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Formulation via vector potentials of eddy-current problems with voltage or current excitation

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Abstract

The aim of this work is to analyze different formulations of the voltage excitation problem and the current intensity excitation problem for the time-harmonic eddy-current approximation of Maxwell equations in the case of a conductor with electric ports. Two formulations based on the introduction of a vector magnetic potential and their finite element approximation are analyzed.

Keywords: `Eddy current model, voltage and current excitation, vector magnetic potential, continuous vector nodal elements, edge elements.`

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1. Statement of the problem.

Let us consider the eddy-current approximation of Maxwell equations in the time-harmonic case:

$$\begin{array}{ll} \text{Ampère law:} & \operatorname{curl} \mathbf{H} = \mathbf{J} \\ \text{Faraday law:} & \operatorname{curl} \mathbf{E} + i\omega \mathbf{B} = \mathbf{0}\,. \end{array}$$

Here $\mathbf{H}$ and $\mathbf{E}$ are the magnetic and electric field respectively, $\mathbf{B}$ is the magnetic induction, $\mathbf{J}$ is the electric current density and $\omega \neq 0$ is a given angular frequency. Assuming linear materials $\mathbf{B} = \boldsymbol{\mu}\mathbf{H}$ where $\boldsymbol{\mu}$ is the magnetic permeability that is a symmetric tensor, uniformly positive definite with entries that are bounded functions of the space variable. The classic Ohm law, based on physical observations about electrical circuits, states that $\mathbf{J} = \boldsymbol{\sigma}\mathbf{E}$ where $\boldsymbol{\sigma}$ is the electric conductivity that is vanishing in insulators while in conducting regions it is a symmetric tensor, uniformly positive definite with entries that are bounded functions of the space variable.

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We study the coupling of the eddy-current model

$$
\begin{aligned}
\operatorname{curl} \mathbf{H} - \boldsymbol{\sigma}\mathbf{E} &= \mathbf{0} \\
\operatorname{curl} \mathbf{E} + i\omega\boldsymbol{\mu}\mathbf{H} &= \mathbf{0}
\end{aligned} \tag{1}
$$

with a formulation in terms of a electric circuit through the "electric ports". We consider a computational domain $\Omega \subset \mathbb{R}^3$ simply-connected and bounded with connected boundary $\partial\Omega$. It is composed by two parts, a conductor Ω_C and an insulator $\Omega_D := \Omega \setminus \overline{\Omega_C}$. For the sake of simplicity we will assume that both the conductor and the insulator are connected. The conductor Ω_C is not strictly contained in Ω; we assume that $\Gamma_C := \partial\Omega_C \cap \partial\Omega \neq \emptyset$ has at least two disjoint connected components, the so-called "electric ports": $\cup_{k=0}^{K}\Gamma_{C,k} = \Gamma_C$ with $K \geq 1$. The coupling of the eddy-current model and the electric circuit is modelized assigning a voltage or a current intensity on the electric ports. We impose the following boundary conditions:

$$
\boldsymbol{\mu}\mathbf{H} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega\,, \qquad \mathbf{E} \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_C\,. \tag{2}
$$

Since $\boldsymbol{\mu}\mathbf{H} \cdot \mathbf{n} = 0$ from Faraday law follows that $\operatorname{div}_\tau(\mathbf{E} \times \mathbf{n}) = 0$ on $\partial\Omega$, hence $\mathbf{E} \times \mathbf{n} = \nabla\phi \times \mathbf{n}$ for some $\phi \in H^1(\Omega)$. The second boundary condition ensures that ϕ is constant on each connected component of Γ_C. In particular we can consider $\phi_{|\Gamma_{C,0}} = 0$. In the voltage excitation problem the constant value of ϕ in the electric ports $\Gamma_{C,k}$ is assigned:

$$
\phi_{|\Gamma_{C,k}} = V_k\,, \quad k = 1, \dots, K\,. \tag{3}
$$

On the other hand, the current excitation problem impose the current intensity through the electric ports

$$
\int_{\Gamma_{C,k}} \operatorname{curl} \mathbf{H} \cdot \mathbf{n} = I_k\,, \quad k = 1, \dots, K\,. \tag{4}
$$

(As usually $\mathbf{n}$ indicates the unit outward normal vector on $\partial\Omega$.)

The set of equations (1), (2) and (3) or (4) do not determine univocally the electric field in the insulator Ω_D but the magnetic field $\mathbf{H}$ and the electric field in the conductor $\mathbf{E}_C$ are unique. In fact, integrating by parts Faraday law one has

$$
-i\omega \int_\Omega \boldsymbol{\mu}\mathbf{H} \cdot \overline{\mathbf{H}} = \int_\Omega \operatorname{curl} \mathbf{E} \cdot \overline{\mathbf{H}} = \int_\Omega \mathbf{E} \cdot \operatorname{curl} \overline{\mathbf{H}} - \int_{\partial\Omega} \mathbf{E} \times \mathbf{n} \cdot \overline{\mathbf{H}}\,.
$$

Using Ampère law and the fact that $\mathbf{E} \times \mathbf{n} = \nabla\phi \times \mathbf{n}$ on $\partial\Omega$ for some $\phi \in H^1(\Omega)$

$$
\int_\Omega \mathbf{E} \cdot \operatorname{curl} \overline{\mathbf{H}} - \int_{\partial\Omega} \mathbf{E} \times \mathbf{n} \cdot \overline{\mathbf{H}} = \int_{\Omega_C} \mathbf{E} \cdot \overline{\boldsymbol{\sigma}\mathbf{E}} + \int_{\partial\Omega} \overline{\mathbf{H}} \times \mathbf{n} \cdot \nabla\phi
$$

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$$= \int_{\Omega_C} \mathbf{E} \cdot \overline{\boldsymbol{\sigma}\mathbf{E}} - \int_{\partial\Omega} \operatorname{curl} \overline{\mathbf{H}} \cdot \mathbf{n}\, \phi\,.$$

Finally since $\operatorname{curl} \mathbf{H} = \mathbf{0}$ in Ω_D and ϕ is constant on each connected component of Γ_C with $\phi_{|\Gamma_{C,0}} = 0$, we obtain the power law

$$\text{(5)} \qquad i\omega \int_{\Omega} \boldsymbol{\mu}\mathbf{H} \cdot \overline{\mathbf{H}} + \int_{\Omega_C} \mathbf{E}_C \cdot \overline{\boldsymbol{\sigma}\mathbf{E}_C} = \sum_{k=1}^{K} \phi_{|\Gamma_{C,k}} \int_{\Gamma_{C,k}} \operatorname{curl} \overline{\mathbf{H}} \cdot \mathbf{n}\,.$$

Since $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ are uniformly positive definite in Ω and Ω_C respectively, if the right hand size term is equal zero then $\mathbf{H} = \mathbf{0}$ and $\mathbf{E}_C = \mathbf{0}$.

It is possible to give a characterization of the voltage in terms of $\mathbf{H}$ and $\mathbf{E}_C$. For each $k = 1, \ldots, K$ let γ_k be an oriented path on $\Gamma := \partial\Omega_C \cap \partial\Omega_D$ connecting a point on the boundary of Γ_0 with a point on the boundary of Γ_k. Let as assume that there exists an orientable surface $\Sigma_k \subset \Omega_D$ such that $\gamma_k \subset \partial\Sigma_k$ and $\partial\Sigma_k \setminus \gamma_k \subset \partial\Omega$ (see Figure 1). Then from Faraday law

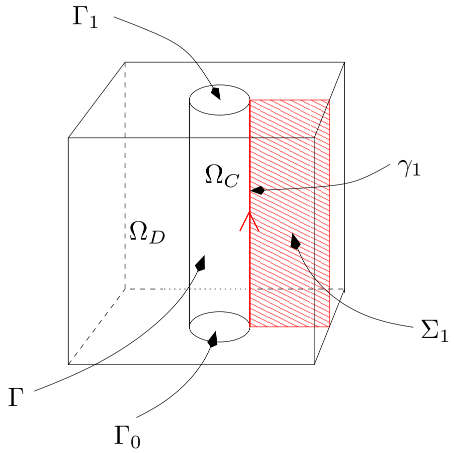

Figure 1. The computational domain with electric ports.

and the Stokes theorem

$$-i\omega \int_{\Sigma_k} \boldsymbol{\mu}\mathbf{H} \cdot \mathbf{n} = \int_{\Sigma_k} \operatorname{curl} \mathbf{E} \cdot \mathbf{n} = \int_{\partial\Sigma_k} \mathbf{E} \cdot \boldsymbol{\tau} = \int_{\gamma_k} \mathbf{E}_C \cdot \boldsymbol{\tau} - V_k\,.$$

Hence

$$V_k = \int_{\gamma_k} \mathbf{E}_C \cdot \boldsymbol{\tau} + i\omega \int_{\Sigma_k} \boldsymbol{\mu}\mathbf{H} \cdot \mathbf{n}\,.$$

So we can consider a reduced problems where Faraday law is required only in Ω_C, namely, we look for $(\mathbf{H}, \mathbf{E}_C)$ such that

$$
\text{(6)} \qquad \begin{array}{ll} \operatorname{curl} \mathbf{H} - \boldsymbol{\sigma} \mathbf{E}_C = \mathbf{0} & \text{in } \Omega \\ \operatorname{curl} \mathbf{E}_C + i\omega \boldsymbol{\mu} \mathbf{H} = \mathbf{0} & \text{in } \Omega_C \\ \boldsymbol{\mu} \mathbf{H} \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \\ \mathbf{E}_C \times \mathbf{n} = \mathbf{0} & \text{on } \Gamma_C \end{array}
$$

and

$$
\text{(7)} \qquad \int_{\gamma_k} \mathbf{E}_C \cdot \boldsymbol{\tau} + i\omega \int_{\Sigma_k} \boldsymbol{\mu} \mathbf{H} \cdot \mathbf{n} = V_k \,. \quad k = 1, \ldots, K
$$

being $\mathbf{V}$ a given vector in $\mathbb{C}^K$, or

$$
\text{(8)} \qquad \int_{\Gamma_k} \operatorname{curl} \mathbf{H} \cdot \mathbf{n} = I_k \,, \quad k = 1, \ldots, K
$$

with $\mathbf{I}$ a given vector in $\mathbb{C}^K$.

Each one of these two problems has a unique solution. It will be part of solution of the complete eddy current problem if there exists an electric field in Ω_D that extends the electric field defined in the conductor and such that Faraday law holds in the whole computational domain Ω, i.e., if there exists $\mathbf{E}_D$ such that

$$
\text{(9)} \qquad \begin{array}{ll} \operatorname{curl} \mathbf{E}_D = -i\omega \boldsymbol{\mu} \mathbf{H}_D & \text{on } \Omega_D \\ \mathbf{E}_D \times \mathbf{n} = \mathbf{E}_C \times \mathbf{n} & \text{on } \Gamma \,. \end{array}
$$

Clearly a necessary condition for the existence of solution of (9) is that for any function $\mathbf{w}_D \in (L^2(\Omega_D))^3$ such that $\operatorname{curl} \mathbf{w}_D = \mathbf{0}$ and $\mathbf{w}_D \times \mathbf{n} = \mathbf{0}$ in $\partial\Omega_D \setminus \Gamma$

$$
\text{(10)} \qquad -i\omega \int_{\Omega_D} \boldsymbol{\mu} \mathbf{H}_D \cdot \overline{\mathbf{w}}_D = \int_{\Gamma} \mathbf{E}_C \times \mathbf{n}_C \cdot \overline{\mathbf{w}}_D \,.
$$

($\mathbf{n}_C = -\mathbf{n}_D$ denotes the unit normal vector on Γ, directed towards Ω_D.) This if in fact also a sufficient condition for the existence of solution of (9). To show it let us introduce some notation. The space $H(\operatorname{curl};\Omega)$ (respectively $H(\operatorname{div};\Omega)$) indicates the set of functions $\mathbf{w} \in (L^2(\Omega))^3$ such that $\operatorname{curl} \mathbf{w} \in (L^2(\Omega))^3$ (respectively $\operatorname{div} \mathbf{w} \in L^2(\Omega)$). $H_0(\operatorname{curl};\Omega)$ denotes the space of functions $\mathbf{w} \in H(\operatorname{curl};\Omega)$ with vanishing tangential trace in $\partial\Omega$. By $H^0(\operatorname{div};\Omega)$ we denote the set of functions belonging to $H(\operatorname{div};\Omega)$ with vanish divergence in Ω. Given a certain subset $\Lambda \subset \partial\Omega$, we denote by $H_{0,\Lambda}(\operatorname{curl};\Omega)$ the space of function belonging to $H(\operatorname{curl};\Omega)$ with vanishing tangential trace in Λ and by $H_{0,\Lambda}(\operatorname{div};\Omega)$ the space of function belonging to $H(\mathrm{div};\Omega)$ with vanishing normal component in Λ. Here and in the follow, for easy of reading we express duality pairing by (surface) integrals. In particular, for all $\mathbf{s}$, $\mathbf{w} \in H(\mathrm{curl};\Omega)$

$$\int_\Omega (\mathbf{s}\cdot \mathrm{curl}\,\mathbf{w} - \mathrm{curl}\,\mathbf{s}\cdot\mathbf{w}) = \int_{\partial\Omega} \mathbf{s}\times\mathbf{n}\cdot\mathbf{w}\,.$$

If $\int_{\Omega_D} \mathbf{F}_D \cdot \overline{\mathbf{w}_D} = -\int_\Gamma \boldsymbol{\lambda}\cdot\overline{\mathbf{w}_D}$ for all $\mathbf{w}_D \in H^0_{0,\Gamma_D}(\mathrm{curl},\Omega_D) := H^0(\mathrm{curl};\Omega_D)\cap H_{0,\Gamma_D}(\mathrm{curl};\Omega_D)$ then the system

$$\begin{aligned}
&\mathrm{curl}\,\mathbf{v}_D = \mathbf{F}_D \text{ in } \Omega_D\\
&\mathrm{div}\,\mathbf{v}_D = 0 \quad\ \text{ in } \Omega_D\\
&\mathbf{v}_D\times\mathbf{n}_D = \boldsymbol{\lambda} \text{ on } \Gamma\\
&\mathbf{v}_D\cdot\mathbf{n}_D = 0 \quad \text{ on } \Gamma_D
\end{aligned}$$

has a unique solution $\mathbf{v}_D \in H(\mathrm{curl};\Omega_D)\cap H^0_{0,\Gamma_D}(\mathrm{div};\Omega_D)\cap[H^0_{0,\Gamma}(\mathrm{curl};\Omega_D)\cap H^0_{0,\Gamma_D}(\mathrm{div};\Omega_D)]^\perp$. For the proof see, e.g., [1]; the main idea is to write $\mathbf{v}_D = \mathrm{curl}\,\mathbf{q}_D$ with $\mathbf{q}_D \in V := H_{0,\Gamma_D}(\mathrm{curl};\Omega_D)\cap H_{0,\Gamma}(\mathrm{div};\Omega_D)\cap[H^0_{0,\Gamma_D}(\mathrm{curl};\Omega_D)\cap H^0_{0,\Gamma}(\mathrm{div};\Omega_D)]^\perp$ such that

$$\int_{\Omega_D} [\mathrm{curl}\,\mathbf{q}_D\cdot\mathrm{curl}\,\overline{\mathbf{p}_D} + \mathrm{div}\,\mathbf{q}_D\ \mathrm{div}\,\overline{\mathbf{p}_D}] = \int_{\Omega_D}\mathbf{F}_D\cdot\overline{\mathbf{p}_D} - \int_\Gamma \boldsymbol{\lambda}\cdot\overline{\mathbf{p}_D}$$

for all $\mathbf{p}_D \in V$.

Notice that $H^0_{0,\Gamma_D}(\mathrm{curl};\Omega_D) = \nabla H^1_{0,\Gamma_D}(\Omega_D)\oplus\mathcal{H}(\Gamma_D,\Gamma;\Omega_D)$ where $\mathcal{H}(\Gamma_D,\Gamma;\Omega_D) := H^0_{0,\Gamma_D}(\mathrm{curl};\Omega_D)\cap H^0_{0,\Gamma}(\mathrm{div};\Omega_D)$. Hence (10) is equivalent to

$$\mathrm{div}(\boldsymbol{\mu}\mathbf{H}_D) = 0\,,\qquad \boldsymbol{\mu}\mathbf{H}_C\cdot\mathbf{n} = \boldsymbol{\mu}\mathbf{H}_C\cdot\mathbf{n}\quad \text{on } \Gamma \tag{11}$$

and

$$-i\omega\int_{\Omega_D}\boldsymbol{\mu}\mathbf{H}_D\cdot\overline{\boldsymbol{\rho}}_D = \int_\Gamma \mathbf{E}_C\times\mathbf{n}_C\cdot\overline{\boldsymbol{\rho}}_D \quad \text{for all } \boldsymbol{\rho}_D\in\mathcal{H}(\Gamma_D,\Gamma;\Omega_D)\,. \tag{12}$$

In conclusion, the solution of (6) and (7) or (8) can be extended to a solution of (1), (2) and (3) or (4) if and only if $\mathrm{div}(\boldsymbol{\mu}\mathbf{H}) = \mathbf{0}$ in Ω and (12).

Several formulations have been proposed in recent years for the voltage and current excitation problem. For instance in [2] the problem with electric ports is formulated in terms of the magnetic field and the input current intensity is imposed by means of a Lagrange multiplier. In [3] the main unknowns are the electric field in the conductor an the magnetic field in the insulator. In [4] and [5] the problem is described in terms of a current vector potential and a magnetic scalar potential, using the so-called $\mathbf{T} - \mathbf{T}_0 - \phi$ formulation.

Formulations in terms of a magnetic vector potential have been proposed in [6], [7] and [8]. Hiptmair and Sterz present in [6] a systematic study of how to take into account voltage and current excitations in the eddy current model. They consider two different formulations of the time-dependent eddy current model: a formulation in terms of the magnetic field and also a formulation in terms of a magnetic vector potential. In the harmonic regime Bermudez propose in [7] a formulation for the current intensity excitation problem in terms of a vector magnetic potential and a scalar electric potential defined in the whole computational domain. More recently Chen *et al.* analyze in [8] a formulation for the voltage excitation problem in the harmonic regime in terms of a vector magnetic potential that do not need to compute a scalar electric potential.

The aim of this work is to discuss different formulations of the voltage and the current intensity excitation problem in terms of a vector magnetic potential defined in Ω. For both problems we analyze in detail a classical formulation, similar to the one proposed in [7] for the current intensity excitation problem, but that use a scalar electric potential defined only in the conductor. We also extend to the current intensity excitation problem the formulation analyzed in [8] that do not compute the scalar electric potential. From the computational point of view the formulation by Chen *et al.* of the voltage excitation problem is the one with the minor number of unknowns. It extension to the current excitation problem is still the one with the minor number of unknowns if the conductor has only two electric ports. In the case of $K > 1$ the formulation that computes also a scalar electric potential in the conductor seems to be more convenient.

2. Weak formulation.

For the sake of simplicity in the following we consider a simply connected conductor Ω_C contained in Ω with two electric ports like in Figure 1. It is worth noting that since Ω_C is simply connected then $\mathcal{H}(\Gamma_D, \Gamma; \Omega_D)$ is trivial. So we are interested in solving

$$
\begin{array}{ll}
\operatorname{curl} \mathbf{H} - \boldsymbol{\sigma} \mathbf{E}_C = \mathbf{0} & \text{in } \Omega \\
\operatorname{curl} \mathbf{E}_C + i\omega \boldsymbol{\mu} \mathbf{H} = \mathbf{0} & \text{in } \Omega_C \\
\operatorname{div}(\boldsymbol{\mu} \mathbf{H}) = 0 & \text{in } \Omega \\
\boldsymbol{\mu} \mathbf{H} \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \\
\mathbf{E}_C \times \mathbf{n} = \mathbf{0} & \text{on } \Gamma_C
\end{array}
\tag{13}
$$

and

$$(14)\qquad \int_{\gamma_1} \mathbf{E}_C \cdot \boldsymbol{\tau} + i\omega \int_{\Sigma_1} \boldsymbol{\mu}\mathbf{H} \cdot \mathbf{n} = V$$

or

$$(15)\qquad \int_{\Gamma_1} \operatorname{curl} \mathbf{H} \cdot \mathbf{n} = I$$

where V and I are assigned complex numbers.

Since $\operatorname{div}(\boldsymbol{\mu}\mathbf{H}) = 0$, a classical approach introduces a vector magnetic potential $\mathbf{A}$ such that $\operatorname{curl} \mathbf{A} = \boldsymbol{\mu}\mathbf{H}$. This is also accompanied by the use of a scalar electric potential U_C in the conductor Ω_C satisfying $\mathbf{E}_C = -i\omega\mathbf{A}_C + \nabla U_C$ where $\mathbf{E}_C := \mathbf{E}_{|\Omega_C}$ and $\mathbf{A}_C := \mathbf{A}_{|\Omega_C}$. In this way Faraday law in Ω_C is clearly verified and Ampère law turns to be

$$\operatorname{curl}(\boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{A}) + \boldsymbol{\sigma}(i\omega\mathbf{A}_C - \nabla U_C) = \mathbf{0}.$$

Concerning the boundary conditions, we impose $\mathbf{A} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$ to ensure $\boldsymbol{\mu}\mathbf{H} \cdot \mathbf{n} = 0$ on $\partial\Omega$. Then, in order to have $\mathbf{E}_C \times \mathbf{n} = \mathbf{0}$ on Γ_C it must be U_C constant on Γ_0 and constant also on Γ_1. In particular we take $U_{C|\Gamma_0} \equiv 0$ and in this way

$$\int_{\gamma_1} (-i\omega\mathbf{A}_C + \nabla U_C) \cdot \boldsymbol{\tau} + i\omega \int_{\Sigma_1} \operatorname{curl} \mathbf{A} = \int_{\gamma_1} \nabla U_C \cdot \boldsymbol{\tau} = U_{C|\Gamma_1}\,.$$

So, we look for $(\mathbf{A}, U_C)$ such that

$$(16)\qquad \begin{cases} \operatorname{curl}(\boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{A}) + \boldsymbol{\sigma}(i\omega\mathbf{A}_C - \nabla U_C) = \mathbf{0} & \text{in } \Omega \\ \mathbf{A} \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega \\ U_C = 0 & \text{on } \Gamma_0 \\ U_C \equiv \text{ constant} & \text{on } \Gamma_1\,. \end{cases}$$

In the voltage excitation problem the constant value of U_C on Γ_1 is assigned so $(16)_4$ is replaced by

$$U_C = V \text{ on } \Gamma_1\,.$$

In the current excitation problem, given $I \in \mathbb{C}$ it must be

$$\int_{\Gamma_1} \boldsymbol{\sigma}(-i\omega\mathbf{A}_C + \nabla U_C) \cdot \mathbf{n} = I$$

and this equation must be added to (16).

Let us consider the space

$$H^1_\sharp(\Omega_C) := \{Q_C \in H^1_{0,\Gamma_E}(\Omega_C) \,|\, Q_{C|\Gamma_J} \text{ is constant } \}\,.$$

We are looking for $(\mathbf{A}, U_C) \in H_0(\mathrm{curl};\Omega) \times H^1_\sharp(\Omega_C)$ such that for each $\mathbf{w} \in H_0(\mathrm{curl};\Omega)$

$$
\begin{aligned}
0 &= \textstyle\int_\Omega \left[\mathrm{curl}(\boldsymbol{\mu}^{-1}\,\mathrm{curl}\,\mathbf{A}) + \boldsymbol{\sigma}\,(i\omega\mathbf{A}_C - \nabla U_C)\right]\cdot\overline{\mathbf{w}}\\
&= \textstyle\int_\Omega \left[\boldsymbol{\mu}^{-1}\,\mathrm{curl}\,\mathbf{A}\cdot\mathrm{curl}\,\overline{\mathbf{w}} + \boldsymbol{\sigma}\,(i\omega\mathbf{A}_C - \nabla U_C)\cdot\overline{\mathbf{w}}\right]\\
&= \textstyle\int_\Omega \boldsymbol{\mu}^{-1}\,\mathrm{curl}\,\mathbf{A}\cdot\mathrm{curl}\,\overline{\mathbf{w}} + i\omega^{-1}\int_{\Omega_C}\boldsymbol{\sigma}\,(i\omega\mathbf{A}_C - \nabla U_C)\cdot\overline{i\omega\mathbf{w}}\,.
\end{aligned}
\tag{17}
$$

Moreover, since $\mathrm{div}\,[\boldsymbol{\sigma}(i\omega\mathbf{A}_C - \nabla U_C)] = 0$ in Ω_C and $\boldsymbol{\sigma}(i\omega\mathbf{A}_C - \nabla U_C)\cdot\mathbf{n} = 0$ on Γ, hence for each $Q_C \in H^1_\sharp(\Omega_C)$

$$
\int_{\Omega_C}\boldsymbol{\sigma}\,(i\omega\mathbf{A}_C - \nabla U_C)\cdot\nabla\overline{Q}_C = \left(\int_{\Gamma_1}\boldsymbol{\sigma}\,(i\omega\mathbf{A}_C - \nabla U_C)\cdot\mathbf{n}\right)\overline{Q}_{C|\Gamma_1}\,.
\tag{18}
$$

So we are looking for $(\mathbf{A}, U_C) \in H_0(\mathrm{curl};\Omega) \times H^1_\sharp(\Omega_C)$ such that for all $(\mathbf{w}, Q_C) \in H_0(\mathrm{curl};\Omega) \times H^1_\sharp(\Omega_C)$

$$
\begin{aligned}
\int_\Omega \boldsymbol{\mu}^{-1}\,\mathrm{curl}\,\mathbf{A}\cdot\mathrm{curl}\,\overline{\mathbf{w}} + i\omega^{-1}\int_{\Omega_C}&\boldsymbol{\sigma}\,(i\omega\mathbf{A}_C - \nabla U_C)\cdot\overline{(i\omega\mathbf{w} - \nabla Q_C)}\\
&= -i\omega^{-1} I\,\overline{Q}_{C|\Gamma_1}
\end{aligned}
$$

with $I = \int_{\Gamma_1}\boldsymbol{\sigma}\,(-i\omega\mathbf{A}_C + \nabla U_C)\cdot\mathbf{n}$.

Let us introduce the sesquilinear form in $H(\mathrm{curl};\Omega) \times H^1(\Omega_C)$

$$
\begin{aligned}
\mathcal{A}[(\mathbf{s}, Z_C), (\mathbf{w}, Q_C)] := &\int_\Omega \boldsymbol{\mu}^{-1}\,\mathrm{curl}\,\mathbf{s}\cdot\mathrm{curl}\,\overline{\mathbf{w}}\\
&+ i\omega^{-1}\int_{\Omega_C}\boldsymbol{\sigma}(i\omega\mathbf{s} - \nabla Z_C)\cdot\overline{(i\omega\mathbf{w} - \nabla Q_C)}\,.
\end{aligned}
\tag{19}
$$

In the voltage excitation problem given $V \in \mathbb{C}$ we seek $(\mathbf{A}, U_C) \in H_0(\mathrm{curl};\Omega) \times H^1_\sharp(\Omega_C)$ such that

$$
\begin{aligned}
&U_{C|\Gamma_1} = V\\
&\mathcal{A}[(\mathbf{A}, U_C), (\mathbf{w}, Q_C)] = 0
\end{aligned}
$$

for all $(\mathbf{w}, Q_C) \in H_0(\mathrm{curl};\Omega) \times H^1_{0,\Gamma_C}(\Omega_C)$.

In the current intensity excitation problem given $I \in \mathbb{C}$ we seek $(\mathbf{A}, U_C) \in H_0(\mathrm{curl};\Omega) \times H^1_\sharp(\Omega_C)$ such that

$$
\mathcal{A}[(\mathbf{A}, U_C), (\mathbf{w}, Q_C)] = -i\omega^{-1} I\,\overline{Q}_{C|\Gamma_1}\,.
$$

for all $(\mathbf{w}, Q_C) \in H_0(\mathrm{curl};\Omega) \times H^1_\sharp(\Omega_C)$.

It is worth noting that in both problems the vector magnetic potential is not unique. Different gauge conditions to identify a unique $\mathbf{A}$ can

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be imposed. One possibility is to look for a vector magnetic potential $\mathbf{A}$ such that $\operatorname{div} \mathbf{A} = 0$ (Coulomb gauge). Another possibility is to choose a particular function $U_C^* \in H^1(\Omega_C)$ such that $U^*_{C|\Gamma_0} = 0$ and $U^*_{C|\Gamma_1} = 1$ and to look for $U_C = V\, U_C^*$ and $\mathbf{A} \in H_0(\operatorname{curl}; \Omega)$ such that $\operatorname{div} \mathbf{A} = 0$ on Ω_D. In the voltage excitation problem V is a data while in the current intensity excitation problem it is an unknown.

2.1. *Coulomb gauge in the whole computational domain.*

Theorem 2.1. *The sesquilinear form $\mathcal{A}(\cdot,\cdot)$ defined by (19) is continuous and coercive in $[H_0(\operatorname{curl}; \Omega) \cap H^0(\operatorname{div}; \Omega)] \times H^1(\Omega_C)$.*

Proof. First we will show that there exists a positive constant α such that

$$|\mathcal{A}[(\mathbf{w}, Q_C), (\mathbf{w}, Q_C)]| \geq \alpha \left(\|\operatorname{curl} \mathbf{w}\|^2_{0,\Omega} + \|\nabla Q_C\|^2_{0,\Omega_C} \right)$$

for all $(\mathbf{w}, Q_C) \in [H_0(\operatorname{curl}; \Omega) \cap H^0(\operatorname{div}; \Omega)] \times H^1(\Omega_C)$.

Since $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ are symmetric tensors with entries in $L^\infty(\Omega)$ and $L^\infty(\Omega_C)$ and uniformly positive definite in Ω and Ω_C respectively, there exists a positive constant K such that for all $(\mathbf{w}, Q_C) \in H(\operatorname{curl}; \Omega) \times H^1(\Omega_C)$

$$\begin{aligned} &|\mathcal{A}[(\mathbf{w}, Q_C), (\mathbf{w}, Q_C)]| \\ &\qquad \geq K \left[\|\operatorname{curl} \mathbf{w}\|^2_{0,\Omega} + |\omega|^{-1} \|i\omega \mathbf{w} - \nabla Q_C\|^2_{0,\Omega_C} \right]. \end{aligned} \tag{20}$$

Using that

$$\|i\omega \mathbf{w} + \nabla Q_C\|^2_{0,\Omega_C} \geq \omega^2 \left(1 - \frac{1}{\gamma}\right) \|\mathbf{w}\|^2_{0,\Omega_C} + (1-\gamma) \|\nabla Q_C\|^2_{0,\Omega_C}$$

for any $\gamma > 0$ one has

$$\begin{aligned} &|\mathcal{A}[(\mathbf{w}, Q_C), (\mathbf{w}, Q_C)]| \\ &\qquad \geq K \left[\|\operatorname{curl} \mathbf{w}\|^2_{0,\Omega} + |\omega| \left(1 - \frac{1}{\gamma}\right) \|\mathbf{w}\|^2_{0,\Omega_C} + |\omega|^{-1} (1-\gamma) \|\nabla Q_C\|^2_{0,\Omega_C} \right]. \end{aligned}$$

Since $\partial\Omega$ is connected there exists a positive constant C such that

$$\|\mathbf{w}\|^2_{0,\Omega} \leq C \left(\|\operatorname{curl} \mathbf{w}\|^2_{0,\Omega} + \|\operatorname{div} \mathbf{w}\|^2_{0,\Omega} \right) \tag{21}$$

for all $\mathbf{w} \in H_0(\operatorname{curl}; \Omega) \cap H(\operatorname{div}; \Omega)$ (see [9]). Hence if $\mathbf{w} \in H_0(\operatorname{curl}; \Omega) \cap H^0(\operatorname{div}; \Omega)$ and $\gamma \in (0, 1)$

$$\begin{aligned} &|\mathcal{A}[(\mathbf{w}, Q_C), (\mathbf{w}, Q_C)]| \\ &\qquad \geq K \left[1 + C|\omega| \left(1 - \frac{1}{\gamma}\right) \right] \|\operatorname{curl} \mathbf{w}\|^2_{0,\Omega} + K|\omega|^{-1} (1-\gamma) \|\nabla Q_C\|^2_{0,\Omega_C} \end{aligned}$$

In particular taking γ such that $\frac{C|\omega|}{1+C|\omega|} < \gamma < 1$ one has

$$\alpha := K \min \left[1 + C|\omega|(1 - 1/\gamma), |\omega|^{-1}(1 - \gamma)\right] > 0\,.$$

From (21) and Poincarè inequality follows that the sesquilinear form $\mathcal{A}(\cdot,\cdot)$ is coercive in $[H_0(\mathrm{curl};\Omega) \cap H^0(\mathrm{div};\Omega)] \times H^1_{0,\Gamma_C}(\Omega_C)$ and also in $[H_0(\mathrm{curl};\Omega) \cap H^0(\mathrm{div};\Omega)] \times H^1_\sharp(\Omega_C)$ (because $H^1_\sharp(\Omega_C)$ is a closed subspace of $H^1_{0,\Gamma_0}(\Omega_C)$).

The continuity follows from the fact that both $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ are symmetric tensors with entries that are bounded functions of the space variable in Ω and Ω_C respectively. □

As a direct consequence of Lax-Milgram lemma, given $V \in \mathbb{C}$ there exists a unique $(\mathbf{A}, U_C) \in [H_0(\mathrm{curl};\Omega) \cap H^0(\mathrm{div};\Omega)] \times H^1_\sharp(\Omega_C)$ such that

$$\begin{aligned} &U_{C|\Gamma_1} = V \\ &\mathcal{A}[(\mathbf{A}, U_C), (\mathbf{w}, Q_C)] = 0 \end{aligned} \tag{22}$$

for all $(\mathbf{w}, Q_C) \in [H_0(\mathrm{curl};\Omega) \cap H^0(\mathrm{div};\Omega)] \times H^1_{0,\Gamma_C}(\Omega_C)$.

Analogously given $I \in \mathbb{C}$ there exists a unique $(\mathbf{A}, U_C) \in [H_0(\mathrm{curl};\Omega) \cap H^0(\mathrm{div};\Omega)] \times H^1_\sharp(\Omega_C)$ such that

$$\mathcal{A}[(\mathbf{A}, U_C), (\mathbf{w}, Q_C)] = -i\omega^{-1}\, I\, \overline{Q}_{C|\Gamma_1}\,. \tag{23}$$

for all $(\mathbf{w}, Q_C) \in [H_0(\mathrm{curl};\Omega) \cap H^0(\mathrm{div};\Omega)] \times H^1_\sharp(\Omega_C)$.

Now we will show that if $(\mathbf{A}, U_C)$ is the solution of (22) or (23) then

$$\mathrm{curl}(\boldsymbol{\mu}^{-1}\,\mathrm{curl}\,\mathbf{A}) + \boldsymbol{\sigma}\,(i\omega\mathbf{A}_C - \nabla U_C) = 0\,.$$

Notice that in both cases $\mathcal{A}[(\mathbf{A}, U_C), (\mathbf{w}, Q_C)] = 0$ for all $(\mathbf{w}, Q_C) \in H_0(\mathrm{curl};\Omega) \times H^1_{0,\Gamma_C}(\Omega_C)$. In fact if $\mathbf{w} \in H_0(\mathrm{curl};\Omega)$ let us consider $\phi \in H^1_0(\Omega)$ such that $\int_\Omega \nabla\phi \cdot \nabla\overline{\psi} = \int_\Omega \mathbf{w} \cdot \nabla\overline{\psi}$ for all $\psi \in H^1_0(\Omega)$. Then $\mathbf{w} - \nabla\phi \in H_0(\mathrm{curl};\Omega) \cap H^0(\mathrm{div};\Omega)$. It is easy to see that for each $(\mathbf{s}, Z_C) \in H(\mathrm{curl};\Omega) \times H^1(\Omega_C)$ and $\psi \in H^1_0(\Omega)$

$$\mathcal{A}[(\mathbf{s}, Z_C), (\nabla\psi, 0)] = \int_{\Omega_C} \boldsymbol{\sigma}(i\omega\mathbf{s} - \nabla Z_C) \cdot \overline{\nabla\psi} = i\omega\mathcal{A}[(\mathbf{s}, Z_C), (\mathbf{0}, \psi_C)]$$

where $\psi_C := \psi_{|\Omega_C} \in H^1_{0,\Gamma_C}(\Omega)$. Then for each $(\mathbf{w}, Q_C) \in H_0(\mathrm{curl};\Omega) \times H^1_{0,\Gamma_C}(\Omega_C)$

$$\begin{aligned} 0 &= \mathcal{A}[(\mathbf{A}, U_C), (\mathbf{w} - \nabla\phi, Q_C)] \\ &= \mathcal{A}[(\mathbf{A}, U_C), (\mathbf{w}, Q_C)] - \mathcal{A}[(\mathbf{A}, U_C), (\nabla\phi, 0)] \\ &= \mathcal{A}[(\mathbf{A}, U_C), (\mathbf{w}, Q_C)] - i\omega\mathcal{A}[(\mathbf{A}, U_C), (\mathbf{0}, \phi_C)] \\ &= \mathcal{A}[(\mathbf{A}, U_C), (\mathbf{w}, Q_C)]\,. \end{aligned}$$

Since $(C_0^\infty(\Omega))^3 \subset H_0(\mathrm{curl};\Omega)$ on has

$$\int_\Omega \boldsymbol{\mu}^{-1}\,\mathrm{curl}\,\mathbf{A}\cdot\mathrm{curl}\,\overline{\mathbf{w}} + \int_{\Omega_C} \boldsymbol{\sigma}\,(i\omega\mathbf{A}_C - \nabla U_C)\cdot\overline{\mathbf{w}} = 0$$

for all $\mathbf{w} \in (C_0^\infty(\Omega))^3$, then $\mathrm{curl}(\boldsymbol{\mu}^{-1}\,\mathrm{curl}\,\mathbf{A}) + \boldsymbol{\sigma}\,(i\omega\mathbf{A}_C - \nabla U_C) = 0$ in the sense of distributions and also in $(L^2(\Omega))^3$.

2.2. *Coulomb gauge in the insulator.*

Let us introduce the space

$$\mathbf{X}^0 := \{\mathbf{w} \in H_0(\mathrm{curl};\Omega) \,:\, \mathbf{w}_D \in H^0(\mathrm{div};\Omega_D)\}$$

and a function $U_C^* \in H^1(\Omega_C)$ such that $U^*_{C|\Gamma_0} = 0$ and $U^*_{C|\Gamma_1} = 1$. The formulation presented and analyzed in [8] of the voltage excitation problem reads: find $\mathbf{A} \in \mathbf{X}^0$ such that

$$\int_\Omega \boldsymbol{\mu}^{-1}\,\mathrm{curl}\,\mathbf{A}\cdot\mathrm{curl}\,\overline{\mathbf{w}} + i\omega\int_{\Omega_C} \boldsymbol{\sigma}\mathbf{A}\cdot\overline{\mathbf{w}} = \int_{\Omega_C} \boldsymbol{\sigma} V\,\nabla U_C^*\cdot\overline{\mathbf{w}} \tag{24}$$

for all $\mathbf{w} \in \mathbf{X}^0$.

Since the sesquilinear form $a(\cdot,\cdot)$ defined

$$a(\mathbf{s},\mathbf{w}) := \int_\Omega \boldsymbol{\mu}^{-1}\mathrm{curl}\,\mathbf{s}\cdot\mathrm{curl}\,\overline{\mathbf{w}} + i\omega\int_{\Omega_C} \boldsymbol{\sigma}\mathbf{s}\cdot\overline{\mathbf{w}}$$

is continuous and coercive in $\mathbf{X}^0$ (for the proof see [8], [10], [11]), from Lax-Milgram lemma problem (24) has a unique solution. Moreover is easy to see that (24) holds in fact for all $\mathbf{w} \in H_0(\mathrm{curl};\Omega)$ hence $\mathbf{H} = \boldsymbol{\mu}^{-1}\,\mathrm{curl}\,\mathbf{A}$ and $\mathbf{E}_C = -i\omega\mathbf{A}_C + V\nabla U_C^*$ satisfy Ampère law and they are the solution of (13) and (14).

The same approach can be used for the current excitation problem. The weak formulation reads: find $(\mathbf{A}, V) \in \mathbf{X}^0 \times \mathbb{C}$ such that

$$\int_\Omega \boldsymbol{\mu}^{-1}\,\mathrm{curl}\,\mathbf{A}\cdot\mathrm{curl}\,\overline{\mathbf{w}} + \int_{\Omega_C} \boldsymbol{\sigma}(i\omega\mathbf{A} - \boldsymbol{\sigma} V\,\nabla U_C^*)\cdot\overline{\mathbf{w}} = 0 \tag{25}$$

for all $\mathbf{w} \in \mathbf{X}^0$ and

$$\int_{\Gamma_J} \boldsymbol{\sigma}\,(-i\omega\mathbf{A}_C + V\,\nabla U_C^*)\cdot\mathbf{n} = I\,. \tag{26}$$

We notice that there exists a unique function $\mathbf{A}^* \in \mathbf{X}^0$ such that

$$\int_\Omega \boldsymbol{\mu}^{-1}\,\mathrm{curl}\,\mathbf{A}^*\cdot\mathrm{curl}\,\overline{\mathbf{w}} + i\omega\int_{\Omega_C} \boldsymbol{\sigma}\mathbf{A}^*\cdot\overline{\mathbf{w}} = i\omega\int_{\Omega_C} \boldsymbol{\sigma}\,\nabla U_C^*\cdot\overline{\mathbf{w}}$$

for all $\mathbf{w} \in \mathbf{X}^0$ and $\mathbf{H}^* := \boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{A}^*$ and $\mathbf{E}_C^* := -i\omega \mathbf{A}_C^* + \nabla U_C^*$ are solution of the eddy current problem with assigned voltage $V = 1$ that is not trivial; hence from the power law (5)

$$I^* := \int_{\Gamma_1} \boldsymbol{\sigma} \left(-i\omega \mathbf{A}^* + \nabla U_C^*\right) \cdot \mathbf{n} \neq 0 \,.$$

Then the solution of (25) and (26) is $(V\mathbf{A}^*, V)$ with $V = I/I^*$.

Remark 2.1. This approach can be extended to the case of many electric ports considering K functions $U_{C,k}^* \in H^1(\Omega_C)$, $k = 1, \ldots, K$, such that $U_{C,k|\Gamma_k}^* = 1$ and $U_{C,k|\Gamma_C \setminus \Gamma_k}^* = 0$.

In the current intensity excitation problem we need also the corresponding functions $\mathbf{A}_k^* \in \mathbf{X}^0$ such that

$$\int_\Omega \boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{A}_k^* \cdot \operatorname{curl} \overline{\mathbf{w}} + i\omega \int_{\Omega_C} \boldsymbol{\sigma} \mathbf{A}_k^* \cdot \overline{\mathbf{w}} = \int_{\Omega_C} \boldsymbol{\sigma} \nabla U_{C,k}^* \cdot \overline{\mathbf{w}}$$

for all $\mathbf{w} \in \mathbf{X}^0$. Problem (25), (26) now reads: given $\mathbf{I} \in \mathbb{C}^K$ find $(\mathbf{A}, \mathbf{V}) \in \mathbf{X}^0 \times \mathbb{C}^K$ such that

$$\int_\Omega \boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{A} \cdot \operatorname{curl} \overline{\mathbf{w}} + i\omega \int_{\Omega_C} \boldsymbol{\sigma} \mathbf{A} \cdot \overline{\mathbf{w}} = \int_{\Omega_C} \boldsymbol{\sigma} \left(\sum_{j=1}^K V_j \nabla U_{C,j}^* \right) \cdot \overline{\mathbf{w}}$$

for all $\mathbf{w} \in \mathbf{X}^0$ and

$$\int_{\Gamma_k} \boldsymbol{\sigma} \left(-i\omega \mathbf{A}_C + \sum_{j=1}^K V_j \nabla U_{C,j}^* \right) \cdot \mathbf{n} = I_k \,, \quad k = 1, \ldots, K \,.$$

The solution is given by $(\sum_{j=1}^K V_j \mathbf{A}_j^*, \mathbf{V})$ with $\mathbf{V}$ solution of the linear system

$$\sum_{j=1}^K V_j \int_{\Gamma_k} \boldsymbol{\sigma} \left(-i\omega \mathbf{A}_{C,j}^* + \nabla U_{C,j}^*\right) \cdot \mathbf{n} = I_k \,, \quad k = 1, \ldots, K \,.$$

It remains to verify that the matrix I^* with coefficients

$$I_{k,j}^* = \int_{\Gamma_k} \boldsymbol{\sigma} \left(-i\omega \mathbf{A}_{C,j}^* + \nabla U_{C,j}^*\right) \cdot \mathbf{n}$$

is not singular but this is again a consequence of the power law (5) because if $I^* \mathbf{x} = \mathbf{0}$ then $\sum_{j=1}^K x_j \int_{\Gamma_k} \boldsymbol{\sigma} \left(-i\omega \mathbf{A}_{C,j}^* + \nabla U_{C,j}^* \right) \cdot \mathbf{n} = 0$ for all $k = 1, \ldots, K$. Notice that this is the current intensity on Γ_k of the solution of the eddy current problem with voltage data given by $\mathbf{x}$ hence if the current intensity through each electric port is equal zero, from (5) both $\mathbf{H}$ and $\mathbf{E}_C$ solution of the eddy current are equal zero but then also $\mathbf{x} = \mathbf{0}$.

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3. Finite element approximation.

We analyze two different ways to deal with the divergence free constrain in the case of Coulomb gauge in the whole computational domain. The first one is to impose it by penalization adding a term to the sesquilinear form. When adopting this approach the natural finite element space for the approximation of the vector potential $\mathbf{A}$ is the space of continuous nodal elements. The second one is to introduce a Lagrange multiplier in which case the finite element space used for the approximation of $\mathbf{A}$ is the space of curl-conforming edge elements. We will present the results for the current intensity excitation problem but the same results hold true for the voltage excitation problem. The finite element approximation of the voltage excitation problem with Coulomb gauge in the insulator has been analyzed in [8] where the authors derive also a posteriori error estimates for the ungauged formulation.

3.1. *Approximation using nodal finite elements.*

In order to work with unconstrained spaces, the gauge condition can be incorporated in the Ampère equation adding a penalization term: let $\mu_* > 0$ be a constant representing a suitable average in Ω of the entries of the matrix $\boldsymbol{\mu}$ then

$$\operatorname{curl}(\boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{A}) - \mu_*^{-1} \nabla \operatorname{div} \mathbf{A} + \boldsymbol{\sigma}(i\omega \mathbf{A}_C - \nabla U_C) = 0 \,.$$

Let us consider the sesquilinear form in $[H(\operatorname{curl};\Omega) \cap H(\operatorname{div};\Omega)] \times H^1(\Omega_C)$

$$\widetilde{\mathcal{A}}[(\mathbf{s}, Z_C), (\mathbf{w}, Q_C)] := \mathcal{A}[(\mathbf{s}, Z_C), (\mathbf{w}, Q_C)] + \int_\Omega \mu_*^{-1} \operatorname{div} \mathbf{s} \operatorname{div} \overline{\mathbf{w}} \,.$$

Proceeding as in Theorem 2.1 it is easy to prove the following result.

Theorem 3.1. *There exists a positive constant* $\widetilde{\alpha}$ *such that*

$$\left| \widetilde{\mathcal{A}}[(\mathbf{w}, Q_C), (\mathbf{w}, Q_C)] \right| \geq \widetilde{\alpha} \left(\|\operatorname{curl} \mathbf{w}\|_{0,\Omega}^2 + \|\operatorname{div} \mathbf{w}\|_{0,\Omega}^2 + \|\nabla Q_C\|_{0,\Omega_C}^2 \right) \tag{27}$$

for all $(\mathbf{w}, Q_C) \in [H_0(\operatorname{curl};\Omega) \cap H(\operatorname{div};\Omega)] \times H^1(\Omega_C)$.

Let us now consider the following problem:

Given $I \in \mathbb{C}$ find $(\mathbf{A}, U_C) \in [H_0(\operatorname{curl};\Omega) \cap H(\operatorname{div};\Omega)] \times H^1_\sharp(\Omega_C)$ such that

$$\widetilde{\mathcal{A}}[(\mathbf{A}, U_C), (\mathbf{w}, Q_C)] = -i\omega^{-1} I \, \overline{Q}_{C|\Gamma_1} \,. \tag{28}$$

for all $(\mathbf{w}, Q_C) \in [H_0(\operatorname{curl};\Omega) \cap H(\operatorname{div};\Omega)] \times H^1_\sharp(\Omega_C)$.

It is easy to see that this problem is equivalent to (23). We only need to verify that if $\mathbf{A}$ is solution of (28) then $\operatorname{div}\mathbf{A} = 0$. Taking test functions of the form $(\mathbf{0}, Q_C)$ with $Q_C \in H^1_{0,\Gamma_C}(\Omega_C)$ in (28) we obtain

$$0 = \widetilde{\mathcal{A}}[(\mathbf{A}, U_C), (\mathbf{0}, Q_C)] = -i\omega^{-1}\int_{\Omega_C} \boldsymbol{\sigma}^{-1}(i\omega\mathbf{A} - \nabla U_C)\nabla\overline{Q_C}\,.$$

Given $f \in L^2(\Omega)$ let $\phi \in H^1_0(\Omega)$ be such that $\Delta\phi = f$. Then $\nabla\phi \in H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div};\Omega)$ hence we can take $(\nabla\phi, 0)$ as test function in (28) obtaining

$$\int_\Omega \operatorname{div}\mathbf{A}\, f + i\omega^{-1}\int_{\Omega_C} \boldsymbol{\sigma}^{-1}(i\omega\mathbf{A} - \nabla U_C)\overline{i\omega\nabla\phi} = 0\,.$$

Since $\phi_{|\Omega_C} \in H^1_{0,\Gamma_C}(\Omega_C)$ the second integral on the left is equal zero and one has $\int_\Omega \operatorname{div}\mathbf{A}\, f = 0$ for all $f \in L^2(\Omega)$ hence $\operatorname{div}\mathbf{A} = 0$.

The finite element approximation of (28) is naturally based on nodal finite elements. In the sequel we assume that Ω and Ω_C are Lipschitz polyhedral and that $\mathcal{T}_h$ is a regular family of triangulations of Ω that induces a regular family of triangulations of Ω_C. Let $\mathbb{P}_k$, $k \geq 1$, be the space of polynomials of degree less than or equal to k. For $r \geq 1$ and $s \geq 1$ we introduce the discrete space of Lagrange nodal elements defined as

$$W^r_h := \{\mathbf{w}_h \in (C^0(\Omega))^3 \,|\, \mathbf{w}_{h|K} \in (\mathbb{P}_r)^3\, \forall\, K \in \mathcal{T}_h\,,\ \mathbf{w}_h \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\}\,,$$

and

$$X^s_{C,h} := \{Q_{C,h} \in C^0(\Omega_C) \,|\, Q_{C,h|K} \in \mathbb{P}_s\, \forall\, K \in \mathcal{T}_{C,h}\}$$

Let us denote $L^s_{C,h} := X^s_{C,h} \cap H^1_{0,\Gamma_C}(\Omega_C)$ and $z^*_{C,h}$ the function in $X^s_{C,h}$ that take the value one in all the nodes of $\overline{\Gamma_1}$ and the value zero in all the remaining nodes.

The finite element approximation of (28) is:

Given $I \in \mathbb{C}$ find $(\mathbf{A}_h, U^0_{C,h}, V^h) \in W^r_h \times L^s_{C,h} \times \mathbb{C}$ such that

$$\widetilde{\mathcal{A}}[(\mathbf{A}_h, U^0_{C,h} + V^h\, z^*_{C,h}), (\mathbf{w}_h, Q^0_{C,h} + R\, z^*_{C,h})] = -i\omega^{-1}\, I\, \overline{R} \tag{29}$$

for all $(\mathbf{w}_h, Q^0_{C,h}, R) \in W^r_h \times L^s_{C,h} \times \mathbb{C}$.

Let us denote $U_{C,h} = U^0_{C,h} + V^h z^*_{C,h}$. Since

$$\widetilde{\mathcal{A}}[(\mathbf{A} - \mathbf{A}_h, U_C - U_{C,h}), (\mathbf{w}_h, Q^0_{C,h} + Rz^*_{C,h})] = 0$$

for all $(\mathbf{w}_h, Q^0_{C,h}, R) \in W^r_h \times L^s_{C,h} \times \mathbb{C}$ from Céa Lemma one has

$$\begin{aligned}\|\operatorname{curl}(\mathbf{A}-\mathbf{A}_h)\|^2_{0,\Omega} + \|\operatorname{div}(\mathbf{A}-\mathbf{A}_h)\|^2_{0,\Omega} + \|\nabla(U_C - U_{C,h})\|^2_{0,\Omega_C} \\ \leq C\Big(\|\operatorname{curl}(\mathbf{A}-\mathbf{w}_h)\|^2_{0,\Omega} + \|\operatorname{div}(\mathbf{A}-\mathbf{w}_h)\|^2_{0,\Omega} \\ +\|\nabla[U_C - (Q^0_{C,h} + Rz^*_{C,h})]\|^2_{0,\Omega_C}\Big)\end{aligned}$$

for all $(\mathbf{w}_h, Q^0_{C,h}, R) \in W^r_h \times L^s_{C,h} \times \mathbb{C}$.

Provided that $\mathbf{A}$ and U_C solution of (28) are regular enough, taking $\mathbf{w}_h$ the interpolation of $\mathbf{A}$, $Q^0_{C,h}$ the interpolation of $U_C - U_{C|\Gamma_1} z^*_{c,h}$ and $R = U_{C|\Gamma_1}$, from well know interpolation results we obtain optimal error estimates. However the regularity of $\mathbf{A}$ is not ensured if Ω is a non-convex polyhedron. In that case the space $(H^1(\Omega))^3 \cap H_0(\operatorname{curl};\Omega)$ turns out to be a proper closed subspace of $H_0(\operatorname{curl};\Omega) \cap H(\operatorname{div};\Omega)$ (see [12]). Hence the nodal finite element approximation $\mathbf{A}_h$ cannot approach the exact solution if it does not belong to $(H^1(\Omega))^3 \cap H_0(\operatorname{curl};\Omega)$.

3.2. *Approximation using edge elements.*

Another way to impose the gauge condition is to introduce a Lagrange multiplier. The current intensity excitation problem can also be formulated in the following way:

Find $(\mathbf{A}, U_C, \Psi) \in H_0(\operatorname{curl};\Omega) \times H^1_\sharp(\Omega_C) \times H^1_0(\Omega)$ such that

$$(30) \qquad \begin{aligned} &\mathcal{A}[(\mathbf{A}, U_C), (\mathbf{w}, Q_C)] + \int_\Omega \nabla\Psi \cdot \overline{\mathbf{w}} = -i\omega^{-1} I\, \overline{Q}_{C|\Gamma_1} \\ &\int_\Omega \mathbf{A} \cdot \nabla\overline{\Phi} = 0 \end{aligned}$$

for all $(\mathbf{w}, Q_C, \Phi) \in H_0(\operatorname{curl};\Omega) \times H^1_\sharp(\Omega_C) \times H^1_0(\Omega)$.

It is easy to see that the following inf-sup condition is verified: there exists a constant $\beta > 0$ such that for all $\Phi \in H^1_0(\Omega)$

$$\sup_{\mathbf{w} \in H_0(\operatorname{curl};\Omega)} \frac{\left|\int_\Omega \nabla\Phi \cdot \overline{\mathbf{w}}\right|}{\|\mathbf{w}\|_{H(\operatorname{curl};\Omega)}} \geq \beta \|\Phi\|_{1,\Omega};$$

it follows from Poincarè inequality taking $\mathbf{w} = \nabla\Phi$. Hence (30) have a unique solution. Moreover is clear that if $(\mathbf{A}, U_C)$ is the solution of (23) then $(\mathbf{A}, U_C, 0)$ is the solution of (30). This means that the Lagrange multiplier Ψ is equal zero.

The natural spaces for the finite element approximation are edge elements for $H_0(\operatorname{curl};\Omega)$ and scalar nodal elements for $H^1_{0,\Gamma_C}(\Omega_C)$ and $H^1_0(\Omega)$.

Let us consider the first family of Nédélec curl-conforming finite elements

$$N_h^r := \{\mathbf{w}_h \in H_0(\mathrm{curl};\Omega) \,|\, \mathbf{w}_{h|K} \in R_r \,\forall\, K \in \mathcal{T}_h\}$$

where $R_r := (\mathbb{P}_{r-1})^3 \oplus S_r$ and $S_r := \{\mathbf{q} \in (\widetilde{\mathbb{P}}_{r-1})^3 \,|\, \mathbf{q}(\mathbf{x}) \cdot \mathbf{x} = 0\}$ (see [13]). And let us denote

$$L_h^s := \{\phi_h \in H_0^1(\Omega) \,|\, \phi_{h|K} \in \mathbb{P}_s \,\forall\, K \in \mathcal{T}_h\} \,.$$

The finite element approximation of problem (30) reads: find $(\mathbf{A}_h, U^0_{C,h}, V^h, \Psi_h) \in N_h^r \times L^s_{C,h} \times \mathbb{C} \times L_h^r$ such that

$$(31)\quad \begin{aligned} &\mathcal{A}[(\mathbf{A}_h, U^0_{C,h} + V^h z^*_{C,h}), (\mathbf{w}_h, Q^0_{C,h} + R z^*_{C,h})] \\ &\qquad\qquad + \textstyle\int_\Omega \nabla\Psi_h \cdot \overline{\mathbf{w}}_h = -i\omega^{-1}\, I\,\overline{R} \\ &\textstyle\int_\Omega \mathbf{A}_h \cdot \nabla\overline{\Phi}_h \qquad\qquad\qquad\quad = 0 \end{aligned}$$

for all $(\mathbf{w}_h, Q_{C,h}, R, \Phi_h) \in N_h^r \times L^s_{C,h} \times \mathbb{C} \times L_h^r$.

If $r \le s$ it is easy to see that this problem has a unique solution; in fact it is enough to prove the uniqueness. First we show that for any $I \in \mathbb{C}$ the discrete Lagrange multiplier is equal zero: for all $Q_{C,h} \in L^s_{C,h}$, testing the first equation with $(\mathbf{0}, Q_{C,h}, 0)$ we obtain that

$$-i\omega \int_{\Omega_C} \Big[i\omega \mathbf{A}_{C,h} - \nabla(U^0_{C,h} + V^h\, z^*_{C,h})\Big] \cdot \nabla\overline{Q_{C,h}} = 0\,.$$

Hence testing with $(\nabla\Psi_h, 0, 0)$ we obtain

$$i\omega \int_{\Omega_C} \Big[i\omega \mathbf{A}_{C,h} - \nabla(U^0_{C,h} + V^h\, z^*_{C,h})\Big] \cdot \overline{i\omega\nabla\Psi_h} + \int_\Omega \nabla\Psi_h \cdot \overline{\nabla\Psi_h} = 0\,.$$

Since $\Psi_{h|\Omega_C} \in L^r_{C,h}$, if $r \le s$ the first integral on the left is equal zero hence follows that $\int_\Omega \nabla\Psi_h \cdot \overline{\nabla\Psi_h} = 0$.

If $I = 0$, from (20) follows $\|i\omega\mathbf{A}_{C,h} - \nabla(U^0_{C,h} + V^h\, z^*_{C,h})\|_{0,\Omega_C} = 0$ and $\|\,\mathrm{curl}\,\mathbf{A}_h\| = 0$. This means that there exists $\varphi_h \in L_h^r$ such that $\mathbf{A}_h = \nabla\varphi_h$ (see, e.g., [14], Lemma 5.28) but then $\mathbf{A}_h = \mathbf{0}$ because $\int_\Omega \mathbf{A}_h \cdot \nabla\overline{\Phi}_h = 0$ for all $\Phi_h \in L_h^r$. As a consequence $U^0_{C,h} + V^h\, z^*_{C,h} = 0$.

Now we will show that the solution of the discrete problem converges to the solution of the continuous problem. Since $\Psi_h = 0$, denoting $U_{C,h} = U^0_{C,h} + V^h z^*_{C,h}$ one has

$$\mathcal{A}[(\mathbf{A}_h, U_{C,h}), (\mathbf{w}_h, Q^0_{C,h} + R z^*_{C,h})] = -i\omega^{-1} I\overline{R}$$

for all $(\mathbf{w}_h, Q^0_{C,h}, R) \in N_h^r \times L_h^s \times \mathbb{C}$. Recalling that

$$\mathcal{A}[(\mathbf{A}, U_C), (\mathbf{w}, Q^0_C + R z^*_{C,h})] = -i\omega^{-1} I\overline{R}$$

for all $(\mathbf{w}_h, Q^0_{C,h}, R) \in N^r_h \times L^s_h \times \mathbb{C}$ one has

$$\mathcal{A}[(\mathbf{A} - \mathbf{A}_h, U_C - U_{C,h}), (\mathbf{w}_h, Q^0_{C,h} + Rz^*_{C,h})] = 0$$

and then using (20)

$$\begin{aligned} K &\left[\| \operatorname{curl}(\mathbf{A} - \mathbf{A}_h)\|^2_{0,\Omega} + |\omega|^{-1}\|i\omega\mathbf{A}_C - \nabla U_C - (i\omega\mathbf{A}_{C,h} - \nabla U_{C,h})\|^2_{0,\Omega_C}\right] \\ &\le |\mathcal{A}[(\mathbf{A} - \mathbf{A}_h, U_C - U_{C,h}), (\mathbf{A} - \mathbf{A}_h, U_C - U_{C,h})]| \\ &= \left|\mathcal{A}[(\mathbf{A} - \mathbf{A}_h, U_C - U_{C,h}), (\mathbf{A} - \mathbf{w}_h, U_C - (Q^0_{C,h} + Rz^*_{C,h})]\right| \end{aligned}$$

for all $(\mathbf{w}_h, Q^0_{C,h}, R) \in N^r_h \times L^s_{C,h} \times \mathbb{C}$. From the definition of the sesquilinear form $\mathcal{A}[\cdot,\cdot]$ is easy to see that there exist a positive constant C such that

$$\begin{aligned} &\left|\mathcal{A}[(\mathbf{A} - \mathbf{A}_h, U_C - U_{C,h}), (\mathbf{A} - \mathbf{w}_h, U_C - (Q^0_{C,h} + Rz^*_{C,h}))]\right| \\ &\quad\le C\Big[\| \operatorname{curl}(\mathbf{A} - \mathbf{A}_h)\|_{0,\Omega}\| \operatorname{curl}(\mathbf{A} - \mathbf{w}_h)\|_{0,\Omega} \\ &\qquad + |\omega|^{-1}\|i\omega(\mathbf{A}_C - \mathbf{A}_{C,h}) - \nabla(U_C - U_{C,h})\|_{0,\Omega_C} \\ &\qquad\quad \|i\omega(\mathbf{A}_C - \mathbf{w}_{C,h}) - \nabla[U_C - (Q^0_{C,h} + Rz^*_{C,h})]\|_{0,\Omega_C}\Big] \end{aligned}$$

where $\mathbf{w}_{C,h} = \mathbf{w}_{h|\Omega_C}$. Hence

$$\begin{aligned} &\| \operatorname{curl}(\mathbf{A} - \mathbf{A}_h)\|^2_{0,\Omega} + |\omega|^{-1}\|i\omega\mathbf{A}_C - \nabla U_C - (i\omega\mathbf{A}_{C,h} - \nabla U_{C,h})\|^2_{0,\Omega_C} \\ &\quad \le \tfrac{C^2}{K^2}\Big(\| \operatorname{curl}(\mathbf{A} - \mathbf{w}_h)\|^2_{0,\Omega} \\ &\qquad + |\omega|^{-1}\|i\omega(\mathbf{A}_C - \mathbf{w}_{C,h}) - \nabla(U_C - (Q^0_{C,h} + Rz^*_{C,h}))\|^2_{0,\Omega_C}\Big)\,. \end{aligned}$$

for all $(\mathbf{w}_h, Q^0_{C,h}, R) \in N^r_h \times L^s_{C,h} \times \mathbb{C}$.

Provided that $\mathbf{A}$ and U_C are regular enough, namely if $\mathbf{A}$ and $\operatorname{curl}\mathbf{A}$ belong to $(H^q(\Omega))^3$ with $1/2 < q \le r$ and $U_C \in H^p(\Omega_C)$ with $3/2 < p \le s$, we can take $\mathbf{w}_h$ the interpolation of $\mathbf{A}$, $Q^0_{C,h}$ the interpolation of $U_C - U_{C|\Gamma_1} z^*_{C,h}$ and $R = U_{C|\Gamma_1}$. Denoting $\mathbf{B}_h := \operatorname{curl}\mathbf{A}_h$ and $\mathbf{E}_{C,h} := -i\omega\mathbf{A}_{C,h} + \nabla U_{C,h}$ one has:

$$\|\mathbf{B} - \mathbf{B}_h\|^2_{0,\Omega} + |\omega|^{-1}\|\mathbf{E}_C - \mathbf{E}_{C,h}\|^2_{0,\Omega_C} \le \widehat{C}(h^{2q} + |\omega|^{-1}h^{2p})\,.$$

Comparing with the finite element approximation using continuous vector nodal elements this approach has an additional unknown, the Lagrange multiplier, defined in the whole computational domain Ω. However, since it is zero it is possible to eliminate it. In fact, let $\{\mathbf{z}_j\}_{j=1}^N$ be a real base for the space N^r_h, $\{q_j\}_{j=1}^{M_C}$ a real base for $L^s_{C,h}$ and $\{\phi_j\}_{j=1}^M$ a real base for L^s_h.

Let us set $q_{M_C+1} = z^*_{C,h}$. The linear system obtaining from (30) has the form

$$\begin{bmatrix} S + i\omega M & -B^T & D^T \\ B & i\omega^{-1}R & \\ D & & \end{bmatrix} \begin{bmatrix} A \\ U \\ \psi \end{bmatrix} = \begin{bmatrix} 0 \\ G \\ 0 \end{bmatrix}$$

where, with obvious notation, $s_{k,j} = \int_\Omega \boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{z}_j \cdot \operatorname{curl} \mathbf{z}_k$, $m_{k,j} = \int_{\Omega_C} \boldsymbol{\sigma} \mathbf{z}_j \cdot \mathbf{z}_k$, $1 \le k, j \le N$; $r_{k,j} = \int_{\Omega_C} \boldsymbol{\sigma} \nabla q_j \cdot \nabla q_k$, $1 \le k, j \le M_C + 1$; $b_{k,j} = \int_{\Omega_C} \boldsymbol{\sigma} \mathbf{z}_j \cdot \nabla q_k$, $1 \le k \le M_C + 1$, $1 \le j \le N$; $d_{k,j} = \int_\Omega \mathbf{z}_j \cdot \nabla \psi_k$ $1 \le k \le M$, $1 \le j \le N$. Concerning the right hand size term $g_k = 0$ for $k = 1, \ldots, M_C$ and $g_{M_C+1} = -i\omega^{-1} I$.

The unique solution of this system has $\psi = 0$. Moreover A and U are also solution of the reduced system

$$\begin{bmatrix} S + i\omega M + \gamma D^T D & -B^T \\ B & i\omega^{-1}R \end{bmatrix} \begin{bmatrix} A \\ U \end{bmatrix} = \begin{bmatrix} F \\ G \end{bmatrix}$$

for any $\gamma > 0$. Notice that this reduced system has a unique solution. In fact, if W and P are solution of the homogeneous problem then both the real part and the imaginary part of

$$\begin{bmatrix} \overline{W} \; \overline{P} \end{bmatrix} \begin{bmatrix} S + i\omega M + \gamma D^T D & -B^T \\ B & i\omega^{-1}R \end{bmatrix} \begin{bmatrix} W \\ P \end{bmatrix}$$

are equal zero, that is

$$\overline{W}(S + \gamma D^T D)W = 0 \tag{32}$$

and

$$\begin{bmatrix} \overline{W} \; \overline{P} \end{bmatrix} \begin{bmatrix} i\omega M & -B^T \\ B & i\omega^{-1}R \end{bmatrix} \begin{bmatrix} W \\ P \end{bmatrix} = 0\,. \tag{33}$$

From (32) one obtains $\operatorname{curl} \mathbf{w}_h = \mathbf{0}$ and $\int_\Omega \mathbf{w}_h \cdot \nabla \overline{\psi}_h = 0$ for all $\psi_h \in L^r_h$. Since Ω is simply connected

$$\{\mathbf{z}_h \in N^r_h \,|\, \operatorname{curl} \mathbf{z}_h = 0\} = \{\nabla \psi_h \,|\, \psi_h \in L^r_h\}$$

hence $W = 0$. Then, from (33) one has $i\omega^{-1} \int_{\Omega_C} \boldsymbol{\sigma} \nabla P_{C,h} \cdot \overline{\nabla P_{C,h}} = 0$ and then also P is equal zero.

The elimination of the Lagrange multiplier has the obvious advantage of reduce the dimension of the linear system to be solved but the choice of the parameter γ is critical because it influence the condition number of the reduced matrix. In [15] a similar approach has been adopted for the formulation of the eddy current problem driven by an applied current in terms of the magnetic field in the conductor and the electric field in the insulator. In that work some numerical results show the sensitivity of the condition number of the reduced system to the choice of the parameter γ.

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