdots & \ddots & \vdots \\ B_{t1} & B_{t2} & \cdots & B_{tu} \end{pmatrix},$$

where each A_{ij} ($i = 1, 2, \cdots, r$ and $j = 1, 2, \cdots, t$) is an $m \times p$ matrix, and each B_{ij} ($i = 1, 2, \cdots, t$ and $j = 1, 2, \cdots, u$) is a $n \times q$ matrix. In [29] the *strong Kronecker* product is defined for two matrices A and B of dimensions $r \times t$ and $t \times u$ respectively as the matrix:

$$C = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1u} \\ C_{21} & C_{22} & \cdots & C_{2u} \\ \vdots & \vdots & \ddots & \vdots \\ C_{r1} & C_{r2} & \cdots & C_{ru} \end{pmatrix},$$

where each

$$C_{ij} = A_{i1} \otimes B_{1j} + A_{i2} \otimes B_{2j} + \cdots + A_{it} \otimes B_{tj},$$

is an $mn \times pq$ matrix. It is important to note that the operation is fully determined only after the parameters r, t, and u are fixed. Generally, the partitioning of the matrices will be clear from the context, and then we call C the strong Kronecker product of A and B, denoted by $A \circledast B$. The strong Kronecker product, developed in [29], supportes the analysis of certain orthogonal matrix multiplication problems. The strong Kronecker product is considered a powerful matrix multiplication tool for Hadamard and other orthogonal matrices from combinatorial theory [16]. In [25] the strong Kronecker product is shown to be a matrix multiplication in a permuted space. Similarly, if $m = n$ and $p = q$, the *strong Hadamard* product $A \odot B$ of A and B is defined in [4] as

$$A \odot B = \begin{pmatrix} D_{11} & D_{12} & \cdots & D_{1u} \\ D_{21} & D_{22} & \cdots & D_{2u} \\ \vdots & \vdots & \ddots & \vdots \\ D_{r1} & D_{r2} & \cdots & D_{ru} \end{pmatrix},$$

where each

$$D_{ij} = A_{i1} \circ B_{1j} + A_{i2} \circ B_{2j} + \cdots + A_{it} \circ B_{tj} \,,$$

is an $m \times p$ matrix.

Let $A = (A_{ij})$ and $B = (B_{ij})$ be $p \times p$ block matrices in which each block is an $n \times n$ matrix. In [13] a *block Hadamard* product $A \Box B$ is defined by $A \Box B := (A_{ij} B_{ij})$, where $A_{ij} B_{ij}$ denotes the usual matrix product of A_{ij} and B_{ij}.

There are other definitions of partitioned matrix products, see for instance [11] where a generalized Kronecker product for block matrices is defined.

3 Irreducibility and uniqueness of Block Perron Vectors through properties of the blocks

In this section we will discuss irreducibility of the block matrices that appear in our different descriptions of multiplex networks. Let us start by introducing some notation.

3.1 Products of block matrices

In the sequel we will consider block matrices consisting of m^2 blocks of dimensions $n \times n$ with real nonnegative coefficients:

$$P = \left(\begin{array}{c|c|c|c} P_{11} & P_{12} & \cdots & P_{1m} \\ \hline P_{21} & P_{22} & \cdots & P_{2m} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline P_{m1} & P_{m2} & \cdots & P_{mm} \end{array}\right), \quad P_{ij} \in \mathbb{R}^{n\times n}.$$

The set of all this matrices will be denoted by $M^+_{nm,n}(\mathbb{R})$, or simply $M^+_{nm,n}$.

For two such block matrices P and P', let us consider the strong Hadamard product defined above:

$$(P \odot P')_{ij} = \sum_{k=1}^{m} P_{ik} \circ P'_{kj},$$

where $P_{ik} \circ P'_{kj}$ denotes the Hadamard product (i.e. the componentwise product) of the blocks P_{ik} and P'_{kj}.

For a given a sequence of $n \times n$ matrices $(A_1, \ldots, A_m)$ we can consider the diagonal block matrix $\underline{A}$ matrix defined by:

$$\underline{A} = \left(\begin{array}{c|c|c|c} A_1 & 0 & \cdots & 0 \\ \hline 0 & A_2 & \cdots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \cdots & A_m \end{array}\right)$$

We will denote by I_n the $n \times n$ identity matrix, and by 1_n the matrix $n \times n$ whose components are all equal to one. Then the identity element of the product $\odot$ is $\underline{1}$, that is, the diagonal block matrix given by the sequence $(1_n, \ldots, 1_n)$.

Let us denote by R_2 the Boolean algebra with two elements $\{0, 1\}$, on which we have two operations, namely:

+	0	1
0	0	1
1	1	1

·	0	1
0	0	0
1	0	1

Then, for every nonnegative matrix $P \in M^+_{nm,n}$ we may define its *booleanization* $\beta(P)$ as the nm block matrix with coefficients in R_2 given by:

$$(\beta(P)_{ij})_{kr} = \begin{cases} 1 & \text{si } (P_{ij})_{kr} \neq 0 \\ 0 & \text{si } (P_{ij})_{kr} = 0 \end{cases}$$

for all $i, j = 1, \ldots, m$, $k, r = 1, \ldots, n$.

Notice that the map $\beta : M^+_{nm,n} \to M_{nm,n}(R_2)$ preserves, by definition, sums, and the usual, Hadamard and strong Hadamard products; notice also that the irreduciblity of a nonnegative matrix, which is the main topic of this section, depends only on its booleanization, which can be thought of as a matrix-representation of the graph defined by the matrix.

A partial order can be defined in $M_{nm,n}(R_2)$ as $B \le B'$ if and only if there exists $B'' \in M_{nm,n}(R_2)$ such that $B + B'' = B'$. It becomes obvious that, if $P \in M^+_{nm,n}$ is irreducible, then any other matrix $P' \in M^+_{nm,n}$ satisfying $\beta(P) \le \beta(P')$ must be irreducible as well.

Finally we note that for every block matrix $P \in M^+_{nm,n}$ a new block matrix $\widehat{P} \in M^+_{nm,m}$ can be defined by reordering the coefficients as follows:

$$(\widehat{P}_{kr})_{ij} = (P_{ij})_{kr}, \quad i,j = 1,\dots,n, \quad k,r = 1,\dots,m$$

This new matrix is formed by n^2 blocks of dimension $m \times m$.

3.2 Block matrices for multiplex networks

In order to model multiplex networks as they appear in nature, scientists have introduced several types of special block matrices. Generally speaking, they are all constructed upon the following data:

- A set of m nonnegative $n \times n$ matrices $\{A_1, \dots, A_m\}$, each A_i is the the adjacency matrix of the i-layer belonging to the multiplex network. In this context the matrix $\overline{A} := \frac{1}{m}\sum_{i=1}^m A_i$, whose associated graph is the projection network of the complex network under study, is considered.
- Two $nm \times nm$ nonnegative block matrices, encoding the interrelation between layers:

$$W = \left(\begin{array}{c|c|c|c} W_{11} & W_{12} & \cdots & W_{1m} \\ \hline W_{21} & W_{22} & \cdots & W_{2m} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline W_{m1} & W_{m2} & \cdots & W_{mm} \end{array}\right), \quad V = \left(\begin{array}{c|c|c|c} V_{11} & V_{12} & \cdots & V_{1m} \\ \hline V_{21} & V_{22} & \cdots & V_{2m} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline V_{m1} & V_{m2} & \cdots & V_{mm} \end{array}\right)$$

We may think of W as the matrix encoding interrelations between layers (influence matrix), whereas V represents interrelations between layers arising from the set of all the specific influences that a node in a layer has over a node in another (not necessarily different) layer.

Then, upon this data, we consider the matrices:

$$\mathbb{B} = \underline{A} \odot W + V \text{ and } \mathbb{B}' = W \odot \underline{A} + V,$$

Notice that both $\mathbb{B}$ and $\mathbb{B}'$ have their own eigenvector centrality.

Two particular cases of the previous general scheme have a clear interest.

1. The term V is identically zero. Then we have two block matrices

$$\mathbb{B}_1 = \underline{A} \odot W \text{ and } \mathbb{B}_1' = W \odot \underline{A}$$

(this is the situation when modelling random walkers with no cost for the sate transition).

2. The term W is equal to $\underline{1}$, so that our two block matrices are equal:

$$\mathbb{B}_2 = \underline{A} \odot \underline{1} + V = \underline{A} + V = \underline{1} \odot \underline{A} + V = \mathbb{B}_2'.$$

Typically in this case one would ask V to satisfy the following property:

$$(\star) \;\; V_{ij} \text{ diagonal, for all } i, j$$

In other words, the property $(\star)$ is satisfied whenever $\widehat{V} = \underline{B}$ being $B = (B_1, \ldots, B_n)$ a sequence of $m \times m$ nonnegative matrices.

Each matrix B_j represent the way in which one may switch between layers, while staying at node j (this is the situation when modelling random walkers with no cost for the sate transition).

In search of irreducibility conditions we will work on this general scheme; this is the content of the next subsection.

3.3 Irreducibility conditions

As announced the rest of the section is devoted to describing irreducibility conditions of the matrices described above. Since we are going to discuss irreducibility through is graph-theoretical counterpart –strong connectedness– we need to introduce first some notation.

Given a multiplex network determined by one of the matrices $\mathbb{B}$ (or $\mathbb{B}'$) described above, we will write $i \overset{k}{\to} j$ when the node i is linked to the node j in layer k, i.e. when the coefficient $(A_k)_{ij}$ is different from zero. We will now consider a new monolayer network with nodes $\tilde{X} = \{(i,k) |\ i = 1, \ldots, n,\ k = 1, \ldots, m\}$ and write $(i,k) \to (j,\ell)$ when the coefficient in the position ij of the block $k\ell$ of $\mathbb{B}$ (or $\mathbb{B}'$) is different from zero. In other words, we consider the weighted graph $(\tilde{X}, \mathbb{B})$ (or $(\tilde{X}, \mathbb{B}')$) supported on the monolayer network $\tilde{\mathcal{M}}$.

In the case 1, we will start by analizing the case in which the projected network is strongly connected, that is, in which $\overline{A}$ is irreducible. Unfortunately, in this case, even if W is positive, very simple examples show that $\mathbb{B}_1$ and $\mathbb{B}_1'$ are not necessarily irreducible. However we may state that there exists a unique Perron vector for them.

Theorem 3.1. *With the same notation as above, assume that $\overline{A}$ is irreducible and W is positive. Then $\mathbb{B}_1$ and $\mathbb{B}_1'$ have a unique Perron vector.*

Proof. We will present the proof of the uniqueness for $\mathbb{B}_1$, being the proof for $\mathbb{B}'_1$ analogous.

Note that the matrix $\mathbb{B}_1$ may have rows completely equal to zero, preventing it from being irreducible. If W is strictly positive, this happens precisely if there exists a sink in the graph of one of the layers. In order to deal with this situation, we consider a permutation matrix P that reorders the rows of $\mathbb{B}_1$ so that all the rows equal to zero appear in the first positions. Then the product $P \cdot \mathbb{B}_1 \cdot P^t$ takes the form:

$$P \cdot \mathbb{B}_1 \cdot P^t = \left(\begin{array}{ccc|ccc} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \hline \star & \cdots & \star & & & \\ \vdots & & \vdots & & \text{R} & \\ \star & \cdots & \star & & & \end{array}\right)$$

and it suffices to show that R is an irreducible matrix, because in this case the algebraic multiplicities of the spectral radius of $\mathbb{B}_1$ as an eigenvalue of $\mathbb{B}_1$ equals its multiplicity as an eigenvalue for R, which is equal to one.

In order to check the irreducibility of R note first that, by the positivity of W:

$$\big(i \overset{k}{\to} j\big) \iff \big((i,k) \to (j,k)\big) \iff \big((i,k) \to (j,\ell)\big) \text{ for all } \ell = 1, \ldots, m \tag{3.2}$$

Considering then the weighted subgraph of $(\tilde{X}, \mathbb{B}_1)$ associated to R, and denoting by $\tilde{X}_R$ its set of nodes, that is:

$$\tilde{X}_R = \{(\, i,k)|\; i \overset{k}{\to} j \text{ for some } j\},$$

it suffices to show that $(\tilde{X}_R, R)$ is strongly connected.

Let then $(i,k), (i',k')$ be two nodes of this subgraph. Since $(i,k) \in \tilde{X}_R$, there exist $j_1 \in \{1, \ldots, n\}$ such that

$$(i,k) \to (j_1, \ell) \text{ for all } \ell.$$

Moreover, by hypothesis on $\overline{A}$, we know that there exist two sequences of indices $(j_1, \ldots, j_r = i')$, $j_p \in \{1, \ldots, n\}$, and $(k_2, \ldots, k_r)$, $k_p \in \{1, \ldots, m\}$, such that:

$$j_1 \overset{k_2}{\to} j_2 \overset{k_3}{\to} \cdots \overset{k_r}{\to} j_r = i',$$

and so $(j_p, k_{p+1}) \to (j_{p+1}, \ell)$ for all ℓ. Summing up, we have a sequence of edges linking (i,k) to (i',k'):

$$(i,k) \to (j_1,k_2) \to (j_2,k_3) \to \cdots \to (j_{r-1},k_r) \to (i',k').$$

□

Remark 3.3. Note that, denoting by $1_{nm} \in M^{+}_{nm,n}$ the matrix whose coefficients are all ones, the proof holds for every nonnegative block matrix W satisfying $\beta(W) \geq \beta(\underline{A} \odot 1_{nm})$ (or $\beta(W) \geq \beta(1_{nm} \odot \underline{A})$, when we are dealing with $\mathbb{B}'_1$).

The next corollary is an immediate consequence of the previous proof:

Corollary 3.4. *With the same notation as above, assume that $\overline{A}$ is irreducible and that W is strictly positive. Assume moreover that each layer A_k of the network has no sinks (respectively, no sources). Then $\mathbb{B}_1$ (resp. $\mathbb{B}'_1$) is irreducible.*

Let us consider now the case 2. Here we will infer the irreducibility of $\mathbb{B}_2 = \mathbb{B}'_2$ from properties of $(A_1, \ldots, A_m)$ and $(B_1, \ldots, B_n)$.

Proposition 3.5. *With the same notation as above, assume that one of the following properties holds:*

(i) $\overline{A}$ and every B_i are irreducible.

(ii) Every A_k and $\overline{B}$ are irreducible.

Then $\mathbb{B}_2$ is irreducible.

Proof. As usual, we will discuss the proof in terms of the subjacent networks. In the first case, given two pairs $(i, k), (i', k') \in \tilde{X}$, the irreducibility of $\overline{A}$ provides a sequences of edges:

$$i = j_0 \overset{k_1}{\to} j_1 \overset{k_2}{\to} \cdots \overset{k_r}{\to} j_r = i'.$$

That is, we have links

$$(i = j_0, k_1) \to (j_1, k_1), \quad (j_1, k_2) \to (j_2, k_2), \quad \ldots, \quad (j_{r-1}, k_r) \to (i' = j_r, k_r).$$

Denote $k_0 := k$, $k_{r+1} := k'$. Then, the irreducibility of the B_i's provides sequences of edges joining (j_p, k_p) with (j_p, k_{p+1}) for all $p = 0, \ldots, r$. Joining all these sequence conveniently, we have a sequence of edges joining (i, k) and (i', k'). The irreducibility of $\mathbb{B}_2$ under the second set of hypotheses is analogous. □

Remark 3.6. As we may see in this Proposition, in this second setup, the links within layers and between layers play a symmetric role. In this way, every theorem about $\mathbb{B}_2$ written in terms of A and B will always have a symmetric counterpart.

4 Computation of Block Perron Vectors in terms of low-dimensional vectors

Our approach is based on the Perron complementation method for finding the Perron eigenvector of a nonnegative irreducible matrix $A_{m\times m}$ with spectral radius ρ, see [22]. This method consists of uncoupling A into smaller matrices whose Perron eigenvectors are coupled together in order to recover the Perron eigenvector of A and it is described in Appendix A. The Perron eigenvector $\pi = \begin{pmatrix} \pi^1 \\ \hline \pi^2 \\ \hline \vdots \\ \hline \pi^k \end{pmatrix} > 0$ of each of $\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3$ is of the form $\pi = \begin{pmatrix} \xi^1 p_1 \\ \hline \xi^2 p_2 \\ \hline \vdots \\ \hline \xi^k p_k \end{pmatrix} > 0$ where each p_i is the Perron eigenvector of the Perron complement P_{ii}, and will be calculated for all the three cases, and the normalizing scalars or coupling factors ξ_i turn to be the i^{th}-components of the Perron eigenvector of W^t.

Our only assumption is that W is row-stochastic and that no i^{th}-row of W equals the i^{th}-vector of the canonical basis e_i of $\mathbb{R}^m$ (this means that all layers have influence at least on some other layer).

Block matrix of type $\mathbb{B}_1$: The obtention of the Perron eigenvector π of $\mathbb{B}_1$ follows from combining the $p_i's$ with the coupling factor, which is the Perron eigenvector of W^t. Remember that

$$\mathbb{B}_1 = \left(\begin{array}{c|c|c|c} w_{11}L_1^t & w_{21}L_2^t & \cdots & w_{m1}L_m^t \\ \hline w_{12}L_1^t & w_{22}L_2^t & \cdots & w_{m2}L_m^t \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline w_{1m}L_1^t & w_{2m}L_2^t & \cdots & w_{mm}L_m^t \end{array}\right) \in \mathbb{R}^{nm\times nm}.$$

Let us calculate the Perron eigenvector p_1 of the Perron complement P_{11}. First calculate $\begin{pmatrix} Q_2 \\ \hline Q_3 \\ \hline \vdots \\ \hline Q_m \end{pmatrix}$, which is an eigenvector associated to 1 of the matrix

$$\mathcal{A}_1{}^{p_1} = w_{11}L + \tilde{W}_{11}^{(1)} - w_{11}L\tilde{W}_{11}^{(1)} + \begin{pmatrix} w_{12}L_1^t \\ \hline w_{13}L_1^t \\ \hline \vdots \\ \hline w_{1m}L_1^t \end{pmatrix} (w_{21}L_2^t \dots w_{m1}L_m^t)$$

where $L = \left(\begin{array}{c|c|c} L_1^t & \cdots & 0 \\ \hline 0 & L_1^t & \cdots \\ \hline \vdots & \ddots & \vdots \\ \hline 0 & \cdots & L_1^t \end{array}\right)$ and $\tilde{W}_{11}^{(1)} = \left(\begin{array}{c|c|c} w_{22}L_2^t & \cdots & w_{m2}L_m^t \\ \hline w_{23}L_2^t & \cdots & w_{m2}L_m^t \\ \hline \vdots & \ddots & \vdots \\ \hline w_{2m}L_2^t & \cdots & w_{mm}L_m^t \end{array}\right)$.

Once the $Q_i's$ are obtained use

$$\left(\begin{array}{c} w_{12}L_1^t \\ \hline w_{13}L_1^t \\ \hline \vdots \\ \hline w_{1m}L_1^t \end{array}\right) p_1 = \left(Id - \left(\begin{array}{c|c|c} w_{22}L_2^t & \cdots & w_{m2}L_m^t \\ \hline w_{23}L_2^t & \cdots & w_{m2}L_m^t \\ \hline \vdots & \ddots & \vdots \\ \hline w_{2m}L_2^t & \cdots & w_{mm}L_m^t \end{array}\right)\right) \left(\begin{array}{c} Q_2 \\ \hline Q_3 \\ \hline \vdots \\ \hline Q_m \end{array}\right)$$

to get $L_1^t p_1$ (remember that some of the $w_{1i} \neq 0$), and then the equality

$$p_1 = w_{11}L_1^t p_1 + (w_{21}L_2^t \dots w_{m1}L_m^t) \left(\begin{array}{c} Q_2 \\ \hline Q_3 \\ \hline \vdots \\ \hline Q_m \end{array}\right)$$

to recover p_1.

The remaining $p_i's$ are analogously calculated.

<u>Block matrix of type $\mathbb{B}_2$</u>: The obtention of the Perron eigenvector π of $\mathbb{B}_2$ follows from combining the $p_i's$ with the coupling factor, which is the Perron eigenvector of W^t. Remember that

$$\mathbb{B}_2 = \left(\begin{array}{c|c|c|c} w_{11}L_1^t & w_{21}L_1^t & \cdots & w_{m1}L_1^t \\ \hline w_{12}L_2^t & w_{22}L_2^t & \cdots & w_{m2}L_2^t \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline w_{1m}L_m^t & w_{2m}L_m^t & \cdots & w_{mm}L_m^t \end{array}\right) \in \mathbb{R}^{nm \times nm}.$$

Let us calculate the Perron eigenvector p_1 of the Perron complement P_{11}. First calculate $\left(\begin{array}{c} Q_2 \\ \hline Q_3 \\ \hline \vdots \\ \hline Q_m \end{array}\right)$, which is an eigenvector associated to 1 of the matrix

$$\mathcal{A}_2{}^{p_1} = w_{11}L + \tilde{W}_{11}^{(2)} - w_{11}\tilde{W}_{11}^{(2)}L + \left(\begin{array}{c} w_{12}L_2^t \\ \hline w_{13}L_3^t \\ \hline \vdots \\ \hline w_{1m}L_m^t \end{array}\right) (w_{21}L_1^t, \dots, w_{m1}L_1^t)$$

where $L = \left(\begin{array}{c|c|c} L_1^t & \cdots & 0 \\ \hline 0 & L_1^t & \cdots \\ \hline \vdots & \ddots & \vdots \\ \hline 0 & \cdots & L_1^t \end{array}\right)$ and $\tilde{W}_{11}^{(2)} = \left(\begin{array}{c|c|c} w_{22}L_2^t & \cdots & w_{m2}L_2^t \\ \hline w_{23}L_3^t & \cdots & w_{m2}L_3^t \\ \hline \vdots & \ddots & \vdots \\ \hline w_{2m}L_m^t & \cdots & w_{mm}L_m^t \end{array}\right)$.

Once the $Q_i's$ are obtained,

$$p_1 = (w_{21}L_1, \ldots, w_{m1}L_1)\begin{pmatrix} Q_2 \\ \hline Q_3 \\ \hline \vdots \\ \hline Q_m \end{pmatrix}.$$

The remaining $p_i's$ are analogously obtained.
<u>Block matrix of type $\mathbb{B}_3$</u>: The obtention of the Perron eigenvector π of $\mathbb{B}_3$ follows from combining the $p_i's$ with the coupling factor, which is the Perron eigenvector of W^t. Remember that

$$\mathbb{B}_3 = \left(\begin{array}{c|c|c|c} w_{11}L_1^t & w_{21}Id & \cdots & w_{m1}Id \\ \hline w_{12}Id & w_{22}L_2^t & \cdots & w_{m2}Id \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline w_{1m}Id & w_{2m}Id & \cdots & w_{mm}L_m^t \end{array}\right) \in \mathbb{R}^{nm\times nm}.$$

The calculation of the Perron eigenvector p_1 of the Perron complement P_{11} can be done as follows: calculate $\begin{pmatrix} Q_2 \\ \hline Q_3 \\ \hline \vdots \\ \hline Q_m \end{pmatrix}$, which is an eigenvector associated to 1 of the matrix

$$\mathcal{A}_3{}^{p_1} = w_{11}L + \tilde{W}_{11}^{(3)} - w_{11}L\tilde{W}_{11}^{(3)} + \begin{pmatrix} w_{12}Id \\ \hline w_{13}Id \\ \hline \vdots \\ \hline w_{1m}Id \end{pmatrix}(w_{21}Id \ldots w_{m1}Id)$$

where $L = \left(\begin{array}{c|c|c} L_1^t & \cdots & 0 \\ \hline 0 & L_1^t & \cdots \\ \hline \vdots & \ddots & \vdots \\ \hline 0 & \cdots & L_1^t \end{array}\right)$ and $\tilde{W}_{11}^{(3)} = \left(\begin{array}{c|c|c|c} w_{22}L_2^t & w_{32}Id & \cdots & w_{m2}Id \\ \hline w_{23}Id & w_{33}L_3^t & \cdots & w_{m2}Id \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline w_{2m}Id & w_{3m}Id & \cdots & w_{mm}L_m^t \end{array}\right)$.

Once the $Q_i's$ are obtained use

$$\begin{pmatrix} w_{12}Id \\ \hline w_{13}Id \\ \hline \vdots \\ \hline w_{1m}Id \end{pmatrix} p_1 = \left(Id - \left(\begin{array}{c|c|c|c} w_{22}L_2^t & w_{32}Id & \cdots & w_{m2}Id \\ \hline w_{23}Id & \cdots & \cdots & w_{m2}L_m^t \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline w_{2m}Id & w_{3m}Id & \cdots & w_{mm}L_m^t \end{array}\right)\right)\begin{pmatrix} Q_2 \\ \hline Q_3 \\ \hline \vdots \\ \hline Q_m \end{pmatrix}$$

to recover p_1 (remember that some $w_{1j} \neq 0$).

The remaining $p_i's$ are analogously calculated.

4.1 Particular case of two layers ($m = 2$)

We will show that the eigenvectors associated to the principal eigenvalue 1 can be computed in terms of the eigenvectors associated to 1 of certain matrices related to L_1^t, L_2^t and the elements of W. The only assumption on W is that it is row-stochastic. The details of the calculations will be shown in §A.

Block matrix of type $\mathbb{B}_1$, $m = 2$:

$$\mathbb{B}_1 = \left(\begin{array}{c|c} w_{11} L_1^t & w_{21} L_2^t \\ \hline w_{12} L_1^t & w_{22} L_2^t \end{array} \right),$$

where L_ℓ^t is the transpose of the row normalization of the adjacency matrix of layer S_ℓ.

(a) If both $w_{11} \neq 1$ and $w_{22} \neq 1$ then if $\left(\begin{array}{c} \pi_1 \\ \hline \pi_2 \end{array} \right)$ is an eigenvector associated to the eigenvalue 1, we get that π_1 and π_2 are eigenvectors associated to 1 to the column stochastic matrices

$$\begin{aligned} \mathcal{A}_1^{\pi_1} &= (w_{11} L_1^t + w_{22} L_2^t + (1 - w_{11} - w_{22}) L_2^t L_1^t), \text{ and} \\ \mathcal{A}_1^{\pi_2} &= (w_{11} L_1^t + w_{22} L_2^t + (1 - w_{11} - w_{22}) L_1^t L_2^t). \end{aligned}$$

(b) If $w_{11} = 1$ then $w_{12} = 0$ and if the vector $\left(\begin{array}{c} \pi_1 \\ \hline \pi_2 \end{array} \right)$ is associated to the eigenvalue 1 then we have one of the three following situations:

(b.1) $0 < w_{22} < 1$: the eigenvectors associated to 1 of $\mathbb{B}_1$ have the form $\left(\begin{array}{c} \pi_1 \\ \hline 0 \end{array} \right)$ where π_1 is an eigenvector of L_1^t associated to 1.

(b.2) $w_{22} = 0$: the eigenvectors associated to $\mathbb{B}_1$ have the form $\left(\begin{array}{c} \pi_1 \\ \hline 0 \end{array} \right)$ where π_1 is an eigenvector of L_1^t associated to 1.

(b.3) $w_{22} = 1$: the eigenvectors of $\mathbb{B}_1$ associated to 1 have the form $\left(\begin{array}{c} \pi_1 \\ \hline \pi_2 \end{array} \right)$ where π_1 is an eigenvector of L_1^t associated to 1 and π_2 an eigenvector of L_2^t associated to 1.

(c) If $w_{22} = 1$ then, arguing as in case (b) either $w_{11} = 1$ and we are again in the situation of (b.3) or the eigenvector of $\mathbb{B}_1$ associated to 1 are of the form $\left(\begin{array}{c} 0 \\ \hline \pi_2 \end{array} \right)$ where π_2 is an eigenvector of L_2^t associated to 1.

Block matrix of type $\mathbb{B}_2$, $m = 2$:

$$\mathbb{B}_2 = \left(\begin{array}{c|c} w_{11} L_1^t & w_{21} L_1^t \\ \hline w_{12} L_2^t & w_{22} L_2^t \end{array} \right),$$

where L_ℓ^t is the transpose of the row normalization of the adjacency matrix of layer S_ℓ.

(a) If both $w_{11} \neq 1$ and $w_{22} \neq 1$ then if $\left(\frac{\pi_1}{\pi_2}\right)$ is an eigenvector associated to the eigenvalue 1 and defining $\pi_1^{aux} = (I - w_{11}L_1^t)^{-1}(I - w_{22}L_2^t)^{-1}L_2^t\pi_1$ and $\pi_2^{aux} = (I - w_{22}L_2^t)^{-1}(I - w_{11}L_1^t)^{-1}L_1^t\pi_2$, we get that π_1^{aux} and π_2^{aux} are eigenvectors associated to 1 of the column stochastic matrices

$$\begin{array}{l} \mathcal{A}_2^{\pi_1^{aux}} = (w_{11}L_1^t + w_{22}L_2^t - w_{11}w_{22}L_2^tL_1^t + w_{12}w_{21}L_2^tL_1^t), \text{ and} \\ \mathcal{A}_2^{\pi_2^{aux}} = (w_{11}L_1^t + w_{22}L_2^t - w_{11}w_{22}L_1^tL_2^t + w_{12}w_{21}L_1^tL_2^t). \end{array}$$

After computing π_1^{aux} and π_2^{aux},

$$\begin{cases} \pi_1 = w_{12}w_{21}L_1^t\pi_1^{aux}, \\ \pi_2 = w_{12}w_{21}L_2^t\pi_2^{aux}. \end{cases}$$

(b) $(w_{11} = 1)$ and (c) $(w_{22} = 1)$ give the same results as for matrices of type $\mathbb{B}_1$.

Block matrix of type $\mathbb{B}_3$, $m = 2$:

$$\mathbb{B}_2 = \left(\begin{array}{c|c} w_{11}L_1^t & w_{21}I_2 \\ \hline w_{12}I_2 & w_{22}L_2^t \end{array}\right),$$

where L_ℓ^t is the transpose of the row normalization of the adjacency matrix of layer S_ℓ.

(a) If both $w_{11} \neq 1$ and $w_{22} \neq 1$ then if $\left(\frac{\pi_1}{\pi_2}\right)$ is an eigenvector associated to the eigenvalue 1, we get that π_1 and π_2 are eigenvectors associated to 1 to the column stochastic matrices

$$\begin{array}{l} \mathcal{A}_2^{\pi_1} = (w_{11}L_1^t + w_{22}L_2^t - w_{11}w_{22}L_2^tL_1^t + w_{12}w_{21}I_2), \text{ and} \\ \mathcal{A}_2^{\pi_2} = (w_{11}L_1^t + w_{22}L_2^t - w_{11}w_{22}L_1^tL_2^t + w_{12}w_{21}I_2). \end{array}$$

(b) $(w_{11} = 1)$ and (c) $(w_{22} = 1)$ give the same results as for matrices of type $\mathbb{B}_1$.

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A Mathematical proof of the results of section 4

Perron complementation method for finding the Perron vector of a nonnegative irreducible matrix $A_{m\times m}$ with spectral radius ρ ([22]): This method consists of uncoupling A into smaller matrices whose Perron vectors are coupled together in order to recover the Perron vector of A. Let us briefly recall it:

Given a k-level partition

$$A = \left(\begin{array}{c|c|c|c} A_{11} & A_{12} & \cdots & A_{1k} \\ \hline A_{21} & A_{22} & \cdots & A_{2k} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline A_{k1} & A_{k2} & \cdots & A_{kk} \end{array}\right)$$

where all the diagonal blocks A_{ii} are square, we consider the principal block submatrices A_i of A obtained by deleting the i^{th}-row of blocks and the i^{th}-column of blocks from A. We also consider

$$A_{i*} = (A_{i1}A_{i2}\cdots A_{i,i-1}\ A_{i,i+1}\cdots A_{ik})$$

and

$$A_{*i} = \left(\begin{array}{c} A_{1i} \\ \hline \vdots \\ \hline A_{i-1,i} \\ \hline A_{i+1,i} \\ \hline \vdots \\ \hline A_{ki} \end{array}\right).$$

The Perron complement of A_{ii} in A is defined as the matrix

$$P_{ii} = A_{ii} + A_{i*}(\rho Id - A_i)^{-1}A_{*i}.$$

The importance of the Perron complements stems from the fact that if A is nonnegative and irreducible with spectral radius ρ, then P_{ii} is also nonnegative and irreducible with spectral radius ρ. In addition, if $\pi = \left(\begin{array}{c} \pi^1 \\ \hline \pi^2 \\ \hline \vdots \\ \hline \pi^k \end{array}\right) > 0$ is the Perron vector of A, partitioned accordingly, then $P_{ii}\pi^i = \rho\pi^i$, that is, π^i is a positive eigenvector of P_{ii} associated to ρ ([22, Thm 2.1 and 2.2]). Call $p_i \equiv \dfrac{\pi^i}{\|\pi^i\|_1}$, the Perron vector of P_{ii}. The normalizing scalar $\xi^i \equiv \|\pi^i\|_1$, or *coupling factor*, turns out to be the i^{th}-component of the

Perron eigenvector $\begin{pmatrix} \xi^1 \\ \hline \xi^2 \\ \hline \vdots \\ \hline \xi^k \end{pmatrix}$ of the *coupling matrix* $C \equiv (c_{ij})$, where $c_{ij} = \|A_{ij}p_j\|_1$. Thus, the Perron vector π can be expressed as $\pi = \begin{pmatrix} \xi^1 p_1 \\ \hline \xi^2 p_2 \\ \hline \vdots \\ \hline \xi^k p_k \end{pmatrix}$.

Our immediate task is to identify the Perron complements for each of the three types of matrices considered and proceed accordingly. Each L_ℓ is row stochastic and therefore L_ℓ^t is column stochastic; similarly W is row stochastic, hence each of the matrices $\mathbb{B}_1, \mathbb{B}_2$ and $\mathbb{B}_3$ given in Section 1 is also column stochastic and its maximal eigenvalue is one.

It will be assumed that no i^{th}-row of W equals the i^{th}-vector of the canonical basis e_i of $\mathbb{R}^m$ (this means that all layers have influence at least on some other layer).

As for the coupling matrix C, since L_ℓ^t are column stochastic, in each of the three cases we get that $C = W^t$ and therefore the coupling factors correspond to the Perron eigenvector of W^t.

<u>Block matrix of type $\mathbb{B}_1$</u>: The obtention of the Perron vector π of $\mathbb{B}_1$ follows from combining the $p_i's$ with the coupling factor, which is the Perron vector of W^t.

$$\mathbb{B}_1 = \left(\begin{array}{c|c|c|c} w_{11}L_1^t & w_{21}L_2^t & \cdots & w_{m1}L_m^t \\ \hline w_{12}L_1^t & w_{22}L_2^t & \cdots & w_{m2}L_m^t \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline w_{1m}L_1^t & w_{2m}L_2^t & \cdots & w_{mm}L_m^t \end{array}\right) \in \mathbb{R}^{nm\times nm}.$$

Let us calculate the Perron vector p_1 of the Perron complement P_{11}. It satisfies

$$p_1 = w_{11}L_1^t p_1 + (w_{21}L_2^t \dots w_{m1}L_m^t)\left(Id - \left(\begin{array}{c|c|c} w_{22}L_2^t & \cdots & w_{m2}L_m^t \\ \hline w_{23}L_2^t & \cdots & w_{m2}L_m^t \\ \hline \vdots & \ddots & \vdots \\ \hline w_{2m}L_2^t & \cdots & w_{mm}L_m^t \end{array}\right)\right)^{-1} \begin{pmatrix} w_{12}L_1^t \\ \hline w_{13}L_1^t \\ \hline \vdots \\ \hline w_{1m}L_1^t \end{pmatrix} p_1$$

Then

$$\begin{pmatrix} w_{12}L_1^t \\ \hline w_{13}L_1^t \\ \hline \vdots \\ \hline w_{1m}L_1^t \end{pmatrix} p_1 = w_{11} \begin{pmatrix} w_{12}L_1^t \\ \hline w_{13}L_1^t \\ \hline \vdots \\ \hline w_{1m}L_1^t \end{pmatrix} L_1^t p_1 + \begin{pmatrix} w_{12}L_1^t \\ \hline w_{13}L_1^t \\ \hline \vdots \\ \hline w_{1m}L_1^t \end{pmatrix} (w_{21}L_2^t \dots w_{m1}L_m^t) \begin{pmatrix} Q_2 \\ \hline Q_3 \\ \hline \vdots \\ \hline Q_m \end{pmatrix}$$

where the following change of variables is used

$$\begin{pmatrix} Q_2 \\ \hline Q_3 \\ \hline \vdots \\ \hline Q_m \end{pmatrix} = \left(Id - \begin{pmatrix} w_{22}L_2^t & \cdots & w_{m2}L_m^t \\ \hline w_{23}L_2^t & \cdots & w_{m2}L_m^t \\ \hline \vdots & \ddots & \vdots \\ \hline w_{2m}L_2^t & \cdots & w_{mm}L_m^t \end{pmatrix} \right)^{-1} \begin{pmatrix} w_{12}L_1^t \\ \hline w_{13}L_1^t \\ \hline \vdots \\ \hline w_{1m}L_1^t \end{pmatrix} p_1.$$

Equivalently

$$\left(Id - \begin{pmatrix} w_{22}L_2^t & \cdots & w_{m2}L_m^t \\ \hline w_{23}L_2^t & \cdots & w_{m2}L_m^t \\ \hline \vdots & \ddots & \vdots \\ \hline w_{2m}L_2^t & \cdots & w_{mm}L_m^t \end{pmatrix} \right) \begin{pmatrix} Q_2 \\ \hline Q_3 \\ \hline \vdots \\ \hline Q_m \end{pmatrix} = w_{11} \begin{pmatrix} L_1^t & 0 & \cdots & 0 \\ \hline 0 & L_1^t & \cdots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \cdots & L_1^t \end{pmatrix} \begin{pmatrix} w_{12}L_1^t \\ \hline w_{13}L_1^t \\ \hline \vdots \\ \hline w_{1m}L_1^t \end{pmatrix} p_1 +$$

$$\begin{pmatrix} w_{12}L_1^t \\ \hline w_{13}L_1^t \\ \hline \vdots \\ \hline w_{1m}L_1^t \end{pmatrix} (w_{21}L_2^t \dots w_{m1}L_m^t) \begin{pmatrix} Q_2 \\ \hline Q_3 \\ \hline \vdots \\ \hline Q_m \end{pmatrix},$$

or

$$\left(Id - \begin{array}{c|c|c} w_{22}L_2^t & \cdots & w_{m2}L_m^t \\ \hline w_{23}L_2^t & \cdots & w_{m2}L_m^t \\ \hline \vdots & \ddots & \vdots \\ \hline w_{2m}L_2^t & \cdots & w_{mm}L_m^t \end{array} \right) \begin{pmatrix} Q_2 \\ \hline Q_3 \\ \hline \vdots \\ \hline Q_m \end{pmatrix} =$$

$$w_{11} \begin{pmatrix} L_1^t & 0 & \cdots & 0 \\ \hline 0 & L_1^t & \cdots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \cdots & L_1^t \end{pmatrix} \left(Id - \begin{array}{c|c|c} w_{22}L_2^t & \cdots & w_{m2}L_m^t \\ \hline w_{23}L_2^t & \cdots & w_{m2}L_m^t \\ \hline \vdots & \ddots & \vdots \\ \hline w_{2m}L_2^t & \cdots & w_{mm}L_m^t \end{array} \right) \begin{pmatrix} Q_2 \\ \hline Q_3 \\ \hline \vdots \\ \hline Q_m \end{pmatrix} +$$

$$+ \begin{pmatrix} w_{12}L_1^t \\ \hline w_{13}L_1^t \\ \hline \vdots \\ \hline w_{1m}L_1^t \end{pmatrix} (w_{21}L_2^t \dots w_{m1}L_m^t) \begin{pmatrix} Q_2 \\ \hline Q_3 \\ \hline \vdots \\ \hline Q_m \end{pmatrix}$$

. This is equivalent to $\begin{pmatrix} Q_2 \\ \hline Q_3 \\ \hline \vdots \\ \hline Q_m \end{pmatrix}$ being an eigenvector associated to 1 of the matrix

$$\mathcal{A}_1{}^{p_1} = w_{11}L + \tilde{W}_{11}^{(1)} - w_{11}L\tilde{W}_{11}^{(1)} + \begin{pmatrix} w_{12}L_1^t \\ \hline w_{13}L_1^t \\ \hline \vdots \\ \hline w_{1m}L_1^t \end{pmatrix} (w_{21}L_2^t \dots w_{m1}L_m^t)$$

$$\text{where } L = \left(\begin{array}{c|c|c} L_1^t & \cdots & 0 \\ \hline 0 & L_1^t & \cdots \\ \hline \vdots & \ddots & \vdots \\ \hline 0 & \cdots & L_1^t \end{array}\right) \text{ and } \tilde{W}_{11}^{(1)} = \left(\begin{array}{c|c|c} w_{22}L_2^t & \cdots & w_{m2}L_m^t \\ \hline w_{23}L_2^t & \cdots & w_{m2}L_m^t \\ \hline \vdots & \ddots & \vdots \\ \hline w_{2m}L_2^t & \cdots & w_{mm}L_m^t \end{array}\right).$$

Once the $Q_i's$ are obtained we use

$$\left(\begin{array}{c} w_{12}L_1^t \\ \hline w_{13}L_1^t \\ \hline \vdots \\ \hline w_{1m}L_1^t \end{array}\right) p_1 = \left(Id - \left(\begin{array}{c|c|c} w_{22}L_2^t & \cdots & w_{m2}L_m^t \\ \hline w_{23}L_2^t & \cdots & w_{m2}L_m^t \\ \hline \vdots & \ddots & \vdots \\ \hline w_{2m}L_2^t & \cdots & w_{mm}L_m^t \end{array}\right)\right) \left(\begin{array}{c} Q_2 \\ \hline Q_3 \\ \hline \vdots \\ \hline Q_m \end{array}\right)$$

to get $L_1^t p_1$ (since some $w_{1i} \neq 0$), and then the equality

$$p_1 = w_{11}L_1^t p_1 + (w_{21}L_2^t \dots w_{m1}L_m^t) \left(\begin{array}{c} Q_2 \\ \hline Q_3 \\ \hline \vdots \\ \hline Q_m \end{array}\right)$$

to recover p_1.

The remaining $p_i's$ are analogously calculated.

<u>Block matrix of type $\mathbb{B}_2$</u>: The obtention of the Perron vector π of $\mathbb{B}_2$ follows from combining the $p_i's$ with the coupling factor, which is the Perron vector of W^t.

$$\mathbb{B}_2 = \left(\begin{array}{c|c|c|c} w_{11}L_1^t & w_{21}L_1^t & \cdots & w_{m1}L_1^t \\ \hline w_{12}L_2^t & w_{22}L_2^t & \cdots & w_{m2}L_2^t \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline w_{1m}L_m^t & w_{2m}L_m^t & \cdots & w_{mm}L_m^t \end{array}\right) \in \mathbb{R}^{nm \times nm}.$$

Let us calculate the Perron vector p_1 of the Perron complement P_{11}. It satisfies

$$p_1 = w_{11}L_1^t p_1 + (w_{21}L_1^t \dots w_{m1}L_1^t) \left(Id - \left(\begin{array}{c|c|c} w_{22}L_2^t & \cdots & w_{m2}L_2^t \\ \hline w_{23}L_3^t & \cdots & w_{m2}L_3^t \\ \hline \vdots & \ddots & \vdots \\ \hline w_{2m}L_m^t & \cdots & w_{mm}L_m^t \end{array}\right)\right)^{-1} \left(\begin{array}{c} w_{12}L_2^t \\ \hline w_{13}L_3^t \\ \hline \vdots \\ \hline w_{1m}L_m^t \end{array}\right) p_1$$

$$\text{so } (Id - w_{11}L_1^t)p_1 = (w_{21}L_1^t \dots w_{m1}L_1^t) \left(Id - \left(\begin{array}{c|c|c} w_{22}L_2^t & \cdots & w_{m2}L_2^t \\ \hline w_{23}L_3^t & \cdots & w_{m2}L_3^t \\ \hline \vdots & \ddots & \vdots \\ \hline w_{2m}L_m^t & \cdots & w_{mm}L_m^t \end{array}\right)\right)^{-1} \left(\begin{array}{c} w_{12}L_2^t \\ \hline w_{13}L_3^t \\ \hline \vdots \\ \hline w_{1m}L_m^t \end{array}\right) p_1$$

or, as $w_{11} \neq 1$,

$$p_1 = (Id - w_{11}L_1^t)^{-1}(w_{21}L_1^t \dots w_{m1}L_1^t) \left(Id - \left(\begin{array}{c|c|c} w_{22}L_2^t & \cdots & w_{m2}L_2^t \\ \hline w_{23}L_3^t & \cdots & w_{m2}L_3^t \\ \hline \vdots & \ddots & \vdots \\ \hline w_{2m}L_m^t & \cdots & w_{mm}L_m^t \end{array} \right) \right)^{-1} \left(\begin{array}{c} w_{12}L_2^t \\ \hline w_{13}L_3^t \\ \hline \vdots \\ \hline w_{1m}L_m^t \end{array} \right) p_1.$$

Now, calling

$$\tilde{C} = Id - \left(\begin{array}{c|c|c|c} (Id - w_{11}L_1^t)^{-1} & 0 & \cdots & 0 \\ \hline 0 & (Id - w_{11}L_1^t)^{-1} & \cdots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \cdots & (Id - w_{11}L_1^t)^{-1} \end{array} \right)$$

we get by matrix commutation

$$p_1 = (w_{21}L_1^t \dots w_{m1}L_1^t)\, \tilde{C} \left(Id - \left(\begin{array}{c|c|c} w_{22}L_2^t & \cdots & w_{m2}L_2^t \\ \hline w_{23}L_3^t & \cdots & w_{m2}L_3^t \\ \hline \vdots & \ddots & \vdots \\ \hline w_{2m}L_m^t & \cdots & w_{mm}L_m^t \end{array} \right) \right)^{-1} \left(\begin{array}{c} w_{12}L_2^t \\ \hline w_{13}L_3^t \\ \hline \vdots \\ \hline w_{1m}L_m^t \end{array} \right) p_1.$$

Multiplyig in both sides by $\left(\begin{array}{c} w_{12}L_2^t \\ \hline w_{13}L_3^t \\ \hline \vdots \\ \hline w_{1m}L_m^t \end{array} \right)$ and using the change of variables

$\left(\begin{array}{c} Q_2 \\ \hline Q_3 \\ \hline \vdots \\ \hline Q_m \end{array} \right) \equiv \tilde{C} \left(Id - \begin{array}{c|c|c} w_{22}L_2^t & \cdots & w_{m2}L_2^t \\ \hline w_{23}L_3^t & \cdots & w_{m2}L_3^t \\ \hline \vdots & \ddots & \vdots \\ \hline w_{2m}L_m^t & \cdots & w_{mm}L_m^t \end{array} \right)^{-1} \left(\begin{array}{c} w_{12}L_2^t \\ \hline w_{13}L_3^t \\ \hline \vdots \\ \hline w_{1m}L_m^t \end{array} \right) p_1$ we get

that $\left(\begin{array}{c} Q_2 \\ \hline Q_3 \\ \hline \vdots \\ \hline Q_m \end{array} \right)$ is an eigenvector associated to 1 of the matrix

$$\mathcal{A}_2{}^{p_1} = w_{11}L + \tilde{W}_{11}^{(2)} - w_{11}\tilde{W}_{11}^{(2)}L + \left(\begin{array}{c} w_{12}L_2^t \\ \hline w_{13}L_3^t \\ \hline \vdots \\ \hline w_{1m}L_m^t \end{array} \right) (w_{21}L_1^t, \dots, w_{m1}L_1^t)$$

where $L = \left(\begin{array}{c|c|c} L_1^t & \cdots & 0 \\ \hline 0 & L_1^t & \cdots \\ \hline \vdots & \ddots & \vdots \\ \hline 0 & \cdots & L_1^t \end{array} \right)$ and $\tilde{W}_{11}^{(2)} = \left(\begin{array}{c|c|c} w_{22}L_2^t & \cdots & w_{m2}L_2^t \\ \hline w_{23}L_3^t & \cdots & w_{m2}L_3^t \\ \hline \vdots & \ddots & \vdots \\ \hline w_{2m}L_m^t & \cdots & w_{mm}L_m^t \end{array} \right)$.

Once the $Q_i's$ are obtained we use

$$p_1 = (w_{21}L_1, \ldots, w_{m1}L_1)\begin{pmatrix} Q_2 \\ \hline Q_3 \\ \hline \vdots \\ \hline Q_m \end{pmatrix}$$

to get p_1. The remaining $p_i's$ are analogously obtained.
<u>Block matrix of type $\mathbb{B}_3$</u>: The obtention of the Perron vector π of $\mathbb{B}_3$ follows from combining the $p_i's$ with the coupling factor, which is the Perron vector of W^t.

$$\mathbb{B}_3 = \left(\begin{array}{c|c|c|c} w_{11}L_1^t & w_{21}Id & \cdots & w_{m1}Id \\ \hline w_{12}Id & w_{22}L_2^t & \cdots & w_{m2}Id \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline w_{1m}Id & w_{2m}Id & \cdots & w_{mm}L_m^t \end{array}\right) \in \mathbb{R}^{nm\times nm}.$$

In this case the Perron vector p_1 of the Perron complement P_{11} satisfies

$$p_1 = w_{11}L_1^t p_1 + (w_{21}Id \ldots w_{m1}Id)\left(Id - \left(\begin{array}{c|c|c|c} w_{22}L_2^t & w_{32}Id & \cdots & w_{m2}Id \\ \hline w_{23}Id & \cdots & \cdots & w_{m2}L_m^t \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline w_{2m}Id & w_{3m}Id & \cdots & w_{mm}L_m^t \end{array}\right)\right)^{-1}\begin{pmatrix} w_{12}Id \\ \hline w_{13}Id \\ \hline \vdots \\ \hline w_{1m}Id \end{pmatrix} p_1$$

Then, multiplying in both sides by $\begin{pmatrix} w_{12}Id \\ \hline w_{13}Id \\ \hline \vdots \\ \hline w_{1m}Id \end{pmatrix}$ and using the change of variables

$$\begin{pmatrix} Q_2 \\ \hline Q_3 \\ \hline \vdots \\ \hline Q_m \end{pmatrix} = \left(Id - \left(\begin{array}{c|c|c|c} w_{22}L_2^t & w_{32}Id & \cdots & w_{m2}Id \\ \hline w_{23}Id & \cdots & \cdots & w_{m2}L_m^t \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline w_{2m}Id & w_{3m}Id & \cdots & w_{mm}L_m^t \end{array}\right)\right)^{(-1)}\begin{pmatrix} w_{12}Id \\ \hline w_{13}Id \\ \hline \vdots \\ \hline w_{1m}Id \end{pmatrix} p_1$$

so

$$\begin{pmatrix} w_{12}Id \\ \hline w_{13}Id \\ \hline \vdots \\ \hline w_{1m}Id \end{pmatrix} p_1 = w_{11}\begin{pmatrix} w_{12}Id \\ \hline w_{13}Id \\ \hline \vdots \\ \hline w_{1m}Id \end{pmatrix} L_1^t p_1 + \begin{pmatrix} w_{12}Id \\ \hline w_{13}Id \\ \hline \vdots \\ \hline w_{1m}Id \end{pmatrix}(w_{21}Id \ldots w_{m1}Id)\begin{pmatrix} Q_2 \\ \hline Q_3 \\ \hline \vdots \\ \hline Q_m \end{pmatrix}$$

or $$\begin{pmatrix} w_{12}Id \\ \hline w_{13}Id \\ \hline \vdots \\ \hline w_{1m}Id \end{pmatrix} p_1 = w_{11}\left(Id - \begin{array}{c|c|c|c} L_1^t & 0 & \cdots & 0 \\ \hline 0 & L_1^t & \cdots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \cdots & L_1^t \end{array}\right)\begin{pmatrix} w_{12}Id \\ \hline w_{13}Id \\ \hline \vdots \\ \hline w_{1m}Id \end{pmatrix} L_1^t p_1 +$$

$\left(\begin{array}{c} w_{12}Id \\ \hline w_{13}Id \\ \hline \vdots \\ \hline w_{1m}Id \end{array}\right) (w_{21}Id \ldots w_{m1}Id) \left(\begin{array}{c} Q_2 \\ \hline Q_3 \\ \hline \vdots \\ \hline Q_m \end{array}\right)$. This is equivalent, by the change of variables above, to $\left(\begin{array}{c} Q_2 \\ \hline Q_3 \\ \hline \vdots \\ \hline Q_m \end{array}\right)$ being an eigenvector associated to 1 of the matrix

$$\mathcal{A}_3{}^{p_1} = w_{11}L + \tilde{W}_{11}^{(3)} - w_{11}L\tilde{W}_{11}^{(3)} + \left(\begin{array}{c} w_{12}Id \\ \hline w_{13}Id \\ \hline \vdots \\ \hline w_{1m}Id \end{array}\right) (w_{21}Id \ldots w_{m1}Id)$$

where $L = \left(\begin{array}{c|c|c} L_1^t & \cdots & 0 \\ \hline 0 & L_1^t & \cdots \\ \hline \vdots & \ddots & \vdots \\ \hline 0 & \cdots & L_1^t \end{array}\right)$ and $\tilde{W}_{11}^{(3)} = \left(\begin{array}{c|c|c|c} w_{22}L_2^t & w_{32}Id & \cdots & w_{m2}Id \\ \hline w_{23}Id & w_{33}L_3^t & \cdots & w_{m2}Id \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline w_{2m}Id & w_{3m}Id & \cdots & w_{mm}L_m^t \end{array}\right)$

Once the $Q_i's$ are obtained we use the change of variables above to recover p_1 (since some of the $w_1 i \neq 0$):

$$\left(\begin{array}{c} w_{12}Id \\ \hline w_{13}Id \\ \hline \vdots \\ \hline w_{1m}Id \end{array}\right) p_1 = \left(Id - \left(\begin{array}{c|c|c|c} w_{22}L_2^t & w_{32}Id & \cdots & w_{m2}Id \\ \hline w_{23}Id & \cdots & \cdots & w_{m2}L_m^t \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline w_{2m}Id & w_{3m}Id & \cdots & w_{mm}L_m^t \end{array}\right)\right) \left(\begin{array}{c} Q_2 \\ \hline Q_3 \\ \hline \vdots \\ \hline Q_m \end{array}\right).$$

The remaining $p_i's$ are analogously calculated.

A.1 Particular case of two layers ($m = 2$)

We will show that the eigenvectors associated to the principal eigenvalue 1 can be computed in terms of the eigenvectors associated to 1 of certain matrices related to L_1^t, L_2^t and the elements of W. Instead of using the techniques of [22] we will do all the calculations directly. Moreover, we will deal with all possible cases of W under the only hypothesis that this matrix is row-stochastic.
<u>Block matrix of type $\mathbb{B}_1$, $m = 2$</u>:

$$\mathbb{B}_1 = \left(\begin{array}{c|c} w_{11}L_1^t & w_{21}L_2^t \\ \hline w_{12}L_1^t & w_{22}L_2^t \end{array}\right),$$

where L_ℓ^t is the transpose of the row normalization of the adjacency matrix of layer S_ℓ.

(a) If both $w_{11} \neq 1$ and $w_{22} \neq 1$ then if $\left(\frac{\pi_1}{\pi_2}\right)$ is an eigenvector associated to the eigenvalue 1, we have

$$\begin{cases} \pi_1 = w_{11}L_1^t\pi_1 + w_{21}L_2^t\pi_2, \\ \pi_2 = w_{12}L_1^t\pi_1 + w_{22}L_2^t\pi_2. \end{cases}$$

From here, taking into account that both $(I - w_{11}L_1^t)$ and $(I - w_{22}L_2^t)$ are invertible matrices, we get that $\pi_1 = w_{21}(I - w_{11}L_1^t)^{-1}L_2^t\pi_2$, and $\pi_2 = w_{12}(I - w_{22}L_2^t)^1 L_1^t\pi_1$. Substituting in the above equations we get

$$\begin{cases} \pi_1 = (w_{11}L_1^t + w_{12}w_{21}L_2^t(I - w_{22}L_2^t)^{-1}L_1^t)\pi_1, \\ \pi_2 = (w_{22}L_2^t + w_{12}w_{21}L_1^t(I - w_{11}L_1^t)^{-1}L_2^t)\pi_2. \end{cases}$$

Now multiplying the first equation by the matrix $(I - w_{22}L_2^t)$ on the left, and the second equation by the matrix $(I - w_{11}L_1^t)$ on the left we get

$$\begin{cases} \pi_1 = (w_{11}L_1^t + w_{22}L_2^t + (1 - w_{11} - w_{22})L_2^tL_1^t)\pi_1, \\ \pi_2 = (w_{11}L_1^t + w_{22}L_2^t + (1 - w_{11} - w_{22})L_1^tL_2^t)\pi_2, \end{cases}$$

i.e., π_1 and π_2 are eigenvectors associated to 1 to the column stochastic matrices

$$\begin{array}{l} \mathcal{A}_1^{\pi_1} = (w_{11}L_1^t + w_{22}L_2^t + (1 - w_{11} - w_{22})L_2^tL_1^t), \text{ and} \\ \mathcal{A}_1^{\pi_2} = (w_{11}L_1^t + w_{22}L_2^t + (1 - w_{11} - w_{22})L_1^tL_2^t). \end{array}$$

(b) If $w_{11} = 1$ then $w_{12} = 0$, in which case $\mathbb{B}_1$ is of the form,

$$\mathbb{B}_1 = \left(\begin{array}{c|c} L_1^t & w_{21}L_2^t \\ \hline 0 & w_{22}L_2^t \end{array}\right),$$

and if the vector $\left(\frac{\pi_1}{\pi_2}\right)$ is associated to the eigenvalue 1 then

$$\begin{cases} \pi_1 = L_1^t\pi_1 + w_{21}L_2^t\pi_2, \\ \pi_2 = w_{22}L_2^t\pi_2. \end{cases}$$

We have one of the three following situations:

(b.1) $0 < w_{22} < 1$: in this case $\pi_2 = 0$ since L_2^t is column stochastic and cannot have nonzero eigenvectors with associated to an eigenvalue $1/w_{22} > 1$. Therefore the eigenvectors associated to 1 of $\mathbb{B}_1$ have the form $\left(\frac{\pi_1}{0}\right)$ where π_1 is an eigenvector of L_1^t associated to 1.

(b.2) $w_{22} = 0$: in this case $w_{21} = 1$ and we have that the eigenvectors associated to $\mathbb{B}_1$ have the form $\left(\frac{\pi_1}{0}\right)$ where π_1 is an eigenvector of L_1^t associated to 1.

(b.3) $w_{22} = 1$: in this case W is the identity (there is no influence of a layer into another layer) and the eigenvectors of $\mathbb{B}_1$ associated to 1 have the form $\left(\frac{\pi_1}{\pi_2}\right)$ where π_1 is an eigenvector of L_1^t associated to 1 and π_2 an eigenvector of L_2^t associated to 1.
(c) If $w_{22} = 1$ then, arguing as in case (b) either $w_{11} = 1$ and we are again in the situation of (b.3) or the eigenvector of $\mathbb{B}_1$ associated to 1 are of the form $\left(\frac{0}{\pi_2}\right)$ where π_2 is an eigenvector of L_2^t associated to 1.
Block matrix of type $\mathbb{B}_2$, $m = 2$:

$$\mathbb{B}_2 = \left(\begin{array}{c|c} w_{11}L_1^t & w_{21}L_1^t \\ \hline w_{12}L_2^t & w_{22}L_2^t \end{array}\right),$$

where L_ℓ^t is the transpose of the row normalization of the adjacency matrix of layer S_ℓ.
(a) If both $w_{11} \neq 1$ and $w_{22} \neq 1$ then if $\left(\frac{\pi_1}{\pi_2}\right)$ is an eigenvector associated to the eigenvalue 1, we have

$$\begin{cases} \pi_1 = w_{11}L_1^t\pi_1 + w_{21}L_1^t\pi_2, \\ \pi_2 = w_{12}L_2^t\pi_1 + w_{22}L_2^t\pi_2. \end{cases}$$

From here, taking into account that both $(I - w_{11}L_1^t)$ and $(I - w_{22}L_2^t)$ are invertible matrices, we get that $\pi_1 = w_{21}(I - w_{11}L_1^t)^{-1}L_1^t\pi_2$, and $\pi_2 = w_{12}(I - w_{22}L_2^t)^{-1}L_2^t\pi_1$. Substituting in the above equations we get

$$\begin{cases} (I - w_{11}L_1^t)\pi_1 = w_{12}w_{21}L_1^t(I - w_{22}L_2^t)^{-1}L_2^t\pi_1, \\ (I - w_{22}L_2^t)\pi_2 = w_{12}w_{21}L_2^t(I - w_{11}L_1^t)^{-1}L_1^t\pi_2, \end{cases}$$

so using that L_1^t and $(I - w_{11}L_1^t)^{-1}$ commute and L_2^t and $(I - w_{22}L_2^t)^{-1}$ commute we have

$$\begin{cases} \pi_1 = w_{12}w_{21}L_1^t(I - w_{11}L_1^t)^{-1}(I - w_{22}L_2^t)^{-1}L_2^t\pi_1, \\ \pi_2 = w_{12}w_{21}L_2^t(I - w_{22}L_2^t)^{-1}(I - w_{11}L_1^t)^{-1}L_1^t\pi_2. \end{cases} \tag{1}$$

Let us define $\pi_1^{aux} = (I - w_{11}L_1^t)^{-1}(I - w_{22}L_2^t)^{-1}L_2^t\pi_1$ and $\pi_2^{aux} = (I - w_{22}L_2^t)^{-1}(I - w_{11}L_1^t)^{-1}L_1^t\pi_2$. . By the equations (1)

$$\begin{cases} \pi_1 = w_{12}w_{21}L_1^t\pi_1^{aux}, \\ \pi_2 = w_{12}w_{21}L_2^t\pi_2^{aux}, \end{cases} \tag{2}$$

and from (1) and (2)

$$\begin{cases} (I - w_{22}L_2^t)(I - w_{11}L_1^t)\pi_1^{aux} = L_2^t\pi_1 = L_2^t w_{12}w_{21}L_1^t\pi_1^{aux}, \\ (I - w_{22}L_2^t)(I - w_{11}L_1^t)\pi_2^{aux} = L_1^t\pi_2 = L_1^t w_{12}w_{21}L_2^t\pi_2^{aux}, \end{cases}$$

i.e., π_1^{aux} and π_2^{aux} are eigenvectors associated to 1 of the column stochastic matrices

$$\begin{aligned} \mathcal{A}_2^{\pi_1^{aux}} &= (w_{11}L_1^t + w_{22}L_2^t - w_{11}w_{22}L_2^tL_1^t + w_{12}w_{21}L_2^tL_1^t), \text{ and} \\ \mathcal{A}_2^{\pi_2^{aux}} &= (w_{11}L_1^t + w_{22}L_2^t - w_{11}w_{22}L_1^tL_2^t + w_{12}w_{21}L_1^tL_2^t). \end{aligned}$$

After computing π_1^{aux} and π_2^{aux},

$$\begin{cases} \pi_1 = w_{12}w_{21}L_1^t\pi_1^{aux}, \\ \pi_2 = w_{12}w_{21}L_2^t\pi_2^{aux}. \end{cases}$$

(b) ($w_{11} = 1$) and (c) ($w_{22} = 1$) give the same results as for matrices of type $\mathbb{B}_1$.

<u>Block matrix of type $\mathbb{B}_3$, $m = 2$</u>:

$$\mathbb{B}_2 = \left(\begin{array}{c|c} w_{11}L_1^t & w_{21}I_2 \\ \hline w_{12}I_2 & w_{22}L_2^t \end{array} \right),$$

where L_ℓ^t is the transpose of the row normalization of the adjacency matrix of layer S_ℓ.

(a) If both $w_{11} \neq 1$ and $w_{22} \neq 1$ then if $\left(\begin{array}{c} \pi_1 \\ \hline \pi_2 \end{array} \right)$ is an eigenvector associated to the eigenvalue 1, we have

$$\begin{cases} \pi_1 = w_{11}L_1^t\pi_1 + w_{21}\pi_2, \\ \pi_2 = w_{12}\pi_1 + w_{22}L_2^t\pi_2. \end{cases}$$

Taking into account that both $(I - w_{11}L_1^t)$ and $(I - w_{22}L_2^t)$ are invertible matrices, we get that $\pi_1 = w_{21}(I - w_{11}L_1^t)^{-1}\pi_2$, and $\pi_2 = w_{12}(I - w_{22}L_2^t)^{-1}\pi_1$. Substituting in the above equations we get

$$\begin{cases} (I - w_{11}L_1^t)\pi_1 = w_{12}w_{21}(I - w_{22}L_2^t)^{-1}\pi_1, \\ (I - w_{22}L_2^t)\pi_2 = w_{12}w_{21}(I - w_{11}L_1^t)^{-1}\pi_2, \end{cases}$$

so multiplying in both sides by $(I - w_{11}L_1^t)$ and $(I - w_{22}L_2^t)$ respectively we have

$$\begin{cases} w_{12}w_{21}\pi_1 = (I - w_{22}L_2^t)(I - w_{11}L_1^t)\pi_1, \\ w_{12}w_{21}\pi_2 = (I - w_{11}L_1^t)(I - w_{22}L_2^t)\pi_2. \end{cases}$$

Therefore, π_1 and π_2 are eigenvectors associated to 1 to the column stochastic matrices

$$\begin{aligned} \mathcal{A}_2^{\pi_1} &= (w_{11}L_1^t + w_{22}L_2^t - w_{11}w_{22}L_2^tL_1^t + w_{12}w_{21}I_2), \text{ and} \\ \mathcal{A}_2^{\pi_2} &= (w_{11}L_1^t + w_{22}L_2^t - w_{11}w_{22}L_1^tL_2^t + w_{12}w_{21}I_2). \end{aligned}$$

(b) ($w_{11} = 1$) and (c) ($w_{22} = 1$) give the same results as for matrices of type $\mathbb{B}_1$.