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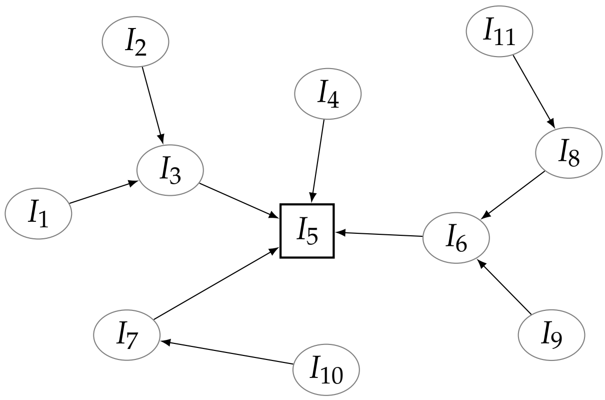

Computable Hilbert schemes

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A mia nonna

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Introduction

The Hilbert scheme $\mathbf{Hilb}^n_{p(t)}$ has been introduced by Grothendieck [40] at the beginning of the '60 and belongs to several objects that arose with the schematic reinterpretation of algebraic geometry. It represents the Hilbert functor the associates to any scheme Z, over a ground field $\mathbb{K}$ of characteristic 0, the set of flat families in a projective space $\mathbb{P}^n$ parametrized by Z. For this reason, usually we say that the Hilbert scheme parametrizes all the subschemes and all the (flat) families of subschemes of $\mathbb{P}^n$ with a fixed Hilbert polynomial $p(t)$. This means that the Hilbert scheme is itself a parameter scheme of a flat family $\mathcal{X}$ of subschemes of $\mathbb{P}^n$ with Hilbert polynomial $p(t)$ such that any other flat family $\mathcal{Y} \to S$ can be seen as pullback of $\mathcal{X} \to \mathbf{Hilb}^n_{p(t)}$ by means of a uniquely defined map $S \to \mathbf{Hilb}^n_{p(t)}$:

$$\begin{tikzcd} \mathcal{Y} = \mathcal{X} \times_{\mathbf{Hilb}^n_{p(t)}} S \arrow[r, dashed] \arrow[d] & \mathcal{X} \arrow[d] \\ S \arrow[r, dashed, "\exists!"] & \mathbf{Hilb}^n_{p(t)} \end{tikzcd}$$

The Hilbert scheme is a projective scheme and it is usually defined as subscheme of a suitable Grassmannian. Following the notation used by Gotzmann in [34], given a subscheme $X \subset \mathbb{P}^n = \operatorname{Proj} \mathbb{K}[x_0, \ldots, x_n]$ and the corresponding saturated ideal $I_X \subset \mathbb{K}[x_0, \ldots, x_n]$ ($\mathbb{K}[x]$ for short), we will say Hilbert polynomial of I_X referring to the Hilbert polynomial $p(t)$ of X, whereas we will say volume polynomial of I_X referring to the polynomial $q(t) = \binom{n+t}{n} - p(t)$ such that $\dim_{\mathbb{K}} I_t = q(t),\ t \gg 0$. By Gotzmann's Regularity Theorem it is well known that for an integer r large enough, for each subscheme $X \subset \mathbb{P}^n$ parametrized by $\mathbf{Hilb}^n_{p(t)}$, the saturated ideal

I_X is generated in degree lower than or equal to r and that I_r is a $q(r)$-dimensional vector subspace of the base vector space of homogeneous polynomials of $\mathbb{K}[x]$ of degree r. Hence any point of $\mathbf{Hilb}^n_{p(t)}$ can be naturally identified with a point of the Grassmannian $\mathbf{Gr}(q(r), \mathbb{K}[x]_r)$ and then embedded by the Plücker embedding in the projective space $\mathbb{P}^E$, $E = \binom{\binom{n+r}{r}}{q(r)}$. The dimension E of the projective space in which we can embed $\mathbf{Hilb}^n_{p(t)}$ clearly becomes very huge just considering non-trivial cases. This fact reveals immediately the great difficulty of studying explicitly and globally the Hilbert scheme, indeed even Hilbert schemes of easy geometric objects give rise to intractable problems of computational algebra.

Dealing with the problem of finding an ideal defining $\mathbf{Hilb}^n_{p(t)}$ as projective scheme, the study can be oriented towards the equations generating such an ideal and particularly to their degree. Iarrobino and Kleiman (1999) [52, Appendix C] proved that there exists an ideal defining the Hilbert scheme $\mathbf{Hilb}^n_{p(t)}$ as subscheme of $\mathbf{Gr}(q(r), \mathbb{K}[x]_r)$ generated by polynomials of degree $q(r+1)+1$ in the Plücker coordinates, and afterwards Haiman and Sturmfels (2004) [41], proving a conjecture by Bayer (1982) [7], defined an ideal generated by polynomials of degree $n+1$.

In Chapter 1, after having recalled some background material about representable functors, Grassmannians and Hilbert schemes, we introduce a set of generators for any exterior power $\wedge^l W$ of a subspace $W \in \mathbf{Gr}(q, N)$ depending *linearly* on the Plücker coordinates of W, given by Plücker embedding. Exploiting this result in the case of Hilbert schemes, we give new and simpler proofs of the theorems about the degree of the equation defining $\mathbf{Hilb}^n_{p(t)} \subset \mathbf{Gr}(q(r), \mathbb{K}[x]_r)$ by Iarrobino-Kleiman and Bayer-Haiman-Sturmfels. Furthermore in Chapter 5, we introduce a new ideal defining the Hilbert scheme as subscheme of a Grassmannian generated by equations of degree smaller than or equal to $\deg p(t) + 2 < n+1 \ll q(r+1)+1$.

The first relevant property proved about the Hilbert scheme is surely its connectedness, proved by Hartshorne (1966) in its PhD thesis [42]. He used a basic idea widely exploited in the field of commutative algebra in the following years, that is to reduce the study on monomial ideals obtained by flat deformation of any ideal defining a point on a Hilbert scheme. This idea has to be carefully managed in this context. Using the modern language of Gröbner degeneration theory, the problem is that any change of coordinates $g \in \mathrm{GL}(n+1)$ defines an isomorphism of $\mathbf{Hilb}^n_{p(t)}$ that identifies the point $\operatorname{Proj} \mathbb{K}[x]/I \in \mathbf{Hilb}^n_{p(t)}$ defined by an ideal I and the point $\operatorname{Proj} \mathbb{K}[x]/(g \,.\, I)$ defined by $g \,.\, I$, whereas the point defined by $\mathsf{in}(I)$ could not be mapped to the point defined by $\mathsf{in}(g \,.\, I)$. To overcome this possible ambiguity, for any ideal I, given the equivalence relation $g \sim g' \Leftrightarrow \mathsf{in}(g \,.\, I) = \mathsf{in}(g' \,.\, I)$, it was proved that one of the equivalence classes corresponds to an open subset U of $\mathrm{GL}(n+1)$. Hence associating to any ideal I the so-called generic initial ideal $\mathsf{in}(g \,.\, I)$ computed considering a change of coordinate g in the open equivalence class turns out to be consistent with the properties of the Hilbert scheme. Generic initial ideals belong to the class of monomial ideals called Borel-fixed, because fixed by the action of the Borel subgroup of $\mathrm{GL}(n+1)$ composed by the upper triangular matrices. They are fundamental in the study of Hilbert schemes for two reasons:

1. each component and each intersection of components of $\mathbf{Hilb}^n_{p(t)}$ contains at least one point defined by a Borel-fixed ideal (roughly speaking they are distributed all over the Hilbert scheme);

2. they have a strong combinatorial characterization that makes them very interesting, also from an algorithmic perspective.

In Chapter 2, after having recalled the main properties of Borel-fixed ideals and showed several ways to represent their combinatorial structure, we expose an algorithm for computing all the (saturated) Borel-fixed ideals in $\mathbb{K}[x]$ with Hilbert polynomial $p(t)$, that is for computing all the points of $\mathbf{Hilb}^n_{p(t)}$ defined by Borel-fixed ideals. Then we discuss how the number of Borel-fixed ideals varies increasing the number of variables of the polynomial ring and changing the Hilbert polynomial. Furthermore we propose new definitions of ideals that generalize the notion of lexicographic ideal, that we call segment ideals.

The basic idea of Hartshorne's proof of the connectedness of the Hilbert scheme is to construct a sequence of deformations and specializations (through distractions) of Borel-fixed ideals (he called it *balanced* ideals) in order to reach the point defined on $\mathbf{Hilb}^n_{p(t)}$ by the unique (saturated) lexicographic ideal associated to $p(t)$ (see [60]).

A second proof was given by Peeva and Stillman (2005) [83], who basically rewrote Hartshorne's idea in terms of Gröbner degenerations. In both cases affine flat deformations are used. In Chapter 3, we introduce a new type of flat deformations involving Borel-fixed ideals simply relying on their combinatorial structure that lead to rational curves on the Hilbert scheme. The idea of giving a "direction" to the deformations works also in this case, so that a new proof of the connectedness of the Hilbert scheme is proposed. Morever we show that all the points defined by segment ideals can play the same role of the lexicographic point in Hartshorne and Peeva-Stillman's proofs. Finally we are able to define families over $(\mathbb{P}^1)^{\times s}$ that can be used to detect set of points defined by Borel-fixed ideals lying on a same component of the Hilbert scheme.

As explained above, the explicit and global study of the Hilbert scheme is unachievable because of the huge number of parameters needed to describe these kind of geometric families. An alternative approach is the local one, i.e. considering the open covering induced on $\mathbf{Hilb}^n_{p(t)}$ by the standard affine open covering of $\mathbb{P}^E$ through the Plücker embedding. This has been the true starting point of the thesis, indeed the first topic I dealt with in my research activity is the construction of families of ideals sharing the same initial ideal. In [82] Notari and Spreafico (2000) proposed to cover set-theoretically the Hilbert scheme with such families. In the first part of Chapter 4, we prove that the family of ideals I such that $\mathsf{in}(I) = J$, that we call Gröbner stratum of J and denote by $\mathcal{S}t(J)$, has a well-defined structure of affine scheme and we determine the conditions in order for a Gröbner stratum to be an open subset of the corresponding Hilbert scheme, namely a local description of its scheme structure. The ideal J is having to be a segment ideal, truncation of a saturated Borel-fixed ideal in degree equal to the degree r used to define the Grassmannian $\mathbf{Gr}(q(r), \mathbb{K}[x]_r)$. $\mathcal{S}t(J)$ can be viewed quite naturally as a homogeneous variety with respect to (w.r.t. for short) a non-standard positive grading. This property has a great relevance in a computational perspective, because it allows to reduce significantly the number of parameters describing $\mathcal{S}t(J)$ (and also the associated open subset of $\mathbf{Hilb}^n_{p(t)}$), indeed we prove that in many cases the degree of the truncation can be lowered while obtaining the same family (an isomorphic one) of ideals.

Unfortunately this technique does not solve the problem of the local study of the Hilbert scheme, because not every Borel-fixed ideal is a segment ideal, so that this method can not be always applied and above all the number of open subsets that in principle we would need to consider, i.e. the number of Gröbner strata we would have to compute, is still enormous. Giving up the segment hypothesis is really costly, because we can no longer use tools provided by Gröbner theory, mainly the *noetherian* Buchberger's algorithm, that are the basis of construction of Gröbner strata. Thus the central part of Chapter 4 is devoted to the ideation and development of a new noetherian algorithm of polynomial reduction based solely on the combinatorial properties of Borel-fixed ideals avoiding any term ordering. With this new procedure we define more general families of ideals, that include Gröbner strata: a family of this type is constructed from a Borel-fixed ideal J, so we call it J-marked family and we denote it by $\mathcal{M}f(J)$. The property common to each ideal $I \in \mathcal{M}f(J)$ is that the set of monomials not belonging to J represents a basis of $\mathbb{K}[x]/I$ as $\mathbb{K}$-vector space.

In the final part of Chapter 4, we moreover answer also to the second problem about the local study of the Hilbert scheme, that is the huge number of open affine subsets to be considered. We prove that it is sufficient to study the J-marked families $\mathcal{M}f(J)$, where J is the truncation in some degree $\leqslant r$ of a saturated Borel-fixed ideal defining a point of $\mathbf{Hilb}^n_{p(t)} \subset \mathbf{Gr}(q(r), \mathbb{K}[x]_r)$, and then to exploit the action of the linear group $\mathrm{GL}(n+1)$ on the Hilbert scheme.

In Chapter 6, we deal with Hilbert schemes of locally Cohen-Macaulay curves in the projective space $\mathbb{P}^3$. A locally Cohen-Macaulay curve is a curve without embedded or isolated points and the set of points of the Hilbert scheme $\mathbf{Hilb}^3_{dt+1-g}$ corresponding to locally Cohen-Macaulay curves of degree d and genus g turns out to be an open subset denoted by $\mathbf{H}_{d,g}$. In turn $\mathbf{H}_{d,g}$ contains an open subset corresponding to the set of smooth curves, that we denoted by $\mathbf{H}^{\mathrm{sm}}_{d,g}$, i.e.

$$\mathbf{H}^{\mathrm{sm}}_{d,g} \subset \mathbf{H}_{d,g} \subset \mathbf{Hilb}^3_{dt+1-g}.$$

As seen in Chapter 3, the full Hilbert scheme is connected, whereas there are known examples of Hilbert schemes of smooth curves in $\mathbb{P}^3$ which are not connected (for instance $\mathbf{H}^{\mathrm{sm}}_{9,10}$ [43, Chapter IV Example 6.4.3]). For the Hilbert scheme of locally Cohen-Macaulay curves nothing is known, in the sense that there are no examples of non-connected Hilbert schemes and neither there is a proof of the connectedness in the general case. An approach similar to that one used in the proof of the connectedness of $\mathbf{Hilb}^n_{p(t)}$, for instance a sequence of Gröbner deformations in a fixed direction, has always seemed unsuitable, because for any term ordering the initial ideal is a monomial ideal, hence except for some rare cases (ACM curves), a Gröbner degeneration would lead to a curve with embedded points.

A Hilbert scheme considered a "good" candidate of being non-connected was $\mathbf{H}_{4,-3}$. In Chapter 6 we show that this Hilbert scheme is indeed connected, by constructing a Gröbner deformation with generic fiber corresponding to four disjoint line on a smooth quadric and special fiber an extremal curve. The key point is to choose appropriately a weight order ω (which is not a total order on the monomials) such that the initial ideal w.r.t. ω is not necessarily a monomial ideal and such that the degeneration "approaches" the component of $\mathbf{H}_{d,g}$ of extremal curves.

In order to strengthen the algorithmic purpose, I wrote lots of lines of code organized in some libraries to explicitly calculate the objects introduced theoretically in the thesis in many non-trivial example. Appendix A contains an handbook for the package `HilbertSchemesEquations` written in the *Macaulay2* language, that covers the topics explained in Chapter 1. As seen the number of parameters is in any case too large, but the equations for the Hilbert scheme $\mathbf{Hilb}^2_2 \subset \mathbf{Gr}(4,6)$ can be computed.

Appendix B presents a library written in `java` that provides an implementation of the combinatorial structure of Borel-fixed ideals. All the algorithms described in Chapter 2 and Chapter 3 are implemented and made available by means of several java applets working on any web browser with java plugins installed.

Finally the algorithms for working on marked families and Gröbner strata introduced in Chapter 4 are made available through the *Macaulay2* package `MarkedSchemes` described in Appendix C.

All this material is available at my web page

www.personalweb.unito.it/paolo.lella/HSC/.

Chapter 1

The Hilbert scheme

In this first chapter we recall the definitions leading to the introduction of the Hilbert scheme and its usual construction as subscheme of a suitable Grassmannian.

1.1 The representability of a functor

Definition 1.1. For any schemes X over $\mathbb{K}$, we define the contravariant *functor of points* from the category $(\text{schemes})_{\mathbb{K}}$ of schemes over an algebraically closed field $\mathbb{K}$ of characteristic 0 to the category (sets) of sets:

$$h_X : \ (\text{schemes})^{\circ}_{\mathbb{K}} \to (\text{sets})$$

such that for any object $Z \in \text{Ob}_{(\text{schemes})_{\mathbb{K}}}$

$$h_X(Z) = \text{Hom}(Z, X)$$

and for any morphism $f : Z \to W \in \text{Mor}_{(\text{schemes})_{\mathbb{K}}}$, by the diagram

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[dr, dashed] & W \arrow[d, "a"] \arrow[dl, phantom, "\circlearrowright"] \\ & X \end{tikzcd}$$

we define

$$\begin{aligned} h_X(f) : \mathrm{Hom}(W, X) &\to \mathrm{Hom}(Z, X) \\ a \quad &\mapsto h_X(f)(a) = a \circ f. \end{aligned}$$

By the definition h_X turns out to be a contravariant functor, indeed given $Z \xrightarrow{f} W \xrightarrow{g} T$ and $a \in \mathrm{Hom}(T, W)$

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[dr] & W \arrow[r, "g"] \arrow[d] \arrow[l, phantom, shift right=3, "\circlearrowright"] \arrow[r, phantom, shift right=3, "\circlearrowright"] & T \arrow[dl, "a"] \\ & X & \end{tikzcd}$$

$h_X(g \circ f)(a) = a \circ g \circ f = h_X(g)(a) \circ f = h_X(f)\big(h_X(g)(a)\big) = \big(h_X(f) \circ h_X(g)\big)(a).$

Given another scheme Y over $\mathbb{K}$ and a morphism $\psi : X \to Y$, it can be defined a natural transformation $h_\psi : h_X \to h_Y$. For any $Z \in \mathrm{Ob}_{(\text{schemes})_{\mathbb{K}}}$, we define $h_\psi(Z) : h_X(Z) \to h_Y(Z)$ through the diagram

$$\begin{tikzcd} X \arrow[r, "\psi"] & Y \\ Z \arrow[u, "a"] \arrow[ur, dashed] \arrow[ur, phantom, shift left=3, "\circlearrowright"] & \end{tikzcd}$$

and for any $f : Z \to W \in \mathrm{Mor}_{(\text{schemes})_{\mathbb{K}}}$,

$$\begin{aligned} h_\psi(f) : \quad h_X(f) \quad &\to \quad h_Y(f) \\ \big(a \mapsto a \circ f\big) &\mapsto \big(\psi \circ a \mapsto \psi \circ a \circ f\big) \end{aligned}$$

as showed in the following diagram

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[dr, "h_X(f)(a)"'] & W \arrow[d, "a"] & \\ & X \arrow[r, "\psi"] & Y \end{tikzcd}$$

Therefore we defined a functor h from the category $(\text{schemes})_{\mathbb{K}}$ of schemes over $\mathbb{K}$ to the category of functors from $(\text{schemes})_{\mathbb{K}}$ to (sets)

$$\begin{aligned} h:\ (\text{schemes})_{\mathbb{K}} &\to \operatorname{Fun}\big((\text{schemes})_{\mathbb{K}}, (\text{sets})\big) \\ X \qquad &\mapsto \qquad h_X \\ (\psi : X \to Y) &\mapsto \quad (h_\psi : h_X \to h_Y) \end{aligned} \tag{1.1}$$

Proposition 1.2. *The functor h described in* (1.1) *is fully faithful.*

Proof. We have to prove that the function $\psi \mapsto h_\psi$ is a bijection, so starting from a map $f : h_X \to h_Y$ we will define a morphism between X and Y and we will prove that this correspondence is the inverse function.

$h_X(X)$ obviously contains the identity map id_X of X, so by means of $f(X) : h_X(X) \to h_Y(X)$, we can define the map

$$\varphi = f(X)(\mathrm{id}_X) : X \to Y,$$

and we want to show that $h_\varphi = f$.

For any scheme $Z \in \mathrm{Ob}_{(\text{schemes})_{\mathbb{K}}}$ and any morphism $g : Z \to X$, we have the diagram

$$\begin{tikzcd} h_X(Z) \arrow[r, "f(Z)"] & h_Y(Z) \\ h_X(X) \arrow[u, "h_X(g)"] \arrow[r, "f(X)"] \arrow[ur, phantom, "\circlearrowright"] & h_Y(X) \arrow[u, "h_Y(g)"'] \end{tikzcd}$$

Note that $g = h_X(g)(\mathrm{id}_X)$, $\forall\, g \in h_X(Z)$, so

$$f(Z)(g) = h_Y(g)\big(f(X)(\mathrm{id}_X)\big) = h_Y(g)(\varphi) = \varphi \circ g = h_\varphi(Z)(g). \qquad \square$$

Theorem 1.3 (Yoneda's Lemma [29, Lemma VI-1]**).** *The functor of points h_X in* $\operatorname{Fun}\big((\text{schemes})_{\mathbb{K}}, (\text{sets})\big)$ *uniquely determines the scheme $X \in (\text{schemes})_{\mathbb{K}}$ up to isomorphism.*

Now we can introduce the notion of representable functor.

Definition 1.4. Let $\mathcal{F} : (\text{schemes})^{\circ}_{\mathbb{K}} \to (\text{sets})$ be a contravariant functor. $\mathcal{F}$ is *representable* if there exists a scheme $X \in \text{Ob}_{(\text{schemes})_{\mathbb{K}}}$ such that $\mathcal{F} \simeq h_X$. By Yoneda's Lemma, we know that such a scheme is uniquely determined, so we will say that the scheme X represents $\mathcal{F}$.

Let us conclude this section recalling some useful results about the representability of a functor.

Proposition 1.5 ([29, Proposition VI-2]). *A scheme X over $\mathbb{K}$ is completely determined by the restriction of its functor of points to affine schemes over $\mathbb{K}$; in fact*

$$h' : (\text{schemes})_{\mathbb{K}} \to \text{Fun}\big((\text{schemes})^{\text{Aff}}_{\mathbb{K}}, (\text{sets})\big)$$

is an equivalence of the category of schemes over $\mathbb{K}$ with a full subcategory of $\text{Fun}\big((\text{schemes})_{\mathbb{K}}, (\text{sets})\big)$.

From now on, in place of the contravariant functor of points (Definition 1.1), we will consider the covariant functor

$$h_A : (\mathbb{K}\text{-algebras}) \to (\text{sets})$$

such that for any object $B \in \text{Ob}_{(\mathbb{K}\text{-algebras})}$

$$h_A(B) = \text{Hom}(\text{Spec}\, B, \text{Spec}\, A)$$

and for any morphism $f : B \to C \in \text{Mor}_{(\mathbb{K}\text{-algebras})}$, by the diagram

$$\begin{tikzcd}
\text{Spec}\, C \arrow[r, "f^*"] \arrow[dr, dashed] & \text{Spec}\, B \arrow[d, "a"] \arrow[dl, phantom, "\circlearrowright"] \\
& \text{Spec}\, A
\end{tikzcd}$$

we define

$$\begin{aligned} h_A(f) : h_A(B) &\to h_A(C) \\ a \quad &\mapsto h_A(f)(a) = a \circ f^*. \end{aligned}$$

1.2 Grassmannians

Let us consider a $\mathbb{K}$-vector space V of dimension N with a basis $\{\underline{v}_1, \ldots, \underline{v}_N\}$. The Grassmannian $\mathbf{Gr}_{\mathbb{K}}(q, N)$ parametrizes the set of all vector subspaces $W \subset V$ of dimension q. Every q-dimensional subspace W can be described as a row span of a $q \times N$ matrix $\mathfrak{M}(W)$ of maximal rank. Furthermore the list of all maximal minors of such a matrix up to scale determines uniquely the space W (see [72, Proposition 14.2]), that is the matrix $\mathfrak{M}(W)$ representing W is unique up to multiplication by invertible $q \times q$ matrices.

By this argument the Grassmannian $\mathbf{Gr}_{\mathbb{K}}(q, N)$ can be embedded in the projective space $\mathbb{P}_{\mathbb{K}}^{\binom{N}{q}-1}$:

$$\begin{array}{rccc} \mathscr{P}: & \mathbf{Gr}_{\mathbb{K}}(q, N) & \rightarrow & \mathbb{P}_{\mathbb{K}}[\wedge^q V] \\ & W = \langle \underline{w}_1, \ldots, \underline{w}_q \rangle & \mapsto & \underline{w}_1 \wedge \cdots \wedge \underline{w}_q \end{array} \tag{1.2}$$

Let $\underline{w}_i = \sum\limits_{j=1}^{N} \delta_{ij}\, \underline{v}_j$, $i = 1, \ldots, q$ be the decomposition of the basis of W with respect the basis of V and let

$$\left\{ \underline{v}_{\mathrm{I}}^{(q)} = \underline{v}_{\mathrm{i}_1} \wedge \cdots \wedge \underline{v}_{\mathrm{i}_q} \quad \middle| \quad \mathrm{I} = (\mathrm{i}_1, \ldots, \mathrm{i}_q),\ 1 \leqslant \mathrm{i}_1 < \cdots < \mathrm{i}_q \leqslant N \right\}$$

the standard basis of $\wedge^q V$. The product $\underline{w}_1 \wedge \cdots \wedge \underline{w}_q$ decomposes as

$$\underline{w}_1 \wedge \cdots \wedge \underline{w}_q = \sum_{|\mathrm{I}|=q} \Delta_{\mathrm{I}}(W)\, \underline{v}_{\mathrm{I}}^{(q)}$$

where the coefficient $\Delta_{\mathrm{I}}(W)$ is just the determinant of the submatrix $\mathfrak{M}_{\mathrm{I}}(W)$ of $\mathfrak{M}(W)$ composed by the columns of the vectors $\underline{v}_{\mathrm{i}_1}, \ldots, \underline{v}_{\mathrm{i}_N}$ and the set $[\ldots : \Delta_{\mathrm{I}}(W) : \ldots]$ is the set of Plücker coordinates of W, hence we will consider $\mathbb{P}_{\mathbb{K}}[\wedge^q V] = \operatorname{Proj} \mathbb{K}[\ldots, \Delta_I, \ldots] = \operatorname{Proj} \mathbb{K}[\Delta]$.

Another way to determine the embedding is considering the short exact sequence

$$0 \longrightarrow W \longrightarrow V \xrightarrow{\pi_W} V/W \longrightarrow 0$$

and the induced epimorphism of the p-th exterior power where $p = N - q$, that is

$$\wedge^p V \xrightarrow{\pi_W^{(p)}} \wedge^p V/W \longrightarrow 0. \tag{1.3}$$

Since $\dim_{\mathbb{K}} V/W = \dim_{\mathbb{K}} V - \dim_{\mathbb{K}} W = p$, the exterior algebra $\wedge^p W$ is isomorphic to the field $\mathbb{K}$ and the map $\pi_W^{(p)}$ is uniquely determined by W up to multiplication by scalar. So we can identify W with the set of the images by $\pi_W^{(p)}$ of the vectors of the basis of $\wedge^p V$

$$\left\{ \underline{v}_{\mathrm{J}}^{(p)} = \underline{v}_{\mathrm{j}_1} \wedge \cdots \wedge \underline{v}_{\mathrm{j}_p} \quad \middle| \quad \mathrm{J} = (\mathrm{j}_1, \ldots, \mathrm{j}_p),\ 1 \leqslant \mathrm{j}_1 < \cdots < \mathrm{j}_p \leqslant N \right\}$$

precisely let us define

$$\Theta_{\mathrm{J}}(W) = \pi_W^{(p)}(\underline{v}_{\mathrm{J}}) \tag{1.4}$$

and the embedding

$$\begin{aligned} \psi : \mathbf{Gr}_{\mathbb{K}}(q,N) \rightarrow &\qquad \mathbb{P}_{\mathbb{K}}^{\binom{N}{p}-1} \\ W \quad &\mapsto [\ldots : \Theta_{\mathrm{J}}(W) : \ldots] \end{aligned} \tag{1.5}$$

considering $\mathbb{P}_{\mathbb{K}}^{\binom{N}{p}-1} = \operatorname{Proj} \mathbb{K}[\ldots, \Theta_{\mathrm{J}}, \ldots] = \operatorname{Proj} \mathbb{K}[\Theta]$ where the variables Θ_{J} are another set of Plücker coordinates.

Proposition 1.6. *The sets of Plücker coordinates* $[\ldots : \Delta_{\mathrm{I}} : \ldots]$ *and* $[\ldots : \Theta_{\mathrm{J}} : \ldots]$ *are equivalent. More precisely, for every* $W \in \mathbf{Gr}_{\mathbb{K}}(q,N)$*, there exists* $c \in \mathbb{K}$ *such that*

$$\Delta_{\mathrm{I}}(W) = c\, \varepsilon_{\mathrm{J}|\mathrm{I}}\, \Theta_{\mathrm{J}}(W), \qquad \mathrm{J} \cup \mathrm{I} = \{1, \ldots, N\} \tag{1.6}$$

where $\varepsilon_{\mathrm{J}|\mathrm{I}}$ *is the signature of the permutation* σ *that orders* $\mathrm{J}|\mathrm{I} = (\mathrm{j}_1, \ldots, \mathrm{j}_p, \mathrm{i}_1, \ldots, \mathrm{i}_q)$*, i.e.* $\sigma(\mathrm{J}|\mathrm{I}) = (1, \ldots, N)$*.*

Proof. Let us suppose W contained in the vector space V with basis $\{\underline{v}_1, \ldots, \underline{v}_N\}$ and let $p = N - q$. Since $\wedge^p(V/W) \simeq \mathbb{K}$, the morphism $\pi_W^{(p)}$ described in (1.3) turns out to be a linear functional over $\wedge^p V$. Let $\{\overline{v}^1, \ldots, \overline{v}^N\}$ be the usual basis of the dual space V^*, that is

$$\overline{v}^j(\underline{v}_i) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$$

Directly by the definition of $\Theta_{\mathrm{J}}(W)$ given in (1.4), we can identify $\pi_W^{(p)}$ with

$$\pi_W^{(p)} \longleftrightarrow \sum_{|\mathrm{J}|=p} \Theta_{\mathrm{J}}(W)\, \overline{v}_{(p)}^{\mathrm{J}} \in \wedge^p(V^*) = (\wedge^p V)^*$$

where $\overline{v}^{\mathrm{J}}_{(p)} = \overline{v}^{\mathrm{j}_1} \wedge \cdots \wedge \overline{v}^{\mathrm{j}_p}$.

Through the standard isomorphism

$$\begin{array}{ccc} \wedge^p(V^*) & \xrightarrow{\sim} & \wedge^q V \\ \bar{t} & \longmapsto & \bar{t}(\underline{v}_1 \wedge \cdots \wedge \underline{v}_N) \end{array}$$

the image of $\pi^{(p)}_W$ is

$$\sum_{|\mathrm{J}|=p} \Theta_{\mathrm{J}}(W)\, \overline{v}^{\mathrm{J}}_{(p)} \big(\underline{v}_1 \wedge \cdots \wedge \underline{v}_N\big)$$

and rewriting in each addend $\underline{v}_1 \wedge \cdots \wedge \underline{v}_N$ as $\underline{v}^{(p)}_{\mathrm{J}} \wedge \varepsilon_{\mathrm{J}|\mathrm{I}}\, \underline{v}^{(q)}_{\mathrm{I}}$ where $\mathrm{J} \cup \mathrm{I} = \{1, \ldots, N\}$ and $\varepsilon_{\mathrm{J}|\mathrm{I}}$ is the signature of the permutation that orders $\mathrm{J}|\mathrm{I}$ (equal obviously also to the signature of the inverse permutation σ such that $\sigma(1, \ldots, N) = \mathrm{J}|\mathrm{I} = (\mathrm{j}_1, \ldots, \mathrm{j}_p, \mathrm{i}_1, \ldots, \mathrm{i}_q)$), we finally obtain

$$\sum_{|\mathrm{J}|=p} \Theta_{\mathrm{J}}(W)\, \overline{v}^{\mathrm{J}}_{(p)} \big(\underline{v}^{(p)}_{\mathrm{J}} \wedge \varepsilon_{\mathrm{J}|\mathrm{I}}\, \underline{v}^{(q)}_{\mathrm{I}}\big) = \sum_{\substack{|\mathrm{I}|=q \\ \mathrm{I}\cup\mathrm{J}=\{1,\ldots,N\}}} \varepsilon_{\mathrm{J}|\mathrm{I}}\, \Theta_{\mathrm{J}}(W)\, \underline{v}^{(q)}_{\mathrm{I}}.$$

By duality the vector subspace generated by this elements has to coincide to the vector subspace identified by the injection

$$\wedge^q W \longrightarrow \wedge^q V.$$

Chosen a basis $\{\underline{w}_1, \ldots, \underline{w}_q\}$ of W, the generator of $\wedge^q W$ in $\wedge^q V$ is

$$\underline{w}_1 \wedge \cdots \wedge \underline{w}_q = \sum_{|\mathrm{I}|=q} \Delta_I(W)\, \underline{v}^{(q)}_{\mathrm{I}}. \qquad \square$$

Remark 1.2.1. Given any (even not-ordered) set of indices $H = (\mathrm{h}_1, \ldots, \mathrm{h}_q)$ (resp. $H = (\mathrm{h}_1, \ldots, \mathrm{h}_p)$), we will denote with $\Delta_H(W)$ (resp. $\Theta_H(W)$) the determinant of the submatrix of $\mathfrak{M}(W)$ obtained considering the columns corresponding to the vectors $\underline{v}_{\mathrm{h}_1}, \ldots, \underline{v}_{h_q}$ (resp. the image of $\underline{v}_{\mathrm{h}_1} \wedge \cdots \wedge \underline{v}_{\mathrm{h}_p}$ using $\pi^{(p)}_W$). It is easy to check that $\Delta_H = \varepsilon_H \Delta_{\mathrm{K}}$ (resp. $\Theta_H = \varepsilon_H \Theta_{\mathrm{K}}$), where ε_H is the signature of the permutation σ that orders H, $\mathrm{K} = \sigma(H)$ is the corresponding ordered set of indices and Δ_{K} (resp. Θ_{K}) is a Plücker coordinate. From now on, if not specified the set of indices are considered in increasing order.

Given two multi-indices $\mathrm{K} = (\mathrm{k}_1, \dots, \mathrm{k}_a)$ and $\mathrm{H} = (\mathrm{h}_1, \dots, \mathrm{h}_b)$, we will denote by $\mathrm{K}|\mathrm{H}$ the set of indices $(\mathrm{k}_1, \dots, \mathrm{k}_a, \mathrm{h}_1, \dots, \mathrm{h}_b)$, that in general will not be ordered, whereas we will denote with the union $\mathrm{K} \cup \mathrm{H}$ the ordered multi-index containing the indices belonging to both K and H. For instance given $\mathrm{K} = (1,5)$ and $\mathrm{H} = (2)$, $\mathrm{K}|\mathrm{H} = (1,5,2)$, $\mathrm{H}|\mathrm{K} = (2,1,5)$ and $\mathrm{K} \cup \mathrm{H} = \mathrm{H} \cup \mathrm{K} = (1,2,5)$. Coming back to Plücker coordinates the following relation holds:

$$\Delta_{\mathrm{K}|\mathrm{H}} = \varepsilon_{\mathrm{K}|\mathrm{H}} \Delta_{\mathrm{K}\cup\mathrm{H}} \qquad (\text{resp. } \Theta_{\mathrm{K}|\mathrm{H}} = \varepsilon_{\mathrm{K}|\mathrm{H}} \Theta_{\mathrm{K}\cup\mathrm{H}}).$$

To determine the equations of the subscheme $\mathscr{P}\big(\mathbf{Gr}_{\mathbb{K}}(q,N)\big) \subset \mathbb{P}[\wedge^q V]$, that is the conditions such that an element $\underline{u}^{(q)} \in \wedge^q V$ can be decomposed as exterior product of q vectors

$$\underline{u}^{(q)} = \underline{w}_1 \wedge \cdots \wedge \underline{w}_q, \qquad \underline{w}_1, \dots, \underline{w}_q \in V$$

let us start considering the *contraction operator* (sometimes also called convolution). For any element $\overline{v}^j$ of the basis of V^*, let us define the operator

$$i^*(\overline{v}^j) : \wedge^s V \to \wedge^{s-1} V$$

as the operator sending the generic element $\underline{v}_{\mathrm{I}}^{(s)} = \underline{v}_{\mathrm{i}_1} \wedge \cdots \wedge \underline{v}_{\mathrm{i}_s}$ of the basis of $\wedge^s V$ to 0 if j does not belong to I whereas if $j = \mathrm{i}_k$ for any k

$$i^*(\overline{v}^j)(\underline{v}_{\mathrm{I}}^{(s)}) = (-1)^{k-1}\, \underline{v}_{\mathrm{i}_1} \wedge \cdots \wedge \underline{v}_{\mathrm{i}_{k-1}} \wedge \underline{v}_{\mathrm{i}_{k+1}} \wedge \cdots \wedge \underline{v}_{\mathrm{i}_s} = (-1)^{k-1}\, \underline{v}_{\mathrm{I}\setminus\{j\}}^{(s-1)}$$

so that for the generic element of $\wedge^s V$

$$i^*(\underline{v}^j) \left(\sum_{|\mathrm{I}|=s} a_{\mathrm{I}}\, \underline{v}_{\mathrm{I}}^{(s)} \right) = \sum_{\substack{|\mathrm{I}|=s \\ \exists\, k \text{ s.t. } \mathrm{i}_k = j}} (-1)^{k-1}\, a_{\mathrm{I}}\, \underline{v}_{\mathrm{I}\setminus\{j\}}^{(s-1)}.$$

Extending by linearity, for any element $\overline{t} = \sum_{j=1}^N b_j\, \overline{v}^j \in V^*$ we define $i^*(\overline{t}) : \wedge^s V \to \wedge^{s-1} V$ as the operator

$$i^*(\overline{t}) = \sum_{j=1}^{N} b_j\, i^*(\overline{v}^j) \tag{1.7}$$

Definition 1.7. For any $\bar{t}_{(r)} = \sum_{|\mathrm{J}|=r} b_{\mathrm{J}}\, \overline{v}^{\mathrm{J}}_{(r)} \in \wedge^r(V^*)$ we define the *contraction operator*

$$i^*(\bar{t}_{(r)}) : \wedge^s V \to \wedge^{s-r} V \tag{1.8}$$

as

$$i^*(\bar{t}_{(r)}) \left(\sum_{|\mathrm{I}|=s} a_{\mathrm{I}}\, \underline{v}^{(s)}_{\mathrm{I}} \right) = \left(\sum_{|\mathrm{J}|=r} b_{\mathrm{J}}\, i^*(\overline{v}^{\mathrm{J}}_{(r)}) \right) \left(\sum_{|\mathrm{I}|=s} a_{\mathrm{I}}\, \underline{v}^{(s)}_{\mathrm{I}} \right) = \\ = \sum_{|\mathrm{J}|=r} \sum_{|\mathrm{I}|=s} b_{\mathrm{J}} a_{\mathrm{I}}\, i^*(\overline{v}^{\mathrm{J}}_{(r)})\big(\underline{v}^{(s)}_{\mathrm{I}}\big)$$

where we define $i^*(\overline{v}^{\mathrm{J}}_{(r)})\big(\underline{v}^{(s)}_{\mathrm{I}}\big) \in \wedge^{s-r} V$ through the composition of maps

$$\wedge^r V \xrightarrow{i^*(\overline{v}^{\mathrm{j}_r})} \wedge^{r-1} V \xrightarrow{i^*(\overline{v}^{\mathrm{j}_{r-1}})} \quad \cdots \quad \xrightarrow{i^*(\overline{v}^{\mathrm{j}_1})} \wedge^{s-r} V$$

as

$$i^*(\overline{v}^{\mathrm{j}_1}) \circ \cdots \circ i^*(\overline{v}^{\mathrm{j}_r})\big(\underline{v}^{(s)}_{\mathrm{I}}\big). \tag{1.9}$$

Proposition 1.8. *A vector $\underline{u}^{(q)} \in \wedge^q V$ is decomposable, i.e. of the form $\underline{w}_{\mathrm{i}_1} \wedge \cdots \wedge \underline{w}_{\mathrm{i}_q}$, if and only if*

$$i^*(\bar{t}_{(q-1)})\big(\underline{u}^{(q)}\big) \wedge \underline{u}^{(q)} = 0, \qquad \forall\, \bar{t}_{(q-1)} \in \wedge^{q-1}(V^*). \tag{1.10}$$

Proof. See [94, Chapter I Section 4.1] and [39, Chapter I Section 5]. □

To compute the Plücker relations, we consider the generic element $\sum_{|\mathrm{I}|=q} \Delta_{\mathrm{I}}\, \underline{v}^{(q)}_{\mathrm{I}}$ of $\wedge^q V$ and we impose the condition (1.10) considering

$$i^*(\overline{v}^{\mathrm{J}}_{(q-1)}) \left(\sum_{|\mathrm{I}|=q} \Delta_{\mathrm{I}}\, \underline{v}^{(q)}_{\mathrm{I}} \right) \wedge \sum_{|\mathrm{I}|=q} \Delta_{\mathrm{I}}\, \underline{v}^{(q)}_{\mathrm{I}}$$

for all $\overline{v}^{\mathrm{J}}_{(q-1)}$ in the standard basis of $\wedge^{q-1}(V^*)$. Firstly

$$i^*(\overline{v}^{\mathrm{J}}_{(q-1)}) \left(\sum_{|\mathrm{I}|=q} \Delta_{\mathrm{I}}\, \underline{v}^{(q)}_{\mathrm{I}} \right) = \sum_{\substack{|\mathrm{I}|=q \\ \mathrm{J}\subset\mathrm{I}}} \Delta_{\mathrm{I}}\, i^*(\overline{v}^{\mathrm{J}}_{(q-1)})\big(\underline{v}^{(q)}_{\mathrm{I}}\big)$$

and supposing that $\mathrm{I} \setminus \mathrm{J} = \{\mathrm{i}_k\}$

$$i^*(\overline{v}^{\mathrm{J}}_{(q-1)})\big(\underline{v}^{(q)}_{\mathrm{I}}\big) = (-1)^{q-k}\, i^*(\overline{v}^{\mathrm{J}}_{(q-1)})\big(\underline{v}^{(q-1)}_{\mathrm{J}} \wedge \underline{v}_{i_k}\big) = (-1)^{\binom{q-1}{2}+q-k}\, \underline{v}_{\mathrm{i}_k}$$

so we can rewrite

$$i^*(\overline{v}^{\mathrm{J}}_{(q-1)})\left(\sum_{|\mathrm{I}|=q}\Delta_{\mathrm{I}}\,\underline{v}^{(q)}_{\mathrm{I}}\right) = (-1)^{\binom{q-1}{2}}\sum_{\substack{|\mathrm{I}|=q\\ \mathrm{J}\subset\mathrm{I}\\ \mathrm{I}\setminus\mathrm{J}=\{\mathrm{i}_k\}}}(-1)^{q-k}\,\Delta_{\mathrm{I}}\,\underline{v}_{\mathrm{i}_k} =$$

$$= (-1)^{\binom{q-1}{2}}\sum_{\substack{1\leqslant \mathrm{j}\leqslant N\\ \mathrm{j}\notin\mathrm{J}}}\varepsilon_{\mathrm{J}|(\mathrm{j})}\,\Delta_{\mathrm{J}\cup(\mathrm{j})}\,\underline{v}_{\mathrm{j}}$$

so that finally

$$\left((-1)^{\binom{q-1}{2}}\sum_{\substack{1\leqslant \mathrm{j}\leqslant N\\ \mathrm{j}\notin\mathrm{J}}}\varepsilon_{\mathrm{J}|(\mathrm{j})}\,\Delta_{\mathrm{J}\cup(\mathrm{j})}\,\underline{v}_{\mathrm{j}}\right)\bigwedge\left(\sum_{|\mathrm{I}|=q}\Delta_{\mathrm{I}}\,\underline{v}^{(q)}_{\mathrm{I}}\right) =$$

$$= (-1)^{\binom{q-1}{2}}\sum_{\substack{1\leqslant \mathrm{j}\leqslant N\\ \mathrm{j}\notin\mathrm{J}}}\sum_{|\mathrm{I}|=q}\varepsilon_{\mathrm{J}|(\mathrm{j})}\,\Delta_{\mathrm{J}\cup(\mathrm{j})}\Delta_{\mathrm{I}}\,\underline{v}_{\mathrm{j}}\wedge\underline{v}^{(q)}_{\mathrm{I}} =$$

$$= (-1)^{\binom{q-1}{2}}\sum_{\substack{1\leqslant \mathrm{j}\leqslant N\\ \mathrm{j}\notin\mathrm{J}}}\sum_{|\mathrm{I}|=q}\varepsilon_{\mathrm{J}|(\mathrm{j})}\varepsilon_{(\mathrm{j})|\mathrm{I}}\,\Delta_{\mathrm{J}\cup(\mathrm{j})}\Delta_{\mathrm{I}}\,\underline{v}^{(q+1)}_{(\mathrm{j})\cup\mathrm{I}}. \tag{1.11}$$

By dimension arguments, each vector $\underline{v}^{(q+1)}_{\mathrm{I}}$ of the basis $\wedge^{q+1}V$ comparing in the previous sum contains at least two vectors $\underline{v}_{\mathrm{i}}$ such that $\mathrm{i}\notin\mathrm{J}$. Let us decompose such a I as

$$\mathrm{I} = (\mathrm{I}\cap\mathrm{J})\cup\mathrm{K}, \qquad |\mathrm{K}|\geqslant 2.$$

For every $\mathrm{i}\in\mathrm{K}$, $\underline{v}^{(q+1)}_{\mathrm{I}}$ compare in the wedge product $\varepsilon_{\mathrm{J}|(\mathrm{i})}\Delta_{\mathrm{J}\cup(\mathrm{i})}\underline{v}_{\mathrm{i}}\wedge\Delta_{\mathrm{I}\setminus(\mathrm{i})}\underline{v}^{(q)}_{\mathrm{I}\setminus(\mathrm{i})}$ therefore the sum (1.11) contains the addend

$$(-1)^{\binom{q-1}{2}}\left(\sum_{\substack{\mathrm{i}\in\mathrm{I}\\ \mathrm{i}\notin\mathrm{J}}}\varepsilon_{\mathrm{J}|(\mathrm{i})}\varepsilon_{(\mathrm{i})|(\mathrm{I}\setminus(\mathrm{i}))}\,\Delta_{\mathrm{J}\cup(\mathrm{i})}\Delta_{\mathrm{I}\setminus(\mathrm{i})}\right)\underline{v}^{(q+1)}_{\mathrm{I}} \tag{1.12}$$

We remark that if $|\mathrm{K}|=2$, i.e. $\mathrm{I}=\mathrm{J}\cup(\mathrm{i}_1,\mathrm{i}_2)$ the coefficient of the term $\underline{v}^{(q+1)}_{\mathrm{I}}$ in (1.12) vanish, indeed

$$\varepsilon_{\mathrm{J}|(\mathrm{i}_1)}\varepsilon_{(\mathrm{i}_1)|(\mathrm{J}\cup(\mathrm{i}_2))}\,\Delta_{\mathrm{J}\cup(\mathrm{i}_1)}\Delta_{\mathrm{J}\cup(\mathrm{i}_2)}+\varepsilon_{\mathrm{J}|(\mathrm{i}_2)}\varepsilon_{(\mathrm{i}_2)|(\mathrm{J}\cup(\mathrm{i}_1))}\,\Delta_{\mathrm{J}\cup(\mathrm{i}_2)}\Delta_{\mathrm{J}\cup(\mathrm{i}_1)}=0$$

because, supposing $\mathrm{i}_1 < \mathrm{i}_2$,

$$\varepsilon_{\mathrm{J}|(\mathrm{i}_1)}\varepsilon_{(\mathrm{i}_1)|(\mathrm{J}\cup(\mathrm{i}_2))} = (-1)^{p_1} \cdot (-1)^{q-p_1-1} = (-1)^{q-1},$$
$$\varepsilon_{\mathrm{J}|(\mathrm{i}_2)}\varepsilon_{(\mathrm{i}_2)|(\mathrm{J}\cup(\mathrm{i}_1))} = (-1)^{p_2} \cdot (-1)^{q-p_2} = (-1)^{q}.$$

Finally the ideal defining the Grassmannian $\mathbf{Gr}_{\mathbb{K}}(q,N)$ as subscheme of $\mathbb{P}_{\mathbb{K}}^{\binom{N}{q}-1}$ is generated by the following set of quadrics

$$\left\{ \sum_{\substack{\mathrm{i}\in\mathrm{I}\\ \mathrm{i}\notin\mathrm{J}}} \varepsilon_{\mathrm{J}|(\mathrm{i})}\varepsilon_{(\mathrm{i})|(\mathrm{I}\setminus(\mathrm{i}))}\, \Delta_{\mathrm{J}\cup(\mathrm{i})}\Delta_{\mathrm{I}\setminus(\mathrm{i})} \;\middle|\; \begin{array}{l} \forall\, \mathrm{J},\ |\mathrm{J}| = q-1 \\ \forall\, \mathrm{I},\ |\mathrm{I}| = q+1,\ |\mathrm{I}\setminus(\mathrm{I}\cap\mathrm{J})| > 2 \end{array} \right\}. \tag{1.13}$$

Remark 1.2.2. Increasing the dimension N of the base vector space and/or the dimension q of the subspaces, the number of Plücker coordinates and above all the number of quadrics become quickly huge, indeed for describing the Plücker embedding of $\mathbf{Gr}_{\mathbb{K}}(q,N)$, we need $\binom{N}{q}$ Plücker coordinates and the Plücker relations among them are $\binom{N}{q-1} \cdot \left[\binom{N}{q+1} - \binom{N-(q-1)}{2}\right]$. If fact these relations are redundant, so the number of quadrics sufficient to describe the ideal is much lower.

Example 1.2.3. Let us consider the Grassmannian $\mathbf{Gr}_{\mathbb{K}}(4,6)$ and given a basis $\{\underline{v}_1, \ldots, \underline{v}_6\}$ of $V \simeq \mathbb{K}^6$ let

$$\mathscr{P} : \mathbf{Gr}_{\mathbb{K}}(4,6) \to \mathbb{P}^{14}_{\mathbb{K}} = \operatorname{Proj} \mathbb{K}[\Delta_{\mathrm{I}}]$$

be the Plücker embedding described in (1.2) where the 15 Plücker coordinates are indexed by ordered subsets $\mathrm{I} = (\mathrm{i}_1, \mathrm{i}_2, \mathrm{i}_3, \mathrm{i}_4)$ of $\{1,2,3,4,5,6\}$. Let $\{\overline{v}^1, \ldots, \overline{v}^6\}$ be the standard basis of the dual space.

$\wedge^3(V^*)$ has a basis containing 20 elements. Let us look for instance at the contraction operator defined by $\overline{v}^2 \wedge \overline{v}^3 \wedge \overline{v}^6$ applied on the vector $\sum_{|\mathrm{I}|=4} \Delta_{\mathrm{I}}\, \underline{v}_{\mathrm{I}}^{(4)}$:

$$\begin{aligned} i^*(\overline{v}^2 \wedge \overline{v}^3 \wedge \overline{v}^6)\left(\sum_{|\mathrm{I}|=4} \Delta_I\, \underline{v}_{\mathrm{I}}^{(4)}\right) &= (-1)^{\binom{3}{2}} \sum_{\mathrm{j}\in\{1,4,5\}} \varepsilon_{236|\mathrm{j}}\, \Delta_{236\cup\mathrm{j}}\, \underline{v}_{\mathrm{j}} = \\ &= \Delta_{1236}\, \underline{v}_1 + \Delta_{2346}\, \underline{v}_4 + \Delta_{2356}\, \underline{v}_5. \end{aligned}$$

Then by

$$(\Delta_{1236}\, \underline{v}_1 + \Delta_{2346}\, \underline{v}_4 + \Delta_{2356}\, \underline{v}_5) \wedge \left(\sum_{|\mathrm{I}|=4} \Delta_{\mathrm{I}}\, \underline{v}_{\mathrm{I}}^{(4)}\right)$$

we obtain the following 3 Plücker relations

$$\begin{aligned}
&\varepsilon_{236|1}\varepsilon_{1|2345}\,\Delta_{1236}\Delta_{2345}+\varepsilon_{236|4}\varepsilon_{4|1235}\,\Delta_{2346}\Delta_{1235}+\varepsilon_{236|5}\varepsilon_{5|1234}\,\Delta_{2356}\Delta_{1234}=\\
&\qquad=-\Delta_{1236}\Delta_{2345}+\Delta_{2346}\Delta_{1235}-\Delta_{2356}\Delta_{1234},\\
&\varepsilon_{236|1}\varepsilon_{1|2456}\,\Delta_{1236}\Delta_{2456}+\varepsilon_{236|4}\varepsilon_{4|1256}\,\Delta_{2346}\Delta_{1256}+\varepsilon_{236|5}\varepsilon_{5|1246}\,\Delta_{2356}\Delta_{1246}=\\
&\qquad=-\Delta_{1236}\Delta_{2456}-\Delta_{2346}\Delta_{1256}+\Delta_{2356}\Delta_{1246},\\
&\varepsilon_{236|1}\varepsilon_{1|3456}\,\Delta_{1236}\Delta_{3456}+\varepsilon_{236|4}\varepsilon_{4|1356}\,\Delta_{2346}\Delta_{1356}+\varepsilon_{236|5}\varepsilon_{5|1346}\,\Delta_{2356}\Delta_{1346}=\\
&\qquad=-\Delta_{1236}\Delta_{3456}-\Delta_{2346}\Delta_{1356}+\Delta_{2356}\Delta_{1346}.
\end{aligned}$$

Repeating this computation for every element of the basis of $\wedge^3(V^*)$, we obtain the ideal defining the Grassmannian $\mathbf{Gr}_{\mathbb{K}}(4,6)$ as subscheme of $\mathbb{P}^{14}_{\mathbb{K}}$ as generated by $3\cdot 20 = 60$ equations of degree 2. In Example A.1.2, we will show that 45 quadrics are redundant, that is we only need 15 polynomials to define this Grassmannian.

Our next task is to understand how we can recover a set of generator of a subspace $W \in \mathbf{Gr}_{\mathbb{K}}(q,N)$ knowing its image by the Plücker embedding $\mathscr{P}(W) \in \mathscr{P}(\mathbf{Gr}_{\mathbb{K}}(q,N)) \subset \mathbb{P}_{\mathbb{K}}^{\binom{N}{q}-1}$. It is well known (see [94, Chapter 4 Section 4.1] and [29, Section III.2.7]) that any point $[\ldots:\Delta_{\mathrm{I}}:\ldots] \in \mathscr{P}(\mathbf{Gr}_{\mathbb{K}}(q,N))$ such that $\Delta_{(1\ldots q)} \neq 0$ corresponds to a subspace $W \subset V$ with a basis of the type

$$\left\{\underline{w}_{\mathrm{i}} = \underline{v}_{\mathrm{i}} + \sum_{\mathrm{j}>q} \delta_{\mathrm{ij}}\,\underline{v}_{\mathrm{j}} \;\middle|\; \mathrm{i}=1,\ldots,q\right\} \qquad \text{where} \qquad \delta_{\mathrm{ij}} = \varepsilon_{(1\ldots q)\setminus(\mathrm{i})|(\mathrm{i})}\frac{\Delta_{(1\ldots q)\setminus(\mathrm{i})|(\mathrm{j})}}{\Delta_{(1\ldots q)}}, \tag{1.14}$$

in fact after multiplying any matrix $\mathfrak{M}(W)$ by the inverse matrix of $\mathfrak{M}_{(1\ldots q)}(W)$ (invertible because $\Delta_{(1\ldots q)} \neq 0$), the coefficient δ_{ij} is equal to the determinant of the matrix composed by the columns with indices in $(1,\ldots,q)\setminus(\mathrm{i})|(\mathrm{j})$. The same reasoning can be applied for every vector space $\wedge^s W$, $s \leqslant q$. Starting from the basis of W described in (1.14), we can consider as basis for $\wedge^s W$ the set $\{\underline{w}^{(s)}_{\mathrm{I}} \mid \forall\, \mathrm{I} = (\mathrm{i}_1,\ldots,\mathrm{i}_s),\ 1 \leqslant \mathrm{i}_1 < \cdots < \mathrm{i}_s \leqslant q\}$. More precisely

$$\underline{w}^{(s)}_{\mathrm{I}} = \underline{w}_{\mathrm{i}_1}\wedge\cdots\wedge\underline{w}_{\mathrm{i}_s} = \underline{v}^{(s)}_{\mathrm{I}} + \sum_{\substack{|\mathrm{J}|=s\\ \mathrm{j}_1>q}} \begin{vmatrix} \delta_{\mathrm{i}_1\mathrm{j}_1} & \cdots & \delta_{\mathrm{i}_1\mathrm{j}_s}\\ \vdots & \ddots & \vdots\\ \delta_{\mathrm{i}_s\mathrm{j}_1} & \cdots & \delta_{\mathrm{i}_s\mathrm{j}_s}\end{vmatrix}\,\underline{v}^{(s)}_{\mathrm{J}}. \tag{1.15}$$

and the coefficient of $\underline{v}_{\mathbb{K}}^{(s)}$ is equal to the determinant of the submatrix of $\mathfrak{M}(W)$ composed by the columns with indices in $(1\dots q) \setminus \mathrm{I}|\mathrm{J}$, i.e.

$$\begin{vmatrix} \delta_{\mathrm{i}_1\mathrm{j}_1} & \cdots & \delta_{\mathrm{i}_1\mathrm{j}_s} \\ \vdots & \ddots & \vdots \\ \delta_{\mathrm{i}_s\mathrm{j}_1} & \cdots & \delta_{\mathrm{i}_s\mathrm{j}_s} \end{vmatrix} = \varepsilon_{(1\dots q)\setminus \mathrm{I}|\mathrm{I}} \frac{\Delta_{(1\dots q)\setminus \mathrm{I}|\mathrm{J}}}{\Delta_{(1\dots q)}}.$$

Now the final step is to determine the vector space $\wedge^s W$ avoiding the hypothesis on the non-vanishing Plücker coordinate.

Definition 1.9. Let $\mathbf{Gr}_{\mathbb{K}}(q,N)$ be the Grassmannian of q-dimensional subspaces of $V = \langle \underline{v}_1,\dots,\underline{v}_N\rangle$ with the usual Plücker embedding $\mathscr{P} : \mathbf{Gr}_{\mathbb{K}}(q,N) \to \mathbb{P}_{\mathbb{K}}^{\binom{N}{q}-1}$. For any point $[\dots : \Delta_I : \dots] \in \mathbb{P}_{\mathbb{K}}^{\binom{N}{q}-1}$ and for any $s < q$, we associate to the ordered multiindex $\mathrm{J} = (\mathrm{j}_1,\dots,\mathrm{j}_{q-s})$ the element of $\wedge^s V$

$$\Lambda_{\mathrm{J}}^{(s)} = \sum_{|\mathrm{K}|=s} \Delta_{\mathrm{J}|\mathrm{K}}\, \underline{v}_{\mathrm{K}}^{(s)}. \tag{1.16}$$

Moreover we denote by $\Gamma^{(s)}$ the set of all possible $\Lambda_{\mathrm{J}}^{(s)}$:

$$\Gamma^{(s)} = \left\{ \Lambda_{\mathrm{J}}^{(s)} \mid \forall\, \mathrm{J} \text{ s.t. } |\mathrm{J}| = q-s \right\}. \tag{1.17}$$

Proposition 1.10. *Let $\mathbf{Gr}_{\mathbb{K}}(q,N)$ be the Grassmannian of q-dimensional subspaces of $V = \langle \underline{v}_1,\dots,\underline{v}_N\rangle$ with the usual Plücker embedding $\mathscr{P} : \mathbf{Gr}_{\mathbb{K}}(q,N) \to \mathbb{P}_{\mathbb{K}}^{\binom{N}{q}-1}$ and let us consider $W \in \mathbf{Gr}_{\mathbb{K}}(q,N)$. The vectors of*

$$\Gamma^{(s)}(W) = \left\{ \Lambda_{\mathrm{J}}^{(s)}(W) = \sum_{|\mathrm{K}|=s} \Delta_{\mathrm{J}|\mathrm{K}}(W)\, \underline{v}_{\mathrm{K}}^{(s)} \;\middle|\; \forall\, \mathrm{J} \textit{ s.t. } |\mathrm{J}| = q-s \right\}$$

generate $\wedge^s W$.

Proof. First of all, we show that for every J, $\Lambda_{\mathrm{J}}^{(s)}(W)$ belongs to $\wedge^s W$. Let us suppose $\Lambda_{\mathrm{J}}^{(s)}(W) \neq 0$, that is there exists a multiindex H, $|\mathrm{H}| = s$ such that the Plücker coordinate $\Delta_{\mathrm{I}} = \Delta_{\mathrm{J}\cup\mathrm{H}}$ does not vanish. We can write

$$\begin{aligned}\Lambda^{(s)}_{\mathrm{J}}(W) &= \Delta_{\mathrm{J}|\mathrm{H}}\,\underline{v}^{(s)}_{\mathrm{H}} + \sum_{\substack{|\mathrm{K}|=s\\ \mathrm{K}\neq\mathrm{H}}} \Delta_{\mathrm{J}|\mathrm{K}}\,\underline{v}^{(s)}_{\mathrm{K}} = \Delta_{\mathrm{J}|\mathrm{H}}\left(\underline{v}^{(s)}_{\mathrm{H}} + \sum_{\substack{|\mathrm{K}|=s\\ \mathrm{K}\neq\mathrm{H}}} \frac{\Delta_{\mathrm{J}|\mathrm{K}}}{\Delta_{\mathrm{J}|\mathrm{H}}}\,\underline{v}^{(s)}_{\mathrm{K}}\right) = \\ &= \Delta_{\mathrm{J}|\mathrm{H}}\left(\underline{v}^{(s)}_{\mathrm{H}} + \sum_{\substack{|\mathrm{K}|=s\\ \mathrm{K}\neq\mathrm{H}}} \frac{\Delta_{\mathrm{J}|\mathrm{K}}}{\varepsilon_{\mathrm{J}|\mathrm{H}}\,\Delta_{\mathrm{J}\cup\mathrm{H}}}\,\underline{v}^{(s)}_{\mathrm{K}}\right) = \\ &= \Delta_{\mathrm{J}|\mathrm{H}}\left(\underline{v}^{(s)}_{\mathrm{H}} + \sum_{\substack{|\mathrm{K}|=s\\ \mathrm{K}\neq\mathrm{H}}} \varepsilon_{\mathrm{I}\setminus\mathrm{H}|\mathrm{H}}\frac{\Delta_{\mathrm{I}\setminus\mathrm{H}|\mathrm{K}}}{\Delta_{\mathrm{I}}}\,\underline{v}^{(s)}_{\mathrm{K}}\right)\end{aligned}$$

obtaining an element of the type (except for a factor) described in (1.15). This element belongs to $\wedge^s W$ because it can be obtained as exterior product (up to multiplication by a scalar) of s elements of th basis of W

$$\left\{\Lambda^{(1)}_{\mathrm{J}'}(W) \;\middle|\; \mathrm{J}' \subset \mathrm{I},\ |\mathrm{J}'| = q-1\right\}.$$

and it is easy to check that this basis is of the same type of that one described in (1.14).

At this point we know that $\langle\Gamma^{(s)}(W)\rangle \subset \wedge^s W$, so to prove the equality it suffices to show that $\dim_{\mathbb{K}}\langle\Gamma^{(s)}(W)\rangle = \dim_{\mathbb{K}} \wedge^s W = \binom{q}{s}$. In fact the set

$$\left\{\Lambda^{(s)}_{\mathrm{K}}(W) \;\middle|\; \mathrm{K} \subset \mathrm{I},\ |\mathrm{K}| = s\right\}$$

contains $\binom{q}{s}$ elements linearly independents, i.e. it represents a basis for $\wedge^s W$. □

Example 1.2.4. We consider again the Grassmannian $\mathbf{Gr}_{\mathbb{K}}(4,6)$ introduced in Example 1.2.3 and we will use the same notation. Obviously $\Gamma^{(4)}$ contains a single element: the vector defining the point in $\mathbb{P}^{14}_{\mathbb{K}}$: $\Lambda^{(4)}_{\emptyset} = \sum \Delta_{\mathrm{I}}\,\underline{v}^{(4)}_{\mathrm{I}}$. We compute as example an element of $\Gamma^{(1)}$, $\Gamma^{(2)}$ and $\Gamma^{(3)}$.

- $\Gamma^{(1)}$, $|\Gamma^{(1)}| = \binom{6}{3} = 20$.

$$\Lambda^{(1)}_{156} = \Delta_{156|2}\,\underline{v}_2 + \Delta_{156|3}\,\underline{v}_3 + \Delta_{156|4}\,\underline{v}_4 = \Delta_{1256}\,\underline{v}_2 + \Delta_{1356}\,\underline{v}_3 + \Delta_{1456}\,\underline{v}_4.$$

- $\Gamma^{(2)}$, $|\Gamma^{(2)}| = \binom{6}{2} = 15$.

$$\begin{aligned}\Lambda^{(2)}_{24} &= \Delta_{24|13}\,\underline{v}^{(2)}_{13} + \Delta_{24|15}\,\underline{v}^{(2)}_{15} + \Delta_{24|16}\,\underline{v}^{(2)}_{16} + \Delta_{24|35}\,\underline{v}^{(2)}_{35} + \Delta_{24|36}\,\underline{v}^{(2)}_{36} + \Delta_{24|56}\,\underline{v}^{(2)}_{56} = \\ &= -\Delta_{1234}\,\underline{v}^{(2)}_{13} + \Delta_{1245}\,\underline{v}^{(2)}_{15} + \Delta_{1246}\,\underline{v}^{(2)}_{16} - \Delta_{2345}\,\underline{v}^{(2)}_{35} - \Delta_{2346}\,\underline{v}^{(2)}_{36} + \Delta_{2456}\,\underline{v}^{(2)}_{56}.\end{aligned}$$

- $\Gamma^{(3)}$, $|\Gamma^{(3)}| = \binom{6}{1} = 6$.

$$\begin{aligned}\Lambda^{(3)}_{3} &= \Delta_{3|124}\,\underline{v}^{(3)}_{124} + \Delta_{3|125}\,\underline{v}^{(3)}_{125} + \Delta_{3|126}\,\underline{v}^{(3)}_{126} + \Delta_{3|145}\,\underline{v}^{(3)}_{145} + \Delta_{3|146}\,\underline{v}^{(3)}_{146} + \\ &\quad + \Delta_{3|156}\,\underline{v}^{(3)}_{156} + \Delta_{3|245}\,\underline{v}^{(3)}_{245} + \Delta_{3|246}\,\underline{v}^{(3)}_{246} + \Delta_{3|256}\,\underline{v}^{(3)}_{256} + \Delta_{3|456}\,\underline{v}^{(3)}_{456} = \\ &= \Delta_{1234}\,\underline{v}^{(3)}_{124} + \Delta_{1235}\,\underline{v}^{(3)}_{125} + \Delta_{1236}\,\underline{v}^{(3)}_{126} - \Delta_{1345}\,\underline{v}^{(3)}_{145} - \Delta_{1346}\,\underline{v}^{(3)}_{146} + \\ &\quad - \Delta_{1356}\,\underline{v}^{(3)}_{156} - \Delta_{2345}\,\underline{v}^{(3)}_{245} - \Delta_{2346}\,\underline{v}^{(3)}_{246} - \Delta_{2356}\,\underline{v}^{(3)}_{256} + \Delta_{3456}\,\underline{v}^{(3)}_{456}.\end{aligned}$$

For the complete lists of elements in $\Gamma^{(1)}$, $\Gamma^{(2)}$ and $\Gamma^{(3)}$ see Example A.1.2.

Remark 1.2.5. In the paper *"Low degree equations defining the Hilbert scheme"* [17], the property stated in Proposition 1.10 is proved using a different approach, more abstract, based on some results of a paper [6] by Barnabei, Brini and Rota. For our computational purpose, we prefer this practical, and in some sense algorithmic, description.

Remark 1.2.6. As Example 1.2.4 suggests, the number of elements in $\Gamma^{(s)}(W)$ is bigger than the dimension of $\wedge^s W$, indeed

$$\dim_{\mathbb{K}} \wedge^s W = \binom{q}{s} \quad \text{and} \quad \left|\Gamma^{(s)}(W)\right| = \binom{N}{q-s}$$

and, set $p = N - q$,

$$\begin{aligned}\binom{N}{q-s} &= \binom{N-1}{q-s} + \binom{N-1}{q-s-1} = \\ &= \binom{N-2}{q-s} + \binom{N-2}{q-s-1} + \binom{N-1}{q-s-1} = \\ &\;\;\vdots \\ &= \binom{N-p}{q-s} + \sum_{i=1}^{p} \binom{N-i}{q-s-1} = \binom{q}{s} + \sum_{i=1}^{p} \binom{N-i}{q-s-1}.\end{aligned}$$

1.2.1 The Grassmann functor

Classically the same construction is extended to the more general setting of $\mathbb{K}$-algebras and direct summands of A^n of rank q.

It is therefore natural to introduce the following functor.

Definition 1.11 ([29, Exercise VI-18])**.** For any couple of integer (q, N), $0 < q < N$, the *Grassmann functor* is the functor

$$\begin{gathered}\mathcal{G}\mathrm{r}_q^N : (\mathbb{K}\text{-algebras}) \to (\text{sets}) \\ \mathcal{G}\mathrm{r}_q^N(A) = \left\{\text{rank } q \text{ direct summands of } A^N\right\}\end{gathered}$$

and for any $f : A \to B \in \mathrm{Mor}_{(\mathbb{K}\text{-algebras})}$ the map $\mathcal{G}\mathrm{r}_q^N(f)$ is defined as

$$\begin{array}{ccc}\mathcal{G}\mathrm{r}_q^N(A) & \to & \mathcal{G}\mathrm{r}_q^N(B) \\ P & \mapsto & P \otimes_A B\end{array}$$

by means of the extension of scalars (see [4, Proposition 2.17]) and using the right exactness of the tensor product.

Theorem 1.12. *The Grassmannian* $\mathbf{Gr}_{\mathbb{K}}(q, N)$ *represents the functor* $\mathcal{G}\mathrm{r}_q^N$*.*

To prove this result we need to say some more words about the Grassmannian. Let us consider the ring homomorphism

$$f : \mathbb{K}[\ldots, \Delta_{\mathrm{J}}, \ldots] \to \mathbb{K}[\ldots, \delta_{\mathrm{ij}}, \ldots]$$

where variables Δ_{J} are indexed over ordered subsets $\mathrm{J} \subset \{1, \ldots, N\}$, $|\mathrm{I}| = q$ and variables δ_{ij} over $1 \leqslant \mathrm{i} \leqslant q$, $1 \leqslant \mathrm{j} \leqslant N$, such that $f(\Delta_{\mathrm{J}}) = \det(\delta_{\mathrm{ij}})|_{\mathrm{j}\in\mathrm{J}}$. f induces the affine homomorphism

$$f^* : \mathbb{A}_{\mathbb{K}}^{qN} = \operatorname{Spec} \mathbb{K}[\ldots, \delta_{\mathrm{ij}}, \ldots] \to \mathbb{A}_{\mathbb{K}}^{\binom{N}{q}} = \operatorname{Spec} \mathbb{K}[\ldots, \Delta_{\mathrm{J}}, \ldots],$$

and considering its restriction to the Zariski open subset

$$U = \mathbb{A}_{\mathbb{K}}^{qN} \setminus Z(\langle \det(\delta_{\mathrm{ij}})|_{\mathrm{j}\in\mathrm{J}},\ \forall\, |\mathrm{J}| = q\rangle)$$

of the matrices (δ_{ij}) of rank q, also the morphism

$$\phi : U \to \mathbb{P}_{\mathbb{K}}^{\binom{N}{q}-1} = \operatorname{Proj}\mathbb{K}[\dots,\Delta_{\mathrm{J}},\dots]$$

turns out to be well defined.

Proposition 1.13. *The morphism $\phi : U \to \mathbb{P}_{\mathbb{K}}^{\binom{N}{q}-1}$ factors through $\mathbf{Gr}_{\mathbb{K}}(q,N)$:*

$$\begin{tikzcd} U \arrow[rr, "\phi"] \arrow[dr, dashed] & & \mathbb{P}_{\mathbb{K}}^{\binom{N}{q}-1} \\ & \mathbf{Gr}_{\mathbb{K}}(q,N) \arrow[ur, hook] & \end{tikzcd}$$

Proof. Firstly ϕ is equivariant with respect the action of $\mathrm{GL}_{\mathbb{K}}(q)$ over U given by left multiplication $\mu : \mathrm{GL}_{\mathbb{K}}(q) \times U \to U$, that is

$$\begin{tikzcd} \mathrm{GL}_{\mathbb{K}}(q) \times U \arrow[r, "\mu"] \arrow[d, "\pi_U"'] \arrow[dr, phantom, "\circlearrowright"] & U \arrow[d, "\phi"] \\ U \arrow[r, "\phi"] & \mathbb{P}_{\mathbb{K}}^{\binom{N}{q}-1} \end{tikzcd}$$

Then for any J ($|\mathrm{J}| = q$), let U_{J} the closed subscheme of U such that the submatrix $(\delta_{\mathrm{ij}})|_{\mathrm{j}\in\mathrm{J}}$ is equal to the identity matrix. Obviously $U_{\mathrm{J}} \simeq \mathbb{A}_{\mathbb{K}}^{q(N-q)}$. Moreover consider the standard open covering on $\mathbb{P}_{\mathbb{K}}^{\binom{N}{q}-1}$

$$\bigcup_{\mathrm{J}} \mathcal{U}_{\mathrm{J}}, \qquad \mathcal{U}_{\mathrm{J}} = \mathbb{P}_{\mathbb{K}}^{\binom{N}{q}-1} \setminus Z(\Delta_{\mathrm{J}}).$$

To prove that ϕ factors through $\mathbf{Gr}_{\mathbb{K}}(q,N)$ we remark that

(i) $\mu|_{U_{\mathrm{J}}} : \mathrm{GL}_{\mathbb{K}}(q) \times U_{\mathrm{J}} \to \phi^{-1}(\mathcal{U}_{\mathrm{J}})$ is an isomorphism;

(ii) $\phi|_{U_{\mathrm{J}}} : U_{\mathrm{J}} \to \mathcal{U}_{\mathrm{J}} \cap \mathbf{Gr}_{\mathbb{K}}(q,N)$ is an isomorphism;

(iii) for each Plücker relation Q of the set in (1.13), the image of Q/Δ_{J}^2 in $\mathbb{K}[\dots,\delta_{\mathrm{ij}},\dots]$ belongs to the ideal $I(U_{\mathrm{J}})$ defining U_{J}. □

We rewrite the statement of Theorem 1.12 through the following proposition.

Proposition 1.14. *There exists an invertible natural transformation*

$$\mathscr{F} : \mathcal{G}r_q^N \to h_{\mathbf{Gr}_{\mathbb{K}}(q,N)}. \tag{1.18}$$

Proof. Let us consider an element $P \in \mathcal{G}r_q^N(A)$ and let us define a morphism $\operatorname{Spec} A \to \mathbf{Gr}_{\mathbb{K}}(q,N)$, i.e. an element in $h_{\mathbf{Gr}_{\mathbb{K}}(q,N)}(A)$. Being P a direct summand of A^N of rank q, the injective map

$$i_P : P \hookrightarrow A^N$$

is described by a $N \times q$ matrix with coefficients in A. Therefore the chain of morphism

$$\mathbb{K}[\ldots,\delta_{\mathrm{ij}},\ldots] \longrightarrow A[\ldots,\delta_{\mathrm{ij}},\ldots] = \mathbb{K}[\ldots,\delta_{\mathrm{ij}},\ldots] \otimes_{\mathbb{K}} A \longrightarrow A, \tag{1.19}$$

where the second map is the evaluation map over the transposed matrix ${}^T i_P$, induces a morphism $f_P : \operatorname{Spec} A \to \mathbb{A}_{\mathbb{K}}^{qN}$ that factors over U. Finally $\phi \circ f_P$ factors over $\mathbf{Gr}_{\mathbb{K}}(q,N)$ giving a morphism $\phi \circ f_p : \operatorname{Spec} A \to \mathbf{Gr}_{\mathbb{K}}(q,N) \in h_{\mathbf{Gr}_{\mathbb{K}}(q,N)}(A)$. The transformation between morphisms of the two category follows directly from (1.19) and extension of scalars.

To invert $\mathscr{F}$, we look for the universal family over $\mathbf{Gr}_{\mathbb{K}}(q,N)$, that is we want to construct a sub-bundle $\mathcal{K} \hookrightarrow \mathcal{O}^N_{\mathbf{Gr}_{\mathbb{K}}(q,N)}$. We consider again the action $\mu : \mathrm{GL}_{\mathbb{K}}(q) \times U \to U$ and let $\mathcal{V}_{\mathrm{I}} = \mathcal{U}_{\mathrm{I}} \cap \mathbf{Gr}_{\mathbb{K}}(q,N)$ and $\mathcal{V}_{\mathrm{J}} = \mathcal{U}_{\mathrm{J}} \cap \mathbf{Gr}_{\mathbb{K}}(q,N)$. By the property (i) exposed in the proof of Proposition 1.13, $\mathcal{V}_{\mathrm{I}} \simeq U_{\mathrm{I}} \simeq \mathbb{A}_{\mathbb{K}}^{q(N-q)}$ and $\mathcal{V}_{\mathrm{J}} \simeq U_{\mathrm{J}} \simeq \mathbb{A}_{\mathbb{K}}^{q(N-q)}$ and using the property (iii) we can define the isomorphism

$$\rho_{\mathrm{IJ}} : \mathrm{GL}_{\mathbb{K}}(q) \times (\mathcal{V}_{\mathrm{I}} \cap \mathcal{V}_{\mathrm{J}}) \xrightarrow{\sim} \phi^{-1}(\mathcal{V}_{\mathrm{I}} \cap \mathcal{V}_{\mathrm{J}}) \xrightarrow{\sim} \mathrm{GL}_{\mathbb{K}}(q) \times (\mathcal{V}_{\mathrm{I}} \cap \mathcal{V}_{\mathrm{J}})$$

where the intersection on the left $\mathcal{V}_{\mathrm{I}} \cap \mathcal{V}_{\mathrm{J}}$ is considered as open subset of $\mathcal{V}_{\mathrm{I}}$ whereas the intersection on the right as open subset of $\mathcal{V}_{\mathrm{J}}$. Thus the morphism to $\mathrm{GL}_{\mathbb{K}}(q)$ induced by ρ_{IJ} determines a transition function $g_{\mathrm{IJ}} : \mathcal{V}_{\mathrm{I}} \cap \mathcal{V}_{\mathrm{J}} \to \mathrm{GL}_K(q)$.

For every I, the embedding $\mathcal{V}_{\mathrm{I}} \simeq U_{\mathrm{I}} \simeq \mathbb{A}_{\mathbb{K}}^{q(N-q)} \hookrightarrow \mathbb{A}_{\mathbb{K}}^{qN}$ induces the map $\mathcal{O}^q_{\mathcal{V}_{\mathrm{I}}} \to \mathcal{O}^N_{\mathcal{V}_{\mathrm{I}}}$. This collection of maps glue together by means of the transition functions g_{IJ}

to the injective map $\mathcal{K} \to \mathcal{O}^N_{\mathbf{Gr}_{\mathbb{K}}(q,N)}$, whose cokernel turns out to be a locally free sheaf of rank $N-q$.

Finally for every morphism $f : \operatorname{Spec} A \to \mathbf{Gr}_{\mathbb{K}}(q,N) \in h_{\mathbf{Gr}_{\mathbb{K}}(q,N)}(A)$, we define an element $\mathscr{F}^{-1}(f) \in \mathcal{G}r^N_q(A)$ starting from the family

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{O}^N_{\mathbf{Gr}_{\mathbb{K}}(q,N)} \longrightarrow \mathcal{Q} \longrightarrow 0$$

via pull-back

$$0 \longrightarrow f^*\mathcal{K} \longrightarrow f^*\mathcal{O}^N_{\mathbf{Gr}_{\mathbb{K}}(q,N)} \simeq A^N \longrightarrow f^*\mathcal{Q} \longrightarrow 0. \qquad \square$$

By Yoneda's Lemma (Theorem 1.3) the schemes representing $\mathcal{G}r^N_q$ and $h_{\mathbf{Gr}_{\mathbb{K}}(q,N)}$ have to be isomorphic, so $\mathbf{Gr}_{\mathbb{K}}(q,N)$ represents also $\mathcal{G}r^N_q$.

1.3 The Hilbert functor

Definition 1.15. Let us define the *Hilbert functor* as the covariant functor

$$\mathcal{H}\mathrm{ilb}^n : (\text{schemes})_{\mathbb{K}} \to (\text{sets})$$

such that for $X \in \mathrm{Ob}_{(\text{schemes})_{\mathbb{K}}}$

$$\mathcal{H}\mathrm{ilb}^n(X) = \{\mathcal{Z} \subset \mathbb{P}^n \times X \mid \mathcal{Z} \text{ flat over } X \text{ via } \pi : \mathcal{Z} \hookrightarrow \mathbb{P}^n \times X \to X\}$$

and for $f : X \to Y \in \mathrm{Mor}_{(\text{schemes})_{\mathbb{K}}}$

$$\begin{aligned} \mathcal{H}\mathrm{ilb}^n(f) : \mathcal{H}\mathrm{ilb}^n(Y) &\to \mathcal{H}\mathrm{ilb}^n(X) \\ \mathcal{Z} &\mapsto f^*(\mathcal{Z}) \end{aligned}$$

that is well defined because the pullback preserves the flatness:

$$\begin{array}{ccccc} f^*(\mathcal{Z}) & \longrightarrow & \mathbb{P}^n_{\mathbb{K}} \times X & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle \mathrm{id}\times f} & & \downarrow{\scriptstyle f} \\ \mathcal{Z} & \longrightarrow & \mathbb{P}^n_{\mathbb{K}} \times Y & \longrightarrow & Y \end{array}$$

For the properties of flatness, we know that the Hilbert polynomial is locally constant for $x \in X$, so we can decompose the functor $\mathcal{H}\mathrm{ilb}^n$ as

$$\mathcal{H}\mathrm{ilb}^n = \coprod_{p(t)\in\mathbb{Q}[t]} \mathcal{H}\mathrm{ilb}^n_{p(t)}.$$

where $p(t) \in \mathbb{Q}[t]$ is a numerical polynomial and $\mathcal{H}\mathrm{ilb}^n_{p(t)}$ is the subfunctor of $\mathcal{H}\mathrm{ilb}^n$ such that

$$\mathcal{H}\mathrm{ilb}^n_{p(t)}(X) = \left\{ \mathcal{Z} \subset \mathbb{P}^n_{\mathbb{K}} \times X \;\middle|\; \begin{array}{l} \mathcal{Z} \text{ flat over } X \text{ with fibers} \\ \text{having Hilbert polynomial } p(t) \end{array} \right\}.$$

Theorem 1.16 (Grothendieck [40]). *The functor $\mathcal{H}\mathrm{ilb}^n_{p(t)}$ is representable.*

Definition 1.17. We call *Hilbert scheme* and denote by $\mathbf{Hilb}^n_{p(t)}$ the scheme representing the functor $\mathcal{H}\mathrm{ilb}^n_{p(t)}$.

There is another useful meaning of the representability of a functor.

Proposition 1.18. *The Hilbert scheme $\mathbf{Hilb}^n_{p(t)}$ representing $\mathcal{H}\mathrm{ilb}^n_{p(t)}$ is the parameter scheme of a flat family $\mathcal{X}$ with Hilbert polynomial $p(t)$*

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[dr] & \mathbb{P}^n_{\mathbb{K}} \times \mathbf{Hilb}^n_{p(t)} \arrow[d] \\ & \mathbf{Hilb}^n_{p(t)} \end{tikzcd}$$

such that any subscheme $\mathcal{Y} \subset \mathbb{P}^n_{\mathbb{K}} \times S$ flat over S with Hilbert polynomial $p(t)$ coincides with the fiber product $\mathcal{X} \times_{\mathbf{Hilb}^n_{p(t)}} S \subset \mathbb{P}^n_{\mathbb{K}} \times S$ for a unique map $S \to \mathbf{Hilb}^n_{p(t)}$:

$$\begin{tikzcd} \mathcal{Y} = \mathcal{X} \times_{\mathbf{Hilb}^n_{p(t)}} S \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ S \arrow[r] & \mathbf{Hilb}^n_{p(t)} \end{tikzcd}$$

The family $\mathcal{X} \to \mathbf{Hilb}^n_{p(t)}$ is called universal family *and we will refer to this property as* universal property *of the Hilbert scheme.*

To simplify the discussion, we use again Proposition 1.5 for rewriting the Hilbert functor as a functor over the category of affine schemes:

$$\mathcal{H}\mathrm{ilb}^n_{p(t)} : (\text{schemes})^{\mathrm{Aff}}_{\mathbb{K}} \to (\text{sets}),$$
$$\mathcal{H}\mathrm{ilb}^n_{p(t)}(\operatorname{Spec} A) = \left\{\mathcal{Z} \subset \mathbb{P}^n_A \mid \mathcal{Z} \text{ flat over } \operatorname{Spec} A \text{ with Hilbert polynomial } p(t)\right\}.$$

It is well known that saying $\mathcal{Z}$ flat over $\operatorname{Spec} A$ means that the structure sheaf $\mathcal{O}_{\mathcal{Z}}$ is flat over $\operatorname{Spec} A$. Being in the affine case we can consider the graded module

$$M = \bigoplus_{t \geqslant 0} H^0(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}(t)) \qquad (\text{s.t. } \widetilde{M} = \mathcal{O}_{\mathcal{Z}})$$

over $A[x_0, \ldots, x_n]$ and $\mathcal{O}_{\mathcal{Z}}$ flat over $\operatorname{Spec} A$ is equivalent to M flat over A (see [43, Chapter III Proposition 9.2]). The flatness is preserved by localization ([43, Chapter III Proposition 9.1]), i.e. M is flat over A if and only if $M_{\mathfrak{p}}$ is flat over $A_{\mathfrak{p}}$, for all $\mathfrak{p} \in \operatorname{Spec} A$. Moreover for any local algebra $A_{\mathfrak{p}}$, the flatness of a finite $A_{\mathfrak{p}}$-module is equivalent to its freeness ([69, Proposition 3.G]), so denoted by $k(\mathfrak{p})$ the residue field of $A_{\mathfrak{p}}$, the Hilbert polynomial $p(t)$ of $M_{\mathfrak{p}}$ is well defined as the Hilbert polynomial of the module $M_{\mathfrak{p}} \otimes k(\mathfrak{p})$, that is

$$p(t) = \dim_{k(\mathfrak{p})} (M_{\mathfrak{p}})_t \otimes k(\mathfrak{p}), \qquad t \gg 0. \tag{1.20}$$

Finally the flatness ensures that the Hilbert polynomial does not depend on the point $\mathfrak{p} \in \operatorname{Spec} A$ for which we localize ([43, Chapter III Theorem 9.9] and [29, Exercise 6.11]).

Hence we can redefine the Hilbert functor as follows

$$\mathcal{H}\mathrm{ilb}^n_{p(t)} : (\mathbb{K}\text{-algebras}) \to (\text{sets})$$
$$\mathcal{H}\mathrm{ilb}^n_{p(t)}(A) = \left\{ \begin{array}{c} M, \text{ graded module over } A[x_0, \ldots, x_n] \text{ flat over } A \\ \text{with Hilbert polynomial } p(t) \end{array} \right\}$$

and for any $f : A \to B$,

$$\begin{array}{rcl} \mathcal{H}\mathrm{ilb}^n_{p(t)}(f) : \mathcal{H}\mathrm{ilb}^n_{p(t)}(A) & \to & \mathcal{H}\mathrm{ilb}^n_{p(t)}(B) \\ M & \mapsto & M \otimes_A B \end{array}$$

using the fact that the extension of scalars preserves the flatness (see [69, (3.C)]).

Definition 1.19. The elements of the set $\mathcal{H}\mathrm{ilb}^n_{p(t)}(A)$ are called A-valued points of the Hilbert scheme $\mathbf{Hilb}^n_{p(t)}$. With this terminology, the universal property of $\mathbf{Hilb}^n_{p(t)}$ can be described saying that any flat family $\mathcal{Y} \to \operatorname{Spec} A$ defines a unique A-valued point of $\mathbf{Hilb}^n_{p(t)}$.

Remark 1.3.1. The set of $\mathbb{K}$-valued points $\mathcal{H}\mathrm{ilb}^n_{p(t)}(\mathbb{K})$ contains exactly the subschemes $X \subset \mathbb{P}^n_{\mathbb{K}}$ with Hilbert polynomial $p(t)$, being $\operatorname{Spec} \mathbb{K}$ a single point.

1.4 The Hilbert scheme as subscheme of the Grassmannian

Definition 1.20. An admissible Hilbert polynomial $p(t)$ has a unique *Gotzmann representation*

$$p(t) = \binom{t+a_1}{a_1} + \binom{t+a_2-1}{a_2} + \ldots + \binom{t+a_r-(r-1)}{a_r}, \qquad a_1 \geqslant \cdots \geqslant a_r. \tag{1.21}$$

The number r of terms in this sum is said *Gotzmann number* of $p(t)$.

Example 1.4.1. The Hilbert polynomial $p(t) = 4t$ has Gotzmann number equal to 6, indeed

$$4t = \binom{t+1}{1} + \binom{t}{1} + \binom{t-1}{1} + \binom{t-2}{1} + \binom{t-4}{0} + \binom{t-5}{0}.$$

Theorem 1.21 (Gotzmann's Regularity Theorem [38, Theorem 3.11]**).** *Let A be any $\mathbb{K}$-algebra and let $Z \subset \operatorname{Proj} A[x_0,\ldots,x_n]$ be any subscheme with Hilbert polynomial $p(t)$, whose Gotzmann number is r. Then the sheaf of ideals $\mathcal{I}_Z$ is r-regular.*

As usual to any subscheme $Z \subset \mathbb{P}^n_A$, we can associate the saturated ideal $I(Z) = \bigoplus_t H^0(Z, \mathcal{I}_Z(t))$. Gotzmann's Regularity Theorem ensures that the truncated ideal $I(Z)_{\geqslant r}$ is generated by its homogenous piece of degree r, that is

$$I(Z)_{\geqslant r} = \langle I(Z)_r \rangle\,. \tag{1.22}$$

This result suggests to associate to any subscheme $Z \subset \operatorname{Proj} A[x_0,\ldots,x_n]$ the truncation $I(Z)_{\geqslant r}$ instead of the saturated ideal $I(Z)$, with the main advantage of a more uniform description, indeed with this approach any ideal associated to such subschemes is generated by the same number of linearly independent polynomials of the same degree.

For any $M \in \mathcal{H}\mathrm{ilb}^n_{p(t)}(A)$, called Z the affine subscheme flat over $\operatorname{Spec} A$ such that $\widetilde{M} = \mathcal{O}_Z$, $M = A[x_0,\ldots,x_n]/I(Z)$. We are interested in the homogeneous piece of degree r: $M_r = A[x_0,\ldots,n_n]_r/I(Z)_r$. For each $\mathfrak{p} \in \operatorname{Spec} A$, we have already said that $(M_{\mathfrak{p}})_r$ is a free $A_{\mathfrak{p}}$-module and this property implies that M_r has to be a projective A-module ([29, Theorem A3.2]). As explained in Appendix C of [52, pp. 302-303] (see also [29, Section A3.2]), being $A[x_0,\ldots,n_n]_r/I(Z)_r$ projective ensures that $I(Z)_r$ is a direct summand of $A[x_0,\ldots,x_n]_r$.

Therefore, set $N(t) = \binom{n+t}{n}$ and $q(t) = N(t) - p(t)$, we can define a natural transformation of functors from $\mathcal{H}\mathrm{ilb}^n_{p(t)}$ and $\mathcal{G}\mathrm{r}^{N(r)}_{q(r)}$

$$\begin{array}{ccc} \mathcal{H}\mathrm{ilb}^n_{p(t)}(A) & \to & \mathcal{G}\mathrm{r}^{N(r)}_{q(r)}(A) \\ M = A[x_0,\ldots,x_n]/I(Z) & \mapsto & I(Z)_r \subset A^{N(r)} \simeq A[x_0,\ldots,x_n]_r \end{array} \tag{1.23}$$

that suggests to determine the Hilbert scheme $\mathbf{Hilb}^n_{p(t)}$ representing $\mathcal{H}\mathrm{ilb}^n_{p(t)}$ as subscheme of the Grassmannian $\mathbf{Gr}_{\mathbb{K}}(q(r), N(r))$ (representing $\mathcal{G}\mathrm{r}^{N(r)}_{q(r)}$). To accomplish this purpose, we need to understand under which conditions an ideal $I = \langle I_r \rangle \subset A[x_0,\ldots,x_n]$ generated by $q(r)$ linearly independent homogeneous polynomials of degree r determines a module $A[x_0,\ldots,x_n]/I$ with Hilbert polynomial $p(t)$. One of the best characterization is stated in the following theorem.

Theorem 1.22 (**Gotzmann's Persistence Theorem** [52, Theorem C.17]). *Let $I \subset A[x_0,\ldots,x_n]$ be a homogeneous ideal generated by its piece of degree r, i.e. $I = \langle I_r \rangle$. If $A[x_0,\ldots,x_n]_r/I_r$ and $A[x_0,\ldots,x_n]_{r+1}/I_{r+1}$ are flat A-modules of rank $p(r)$ and $p(r+1)$, then $A[x_0,\ldots,x_n]_t/I_t$ is a A-flat module of rank $p(t)$ for all $t \geqslant r$.*

There is another important results by Macaulay that further simplifies the characterization.

Theorem 1.23 (**Macaulay's Estimate on the Growth of Ideals** [38, Theorem 3.3]). *Let $I \subset A[x_0,\ldots,x_n]$ be a homogeneous ideal and let $p(t)$ be an admissible Hilbert polynomial. If* $\operatorname{rank} A[x_0,\ldots,x_n]_r/I_r = p(r)$, *then* $\operatorname{rank} A[x_0,\ldots,x_n]_{r+1}/I_{r+1} \leqslant p(r+1)$.

Putting together Theorem 1.22 and Theorem 1.23, the condition we have to impose on $I \subset A[x_0,\dots,x_n]$ is $\operatorname{rank} A[x_0,\dots,x_n]_{r+1}/I_{r+1} \geqslant p(r+1)$ or $\operatorname{rank} I_{r+1} \leqslant q(r+1)$.

1.5 Known sets of equations

In this section we will consider a fixed Hilbert polynomial $p(t)$ with Gotzmann number r, a fixed projective space $\mathbb{P}^n_{\mathbb{K}} = \operatorname{Proj}\mathbb{K}[x_0,\dots,x_n]$. Let $N = N(r)$ and $q = q(r) = N(r) - p(r)$. In the previous section we embedded the Hilbert scheme $\mathbf{Hilb}^n_{p(t)}$ in the Grassmannian $\mathbf{Gr}_{\mathbb{K}}(q,N)$ parametrizing the vector subspaces of dimension q in the base vector space $\mathbb{K}[x_0,\dots,x_n]_r \simeq \mathbb{K}^N$.

To determine equations defining $\mathbf{Hilb}^n_{p(t)}$, we will use tools introduced in Section 1.2 and for this reason we slightly modify the notation, adapting it to this special case. We consider $\mathbb{K}[x_0,\dots,x_n]_r$ equipped with its standard monomial basis $\{x^\beta \text{ s.t. } |\beta| = r\}$ and the bijective function

$$\begin{gathered}\alpha : \{1,\dots,N\} \to \left\{(a_0,\dots,a_n) \in \mathbb{Z}^{n+1} \;\middle|\; a_i \geqslant 0,\ \forall\, i \text{ and } \textstyle\sum_{i=0}^n a_i = r\right\}\\ x^{\alpha(1)} >_{\texttt{DegRevLex}} x^{\alpha(2)} >_{\texttt{DegRevLex}} \cdots >_{\texttt{DegRevLex}} x^{\alpha(N)}\end{gathered} \tag{1.24}$$

so that the Plücker coordinate Δ_{I} corresponds to the vector $\underline{x}^{(s)}_{\mathrm{I}} = x^{\alpha(\mathrm{i}_1)} \wedge \cdots \wedge x^{\alpha(\mathrm{i}_q)}$ of the basis of $\wedge^q \mathbb{K}[x_0,\dots,x_n]_r$ and for all $\mathrm{J} \subset \{1,\dots,N\}$, $|\mathrm{J}| = q - s$,

$$\Lambda^{(s)}_{\mathrm{J}} = \sum_{|\mathrm{K}|=s} \Delta_{\mathrm{J}|\mathrm{K}}\, \underline{x}^{(s)}_{\mathrm{K}} = \sum_{|\mathrm{K}|=s} \Delta_{\mathrm{J}|\mathrm{K}}\, x^{\alpha(\mathrm{k}_1)} \wedge \cdots \wedge x^{\alpha(\mathrm{k}_s)}.$$

Moreover we define for each variable x_i

$$x_i\Lambda^{(s)}_{\mathrm{J}} = \sum_{|\mathrm{K}|=s} \Delta_{\mathrm{J}|\mathrm{K}}\, x_i\underline{x}^{(s)}_{\mathrm{K}} = \sum_{|\mathrm{K}|=s} \Delta_{\mathrm{J}|\mathrm{K}}\, x_i x^{\alpha(\mathrm{k}_1)} \wedge \cdots \wedge x_i x^{\alpha(\mathrm{k}_s)}, \tag{1.25}$$

$$x_i\Gamma^{(s)} = \left\{ x_i\Lambda^{(s)}_{\mathrm{J}} \mid \forall\, \mathrm{J} \subset \{1,\dots,N\},\ |\mathrm{J}| = q-s \right\}. \tag{1.26}$$

1.5.1 Gotzmann equations

Moving from the Persistence Theorem (Theorem 1.22), the idea of Gotzmann [34] was to consider the natural transformation of functors $\mathcal{H}\mathrm{ilb}^n_{p(t)} \to \mathcal{G}\mathrm{r}^{N(r)}_{q(r)} \times \mathcal{G}\mathrm{r}^{N(r+1)}_{q(r+1)}$

$$\begin{array}{rcc} \mathcal{H}\mathrm{ilb}^n_{p(t)}(A) & \to & \mathcal{G}\mathrm{r}^{N(r)}_{q(r)}(A) \times \mathcal{G}\mathrm{r}^{N(r+1)}_{q(r+1)}(A) \\ M = A[x]/I(Z) & \mapsto & I(Z)_r \times I(Z)_{r+1} \end{array} \tag{1.27}$$

in order to translate the Hilbert functor as

$$\mathcal{H}\mathrm{ilb}^n_{p(t)}(A) = \left\{ \begin{array}{c} (I,J) \in \mathcal{G}\mathrm{r}^{N(r)}_{q(r)}(A) \times \mathcal{G}\mathrm{r}^{N(r+1)}_{q(r+1)}(A) \text{ s.t.} \\ I \subset A[x]_r,\ J \subset A[x]_{r+1} \text{ and } I \cdot A[x]_1 = J \end{array} \right\} \tag{1.28}$$

and applying Theorem 1.23 we can write

$$\mathcal{H}\mathrm{ilb}^n_{p(t)}(A) = \left\{ \begin{array}{c} (I,J) \in \mathcal{G}\mathrm{r}^{N(r)}_{q(r)}(A) \times \mathcal{G}\mathrm{r}^{N(r+1)}_{q(r+1)}(A) \text{ s.t.} \\ I \subset A[x]_r,\ J \subset A[x]_{r+1} \text{ and } I \cdot A[x]_1 \subset J \end{array} \right\}. \tag{1.29}$$

Theorem 1.24 ([52, Proposition C.28, Theorem C.29]). *The Hilbert scheme* $\mathbf{Hilb}^n_{p(t)}$ *is defined by quadric equations in the product of Grassmannians*

$$\mathbf{Gr}_{\mathbb{K}}(q(r), N(r)) \times \mathbf{Gr}_{\mathbb{K}}(q(r+1), N(r+1))$$

and then can be embedded in $\mathbf{Gr}_{\mathbb{K}}(q(r), N(r))$ *through the projection on the first factor.*

Proof. We want to find closed conditions to describe the set

$$\big\{(I,J) \in \mathbf{Gr}_{\mathbb{K}}(q(r), N(r)) \times \mathbf{Gr}_{\mathbb{K}}(q(r+1), N(r+1)) \mid I \cdot \mathbb{K}[x_0,\dots,x_n]_1 \subset J\big\}.$$

Through the isomorphism

$$\begin{array}{ccc} \mathbf{Gr}_{\mathbb{K}}(q(r+1), N(r+1)) & \to & \mathbf{Gr}_{\mathbb{K}}(p(r+1), N(r+1)) \\ J & \mapsto & J^{\perp} \end{array}$$

we redefine the condition as

$$\big\{(I,J) \in \mathbf{Gr}_{\mathbb{K}}(q(r), N(r)) \times \mathbf{Gr}_{\mathbb{K}}(p(r+1), N(r+1)) \mid I \cdot \mathbb{K}[x]_1 \subset J^{\perp}\big\}.$$

We consider the Grassmannian $\mathbf{Gr}_{\mathbb{K}}(q(r), N(r))$ equipped with the Plücker coordinates $[\ldots, \Delta_{\mathrm{I}}, \ldots]$. The ideal I is generated by the set $\Gamma^{(1)}$ and so

$$I \cdot \mathbb{K}[x]_1 = \left\langle x_i \Gamma^{(1)} \mid i = 0, \ldots, n \right\rangle.$$

Whereas for $\mathbf{Gr}_{\mathbb{K}}(p(r+1), N(r+1))$, we denote the Plücker coordinates by $[\ldots, \nabla_{\mathrm{J}}, \ldots]$, the generators of J by $\widetilde{\Lambda}^{(1)}_{\mathrm{K}}$ and the whole set of them by $\widetilde{\Gamma}^{(1)}$.

The condition $I \cdot \mathbb{K}[x]_1 \subset J^{\perp}$ is equivalent to the vanishing of the scalar products

$$\begin{aligned} x_i \Lambda^{(1)}_{\mathrm{H}} \cdot \widetilde{\Lambda}^{(1)}_{\mathrm{K}} &= \left(\sum_{\mathrm{h}} \Delta_{\mathrm{H}|(\mathrm{h})}\, x_i x^{\alpha(\mathrm{h})} \right) \cdot \left(\sum_{\mathrm{k}} \nabla_{\mathrm{K}|(\mathrm{k})}\, x^{\alpha(\mathrm{k})} \right) = \\ &= \sum_{\substack{\mathrm{h},\,\mathrm{k} \\ x_i x^{\alpha(\mathrm{h})} = x^{\alpha(\mathrm{k})}}} \Delta_{\mathrm{H}|(\mathrm{h})} \nabla_{\mathrm{K}|(\mathrm{k})}, \end{aligned}$$

for each $i = 0, \ldots, n$ and for all $x_i \Lambda^{(1)}_{\mathrm{H}} \in x_i \Gamma^{(1)}$ and $\widetilde{\Lambda}^{(1)}_{\mathrm{K}} \in \widetilde{\Gamma}^{(1)}$. Denoted by $\mathcal{I}_{\mathcal{H}}$ the ideal generated by the set of quadrics

$$\left\{ \sum_{\substack{\mathrm{h},\,\mathrm{k} \\ x_i x^{\alpha(\mathrm{h})} = x^{\alpha(\mathrm{k})}}} \Delta_{\mathrm{H}|(\mathrm{h})} \nabla_{\mathrm{K}|(\mathrm{k})} \;\middle|\; \forall\, \mathrm{H}, \mathrm{K}, i \right\}$$

and by the Plücker relations of both $\mathbf{Gr}_{\mathbb{K}}(q(r), N(r))$ and $\mathbf{Gr}_{\mathbb{K}}(p(r+1), N(r+1))$, the Hilbert scheme is defined as

$$\mathbf{Hilb}^n_{p(t)} \simeq \operatorname{Proj}\left(\mathbb{K}[\ldots, \Delta_{\mathrm{I}}, \ldots] \times \mathbb{K}[\ldots, \nabla_{\mathrm{J}}, \ldots] \right) / \mathcal{I}_{\mathcal{H}}. \qquad \square$$

Example 1.5.1. Let us compute Gotzmann's equations of the Hilbert scheme $\mathbf{Hilb}^2_2$. Since the Gotzmann number of the Hilbert polynomial $p(t) = 2$ is 2 and $\binom{2+2}{2} = 6$, $\binom{2+3}{2} = 10$, we embed $\mathbf{Hilb}^2_2$ in

$$\mathbf{Gr}_{\mathbb{K}}(4, 6) \times \mathbf{Gr}_{\mathbb{K}}(8, 10) \simeq \mathbf{Gr}_{\mathbb{K}}(4, 6) \times \mathbf{Gr}_{\mathbb{K}}(2, 10)$$

The first Grassmannian is the same already introduced in Examples 1.2.3 and 1.2.4, but in this case we consider the monomial basis $\{x_0^2, x_0x_1, x_1^2, x_0x_2, x_1x_2, x_2^2\}$ of $\mathbb{K}[x_0, x_1, x_2]_2$. By Example 1.2.4, we know that among the generators of $I \in \mathbf{Gr}_{\mathbb{K}}(4, 6)$ there is

$$\Lambda^{(1)}_{156} = \Delta_{1256}\, x_0x_1 + \Delta_{1356}\, x_1^2 + \Delta_{1456}\, x_0x_2.$$

Fixed the monomial basis $\{x_0^3, x_0^2x_1, x_0x_1^2, x_1^3, x_0^2x_2, x_0x_1x_2, x_1^2x_2, x_0x_2^2, x_1x_2^2, x_2^3\}$ of $\mathbb{K}[x_0, x_1, x_2]_3$ among the generators $\widetilde{\Gamma}^{(1)}$ of $J \in \mathbf{Gr}_{\mathbb{K}}(2, 10)$ there is

$$\begin{aligned}\widetilde{\Lambda}_7^{(1)} = &-\nabla_{1,7}\, x_0^3 - \nabla_{2,7}\, x_0^2x_1 - \nabla_{3,7}\, x_0x_1^2 - \nabla_{4,7}\, x_1^3 - \nabla_{5,7}\, x_0^2x_2 + \\ &- \nabla_{6,7}\, x_0x_1x_2 + \nabla_{7,8}\, x_0x_2^2 + \nabla_{7,9}\, x_1x_2^2 + \nabla_{7,10}\, x_2^3.\end{aligned}$$

Thus among the equations described in Theorem 1.24, we will find

$$\begin{aligned}x_0\Lambda_{156}^{(1)} \cdot \widetilde{\Lambda}_7^{(1)} &= \left(\Delta_{1256}\, x_0^2x_1 + \Delta_{1356}\, x_0x_1^2 + \Delta_{1456}\, x_0^2x_2\right) \cdot \widetilde{\Lambda}_7^{(1)} = \\ &= -\Delta_{1256}\nabla_{2,7} - \Delta_{1356}\nabla_{3,7} - \Delta_{1456}\nabla_{5,7}, \\ x_1\Lambda_{156}^{(1)} \cdot \widetilde{\Lambda}_7^{(1)} &= \left(\Delta_{1256}\, x_0x_1^2 + \Delta_{1356}\, x_1^3 + \Delta_{1456}\, x_0x_1x_2\right) \cdot \widetilde{\Lambda}_7^{(1)} = \\ &= -\Delta_{1256}\nabla_{3,7} - \Delta_{1356}\nabla_{4,7} - \Delta_{1456}\nabla_{6,7}, \\ x_2\Lambda_{156}^{(1)} \cdot \widetilde{\Lambda}_7^{(1)} &= \left(\Delta_{1256}\, x_0x_1x_2 + \Delta_{1356}\, x_1^2x_2 + \Delta_{1456}\, x_0x_2^2\right) \cdot \widetilde{\Lambda}_7^{(1)} = \\ &= -\Delta_{1256}\nabla_{6,7} + \Delta_{1456}\nabla_{7,8}.\end{aligned}$$

Since $|\Gamma^{(1)}| = \binom{6}{3} = 20$ and $|\widetilde{\Gamma}^{(1)}| = \binom{10}{1} = 10$, the condition $I \subset J^\perp$ is represented by $3 \cdot 20 \cdot 10 = 600$ bilinear equations (the same number of equation was determined by Haiman and Sturmfels [41, Example 4.3]) to whom we must add $60 + 840 = 900$ Plücker relations. See Example A.2.1 in Appendix A for further details.

1.5.2 Iarrobino-Kleiman equations

As Example 1.5.1 shows the transformation $\mathcal{H}\mathrm{ilb}^n_{p(t)} \to \mathcal{G}\mathrm{r}^{N(r)}_{q(r)} \times \mathcal{G}\mathrm{r}^{N(r+1)}_{q(r+1)}$ allows to describe the Hilbert scheme by quadric equation, i.e. of low degree, but requiring lots of Plücker coordinates and consequently lots of Plücker relations, even in one of the simplest cases. Therefore we come back to the transformation of functors described in (1.23), in order to interpret the Hilbert functor as

$$\mathcal{H}\mathrm{ilb}^n_{p(t)}(A) = \left\{ \begin{array}{c} I \in \mathcal{G}\mathrm{r}^{N(r)}_{q(r)}(A) \text{ s.t. } I \subset A[x]_r \\ \text{and rank } A[x]_1 \cdot I = q(r+1) \end{array} \right\}$$

and again by Macaulay's Estimate (Theorem 1.23)

$$\mathcal{H}\mathrm{ilb}^n_{p(t)}(A) = \left\{ \begin{array}{c} I \in \mathcal{G}\mathrm{r}^{N(r)}_{q(r)}(A) \text{ s.t. } I \subset A[x]_r \\ \text{and rank } A[x]_1 \cdot I \leqslant q(r+1) \end{array} \right\}.$$

Theorem 1.25 ([52, Proposition C.30])**.** *The Hilbert scheme* $\mathbf{Hilb}^n_{p(t)}$ *can be defined as subscheme of the Grassmannian* $\mathbf{Gr}_{\mathbb{K}}(q,N)$ *by equations of degree* $q(r+1)+1$*.*

Proof. Associated to any point of $\mathbf{Gr}_{\mathbb{K}}(q,N)$, we consider the ideal I generated by $\Gamma^{(1)}$, so that the vector space I_{r+1} is spanned by $\left\langle x_i\Gamma^{(1)} \mid i=0,\dots,n\right\rangle$. Requiring that $\dim_{\mathbb{K}} I_{r+1} \leqslant q(r+1)$ is equivalent to ask that the $\big(q(r+1)+1\big)$-th exterior power of I_{r+1} vanishes:

$$\dim_{\mathbb{K}} I_{r+1} \leqslant q(r+1) \quad \Longleftrightarrow \quad \bigwedge^{q(r+1)+1} I_{r+1} = 0. \tag{1.30}$$

The vector space $\wedge^{q(r+1)+1} I_{r+1}$ is spanned by the set

$$\left\{ \bigwedge_{j=1}^{q(r+1)+1} x_{i_j}\Lambda^{(1)}_{\mathrm{J}_j} \;\middle|\; \forall\, i_j \in \{0,\dots,n\},\ \forall\, x_{i_j}\Lambda^{(1)}_{\mathrm{J}_j} \in x_{i_j}\Gamma^{(1)} \right\}$$

whose elements have coefficient in the Plücker coordinates of $\mathbf{Gr}_{\mathbb{K}}(q,N)$ of degree $q(r+1)+1$. Considering the ideal $\mathcal{I}_{\mathcal{H}}$ generated by all the coefficients of these generators of $\wedge^{q(r+1)+1} I_{r+1}$ and the ideal $\mathcal{Q}$ of the Plücker relations, we define the Hilbert scheme as

$$\mathbf{Hilb}^n_{p(t)} \simeq \operatorname{Proj} \mathbb{K}[\dots,\Delta_{\mathrm{I}},\dots]/(\mathcal{Q},\mathcal{I}_{\mathcal{H}}).$$ □

Remark 1.5.2. Iarrobino and Kleiman in [52] proved this result in affine coordinates. In fact considering the standard affine covering of $\operatorname{Proj}\mathbb{K}[\dots,\Delta_{\mathrm{I}},\dots]$, on each affine chart we can consider as basis the ideal I that one described in (1.14), thus working with an optimal set of generators, and then we can glue together these affine subschemes via transition maps.

Remark 1.5.3. From a computational perspective, the set spanning $\wedge^{q(r+1)+1} I_{r+1}$ considered in the proof of Theorem 1.25 contains many vanishing elements, indeed any set of $s > q$ polynomials, coming from s polynomials of degree r multiplied by the same variable, is surely dependent. A better set spanning $\wedge^{q(r+1)+1} I_{r+1}$ is

$$\left\{ \bigwedge_{j=0}^{n} \left(\bigwedge_{\mathrm{i}=1}^{s_j} x_j\Lambda^{(1)}_{\mathrm{J}_{\mathrm{i}}} \right) \;\middle|\; \begin{array}{c} \forall\, 0 \leqslant s_j \leqslant q \text{ s.t. } \sum_{j=0}^n s_j = q(r+1)+1 \\ \forall\, x_j\Lambda^{(1)}_{\mathrm{J}_{\mathrm{i}}} \in x_j\Gamma^{(1)},\ j=0,\dots,n \end{array} \right\} \tag{1.31}$$

Example 1.5.4. Let us look at the Iarrobino-Kleiman equation of the Hilbert scheme $\mathbf{Hilb}_2^2$ already introduced in Example 1.5.1. In the proof of Theorem 1.25, we are asking that any set of $q(r+1)+1$ polynomials in $\{x_i\Gamma^{(1)} \mid i = 0,\dots,n\}$ is linearly dependent. Hence in this special case we have to impose the dependency of any set of 9 polynomials in $\{x_0\Gamma^{(1)}, x_1\Gamma^{(1)}, x_2\Gamma^{(1)}\}$. For instance, the polynomials represented in the following matrix

$$\begin{array}{c|cccccccccc|}
 & x_0^3 & x_0^2x_1 & x_0x_1^2 & x_1^3 & x_0^2x_2 & x_0x_1x_2 & x_1^2x_2 & x_0x_2^2 & x_1x_2^2 & x_2^3 \\ \hline
x_0\Lambda^{(1)}_{126} & 0 & 0 & -\Delta_{1236} & 0 & -\Delta_{1246} & -\Delta_{1256} & 0 & 0 & 0 & 0 \\
x_0\Lambda^{(1)}_{156} & 0 & \Delta_{1256} & \Delta_{1356} & 0 & \Delta_{1456} & 0 & 0 & 0 & 0 & 0 \\
x_0\Lambda^{(1)}_{234} & -\Delta_{1234} & 0 & 0 & 0 & 0 & \Delta_{2345} & 0 & \Delta_{2346} & 0 & 0 \\
x_0\Lambda^{(1)}_{356} & -\Delta_{1356} & -\Delta_{2356} & 0 & 0 & \Delta_{3456} & 0 & 0 & 0 & 0 & 0 \\
x_1\Lambda^{(1)}_{123} & 0 & 0 & 0 & 0 & 0 & \Delta_{1234} & \Delta_{1235} & 0 & \Delta_{1236} & 0 \\
x_1\Lambda^{(1)}_{245} & 0 & -\Delta_{1245} & 0 & \Delta_{2345} & 0 & 0 & 0 & 0 & \Delta_{2456} & 0 \\
x_2\Lambda^{(1)}_{146} & 0 & 0 & 0 & 0 & 0 & \Delta_{1246} & \Delta_{1346} & 0 & -\Delta_{1456} & 0 \\
x_2\Lambda^{(1)}_{234} & 0 & 0 & 0 & 0 & -\Delta_{1234} & 0 & 0 & 0 & \Delta_{2345} & \Delta_{2346} \\
x_2\Lambda^{(1)}_{456} & 0 & 0 & 0 & 0 & -\Delta_{1456} & -\Delta_{2456} & -\Delta_{3456} & 0 & 0 & 0 \\ \hline
\end{array}$$

are linearly dependent if the rank of the matrix is not maximal, that is if the following polynomials of degree 9 in the Plücker coordinates, corresponding to the minors of dimension 9, vanish

- $\Delta_{1235}\Delta_{1236}\Delta_{1246}\Delta_{1256}\Delta_{1356}\Delta_{1456}\Delta_{2345}^2\Delta_{2346} - \Delta_{1234}\Delta_{1236}\Delta_{1256}\Delta_{1346}\Delta_{1356}\Delta_{1456}\Delta_{2345}^2\Delta_{2346} +$
  $+ \Delta_{1234}\Delta_{1236}^2\Delta_{1256}\Delta_{1346}\Delta_{1356}\Delta_{2345}\Delta_{2346}\Delta_{2456} + \Delta_{1234}\Delta_{1235}\Delta_{1236}\Delta_{1256}\Delta_{1356}\Delta_{1456}\Delta_{2345}\Delta_{2346}\Delta_{2456} +$
  $- \Delta_{1234}\Delta_{1236}^2\Delta_{1246}\Delta_{1256}\Delta_{1356}\Delta_{2345}\Delta_{2346}\Delta_{3456} - \Delta_{1234}^2\Delta_{1236}\Delta_{1256}\Delta_{1356}\Delta_{1456}\Delta_{2345}\Delta_{2346}\Delta_{3456},$
- $\Delta_{1235}\Delta_{1236}\Delta_{1246}\Delta_{1256}\Delta_{1356}\Delta_{1456}\Delta_{2345}\Delta_{2346}^2 - \Delta_{1234}\Delta_{1236}\Delta_{1256}\Delta_{1346}\Delta_{1356}\Delta_{1456}\Delta_{2345}\Delta_{2346}^2,$
- $\Delta_{1236}^2\Delta_{1256}\Delta_{1346}\Delta_{1356}\Delta_{1456}\Delta_{2345}^2\Delta_{2346} + \Delta_{1235}\Delta_{1236}\Delta_{1256}\Delta_{1356}\Delta_{1456}^2\Delta_{2345}^2\Delta_{2346} +$
  $+ \Delta_{1234}\Delta_{1236}\Delta_{1256}\Delta_{1346}\Delta_{1356}\Delta_{1456}\Delta_{2345}\Delta_{2346}\Delta_{2356} + \Delta_{1234}\Delta_{1235}\Delta_{1256}\Delta_{1356}\Delta_{1456}^2\Delta_{2345}\Delta_{2346}\Delta_{2356} +$
  $- \Delta_{1234}\Delta_{1236}\Delta_{1246}\Delta_{1346}\Delta_{1356}\Delta_{2345}\Delta_{2346}\Delta_{2356}\Delta_{2456} + \Delta_{1234}\Delta_{1236}^2\Delta_{1346}\Delta_{1456}\Delta_{2345}\Delta_{2346}\Delta_{2356}\Delta_{2456} +$
  $- \Delta_{1234}\Delta_{1235}\Delta_{1246}\Delta_{1356}\Delta_{1456}\Delta_{2345}\Delta_{2346}\Delta_{2356}\Delta_{2456} + \Delta_{1234}\Delta_{1235}\Delta_{1236}\Delta_{1456}^2\Delta_{2345}\Delta_{2346}\Delta_{2356}\Delta_{2456} +$
  $+ \Delta_{1234}\Delta_{1236}\Delta_{1246}^2\Delta_{1356}\Delta_{2345}\Delta_{2346}\Delta_{2356}\Delta_{3456} - \Delta_{1234}\Delta_{1236}^2\Delta_{1246}\Delta_{1456}\Delta_{2345}\Delta_{2346}\Delta_{2356}\Delta_{3456} +$
  $+ \Delta_{1234}^2\Delta_{1246}\Delta_{1356}\Delta_{1456}\Delta_{2345}\Delta_{2346}\Delta_{2356}\Delta_{3456} - \Delta_{1234}^2\Delta_{1236}\Delta_{1456}^2\Delta_{2345}\Delta_{2346}\Delta_{2356}\Delta_{3456} +$
  $+ \Delta_{1234}\Delta_{1236}^2\Delta_{1256}\Delta_{1346}\Delta_{2345}\Delta_{2346}\Delta_{2456}\Delta_{3456} + \Delta_{1234}\Delta_{1235}\Delta_{1236}\Delta_{1256}\Delta_{1456}\Delta_{2345}\Delta_{2346}\Delta_{2456}\Delta_{3456} +$
  $- \Delta_{1234}\Delta_{1236}^2\Delta_{1246}\Delta_{1256}\Delta_{2345}\Delta_{2346}\Delta_{3456}^2 - \Delta_{1234}^2\Delta_{1236}\Delta_{1256}\Delta_{1456}\Delta_{2345}\Delta_{2346}\Delta_{3456}^2,$
- $- \Delta_{1236}^2\Delta_{1246}\Delta_{1256}\Delta_{1356}\Delta_{1456}\Delta_{2345}\Delta_{2346}^2 - \Delta_{1234}\Delta_{1236}\Delta_{1256}\Delta_{1356}\Delta_{1456}^2\Delta_{2345}\Delta_{2346}^2,$

- $-\Delta_{1236}^2\Delta_{1256}\Delta_{1346}\Delta_{1356}\Delta_{1456}\Delta_{2345}\Delta_{2346}^2 - \Delta_{1235}\Delta_{1236}\Delta_{1256}\Delta_{1356}\Delta_{1456}^2\Delta_{2345}\Delta_{2346}^2$,
- $-\Delta_{1236}^2\Delta_{1256}\Delta_{1346}\Delta_{1356}\Delta_{2345}\Delta_{2346}^2\Delta_{2456} - \Delta_{1235}\Delta_{1236}\Delta_{1256}\Delta_{1356}\Delta_{1456}\Delta_{2345}\Delta_{2346}^2\Delta_{2456} +$
  $+\Delta_{1236}^2\Delta_{1246}\Delta_{1256}\Delta_{1356}\Delta_{2345}\Delta_{2346}^2\Delta_{3456} + \Delta_{1234}\Delta_{1236}\Delta_{1256}\Delta_{1356}\Delta_{1456}\Delta_{2345}\Delta_{2346}^2\Delta_{3456}$,
- $-\Delta_{1236}\Delta_{1256}^2\Delta_{1346}\Delta_{1356}\Delta_{1456}\Delta_{2345}\Delta_{2346}^2 - \Delta_{1235}\Delta_{1256}^2\Delta_{1356}\Delta_{1456}^2\Delta_{2345}\Delta_{2346}^2 +$
  $+\Delta_{1236}\Delta_{1246}\Delta_{1256}\Delta_{1346}\Delta_{1356}\Delta_{2345}\Delta_{2346}^2\Delta_{2456} + \Delta_{1235}\Delta_{1246}\Delta_{1256}\Delta_{1356}\Delta_{1456}\Delta_{2345}\Delta_{2346}^2\Delta_{2456} +$
  $-\Delta_{1236}\Delta_{1246}^2\Delta_{1256}\Delta_{1356}\Delta_{2345}\Delta_{2346}^2\Delta_{3456} - \Delta_{1234}\Delta_{1246}\Delta_{1256}\Delta_{1356}\Delta_{1456}\Delta_{2345}\Delta_{2346}^2\Delta_{3456}$,
- $-\Delta_{1236}\Delta_{1245}\Delta_{1256}\Delta_{1346}\Delta_{1356}^2\Delta_{1456}\Delta_{2346}^2 - \Delta_{1235}\Delta_{1245}\Delta_{1256}\Delta_{1356}^2\Delta_{1456}^2\Delta_{2346}^2 +$
  $+\Delta_{1236}\Delta_{1245}\Delta_{1246}\Delta_{1346}\Delta_{1356}^2\Delta_{2346}^2\Delta_{2456} + \Delta_{1235}\Delta_{1236}\Delta_{1246}\Delta_{1256}\Delta_{1356}\Delta_{1456}\Delta_{2346}^2\Delta_{2456} +$
  $-\Delta_{1236}^2\Delta_{1245}\Delta_{1346}\Delta_{1356}\Delta_{1456}\Delta_{2346}^2\Delta_{2456} - \Delta_{1234}\Delta_{1236}\Delta_{1256}\Delta_{1346}\Delta_{1356}\Delta_{1456}\Delta_{2346}^2\Delta_{2456} +$
  $+\Delta_{1235}\Delta_{1245}\Delta_{1246}\Delta_{1356}^2\Delta_{1456}\Delta_{2346}^2\Delta_{2456} - \Delta_{1235}\Delta_{1236}\Delta_{1245}\Delta_{1356}\Delta_{1456}^2\Delta_{2346}^2\Delta_{2456} +$
  $-\Delta_{1236}\Delta_{1245}\Delta_{1246}^2\Delta_{1356}^2\Delta_{2346}^2\Delta_{3456} + \Delta_{1236}^2\Delta_{1245}\Delta_{1246}\Delta_{1356}\Delta_{1456}\Delta_{2346}^2\Delta_{3456} +$
  $-\Delta_{1234}\Delta_{1245}\Delta_{1246}\Delta_{1356}^2\Delta_{1456}\Delta_{2346}^2\Delta_{3456} + \Delta_{1234}\Delta_{1236}\Delta_{1245}\Delta_{1356}\Delta_{1456}^2\Delta_{2346}^2\Delta_{3456}$,
- $-\Delta_{1236}\Delta_{1256}\Delta_{1346}\Delta_{1356}^2\Delta_{1456}\Delta_{2345}\Delta_{2346}^2 - \Delta_{1235}\Delta_{1256}\Delta_{1356}^2\Delta_{1456}^2\Delta_{2345}\Delta_{2346}^2 +$
  $+\Delta_{1236}\Delta_{1246}\Delta_{1346}\Delta_{1356}^2\Delta_{2345}\Delta_{2346}^2\Delta_{2456} - \Delta_{1236}^2\Delta_{1346}\Delta_{1356}\Delta_{1456}\Delta_{2345}\Delta_{2346}^2\Delta_{2456} +$
  $+\Delta_{1235}\Delta_{1246}\Delta_{1356}^2\Delta_{1456}\Delta_{2345}\Delta_{2346}^2\Delta_{2456} - \Delta_{1235}\Delta_{1236}\Delta_{1356}\Delta_{1456}^2\Delta_{2345}\Delta_{2346}^2\Delta_{2456} +$
  $-\Delta_{1236}\Delta_{1246}^2\Delta_{1356}^2\Delta_{2345}\Delta_{2346}^2\Delta_{3456} + \Delta_{1236}^2\Delta_{1246}\Delta_{1356}\Delta_{1456}\Delta_{2345}\Delta_{2346}^2\Delta_{3456} +$
  $-\Delta_{1234}\Delta_{1246}\Delta_{1356}^2\Delta_{1456}\Delta_{2345}\Delta_{2346}^2\Delta_{3456} + \Delta_{1234}\Delta_{1236}\Delta_{1356}\Delta_{1456}^2\Delta_{2345}\Delta_{2346}^2\Delta_{3456}$,
- $-\Delta_{1236}\Delta_{1256}\Delta_{1346}\Delta_{1356}\Delta_{1456}\Delta_{2345}\Delta_{2346}^2\Delta_{2356} - \Delta_{1235}\Delta_{1256}\Delta_{1356}\Delta_{1456}^2\Delta_{2345}\Delta_{2346}^2\Delta_{2356} +$
  $+\Delta_{1236}\Delta_{1246}\Delta_{1346}\Delta_{1356}\Delta_{2345}\Delta_{2346}^2\Delta_{2356}\Delta_{2456} - \Delta_{1236}^2\Delta_{1346}\Delta_{1456}\Delta_{2345}\Delta_{2346}^2\Delta_{2356}\Delta_{2456} +$
  $+\Delta_{1235}\Delta_{1246}\Delta_{1356}\Delta_{1456}\Delta_{2345}\Delta_{2346}^2\Delta_{2356}\Delta_{2456} - \Delta_{1235}\Delta_{1236}\Delta_{1456}^2\Delta_{2345}\Delta_{2346}^2\Delta_{2356}\Delta_{2456} +$
  $-\Delta_{1236}\Delta_{1246}^2\Delta_{1356}\Delta_{2345}\Delta_{2346}^2\Delta_{2356}\Delta_{3456} + \Delta_{1236}^2\Delta_{1246}\Delta_{1456}\Delta_{2345}\Delta_{2346}^2\Delta_{2356}\Delta_{3456} +$
  $-\Delta_{1234}\Delta_{1246}\Delta_{1356}\Delta_{1456}\Delta_{2345}\Delta_{2346}^2\Delta_{2356}\Delta_{3456} + \Delta_{1234}\Delta_{1236}\Delta_{1456}^2\Delta_{2345}\Delta_{2346}^2\Delta_{2356}\Delta_{3456} +$
  $-\Delta_{1236}^2\Delta_{1256}\Delta_{1346}\Delta_{2345}\Delta_{2346}^2\Delta_{2456}\Delta_{3456} - \Delta_{1235}\Delta_{1236}\Delta_{1256}\Delta_{1456}\Delta_{2345}\Delta_{2346}^2\Delta_{2456}\Delta_{3456} +$
  $+\Delta_{1236}^2\Delta_{1246}\Delta_{1256}\Delta_{2345}\Delta_{2346}^2\Delta_{3456}^2 + \Delta_{1234}\Delta_{1236}\Delta_{1256}\Delta_{1456}\Delta_{2345}\Delta_{2346}^2\Delta_{3456}^2$.

If we consider any possible subset of 9 elements of $\{x_0\Gamma^{(1)}, x_1\Gamma^{(1)}, x_2\Gamma^{(1)}\}$, we would need to examine $\binom{60}{9}$ exterior products, each of them containing at most 10 terms ($\dim_{\mathbb{K}} \wedge^9 \mathbb{K}[x_0, x_1, x_2]_3 = 10$) so that an overestimate of the number of equations would be $10 \cdot \binom{60}{9} = 147831426600$. Following Remark 1.5.3, it suffices to consider the subsets of 9 polynomials subdivided according to the multiplication variable in 3 subsets of cardinality 4,4,1 or 4,3,2 or 3,3,3. Of the type (4,4,1), there are $3 \cdot \binom{20}{4}^2 \cdot \binom{20}{1} = 1408441500$ possibilities, $6 \cdot \binom{20}{4} \cdot \binom{20}{3} \cdot \binom{20}{2} = 6296562000$ of the type (4,3,2) and $\binom{20}{3}^3 = 1481544000$ corresponding to (3,3,3), for a total of 9186547500:

still a huge number just under the half of $\binom{60}{9}$. It is clear that the set of equations defining the Hilbert scheme provided by Theorem 1.25 can not be use to project an effective algorithm (see Section A.2 of Appendix A).

1.5.3 Bayer-Haiman-Sturmfels equations

Even in the very simple case of $\mathbf{Hilb}^2_2$, the equations determined by Iarrobino and Kleiman have a large degree and they are too many to think about using them for a methodical study of Hilbert schemes through computational software. Then one of the first goal is to lower the degree of the equations. The idea introduced by Bayer in his thesis [7] is to put together polynomials of degree $r+1$ coming from polynomials of degree r multiplied for the same variable, that is to associate to any subproduct in (1.31) a generator of an exterior power of I_{r+1} having as coefficients linear polynomials in the Plücker coordinates:

$$\bigwedge_{\mathrm{i}=1}^{s_j} x_j \Lambda^{(1)}_{\mathrm{J_i}} \qquad \stackrel{?}{\longleftrightarrow} \qquad x_j \Lambda^{(s_j)}_{\mathrm{J}}.$$

Theorem 1.26 ([41, Theorem 4.4]). *The Hilbert scheme $\mathbf{Hilb}^n_{p(t)}$ can be defined as subscheme of the Grassmannian $\mathbf{Gr}_{\mathbb{K}}(q,N)$ by equations of degree equal to or less than $n+1$.*

Proof. We consider again the equivalent condition showed in (1.30) to impose $\dim_{\mathbb{K}} I_{r+1} \leqslant q(r+1)$, but we construct a different set of generators for $\wedge^{q(r+1)+1} I_{r+1}$ taking advantage of the elements of $x_i\Gamma^{(s)}$, $\forall\ 1 \leqslant s \leqslant q$, indeed $\wedge^{q(r+1)+1} I_{r+1}$ is spanned by the set

$$\left\{ \bigwedge_{j=0}^{n} x_j \Lambda^{(s_j)}_{\mathrm{J}_j} \;\middle|\; \begin{array}{l} \forall\ s_j \leqslant q \text{ s.t. } \sum_{j=0}^{n} s_j = q(r+1)+1 \\ \forall\ x_0\Lambda^{(s_0)}_{\mathrm{J}_0} \in x_0\Gamma^{(s_0)},\ \ldots,\ \forall\ x_n\Lambda^{(s_n)}_{\mathrm{J}_n} \in x_n\Gamma^{(s_n)} \end{array} \right\}.$$

Denoted by $\mathcal{I}_{\mathcal{H}}$ the ideal generated by all the coefficients of the elements spanning $\wedge^{q(r+1)+1} I_{r+1}$, that are polynomials of degree $n+1$ in the Plücker coordinates, and by $\mathcal{Q}$ the ideal of the Plücker relations, the Hilbert scheme $\mathbf{Hilb}^n_{p(t)}$ can be defined as

$$\mathbf{Hilb}^n_{p(t)} \simeq \operatorname{Proj} \mathbb{K}[\ldots,\Delta_{\mathrm{I}},\ldots]/(\mathcal{Q},\mathcal{I}_{\mathcal{H}}). \qquad \square$$

Example 1.5.5. Let us consider again $\mathbf{Hilb}^2_2$ as in Examples 1.5.1 and 1.5.4. Applying Theorem 1.26, we have to impose the vanishing of the exterior products

$$x_0\Lambda^{(s_0)}_{J_0} \wedge x_1\Lambda^{(s_1)}_{J_1} \wedge x_2\Lambda^{(s_2)}_{J_2}, \qquad \forall\ \Lambda^{(s_0)}_{J_0} \in \Gamma^{(s_0)},\ \Lambda^{(s_1)}_{J_1} \in \Gamma^{(s_1)},\ \Lambda^{(s_2)}_{J_2} \in \Gamma^{(s_2)},$$

where (s_0, s_1, s_2) is a triple chosen among the set $\{(4,4,1),(4,3,2),(4,2,3),(4,1,4),$ $(3,4,2),(3,3,3),(3,2,4),(2,4,3),(2,3,4),(1,4,4)\}$. For instance let us compute explicitly the product $x_0\Lambda^{(4)}_{\emptyset} \wedge x_1\Lambda^{(2)}_{25} \wedge x_2\Lambda^{(3)}_{1}$:

$$\begin{aligned}
x_0\Lambda^{(4)}_{\emptyset} = {} & \Delta_{1234}\, x_0^3 \wedge x_0^2x_1 \wedge x_0x_1^2 \wedge x_0^2x_2 + \Delta_{1235}\, x_0^3 \wedge x_0^2x_1 \wedge x_0x_1^2 \wedge x_0x_1x_2 + \\
& + \Delta_{1236}\, x_0^3 \wedge x_0^2x_1 \wedge x_0x_1^2 \wedge x_0x_2^2 + \Delta_{1245}\, x_0^3 \wedge x_0^2x_1 \wedge x_0^2x_2 \wedge x_0x_1x_2 + \\
& + \Delta_{1246}\, x_0^3 \wedge x_0^2x_1 \wedge x_0^2x_2 \wedge x_0x_2^2 + \Delta_{1256}\, x_0^3 \wedge x_0^2x_1 \wedge x_0x_1x_2 \wedge x_0x_2^2 + \\
& + \Delta_{1345}\, x_0^3 \wedge x_0x_1^2 \wedge x_0^2x_2 \wedge x_0x_1x_2 + \Delta_{1346}\, x_0^3 \wedge x_0x_1^2 \wedge x_0^2x_2 \wedge x_0x_2^2 + \\
& + \Delta_{1356}\, x_0^3 \wedge x_0x_1^2 \wedge x_0x_1x_2 \wedge x_0x_2^2 + \Delta_{1456}\, x_0^3 \wedge x_0^2x_2 \wedge x_0x_1x_2 \wedge x_0x_2^2 + \\
& + \Delta_{2345}\, x_0^2x_1 \wedge x_0x_1^2 \wedge x_0^2x_2 \wedge x_0x_1x_2 + \Delta_{2346}\, x_0^2x_1 \wedge x_0x_1^2 \wedge x_0^2x_2 \wedge x_0x_2^2 + \\
& + \Delta_{2356}\, x_0^2x_1 \wedge x_0x_1^2 \wedge x_0x_1x_2 \wedge x_0x_2^2 + \Delta_{2456}\, x_0^2x_1 \wedge x_0^2x_2 \wedge x_0x_1x_2 \wedge x_0x_2^2 + \\
& + \Delta_{3456}\, x_0x_1^2 \wedge x_0^2x_2 \wedge x_0x_1x_2 \wedge x_0x_2^2, \\
x_1\Lambda^{(2)}_{25} = {} & -\Delta_{1235}\, x_0^2x_1 \wedge x_1^3 - \Delta_{1245}\, x_0^2x_1 \wedge x_0x_1x_2 + \Delta_{1256}\, x_0^2x_1 \wedge x_1x_2^2 + \Delta_{2345}\, x_1^3 \wedge x_0x_1x_2 + \\
& - \Delta_{2356}\, x_1^3 \wedge x_1x_2^2 - \Delta_{2456}\, x_0x_1x_2 \wedge x_1x_2^2, \\
x_2\Lambda^{(3)}_{1} = {} & \Delta_{1234}\, x_0x_1x_2 \wedge x_1^2x_2 \wedge x_0x_2^2 + \Delta_{1235}\, x_0x_1x_2 \wedge x_1^2x_2 \wedge x_1x_2^2 + \Delta_{1236}\, x_0x_1x_2 \wedge x_1^2x_2 \wedge x_2^3 + \\
& + \Delta_{1245}\, x_0x_1x_2 \wedge x_0x_2^2 \wedge x_1x_2^2 + \Delta_{1246}\, x_0x_1x_2 \wedge x_0x_2^2 \wedge x_2^3 + \Delta_{1256}\, x_0x_1x_2 \wedge x_1x_2^2 \wedge x_2^3 + \\
& + \Delta_{1345}\, x_1^2x_2 \wedge x_0x_2^2 \wedge x_1x_2^2 + \Delta_{1346}\, x_1^2x_2 \wedge x_0x_2^2 \wedge x_2^3 + \Delta_{1356}\, x_1^2x_2 \wedge x_1x_2^2 \wedge x_2^3 + \\
& + \Delta_{1456}\, x_0x_2^2 \wedge x_1x_2^2 \wedge x_2^3.
\end{aligned}$$

We obtain the following 10 equations:

- $\Delta_{1235}\Delta_{1345}^2 - \Delta_{1234}\Delta_{1345}\Delta_{2345} + \Delta_{1235}^2\Delta_{1346} - \Delta_{1234}^2\Delta_{2356}$,
- $\Delta_{1235}\Delta_{1345}\Delta_{1346} - \Delta_{1234}\Delta_{2345}\Delta_{1346} + \Delta_{1235}\Delta_{1236}\Delta_{1346}$,
- $\Delta_{1235}\Delta_{1345}\Delta_{1356} - \Delta_{1234}\Delta_{2345}\Delta_{1356} + \Delta_{1234}\Delta_{1236}\Delta_{2356}$,
- $-\Delta_{1235}\Delta_{1346}\Delta_{1256} + \Delta_{1234}\Delta_{1246}\Delta_{2356} + \Delta_{1235}\Delta_{1345}\Delta_{1456} - \Delta_{1234}\Delta_{2345}\Delta_{1456}$,
- $-\Delta_{1235}\Delta_{1346}\Delta_{1356} + \Delta_{1234}\Delta_{1346}\Delta_{2356}$,
- $-\Delta_{2345}\Delta_{1236}\Delta_{1356} - \Delta_{1235}\Delta_{1356}^2 + \Delta_{1236}^2\Delta_{2356} + \Delta_{1235}\Delta_{1346}\Delta_{2356}$,
- $-\Delta_{1345}\Delta_{1346}\Delta_{1256} - \Delta_{1236}\Delta_{1346}\Delta_{1256} + \Delta_{1245}\Delta_{1346}\Delta_{1356} - \Delta_{1234}\Delta_{1346}\Delta_{2456}$,
- $\Delta_{2345}\Delta_{1246}\Delta_{1356} - \Delta_{1236}\Delta_{1246}\Delta_{2356} - \Delta_{1245}\Delta_{1346}\Delta_{2356} + \Delta_{1235}\Delta_{1356}\Delta_{1456}$,
- $\Delta_{2345}\Delta_{1346}\Delta_{1356} - \Delta_{1345}\Delta_{1346}\Delta_{2356} - \Delta_{1236}\Delta_{1346}\Delta_{2356}$,
- $\Delta_{2345}\Delta_{2346}\Delta_{1356} - \Delta_{2345}\Delta_{1346}\Delta_{2356} - \Delta_{1236}\Delta_{2346}\Delta_{2356} - \Delta_{1235}\Delta_{1356}\Delta_{3456}$.

Globally for the tern $(4, 4, 1)$ there are $1 \cdot 1 \cdot 20$ possibilities, there are $1 \cdot 6 \cdot 15 = 90$ to $(4, 3, 2)$ and $3 \cdot 6 = 18$ to $(3, 3, 3)$, so we have to examine $3 \cdot 20 + 6 \cdot 90 + 18 = 618$ products. Since $\dim_{\mathbb{K}} \wedge^9 \mathbb{K}[x_0, x_1, x_2]_3 = 10$, for each product we could have at most 10 coefficients and an upper bound of the number of equations for $\mathcal{I}_{\mathcal{H}}$ is 6180. Comparing this set of equations with that discussed in Example 1.5.4, we discover that the number of exterior products to be examined is reduced by about a factor of 10^7.

Let us examine in the general case the difference between the number of exterior products to be considered according to Theorem 1.25 (Remark 1.5.3) and Theorem 1.26. We saw that in both cases every exterior product can be subdivided in $n + 1$ parts depending on the variable used to move by multiplication from a element of degree r to an element of degree $r + 1$, so let us consider any sequence of integers $(s_0, \ldots, s_n)$ such that $0 \leqslant s_i \leqslant q$, $\forall\, i = 0, \ldots, n$ and $\sum_{i=0}^{n} s_i = q(r + 1) + 1$.

To compute Iarrobino-Kleiman equations, we have to choose in every possible way s_i polynomials among $x_i\Gamma^{(1)}$ for each i, hence the possibilities are

$$\prod_{i=0}^{n} \binom{|x_i\Gamma^{(1)}|}{s_i} = \prod_{i=0}^{n} \binom{\binom{N}{q-1}}{s_i}.$$

Instead for Bayer-Haiman-Sturmfels equations, for every i we need to pick a single element among $x_i\Gamma^{(s_i)}$, so that the possibilities are

$$\prod_{i=0}^{n} \left| x_i\Gamma^{(s_i)} \right| = \prod_{i=0}^{n} \binom{N}{q - s_i}.$$

It would be interesting to determine a good underestimation of the ratio between these two numbers, independent of s_i, i.e. a formula

$$\frac{\prod_{i=0}^{n} \binom{\binom{N}{q-1}}{s_i}}{\prod_{i=0}^{n} \binom{N}{q-s_i}} = \prod_{i=0}^{n} \frac{\binom{\binom{N}{q-1}}{s_i}}{\binom{N}{q-s_i}} \geqslant F(q, N). \tag{1.32}$$

Chapter 2

Borel-fixed ideals

In this chapter we introduce the most important objects used in this thesis, i.e. the Borel-fixed ideals.

2.1 Definition

Let us consider the usual action of the linear group $\mathrm{GL}_{\mathbb{K}}(n+1)$ of the square matrix of dimension $n+1$ on the polynomial ring $\mathbb{K}[x_0,\ldots,n_n]$, i.e. for any invertible matrix $g = (g_{ij}) \in \mathrm{GL}_{\mathbb{K}}(n+1)$, the variable x_i is mapped to the linear form $g \mathbin{.} x_i = \sum_j g_{ij}x_j$, so that any polynomial $P(x) \in \mathbb{K}[x]$ is mapped to

$$g \mathbin{.} P(x) = g \mathbin{.} P(x_0,\ldots,x_n) = P(g \mathbin{.} x_0,\ldots,g \mathbin{.} x_n) = P(g \mathbin{.} x).$$

Hence given an ideal $I \subset \mathbb{K}[x]$, it is well define the ideal

$$g \mathbin{.} I = \langle g \mathbin{.} P \mid \forall\, P \in I\rangle.$$

For any ideal $I \subset \mathbb{K}[x]$ and for any term ordering σ, we can define the following equivalence relation on $\mathrm{GL}_{\mathbb{K}}(n+1)$:

$$g \sim g' \qquad \Longleftrightarrow \qquad \mathrm{in}_\sigma(g \mathbin{.} I) = \mathrm{in}_\sigma(g' \mathbin{.} I) \tag{2.1}$$

Viewing each matrix $g \in \mathrm{GL}_{\mathbb{K}}(n+1)$ as a element of the affine space $\mathbb{A}^{(n+1)^2} = \operatorname{Spec}\mathbb{K}[\ldots,g_{ij},\ldots]$, the equivalence classes correspond to a stratification of $\mathbb{A}^{(n+1)^2}$.

Lemma 2.1 ([72, Lemma 2.6]). *For a fixed ideal I and a term order σ, the number of equivalence classes in $\mathrm{GL}_{\mathbb{K}}(n+1)$ is finite and one of these classes is a nonempty Zariski open subset U of $\mathrm{GL}_{\mathbb{K}}(n+1)$*

Definition 2.2. Let I be an ideal of $\mathbb{K}[x]$ and let σ be a fixed term ordering. The initial ideal $\mathsf{in}_\sigma(g \centerdot I)$ obtained considering a matrix g in the open subset $U \subset \mathrm{GL}_{\mathbb{K}}(n+1)$ corresponding to an equivalence class as in Lemma 2.1 is called *generic initial ideal* of I w.r.t. σ and it is denote by $\mathsf{gin}_\sigma(I)$.

To better understand the interaction between the change of coordinates and the computation of initial ideals, let us now recall the LU decomposition of a matrix.

Theorem 2.3. *Given any square matrix $g \in \mathrm{GL}_{\mathbb{K}}(n+1)$, if g can be turned into an upper triangular matrix by means of Gaussian elimination without swapping rows, the g can be decomposed as the product $l \cdot u$, where l is a lower triangular matrix and u is an upper triangular matrix with all entries on the diagonal equal to 1:*

$$\begin{pmatrix} g_{00} & \cdots & g_{0n} \\ & & \\ \vdots & g_{ij} & \vdots \\ & & \\ g_{n0} & \cdots & g_{nn} \end{pmatrix} = \begin{pmatrix} l_{00} & 0 & 0 \\ & \ddots & \ddots & \\ l_{i0} & l_{ii} & 0 \\ & \ddots & \ddots & \\ l_{n0} & l_{ni} & l_{nn} \end{pmatrix} \begin{pmatrix} 1 & u_{0j} & u_{0n} \\ & \ddots & \ddots & \\ 0 & 1 & u_{jn} \\ & \ddots & \ddots & \\ 0 & 0 & 1 \end{pmatrix}$$

We remark that always considering matrices as points of $\mathbb{A}^{(n+1)^2}$, the matrices having a LU decomposition correspond to an open subset, indeed any matrix that need row interchanges to be made upper triangular satisfies linear realations among the variables.

Lemma 2.4. *Let σ be any term ordering. For each homogeneous polynomial $P \in \mathbb{K}[x]_r$*

$$\mathsf{in}_\sigma(P) = \mathsf{in}_\sigma(l \centerdot P) \tag{2.2}$$

for each lower triangular matrix l.

Proof. It suffices to prove the statement for any monomial $x^\alpha = x_n^{\alpha_n} \cdots x_0^{\alpha_0}$, $|\alpha| = r$.

$$\begin{aligned} l \centerdot x^\alpha &= (l \centerdot x_n)^{\alpha_n} \cdots (l \centerdot x_i)^{\alpha_i} \cdots (l \centerdot x_0^{\alpha_0}) = \\ &= (l_{nn}x_n + \ldots + l_{n0}x_0)^{\alpha_n} \cdots (l_{ii}x_i + \ldots + u_{j0}x_0)^{\alpha_i} \cdots (l_{00}x_0)^{\alpha_0} = \\ &= (l_{nn}^{\alpha_n} x_n^{\alpha_n} + \text{ lower terms}) \cdots (l_{ii}^{\alpha_i} x_i^{\alpha_i} + \text{ lower terms}) \cdots (l_{00}^{\alpha_0} x_0^{\alpha_0}) = \\ &= \left(\prod_{i=0}^{n} l_{ii}^{\alpha_i} \right) x^\alpha + \text{ lower terms.} \end{aligned}$$
□

From this lemma, it is clear that to understand which ideals have a nice behaviour under change of coordinates and initial ideal computation the key point is analyzing the action of the Borel subgroup $\mathrm{B}_{\mathbb{K}}^{\mathrm{GL}}(n+1)$ of $\mathrm{GL}_{\mathbb{K}}(n+1)$ of the upper triangular matrices.

Definition 2.5. An ideal $I \subset \mathbb{K}[x]$ is called *Borel-fixed* if it is fixed by the action of the Borel subgroup, i.e.

$$g \centerdot I = I, \qquad \forall\, g \in \mathrm{B}_{\mathbb{K}}^{\mathrm{GL}}(n+1).$$

First of all, a Borel-fixed ideal has to be a monomial ideal because the Borel subgroup contains the algebraic torus group $\mathrm{T}_{\mathbb{K}}^{\mathrm{GL}}(n+1)$ of the diagonal matrices that fixes all (and only) the monomials ideals. In the case of our interest, i.e. with a ground field $\mathbb{K}$ of characteristic 0, there is the following characterization.

Proposition 2.6 ([38, Proposition 1.25], [72, Proposition 2.3])**.** *Let $I \subset \mathbb{K}[x]$ be a monomial ideal. The following statements are equivalent*

(1) I is Borel-fixed;

(2) if $x^\alpha \in I$, then $\frac{x_j}{x_i} x^\alpha \in I$, $\forall\, x_i \mid x^\alpha, j > i$;

Proof. *(1)⇒(2).* Let us consider the action on x^α of the matrix $g \in \mathrm{B}_{\mathbb{K}}^{\mathrm{GL}}(n+1)$ sending the variable x_i to $x_j + x_i$ (i.e. $j > i$) and leaving fixed the others:

$$\begin{aligned} g \centerdot x^\alpha &= g \centerdot \prod_{k=0}^{n} x_k^{\alpha_k} = x_n^{\alpha_n} \cdots x_{i+1}^{\alpha_{i+1}} \cdot (x_j + x_i)^{\alpha_i} \cdot x_{i-1}^{\alpha_{i-1}} \cdots x_0^{\alpha_0} = \\ &= \left(\prod_{k \neq i} x_k^{\alpha_k} \right) \cdot \sum_{h=0}^{\alpha_i} \binom{\alpha_i}{h} x_j^h x_i^{\alpha_i - h}. \end{aligned}$$

Since $g \centerdot I = I$ and I is a monomial ideal, each monomial appearing in $g \centerdot x^\alpha$ belongs to I and $\left(\prod_{k\neq i} x_k^{\alpha_k}\right) x_j x_i^{\alpha_i - 1} = \frac{x_j}{x_i} x^\alpha$.

(2)⇒(1). Let $x^\alpha \in I$. For all $g \in \mathrm{B}_{\mathbb{K}}^{\mathrm{GL}}(n+1)$, each monomial in $g \centerdot x^\alpha$ can be obtained from x^α through a sequence of multiplication by $\frac{x_j}{x_i}$, $j > i$. By the hypothesis *(2)* each monomial belongs to I, so that $g \centerdot x^\alpha \in I$. □

Now we can state the theorem providing the link between Borel-fixed ideals and Hilbert schemes due to Galligo [33] in characteristic 0 and generalized by Bayer and Stillman [11] in any characteristic.

Theorem 2.7. *The generic initial ideal* $\mathsf{gin}_\sigma(I)$ *is Borel-fixed.*

Proof. See [27, Theorem 15.20] or [38, Theorem 1.27]. □

Let $I \subset \mathbb{K}[x]$ be an ideal and σ any term ordering. It is well known that the ideal $\mathsf{in}_\sigma(I)$ has the same Hilbert function of I and that there is a family over $\mathbb{A}^1_{\mathbb{K}}$ having as fibers both I and $\mathsf{in}_\sigma(I)$. In the Hilbert schemes context, this means that the ideals I and $\mathsf{in}_\sigma(I)$ defines two $\mathbb{K}$-rational point of the Hilbert scheme $\mathbf{Hilb}^n_{p(t)}$, where $p(t)$ is the Hilbert polynomial of the subscheme $\operatorname{Proj} \mathbb{K}[x]/I$. Since there is a flat deformation of I which specializes to $\mathsf{in}_\sigma(I)$, there is necessarily a component of $\mathbf{Hilb}^n_{p(t)}$ containing both points. Being embedded in a suitable Grassmannian, the Hilbert scheme turns out to be invariant under the action of the linear group (Proposition 1.13), so it is interesting to face again with the problem of the relation between change of coordinates and initial ideal on the Hilbert scheme.

Any change of coordinates $g \in \mathrm{GL}_{\mathbb{K}}(n+1)$ of $\mathbb{K}[x]$ induces a linear action on the vector space $\mathbb{K}[x]_r$ of the homogeneous polynomials of degree r, $\forall\, r$ so also on the Hilbert scheme $\mathbf{Hilb}^n_{p(t)}$. Fixed any term ordering σ, the correspondence $I \mapsto \mathsf{in}_\sigma(I)$ in general is not well-defined, indeed I and $g.I$ define the same $\mathbb{K}$-rational point on $\mathbf{Hilb}^n_{p(t)}$ (up to isomorphism) but $\mathsf{in}_\sigma(I)$ and $\mathsf{in}_\sigma(g.I)$ could not. But by Lemma 2.1, we know that the generic initial ideal $\mathsf{gin}_\sigma(I)$ is stable for change of coordinates in an open subset $U \subset \mathrm{GL}_{\mathbb{K}}(n+1)$, and the same holds for the corresponding points on the Hilbert schemes. Hence the correspondence $I \mapsto \mathsf{gin}_\sigma(I)$ results to be well-defined also in the context of Hilbert schemes.

$\mathsf{gin}_\sigma(I)$ lies on the same component of I and if I belongs to an intersection of components then $\mathsf{gin}_\sigma(I)$ does too. So each components and each intersection of components of the Hilbert schemes has to contain at least one $\mathbb{K}$-rational point defined by a Borel-fixed ideal. The key role played by Borel ideals in the study of Hilbert schemes spring out from this remark, indeed we can consider the points defined by Borel ideals as distributed throughout the Hilbert scheme. Each Borel-fixed ideal can be used as the starting point of a local study of the Hilbert scheme (see Chapter 4), whereas as a whole they can be used to investigate global properties (see Chapter 3).

2.2 Basic properties

We now recall some of the main properties of Borel-fixed ideals that will be useful hereinafter. For this part we mainly refer to the first two sections of [38].

Definition 2.8. Denoted by $\mathfrak{m}$ the irrelevant ideal $(x_0, \ldots, x_n)$, a homogeneous ideal $I \subset \mathbb{K}[x]$ is *saturated* if $(I : \mathfrak{m}) = I$. Given a non-saturated ideal $J \subset \mathbb{K}[x]$, its saturation is the ideal

$$J^{\text{sat}} = \bigcup_{k \geqslant 0} (J : \mathfrak{m}^k). \tag{2.3}$$

Proposition 2.9. *For a Borel-fixed ideal $I \subset \mathbb{K}[x]$, $(I : \mathfrak{m}) = (I : x_0)$.*

Proof. The inclusion $(I : \mathfrak{m}) \subseteq (I : x_0)$ is obvious. For any monomial $x^\alpha \in (I : x_0)$, $x^\alpha x_0$ belongs to I and so by Proposition 2.6 $\frac{x_i}{x_0} x^\alpha x_0 = x_i x^\alpha,\ i = 1, \ldots, n$ belongs to I too. Finally $x^\alpha \in (I : \mathfrak{m})$. $\square$

Corollary 2.10. *A Borel-fixed ideal $I \subset \mathbb{K}[x]$ is saturated if the variable x_0 does not appear in any generator of I.*

Therefore from a operative point of view, given a Borel ideal I, to compute its saturation we can consider any set of generators and impose $x_0 = 1$, indeed if the monomial $x^\alpha x_0^k$ $(x_0 \nmid x^\alpha)$ belongs to I, the monomial x^α belongs to $(I : x_0^k) = (I : \mathfrak{m}^k)$ and so to I^{sat}.

Definition 2.11. Given an ideal $I \subset \mathbb{K}[x]$ and a minimal free resolution

$$0 \to M_n \to \cdots \to M_1 \to M_0 \to I \to 0,$$

where $M_i = \oplus_j \mathbb{K}[x](-m_{ij})$, the *regularity* of I is $\max\{m_{ij} - i\}$.

Proposition 2.12 ([38, Proposition 2.11], [10, Proposition 2.9])**.** *The regularity of a Borel-fixed ideal $I \subset \mathbb{K}[x]$ is equal to the maximal degree of one of its generators.*

Now we state a proposition giving a meaning of these properties of ideals in the context of schemes.

Proposition 2.13 ([38, Proposition 2.6])**.** *Let $I \subset \mathbb{K}[x]$ be a saturated ideal. The regularity of I (as in Definition 2.11) is equal to the Castelnuovo-Mumford regularity of the sheaf of ideals $\mathscr{I}$ obtained from the sheafification of I.*

Another important property of the last variable x_0 is the following.

Proposition 2.14. *Let $I \subset \mathbb{K}[x]$ be a Borel-fixed ideal. The linear form x_0 is regular for I, i.e. the hyperplane $H = \{x_0 = 0\}$ does not contain any irreducible component of* $\operatorname{Proj} \mathbb{K}[x]/I$*. Thus there is the short exact sequence induced by the multiplication by x_0*

$$0 \longrightarrow \frac{\mathbb{K}[x]}{I}(t-1) \xrightarrow{\cdot x_0} \frac{\mathbb{K}[x]}{I}(t) \longrightarrow \frac{\mathbb{K}[x]}{(I, x_0)}(t) \longrightarrow 0 \tag{2.4}$$

Called $p(t)$ the Hilbert polynomial of the scheme $X = \operatorname{Proj} \mathbb{K}[x]/I$ of dimension d, it is well-known that the scheme obtained through the generic hyperplane section $X_H = X \cap H$ and defined by the ideal (I, x_0) has dimension $d-1$, indeed its Hilbert polynomial is

$$p_H(t) = \Delta p(t) = p(t) - p(t-1) \quad \text{and} \quad \deg p(t) = d \Rightarrow \deg p_H(t) = d-1.$$

Moreover we remark that the ideal $(I, x_0) \cap \mathbb{K}[x_1, \ldots, x_n]$ is still Borel-fixed, because its monomials satisfy the characterization of Proposition 2.6.

Proposition 2.15. *Let $I \subset \mathbb{K}[x]$ be a Borel-fixed ideal and let $p(t)$ be the Hilbert polynomial of* $\operatorname{Proj} \mathbb{K}[x]/I$*. The degree of $p(t)$ is equal to* $\max\{i \mid x_i^m \notin I,\ m \geqslant \operatorname{reg}(I)\} = \min\{j \mid x_j^m \in I,\ m \geqslant \operatorname{reg}(I)\} - 1$.

Proof. Let us proceed by induction on the degree d of the Hilbert polynomial. x_0^m does not belong to the ideal I for all m, because if it does, applying repeatedly Proposition 2.6 any other monomial should belong to the ideal, i.e. $I_{\geqslant m} = \mathbb{K}[x]_{\geqslant m}$. Considering the sequence in (2.4), if $\deg p(t) = 0$, the intersection between points and a generic hyperplane section should be empty, so that the ideal $(I, x_0)_{\geqslant m}$ coincides with $\mathbb{K}[x]_{\geqslant m}$. Hence $x_1^m, \ldots, x_n^m$ belong to I and $\deg p(t) = 0 = \max\{i \mid x_i^m \notin I\} = \min\{j \mid x_j^m \in I\} - 1$.

Let us suppose that the statement is true for $\deg p(t) = d - 1$ and let us consider an ideal I defining a scheme of dimension d. Again through the exact sequence in (2.4), in order to obtain the ideal (I, x_0) defining a subscheme of dimension $d - 1$. Let us consider the map

$$\begin{aligned} \varphi : \mathbb{K}[x_0, \ldots, x_n] &\to \mathbb{K}[y_0, \ldots, y_{n-1}] \\ x_0 &\mapsto 0 \\ x_i &\mapsto y_{i-1}, \quad i = 1, \ldots, n. \end{aligned}$$

The ideal $\widetilde{I} = \phi((I, x_0))$ is still Borel-fixed and defines a subscheme isomorphic to the subscheme defined by (I, x_0) and $\operatorname{reg}(\widetilde{I}) \leqslant \operatorname{reg}(I)$. By the inductive hypothesis we know that $d - 1 = \max\{i \mid y_i^m \notin \widetilde{I}\} = \min\{j \mid y_j^m \in \widetilde{I}\}$ so that $x_{d+1}^m = \varphi^{-1}(y_d^m) \in I$ and $x_d^m = \varphi^{-1}(y_{d-1}^m) \notin I$. □

Definition 2.16. For any (non-constant) monomial $x^\alpha \in \mathbb{K}[x]$ we define

- $\min x^\alpha = \min\{i \text{ s.t. } x_i \mid x^\alpha\}$;
- $\max x^\alpha = \max\{j \text{ s.t. } x_j \mid x^\alpha\}$.

As we chose x_n as the greatest variable and x_0 as the smallest one, the definition makes sense also in the following way

- $\min x^\alpha = \min\{x_i \text{ s.t. } x_i \mid x^\alpha\}$;
- $\max x^\alpha = \max\{x_j \text{ s.t. } x_j \mid x^\alpha\}$.

From now on we will use both definitions interchangeably.

Lemma 2.17 ([30, Lemma 1.1],[72, Lemma 2.11]). *Let $I = (x^{\alpha_1}, \dots, x^{\alpha_s}) \subset \mathbb{K}[x]$ be a Borel-fixed ideal. Each monomial $x^\beta \in I$ can be written uniquely as $x^{\alpha_i} x^\gamma$ so that* $\min x^{\alpha_i} \geqslant \max x^\gamma$.

Proof. (Existence) Let us consider any decomposition of $x^\beta = x^{\alpha_i} x^\gamma$ and let us determine one of the type described in the statement. Suppose that $\min x^{\alpha_i} < \max x^\gamma$. By Proposition 2.6 we know that also $\dfrac{\max x^\gamma}{\min x^{\alpha_i}} x^{\alpha_i}$ belongs to I, so we can consider another minimal generator x^{α_j} of I dividing $\frac{\max x^\gamma}{\min x^{\alpha_i}} x^{\alpha_i}$. By construction $\min x^{\alpha_j} \geqslant \min x^{\alpha_i}$, and either $\min x^{\alpha_j} > \min x^{\alpha_i}$ or $\min x^{\alpha_j} = \min x^{\alpha_i}$ and the degree of $\min x^{\alpha_j}$ in x^{α_j} is lower than the degree of $\min x^{\alpha_j}$ in x^{α_i}. This procedure can not be repeated infinitely so at the end we will find a good decomposition.

(Uniqueness) Let us suppose that x^β has two decomposition $x^{\alpha_i} x^\gamma = x^{\alpha_j} x^{\gamma'}$ such that $\min x^{\alpha_i} \geqslant \max x^\gamma$ and $\min x^{\alpha_j} \geqslant \max x^{\gamma'}$. If we suppese that $\min x^{\alpha_i} \geqslant \min x^{\alpha_j}$, then x^{α_i} and x^{α_j} are divided by the same power of each variable $x_k > \min x^{\alpha_i}$. If $\min x^{\alpha_i}$ divides x^{α_j}, then $\min x^{\alpha_i} = \min x^{\alpha_j}$, hence either x^{α_i} divides x^{α_j} or viceversa and being both minimal generators they coincide. If $\max x^{\alpha_i}$ does not divide x^{α_j}, then the degree of $\max x^{\alpha_i}$ in x^{α_i} is smaller than the degree of $\max x^{\alpha_i}$ in x^{α_j}, i.e. the degree of $\max x^{\alpha_i}$ in x^β. Again x^{α_i} divides x^{α_j} but being both minimal generators of I, $x^{\alpha_i} = x^{\alpha_j}$. □

Definition 2.18. Given a Borel-fixed ideal $I = (x^{\alpha_1}, \dots, x^{\alpha_s}) \subset \mathbb{K}[x]$ and a monomial $x^\beta \in I$, the unique decomposition described in Lemma 2.17 is called *canonical decomposition* of x^β over I or *canonical I-decomposition* of x^β and we will denote it by

$$x^\beta = \langle x^{\alpha_i} | x^\gamma \rangle^I, \qquad \min x^{\alpha_i} \geqslant \max x^\gamma$$

In [30] Eliahou and Kervaire introduce the *decomposition function* ∂_I from the set of monomials $M(I)$ of the Borel-fixed ideal I to the minimal set of generators $G(I) = \{x^{\alpha_1}, \dots, x^{\alpha_s}\}$ of I:

$$\begin{aligned} \partial_I : \qquad M(I) \qquad &\longrightarrow G(I) \\ x^\beta = \langle x^{\alpha_i} | x^\gamma \rangle^I &\longmapsto x^{\alpha_i} \end{aligned}$$

We recall the main properties of this function.

Proposition 2.19. *Let $I \subset \mathbb{K}[x]$ be a Borel-fixed ideal and let $\partial_I : M(I) \to G(I)$ be its decomposition function.*

1. *For any pairs of monomials $x^\beta, x^\gamma \in I$,*

$$\partial_I(x^\beta x^\gamma) = \partial_I(x^\beta) \quad \Longleftrightarrow \quad \min \partial_I(x^\beta) \geqslant \max x^\gamma. \tag{2.5}$$

2. *For any pairs of monomials $x^\beta \in I$ and $x^\gamma \in \mathbb{K}[x]$,*

$$\partial_I\big(x^\gamma \partial_I(x^\beta)\big) = \partial_I(x^\beta x^\gamma), \tag{2.6}$$

$$\min \partial_I(x^\beta x^\gamma) \geqslant \min \partial_I(x^\beta), \tag{2.7}$$

$$\partial_I(x^\beta x^\gamma) \leq_{\texttt{DegRevLex}} \partial_I(x^\beta). \tag{2.8}$$

We finish this section describing the set of generators for the module of syzygies of a Borel-fixed ideal given in the free resolution constructed by Eliahou and Kervaire.

Theorem 2.20 ([30, Theorem 2.1]). *Let $I \subset \mathbb{K}[x]$ be a Borel-fixed ideal generated by $(x^{\alpha_1}, \ldots, x^{\alpha_s})$. The module of syzygies* $\mathrm{Syz}(I)$*, i.e. the kernel of the map*

$$\begin{array}{ccc} \bigoplus_{i=1,\ldots,s} \mathbb{K}[x]\big(-|\alpha_i|\big) & \to & \mathbb{K}[x] \\ \mathbf{e}_i & \mapsto & x^{\alpha_i} \end{array}$$

is generated by the elements

$$x_k \mathbf{e}_i - x^\eta \mathbf{e}_j, \quad \forall\, i = 1, \ldots, s,\ \forall\, x_k > \min x^{\alpha_i} \text{ s.t. } x_k x^{\alpha_i} = \langle x^{\alpha_j} | x^\eta \rangle^I. \tag{2.9}$$

Example 2.2.1. Let us consider the Borel ideal $I = (x_3^2, x_3x_2, x_3x_1^2, x_3x_1x_0, x_2^5, x_2^4x_1)$ in $\mathbb{K}[x_0, x_1, x_2, x_3]$. The kernel of the map

$$\mathbb{K}[x](-2)^2 \oplus \mathbb{K}[x](-3)^2 \oplus \mathbb{K}[x](-5)^2 \to \mathbb{K}[x],$$

sending $\mathbf{e}_1 \mapsto x_3^2, \mathbf{e}_2 \mapsto x_3x_2, \mathbf{e}_3 \mapsto x_3x_1^2, \mathbf{e}_4 \mapsto x_3x_1x_0, \mathbf{e}_5 \mapsto x_2^5, \mathbf{e}_6 \mapsto x_2^4x_1$, is generated by 9 elements.

- $\min x_3^2 = x_3$, so no generators of the type (2.9) can be found.

- $\min x_3x_2 = x_2$, so there is the generator

    1. $x_3\mathbf{e}_2 - x_2\mathbf{e_1}$, since $x_3(x_3x_2) = \langle x_3^2 | x_2\rangle^I$.

- $\min x_3x_1^2 = x_1$, so there are

    2. $x_2\mathbf{e}_3 - x_1^2\mathbf{e}_2$, since $x_2(x_3x_1^2) = \langle x_3x_2 | x_1^2\rangle^I$;
    3. $x_3\mathbf{e}_3 - x_1^2\mathbf{e}_1$, since $x_3(x_3x_1^2) = \langle x_3^2 | x_1^2\rangle^I$.

- $\min x_3x_1x_0 = x_0$, so there are

    4. $x_1\mathbf{e}_4 - x_1^2\mathbf{e}_3$, since $x_1(x_3x_1x_0) = \langle x_3x_1^2 | x_0\rangle^I$;
    5. $x_2\mathbf{e}_4 - x_1^2\mathbf{e}_2$, since $x_2(x_3x_1x_0) = \langle x_3x_2 | x_1x_0\rangle^I$;
    6. $x_3\mathbf{e}_4 - x_1^2\mathbf{e}_1$, since $x_3(x_3x_1x_0) = \langle x_3^2 | x_1x_0\rangle^I$.

- $\min x_2^5 = x_2$, so there is

    7. $x_3\mathbf{e}_5 - x_2^4\mathbf{e}_2$, since $x_3(x_2^5) = \langle x_3x_2 | x_2^4\rangle^I$.

- $\min x_2^4x_1 = x_1$, so there are

    8. $x_2\mathbf{e}_6 - x_1\mathbf{e}_5$, since $x_2(x_2^4x_1) = \langle x_2^4 | x_1\rangle^I$;
    9. $x_3\mathbf{e}_6 - x_2^3x_1\mathbf{e}_2$, since $x_3(x_2^4x_1) = \langle x_3x_2 | x_2^3x_1\rangle^I$.

2.2.1 Basic manipulations of Hilbert polynomials

Definition 2.21. Let $p(t)$ be an admissible Hilbert polynomial. We define $\Delta^0 p(t) = p(t)$ and recursively

$$\Delta^i p(t) = \Delta^{i-1}p(t) - \Delta^{i-1}p(t-1). \tag{2.10}$$

Directly from the definition, $\Delta p(t) = \Delta^1 p(t)$ and if $\deg p(t) = d$, $\Delta^{d+1}p(t) = 0$.

Proposition 2.22. *Let $p(t)$ be an admissible Hilbert polynomial. Let r be its Gotzmann number and $\overline{r}$ the Gotzmann number of $\Delta p(t)$. Then $\overline{r} \leqslant r$.*

Proof. Considered the Gotzmann representation (1.21) of $p(t)$, the representation of $\Delta p(t)$ is

$$\begin{aligned}\Delta p(t) = p(t) - p(t-1) = &\binom{t+a_1}{a_1} + \ldots + \binom{t+a_r-(r-1)}{a_r}\\ &- \left(\binom{t-1+a_1}{a_1} + \ldots + \binom{t-1+a_r-(r-1)}{a_r}\right) =\\ &= \binom{t+(a_1-1)}{(a_1-1)} + \ldots + \binom{t+(a_r-1)-(r-1)}{a_r-1}\end{aligned}$$

where all the binomial coefficients with $a_i - 1 < 0$ vanish, so the Gotzmann number $\bar{r}$ of $\Delta p(t)$ is equal to the number of coefficient $a_i \geqslant 1$. □

Taking inspiration from the proof of the previous proposition we introduce a new manipulation of Hilbert polynomials.

Definition 2.23. Given an admissible Hilbert polynomial $p(t)$ with Gotzmann representation as in (1.21), we define

$$\Sigma p(t) = \binom{t+(a_1+1)}{a_1+1} + \ldots + \binom{t+(a_r+1)-(r-1)}{a_r+1}. \tag{2.11}$$

Obviously $\Sigma p(t)$ and $p(t)$ have the same Gotzmann number. Furthermore $\Delta(\Sigma p)(t) = p(t)$, whereas $\Sigma(\Delta p)(t) = p(t) - c,\ c \geqslant 0$, indeed in the second case we lose the constant part corresponding in the Gotzmann decomposition to the binomial coefficients with $a_i = 0$.

Definition 2.24. Let $\overline{p}(t)$ be an admissible Hilbert polynomial and let us define the set

$$\mathrm{HP}(\overline{p}(t)) = \{p(t) \mid \Delta p(t) = \overline{p}(t)\}. \tag{2.12}$$

The polynomial $\Sigma\overline{p}(t)$ belongs to $\mathrm{HP}(\overline{p}(t))$ and

$$p(t) = \Sigma\overline{p}(t) + c,\ c \geqslant 0, \qquad \forall\ p(t) \in \mathrm{HP}(\overline{p}(t));$$

thus we call $\Sigma\overline{p}(t)$ *minimal polynomial* of $\mathrm{HP}(\overline{p}(t))$.

Remark 2.2.2. By Proposition 2.22, we deduce that $\Sigma\overline{p}(t)$ is the Hilbert polynomial in $\mathrm{HP}(\overline{p}(t))$ with lowest Gotzmann number: let us denote it with $\bar{r}$. For any other polynomial $p(t) \in \mathrm{HP}(\overline{p}(t))$ with Gotzmann number r

$$p(t) = \Sigma\overline{p}(t) + r - \bar{r}.$$

2.3 The combinatorial interpretation

In this section we provide a more combinatorial way to look at Borel-fixed ideals, that will turn out very useful in our algorithmic perspective.

Definition 2.25 ([38, Definition 1.24])**.** Let $\mathbb{K}[x_0, \dots, x_n]$ be a polynomial ring and let $\mathbb{K}(x)$ its field of fraction. We define

- the *i-th increasing elementary move* e_i^+ as the element $\dfrac{x_{i+1}}{x_i} \in \mathbb{K}(x)$, $\forall\, i < n$;
- the *j-th decreasing elementary move* e_j^- as the element $\dfrac{x_{j-1}}{x_j} \in \mathbb{K}(x)$, $\forall\, j > 0$.

Given a monomial $x^\alpha \in \mathbb{K}[x]$, we will say that the elementary move e_i^+ (resp. e_j^-) is *admissible* on x^α if $x_i \mid x^\alpha$ (resp. $x_j \mid x^\alpha$), i.e. $\mathsf{e}_i^+(x^\alpha) = \frac{x_{i+1}}{x_i} x^\alpha \in \mathbb{K}[x]$ (resp. $\mathsf{e}_j^-(x^\alpha) = \frac{x_{j-1}}{x_j} x^\alpha \in \mathbb{K}[x]$).

In the following, we will use an additive notation to denote the composition of an elementary move with itself, that is

$$2\mathsf{e}_i^+ = \mathsf{e}_i^+ \circ \mathsf{e}_i^+ = \left(\frac{x_{i+1}}{x_i}\right)^2 \quad \text{and} \quad 2\mathsf{e}_j^- := \mathsf{e}_j^- \circ \mathsf{e}_j^- = \left(\frac{x_{j-1}}{x_j}\right)^2 \tag{2.13}$$

and so on.

In general, a composition of elementary moves turns out to be a “monomial” x^γ ($\gamma \in \mathbb{Z}^{n+1}$) of degree 0 in $\mathbb{K}(x)$ and we will say that it is admissible on x^α if the product $x^\gamma x^\alpha$ belongs to $\mathbb{K}[x]$. Going by the commutativity of the product, given a composition $F = \lambda_a \mathsf{e}_{i_a}^+ \circ \cdots \circ \lambda_1 \mathsf{e}_{i_1}^+$, we can suppose $i_1 < \ldots < i_a$ so that whenever F is admissible, each elementary move in the written order is admissible. Similarly for any composition $G = \mu_b \mathsf{e}_{j_b}^- \circ \cdots \circ \mu_1 \mathsf{e}_{j_1}^-$, we will suppose $j_1 > \ldots > j_b$.

Remark 2.3.1. Rewriting the characterization of Borel-fixed ideals given in 2.6, we can say that an ideal I is Borel-fixed if and only if its set of monomials is closed w.r.t. increasing elementary moves, indeed $\forall\, j > i$

$$\frac{x_j}{x_i} = \frac{x_j}{x_{j-1}} \cdot \frac{x_{j-1}}{x_{j-2}} \cdots \frac{x_{i+2}}{x_{i+1}} \cdot \frac{x_{i+1}}{x_i} = \mathsf{e}_{j-1}^+ \circ \cdots \circ \mathsf{e}_i^+.$$

Definition 2.26. We call *Borel order*, and we denote it by $\leq_B$, the partial order defined on the set of monomials of a fixed degree by the transitive closure of the relations

$$\mathsf{e}_i^+(x^\alpha) >_B x^\alpha >_B \mathsf{e}_j^-(x^\alpha). \tag{2.14}$$

We note that the Borel order can be also obtained imposing the compatibility of the assumption $x_n > \ldots > x_0$ with the multiplication, because for any admissible elementary move e_i^+ on x^α, set $x^{\overline{\alpha}} = \frac{x^\alpha}{x_i}$, we have that $x^\alpha = x_i x^{\overline{\alpha}}$, $\mathsf{e}_i^+(x^\alpha) = x_{i+1}x^{\overline{\alpha}}$ and

$$x_{i+1} > x_i \qquad \Longrightarrow \qquad x_{i+1}x^{\overline{\alpha}} > x_i x^{\overline{\alpha}} \qquad \Longleftrightarrow \qquad \mathsf{e}_i^+(x^\alpha) >_B x^\alpha.$$

In the definition of a monomial order (see [54, Definition 1.4.1], [24, Definition 1]), the compatibility between the order relation and the multiplication is always required, therefore any graded term ordering σ is a total order on the monomials of fixed degree that refines the Borel partial order, that is

$$x^\alpha >_B x^\beta \qquad \Longrightarrow \qquad x^\alpha >_\sigma x^\beta.$$

We now characterize the Borel order by means of an analysis on the sets of exponents of monomials. Firstly, for any pair of multiindices $\alpha, \beta \in \mathbb{N}^{n+1}$, $|\alpha| = |\beta|$ and for any $0 \leqslant i \leqslant n$, we define the integer

$$\rho(\alpha, \beta, i) = \sum_{j=i}^{n} (\alpha_j - \beta_j). \tag{2.15}$$

Lemma 2.27. *Let x^α and x^β be two monomials in $\mathbb{K}[x]_m$.*

$$x^\alpha >_B x^\beta \qquad \Longleftrightarrow \qquad \rho(\alpha,\beta,i) \geqslant 0,\ \forall\, i = 0, \ldots, n. \tag{2.16}$$

Proof. ($\Rightarrow$) $x^\alpha >_B x^\beta$ means $x^\beta = \mu_b \mathsf{e}_{j_b}^- \circ \cdots \circ \mu_1 \mathsf{e}_{j_1}^-(x^\alpha)$, $j_1 > \ldots > j_b$. Obviously $\rho(\alpha,\beta,0) = |\alpha| - |\beta| = 0$. Moreover $\rho(\alpha,\beta,i) = 0$, $\forall\, i > j_1$ and $\rho(\alpha,\beta,j_1)$ has to be positive because $\alpha_{j_1} > \beta_{j_1}$. Let $x^\gamma = \mu_1 \mathsf{e}_{j_1}^-(x^\alpha)$. By definition $\gamma = (\alpha_0, \ldots, \alpha_{j_1-1} + (\alpha_{j_1} - \beta_{j_1}), \beta_{j_1}, \ldots, \alpha_n)$, i.e. $\mu_1 = \alpha_{j_1} - \beta_{j_1}$, so that $\rho(\alpha,\beta,i) = \rho(\gamma,\beta,i)$, $\forall\, i < j_1$. Repeating the reasoning on $x^\gamma >_B x^\beta = \mu_b \mathsf{e}_{j_b}^- \circ \cdots \circ \mu_2 \mathsf{e}_{j_2}^-(x^\gamma)$, we prove $\rho(\alpha,\beta,i) \geqslant 0$, $\forall\, i$.

($\Leftarrow$) It suffices to consider the composition of decreasing moves

$$G = \rho(\alpha,\beta,1)\mathsf{e}_1^- \circ \cdots \circ \rho(\alpha,\beta,n)\mathsf{e}_n^-. \qquad \square$$

Corollary 2.28. *Let x^α and x^β be two monomials in $\mathbb{K}[x]_m$. They are not comparable w.r.t. the Borel order if there exists two integer i, j such that $\rho(\alpha,\beta,i)\cdot\rho(\alpha,\beta,j) < 0$.*

Example 2.3.2. Consider the polynomial ring $\mathbb{K}[x_0,x_1,x_2,x_3,x_4]$ and the monomial $x_4^3x_3x_2^3x_0$. By definition

$$x_4^3x_3x_2^3x_0 >_B \mathsf{e}_1^- \circ 3\mathsf{e}_2^- \circ 2\mathsf{e}_4^-(x_4^3x_3x_2^3x_0) = x_4x_3^3x_1^2x_0^2$$

and, set $\alpha = (1,0,3,1,3)$ and $\beta = (2,2,0,3,1)$,

$$\rho(\alpha,\beta,0) = 0,\ \rho(\alpha,\beta,1) = 1,\ \rho(\alpha,\beta,2) = 3,\ \rho(\alpha,\beta,3) = 0,\ \rho(\alpha,\beta,4) = 2.$$

Furthermore the monomials $x_4x_3^2x_2^3x_0$ and $x_4^3x_2x_1^3$ are not comparable, indeed, set $\gamma = (1,0,3,2,1)$ and $\delta = (0,3,1,3,0)$,

$$\rho(\gamma,\delta,0) = 0,\ \rho(\gamma,\delta,1) = -1,\ \rho(\gamma,\delta,2) = 2,\ \rho(\gamma,\delta,3) = 0,\ \rho(\gamma,\delta,4) = 1.$$

Definition 2.29. We denote by $\mathcal{P}(n,m)$ the *Partially Ordered SET* (*poset* for short) of the monomials of degree m in the polynomial ring $\mathbb{K}[x_0,\dots,x_n]$ with the Borel partial order $\leq_B$.

Definition 2.30. Following the characterization of Borel-fixed ideals in terms of elementary moves, we call *Borel set* any subset $\mathscr{B} \subset \mathcal{P}(n,m)$ closed w.r.t. increasing elementary moves, i.e.

$$x^\alpha \in \mathscr{B} \qquad \Longrightarrow \qquad \mathsf{e}_i^+(x^\alpha) \in \mathscr{B},\ \forall\, \mathsf{e}_i^+ \text{ admissible on } x^\alpha.$$

With the terminology of orderings on sets, a Borel set represents a *filter* of $\mathcal{P}(n,m)$ for the Borel partial order. Given a Borel-fixed ideal I, we will write $\{I_m\}$ referring to the Borel set defined by the piece of degree m of the ideal I in the poset $\mathcal{P}(n,m)$.

Obviously the complement $\mathscr{N} = \mathcal{P}(n,m) \setminus \mathscr{B}$, that we will also denote by $\mathscr{B}^\mathcal{C}$, is closed w.r.t. decreasing elementary moves. We will call such a subset an *order set*, taking inspiration from the definition of order ideals, since the dehomogeneization of the complement of a Borel set (imposing $x_0 = 1$) turns out to be exactly an order ideal.

Given any subset $S \subset \mathcal{P}(n,m)$, we will denote with $S_{(\geqslant i)}$ the subset of S

$$S_{(\geqslant i)} = \{x^\alpha \in S \mid \min x^\alpha \geqslant i\}. \tag{2.17}$$

Obviously $S_{(\geqslant 0)} = S$. Now we introduce some further definitions borrowed from the terminology of ordering on sets.

Definition 2.31. Let $\mathscr{B} \subset \mathcal{P}(n,m)$ be a Borel set and let $\mathscr{N} = \mathscr{B}^\mathcal{C}$ the corresponding order set.

- $x^\alpha \in \mathscr{B}$ will be called *minimal* if for any admissible decreasing move e_j^-, $\mathsf{e}_j^-(x^\alpha)$ does not belong to $\mathscr{B}$.
- $x^\beta \in \mathscr{N}$ will be called *maximal* if for any admissible increasing move e_i^+, $\mathsf{e}_i^+(x^\beta)$ belongs to $\mathscr{B}$.

Moreover we will say that

- $x^\alpha \in \mathscr{B}$ is *k-minimal* if $\mathsf{e}_j^-(x^\alpha) \in \mathscr{N}$ for any admissible e_j^-, $j > k$;
- $x^\beta \in \mathscr{N}$ is *k-maximal* if $\mathsf{e}_i^+(x^\beta) \in \mathscr{B}$ for any admissible e_i^+, $i \geqslant k$.

Example 2.3.3. Let us consider the poset $\mathcal{P}(2,3)$ and its Borel subset

$$\mathscr{B} = \{x_2^3, x_2^2x_1, x_2x_1^2, x_2^2x_0, x_2x_1x_0\},$$

so that

$$\mathscr{N} = \mathscr{B}^\mathcal{C} = \{x_1^3, x_1^2x_0, x_2x_0^2, x_1x_0^2, x_0^3\}.$$

There is a single minimal element $x_2x_1x_0$ and two maximal elements: $x_2x_0^2$ and x_1^3. Moreover the 1-minimal monomials are $x_2x_1^2$ and $x_2x_1x_0$ and the 1-maximal ones are x_1^3 and $x_1^2x_0$.

Remark 2.3.4. For any term ordering σ, refinement of the Borel order, and for any Borel set $\mathscr{B}$

- $\min_\sigma \mathscr{B}_{(\geqslant k)}$ is a minimal element of $\mathscr{B}_{(\geqslant k)}$;
- $\max_\sigma \mathscr{B}^\mathcal{C}_{(\geqslant k)}$ is a maximal element of $\mathscr{B}^\mathcal{C}_{(\geqslant k)}$.

2.4 Graphical representations

The combinatorial interpretation of Borel-fixed ideals leads up to nice representations of the posets of monomials of the same degree and their Borel subsets. We now briefly describe some different approaches emphasizing positive and negative aspects.

Green's diagrams We mainly refer to Section 4 of [38]. Green's diagrams can be used to describe few situations, indeed through this approach we can describe only Borel-fixed ideals defining points in $\mathbb{P}^2$ or curves in $\mathbb{P}^3$.

Let us begin with $\mathbb{P}^2 = \operatorname{Proj} \mathbb{K}[x_0, x_1, x_2]$, i.e. looking at posets of the type $\mathcal{P}(2, m)$. Green arranges the monomials of degree m in a triangle shape with the top vertix corresponding to x_0^m and completing the diagram moving down with the rule described in the following picture

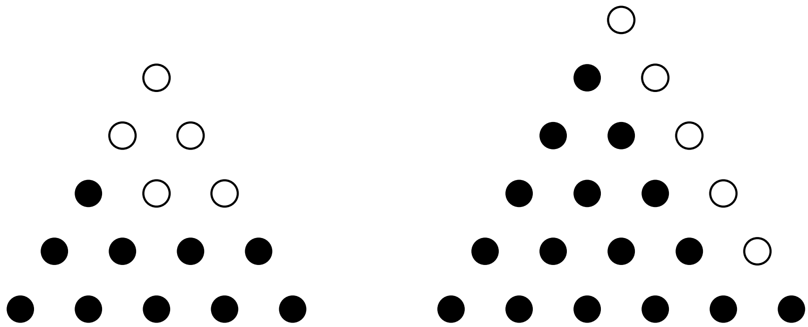
$$\begin{tikzcd}
 & x^\alpha \arrow[dl, "\mathsf{e}_1^+ \circ \mathsf{e}_0^+"'] \arrow[dr, "\mathsf{e}_0^+"] & \\
x^\beta & & x^\gamma \arrow[ll, "\mathsf{e}_1^+"]
\end{tikzcd}$$

so that the base of the triangle contains the monomials of degree m in $\mathbb{K}[x_1, x_2]$ (see Figure 2.1).

$$\begin{array}{ccccccccc}
 & & & & x_0^4 & & & & \\
 & & & x_2x_0^3 & & x_1x_0^3 & & & \\
 & & x_2^2x_0^2 & & x_2x_1x_0^2 & & x_1^2x_0^2 & & \\
 & x_2^3x_0 & & x_2^2x_1x_0 & & x_2x_1^2x_0 & & x_1^3x_0 & \\
x_2^4 & & x_2^3x_1 & & x_2^2x_1^2 & & x_2x_1^3 & & x_1^4
\end{array}$$

Figure 2.1: An example of Green's diagram for $\mathbb{P}^2$: the poset $\mathcal{P}(2, 4)$.

Afterwards he does not write explicitly the monomials, and given an ideal $I \subset \mathbb{K}[x_0, x_1, x_2]$ he uses a black circle to denote a monomial of the ideal and an empty circle to denote a monomial not belonging to I. In Figure 2.2, there are two examples of Borel sets defined by Borel-fixed ideals.



(a) The Borel set $\{I_4\} \subset \mathcal{P}(2,4)$, where $I = (x_2^2, x_2x_1^2, x_1^3)$.

(b) The Borel set $\{L_5\} \subset \mathcal{P}(2,5)$, defined by $L = (x_2, x_1^5)$.

Figure 2.2: Example of Green's diagrams of Borel sets defined by Borel-fixed ideals of points in $\mathbb{K}[x_0, x_1, x_2]$.

To describe a Borel set $\mathscr{B} \subset \mathcal{P}(3,m)$, Green thinks a trihedron (drawn with orthographic projections in Figure 2.3) described looking at its plane view with top vertix corresponding to the monomial x_0^m and completed with the following rule

$$\begin{tikzcd}[column sep=small] & x^\alpha \arrow[dl, "\mathsf{e}_2^+ \circ \mathsf{e}_1^+ \circ \mathsf{e}_0^+"'] \arrow[dr, "\mathsf{e}_1^+ \circ \mathsf{e}_0^+"] & \\ x^\beta & & x^\gamma \arrow[ll, "\mathsf{e}_2^+"] \end{tikzcd}$$

Then the monomials are marked according to the following notation:

- a black circle denotes a monomial in $\mathscr{B}$ such that also all the other monomials under it belong to $\mathscr{B}$;
- a empty circle denotes a monomial in $\mathscr{B}^{\mathcal{C}}$ such that all the monomials under it do not belong to $\mathscr{B}$;

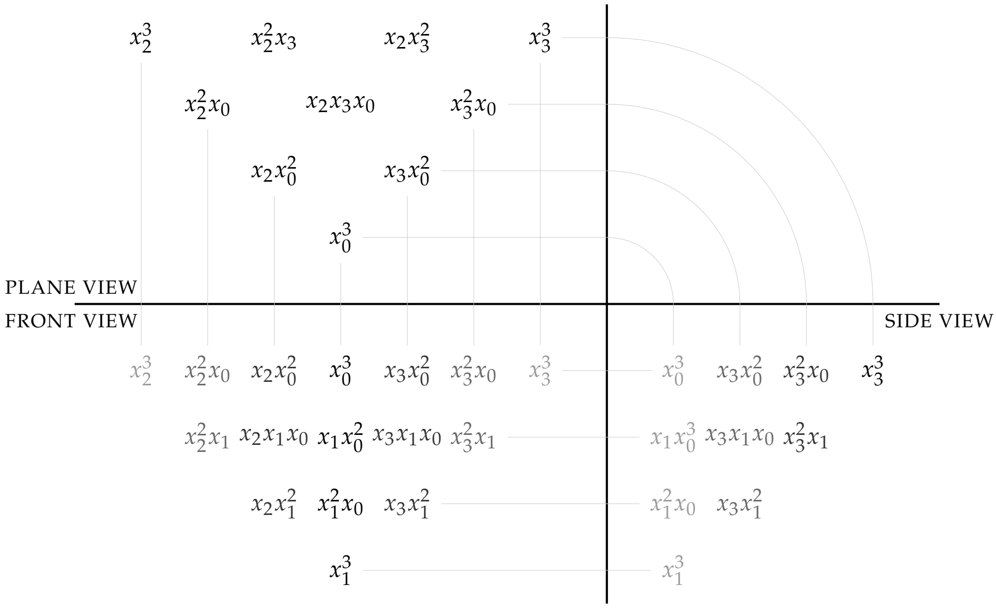
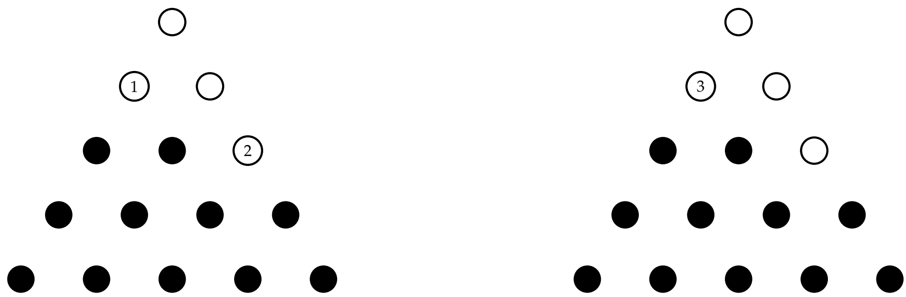


Figure 2.3: The trihedron describing the poset $\mathcal{P}(3,3)$ with orthographic projections.

- a empty circle with inside a positive integer λ denotes a monomial x^α in $\mathscr{B}^C$ such that the monomial (under it) $\lambda\mathsf{e}_0^+(x^\alpha)$ belongs to $\mathscr{B}$ and $(\lambda-1)\mathsf{e}_0^+(x^\alpha)$ does not.

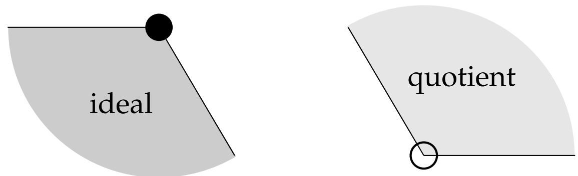
We remark that in the diagram corresponding to a saturated monomial ideal I, a monomial x^α marked with a black circle imposes that also every other monomial under it and contained in a triangle with x^α as top vertix belongs to I. Indeed splitting x^α as $x^{\overline{\alpha}}x_0^a$ ($x_0 \nmid x^{\overline{\alpha}}$), the black circle means $x^{\overline{\alpha}} \in I$ and any monomial in the triangle under it is divided by $x^{\overline{\alpha}}$. From now on, we will only draw the black circles defining the saturation of an ideal. Moreover if the ideal I is Borel-fixed also any monomial at the left of x^α has to be marked with a black circle. Thinking about the quotient, any monomial above or at the right of a monomial marked with a empty circle does not belong to the ideal. We can summarize this characterization with the following diagram



(a) The Borel set $\{I_4\} \subset \mathcal{P}(3,4)$, defined by the ideal $I = (x_3^2, x_3x_2, x_3x_1, x_2^3, x_2^2x_1^2)$.

(b) The Borel set $\{J_4\} \subset \mathcal{P}(3,4)$, defined by the ideal $J = (x_3^2, x_3x_2, x_2^3, x_3x_1^3)$.

Figure 2.4: Example of Green's diagrams of Borel sets defined by Borel-fixed ideals of curves in $\mathbb{K}[x_0, x_1, x_2, x_3]$.



This type of diagram works very well in the context of curves in $\mathbb{P}^3$, because we can understand many geometrical information about a curve simply looking at its diagram. For instance the number of empty circles corresponds to the degree of the curve, indeed it is easy to check that from the diagram of a curve in $\mathbb{P}^3$, the diagram of the hyperplane section with $H|_{x_0=0}$, i.e. points in $\mathbb{P}^2$, can be obtained substituting the empty circles with an integer inside with black circles.

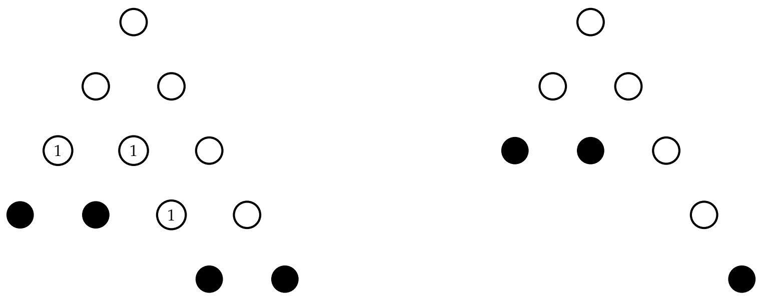
Example 2.4.1. Let us consider the curve defined by the ideal

$$I = (x_3^3, x_3^2x_2, x_3^2x_1, x_3x_2^3, x_2^4, x_3x_2^2x_1, x_3x_2x_1^2) \subset \mathbb{K}[x_0, x_1, x_2, x_3].$$

Its Hilbert polynomial is $p(t) = 5t + 2$, i.e. the curve has degree 5 and genus -1. The ideal defining the plane section of the curve with the plane $H|_{x_0=0}$ turns out to be

$$J = (x_3^2, x_3x_2, x_2^4) \subset \mathbb{K}[x_1, x_2, x_3]$$

and defines 5 points in $\mathbb{P}^2$ as expected. The two diagram are drawn in Figure 2.5.



(a) The curve in $\mathbb{P}^3$. (b) The plane section in $\mathbb{P}^2$.

Figure 2.5: Green's diagrams of the curve in $\mathbb{P}^3$ and its plane section in $\mathbb{P}^2$ described in Example 2.4.1.

Marinari's lattices Another way to represent posets in 3 or 4 variables was taught to me by Maria Grazia Marinari, that with some collegues has worked extensively on Borel-fixed ideals (see [62–65]). As for Green's diagrams, this approach works only for ideals defining subschemes in $\mathbb{P}^2$ and $\mathbb{P}^3$, but without any restriction on the degree of the Hilbert polynomial.

The poset $\mathcal{P}(2,m)$ is described again through a triangle shape with the bottom right vertix corresponding to the monomial x_0^m and then moving up and left with the following rule

$$\begin{tikzcd} x^\gamma & x^\beta \arrow[l, "\mathsf{e}_1^+"'] \\ & x^\alpha \arrow[u, "\mathsf{e}_0^+"] \end{tikzcd}$$

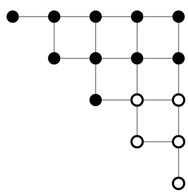
In the following we will not write explicitly the monomials and we will consider the lattice without the verse of the arrows. Given a Borel set $\mathscr{B} \subset \mathcal{P}(2,m)$, we will denote again with a black circle a monomial belonging to $\mathscr{B}$ and with a empty circle a monomial not belonging.

To describe posets in 4 variables, the idea is to decompose $\mathcal{P}(3,m)$ as

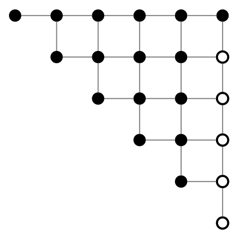
$$\mathcal{P}(3,m) = \mathcal{P}(2,m) \cup x_0 \cdot \mathcal{P}(2,m-1) \cup \cdots \cup x_0^{m-1} \cdot \mathcal{P}(2,1) \cup \{x_0^m\}$$

$$\begin{tikzcd}
x_2^4 & x_2^3x_1 \arrow[l] & x_2^2x_1^2 \arrow[l] & x_2x_1^3 \arrow[l] & x_1^4 \arrow[l] \\
 & x_2^3x_0 \arrow[u] & x_2^2x_1x_0 \arrow[l] \arrow[u] & x_2x_1^2x_0 \arrow[l] \arrow[u] & x_1^3x_0 \arrow[l] \arrow[u] \\
 & & x_2^2x_0^2 \arrow[u] & x_2x_1x_0^2 \arrow[l] \arrow[u] & x_1^2x_0^2 \arrow[l] \arrow[u] \\
 & & & x_2x_0^3 \arrow[u] & x_1x_0^3 \arrow[l] \arrow[u] \\
 & & & & x_0^4 \arrow[u]
\end{tikzcd}$$

Figure 2.6: An example of Marinari's lattice for $\mathbb{P}^2$: the poset $\mathcal{P}(2,4)$ (cf. Figure 2.1).



(a) The Borel set $\{I_4\} \subset \mathcal{P}(2,4)$, where $I = (x_2^2, x_2x_1^2, x_1^3)$.



(b) The Borel set $\{L_5\} \subset \mathcal{P}(2,5)$, defined by $L = (x_2, x_1^5)$.

Figure 2.7: Example of Marinari's lattices representing Borel sets defined by Borel-fixed ideals of points in $\mathbb{K}[x_0, x_1, x_2]$ (cf. Figure 2.2).

and to draw each $\mathcal{P}(2,i)$ one above the other, so that a monomial $x^\alpha \in x_0^{m-i}\mathcal{P}(2,i)$ has above it the monomial $\mathsf{e}_0^+(x^\alpha) \in x_0^{m-i-1}\mathcal{P}(2,i+1)$ and beaneath it $\mathsf{e}_0^-(x^\alpha) \in x_0^{m-i+1}\mathcal{P}(2,i-1)$.

With this representation, given an ideal defining a curve, the ideal of its plane section (with $H|_{x_0=0}$) in $\mathbb{K}[x_1, x_2, x_3]$ is directly described in the highest triangle.

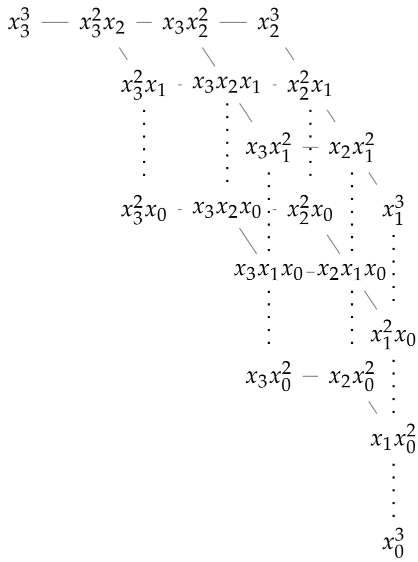
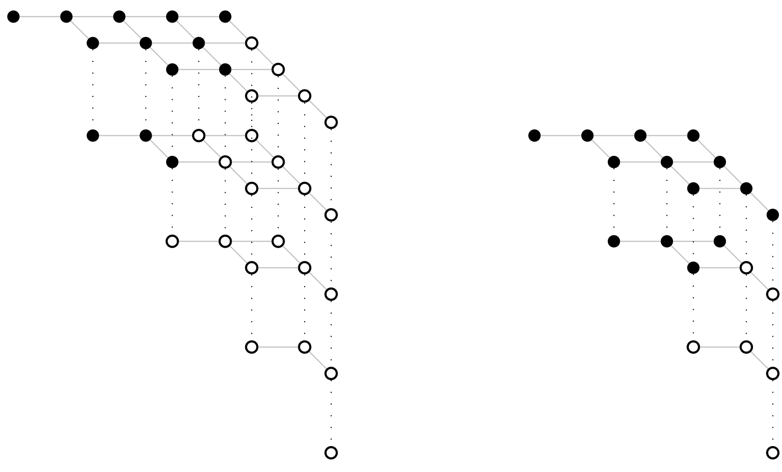


Figure 2.8: The Marinari's lattice describing the poset $\mathcal{P}(3,3)$ (cf. Figure 2.3).

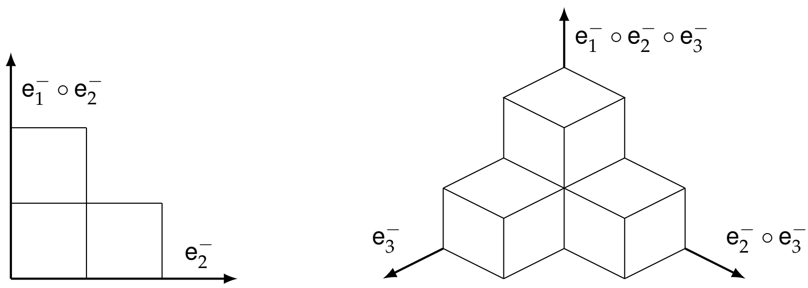


(a) The curve in $\mathbb{P}^3$ introduced in Example 2.4.1.

(b) The Borel set defined by $(x_3^2, x_3x_2, x_2^2, x_3x_1, x_2x_1^2, x_1^3)$ (6 points in $\mathbb{P}^3$).

Figure 2.9: Borel sets of a curve and points in $\mathbb{P}^3$ in the Marinari's lattice.

Gotzmann's pyramids Gotzmann showed to me another slightly different way to draw posets of the type $\mathcal{P}(2,m)$ and $\mathcal{P}(3,m)$ using two and three dimensonal spaces. He (literally) builds two or three dimensional pyramids (see Figure 2.10) representing monomials as bricks (squares for $\mathcal{P}(2,m)$ and cubes for $\mathcal{P}(3,m)$) starting from the monomial x_2^m in the case of $\mathcal{P}(2,m)$ and from x_3^m in the case of $\mathcal{P}(3,m)$ with the following expansion rules:



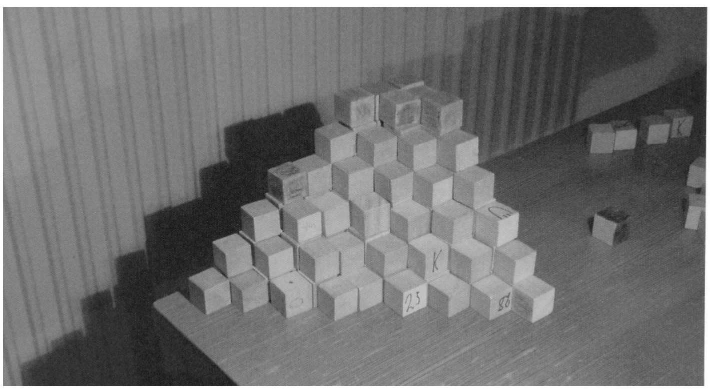
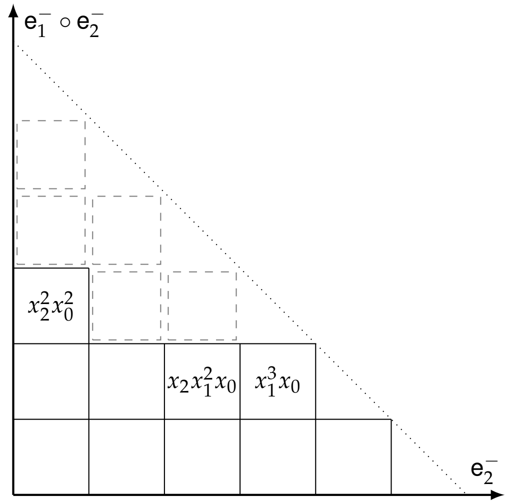
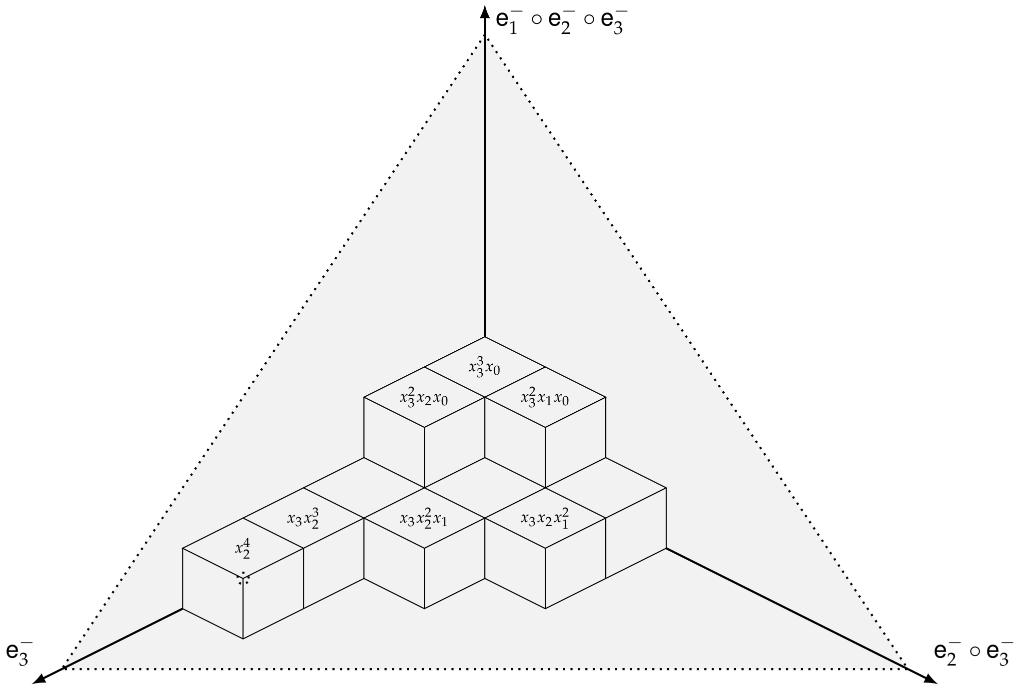


Figure 2.10: A photo of a Borel set sent to me by Gotzmann.

With this technique, the monomials not belonging to the Borel set are not drawn. To understand which monomial are missing, i.e. to have information about the subscheme defined, we can consider for $\mathcal{P}(2,m)$ the line touching the monomials x_1^m and x_0^m, for $\mathcal{P}(3,m)$ the plane touching the monomials x_2^m, x_1^m and x_0^m and look at the "bricks" that we would need to fill the empty space between the pyramid and the line or plane (see Figure 2.11).



(a) The Borel set defined by $(x_2^2, x_2x_1^2, x_1^3)$ in $\mathcal{P}(2,4)$ (cf. Figure 2.7a and Figure 2.2a).



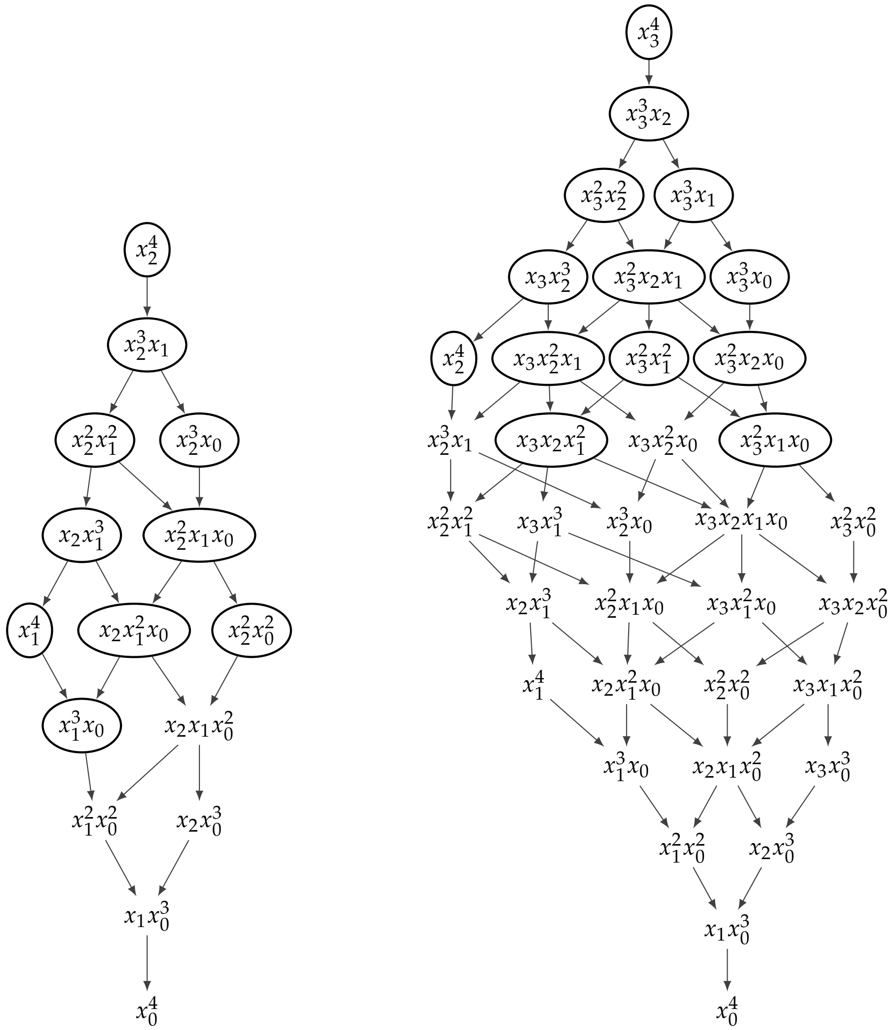
(b) The Borel set corresponding to the curve defined in Example 2.4.1 (cf. Figure 2.5a and Figure 2.9a).

Figure 2.11: An example of Gotzmann's pyramids.

Planar graphs All previous approches result to be intrinsically limitated to the projective 3-space. We would like to overcome this limit and to find a nice representations available for any kind of poset. A very natural way to describe $\mathcal{P}(n,m)$ comes directly from the definition of partially ordered set: we associate to it the graph whose vertices are the monomials of $\mathbb{K}[x_0,\ldots,x_n]_m$ and whose edges correspond to elementary decreasing moves (see Figure 2.12). Given a Borel set $\mathscr{B} \subset \mathcal{P}(n,m)$, we will represent its monomials with vertices with elliptic black boundary and without boundaries the monomials outside $\mathscr{B}$.

This representation, as well as allowing to manage any poset, is very advantageous in an algorithmic perspective, indeed there are many tools to work on graphs. We underline that with this description is very easy to detect minimal and maximal monomials (see Figure 2.13).

In the following we will use Marinari's lattices and planar graphs: the pictures of lattices turn out to be very helpful to understand the main ideas and planar graphs allow to generalize such ideas to posets in any number of variables and to concretely project algorithms, indeed the java class `PosetGraph` of the package `HSC` implements the poset by means of the associated direct graph (see Appendix B).



(a) The Borel set defined by $(x_2^2, x_2x_1^2, x_1^3)$ in $\mathcal{P}(2,4)$ (cf. Figure 2.7a, Figure 2.2a and Figure 2.11a).

(b) The Borel set corresponding to the curve defined in Example 2.4.1 (cf. Figure 2.5a, Figure 2.9a and Figure 2.11b).

Figure 2.12: An example of Borel sets drawn as planar graphs.

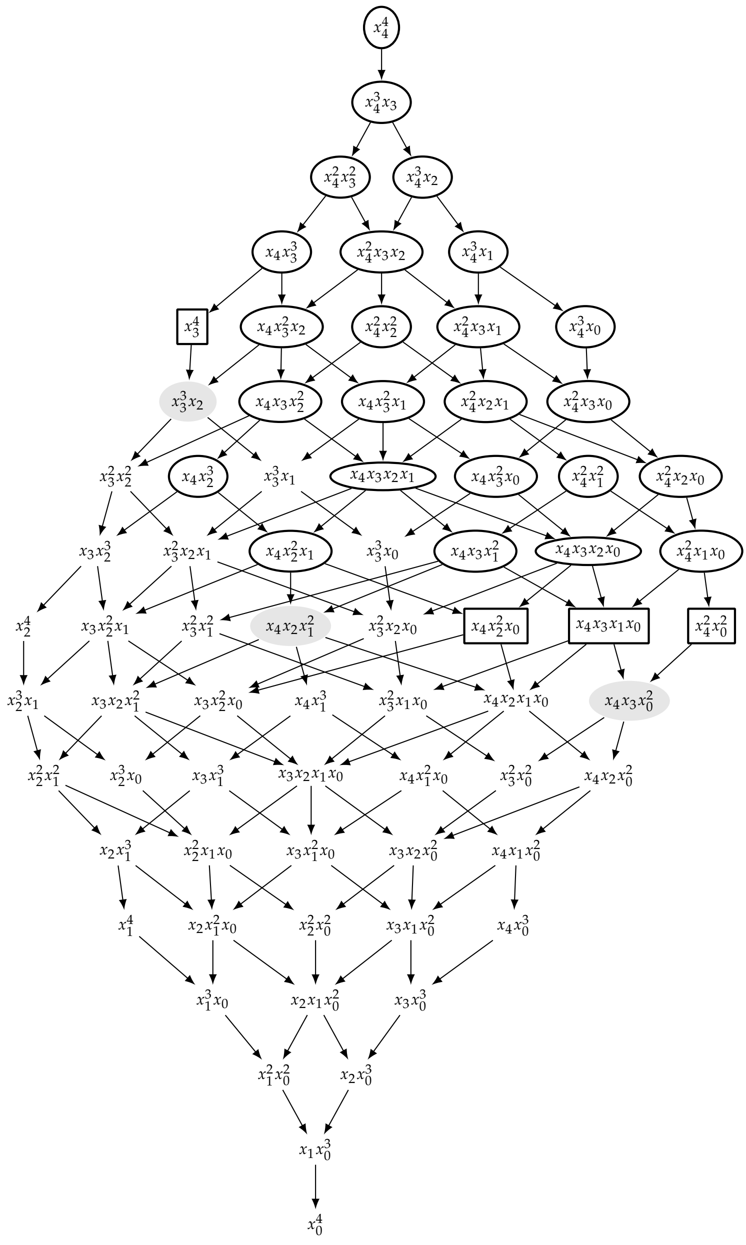


Figure 2.13: The Borel set defined by the ideal $(x_4^2, x_4x_3^2, x_4x_3x_2, x_4x_2^2, x_4x_3x_1, x_3^4)$ in $\mathcal{P}(4,4)$. The monomials with rectangle boundary are minimal elements of the Borel set, whereas the monomials with a light gray background are maximal elements in the complement.

2.5 An algorithm computing Borel-fixed ideals

In this section and in Section 2.7, we will expose some of the results contained in the paper [21] "Segments and Hilbert schemes of points" written in collaboration with Francesca Cioffi, Maria Grazia Marinari and Margherita Roggero.

Let $J \subset \mathbb{K}[x]$ be a Borel-fixed ideal. In this section we denote by J_{x_0} the ideal obtained from J setting $x_0 = 1$. Keeping in mind Corollary 2.10, we know that $J_{x_0} = J^{\mathrm{sat}}$. We extend this notation denoting by $J_{x_0x_1}$ the ideal obtained from J setting both x_0 and x_1 equal to 1. We call $J_{x_0x_1}$ the x_1-*saturation* of J and say that J is x_1-*saturated* if $J = J_{x_0x_1}$. Hence an ideal J that is x_1-saturated is also saturated.

Remark 2.5.1. A x_1-saturated Borel-fixed ideal $J \subset \mathbb{K}[x]$ defining a subscheme with Hilbert polynomial $p(t)$ has the same minimal generators as the saturated Borel ideal $J \cap \mathbb{K}[x_1, \dots, x_n] \subset \mathbb{K}[x_1, \dots, x_n]$, that defines a subscheme of $\mathbb{P}^{n-1}$ with Hilbert polynomial $\Delta p(t)$.

The following result is analogous to Theorem 3 of [86], where the notion of "fan" is used. Here we apply the combinatorial properties of Borel ideals only.

Proposition 2.32. *Let $J \subset \mathbb{K}[x]$ be a saturated Borel-fixed ideal defining a subscheme with Hilbert polynomial $p(t)$ whose Gotzmann number is r. Let $I = J_{x_0x_1}$ be its x_1-saturation and let $\overline{p}(t)$ be the Hilbert polynomial of the subscheme defined by I in $\mathbb{P}^n$. Set $q = \dim_{\mathbb{K}} I_r - \dim_{\mathbb{K}} J_r$,*

(i) $\overline{p}(t) = p(t) - q$;

(ii) q is equal to the sum of the exponents of x_1 in the minimal generators of J.

Proof. *(i)* We show that if $q = \dim_{\mathbb{K}} I_s - \dim_{\mathbb{K}} J_s$ then $q = \dim_{\mathbb{K}} I_{s+1} - \dim_{\mathbb{K}} J_{s+1}$, for every $s \geq r$. Let $x^{\beta_1}, \dots, x^{\beta_q}$ be the terms of $I_s \setminus J_s$. Thus, $x_0x^{\beta_1}, \dots, x_0x^{\beta_q}$ are terms of $I_{s+1} \setminus J_{s+1}$ and so $\dim_{\mathbb{K}} I_{s+1} - \dim_{\mathbb{K}} J_{s+1} \geqslant q$, since $x_0x^{\beta_i}$ would belong to J_{s+1} if and only if x^{β_i} belong to J_s, being J saturated. Now, to obtain the opposite inequality, it is enough to show that every term of $I_{s+1} \setminus J_{s+1}$ is divisible by x_0. Let $x^\gamma \in I_{s+1}$ be such that $\min x^\gamma \geqslant 1$ and let x^α be a minimal generator of I such that $x^\gamma = x^\alpha x^\delta$. Since J is saturated and I is the x_1-saturation of J, $x^\alpha x_1^a$ is a minimal generator of J

for some non negative integer a. Hence, for every $x^{\delta'}$ of degree $s+1-|\alpha|$ and with $\min x^{\delta'} \geqslant 1$, $x^{\delta'} >_B x_1^a$ implies $x^\alpha x^{\delta'} \in J_{s+1}$. In particular, $x^\gamma \in J_{s+1}$.

(ii) Let $x^{\alpha_1}x_1^{s_1}, \dots, x^{\alpha_h}x_1^{s_h}$ be the minimal generators of J, with $\min x^{\alpha_i} > 1$, for every $1 \leq i \leq h$. Since the $\sum s_i$ terms $x^{\alpha_i}x_1^{s_i-t}x_0^{r-|\alpha_i|-s_i+t}, 1 \leq t \leq s_i$, are in $I_r \setminus J_r$, one has $q \geq \sum s_i$. Vice versa, we show that each term x^δ in $I_r \setminus J_r$ is of the previous type. We can write $x^\delta = x^\beta x_1^u x_0^{r-|\beta|-u}$, with $\min x^\beta > 1$ and $u < s_i$. Let s be the minimum non negative integer such that $x^\beta x_1^s$ is in J. Then there exists i such that $x^{\alpha_i}x_1^{s_i} | x^\beta x_1^s$, i.e. $x^{\alpha_i} | x^\beta$ and $s_i \leq s$. By the definition of s, we get $s_i = s$ and there exists x^γ with $\min x^\gamma > 1$ such that $x^\beta = x^{\alpha_i}x^\gamma$. Since x^β does not belong to J we have $|\gamma| < s_i = s$, or otherwise $x^{\alpha_i}x_1^{|\gamma|}$ and hence, by the Borel property, $x^\beta = x^{\alpha_i}x^\gamma$ should belong to J. Now we can take $x^\beta x_1^{s-|\gamma|}$ and observe that this term belongs to J because it follows $x^{\alpha_i}x_1^s$ in the Borel relation. Thus $s \leq s - |\gamma|$, so that $\gamma = 0$, i.e. $x^\beta = x^{\alpha_i}$ as claimed. □

Lemma 2.33. *Let $J \subset \mathbb{K}[x]$ be a saturated Borel-fixed ideal such that $\mathbb{K}[x]/J$ has Hilbert polynomial $p(t)$ whose Gotzmann number is r. Let x^β be a minimal monomial of $\{J_s\} \subset \mathcal{P}(n,s)$ of degree $s \geqslant r$ such that $\min x^\beta = x_0$. Then the ideal $I = \langle \{J_s\} \setminus \{x^\beta\} \rangle$ is Borel-fixed and $\mathbb{K}[x]/I$ has Hilbert polynomial $\overline{p}(t) = p(t) + 1$.*

Proof. First, note that by definition of minimal monomial, $\{I_s\}$ is still a Borel set. Called $q(t)$ the volume polynomial of J, we show that I has volume polynomial $\overline{q}(t) = q(t) - 1$ applying Gotzmann's Persistence Theorem (Theorem 1.22), i.e. proving that $\dim_{\mathbb{K}} J_s - \dim_{\mathbb{K}} I_s = \dim_{\mathbb{K}} J_{s+1} - \dim_{\mathbb{K}} I_{s+1} = 1$. By construction $\dim_{\mathbb{K}} J_s - \dim_{\mathbb{K}} I_s = 1$. The Borel considition ensures that $x^\beta x_0 \in J_{s+1} \setminus I_{s+1}$ and there are no other elements, because $x^\beta x_0$ is the only monomial that cannot be generated from the monomials in I_s by multiplication of a single variable. In fact let us consider the monomial $x_i x^\beta$, $i > 0$. Since $x_0 \mid x^\beta$ the following identity holds:

$$x_i x^\beta = \frac{x_i}{x_{i-1}} \cdot \ldots \cdot \frac{x_1}{x_0} x_0 x^\beta = \mathsf{e}_{i-1}^+ \circ \cdots \circ \mathsf{e}_0^+(x^\beta) x_0$$

and for each i, $\mathsf{e}_{i-1}^+ \circ \cdots \circ \mathsf{e}_0^+(x^\beta)$ belongs to I_s, by the minimality of x^β. □

Proposition 2.34. *Let I and J be Borel-fixed ideals of $\mathbb{K}[x]$. If for every $s \gg 0$ we have $I_s \subset J_s$ and $p_{\mathbb{K}[x]/I}(t) = p_{\mathbb{K}[x]/J}(t) + a$, with $a \in \mathbb{N}$, then I and J have the same x_1-saturation.*

Proof. Let $s \geq \max\{\mathrm{reg}(I), \mathrm{reg}(J)\}$. In case $a = 1$, there exists a unique term in $J_{s+t} \setminus I_{s+t}$, for every $t \geq 0$. Let x^α be the unique term in $J_s \setminus I_s$. Then, both $x^\alpha x_0$ and $x^\alpha x_1$ belong to J_{s+1}. By the Borel property, $x^\alpha x_1$ must be in I_{s+1} and so the unique term in $J_{s+t} \setminus I_{s+t}$ is $x^\alpha {x_0}^t$. This is enough to say that I and J have the same x_1-saturation. If $a > 1$, the thesis follows by induction applying Lemma 2.33. $\square$

Theorem 2.35. *Let $p(t)$ be an admissible Hilbert polynomial in $\mathbb{P}^n$. For any s, there is a bijective function*

$$\left\{ \begin{array}{c} J \subset \mathbb{K}[x] \textit{ saturated Borel-fixed} \\ \textit{ideal s.t. } \mathrm{reg}(J) \leqslant s \textit{ and } \mathbb{K}[x]/J \textit{ has} \\ \textit{Hilbert polynomial } p(t) \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{c} \mathscr{B} \subset \mathcal{P}(n,s) \textit{ Borel set s.t.} \\ \textit{set } \mathscr{N} = \mathcal{P}(n,s) \setminus \mathscr{B} \\ \left|\mathscr{N}_{(\geqslant i)}\right| = \Delta^i p(s),\ \forall\, i \end{array} \right\}$$
$$\begin{array}{rcl} J & \longrightarrow & \{J_s\} \\ \langle \mathscr{B} \rangle^{\mathrm{sat}} & \longleftarrow & \mathscr{B} \end{array} \tag{2.18}$$

Proof. First of all, note that if the two maps are well-defined, keeping in mind 2.12, i.e. for each J, $J_{\geqslant s} = \langle J_s \rangle$,

$$J \longrightarrow \{J_s\} \longrightarrow \big\langle \{J_s\} \big\rangle^{\mathrm{sat}} = J,$$
$$\mathscr{B} \longrightarrow \langle \mathscr{B} \rangle^{\mathrm{sat}} \longrightarrow \big\{ \langle \mathscr{B} \rangle^{\mathrm{sat}}_s \big\} = \mathscr{B}.$$

Let $J \subset \mathbb{K}[x]$ be a Borel-fixed ideal such that the Hilbert polynomial of $\mathbb{K}[x]/J$ is equal to $p(t)$ and let $\mathscr{N} = \mathcal{P}(n,s) \setminus \{J_s\}$. Obviously $\left|\mathscr{N}_{(\geqslant 0)}\right| = |\mathscr{N}| = p(s) = \Delta^0 p(s)$. Using the short exact sequence (2.4), we determine the Borel ideal $I = (J, x_0) \cap \mathbb{K}[x_1, \ldots, x_n]$ with module $\mathbb{K}[x_1, \ldots, x_n]/I$ having Hilbert polynomial $\Delta p(t)$. Thus being $\{I_s\} = \{J_s\}_{(\geqslant 1)} \subset \mathcal{P}(n-1, s)$, $\left|\mathscr{N}_{(\geqslant 1)}\right| = \left|\{I_s\}^{\mathcal{C}}\right| = \Delta p(s)$. Since I is Borel-fixed in the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ we can repeat the reasoning with the hyperplane section defined by $x_1 = 0$ and so on.

Let us now consider a Borel set $\mathscr{B} \subset \mathcal{P}(n,s)$, such that the complement $\mathscr{N} = \mathscr{B}^{\mathcal{C}}$ satisfies the condition $\left|\mathscr{N}_{(\geqslant i)}\right| = \Delta^i p(s)$ for every i. Firstly $\mathrm{reg}\langle \mathscr{B} \rangle^{\mathrm{sat}} \leqslant s$ by Proposition 2.12, so let us prove that $\mathbb{K}[x]/\langle \mathscr{B} \rangle$ has Hilbert polynomial $p(t)$. We proceed by induction on the degree d of the Hilbert polynomial. For any n, if $\deg p(t) = 0$, then

$\mathcal{N}_{(\geqslant i)} = \emptyset$, for every $i \geqslant 1$, since $\Delta p(t) = 0$, that is for any $x^\beta \in \mathcal{N}$, $\min x^\beta = 0$. Applying repeatedly Lemma 2.33 starting from the Hilbert polynomial $\overline{p}(t) = 0$ (corresponding to the ideal (1)), we obtain that $\langle \mathscr{B} \rangle^{\text{sat}}$ defines a module $\mathbb{K}[x]/\langle \mathscr{B} \rangle^{\text{sat}}$ having constant Hilbert polynomial $p(t)(= p(s))$. Let us know suppose that the map $\mathscr{B} \to \langle \mathscr{B} \rangle^{\text{sat}}$ is well-defined for any Hilbert polynomial of degree $d-1$ and let $p(t)$ be a Hilbert polynomial of degree d. $\overline{\mathscr{B}} = \mathscr{B}_{(\geqslant 1)} \subset \mathcal{P}(n-1,s)$ realizes the condition of the theorem w.r.t. the Hilbert polynomial $\Delta p(t)$ and $\deg \Delta p(t) = d-1$. Hence by the inductive hypothesis the ideal $\langle \overline{\mathscr{B}} \rangle^{\text{sat}} \subset \mathbb{K}[x_1,\ldots,x_n]$ defines the module $\mathbb{K}[x_1,\ldots,x_n]/\langle \overline{\mathscr{B}} \rangle^{\text{sat}}$ with Hilbert polynomial $\Delta p(t)$. Let $\overline{p}(t)$ be the Hilbert polynomial of $\mathbb{K}[x_0,\ldots,x_n]/\langle \overline{\mathscr{B}} \rangle^{\text{sat}}$: $\overline{p}(t) = p(t) + a$, because $\Delta \overline{p}(t) = \Delta p(t)$. $\langle \overline{\mathscr{B}} \rangle^{\text{sat}}$ turns out to be the x_1-saturation of $\langle \mathscr{B} \rangle^{\text{sat}}$, so by Proposition 2.32 the Hilbert polynomial of $\mathbb{K}[x_0,\ldots,x_n]/\langle \mathscr{B} \rangle^{\text{sat}}$ differs by a constant from $\overline{p}(t)$ and since $|\mathcal{N}| = |\mathcal{N}_{(\geqslant 0)}| = p(r)$ it coincides with $p(t)$. $\square$

Corollary 2.36. *Let $p(t)$ be an admissible Hilbert polynomial in $\mathbb{P}^n$ whose Gotzmann number is r. There is a bijective function*

$$\begin{array}{ccc} \left\{ \begin{array}{c} J \subset \mathbb{K}[x] \textit{ saturated Borel-fixed} \\ \textit{ideal s.t. } \mathbb{K}[x]/J \textit{ has} \\ \textit{Hilbert polynomial } p(t) \end{array} \right\} & \xleftrightarrow{1:1} & \left\{ \begin{array}{c} \mathscr{B} \subset \mathcal{P}(n,r) \textit{ Borel set s.t.} \\ \textit{set } \mathcal{N} = \mathcal{P}(n,r) \setminus \mathscr{B} \\ \left|\mathcal{N}_{(\geqslant i)}\right| = \Delta^i p(r),\ \forall\, i \end{array} \right\} \\ J & \longrightarrow & \{J_r\} \\ \langle \mathscr{B} \rangle^{\text{sat}} & \longleftarrow & \mathscr{B} \end{array} \tag{2.19}$$

Proof. By Proposition 2.12, Proposition 2.13 ant Theorem 1.21, any saturated Borel-fixed ideal J defining a module $\mathbb{K}[x]/J$ with Hilbert polynomial $p(t)$ has regularity lower than or equal to the Gotzmann number of $p(t)$. $\square$

Therefore to compute the saturated Borel-fixed ideals we can construct Borel sets with the prescribed property. The proof of Theorem 2.35 suggests to use a recursive algorithm: i.e. to determine the Borel sets in $\mathcal{P}(n,r)$ corresponding to the Hilbert polynomial $p(t)$, we begin computing the Borel sets in $\mathcal{P}(n-1,r)$ corresponding to the Hilbert polynomial $\Delta p(t)$.

Let us examine more precisely this idea. Let $\overline{\mathscr{B}} \subset \mathcal{P}(n-1,r)$ a Borel set corresponding to the Hilbert polynomial $\Delta p(t)$ and let $\overline{\mathscr{N}} = \overline{\mathscr{B}}^{\mathcal{C}}$. In order for $\overline{\mathscr{B}}$ to be the restriction $\mathscr{B}_{(\geqslant 1)}$ of a Borel set $\mathscr{B} \subset \mathcal{P}(n,r)$ (where $\mathcal{P}(n,r)$ contains one more variable smaller than variables in $\mathcal{P}(n-1,r)$), each monomial that can be obtained by decreasing moves from a monomial in $\overline{\mathscr{N}}$ has to belong to $\mathscr{N} = \mathscr{B}^{\mathcal{C}}$. This extension of an order set $\overline{\mathscr{N}} \subset \mathcal{P}(n-1,r)$ to an order set $\mathscr{N} \subset \mathcal{P}(n,r)$ has an ideal interpretation.

Lemma 2.37. *Let $\overline{\mathscr{B}} \subset \mathcal{P}(n-1,r)$ be a Borel set and let $\overline{\mathscr{N}} = \overline{\mathscr{B}}^{\mathcal{C}}$. Moreover let $\mathscr{N} \subset \mathcal{P}(n,r)$ be the order set containing the monomials in $\overline{\mathscr{N}}$ and all those obtained by descreasing moves from them. Then,*

$$\mathscr{N} = \mathcal{P}(n,r) \setminus \left\{ \left(\langle \overline{\mathscr{B}} \rangle^{\mathrm{sat}} \cdot \mathbb{K}[x_0,\ldots,x_n]\right)_r \right\}. \tag{2.20}$$

Proof. Let us call $\mathscr{B}$ the Borel set $\{(\langle \overline{\mathscr{B}} \rangle^{\mathrm{sat}} \cdot \mathbb{K}[x_0,\ldots,x_n])_r\}$. Let $x^\alpha = x_n^{\alpha_n} \cdots x_0^{\alpha_0}$ be a monomial of $\mathcal{P}(n,r)$ and suppose $\min x^\alpha = 0$, i.e. $\alpha_0 > 0$. The monomial $\alpha_0 \mathsf{e}_0^+(x^\alpha) = x^{\alpha_n} \cdots x_1^{\alpha_1+\alpha_0}$ belongs to $\mathcal{P}(n-1,r)$, so either $\alpha_0 \mathsf{e}_0^+(x^\alpha) \in \overline{\mathscr{B}}$ or $\alpha_0 \mathsf{e}_0^+(x^\alpha) \in \overline{\mathscr{N}}$. If $x^{\alpha_n} \cdots x_1^{\alpha_1+\alpha_0} \in \overline{\mathscr{B}}$, then $x_n^{\alpha_n} \cdots x_2^{\alpha_2}$ is in $\langle \overline{\mathscr{B}} \rangle^{\mathrm{sat}}$ and so $x^\alpha \in \langle \overline{\mathscr{B}} \rangle^{\mathrm{sat}} \cdot \mathbb{K}[x_0,\ldots,x_n]$, otherwise $x^{\alpha_n} \cdots x_1^{\alpha_1+\alpha_0} \in \overline{\mathscr{N}}$ implies $x^\alpha \in \mathscr{N}$. $\square$

At this point, by Proposition 2.32, we know that the Hilbert polynomial corresponding to a Borel set $\mathscr{B}$ of the type $\{(\langle \overline{\mathscr{B}} \rangle^{\mathrm{sat}} \cdot \mathbb{K}[x])_r\}$ differs from the target Hilbert polynomial by a constant: to determine this constant we compare the value $p(r)$ of the Hilbert polynomial $p(t)$ in degree r with the cardinality of the order set $\mathscr{N}$ obtained by decreasing moves from $\overline{\mathscr{N}}$.

Lemma 2.38. *Let $\overline{\mathscr{N}} \subset \mathcal{P}(n-k,r)$ be an order set and let $\mathscr{N} \subset \mathcal{P}(n,r)$ be the order set defined from $\overline{\mathscr{N}}$ by decreasing moves. Then,*

$$|\mathscr{N}| = \sum_{\substack{x^\alpha \in \overline{\mathscr{N}} \\ x^\alpha = x_n^{\alpha_n} \cdots x_k^{\alpha_k}}} \binom{\alpha_k + k}{k}. \tag{2.21}$$

Proof. Each monomial $x^\alpha \in \overline{\mathscr{N}}$ imposes the belonging to $\mathscr{N}$ of any monomials obtained from it applying a composition of decreasing moves $\lambda_1 \mathsf{e}_1^- \circ \cdots \circ \lambda_k \mathsf{e}_k^-$.

These type of moves act on the maximal power of x_k in x^α and so they describe a poset isomoprhic to $\mathcal{P}(k,\alpha_k)$ and

$$|\mathcal{P}(k,\alpha_k)| = \dim_{\mathbb{K}} \mathbb{K}[x_0,\ldots,x_k]_{\alpha_k} = \binom{k+\alpha_k}{k}. \qquad \square$$

There are three possibilities:

- $p(r) - |\mathscr{N}| < 0$, $\overline{\mathscr{N}}$ imposes too many monomials ouside the ideal, so the hyperplane section defined by $\langle\overline{\mathscr{B}}\rangle^{\text{sat}}$ has to be discarded (there exist no Borel-fixed ideals corresponding to $p(t)$ with such a hyperplane section);
- $p(r) - |\mathscr{N}| = 0$, $\langle\overline{\mathscr{B}}\rangle^{\text{sat}} \subset \mathbb{K}[x_0,\ldots,x_n]$ is one of the ideals sought;
- $p(r) - |\mathscr{N}| > 0$, applying repeatedly Lemma 2.33 we determine the ideals we are looking for.

Putting together Remark 2.2.2 with Lemma 2.37, we can establish a sharp upper bound of the difference $p(r) - |\mathscr{N}|$.

Proposition 2.39. *Let $p(t)$ be an admissible Hilbert polynomial with Gotzmann number r and let r_1 be the Gotzmann number of $\Delta p(t)$.*

(i) Given saturated Borel-fixed ideal $J \subset \mathbb{K}[x_1,\ldots,x_n]$ such that $\mathbb{K}[x_1,\ldots,x_n]/J$ has Hilbert polynomial $\Delta p(t)$, to pass from $\{(J\cdot\mathbb{K}[x_0,\ldots,x_n])_r\} \subset \mathcal{P}(n,r)$ to a Borel set corresponding to $p(t)$, we need to remove at most $r - r_1$ monomials.

(ii) We need to remove exactly $r - r_1$ monomials if we consider the lexicographic ideal $L \subset \mathbb{K}[x_1,\ldots,x_n]$ corresponding to the polynomial $\Delta p(t)$.

(iii) Let $I \subset \mathbb{K}[x]$ be a Borel-fixed ideal, such that the Hilbert polynomial of $\mathbb{K}[x]/I$ is $p(t)$. To construct the Borel set $\mathscr{B} \subset \mathcal{P}(r,n)$ corresponding to I, we need to remove at most r monomials.

Proof. (i) Straightforward from Remark 2.2.2, because the Hilbert polynomial of $\mathbb{K}[x_0,\ldots,x_n]/(J\cdot\mathbb{K}[x_0,\ldots,x_n])$ belongs to $\text{HP}(\Delta p(t))$.

(ii) There are more than one way to prove this point. We exploit Lemma 2.38 in order to show that the ideal $L\cdot\mathbb{K}[x_0,\ldots,x_n]$ corresponds to the minimal polynomial in $\text{HP}(\Delta p(t))$. By definition, the order set $\overline{\mathscr{N}} = \{L_r\}^{\mathcal{C}} \subset \mathcal{P}(n-1,r)$ contains

the smallest $\Delta p(r)$ monomials in $\mathcal{P}(n-1,r)$ w.r.t. the degree lexicographic order. To construct $\overline{\mathscr{N}}$ we can think to start from the order set $\{x_1^r\} \subset \mathcal{P}(n-1,r)$ and to remove successively the minimum w.r.t. the lexicographic order (see Remark 2.3.4) among the minimal monomials of the complement. This minimum can be detected also looking at the elementary increasing move by which we can reach it: it will correspond to the lowest index possible. So whenever it is possible we add a monomials reached with e_1^+ so that the number of monomials smaller than it in $\mathcal{P}(n,r)$ decreases. Any other choice will generate an order set $\mathscr{N}$ with more elements.

(iii) It comes directly applying *(i)* recursively on the construction of the Borel sets corresponding to any $\Delta^i p(t)$. □

2.5.1 The pseudocode description of the algorithm

In Algorithm 2.1, we give a pseudocode description of the algorithm just designed. Of some auxiliary methods implementing basic operations, we only describe the requirements on the input and the result returned in the output. In Table 2.1, we simulate an execution of BORELGENERATOR on $\mathbb{K}[x_0,x_1,x_2,x_3]$ and $p(t) = 3t+2$.

GOTZMANNNUMBER$\big(p(t)\big)$
Input: $p(t)$, admissible Hilbert polynomial.
Output: the Gotzmann number of $p(t)$.

MINIMALELEMENTS($\mathscr{B}$)
Input: $\mathscr{B}$, Borel set.
Output: the set of minimal elements of $\mathscr{B}$ w.r.t. $\leq_B$.

1: BORELGENERATOR$\big(\mathbb{K}[x_k,\ldots,x_h], p(t)\big)$
Input: $\mathbb{K}[x_k,\ldots,x_h]$, polynomial ring.
Input: $p(t)$, admissible Hilbert polynomial in $\mathbb{P}^{h-k} = \operatorname{Proj}\mathbb{K}[x_k,\ldots,x_h]$.
Output: the set of all Borel-fixed ideals in $\mathbb{K}[x_k,\ldots,x_h]$ defining subschemes of $\mathbb{P}^{h-k}$ with Hilbert polynomial $p(t)$.
2: **if** $p(t) = 0$ **then**
3: **return** $\big\{(1)\big\}$;
4: **end if**

hyperplaneSections $\leftarrow$ BORELGENERATOR$\big(\mathbb{K}[x_{k+1},\ldots,x_h],\Delta p(t)\big)$;
borelFixedIdeals $\leftarrow \emptyset$;
$r \leftarrow$ GOTZMANNNUMBER$\big(p(t)\big)$;
for all $J \in$ hyperplaneSections **do**
    $\mathscr{B} \leftarrow \big\{(J\cdot\mathbb{K}[x_k,\ldots,x_h])_r\big\} \subset \mathcal{P}(h-k,r)$;
    $q \leftarrow p(r) - |\mathscr{B}^{\mathcal{C}}|$;
    if $q \geqslant 0$ **then**
        borelFixedIdeals $\leftarrow$ borelFixedIdeals $\cup$ REMOVE$(\mathscr{B},q)$;
    end if
end for
return borelFixedIdeals;

REMOVE($\mathscr{B}$,q)
Input: $\mathscr{B}$, a Borel set.
Input: q, the number of monomials to remove from $\mathscr{B}$.
Output: the set of saturated Borel-fixed ideals obtained from Borel sets constructed from $\mathscr{B}$ removing in all the possible ways q monomials.
if $q = 0$ **then**
    return $\{\langle\mathscr{B}\rangle^{\mathrm{sat}}\}$;
else
    borelIdeals $\leftarrow \emptyset$;
    minimalMonomials $\leftarrow$ MINIMALELEMENTS$(\mathscr{B})$;
    for all $x^\alpha \in$ minimalMonomials **do**
        borelIdeals $\leftarrow$ borelIdeals $\cup$ REMOVE$\big(\mathscr{B}\setminus\{x^\alpha\},q-1\big)$;
    end for
    return borelIdeals;
end if

Algorithm 2.1: The algorithm computing the set of all saturated Borel-fixed ideals in a fixed polynomial ring with a fixed Hilbert polynomial.

$\textsc{BorelGenerator}\big(\mathbb{K}[x_0,x_1,x_2,x_3],3t+2\big)$

- $\textsc{BorelGenerator}\big(\mathbb{K}[x_1,x_2,x_3],3\big)$ (because $3t+2\neq 0$)
  - $\textsc{BorelGenerator}\big(\mathbb{K}[x_2,x_3],0\big)$ (because $3\neq 0$)
    - **return** $\{(1)\}$ in $\mathbb{K}[x_2,x_3]$
  - The Gotzmann number of $\Delta p(t)=3$ is 3
  - $\mathscr{B}_0=\big\{\big((1)\cdot\mathbb{K}[x_1,x_2,x_3]\big)_3\big\}=\mathcal{P}(2,3)$ and $q_0=3-|\mathscr{B}_0^{\mathcal{C}}|=3$
    - $\textsc{Remove}(\mathscr{B}_0,3)$
      - $\textsc{Remove}(\mathscr{B}_0\setminus\{x_1^3\},2)$
        - $\textsc{Remove}(\mathscr{B}_0\setminus\{x_1^3,x_2x_1^2\},1)$
          - $\textsc{Remove}(\mathscr{B}_0\setminus\{x_1^3,x_2x_1^2,x_2^2x_1\},0)$
          - $\textsc{Remove}(\mathscr{B}_0\setminus\{x_1^3,x_2x_1^2,x_3x_1^2\},0)$
  - **return** $\{(x_3,x_2^3),(x_3^2,x_3x_2,x_2^2)\}$ in $\mathbb{K}[x_1,x_2,x_3]$
- The Gotzmann number of $p(t)=3t+2$ is 5
- $\mathscr{B}_{1,1}=\big\{\big((x_3,x_2^3)\cdot\mathbb{K}[x_0,x_1,x_2,x_3]\big)_5\big\}\subset\mathcal{P}(3,5)$ and $q_{1,1}=17-|\mathscr{B}_{1,1}^{\mathcal{C}}|=2$
  - $\textsc{Remove}(\mathscr{B}_{1,1},2)$
    - $\textsc{Remove}(\mathscr{B}_{1,1}\setminus\{x_2^3x_0^2\},1)$
      - $\textsc{Remove}(\mathscr{B}_{1,1}\setminus\{x_2^3x_0^2,x_2^3x_1x_0\},0)$
      - $\textsc{Remove}(\mathscr{B}_{1,1}\setminus\{x_2^3x_0^2,x_3x_0^4\},0)$
    - $\textsc{Remove}(\mathscr{B}_{1,1}\setminus\{x_3x_0^4\},1)$
      - $\textsc{Remove}(\mathscr{B}_{1,1}\setminus\{x_3x_0^4,x_2^3x_0^2\},0)$ (already found)
      - $\textsc{Remove}(\mathscr{B}_{1,1}\setminus\{x_3x_0^4,x_3x_1x_0^3\},0)$
- $\mathscr{B}_{1,2}=\big\{\big((x_3^2,x_3x_2,x_2^2)\cdot\mathbb{K}[x_0,x_1,x_2,x_3]\big)_5\big\}\subset\mathcal{P}(3,5)$ and
  $q_{1,2}=17-|\mathscr{B}_{1,2}^{\mathcal{C}}|=1$
  - $\textsc{Remove}(\mathscr{B}_{1,2},1)$
    - $\textsc{Remove}(\mathscr{B}_{1,2}\setminus\{x_2^2x_0^3\},0)$
- **return** $\{(x_3,x_2^4,x_2^3x_1^2),(x_3^2,x_3x_2,x_3x_1,x_2^4,x_2^3x_1),(x_3^2,x_3x_2,x_2^3,x_3x_1^2),$
  $(x_3^2,x_3x_2,x_2^3,x_2^2x_1)\}$ in $\mathbb{K}[x_0,x_1,x_2,x_3]$

Table 2.1: The diagram of the execution of BorelGenerator with as inputs the polynomial ring $\mathbb{K}[x_0,x_1,x_2,x_3]$ and the Hilbert polynomial $p(t)=3t+2$.

The example described in Table 2.1 shows a first inaccuracy of the strategy, in fact BorelGenerator could compute many times the same ideal (the Borel set $\mathscr{B}_{1,1} \setminus \{x_3x_0^4, x_2^3x_0^2\}$ corresponding to the ideal $(x_3^2, x_3x_2, x_3x_1, x_2^4, x_2^3x_1)$ is obtained 2 times). To solve this problem, we can use a total order on the monomials, so we fix any term ordering σ, and then keep trace of the computation: we add as argument of the function Remove a monomial (that usually will be the last monomial removed) and we consider as monomials to remove only those greater than it.

RemoveUniqueness($\mathscr{B}, q, x^\beta$)
Input: $\mathscr{B}$, a Borel set.
Input: q, the number of monomials to remove from $\mathscr{B}$.
Input: x^β, monomial (usually it will be a maximal element of $\mathscr{B}$).
Output: the set of saturated Borel-fixed ideals obtained from Borel sets constructed from $\mathscr{B}$ removing in all the possible ways q monomials without repetitions.
if $q = 0$ **then**
  return $\{\langle \mathscr{B} \rangle^{\text{sat}}\}$;
else
  borelIdeals $\leftarrow \emptyset$;
  minimalMonomials $\leftarrow$ MinimalElements($\mathscr{B}$);
  for all $x^\alpha \in$ minimalMonomials **do**
    if $x^\alpha >_{\texttt{DegLex}} x^\beta$ **then**
      borelIdeals $\leftarrow$ borelIdeals $\cup$ RemoveUniqueness$\big(\mathscr{B} \setminus \{x^\alpha\}, q-1, x^\alpha\big)$;
    end if
  end for
  return borelIdeals;
end if

Algorithm 2.2: The modified version of Algorithm 2.1 to avoid repetitions of ideals.

We remark that Algorithm 2.1 could be naturally interpreted as an algorithm visiting a tree. Let us consider any Hilbert polynomial $p(t)$ of degree d, admissible for the projective space $\mathbb{P}^n$ (i.e. $d < n$). We can associate to the pair $(p(t), \mathbb{P}^n)$ the rooted tree defined as follows:

- the nodes are all Borel-fixed ideals of $\mathbb{K}[x_i,\ldots,x_n]$ with Hilbert polynomial $\Delta^i p(t)$, $\forall\, i = 0,\ldots,d+1$;
- the father of $I \subset \mathbb{K}[x_i,\ldots,x_n]$ is the ideal $J \subset \mathbb{K}[x_{i+1},\ldots,x_n]$ such that $J = \big(I|_{x_i=1} \cap \mathbb{K}[x_{i+1},\ldots,x_n]\big)^{\mathrm{sat}}$, that is J represents the hyperplane section of I w.r.t. x_i.

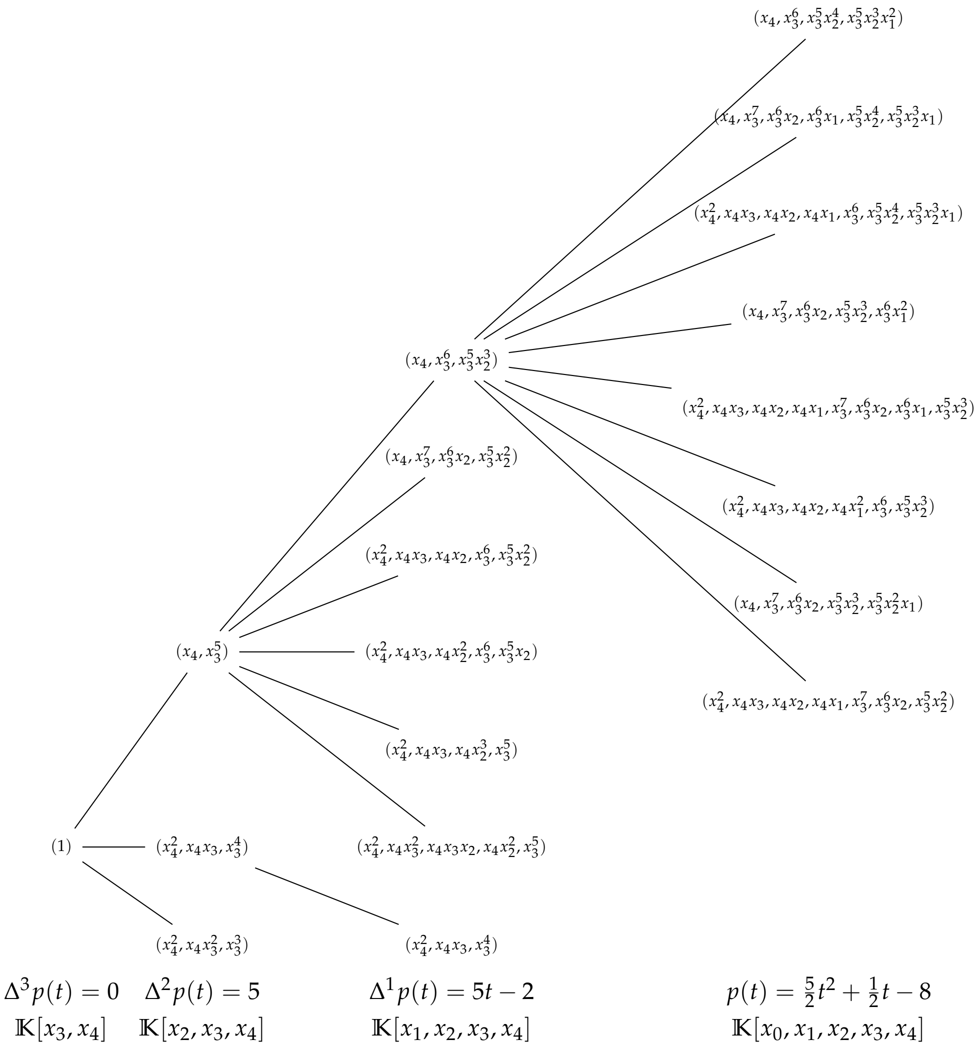


Figure 2.14: The tree of Borel-fixed ideals associated to $\mathbb{P}^4$ and $p(t) = \frac{5}{2}t^2 + \frac{1}{2}t - 8$.

With such a definition, we have that the root of the tree is the ideal $(1) \subset \mathbb{K}[x_{d+1},\ldots,x_n]$ if $d < n-1$, or the ideal $\big(x_n^{\Delta^d p(t)}\big) \subset \mathbb{K}[x_{n-1},x_n]$, if $d = n-1$, and the Borel-fixed ideals defining subschemes of $\mathbb{P}^n$ with Hilbert polynomial $p(t)$ are represented by the leaves at maximal distance from the root (see for an example Figure 2.14). Algorithm 2.1 turns out to be a BFS (Breadth First Search) on the tree, indeed to determine the leaves at maximal distance we have to examine before all the nodes closer to the root, that from a computational point of view means that we need to store in the memory of a computer all the intermediate steps. Figure 2.14 clearly shows that this approach could not be optimal also because generally there are many ideals that will be finally discarded by the algorithm (because imposing too many monomials outside the ideal) but that we keep in mind for a long time before examing them.

Therefore a better approach is to visit the nodes of the tree of Borel-fixed ideals by means of a DFS (Depth First Search) visiting algorithm, so that the algorithm discards an ideal (if necessary) immediatly after having determined it. In Algorithm 2.4 and Algorithm 2.3, there is the description of this strategy, which is used in the effective implementation `BorelIdeals` of the algorithm in the package `HSC` (see Appendix B).

BorelGeneratorHS$\big(\mathbb{K}[x_0,\ldots,x_n], p(t), k, I\big)$

Input: $\mathbb{K}[x_0,\ldots,x_n]$, polynomial ring.

Input: $p(t)$, admissible Hilbert polynomial in $\mathbb{P}^n = \operatorname{Proj}\mathbb{K}[x_0,\ldots,x_n]$.

Input: k, integer s.t. $0 \leqslant k \leqslant \deg p(t)$.

Input: I, Borel-fixed ideal in $\mathbb{K}[x_k,\ldots,x_n]$ s.t. $\mathbb{K}[x_k,\ldots,x_n]/I$ has Hilbert polynomial $\Delta^k p(t)$.

Output: the set of all Borel-fixed ideals J in $\mathbb{K}[x_0,\ldots,x_n]$ defining subschemes of $\mathbb{P}^n$ with Hilbert polynomial $p(t)$ s.t. $\big(J|_{x_0=\ldots=x_k=1}\big)^{\text{sat}} = I$.

if $k = 0$ **then**
  return $\{I\}$;
end if
$r \leftarrow$ GotzmannNumber$\big(\Delta^{k-1}p(t)\big)$;

$\mathscr{B} \leftarrow \big\{(I \cdot \mathbb{K}[x_{k-1},\ldots,x_n])_r\big\} \subset \mathcal{P}(n-k+1,r)$;
$q \leftarrow \Delta^{k-1}p(r) - \big|\mathscr{B}^{\mathcal{C}}\big|$;
if $q \geqslant 0$ **then**
  HS $\leftarrow$ RemoveUniqueness($\mathscr{B}, q, 0$);
  BorelIdeals $\leftarrow \emptyset$;
  for all $\widetilde{I} \in$ HS **do**
    BorelIdeals $\leftarrow$ BorelIdeals $\cup$ BorelGeneratorHS$\big(\mathbb{K}[x_0,\ldots,x_n], p(t), k-1, \widetilde{I}\big)$;
  end for
  return BorelIdeals;
else
  return $\emptyset$;
end if

Algorithm 2.3: The core of the DFS strategy to compute Borel-fixed ideals defining subschemes of $\mathbb{P}^n$ with Hilbert polynomial $p(t)$. This function visits a node and then calls itself on the children of the node.

BorelGeneratorDFS$\big(\mathbb{K}[x_0,\ldots,x_n], p(t)\big)$
Input: $\mathbb{K}[x_0,\ldots,x_n]$, polynomial ring.
Input: $p(t)$, admissible Hilbert polynomial in $\mathbb{P}^n = \operatorname{Proj}\mathbb{K}[x_0,\ldots,x_n]$.
Output: the set of all Borel-fixed ideals in $\mathbb{K}[x_0,\ldots,x_n]$ defining subschemes of $\mathbb{P}^n$ with Hilbert polynomial $p(t)$.
$d \leftarrow \deg p(t)$;
if $d = n-1$ **then**
  $c \leftarrow \Delta^d p(t)$;
  return BorelGeneratorHS$\big(\mathbb{K}[x_0,\ldots,x_n], p(t), d, (x_n^c)\big)$;
else
  return BorelGeneratorHS$\big(\mathbb{K}[x_0,\ldots,x_n], p(t), d+1, (1)\big)$
end if

Algorithm 2.4: This function detects the root of the tree associated to $\big(\mathbb{K}[x], p(t)\big)$ and then starts the DFS visit.

BorelGeneratorDFS$(\mathbb{K}[x_0,x_1,x_2,x_3], 3t+2)$

- $d = \deg 3t+2 = 1 < 2 = 3-1$
- BorelGeneratorHS$(\mathbb{K}[x_0,x_1,x_2,x_3], 3t+2, 2, (1))$
  - $\Delta^1(3t+2) = 3 \Rightarrow r = 3$, $\mathscr{B}_0 = \mathcal{P}(2,3)$ and $q_0 = 3$
    - RemoveUniqueness$(\mathscr{B}_0, 3, 0)$
      - RemoveUniqueness$(\mathscr{B}_0 \setminus \{x_1^3\}, 2, x_1^3)$
        - RemoveUniqueness$(\mathscr{B}_0 \setminus \{x_1^3, x_2x_1^2\}, 1, x_2x_1^2)$
          - RemoveUniqueness$(\mathscr{B}_0 \setminus \{x_1^3, x_2x_1^2, x_2^2x_1\}, 0, x_2^2x_1)$
          - RemoveUniqueness$(\mathscr{B}_0 \setminus \{x_1^3, x_2x_1^2, x_3x_1^2\}, 0, x_3x_1^2)$
    - **return** $\{(x_3, x_2^3), (x_3^2, x_3x_2, x_2^2)\}$
  - BorelGeneratorHS$(\mathbb{K}[x_0,x_1,x_2,x_3], 3t+2, 1, (x_3, x_2^3))$
    - $\Delta^0(3t+2) = 3t+2 \Rightarrow r = 5$,
      $\mathscr{B}_{1,1} = \{((x_3, x_2^3) \cdot \mathbb{K}[x_0,x_1,x_2,x_3])_5\}$ and $q_{1,1} = 17-15 = 2$
      - RemoveUniqueness$(\mathscr{B}_{1,1}, 2, 0)$
        - RemoveUniqueness$(\mathscr{B}_{1,1} \setminus \{x_2^3x_0^2\}, 1, x_2^3x_0^2)$
          - RemoveUniqueness$(\mathscr{B}_{1,1} \setminus \{x_2^3x_0^2, x_2^3x_1x_0\}, 0, x_2^3x_1x_0)$
          - RemoveUniqueness$(\mathscr{B}_{1,1} \setminus \{x_2^3x_0^2, x_3x_0^4\}, 0, x_3x_0^4)$
        - RemoveUniqueness$(\mathscr{B}_{1,1} \setminus \{x_3x_0^4\}, 1, x_3x_0^4)$
          - RemoveUniqueness$(\mathscr{B}_{1,1} \setminus \{x_3x_0^4, x_3x_1x_0^3\}, 0, x_3x_1x_0^3)$
    - **return** $\{(x_3, x_2^4, x_2^3x_1^2), (x_3^2, x_3x_2, x_3x_1, x_2^4, x_2^3x_1), (x_3^2, x_3x_2, x_2^3, x_3x_1^2)\}$
  - BorelGeneratorHS$(\mathbb{K}[x_0,x_1,x_2,x_3], 3t+2, 1, (x_3^2, x_3x_2, x_2^2))$
    - $\Delta^0(3t+2) = 3t+2 \Rightarrow r = 5$,
      $\mathscr{B}_{1,2} = \{((x_3^2, x_3x_2, x_2^2) \cdot \mathbb{K}[x_0,x_1,x_2,x_3])_5\}$ and $q_{1,2} = 17-16 = 1$
      - RemoveUniqueness$(\mathscr{B}_{1,2}, 1, 0)$
        - RemoveUniqueness$(\mathscr{B}_{1,2} \setminus \{x_3x_1x_0^3\}, 0, x_3x_1x_0^3)$
    - **return** $\{(x_3^2, x_3x_2, x_2^3, x_2^2x_1)\}$
- **return** $\{(x_3, x_2^4, x_2^3x_1^2), (x_3^2, x_3x_2, x_3x_1, x_2^4, x_2^3x_1), (x_3^2, x_3x_2, x_2^3, x_3x_1^2),$
  $(x_3^2, x_3x_2, x_2^3, x_2^2x_1)\}$ in $\mathbb{K}[x_0,x_1,x_2,x_3]$

Table 2.2: The diagram of the execution of BorelGeneratorDFS with as arguments $\mathbb{K}[x_0,x_1,x_2,x_3]$ and $3t+2$ (cf. Table 2.1).

2.6 How many Borel-fixed ideals are there?

An interesting question that naturally arises from the algorithm projected in the previous section is if we are able to predict the number of Borel-fixed ideals associated to any pairs $(n, p(t))$ without computing all them. In Table 2.3, there is a summary of the number of Borel-fixed ideals in the case of some constant Hilbert polynomial.

	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$
$p(t)=2$	1	1	1	1	1	1	1	1	1
$p(t)=3$	2	2	2	2	2	2	2	2	2
$p(t)=4$	2	3	3	3	3	3	3	3	3
$p(t)=5$	3	4	5	5	5	5	5	5	5
$p(t)=6$	4	6	7	8	8	8	8	8	8
$p(t)=7$	5	9	11	12	13	13	13	13	13
$p(t)=8$	6	12	16	18	19	20	20	20	20
$p(t)=9$	8	17	24	28	30	31	32	32	32
$p(t)=10$	10	24	35	42	46	48	49	50	50

Table 2.3: The number of Borel-fixed ideals in the case of constant Hilbert polynomials $p(t) = s$ in $\mathbb{P}^n$, for $2 \leqslant n \leqslant 10$ and $2 \leqslant s \leqslant 10$.

Definition 2.40. Given a projective space $\mathbb{P}^n$ and an admissible Hilbert polynomial $p(t)$, we denote by $\mathcal{B}^n_{p(t)}$ the set of all Borel-fixed ideals defining subschemes of $\mathbb{P}^n$ with Hilbert polynomial $p(t)$ and by $\mathcal{N}^n_{p(t)}$ its cardinality

$$\mathcal{N}^n_{p(t)} = \left|\mathcal{B}^n_{p(t)}\right|. \tag{2.22}$$

The dependence of $\mathcal{N}^n_{p(t)}$ on the Hilbert polynomial seems a very hard task to achieve, whereas already the numbers showed in Table 2.3 suggests that the behavior of $\mathcal{N}^n_{p(t)}$ varying n could be more treatable. Thus let us start discussing how the dimension n of the projective space affects $\mathcal{N}^n_{p(t)}$.

Remark 2.6.1. Let $p(t) = \binom{t+d}{d}$ be the Hilbert polynomial of a d-projective space contained in $\mathbb{P}^n$ for any $0 \leqslant d < n$. The Gotzmann number of such a Hilbert polynomial is 1, so that we consider posets of the type $\mathcal{P}(n, 1)$, that turn out to be totally ordered sets.

$$x_n \longrightarrow x_{n-1} \longrightarrow x_{n-2} \longrightarrow \cdots\cdots\cdots \longrightarrow x_1 \longrightarrow x_0$$

Therefore

$$\mathcal{B}^n_{\binom{t+d}{d}} = \big\{(x_n, \ldots, x_{d+1})\big\} \quad \Longrightarrow \quad \mathcal{N}^n_{\binom{t+d}{d}} = 1. \tag{2.23}$$

The inclusion $\mathbb{K}[x_0, \ldots, x_n] \hookrightarrow \mathbb{K}[x_0, \ldots, x_n, x_{x+1}]$ with $x_{n+1} >_B x_n$ naturally extends to the corresponding posets in any degree. Thinking about the characterization of Borel-fixed ideals given in Corollary 2.36 and looking for a setting that allows to consider simultaneously Borel-fixed ideals in any number of variables we introduce the following setting.

Definition 2.41. Let $\{x_0, \ldots, x_n, \ldots\}$ an infinite set of variables. We denote by $\mathcal{P}(m)$ the poset composed by monomials of degree m in the variables $\{x_0, \ldots, x_n, \ldots\}$ and given by the natural extension to an infinite number of variables of the elementary moves. Moreover for any finite poset $\mathcal{P}(n, m)$, we denote by $i_n : \mathcal{P}(n, m) \hookrightarrow \mathcal{P}(m)$ the inclusion map.

Now we need to extend the notion of Borel set.

Definition 2.42. A subset $\mathscr{B} \subset \mathcal{P}(m)$ is called *Borel set* if the complement $\mathscr{B}^{\mathcal{C}} = \mathcal{P}(m) \setminus \mathscr{B}$ is a finite order set, i.e. $\mathscr{B}^{\mathcal{C}}$ is closed w.r.t. decreasing elementary moves.

For any n, we define (and denote again with i_n) the map

$$\begin{array}{rccc} i_n : & \{\text{Borel sets of } \mathcal{P}(n, m)\} & \longrightarrow & \{\text{Borel sets of } \mathcal{P}(m)\} \\ & \mathscr{B} & \longmapsto & \big(i_n(\mathscr{B}^{\mathcal{C}})\big)^{\mathcal{C}} \end{array} \tag{2.24}$$

so that the order sets defined by $\mathscr{B}$ and $i_n(\mathscr{B})$ are equal, and the map

$$\begin{array}{rccc} s_n : & \{\text{Borel sets of } \mathcal{P}(m)\} & \longrightarrow & \{\text{Borel sets of } \mathcal{P}(n, m)\} \\ & \mathscr{B} & \longmapsto & \mathscr{B} \cap \mathcal{P}(n, m) \end{array}. \tag{2.25}$$

Obviously $s_n(i_n(\mathscr{B})) = \mathscr{B}$.

Given a Hilbert polynomial $p(t)$ with Gotzmann number r, let us now consider the inclusion

$$i_n : \left\{ \begin{array}{c} \mathscr{B} \subset \mathcal{P}(n, r) \text{ Borel set s.t.} \\ \text{set } \mathscr{N} = \mathcal{P}(n, r) \setminus \mathscr{B} \\ \left|\mathscr{N}_{(\geqslant i)}\right| = \Delta^i p(r),\ \forall\, i \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \mathscr{B} \subset \mathcal{P}(r) \text{ Borel set s.t.} \\ \text{set } \mathscr{N} = \mathcal{P}(r) \setminus \mathscr{B} \\ \left|\mathscr{N}_{(\geqslant i)}\right| = \Delta^i p(r),\ \forall\, i \end{array} \right\}. \tag{2.26}$$

that results still well-defined. To understand how the number of Borel-fixed ideals of $\mathbb{K}[x]$ with fixed Hilbert polynomial is affected by n, we will discuss the property of the map (2.26).

Definition 2.43. Let $\mathscr{N}$ be a finite order set (either in $\mathcal{P}(n,m)$ or $\mathcal{P}(m)$). We define the maximal variable of $\mathscr{N}$ as

$$\max\operatorname{var}\mathscr{N} = \max\left\{\max x^{\beta} \mid x^{\beta} \in \mathscr{N}\right\}. \tag{2.27}$$

Proposition 2.44. *Let $I \subset \mathbb{K}[x]$ be a saturated Borel-fixed ideal and let $\mathscr{N} = \{I_r\}^{\mathcal{C}} \subset \mathcal{P}(n,r)$ be the corresponding order set, where r is the Gotzmann number of the Hilbert polynomial of $\mathbb{K}[x]/I$. Then*

$$x_j \text{ is a minimal generator of } I \quad \Longleftrightarrow \quad \max\operatorname{var}\mathscr{N} < j.$$

Proof. ($\Leftarrow$) If $j > \max\operatorname{var}\mathscr{N}$, then $x_j x_0^{r-1}$ belongs to $\{I_r\}$, so x_j is a generator of I.

($\Rightarrow$) Since $x_j \in I$, $x_j x_0^{r-1}$ does not belong to $\mathscr{N}$. Moreover any other monomial x^{α} of degree r such that $\max x^{\alpha} \geqslant j$ cannot belong to $\mathscr{N}$, because $x^{\alpha} \geq_B x_j x_0^{r-1}$. In fact $x^{\alpha} \geq_B \frac{x^{\alpha}}{\max x^{\alpha}} x_j \geq_B x_j x_0^{r-1}$ since x_0^{r-1} is the minimum w.r.t. the Borel order among monomials of degree $r-1$. $\square$

Proposition 2.45. *Let $\mathscr{B} \subset \mathcal{P}(m)$ be a Borel set. If $\max\operatorname{var}\mathscr{B}^{\mathcal{C}} = j$, then*

$$i_n\left(s_n\left(\mathscr{B}\right)\right) = \mathscr{B}, \qquad \forall\, n \geqslant j. \tag{2.28}$$

Proof. $\max\operatorname{var}\mathscr{B}^{\mathcal{C}} = j$ implies $\mathscr{B}^{\mathcal{C}} \subset \mathcal{P}(n,m)$, for all $n \geqslant j$. $\square$

Lemma 2.46. *Let $\mathscr{N} \subset \mathcal{P}(m)$ be any order set such that $|\mathscr{N}| = a$. Then*

$$\max\operatorname{var}\mathscr{N} \leqslant a-1. \tag{2.29}$$

Proof. It comes directly from the remark that to reach x_0^m from a monomial x^{α} s.t. $\max x^{\alpha} > a-1$ we need at least a decreasing elementary moves, i.e. any order set containing x^{α} would contain at least $a+1$ elements.

The bound is sharp, indeed for the order set $\overline{\mathscr{N}} = \{x_0^m, x_0^{m-1}x_1, \ldots, x_0^{m-1}x_{a-1}\}$ $|\overline{\mathscr{N}}| = a$ and $\max\operatorname{var}\overline{\mathscr{N}} = a-1$. $\square$

Proposition 2.47. *Let $p(t)$ be an admissible Hilbert polynomial with Gotzmann number r. Then for any $n \geqslant p(r) - 1$*

$$\left\{\begin{array}{c} \mathscr{B} \subset \mathcal{P}(n,r) \text{ Borel set s.t.} \\ \text{set } \mathscr{N} = \mathcal{P}(n,r) \setminus \mathscr{B} \\ \left|\mathscr{N}_{(\geqslant i)}\right| = \Delta^i p(r),\ \forall\, i \end{array}\right\} \xleftrightarrow{1:1} \left\{\begin{array}{c} \mathscr{B} \subset \mathcal{P}(n+1,r) \text{ Borel set s.t.} \\ \text{set } \mathscr{N} = \mathcal{P}(n+1,r) \setminus \mathscr{B} \\ \left|\mathscr{N}_{(\geqslant i)}\right| = \Delta^i p(r),\ \forall\, i \end{array}\right\} \tag{2.30}$$

Proof. Using the Borel sets in the infinite poset $\mathcal{P}(r)$ as intermediate step, the maps giving the bijection are $s_{n+1} \circ i_n$ and $s_n \circ i_{n+1}$ □

From the point of view of saturated Borel-fixed ideals, since $n+1$ is surely greater than $\max \operatorname{var} \mathscr{N}$, the correspondence turns out to be

$$\begin{array}{ccc} \left\{\begin{array}{c} J \text{ saturated Borel-fixed} \\ \text{ideal s.t. } \mathbb{K}[x_0,\ldots,x_n]/J \text{ has} \\ \text{Hilbert polynomial } p(t) \end{array}\right\} & \longrightarrow & \left\{\begin{array}{c} J' \text{ saturated Borel-fixed} \\ \text{ideal s.t. } \mathbb{K}[x_0,\ldots,x_{n+1}]/J' \text{ has} \\ \text{Hilbert polynomial } p(t) \end{array}\right\} \\ J & \longmapsto & (x_{n+1}, J). \end{array}$$

Corollary 2.48. *Let $p(t)$ be an admissible Hilbert polynomial with Gotzmann number r. The number $\mathcal{N}^n_{p(t)}$ is constant for $n \geqslant p(r) - 1$.*

Definition 2.49. We denote by $\mathcal{N}^{\bullet}_{p(t)}$ the sequence of the the number of Borel-fixed ideals associated to a fixed Hilbert polynomial $p(t)$ and varying number of variables n:

$$\mathcal{N}^{\bullet}_{p(t)} = \left(\ldots, \mathcal{N}^n_{p(t)}, \ldots\right) \tag{2.31}$$

If $d = \deg p(t)$, then $p(t)$ is admissible in $\mathbb{P}^n$ for $n > d$, so the first integer of the sequence will be always $\mathcal{N}^{d+1}_{p(t)}$. Moreover since by Corollary 2.48 the sequence at some point becomes constant, we could write the sequence as a finite list of integer

$$\mathcal{N}^{\bullet}_{p(t)} = \left(\mathcal{N}^{d+1}_{p(t)}, \ldots, \mathcal{N}^{A}_{p(t)}\right)$$

meaning that $\mathcal{N}^{A+k}_{p(t)} = \mathcal{N}^{A}_{p(t)},\ \forall\, k \geqslant 0$.

In general, we expect that the bound $p(r) - 1$ is an overestimation of the point of stabilization of $\mathcal{N}^n_{p(t)}$, because the order set $\overline{\mathscr{N}} = \{x_0^r, \ldots, x_{p(r)-1}x_0^{r-1}\}$ constructed in Lemma 2.46 corresponds to a Borel-fixed ideal in any $\mathbb{K}[x_0,\ldots,x_n]$, $n \geqslant p(r)-1$ with *constant* Hilbert polynomial $\overline{p}(t) = p(r)$, indeed the hyperplane section is the ideal $(1) \subset \mathbb{K}[x_1,\ldots,x_n]$. For this reason, we carry on with a more detailed analisys.

Let $p(t)$ be an admissible Hilbert polynomial of degree d with Gotzmann number r. Thinking about the recursive strategy of Algorithm 2.4, we want determine the Borel-fixed ideal defining the order set $\mathscr{N}$ with maximum $\max\operatorname{var}\mathscr{N}$ among the ideals with a given hyperplane section $\overline{I} \subset \mathbb{K}[x_1,\ldots,x_n]$, for some n.

Let $\overline{\mathscr{B}} = \left\{(\overline{I}\cdot\mathbb{K}[x_0,\ldots,x_n])_r\right\} \subset \mathcal{P}(n,r)$ and let $\overline{\mathscr{N}}$ be the associated order set, viewed in the infinite poset $\mathcal{P}(r)$. We saw that the Hilbert polynomial $\overline{p}(t)$ associated to $\overline{\mathscr{N}}$ differs from the Hilbert polynomial $p(t)$ by a constant (Proposition 2.32). Set $c = p(t) - \overline{p}(t)$, to determine an order set corresponding to $p(t)$ we have to add c monomials to $\overline{\mathscr{N}}$ and we want to achieve it using as many variables as possible. Let $A = \max\operatorname{var}\overline{\mathscr{N}}$ ($= \max\operatorname{var}\{\overline{I}_r\}^{\mathcal{C}}$ by construction); by Proposition 2.44, we know that $x_{A+1}x_0^{r-1} \notin \overline{\mathscr{N}}$ and moreover this monomial is minimal in $\overline{\mathscr{N}}^{\mathcal{C}}$, so $\overline{\mathscr{N}} \cup \{x_{A+1}x_0^{r-1}\}$ is still an order set and

$$\max\operatorname{var}\left(\overline{\mathscr{N}} \cup \{x_{A+1}x_0^{r-1}\}\right) = A+1 = \max\operatorname{var}\overline{\mathscr{N}} + 1.$$

Repeating the reasoning c times we construct the order set $\mathscr{N} = \overline{\mathscr{N}} \cup \{x_{A+1}x_0^{r-1}, \ldots, x_{A+c}x_0^{r-1}\}$ and

$$\begin{aligned}\max\operatorname{var}\mathscr{N} &= \max\operatorname{var}\left(\overline{\mathscr{N}} \cup \{x_{A+1}x_0^{r-1},\ldots,x_{A+c}x_0^{r-1}\}\right) = \\ &= A + c = \max\operatorname{var}\overline{\mathscr{N}} + c.\end{aligned}$$

We summarize this construction in the following lemma.

Lemma 2.50. *Let $p(t)$ be an admissible Hilbert polynomial with Gotzmann number r and let $\overline{I} \subset \mathbb{K}[x_1,\ldots,x_n]$ the saturated Borel-fixed ideal of an admissible hyperplane section of $p(t)$. Moreover let $\overline{p}(t)$ be the Hilbert polynomial associated to $\overline{I}\cdot\mathbb{K}[x_0,\ldots,x_n]$ and $c = p(t) - \overline{p}(t)$.*

(i) $\max\left\{\max\mathscr{N} \text{ s.t. } |\mathscr{N}| = p(r) \text{ and } \mathscr{N}_{(\geqslant 1)} = \{\overline{I}_r\}^{\mathcal{C}}\right\} = \max\operatorname{var}\{\overline{I}_r\}^{\mathcal{C}} + c.$

(ii) Any Borel-fixed ideal $I \subset \mathbb{K}[x_0,\ldots,x_n]$ associated to $p(t)$ with hyperplane section $\overline{I}$, contains as minimal generator x_j, $\max\operatorname{var}\{\overline{I}_r\}^{\mathcal{C}} + c < j \leqslant n$.

So to estimate a bound, we can consider among the hyperplane sections defining order sets with same maximum the one with smallest Hilbert polynomial.

Proposition 2.51. *Let $p(t)$ be an admissible Hilbert polynomial of degree d with Gotzmann number r and let $L \subset \mathbb{K}[x_1,\ldots,x_n]$ the saturated lexicographic ideal associated to $\Delta p(t)$. For any ideal $I \subset \mathbb{K}[x_0,\ldots,x_n]$ having L as hyperplane section*

$$\max\operatorname{var}\{I_r\}^{\mathcal{C}} \leqslant \begin{cases} d + p(t) - \Sigma(\Delta p)(t), & \text{if } p(t) = \binom{t+d}{d} + c,\\ d + p(t) - \Sigma(\Delta p)(t) + 1, & \text{otherwise.}\end{cases} \tag{2.32}$$

Proof. To determine the smallest number of variables that we need to determine an order set in $\mathcal{P}(n-1,r)$ associated to $\Delta p(t)$, we think to Macaulauy's Theorem [60]. It says that whenever the Hilbert polynomial is admissible a lexicographic ideal L realizing it exists, i.e. the lexicographic ideal has to be the ideal with the order set involving the smallest number of variables.

If $p(t) = \binom{t+d}{d} + c$, then $\Delta p(t) = \binom{t+d-1}{d-1}$ and by Remark 2.6.1 the lexicographic ideal is the only Borel-fixed ideal $(x_n,\ldots,x_{d+1}) \subset \mathbb{K}[x_1,\ldots,x_n]$, so that by Proposition 2.44

$$\max\operatorname{var}\{(x_n,\ldots,x_{d+1})_r\}^{\mathcal{C}} = d.$$

For any other Hilbert polynomial the condition of admissibility is $n > d$, so the order set $\{L_r\}^{\mathcal{C}} \subset \mathcal{P}(n-1,r)$ will involve the variables $x_1,\ldots,x_{d+1}$ (regardless of n) and

$$\max\operatorname{var}\{L_r\}^{\mathcal{C}} = d+1.$$

Let D be the maximum of the order set defined by the lexicographic ideal.

In the proof of Proposition 2.39 we showed that the ideal $L\cdot\mathbb{K}[x_0,\ldots,x_n]$ is associated to the minimal polynomial among those with first difference equal to $\Delta p(t)$ and by Definition 2.24 such minimal polynomial is $\Sigma(\Delta p)(t)$. Hence called $\mathscr{N}$ the order set defined by $\{(L\cdot\mathbb{K}[x_0,\ldots,x_n])_r\}^{\mathcal{C}}$, to obtain an order set associated to $p(t)$ we need add $p(t) - \Sigma(\Delta p)(t)$ monomials. With the goal of constructing the order ideal involving as many variables as possible, we begin adding $x_{D+1}x_0^{r-1}$, since $\max\operatorname{var}\mathscr{N} = D \Rightarrow x_{d+1}x_0^{r-1} \notin \mathscr{N}$. Repeating the reasoning $p(t) - \Sigma(\Delta p)(t)$ times we can obtain as limit case an order set with maximum variable equal to $D + p(t) - \Sigma(\Delta p)(t)$. $\square$

Now we compare the lexicographic hyperplane section with any other section defining an order set with maximum greater than the maximum of the order set defined by the lexicographic ideal associated to $\Delta p(t)$ (d or $d+1$).

Lemma 2.52. *Let $\overline{I} \subset \mathbb{K}[x_1,\ldots,x_n]$ be an admissible hyperplane sections for the Hilbert polynomial $p(t)$ of degree d with Gotzmann number r and let $\overline{p}(t)$ the Hilbert polynomial associated to $\overline{I} \cdot \mathbb{K}[x_0,\ldots,x_n]$. Moreover let D be the maximum of the order set defined by the lexicographic ideal that realizes $\Delta p(t)$. If $\max\operatorname{var}\{\overline{I}_r\}^{\mathcal{C}} = D + B$, then*

(i) $\overline{p}(t) \geqslant \Sigma\big(\Delta p\big)(t) + B$;

(ii) $\max\operatorname{var}\{\overline{I}_r\}^{\mathcal{C}} + p(t) - \overline{p}(t) \leqslant D + p(t) - \Sigma\big(\Delta p\big)(t)$.

Proof. (i) Preliminarly we can simplify the problem considering the case $B = 1$ and then proceeding iteratively.

Furthermore we suppose that $\{\overline{I}_r\}^{\mathcal{C}}$ contains a single monomial with maximum variable equal to $D+1$, i.e. the monomial $x_{D+1}x_1^{r-1}$, because we want to not move too away from the lexicographic ideal in order to preserve a small polynomial $\overline{p}(t)$ (this idea, at this point almost intuitive, will be clarified in Chapter 3).

Applying Lemma 2.37 and Lemma 2.46 we know that

$$\overline{p}(t) = p(t) - \left(p(r) - \sum_{x^\alpha \in \{\overline{I}_r\}^{\mathcal{C}}} (\alpha_1 + 1)\right).$$

The order sets defined by $\overline{I}$ and by the lexicographic ideal surely differs for at least 1 monomial: $x_{D+1}x_1^{r-1}$. It is replaced necessarly by a monomial divided by a power of x_1 lower than $r-1$ because all the monomials divided by a power of x_1 greater than or equal to $r-1$ already belong to $\{\overline{I}\}^{\mathcal{C}}$, so

$$\sum_{x^\alpha \in \{\overline{I}_r\}^{\mathcal{C}}} (\alpha_1 + 1) > \Sigma\big(\Delta p\big)(r) \qquad \Longrightarrow \qquad (i) \quad \overline{p}(t) > \Sigma\big(\Delta p\big)(t)$$

(ii) By the inequality *(i)*

$$p(t) - \overline{p}(t) < p(t) - \Sigma\big(\Delta p\big)(t) \;\Leftrightarrow\; p(t) - \overline{p}(t) \leqslant p(t) - \Sigma\big(\Delta p\big)(t) - 1$$

and finally

$$\max\operatorname{var}\{\overline{I}_r\}^{\mathcal{C}} + p(t) - \overline{p}(t) = D + 1 + p(t) - \overline{p}(t) \leqslant D + p(t) - \Sigma\big(\Delta p\big)(t). \qquad \square$$

Theorem 2.53. *Let $p(t)$ be an admissible Hilbert polynomial of degree d with Gotzmann number r. For any $\mathcal{N} \subset \mathcal{P}(r)$ order set defined by a Borel-fixed ideal associated to $p(t)$,*

$$\max\operatorname{var}\mathcal{N} \leqslant \begin{cases} d + p(t) - \Sigma(\Delta p)(t), & \text{if } p(t) = \binom{t+d}{d} + c \\ d + p(t) - \Sigma(\Delta p)(t) + 1, & \text{otherwise.} \end{cases} \tag{2.33}$$

Proof. It comes directly applying Lemma 2.52 and Proposition 2.51. Moreover Proposition 2.51 ensures that the bound is sharp. □

Corollary 2.54. *Let $p(t)$ be an admissible Hilbert polynomial of degree d with Gotzmann number r. Then,*

$$\mathcal{N}^{n}_{p(t)} = \mathcal{N}^{n+1}_{p(t)}, \qquad \forall\, n \geqslant \begin{cases} d + p(t) - \Sigma(\Delta p)(t), & \text{if } p(t) = \binom{t+d}{d} + c \\ d + p(t) - \Sigma(\Delta p)(t) + 1, & \text{otherwise.} \end{cases} \tag{2.34}$$

Example 2.6.2. Let us check empirically the statement of Corollary 2.54 on some examples, with the help of the java function `BorelIdeals` of the package `HSC`.
$\boldsymbol{p(t) = 5t+1}$**.** From the Gotzmann representation

$$\begin{aligned} 5t+1 = {} & \binom{t+1}{1} + \binom{t}{1} + \binom{t-1}{1} + \binom{t-2}{1} + \binom{t-3}{1} + \\ & + \binom{t-5}{0} + \binom{t-6}{0} + \binom{t-7}{0} + \binom{t-8}{0} + \binom{t-9}{0} + \binom{t-10}{0} \end{aligned}$$

we can easily compute

$$5t+1 - \Sigma\big(\Delta(5t+1)\big) = 5t+1 - (5t-5) = 6$$

so that the sequence $\mathcal{N}^{\bullet}_{5t+1}$ is supposed to stabilize for $n = 2+6 = 8$. In fact

$$\mathcal{N}^{\bullet}_{5t+1} = (4, 38, 71, 89, 95, 97, 98, 98, \ldots).$$

$\boldsymbol{p(t) = \frac{3}{2}t^2 + \frac{7}{2}t}$**.** The Gotzmann representation is

$$\begin{aligned} \frac{3}{2}t^2 + \frac{7}{2}t = {} & \binom{t+2}{2} + \binom{t+1}{2} + \binom{t}{2} + \binom{t-2}{1} + \binom{t-3}{1} + \\ & + \binom{t-5}{0} + \binom{t-6}{0} + \binom{t-7}{0} + \binom{t-8}{0} \end{aligned}$$

so
$$\frac{3}{2}t^2+\frac{7}{2}t+1-\Sigma\left(\Delta\left(\frac{3}{2}t^2+\frac{7}{2}t\right)\right)=\frac{3}{2}t^2+\frac{7}{2}t-\left(\frac{3}{2}t^2+\frac{7}{2}t-4\right)=4$$
and the sequence $\mathcal{N}^\bullet_{\frac{3}{2}t^2+\frac{7}{2}t}$ levels off for $n=3+4=7$. In fact
$$\mathcal{N}^\bullet_{\frac{3}{2}t^2+\frac{7}{2}t}=(8,27,36,39,40,40,\ldots).$$

$\boldsymbol{p(t)=\frac{1}{3}t^3+\frac{3}{2}t^2+\frac{25}{6}t+3}$**.** The Gotzmann representation is
$$\frac{1}{3}t^3+\frac{3}{2}t^2+\frac{25}{6}t+3=\binom{t+3}{3}+\binom{t+2}{3}+\binom{t-1}{1}+\binom{t-2}{1}+ \\ +\binom{t-4}{0}+\binom{t-5}{0}+\binom{t-6}{0}+\binom{t-7}{0}+\binom{t-8}{0}$$
so
$$\frac{1}{3}t^3+\frac{3}{2}t^2+\frac{25}{6}t+3-\Sigma\left(\Delta\left(\frac{1}{3}t^3+\frac{3}{2}t^2+\frac{25}{6}t+3\right)\right)=5$$
and the sequence $\mathcal{N}^\bullet_{\frac{1}{3}t^3+\frac{3}{2}t^2+\frac{19}{6}t+5}$ levels off for $n=4+5=9$. In fact
$$\mathcal{N}^\bullet_{\frac{1}{3}t^3+\frac{3}{2}t^2+\frac{25}{6}t+3}=(21,84,130,149,155,156,156,\ldots).$$

2.6.1 The special case of constant Hilbert polynomials

We will now discuss in depth the case of constant Hilbert polynomial. The bound given in Corollary 2.54, if $p(t)=s=\binom{t+0}{0}+(s-1)$, turns out to be $s-1$.

Definition 2.55. For any constant Hilbert polynomial $p(t)=s$, we define for $1\leqslant i\leqslant s-2$
$$\Delta\mathcal{N}^i_s=\mathcal{N}^{s-i}_s-\mathcal{N}^{s-i-1}_s \tag{2.35}$$
and
$$\Delta\mathcal{N}^\bullet_s=\left(\Delta\mathcal{N}^1_s,\Delta\mathcal{N}^2_s,\ldots,\Delta\mathcal{N}^{s-2}_s\right) \tag{2.36}$$

Example 2.6.3. Let us consider the Hilbert polynomial $p(t)=15$. Since
$$\mathcal{N}^\bullet_{15}=(1,27,107,206,287,342,377,398,410,417,421,423,424),$$
we obtain
$$\Delta\mathcal{N}^\bullet_{15}=(1,1,2,4,7,12,21,35,55,81,99,80,26).$$

Experimentally we noticed that $\Delta\mathcal{N}_s^i$ for $s \gg 0$ becomes constant. In the following proposition we explain this behavior.

Proposition 2.56. *$\Delta\mathcal{N}_s^i$ is constant for $s \geqslant 2i-1$.*

Proof. Since $\Delta\mathcal{N}_s^i = \mathcal{N}_s^{s-i} - \mathcal{N}_s^{s-i-1}$, we are interested in studying order sets $\mathscr{N} \subset \mathcal{P}(s)$, such that $\max\operatorname{var}\mathscr{N} = s-i$, indeed if $\max\operatorname{var}\mathscr{N} < s-i$, $\mathscr{N}$ can be defined by an ideal in $\mathbb{K}[x_0,\dots,x_n]$ with $n \leqslant s-i-1$.

By Proposition 2.44, if $\max\operatorname{var}\mathscr{N} = s-i$ then $x_{s-i}x_0^{s-1}$ belongs to $\mathscr{N}$ and being $\mathscr{N}$ closed by decreasing elementary moves

$$\{x_{s-i}x_0^{s-1}, x_{s-i-1}x_0^{s-1},\dots,x_0^s\} \subset \mathscr{N}.$$

Thus we have to study the remaining

$$\left|\mathscr{N} \setminus \{x_{s-i}x_0^{s-1}, x_{s-i-1}x_0^{s-1},\dots,x_0^s\}\right| = s-(s-i+1) = i-1$$

monomials (not depending on s!). The point is again to determine which is the greatest number of variables involved in a subset $\mathscr{M}$ such that $\mathscr{N} = \mathscr{M} \cup \{x_{s-i}x_0^{s-1},\dots,x_0^s\}$. The Borel set $\mathcal{P}(s-i,s) \setminus \{x_{s-i}x_0^{s-1},\dots,x_0^s\}$ has only one minimal monomial: $x_1^2x_0^{s-2}$, so surely $x_1^2x_0^{s-2} \in \mathscr{M}$. Then the strategy is always the same: we choose the next monomial to add applying succesively the increasing elementary moves with the index as big as possible, i.e.

$$x_1^2x_0^{s-2} \xrightarrow{\mathsf{e}_1^+} x_2x_1x_0^{s-2} \xrightarrow{\mathsf{e}_2^+} x_3x_1x_0^{s-2} \xrightarrow{\mathsf{e}_3^+} \cdots$$

In order for being able to apply this strategy $i-2$ times, that is being able to construct the order set

$$\{x_{s-i}x_0^{s-1},\dots,x_0^s\} \cup \{x_1^2x_0^{s-2},\dots,x_{i-1}x_1x_0^{s-2}\}$$

we need $i-1 \leqslant s-i \;\Rightarrow\; s \geqslant 2i-1$. □

Definition 2.57. We define

$$\Delta\mathcal{N}^i = \Delta\mathcal{N}_s^i, \qquad s \gg 0 \tag{2.37}$$

and we denote by $\Delta\mathcal{N}^\bullet$ the sequence

$$\left(\Delta\mathcal{N}^1, \Delta\mathcal{N}^2, \dots, \Delta\mathcal{N}^i, \dots\right). \tag{2.38}$$

The optimal way to compute $\Delta\mathcal{N}^i$ is to consider $\Delta\mathcal{N}^i_{2i-1} = \mathcal{N}^{i-1}_{2i-1} - \mathcal{N}^{i-2}_{2i-1}$, that is to count the saturated ideals of $\mathcal{N}^{i-1}_{2i-1}$ not having variables as generator. The first values of the sequence are

$$1, 1, 2, 4, 7, 12, 21, 35, 58, 96, 156, 251, 403, 639, 1008, 1582, 2465, 3821, 5898, 9055, \ldots.$$

Proposition 2.56 (and its proof) suggests a new strategy for designing an algorithm computing Borel-fixed ideals with constant Hilbert polynomial. Let us consider the Hilbert polynomial $p(t) = s$ and the polynomial ring $\mathbb{K}[x]$. The sequences $\Delta\mathcal{N}^\bullet_s$ coincides with $\Delta\mathcal{N}^\bullet$ for the first i values with $i \leqslant C = \lfloor \frac{s+1}{2} \rfloor$. There are three cases.

$n \geqslant s$**.** By Corollary 2.54, we know that all the possible order sets involve at most $s-1$ variables, hence we compute the ideals $\mathcal{B}^{s-1}_s$ and

$$\mathcal{B}^n_s = \left\{ (x_n, \ldots, x_s, I) \mid I \in \mathcal{B}^{s-1}_s \right\}.$$

$n < s - C$**.** We compute $\mathcal{B}^n_s$ using Algorithm 2.4.

$s - C \leqslant n \leqslant s-1$**.** We compute $\mathcal{B}^{s-C-1}_s$ using Algorithm 2.4 and then we apply repeatedly Proposition 2.56. Let us suppose to have computed $\mathcal{B}^k_s$. The set $\mathcal{B}^{k+1}_s$ surely contains the set

$$\left\{ (x_{k+1}, I) \quad \middle| \quad I \in \mathcal{B}^k_s \right\}$$

to which we would add the Borel-ideals defining order set with maximum equal to $k+1$. We are looking at

$$\mathcal{N}^{k+1}_s - \mathcal{N}^k_s = \Delta\mathcal{N}^i_s, \qquad i = s-k-1,$$

new order sets and the best way to study them is considering the Borel ideals in $\mathcal{B}^{i-1}_{2i-1}$. Let $\mathcal{O}$ the set of order sets with maximum equal to $i-1$. The order sets we are looking for are

$$\overline{\mathcal{O}} = \left\{ \{x_{k+1}x_0^{s-1}, \ldots, x_i x_0^{s-1}\} \cup x_0^{s-2i+1} \cdot \mathscr{N} \mid \mathscr{N} \in \mathcal{O} \right\}.$$

BorelGeneratorConstantHP($\mathbb{K}[x_0,\dots,x_n], s$)
Input: $\mathbb{K}[x_0,\dots,x_n]$, polynomial ring.
Input: s, positive integer (the Hilbert polynomial).
Output: the set of all Borel-fixed ideals in $\mathbb{K}[x_0,\dots,x_n]$ with Hilbert polynomial $p(t) = s$.
$i \leftarrow s - n$;
if $i \leqslant 0$ **then**
  idealsLessVars $\leftarrow$ BorelGeneratorConstantHP($\mathbb{K}[x_0,\dots,x_{s-1}], s$);
  borelIdeals $\leftarrow \emptyset$;
  for all $I \in$ idealsLessVars **do**
    borelIdeals $\leftarrow$ borelIdeals $\cup \{(x_n,\dots,x_s, I)\}$;
  end for
  return borelIdeals;
else if $i > \lfloor \frac{s+1}{2} \rfloor$ **then**
  return BorelGeneratorDFS($\mathbb{K}[x_0,\dots,n_n], s$);
else
  idealsLessVars $\leftarrow$ BorelGeneratorConstantHP($\mathbb{K}[x_0,\dots,x_{n-1}], s$);
  idealsLinGen $\leftarrow \emptyset$;
  for all $I \in$ idealsLessVars **do**
    idealsLinGen $\leftarrow$ idealsLinGen $\cup \{(x_n, I)\}$;
  end for
  idealsDelta $\leftarrow$ BorelGeneratorDFS($\mathbb{K}[x_0,\dots,x_{i-1}], 2i-1$);
  idealsWithoutLinGen $\leftarrow \emptyset$;
  for all $J \in$ idealsDelta **do**
    if $\max\{J_s\}^{\mathcal{C}} = i - 1$ **then**
      $\mathscr{N} \leftarrow \{J_s\}^{\mathcal{C}} \cup \{x_i x_0^{s-1},\dots,x_n x_0^{s-1}\}$;
      idealsWithoutLinGen $\leftarrow$ idealsWithoutLinGen $\cup \left\{\langle \mathscr{N}^{\mathcal{C}} \rangle^{\mathrm{sat}}\right\}$;
    end if
  end for
  return idealsLinGen $\cup$ idealsWithoutLinGen;
end if

Algorithm 2.5: A new strategy for computing Borel-fixed ideals with constant Hilbert polynomial.

Example 2.6.4. Let us see how the strategy works in the concrete case of the polynomial ring $\mathbb{K}[x_0,\ldots,x_6]$ and the Hilbert polynomial $p(t) = 10$. Following Algorithm 2.5, we have that $i = 10-6 = 4$ and since $s = 10 > 7 = 2i-1$, we are sure that $\Delta\mathcal{N}^4_{10} = \Delta\mathcal{N}^4$.

Among the Borel-ideals with Hilbert polynomial $2i-1 = 7$ in the polynomial ring $\mathbb{K}[x_0,\ldots,x_{i-1}] = \mathbb{K}[x_0,x_1,x_2,x_3]$, there are 4 saturated ideals without a linear generator, precisely

$$\begin{aligned}
J_1 &= (x_3^2, x_3x_2, x_2^2, x_3x_1, x_2x_1, x_1^5) \ \Rightarrow\ \{(J_1)_{10}\}^{\mathcal{C}} = \mathscr{M} \cup \{x_1^2x_0^8, x_1^3x_0^7, x_1^4x_0^6\}\\
J_2 &= (x_3^2, x_3x_2, x_2^2, x_3x_1, x_2x_1^2, x_1^4) \ \Rightarrow\ \{(J_2)_{10}\}^{\mathcal{C}} = \mathscr{M} \cup \{x_1^2x_0^8, x_2x_1x_0^8, x_1^3x_0^7\}\\
J_3 &= (x_3^2, x_3x_2, x_3x_1, x_2^3, x_2^2x_1, x_2x_1^2, x_1^3) \ \Rightarrow\ \{(J_3)_{10}\}^{\mathcal{C}} = \mathscr{M} \cup \{x_1^2x_0^8, x_2x_1x_0^8, x_1^2x_0^8\}\\
J_4 &= (x_3^2, x_3x_2, x_2^2, x_3x_1^2, x_2x_1^2, x_1^3) \ \Rightarrow\ \{(J_4)_{10}\}^{\mathcal{C}} = \mathscr{M} \cup \{x_1^2x_0^8, x_2x_1x_0^8, x_3x_1x_0^8\}
\end{aligned}$$

where $\mathscr{M} = \{x_3x_0^9, x_2x_0^9, x_1x_0^9, x_0^{10}\}$. Therefore, set $\overline{\mathscr{M}} = \{x_6x_0^9, x_5x_0^9, x_4x_0^9\} \cup \mathscr{M}$, the ideals in $\mathbb{K}[x_0,\ldots,x_6]$ with Hilbert polynomial $p(t) = 10$ without linear generators are

$$\begin{gathered}
\left\langle \left(\overline{\mathscr{M}} \cup \{x_1^2x_0^8, x_1^3x_0^7, x_1^4x_0^6\}\right)^{\mathcal{C}} \right\rangle^{\text{sat}}\\
\left\langle \left(\overline{\mathscr{M}} \cup \{x_1^2x_0^8, x_2x_1x_0^8, x_1^3x_0^7\}\right)^{\mathcal{C}} \right\rangle^{\text{sat}}\\
\left\langle \left(\overline{\mathscr{M}} \cup \{x_1^2x_0^8, x_2x_1x_0^8, x_1^2x_0^8\}\right)^{\mathcal{C}} \right\rangle^{\text{sat}}\\
\left\langle \left(\overline{\mathscr{M}} \cup \{x_1^2x_0^8, x_2x_1x_0^8, x_3x_1x_0^8\}\right)^{\mathcal{C}} \right\rangle^{\text{sat}}
\end{gathered}$$

In addition to these 4 ideals, we have to consider the ideals with the same Hilbert polynomial in $\mathbb{K}[x_0,\ldots,x_5]$ to which we add x_6 as generator.

2.7 Segment ideals

We conclude this chapter, trying to generalize the notion of lexicographic ideal. We recall that by a theorem of Macaulay [60], if a numerical function $f : \mathbb{N} \to \mathbb{N}$ is admissible, i.e. there exists a $\mathbb{K}$-algebra A such that $\mathrm{HF}_A(t) = f(t)$, then the direct sum

$$L = \bigoplus_{t \in \mathbb{N}} L_t, \quad L_t = \left\langle \text{biggest } \binom{n+t}{n} - f(t) \text{ monomials of } \mathbb{K}[x]_t \text{ w.r.t. } \texttt{DegLex} \right\rangle$$

is an ideal of $\mathbb{K}[x]$ and $\mathbb{K}[x]/L$ has $f(t)$ has Hilbert function. L is uniquely determined by $f(t)$, so it is called *lexicographic ideal* associated to $f(t)$. In our context, we are mostly interested in Hilbert polynomials, so we would like to associate uniquely a saturated lexicographic ideal to any Hilbert polynomial $p(t)$. Let us start writing in a slightly different way the Gotzmann representation of $p(t)$ supposing $\deg p(t) = d$:

$$\begin{aligned} p(t) = & \binom{t+d}{d} + \ldots + \binom{t+d-(b_d-1)}{d} + \\ & + \binom{t+d-1-b_d}{d-1} + \ldots + \binom{t+d-1-(b_d+b_{d-1}-1)}{d-1} + \\ & + \ldots + \\ & + \binom{t-(b_d+\ldots+b_1)}{0} + \ldots + \binom{t-(b_d+\ldots+b_1+b_0-1)}{0}. \end{aligned} \tag{2.39}$$

Comparing (2.39) with (1.21), we see that b_j counts the number of a_i equal to j.

We have the following characterization of the saturated lexicographic ideal.

Proposition 2.58. *Let $p(t)$ be an admissible Hilbert polynomial in $\mathbb{K}[x_0, \ldots, x_n]$. The saturated lexicographic ideal L, such that $\mathbb{K}[x]/L$ has Hilbert polynomial $p(t)$, is*

$$L = \left(x_n, \ldots x_{d+2}, x_{d+1}^{b_d+1}, x_{d+1}^{b_d} x_d^{b_{d-1}+1}, \ldots, x_{d+1}^{b_d} \cdots x_2^{b_1+1}, x_{d+1}^{b_d} \cdots x_1^{b_0} \right), \tag{2.40}$$

where the exponents b_j are the integers defined in (2.39).

Example 2.7.1. Let us consider the Hilbert polynomial $p(t) = \frac{1}{3}t^3 + \frac{3}{2}t^2 + \frac{25}{6}t + 3$ in $\mathbb{P}^6$. The Gotzmann representation of $p(t)$ is

$$\begin{aligned} \tfrac{1}{3}t^3 + \tfrac{3}{2}t^2 + \tfrac{25}{6}t + 3 = \tbinom{t+3}{3} + \tbinom{t+2}{3} + & & \Rightarrow \quad & b_3 = 2 \\ & & \Rightarrow \quad & b_2 = 0 \\ + \tbinom{t-1}{1} + \tbinom{t-2}{1} + & & \Rightarrow \quad & b_1 = 2 \\ + \tbinom{t-4}{0} + \tbinom{t-5}{0} + \tbinom{t-6}{0} + \tbinom{t-7}{0} + \tbinom{t-8}{0} & & \Rightarrow \quad & b_0 = 5 \end{aligned}$$

so the corresponding saturated lexicographic ideal is

$$L = (x_6, x_5, x_4^3, x_4^2 x_3, x_4^2 x_2^3, x_4^2 x_2^2 x_1^5).$$

We want to study what happens considering any term ordering σ instead of the lexicographic order.

Definition 2.59. A set S of monomials of degree t is called *segment* w.r.t. the term ordering σ if for any monomial $x^\alpha \in S$, then $x^\beta >_\sigma x^\alpha \Rightarrow x^\beta \in S$.

Definition 2.60. Let $I \subset \mathbb{K}[x]$ be an non-null monomial ideal and let σ be any term ordering.

- I is called *segment ideal* w.r.t. σ, if for every $t \in \mathbb{N}$, I_t is a segment w.r.t. σ.
- I is called *hilb-segment ideal* w.r.t. σ, if I_r is a segment w.r.t. σ, where r is the Gotzmann number of the Hilbert polynomial of $\mathbb{K}[x]/I$.
- I is called *reg-segment ideal* w.r.t. σ, if $I_{\mathrm{reg}(I)}$ is a segment w.r.t. σ.
- I is called *gen-segment ideal* w.r.t. σ, if for every $t \in \mathbb{N}$, the generators of I of degree t, i.e. the generators of $I_t \setminus \langle I_{t-1} \rangle_t$, are the biggest monomials w.r.t. σ among the monomials of degree t non contained in $\langle I_{t-1} \rangle$.

By definition, the lexicographic ideal is a segment ideal w.r.t. the degree lexicographic order. In the following we will call it also lexsegment ideal.

As seen in Definition 2.26, each term ordering σ defines a total order on the monomials of a fixed degree that refines the Borel order. Hence it is clear that in order for a set S to be a segment it is necessary to be a Borel set.

Proposition 2.61. *(i) If S is a segment, then S is a Borel set.*

(ii) If I is an ideal that respects one properties described in Definition 2.60, then I is a Borel-fixed ideal.

Lemma 2.62. *Let $I \subset \mathbb{K}[x]$ be a saturated Borel-fixed ideal and let σ be any term ordering. If I_p is a segment then I_q, $q < p$, is a segment too.*

Proof. Let x^α and x^β be two monomials of degree q, such that $x^\alpha \in I_q$ and $x^\beta >_\sigma x^\alpha$. $x^\alpha x_0^{p-q}$ belongs to I_p and $x^\beta x_0^{p-q} >_\sigma x^\alpha x_0^{p-q}$ implies $x^\beta x_0^{p-q} \in I_p$, because I_p is a segment. Recalling that I is saturated, x^β belongs to I_q. □

Lemma 2.63. *Let $\mathscr{B} \subset \mathcal{P}(n,m)$ be a Borel set. If there exists four terms $x^\alpha, x^\beta \in \mathscr{B}$, $x^\gamma, x^\delta \in \mathscr{B}^C$ such that $x^\alpha x^\beta = x^\gamma x^\delta$, then $\mathscr{B}$ is not a segment w.r.t. any term order σ.*

Proof. If $\mathscr{B}$ were a segment w.r.t some σ, by the given assumptions we would have in particular $x^\alpha >_\sigma x^\gamma$ and $x^\beta >_\sigma x^\delta$. From these it would follow

$$x^\alpha x^\beta >_\sigma x^\gamma x^\beta, \quad x^\beta x^\gamma >_\sigma x^\delta x^\gamma \quad \Rightarrow \quad x^\alpha x^\beta >_\sigma x^\gamma x^\delta$$

contradicting $x^\alpha x^\beta = x^\gamma x^\delta$. □

Clearly this lemma can be used to deduce properties also about ideals. Let I be a Borel-fixed ideal.

- If $\{I_t\}$ for some t realizes the hypothesis of Lemma 2.63, then I could not be a segment ideal.
- If $\{I_r\}$ realizes the hypothesis of Lemma 2.63, then I could not be a hilb-segment ideal.
- If $\{I_{\mathrm{reg}(I)}\}$ realizes the hypothesis of Lemma 2.63, then I could not be a reg-segment ideal.
- If $\{I_t\}$, for some t, and two generators of degree t realize the hypothesis of Lemma 2.63, then I could not be a gen-segment ideal.

Proposition 2.64. *Let $I \subset \mathbb{K}[x]$ be a saturated Borel-fixed ideal and let σ be a term ordering. Then*

(i) I segment ideal $\Rightarrow$ I hilb-segment ideal $\Rightarrow$ I reg-segment ideal $\Rightarrow$ I gen-segment ideal.

(ii) σ is the lexicographic order $\Leftrightarrow$ the implications in (i) are all equivalences, for every ideal I.

(iii) If the projective scheme defined by I has constant Hilbert polynomial, then: I segment ideal $\Leftrightarrow$ I hilb-segment ideal $\Leftrightarrow$ I reg-segment ideal.

Proof. *(i)* The first implication is obvious. For the second one, it is enough to apply Lemma 2.62, because the Gotzmann number is greater than or equal to $reg(I)$. For the third implication, recall that I is generated in degrees $\leqslant \operatorname{reg}(I)$, by definition. Moreover, if I is a reg-segment ideal, by Lemma 2.62 I_t contains the greatest terms of degree t, for every $t \leqslant \operatorname{reg}(I)$. Thus, in particular, minimal generators of I must to be the greatest possible.

(ii) First, suppose that σ is the lexicographic order. Then, by *(i)*, it is enough to show that a gen-segment ideal is also a segment ideal. Indeed, by induction on the degree s of monomials and with $s = 0$ as base of induction, for $s > 0$ suppose that I_{s-1} is a segment. $\langle I_{s-1}\rangle_s$ is still a segment and, since possible minimal generators are always the greatest possible, we are done.

Vice versa, if σ is not the lexicographic order, let s be the minimum degree at which the monomials are ordered in a different way from the lexicographic one. Thus, there exist two terms x^α and x^β with maximum variables x_l and x_h, respectively, such that $x^\beta <_\sigma x^\alpha$ but $x_h >_\sigma x_l$. The ideal $I = (x_h, \ldots, x_n)$ is a gen-segment ideal but not a segment ideal, since x^β belongs to I and x^α does not.

(iii) It is enough to show that, in the case of constant Hilbert polynomial, a reg-segment ideal I is also a segment ideal. By induction on the degree s, if $s \leqslant \operatorname{reg}(I)$, then the thesis follows by the hypothesis and by Lemma 2.62. Suppose that $s > \operatorname{reg}(I)$ and that I_{s-1} is a segment. At degree s there are not minimal generators for I so that a monomial of I_s is always of type $x^\alpha x_h$ with x^α in I_{s-1}. Let x^β be a monomial of degree s such that $x^\beta >_\sigma x^\alpha x_h$, thus $x^\beta >_\sigma x^\alpha x_0$. By Proposition 2.15, we have that $(x_1, \ldots, x_n)^s \subseteq I$. So, if x^β is not divided by x_0, then x^β belongs to I_s, otherwise there exists a monomial x^γ such that $x^\beta = x^\gamma x_0$. Thus $x^\gamma >_\sigma x^\alpha$ and by induction x^γ belongs to I_{s-1} so that $x^\beta = x^\gamma x_0$ belongs to I_s. □

In the following chapters, the relevance of the definition of various type of segment ideals will be clearified. At this point, we are interested in explaining how to determine the term ordering σ that makes an ideal a segment ideal.

First of all, we recall the characterization of term orderings by means of matrices with rational coefficients. Let $T \in \mathrm{GL}_{\mathbb{Q}}(n+1)$ be any invertible matrix. T induces an order relations on the monomials of $\mathbb{K}[x]$ defined as follows

$$x^\alpha >_T x^\beta \quad \Longleftrightarrow \quad \text{the first non-zero entry of the vector } T \cdot (\alpha - \beta)^{\mathrm{tr}} \text{ is positive.}$$

Proposition 2.65 ([54, Proposition 1.4.12]). *Let T be an invertible matrix in $\mathrm{GL}_{\mathbb{Q}}(n+1)$. The order induced by T on the monomials of $\mathbb{K}[x]$ is a term ordering if and only if the first non-zero element in each column of T is positive.*

As well known the matrices representing the `Lex`, `DegLex` and `DegRevLex` are

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & \dots & 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & \dots & 0 & -1 \\ 0 & \iddots & \iddots & 0 \\ 0 & -1 & \dots & 0 \end{pmatrix}.$$

We are working in the projective case, i.e. with homogeneous polynomials, so we are interested in matrices with the first rows represented by the vector $(1, \dots, 1)$ to evaluate the total degree of a monomial. We consider as second row a vector $\omega = (\omega_n, \dots, \omega_0) \in \mathbb{Q}^{n+1}$ and then we complete the matrix with the same rows of the matrix associated to the `DegLex` term ordering:

$$T = \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ \omega_n & \omega_{n-1} & \dots & \omega_1 & \omega_0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix}. \tag{2.41}$$

We want that the induced term ordering agrees with the hypothesis $x_n >_B \dots >_B x_0$, so we will always suppose $\omega_n > \dots > \omega_0$. This fact ensures that the matrix is invertible because the rank of T could be lower than $n+1$ if and only if

$$\begin{vmatrix} 1 & 1 \\ \omega_n & \omega_0 \end{vmatrix} = \omega_0 - \omega_n = 0.$$

Since the term ordering induced by T depends on the vector ω, we will say "the term ordering ω" meaning "the term ordering induced by the matrix (2.41) with second row equal to the vector ω".

Now let us consider a Borel set $\mathscr{B} \subset \mathcal{P}(n,m)$ and let us try to determine the vector ω in order for $\mathscr{B}$ to be a segment. If $x^{\alpha_1},\ldots,x^{\alpha_a}$ are the minimal monomials of $\mathscr{B}$ and $x^{\beta_1},\ldots,x^{\beta_b}$ are the maximal elements of the complement $\mathscr{B}^{\mathcal{C}}$, we need to impose that $x^{\alpha_i} >_\omega x^{\beta_j}$, $\forall\, i = 1,\ldots,a,\ j = 1,\ldots,b$. The first row of the matrix (2.41) compares the degree of the monomials, therefore it does not affect the ordering on the monomials of $\mathcal{P}(n,m)$. We can look for a vector ω that orders the monomials as we want, that is we have to solve the following system of inequalitites

$$\begin{cases} \omega_i > \omega_{i-1}, & i = 1,\ldots,n, \\ \omega \cdot (\alpha_i - \beta_j) > 0, & \forall\, x^{\alpha_i},\ \forall\, x^{\beta_j}. \end{cases} \tag{2.42}$$

Example 2.7.2. Let us consider the Borel set $\{I_4\} \subset \mathcal{P}(2,4)$ defined by the ideal $I = (x_2^2, x_2x_1^2, x_1^4) \subset \mathbb{K}[x_0,x_1,x_2]$. The minimal monomials of $\{I_4\}$ are $x_2^2x_0^2, x_2x_1^2x_0, x_1^4$ and the maximal monomials of $\{I_4\}^{\mathcal{C}}$ are $x_2x_1x_0^2, x_1^3x_0$. So the system of inequalities to solve is

$$\begin{cases} \omega_2 > \omega_1 & (x_2 >_\omega x_1) \\ \omega_1 > \omega_0 & (x_1 >_\omega x_0) \\ 2\omega_2 + 2\omega_0 > \omega_2 + \omega_1 + 2\omega_0 & (x_2^2x_0^2 >_\omega x_2x_1x_0^2) \\ 2\omega_2 + 2\omega_0 > 3\omega_1 + \omega_0 & (x_2^2x_0^2 >_\omega x_1^3x_0) \\ \omega_2 + 2\omega_1 + \omega_0 > \omega_2 + \omega_1 + 2\omega_0 & (x_2x_1^2x_0 >_\omega x_2x_1x_0^2) \\ \omega_2 + 2\omega_1 + \omega_0 > 3\omega_1 + \omega_0 & (x_2x_1^2x_0 >_\omega x_1^3x_0) \\ 4\omega_1 > \omega_2 + \omega_1 + 2\omega_0 & (x_1^4 >_\omega x_2x_1x_0^2) \\ 4\omega_1 > 3\omega_1 + \omega_0 & (x_1^4 >_\omega x_1^3x_0) \end{cases} \Rightarrow \begin{cases} \omega_2 > \omega_1 \\ \omega_1 > \omega_0 \\ \omega_2 > \omega_1 \\ 2\omega_2 + \omega_0 > 3\omega_1 \\ \omega_1 > \omega_0 \\ \omega_2 > \omega_1 \\ 3\omega_1 > \omega_2 + 2\omega_0 \\ \omega_1 > \omega_0 \end{cases}$$

Supposing $\omega_0 = 1$, we obtain

$$\begin{cases} \omega_2 > \omega_1 \\ \omega_1 > 1 \\ 2\omega_2 > 3\omega_1 - 1 \\ \omega_2 < 3\omega_1 + 2 \end{cases} \Rightarrow \quad \omega_2 = 3,\ \omega_1 = 2,\ \omega_0 = 1.$$

The example clearly shows that whenever for the pair $x^{\alpha_i}, x^{\beta_j}$ there exists an elementary decreasing move e_k^- such that $\mathsf{e}_k^-(x^{\alpha_i}) = x^{\beta_j}$, the inequality given is already determined by the Borel order. Thus we can restrict the conditions to be imposed as follows

$$\begin{cases} \omega_i > \omega_{i-1}, & i = 1, \ldots, n, \\ \omega \cdot (\alpha_i - \beta_j) > 0, & \forall\, x^{\alpha_i},\, \forall\, x^{\beta_j},\, \nexists\, \mathsf{e}_k^- \text{ s.t. } \mathsf{e}_k^-(x^{\alpha_i}) = x^{\beta_j}. \end{cases} \tag{2.43}$$

For the simplest cases, a solution of the system of inequalitites can be found by hand with a little bit of work, but just increasing the number of variables and the degree of monomials of a poset, the system can become much more complicated. A natural way to face this problem is to use the theory of linear programming and the simplex algorithm (see [103, Chapter 2] for an introduction to the topic).

The problem of linear programming that best fits our case is the minimization of a linear expression under some linear constraints. In fact the standard problem of minimization requires:

- a set of variables $z_0, \ldots, z_n$ with the hypothesis that they do not assume negative values, i.e.
$$z_0 \geqslant 0, \quad \ldots \quad, z_n \geqslant 0,$$
- a target function (that we want to minimize)
$$c_0 z_0 + \ldots + c_n z_n,$$
- a set of constraints on the variables expressed by linear inequalities of the type
$$a_{i0} z_0 + \ldots + a_{in} z_n \geqslant b_i.$$

The simplex algorithm allows to compute the solution $(\overline{z}_0, \ldots, \overline{z}_n)$ of the system of constraints such that the value $c_0\overline{z}_0 + \ldots + c_n\overline{z}_n$ is minimum.

In our context we are looking for any solution of the system (2.43), not the minimal one, so we choose arbitrarily as target function the sum of the vector components $\omega_0 + \ldots + \omega_n$. The next step is to transform our strict inequalities in non-strict ones, so we rewrite the system (2.43) as

$$\begin{cases} \omega_0 \geqslant 1 \\ \omega_i - \omega_{i-1} \geqslant 1, & i = 1, \ldots, n, \\ \omega \cdot (\alpha_i - \beta_j) \geqslant 1, & \forall\, x^{\alpha_i},\, \forall\, x^{\beta_j},\, \nexists\, \mathsf{e}_k^- \text{ s.t. } \mathsf{e}_k^-(x^{\alpha_i}) = x^{\beta_j}. \end{cases} \tag{2.44}$$

ensuring also $\omega_0 \geqslant 0, \ldots, \omega_n \geqslant 0$.

Example 2.7.3. Considering again the Borel set $\{I_4\}$ introduced in Example 2.44, the system of inequalities written so that it can be solved with the simplex algorithm is

$$\begin{cases} \omega_0 \geqslant 1 \\ \omega_1 - \omega_0 \geqslant 1 \\ \omega_2 - \omega_1 \geqslant 1 \\ 2\omega_2 - 3\omega_1 + \omega_0 \geqslant 1 \\ -\omega_2 + 3\omega_1 - 2\omega_0 \geqslant 1 \end{cases}$$

and the solution we obtain with target fucntion $\omega_2 + \omega_1 + \omega_0$ is again $(3, 2, 1)$.

In Algorithm 2.6, there is the pseudocode description of the strategy just exposed and also used for designing the corresponding functions of the class `BorelInequalitiesSystem` of the package `HSC` (see Appendix B). In Algorithm 2.7, there are the pseudocode descriptions of the methods for determining whenever a Borel-fixed ideal is a hilb-segment or a reg-segment ideal, that can be easily deduced by Algorithm 2.6.

MINIMALELEMENTS($\mathscr{B}$)
Input: $\mathscr{B}$, Borel set.
Output: the set of minimal elements of $\mathscr{B}$ w.r.t. $\leq_B$.

MAXIMALELEMENTS($\mathscr{B}$)
Input: $\mathscr{B}$, Borel set.
Output: the set of maximal elements of $\mathscr{B}$ w.r.t. $\leq_B$.

SIMPLEXALGORITHM(f, S, ω)
Input: f, a target function (to be minimized).
Input: S, a set of constraints.
Input: ω, a vector.
Output: **true** if the system of constraints is solvable, **false** otherwise. If **true** the solution that minimizes f is stored in ω.

Some auxiliary methods for Algorithm 2.6.

IsSegment($\mathscr{B}, \omega$)
Input: $\mathscr{B} \subset \mathcal{P}(n, m)$, Borel set.
Input: $\omega = (\omega_0, \ldots, \omega_n)$, a vector.
Output: **false**, if $\mathscr{B}$ can not be a segment, **true** otherwise. In this case the term ordering realizing $\mathscr{B}$ as a segment is stored in ω.
constraints $\leftarrow \{z_0 \geqslant 1\}$;
for $i = 1, \ldots, n$ **do**
  constraints $\leftarrow$ constraints $\cup \{z_i - z_{i-1} \geqslant 1\}$;
end for
minimalMonomials $\leftarrow$ MinimalElements($\mathscr{B}$);
maximalMonomials $\leftarrow$ MaximalElements($\mathscr{B}$);
for all $x^\alpha \in$ minimalMonomials **do**
  for all $x^\beta \in$ maximalMonomials **do**
    if $\nexists\ \mathrm{e}_k^-$ s.t. $\mathrm{e}_k^-(x^\alpha) = x^\beta$ **then**
      constraints $\leftarrow$ constraints $\cup \{(z_n, \ldots, z_0) \cdot (\alpha - \beta) \geqslant 1\}$;
    end if
  end for
end for
return SimplexAlgorithm($z_0 + \ldots + z_n$, constraints, ω);

Algorithm 2.6: The algorithm computing a vector defining a term ordering that realizes a Borel set as segment.

In the following proposition we use definitions and algorithms just introduced to characterize segment ideals in $\mathbb{K}[x_0, x_1, x_2]$ with constant Hilbert polynomial.

Proposition 2.66. *In $\mathbb{K}[x_0, x_1, x_2]$ every saturated Borel-fixed ideal with Hilbert polynomial $p(t) = s \leqslant 6$ is a segment ideal. Whereas for every $p(t) = s \geqslant 7$, a saturated Borel-fixed ideal, which is not a segment for any term order, always exists.*

Proof. We give a constructive proof of this result, examining the Borel sets defined by the ideals in degree equal to their regularity (Proposition 2.64*iii*).

$s = 1, 2$. There exists a unique saturated Borel-fixed ideal (x_2, x_1^d), which is the lexi-

```
isHilbSegment(I, ω)
Input: I ⊂ K[x_0, ..., x_n], Borel-fixed ideal.
Input: ω = (ω_0, ..., ω_n), a vector.
Output: false, if I can not be a hilb-segment, true otherwise. In this case the term
    ordering realizing I as a hilb-segment is stored in ω.
p(t) ← HilbertPolynomial(K[x_0, ..., x_n]/I);
r ← GotzmannNumber(p(t));
return isSegment({I_r}, ω);

isRegSegment(I, ω)
Input: I ⊂ K[x_0, ..., x_n], Borel-fixed ideal.
Input: ω = (ω_0, ..., ω_n), a vector.
Output: false, if I can not be a reg-segment, true otherwise. In this case the term
    ordering realizing I as a reg-segment is stored in ω.
return isSegment({I_reg(I)}, ω);
```

Algorithm 2.7: The methods for computing the term ordering that realizes an Borel-fixed ideal as hilb-segment or reg-segment.

cographic ideal.

$s = 3, 4$. There are two saturated Borel-fixed ideals: the lexsegment ideal (x_2, x_1^d) and $(x_2^2, x_2x_1, x_1^{d-1})$, segment w.r.t. `DegRevLex`.

$s = 5$. There are three saturated Borel-fixed ideals: the lexsegment ideal (x_2, x_1^5), (x_2^2, x_2x_1, x_1^4) segment w.r.t. $(4, 2, 1)$ and $(x_2^2, x_2x_1^2, x_1^3)$ segment w.r.t. `DegRevLex`.

$s = 6$. There are four saturated Borel ideals: the lexsegment (x_2, x_1^6), (x_2^2, x_2x_1, x_1^5) segment w.r.t. $(5, 2, 1)$, $(x_2^2, x_2x_1^2, x_1^4)$ segment w.r.t. $(3, 2, 1)$ and $(x_2^3, x_2^2x_1, x_2x_1^2, x_1^3)$ segment w.r.t. `DegRevLex`.

$s = 2a + 1$, $a \geqslant 3$. Let us consider the ideal $J = (x_2^2, x_2x_1^a, x_1^{a+1})$. It has constant Hilbert polynomial $p(t) = 2a + 1$, because x_1^{a+1} belongs to J and

$\dim_{\mathbb{K}} \mathbb{K}[x_0, x_1, x_2]_{a+1}/J_{a+1} = 2a + 1$, indeed the monomials of degree $a + 1$ not belonging to J are

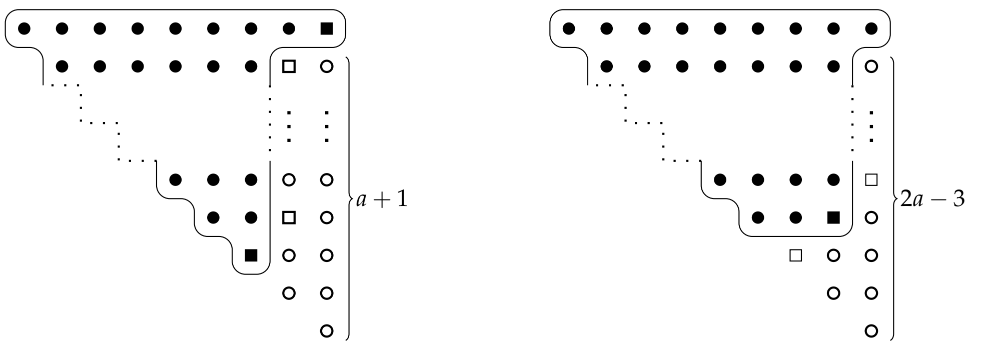
$$\left\{x_0^{a+1}, x_1 x_0^a, \ldots, x_1^a x_0\right\} \cup \left\{x_2 x_0^a, x_2 x_1 x_0^{a-1}, \ldots, x_2 x_1^{a-1} x_0\right\}.$$

By Lemma 2.63, J cannot be a segment because $x_2^2 x_0^{a-1}, x_1^{a+1} \in \{J_{a+1}\}$, $x_2 x_1^2 x_0^{a-2}$, $x_2 x_1^{a-1} x_0 \in \{J_{a+1}\}^{\mathcal{C}}$ and $x_2^2 x_0^{a-1} \cdot x_1^{a+1} = x_2 x_1^2 x_0^{a-2} \cdot x_2 x_1^{a-1} x_0$ (see Figure 2.15a).

$s = 2a$, $a \geqslant 4$. The ideal $J = (x_2^3, x_2^2 x_1, x_2 x_1^2, x_1^{2a-3})$ has Hilbert polynomial $p(t) = 2a$, because $x_1^{2a-3} \in J$ and $\mathbb{K}[x_0, x_1, x_2]_{2a-3}/J_{2a-3}$ is spanned by the $2a$ monomials

$$\left\{x_2^2 x_0^{2a-5}, x_2 x_1 x_0^{2a-5}, x_2 x_0^{2a-4}\right\} \cup \left\{x_0^{2a-3}, x_1 x_0^{2a-4}, \ldots, x_1^{2a-4} x_0\right\}.$$

Finally by Lemma 2.63, J can not be a segment because $x_2 x_1^2 x_0^{2a-6} \in \{J_{2a-3}\}$, $x_2^2 x_0^{2a-5}$, $x_1^4 x_0^{2a-7} \in \{J_{2a-3}\}^{\mathcal{C}}$ and $(x_2 x_1^2 x_0^{2a-6})^2 = x_2^2 x_0^{2a-5} \cdot x_1^4 x_0^{2a-7}$ (see Figure 2.15b). $\square$



(a) The Borel set of $\mathcal{P}(2, a+1)$ defined by $(x_2^2, x_2 x_1^a, x_1^{a+1})$.

(b) The Borel set of $\mathcal{P}(2, 2a-3)$ defined by $(x_2^3, x_2^2 x_1, x_2 x_1^2, x_1^{2a-3})$.

Figure 2.15: The Borel sets defined by Borel-fixed ideals of $\mathbb{K}[x_0, x_1, x_2]$ with Hilbert polynomial $p(t) = s \geqslant 7$, which can not be segment ideals (see Proposition 2.66). With $\square$ and $\blacksquare$ we denoted the monomials realizing the hypothesis of Lemma 2.63.

A slightly different approach is needed to deal with the gen-segment ideals, because the property required in the definition involes more than a single degree of the monomials. Let us consider the truncation $I_{\geqslant m}$ of a saturated Borel-fixed ideal I. In order for $I_{\geqslant m}$ to be a gen-segment ideal, we have to check that $\{I_m\}$ is a segment and that any generator of I of degree $\overline{m} > m$ is bigger than the monomials of $\{I_{\overline{m}}\}^{\mathcal{C}}$:

$$\begin{cases} \omega_0 \geqslant 1, \\ \omega_i - \omega_{i-1} \geqslant 1, \quad i = 1,\ldots,n, \\ \omega\cdot(\alpha-\beta) \geqslant 1, & \begin{cases} \forall\ x^\alpha \text{ minimal monomial of } \{J_m\} \\ \forall\ x^\beta \text{ maximal monomial of } \{J_m\}^{\mathcal{C}} \\ \nexists\ \mathsf{e}_k^- \text{ s.t. } \mathsf{e}_k^-(x^\alpha) = x^\beta \end{cases} \\ \omega\cdot(\gamma-\delta) \geqslant 1, & \begin{cases} \forall\ x^\gamma \text{ minimal generator of } I,\ |\gamma| > m \\ \forall\ x^\delta \text{ maximal monomial of } \{I_{|\alpha|}\}^{\mathcal{C}} \\ \nexists\ \mathsf{e}_k^- \text{ s.t. } \mathsf{e}_k^-(x^\gamma) = x^\delta \end{cases} \end{cases} \tag{2.45}$$

Example 2.7.4. Let us consider the ideal $I = (x_3^3, x_3^2x_2, x_3x_2^2, x_3^2x_1, x_3x_2x_1, x_3x_1^2, x_2^5) \subset \mathbb{K}[x_0,x_1,x_2,x_3]$. Note that I can not be a reg-segment ideal because

$$x_3x_2^2x_0^2 \in I,\ x_2^4x_0, x_3^2x_0^3 \notin I \quad \text{and} \quad \left(x_3x_2^2x_0^2\right)^2 = x_2^4x_0\cdot x_3^2x_0^3 \qquad \text{(Lemma 2.63)}$$

neither a hilb-segment ideal because in degree 9 (the Gotzmann number of the Hilbert polynomial $5t-1$ of I) the same relation multiplied by x_0^4 holds.

Applying Algorithm 2.8 to determine if I could be a gen-segment ideal, we need to impose that $\{I_3\}$ is a segment of $\mathcal{P}(3,3)$ and that x_2^5 is greater than all the monomials of $\{I_5\}^{\mathcal{C}}$. These conditions lead to the following system of inequalities:

$$\begin{cases} \omega_0 \geqslant 1 \\ \omega_1 - \omega_0 \geqslant 1 \\ \omega_2 - \omega_1 \geqslant 1 \\ \omega_3 - \omega_2 \geqslant 1 \\ \omega_3 - 3\omega_2 + 2\omega_1 \geqslant 1 & (x_3x_1^2 >_\omega x_2^3) \\ -\omega_3 2\omega_1 - \omega_0 \geqslant 1 & (x_3x_1^2 >_\omega x_3^2x_0) \\ -2\omega_3 + 5\omega_2 - 3\omega_0 \geqslant 1 & (x_2^5 >_\omega x_3^2x_0^3) \end{cases} \qquad \Rightarrow \qquad \omega = (10,7,6,1).$$

isGenSegment(I, m, ω)
Input: $I \subset \mathbb{K}[x_0, \ldots, x_n]$, saturated Borel-fixed ideal.
Input: m, positive integer.
Input: $\omega = (\omega_0, \ldots, \omega_n)$, a vector.
Output: **false**, if $I_{\geqslant m}$ can not be a gen-segment, **true** otherwise. In this case the term ordering realizing $I_{\geqslant m}$ as a gen-segment ideal is stored in ω.
constraints $\leftarrow \{z_0 \geqslant 1\}$;
for $i = 1, \ldots, n$ **do**
  constraints $\leftarrow$ constraints $\cup \{z_i - z_{i-1} \geqslant 1\}$;
end for
minimalMonomials $\leftarrow$ minimalElements$\big(\{I_m\}\big)$;
maximalMonomials $\leftarrow$ maximalElements$\big(\{I_m\}^{\mathcal{C}}\big)$;
for all $x^\alpha \in$ minimalMonomials **do**
  for all $x^\beta \in$ maximalMonomials **do**
    if $\nexists\ \mathrm{e}_k^-$ s.t. $\mathrm{e}_k^-(x^\alpha) = x^\beta$ **then**
      constraints $\leftarrow$ constraints $\cup \{(z_n, \ldots, z_0) \cdot (\alpha - \beta) \geqslant 1\}$;
    end if
  end for
end for
for $j = m+1, \ldots, \mathrm{reg}(I)$ **do**
  maximalMonomials $\leftarrow$ maximalElements$\big(\{I_j\}\big)$;
  for all x^γ minimal generator of $I_{\geqslant m}$, $|\gamma| = j$ **do**
    for all $x^\delta \in$ maximalMonomials **do**
      if $\nexists\ \mathrm{e}_k^-$ s.t. $\mathrm{e}_k^-(x^\gamma) = x^\delta$ **then**
        constraints $\leftarrow$ constraints $\cup \{(z_n, \ldots, z_0) \cdot (\gamma - \delta) \geqslant 1\}$;
      end if
    end for
  end for
end for
return SimplexAlgorithm$(z_0 + \ldots + z_n$, constraints$, \omega)$;

Algorithm 2.8: The pseudocode description of the algorithm determining if a Borel-fixed ideal is a gen-segment ideal or not.

Chapter 3

Rational curves on the Hilbert scheme

In this chapter we expose the results contained in the preprint [57] "A network of rational curves on the Hilbert scheme".

3.1 Rational deformations of Borel-fixed ideals

The starting idea is expressed in the following remark.

Remark 3.1.1. Let $\mathscr{B} \subset \mathcal{P}(n,m)$ be a Borel set and let x^α and x^β be a minimal monomial of $\mathscr{B}$ and a maximal monomial of $\mathscr{B}$. By definition both $\mathscr{B} \setminus \{x^\alpha\}$ and $\mathscr{B} \cup \{x^\beta\}$ are still Borel sets. Moreover x^α will be a maximal element of $(\mathscr{B} \setminus \{x^\alpha\})^{\mathcal{C}}$ and x^β will be a minimal element of $\mathscr{B} \cup \{x^\beta\}$.

Let us consider a Borel set $\{I_r\} \subset \mathcal{P}(n,r)$, defined by a Borel-fixed ideal I with constant Hilbert polynomial $p(t) = r$, i.e. $|\{I_r\}^{\mathcal{C}}| = r$ and $\{I_r\}^{\mathcal{C}}_{(\geqslant j)} = \varnothing, \forall\ j > 0$. Moreover let us suppose that there exist two monomials $x^\alpha, x^\beta \in \mathcal{P}(n,r)$, such that x^α is a minimal element in $\{I_r\}$, x^β is a maximal element in $\{I_r\}^{\mathcal{C}}$ and assume $\min x^\alpha = \min x^\beta = 0$. If $\mathscr{B} = \{I_r\} \setminus \{x^\alpha\} \cup \{x^\beta\}$ is still a Borel set, the ideal $\langle \mathscr{B} \rangle^{\mathrm{sat}}$ has the same Hilbert polynomial of I (by Corollary 2.36).

Example 3.1.2. Let us consider the ideal $I = (x_2^2, x_2x_1, x_1^3) \subset \mathbb{K}[x_0, x_1, x_2]$ having Hilbert polynomial $p(t) = 4$. The Borel set $\{I_4\} \subseteq \mathcal{P}(2,4)$ has two minimal monomials: $x_2x_1x_0^2$ and $x_1^3x_0$; while the monomials $x_2x_0^3$ and $x_1^2x_0^2$ are maximal elements in the complement $\{I_4\}^{\mathcal{C}}$ (see Figure 3.1a). There are four possibilities of removing a minimal element and adding a maximal one:

(Figure 3.1b) the set $\mathscr{B}_1 = \{I_4\} \setminus \{x_2x_1x_0^2\} \cup \{x_1^2x_0^2\}$ is not Borel, because $x_1^2x_0^2 \in \mathscr{B}_1$ and $x_2x_1x_0^2 = \mathsf{e}_1^+(x_1^2x_0^2) \notin \mathscr{B}_1$;

(Figure 3.1c) the set $\mathscr{B}_2 = \{I_4\} \setminus \{x_1^3x_0\} \cup \{x_1^2x_0^2\}$ is not Borel, because $x_1^2x_0^2 \in \mathscr{B}_2$ and $x_1^3x_0 = \mathsf{e}_0^+(x_1^2x_0^2) \notin \mathscr{B}_2$;

(Figure 3.1d) the set $\mathscr{B}_3 = \{I_4\} \setminus \{x_2x_1x_0^2\} \cup \{x_2x_0^3\}$ is not Borel, because $x_2x_0^3 \in \mathscr{B}_3$ and $x_2x_1x_0^2 = \mathsf{e}_0^+(x_2x_0^3) \notin \mathscr{B}_3$;

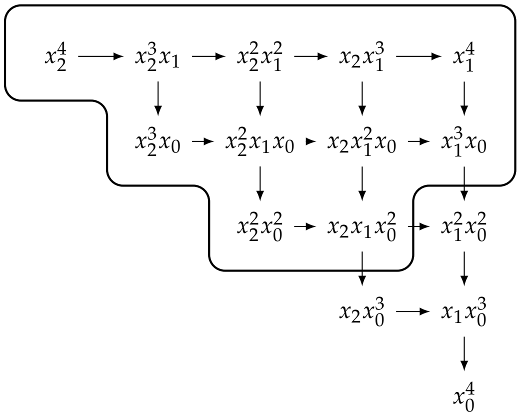
(Figure 3.1e) the set $\mathscr{B}_4 = \{I_4\} \setminus \{x_1^3x_0\} \cup \{x_2x_0^3\}$ is Borel and $\langle \mathscr{B}_4 \rangle^{\mathrm{sat}} = (x_2, x_1^4)$.

By Remark 3.1.1 and Example 3.1.2, we see that swapping two monomials x^α, x^β fails to preserve the Borel property whenever the minimal monomial is mapped by an elementary move to the maximal monomial, in fact in this case if we first remove the minimal monomial x^α from $\mathscr{B}$, x^β is no longer a maximal element of $(\mathscr{B} \setminus \{x^\alpha\})^{\mathcal{C}}$.

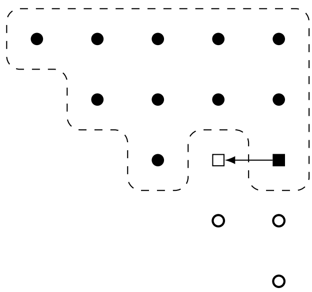
What happens when considering Borel-fixed ideals with Hilbert polynomial $p(t)$ of any degree? The point is to understand how to move in and out monomials for Borel set $\mathscr{B}$ for obtaining another Borel set $\overline{\mathscr{B}}$, with the same Hilbert polynomial, that is by Corollary 2.36 $|\mathscr{B}^{\mathcal{C}}_{(\geqslant j)}| = |\overline{\mathscr{B}}^{\mathcal{C}}_{(\geqslant j)}|$, $\forall\, j$.

The first idea is to exchange a minimal monomial $x^\alpha \in \mathscr{B}$ with a maximal $x^\beta \in \mathscr{B}^{\mathcal{C}}$, but whenever $\min x^\alpha \neq \min x^\beta$ this exchange will not preserve the Hilbert polynomial (see Figure 3.2).

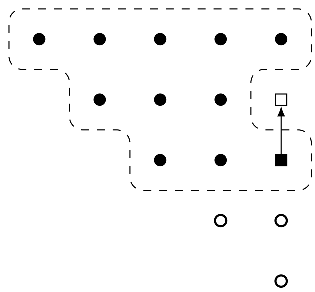
A second idea could be to exchange a minimal monomial x^α in $\mathscr{B}_{(\geqslant k)}$ and a maximal monomial x^β in $\mathscr{B}^{\mathcal{C}}_{(\geqslant k)}$. In this way, the cardinality of the sets $\mathscr{B}_{(\geqslant k)}$ and $\overline{\mathscr{B}}_{(\geqslant k)}$ is preserved, but whenever $\mathsf{e}_k^-(x^\alpha)$ belongs to $\mathscr{B}$, swapping x^α and x^β we do not obtain a Borel set (see Figure 3.3). This fact suggests that the general case is more complicated and that we have to swap more monomials than a minimal and a maximal element.



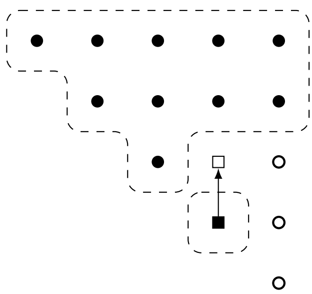
(a) The Borel set $\{I_4\}$ defined by $I = (x_2^2, x_2x_1, x_1^3)$.



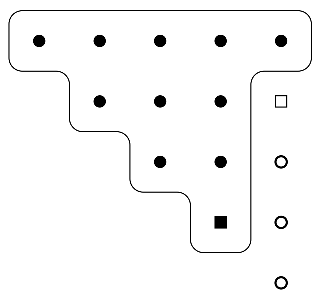
(b) $\{I_4\} \setminus \{x_2x_1x_0^2\} \cup \{x_1^2x_0^2\}$.



(c) $\{I_4\} \setminus \{x_1^3x_0\} \cup \{x_1^2x_0^2\}$.

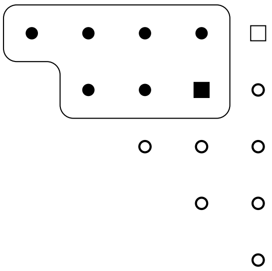


(d) $\{I_4\} \setminus \{x_2x_1x_0^2\} \cup \{x_2x_0^3\}$.

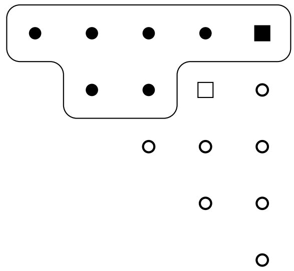


(e) $\{I_4\} \setminus \{x_2x_1x_0^2\} \cup \{x_2x_0^3\}$.

Figure 3.1: The graphical description of Example 3.1.2. In Figure 3.1a there is the Borel set defined by the ideal initially considered. In the other figures, there are the sets that can be obtained swapping a pair of monomials composed by a minimal element of $\{I_4\}$ and a maximal element of the complement. With $\square$ and $\blacksquare$ we denote the monomials exchanged. The arrows correspond to the increasing elementary moves showing that the set of monomials obtained is not Borel.

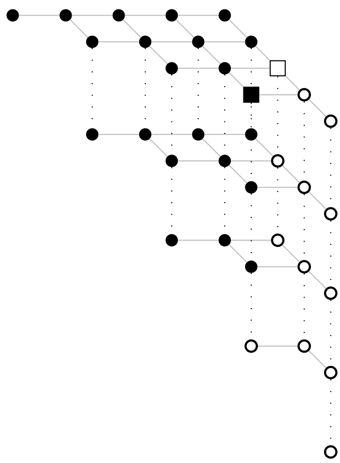


(a) The Borel set $\{I_4\}$ defined by the ideal $I = (x_2^3, x_2^2x_1, x_2x_1^2) \subset \mathbb{K}[x_0, x_1, x_2]$ with Hilbert polynomial $p(t) = t + 4$.

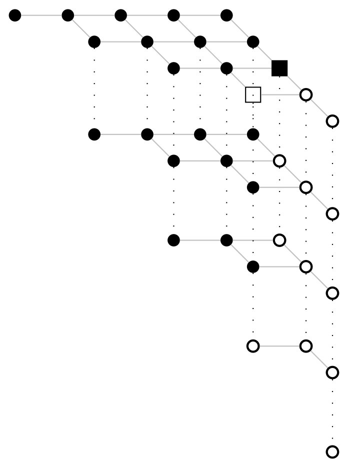


(b) The Borel set $\{I_4\} \setminus \{x_2x_1^2x_0\} \cup \{x_1^4\}$ corresponding to the ideal $(x_2^3, x_2^2x_1, x_2x_1^3, x_1^4) \subset \mathbb{K}[x_0, x_1, x_2]$ with Hilbert polynomial $\overline{p}(t) = 8$.

Figure 3.2: An example of an exchange of monomials that does not preserves the Hilbert polynomial.



(a) The Borel set $\{I_4\}$ defined by $I = (x_3^2, x_3x_2, x_3x_1, x_2^3) \subset \mathbb{K}[x_0, x_1, x_2, x_3]$.



(b) Swapping $x_3x_1^3$ and $x_2^2x_1^2$, we do not obtain a new Borel set, because $x_3x_1^2x_0$ belongs to $\{I_4\} \setminus \{x_3x_1^3\} \cup \{x_2^2x_1^2\}$ and $\mathsf{e}_0^+(x_3x_1^2x_0)$ does not.

Figure 3.3: An example of an exchange of monomials that does not preserves the Borel property.

Definition 3.1. Let $\mathscr{B} \subset \mathcal{P}(n,r)$ be a Borel set and let x^α be a minimal monomial of $\mathscr{B}_{(\geqslant k)}$ with $\min x^\alpha = k$. The set of monomials in $\mathscr{B}$ smaller than x^α

$$\{x^\gamma \in \mathscr{B} \mid x^\gamma \leq_B x^\alpha\} \tag{3.1}$$

is uniquely associated to the composition of decreasing elementary moves

$$\mathcal{F}_\alpha = \left\{\mu_b \mathsf{e}^-_{j_b} \circ \cdots \circ \mu_1 \mathsf{e}^-_{j_1} \;\middle|\; \mu_b \mathsf{e}^-_{j_b} \circ \cdots \circ \mu_1 \mathsf{e}^-_{j_1}(x^\alpha) \in \mathscr{B}\right\}. \tag{3.2}$$

Note that being x^α minimal in the restriction $\mathscr{B}_{(\geqslant k)}$, the elementary moves appearing in the composed moves of $\mathcal{F}_\alpha$ have index smaller than or equal to k. We call $\mathcal{F}_\alpha$ *decreasing set* of x^α and we denote with $\mathcal{F}_\alpha(x^\alpha)$ the set of monomials in (3.1).

Given a maximal element x^β of $\mathscr{B}^{\mathcal{C}}_{(\geqslant k)}$, we will say that the set $\mathcal{F}_\alpha$ is *Borel-admissible* w.r.t. x^β if every $F \in \mathcal{F}_\alpha$ is admissible also for x^β (note that $\min x^\beta = k$) and if $\mathsf{e}^+_i\big(F(x^\beta)\big) \in \mathscr{B}$ for each admissible elementary increasing move e^+_i, $i \geqslant k$. We denote by $\mathcal{F}_\alpha(x^\beta)$ the set of monomials obtained from x^β applying the moves in $\mathcal{F}_\alpha$.

Remark 3.1.3. Note that, recalling Definition 2.31, the set $\mathcal{F}_\alpha(x^\beta)$ contains k-maximal elements (by construction) and $\mathcal{F}_\alpha(x^\alpha)$ k-minimal monomials, indeed for each $x^\gamma = F(x^\alpha)$, $F \in \mathcal{F}_\alpha$ and any admissible e^-_j, $j > k$, $\mathsf{e}^-_j\big(F(x^\alpha)\big) = F\big(\mathsf{e}^-_j(x^\alpha)\big) \in \mathscr{B}^{\mathcal{C}}$, because $\mathsf{e}^-_j(x^\alpha) \in \mathscr{B}^{\mathcal{C}}$. Furthermore these two sets of monomials are always disjoint. Indeed we can write $x^\alpha = x^{\overline{\alpha}} x_k^a$ and $x^{\overline{\beta}} x_k^a$, so that $\min x^{\overline{\alpha}} \geqslant k$ and $\min x^{\overline{\beta}} \geqslant k$. The generic monomials in $\mathcal{F}_\alpha(x^\alpha)$ and $\mathcal{F}_\alpha(x^\beta)$ are of the type $x^{\overline{\alpha}} x^\gamma$ and $x^{\overline{\beta}} x^\delta$ where $\max x^\gamma \leqslant k$ and $\max x^\delta \leqslant k$, so since $x^{\overline{\alpha}} \neq x^{\overline{\beta}}$ we can not obtain the same monomial in the two sets.

Lemma 3.2. *Let $\mathscr{B} \subset \mathcal{P}(n,m)$ be a Borel set and let x^α and x^β be a minimal monomial of $\mathscr{B}_{(\geqslant k)}$ and a maximal monomial of $\mathscr{B}^{\mathcal{C}}_{(\geqslant k)}$ with $\min x^\alpha = \min x^\beta = k$. If there does not exist an elementary move e^-_j, $j > k$ such that $\mathsf{e}^-_j(x^\alpha) = x^\beta$ and if the decreasing set $\mathcal{F}_\alpha$ of x^α is Borel-admissible w.r.t. x^β, then $\overline{\mathscr{B}} = \mathscr{B} \setminus \mathcal{F}_\alpha(x^\alpha) \cup \mathcal{F}_\alpha(x^\beta)$ is a Borel set and $\left|\mathscr{B}^{\mathcal{C}}_{(\geqslant i)}\right| = \left|\overline{\mathscr{B}}^{\mathcal{C}}_{(\geqslant i)}\right|$, $\forall\, i$.*

Proof. The set $\overline{\mathscr{B}} = \mathscr{B} \setminus \mathcal{F}_\alpha(x^\alpha) \cup \mathcal{F}_\alpha(x^\beta)$ turns out to be Borel, mainly for Remark 3.1.3 and because the monomials in the set $\mathcal{F}_\alpha(x^\alpha)$ and $\mathcal{F}_\alpha(x^\beta)$ are not comparable w.r.t. the Borel order $\leq_B$.

The condition $\left|\mathscr{B}^{\mathcal{C}}_{(\geqslant i)}\right| = \left|\overline{\mathscr{B}}^{\mathcal{C}}_{(\geqslant i)}\right|$ follows from the fact that for each $F \in \mathcal{F}_\alpha$, $\min F(x^\alpha) = \min F(x^\beta)$. $\square$

Example 3.1.4. Let us see with some example in the poset $\mathcal{P}(3,4)$ the problems we can meet in choosing sets of monomials to exchange preserving both the Borel property and the Hilbert polynomial.

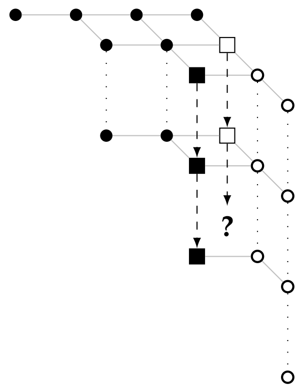
(Figure 3.4a) Let $\{I_3\}$ be the Borel set defined by the ideal $I = (x_3, x_2^3)$. $x_3x_1^2$ is a minimal monomial in $\{I_3\}_{(\geqslant 1)}$ and $x_2^2x_1$ a maximal one in $\{I_3\}^{\mathcal{C}}_{(\geqslant 1)}$. The decreasing set of $x_3x_1^2$ is $\mathcal{F}_{x_3x_1^2} = \{\mathrm{id}, \mathsf{e}_1^-, 2\mathsf{e}_1^-\}$ and it turns out to be not Borel-admissible w.r.t. $x_2^2x_1$, because $2\mathsf{e}_1^-$ is not admissible for $x_2^2x_1$.

(Figure 3.4b) Let now consider the Borel set $\{J_4\}$ defined by $(x_3^2, x_3x_2^2, x_2^3, x_3x_2x_1^2)$. $x_2^3x_1$ is a minimal monomial in $\{J_4\}_{(\geqslant 1)}$ and $x_3x_1^3$ a maximal one in $\{J_4\}^{\mathcal{C}}_{(\geqslant 1)}$. The decreasing set of $x_2^3x_1$ is $\mathcal{F}_{x_2^3x_1} = \{\mathrm{id}, \mathsf{e}_1^-\}$ and each move in it is also admissible w.r.t. $x_3x_1^3$. But $\mathsf{e}_1^+\big(\mathsf{e}_1^-(x_3x_1^3)\big) = x_3x_2x_1x_0 \notin \{J_4\}$, so exchanging $\mathcal{F}_{x_2^3x_1}(x_2^3x_1)$ with $\mathcal{F}_{x_2^3x_1}(x_3x_1^3)$ would not respect the Borel property.

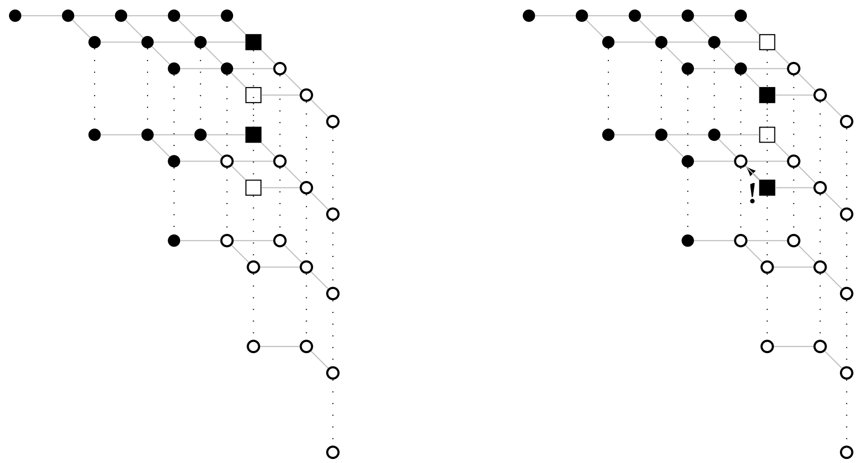
The same decreasing set arises in the Borel set defined by the ideal $(x_3^2, x_3x_2^2, x_2^3, x_3x_2x_1)$, but in this case $\mathsf{e}_1^+\big(\mathsf{e}_1^-(x_3x_1^3)\big) = x_3x_2x_1x_0$ would belong to the ideal, so the decreasing set would be Borel-admissible w.r.t. $x_3x_1^3$.

After having understood how to move monomials out of and in to a Borel set preserving the Borel property and the Hilbert polynomial associated, we carry on trying to determine a flat deformation having among the fibers the Borel-fixed ideals associated to the Borel sets. We will use the following property.

Proposition 3.3 ([3, Chapter 1 Section 3])**.** *Let A be a local $\mathbb{K}$-algebra and let us consider $M = \mathbb{K}[x]/\langle f_1, \dots, f_s\rangle$ and $M_A = A[x]/\langle f_1', \dots, f_s'\rangle$, where f_i' is a lifting of f_i in $A[x]$, i.e. tensoring the ideal $\langle f_1', \dots, f_s'\rangle$ by $\mathbb{K}$, the residue field of A, leads to the ideal $\langle f_1, \dots, f_s\rangle$. Then, M_A is flat over A if and only if any relation among $(f_1, \dots, f_s)$ lifts to a relation among $(f_1', \dots, f_s')$.*



(a) An example of decreasing set not Borel-admissible, because not well defined.



(b) An example of decreasing set not Borel-admissible, because exchanging the sets of monomials we do not preserve the Borel property.

Figure 3.4: The graphical description of the further problems on choosing correctly the monomials to move described in Example 3.1.4.

Theorem 3.4. *Let $\{I_r\} \subset \mathcal{P}(n,r)$ be a Borel set defined by a Borel-fixed ideal I. Moreover let us suppose that there exist two monomials: x^α minimal element of $\{I_r\}_{(\geqslant k)}$ and x^β maximal element of $\{I_r\}^{\mathcal{C}}_{(\geqslant k)}$, such that $\min x^\alpha = \min x^\beta = k$, there does not exist a decreasing move e_j^-, $j > k$ with $\mathsf{e}_j^-(x^\alpha) = x^\beta$ and such that the decreasing set $\mathcal{F}_\alpha$ of x^α is Borel-admissible w.r.t. x^β. Then, the family of subschemes of $\mathbb{P}^n$ parametrized by the ideal*

$$\mathcal{I} = \Big\langle \{I_r\} \setminus \mathcal{F}_\alpha(x^\alpha) \cup \{y_0\, F(x^\alpha) + y_1\, F(x^\beta) \mid F \in \mathcal{F}_\alpha\} \Big\rangle \tag{3.3}$$

is flat over $\mathbb{P}^1$.

Proof. First of all let $\mathscr{B} = \{I_r\}$ and let us call $p(t)$ the Hilbert polynomial of I. By Lemma 3.2, also the ideal $\mathcal{I}|_{[0:1]} = \langle \{I_r\} \setminus \mathcal{F}_\alpha(x^\alpha) \cup \mathcal{F}_\alpha(x^\beta) \rangle$ has Hilbert polynomial $p(t)$.

Let us suppose $y_0 \neq 0$ and set $z = \frac{y_1}{y_0}$ let us denote by I_z the ideal $\mathcal{I}|_{[1:z]} \subset \mathbb{K}[z][x]$. By [43, Chapter III Theorem 9.9] we know that to prove the flatness it suffices to check that for each point $\mathfrak{p} \in \operatorname{Spec} \mathbb{K}[z]$ the Hilbert polynomial of $(I_z)_\mathfrak{p}$ is $p(t)$. Let us start considering the presentation map of $I = I_{\geqslant r}$

$$\begin{aligned} \psi_0 : \big(\mathbb{K}[x](-r)\big)^{|\mathscr{B}|} &\longrightarrow \mathbb{K}[x] \\ \mathbf{e}_\gamma \quad &\longmapsto \quad x^\gamma, \qquad \forall\, x^\gamma \in \mathscr{B} \end{aligned} \tag{3.4}$$

and the set of Eliahou-Kervaire syzygies generating $\ker \psi_0$

$$\mathrm{Syz}_{\mathrm{EK}}(I) = \Big\{ x_i \mathbf{e}_\gamma - \min x^\gamma \mathbf{e}_\delta \mid \forall\, x^\gamma \in \mathscr{B},\ \forall\, x_i >_B \min x^\gamma \text{ s.t. } x_i x^\gamma = \langle x^\delta | \min x^\gamma \rangle^I \Big\}.$$

Now we will show that the kernel of the presentation map of I_z

$$\begin{aligned} \psi : \big(\mathbb{K}[z][x](-r)\big)^{|\mathscr{B}|} &\longrightarrow \mathbb{K}[z][x] \\ \mathbf{f}_\gamma \quad &\longmapsto \begin{cases} x^\gamma & \forall\, x^\gamma \in \mathscr{B} \setminus \mathcal{F}_\alpha(x^\alpha) \\ F(x^\alpha) + zF(x^\beta), & \forall\, x^\gamma = F(x^\alpha), F \in \mathcal{F}_\alpha \end{cases} \end{aligned} \tag{3.5}$$

has a set of generators of syzygies lifted directly from $\mathrm{Syz}_{\mathrm{EK}}(I)$. For any $x^\gamma \in \mathscr{B} \setminus \mathcal{F}_\alpha(x^\alpha)$, since $x^\gamma = \psi(\mathbf{f}_\gamma) = \psi_0(\mathbf{e}_\gamma)$, we consider the same syzygies of I

$$x_i \mathbf{f}_\gamma - \min x^\gamma \mathbf{f}_\delta \qquad \text{in place of} \qquad x_i \mathbf{e}_\gamma - \min x^\gamma \mathbf{e}_\delta,$$

indeed $x^\delta = \frac{x_i}{\min x^\gamma} x^\gamma = \mathsf{e}^+_{i-1} \circ \cdots \circ \mathsf{e}^+_{\min x^\gamma}(x^\gamma)$ belongs to $\mathscr{B} \setminus \mathcal{F}_\alpha(x^\alpha)$ as well.

Let us now look at the generators $x^{\overline{\alpha}} + z x^{\overline{\beta}} = F(x^\alpha) + zF(x^\beta)$. For each $x_i >_B x_k = \min x^\alpha = \min x^\beta >_B \min x^{\overline{\alpha}} = \min x^{\overline{\beta}}$, both $x_i x^{\overline{\alpha}}$ and $x_i x^{\overline{\beta}}$ belong to $\mathscr{B} \setminus \mathcal{F}_\alpha(x^\alpha)$ because the identity $\frac{x_i}{\min x^{\overline{\alpha}}} = \mathsf{e}^+_{i-1} \circ \cdots \circ \mathsf{e}^+_{\min x^{\overline{\alpha}}}$ and from the fact that x^β is k-maximal element and that in order for moving $x^{\overline{\alpha}}$ to any other monomial of $\mathcal{F}_\alpha$ we should only use moves e^+_j with index $0 \leqslant j < k$. Thus we consider the following syzygies

$$x_i \mathbf{f}_{\overline{\alpha}} - \min x^{\overline{\alpha}} \mathbf{f}_\gamma - z \min x^{\overline{\beta}} \mathbf{f}_\delta \qquad \text{(in place of } x_i \mathbf{e}_{\overline{\alpha}} - \min x^{\overline{\alpha}} \mathbf{e}_\gamma)$$

where $x_i x^{\overline{\alpha}} = \langle x^\gamma | \min x^{\overline{\alpha}} \rangle^{I_r}$, $x_i x^{\overline{\beta}} = \langle x^\delta | \min x^{\overline{\beta}} \rangle^{I_r}$ and $x^\gamma, x^\delta \in \mathscr{B} \setminus \mathcal{F}_\alpha$. The last possibility to consider is when multiplying $x^{\overline{\alpha}} + z x^{\overline{\beta}}$ by a variable $\min x^{\overline{\alpha}} <_B x_i \leq_B x_k$. In this case $\frac{x_i}{\min x^{\overline{\alpha}}} = \mathsf{e}^+_{i-1} \circ \cdots \circ \mathsf{e}^+_{\min x^{\overline{\alpha}}}$ involves only elementary decreasing moves with index included in 0 and $k-1$, that is

$$\mathsf{e}^+_{i-1} \circ \cdots \circ \mathsf{e}^+_{\min x^{\overline{\alpha}}}(x^{\overline{\alpha}}) = \mathsf{e}^+_{i-1} \circ \cdots \circ \mathsf{e}^+_{\min x^{\overline{\alpha}}}\big(F(x^\alpha)\big) = x^{\widetilde{\alpha}} \in \mathcal{F}_\alpha(x^\alpha)$$

and

$$\mathsf{e}^+_{i-1} \circ \cdots \circ \mathsf{e}^+_{\min x^{\overline{\beta}}}(x^{\overline{\beta}}) = \mathsf{e}^+_{i-1} \circ \cdots \circ \mathsf{e}^+_{\min x^{\overline{\beta}}}\big(F(x^\beta)\big) = x^{\widetilde{\beta}} \in \mathcal{F}_\alpha(x^\beta).$$

In this case the sygygies are equal to the corresponding syzygies of I:

$$x_i \mathbf{f}_{\overline{\alpha}} - \min x^{\overline{\alpha}} \mathbf{f}_{\widetilde{\alpha}} \qquad \text{in place of} \qquad x_i \mathbf{e}_{\overline{\alpha}} - \min x^{\overline{\alpha}} \mathbf{e}_{\widetilde{\alpha}}.$$

For all the closed point $\mathfrak{p}$ of $\operatorname{Spec} \mathbb{K}[z]$, having lifted the syzygies, we have that $\dim \big((I_z)_{(\mathfrak{p})}\big)_r = \dim I_r$ and $\dim \big((I_z)_{(\mathfrak{p})}\big)_{r+1} = \dim I_{r+1}$, hence by Gotzmann's Persistence Theorem (Theorem 1.22) the Hilbert polynomial of the fiber of the point $\mathfrak{p}$ is equal to the Hilbert polynomial of $\operatorname{Proj} \mathbb{K}[x]/I$, i.e. $p(t)$. For the generic point, after having localized in $(I_z)_{(0)}$, the flatness is ensured by Proposition 3.3.

The same reasoning works under the hypothesis $y_1 \neq 0$, considering as Borel set $\overline{\mathscr{B}} = \mathscr{B} \setminus \mathcal{F}_\alpha(x^\alpha) \cup \mathcal{F}_\alpha(x^\beta)$. x^β will be a minimal monomial of $\overline{\mathscr{B}}_{(\geqslant k)}$, x^α a maximal monomial of $\overline{\mathscr{B}}^{\mathcal{C}}_{(\geqslant k)}$ and the decreasing set $\mathcal{F}_\beta$ will coincide with $\mathcal{F}_\alpha$. $\square$

Example 3.1.5. Let us consider the ideal $I = (x_3^2, x_3x_2, x_3x_1, x_2^3)_{\geqslant 4} \subset \mathbb{K}[x_0, x_1, x_2, x_3]$. Its Hilbert polynomial is $p(t) = 3t + 1$ with Gotzmann number equal to 4. $x_3 x_1^3$ is a minimal monomial of $\{I_4\}_{(\geqslant 1)}$ with decreasing set $\mathcal{F}_{x_3x_1^3} = \{\mathrm{id}, \mathsf{e}_1^-, 2\mathsf{e}_1^-\}$ Borel-admissible w.r.t. $x_2^2x_1^2$, maximal element of $\{I_4\}^{\mathcal{C}}_{(\geqslant 1)}$. Let us consider the monomial $x_3x_1^2x_0 \in I$. The Eliahou-Kervaire syzygies of I involving it are

$$
\begin{aligned}
x_3 \cdot x_3x_1^2x_0 &= \langle x_3^2x_1^2 | x_0\rangle^I \quad \Rightarrow \quad x_3\mathbf{e}_{x_3x_1^2x_0} - x_0\mathbf{e}_{x_3^2x_1^2},\\
x_2 \cdot x_3x_1^2x_0 &= \langle x_3x_2x_1^2 | x_0\rangle^I \quad \Rightarrow \quad x_2\mathbf{e}_{x_3x_1^2x_0} - x_0\mathbf{e}_{x_3x_2x_1^2},\\
x_1 \cdot x_3x_1^2x_0 &= \langle x_3x_1^3 | x_0\rangle^I \quad \Rightarrow \quad x_3\mathbf{e}_{x_3x_1^2x_0} - x_0\mathbf{e}_{x_3x_1^3}.
\end{aligned}
$$

Set $J = \langle \{I_4\} \setminus \{x_3x_1^3, x_3x_1^2x_0, x_3x_1x_0^2\} \cup \{x_3x_1^3 + z\,x_2^2x_1^2, x_3x_1^2x_0 + z\,x_2^2x_1x_0, x_3x_1x_0^2 + z\,x_2^2x_0^2\}\rangle$, the lifted syzygies involving $x_3x_1^2x_0 + z\,x_2^2x_1x_0$ are

$$
\begin{aligned}
x_3 \cdot (x_3x_1^2x_0 + z\,x_2^2x_1x_0) &= \langle x_3^2x_1^2|x_0\rangle^I + z\langle x_3x_2^2x_1|x_0\rangle^I \quad \Rightarrow\\
&\qquad x_3\mathbf{f}_{x_3x_1^2x_0} - x_0\mathbf{f}_{x_3^2x_1^2} - zx_0\mathbf{f}_{x_3x_2^2x_1},\\
x_2 \cdot (x_3x_1^2x_0 + z\,x_2^2x_1x_0) &= \langle x_2x_3x_1^2|x_0\rangle^I + z\langle x_2^3x_1|x_0\rangle^I \quad \Rightarrow\\
&\qquad x_2\mathbf{f}_{x_3x_1^2x_0} - x_0\mathbf{f}_{x_2x_3x_1^2} - zx_0\mathbf{f}_{x_2^3x_1},\\
x_1 \cdot (x_3x_1^2x_0 + z\,x_2^2x_1x_0) &= x_0 \cdot (x_3x_1^3 + zx_2^2x_1^2) \quad \Rightarrow \quad x_3\mathbf{f}_{x_3x_1^2x_0} - x_0\mathbf{f}_{x_3x_1^3}.
\end{aligned}
$$

Theorem 3.5. *Let $\mathbf{Hilb}^n_{p(t)}$ be the Hilbert scheme parametrizing subschemes of $\mathbb{P}^n$ with Hilbert polynomial $p(t)$, whose Gotzmann number is r. Consider two Borel-fixed ideals I and J such that*

(a) I and J define two $\mathbb{K}$-rational points of the Hilbert functor, i.e. $\operatorname{Proj} \mathbb{K}[x]/I, \operatorname{Proj} \mathbb{K}[x]/J$ *in $\mathcal{H}\mathrm{ilb}^n_{p(t)}(\mathbb{K})$;*

(b) there exist a minimal monomial x^α of $\{I_r\}_{(\geqslant k)}$ and a maximal monomial x^β of $\{I_r\}^{\mathcal{C}}_{(\geqslant k)}$ such that $\min x^\alpha = \min x^\beta = k$*, there does not exist any e_j^- for which $\mathsf{e}_j^-(x^\alpha) = x^\beta$, the decreasing set $\mathcal{F}_\alpha$ of x^α is Borel-admissible w.r.t. x^β and*

$$\{J_r\} = \{I_r\} \setminus \mathcal{F}_\alpha(x^\alpha) \cup \mathcal{F}_\alpha(x^\beta).$$

Then, there is a rational curve $\mathcal{C} : \mathbb{P}^1 \to \mathbf{Hilb}^n_{p(t)}$ having two fibers corresponding to the $\mathbb{K}$-rational points defined by I and J.

Moreover, if we consider the construction of the Hilbert scheme $\mathbf{Hilb}^n_{p(t)}$ as subscheme of the Grassmannian $\mathbf{Gr}_{\mathbb{K}}(q(r), N)$, where $N = \binom{n+r}{n}$ and $q(r) = N - p(r)$, described in

Chapter 1, the degree of the curve C via the Plücker embedding (1.2) $\mathscr{P} : \mathbf{Gr}_{\mathbb{K}}(q(r), N) \to \mathbb{P}\big[\wedge^{q(r)} \mathbb{K}[x]_r\big]$ *is* $|\mathcal{F}_\alpha|$ *and C is isomorphic to the rational normal curve in a convenient subspace of* $\mathbb{P}\big[\wedge^{q(r)} \mathbb{K}[x]_r\big]$.

Proof. The Borel sets $\{I_r\}$ and $\{J_r\}$ realize the hypothesis of Theorem 3.4, hence the ideal

$$\mathcal{I} = \Big\langle \{I_r\} \setminus \mathcal{F}_\alpha \cup \{y_0 F(x^\alpha) + y_1 F(x^\beta) \mid F \in \mathcal{F}_\alpha\} \Big\rangle \subset \mathbb{K}[y_0, y_1][x]$$

parametrizes a flat family with Hilbert polynomial $p(t)$. By the universal property of the Hilbert scheme (Proposition 1.18, see also [43, Chapter III Remark 9.8.1]) the flat family $\operatorname{Proj} \mathbb{K}[y_0, y_1][x]/\mathcal{I} \to \mathbb{P}^1$ determines uniquely a map $\mathcal{C} : \mathbb{P}^1 \to \mathbf{Hilb}^n_{p(t)}$ and by construction the fibers over the points $[0 : 1]$ and $[1 : 0]$ correspond to the points defined by I and J.

For the second part of the theorem, let us think to the submodule defined by $\mathcal{I}_r$ in $(\mathbb{K}[y_0, y_1])[x]_r$. As usual we can represent it by a matrix with $q(r)$ rows and N columns with maximal rank; assuming to order the columns with the monomials of $\{I_r\} \setminus \mathcal{F}_\alpha(x^\alpha)$, the monomials of $\mathcal{F}_\alpha(x^\alpha)$, then the monomials $\mathcal{F}_\alpha(x^\beta)$ and finally the remaining ones, the subspace $\mathcal{I}_r$ is represented by the matrix

$$
\begin{array}{c}
\begin{array}{ccc} \{I_r\} \setminus \mathcal{F}_\alpha(x^\alpha) \qquad & \qquad \mathcal{F}_\alpha(x^\alpha) \qquad & \qquad \mathcal{F}_\alpha(x^\beta) \qquad\qquad\qquad\qquad \end{array}\\
\left(\begin{array}{ccc|ccc|ccc|ccc}
1 & \cdots & 0 & & \vdots & & & \vdots & & & & \\
 & \ddots & & & & & & & & & \vdots & \\
\vdots & 1 & \vdots & \cdots & 0 & \cdots & \cdots & 0 & \cdots & \cdots & 0 & \cdots \\
 & & \ddots & & & & & & & & \vdots & \\
0 & \cdots & 1 & & \vdots & & & \vdots & & & & \vdots \\
\cline{1-9}
 & \vdots & & y_0 & & & y_1 & & & & & \cdots\ 0\ \cdots \\
\cdots & 0 & \cdots & & \ddots & & & \ddots & & & & \vdots \\
 & \vdots & & & & y_0 & & & y_1 & & &
\end{array}\right)
\end{array}
$$

The Plücker coordinates via $\mathscr{P}$ are the $q(r)$-minors of this matrix. The non-vanishing coordinates correspond to the submatrices obtained by all columns associated to the monomials of $\{I_r\} \setminus \mathcal{F}_\alpha(x^\alpha)$ plus $|\mathcal{F}_\alpha|$ columns chosen among those associated to the monomials in $\mathcal{F}_\alpha(x^\alpha) \cup \mathcal{F}_\alpha(x^\beta)$, with the constraint of picking a monomial from each pair $(F(x^\alpha), F(x^\beta))$. Called H the set of indices corresponding to the columns labeled with the monomials in $\{I_r\} \setminus \mathcal{F}_\alpha(x^\alpha)$, to any subset J of columns in $\mathcal{F}_\alpha(x^\alpha)$ the subset $\bar{\mathrm{J}}$ of columns of $\mathcal{F}_\alpha(x^\beta)$ giving a nonzero minor is uniquely associated and the Plücker coordinate $\Delta_{\mathrm{H}\cup\mathrm{J}\cup\bar{\mathrm{J}}}$ is equal to $y_0^{|\mathrm{J}|}y_1^{|\bar{\mathrm{J}}|}$. Hence there are $2^{|\mathcal{F}_\alpha|}$ nonzero coordinates, and for any pair J,J' of set of columns of $\mathcal{F}_\alpha(x^\alpha)$ such that $|\mathrm{J}| = |\mathrm{J}'|$, the Plücker coordinates $\Delta_{\mathrm{H}\cup\mathrm{J}\cup\bar{\mathrm{J}}}$ and $\Delta_{\mathrm{H}\cup\mathrm{J}'\cup\bar{\mathrm{J}}'}$ coincide, so that cutting with a suitable sets of hyperplanes we can obtain the curve $\mathbb{P}^1 \to \mathbb{P}^{|\mathcal{F}_\alpha|}$

$$[y_0 : y_1] \longmapsto \left[y_0^{|\mathcal{F}_\alpha|} : y_0^{|\mathcal{F}_\alpha|-1}y_1 : \ldots : y_0y_1^{|\mathcal{F}_\alpha|-1} : y_1^{|\mathcal{F}_\alpha|}\right]. \qquad \square$$

Example 3.1.6. Let us consider again the ideal $I = (x_3^2, x_3x_2, x_3x_1, x_2^3)_{\geqslant 4}$ introduced in Example 3.1.5 and the rational deformation defined by the ideal

$$\begin{aligned}\mathcal{I} = \big\langle &\{I_4\} \setminus \{x_3x_1^3, x_3x_1^2x_0, x_3x_1x_0^2\} \\ &\cup \{y_0\, x_3x_1^3 + y_1\, x_2^2x_1^2, y_0\, x_3x_1^2x_0 + y_1\, x_2^2x_1x_0, y_0\, x_3x_1x_0^2 + y_1\, x_2^2x_0^2\}\big\rangle.\end{aligned}$$

I defines a point of the Hilbert scheme $\mathbf{Hilb}^3_{3t+1}$ that we embed in a projective space by means of the Plücker embedding of the Grassmannian $\mathscr{P} : \mathbf{Gr}_{\mathbb{K}}(22, 35) \to \mathbb{P}^{\binom{35}{22}-1}$. Considered the correspondence (1.24) between indices from 1 up to 35 and multiindices defining monomials in $\mathbb{K}[x_0, x_1, x_2, x_3]_4$, we call H the set containing the indices associated to the monomials in $\{I_4\} \setminus \{x_3x_1^3, x_3x_1^2x_0, x_3x_1x_0^2\}$ and moreover we have

$$\begin{aligned} 13 &\leftrightarrow x_3x_1^3, & 12 &\leftrightarrow x_2^2x_1^2,\\ 23 &\leftrightarrow x_3x_1^2x_0, & 22 &\leftrightarrow x_2^2x_1x_0,\\ 29 &\leftrightarrow x_3x_1x_0, & 28 &\leftrightarrow x_2^2x_0^2.\end{aligned}$$

In order for obtaining a nonzero Plücker coordinate, we have to pick a monomial from each binomial, so there are 2^3 non-vanishing Plücker coordinates:

$$\begin{aligned}&\Delta_{\mathrm{H}\cup(13,23,29)} = y_0^3, \qquad \Delta_{\mathrm{H}\cup(12,23,29)} = \Delta_{\mathrm{H}\cup(13,22,29)} = \Delta_{\mathrm{H}\cup(13,23,28)} = y_0^2y_1,\\ &\Delta_{\mathrm{H}\cup(12,22,29)} = \Delta_{\mathrm{H}\cup(12,23,28)} = \Delta_{\mathrm{H}\cup(13,22,28)} = y_0y_1^2, \qquad \Delta_{\mathrm{H}\cup(12,22,28)} = y_1^3.\end{aligned}$$

Finally in the projective space $\mathbb{P}^3$ obtained by $\mathbb{P}^{\binom{35}{22}-1}$ cutting with the hyperplanes

$$\begin{cases} \Delta_{\mathrm{H}\cup(13,22,29)} - \Delta_{\mathrm{H}\cup(12,23,29)} = 0, \\ \Delta_{\mathrm{H}\cup(13,23,28)} - \Delta_{\mathrm{H}\cup(12,23,29)} = 0, \\ \Delta_{\mathrm{H}\cup(12,23,28)} - \Delta_{\mathrm{H}\cup(12,22,29)} = 0, \\ \Delta_{\mathrm{H}\cup(13,22,28)} - \Delta_{\mathrm{H}\cup(12,22,29)} = 0, \\ \Delta_{\mathrm{I}} = 0, \quad \text{I not appearing in the 8 coordinates listed above} \end{cases}$$

the curve is exactly the rational normal curves of degree 3.

Definition 3.6. Given two Borel-fixed ideals I and J that verify the hypothesis of Theorem 3.4 and Theorem 3.5, we call *Borel rational deformation* the deformation defined by the ideal (3.3) and *Borel rational curve* the corresponding curve on the Hilbert scheme. Moreover we will call J a *Borel degeneration* of I (and viceversa).

Remark 3.1.7. A Borel rational deformation between two Borel-fixed ideals I and $\overline{I}$ corresponds also to an edge connecting the vertices I and $\overline{I}$ in the *graph of monomial ideals* introduced by Altmann and Sturmfels in [2]. The vertices of this graph are the monomial ideals in $\mathbb{K}[x]$ and two ideals $I, \overline{I}$ are connected by an edge if there exists an ideal J such that the set of all its initial ideals (w.r.t. all term orderings) is $\{I, \overline{I}\}$.

Let $J = \mathcal{I}|_{[1:1]}$ be the ideal of the family described in (3.3) defining the deformation from I to $\widetilde{I}$. By construction x^α and x^β are not comparable w.r.t. the Borel partial order $\leq_B$, so for any term orderings σ, $x^\alpha >_\sigma x^\beta$ or $x^\alpha <_\sigma x^\beta$.

Let $k = \min x^\alpha = \min x^\beta$ and let $s = \max\{\mu_k \mid \mu_1 \mathsf{e}_1^- \circ \cdots \circ \mu_k \mathsf{e}_k^- = F \in \mathcal{F}_\alpha\}$. $x_k^s \mid \gcd(x^\alpha, x^\beta)$, then $x^\alpha = x^{\overline{\alpha}} x_k^s$, $x^\beta = x^{\overline{\beta}} x_k^s$ and $F(x^\alpha) = x^{\overline{\alpha}} F(x_k^s)$, $F(x^\beta) = x^{\overline{\beta}} F(x_k^s)$. Finally

$$x^\alpha = x^{\overline{\alpha}} x_k^s \gtrless_\sigma x^{\overline{\beta}} x_k^s = x^\beta \;\Rightarrow\; x^{\overline{\alpha}} \gtrless_\sigma x^{\overline{\beta}} \;\Rightarrow\; F(x^\alpha) = x^{\overline{\alpha}} F(x_k^s) \gtrless_\sigma x^{\overline{\beta}} F(x_k^s) = F(x^\beta),$$

for each $F \in \mathcal{F}_\alpha$. So the ideal J has $\{I, \overline{I}\}$ as set of possible initial ideals.

An algorithm for computing all the possible Borel rational deformations (or degenerations) of a Borel-fixed ideal naturally arises from Theorem 3.4. In Algorithm 3.1 and Algorithm 3.2 there are their pseudocode descriptions: the key point is to look in any restriction of the Borel set $\{I_m\}$ for minimal and maximal elements with the required property.

BorelRationalDeformations(I, m)

Input: $I \subset \mathbb{K}[x]$, Borel-fixed ideal.

Input: m, positive integer.

Output: the set of Borel rational deformations involving I, constructed considering the Borel set $\{I_m\}$.

deformations $\leftarrow \emptyset$;
for $k = 0, \ldots, n-1$ **do**
    minimalMonomials $\leftarrow$ MinimalElements$\big(\{I_m\}_{(\geqslant k)}\big)$;
    maximalMonomials $\leftarrow$ MaximalElements$\big(\{I_m\}^{\mathcal{C}}_{(\geqslant k)}\big)$;
    for all $x^\alpha \in$ minimalMonomials **do**
        for all $x^\beta \in$ maximalMonomials **do**
            if $\nexists\ \mathsf{e}_j^-,\ j > k$ s.t. $\mathsf{e}_j^-(x^\alpha) = x^\beta$ **then**
                $\mathcal{F}_\alpha \leftarrow$ DecreasingSet$(\{I_m\}, x^\alpha)$;
                if $\mathcal{F}_\alpha$ is Borel-admissible w.r.t. x^β **then**
                    $\mathcal{I} \leftarrow \big\langle \{I_m\} \setminus \mathcal{F}_\alpha(x^\alpha) \cup \big\{ y_0 F(x^\alpha) + y_1 F(x^\beta) \mid F \in \mathcal{F}_\alpha \big\} \big\rangle$;
                    deformations $\leftarrow$ deformations $\cup \{\mathcal{I}\}$;
                end if
            end if
        end for
    end for
end for
return deformations;

DecreasingSet$(\mathscr{B}, x^\alpha)$

Input: $\mathscr{B}$, a Borel set.

Input: x^α, a monomial of $\mathscr{B}$.

Output: the set of decreasing moves going from x^α to any other monomial $x^\gamma \in \mathscr{B}$, i.e. such that $x^\alpha >_B x^\gamma$.

Algorithm 3.1: Pseudocode description of the algorithm computing all the possible Borel rational deformations of a given Borel-fixed ideal I. For more details see Appendix B.

1: BORELRATIONALDEGENERATIONS(I, m)
Input: $I \subset \mathbb{K}[x]$, Borel-fixed ideal.
Input: m, positive integer.
Output: the set of Borel rational degenerations of I, constructed considering the Borel set $\{I_m\}$.
2: deformations $\leftarrow$ BORELRATIONALDEFORMATIONS(I, m);
3: degenerations $\leftarrow \emptyset$;
4: **for all** $\mathcal{I} \in$ deformations **do**
5: degenerations $\leftarrow$ degenerations $\cup \left\{\mathcal{I}|_{[0:1]}\right\}$;
6: **end for**
7: **return** deformations;

Algorithm 3.2: Pseudocode description of the algorithm computing all the possible Borel rational degeneration of a given Borel-fixed ideal I. For more details see Appendix B.

Example 3.1.8. Let us simulate an execution of Algorithm 3.1 on the ideal $I = (x_3^2, x_3x_2^2, x_3x_2x_1, x_2^4, x_2^3x_1, x_2^2x_1^2) \subset \mathbb{K}[x_0, x_1, x_2, x_3]$ (see Figure 3.5a for its Green's diagram). The corresponding subscheme $\operatorname{Proj} \mathbb{K}[x_0, x_1, x_2, x_3]/I$ has Hilbert polynomial $p(t) = 3t + 5$, whose Gotzmann number is 8, so we consider the Borel set $\{I_8\}$.

$k = 0$. The minimal monomials of $\{I_8\}$ are $\{x_3^2x_0^6, x_3x_2x_1x_0^5, x_2^2x_1^2x_0^4\}$ and the maximal elements of $\{I_8\}^{\mathcal{C}}$ are $\{x_3x_2x_0^6, x_2^3x_0^5\}$. The pairs $(x_3x_2x_1x_0^5, x_3x_2x_0^6)$ and $(x_3^2x_0^6, x_3x_2x_0^6)$ are discarded because $\mathsf{e}_1^-(x_3x_2x_1x_0^5) = x_3x_2x_0^6$ and $\mathsf{e}_3^-(x_3^2x_0^6) = x_3x_2x_0^6$, so we have four possibilities. In the case $k = 0$, the Borel-admissibility derives directly from the fact that the decreasing set contains only the identity move, so we do not need to check further conditions.

(Figure 3.5b) $x_3^2x_0^6$ and $x_2^3x_0^5$ define the Borel rational deformation

$$\left\langle \{I_8\} \setminus \{x_3^2x_0^6\} \cup \{y_0\, x_3^2x_0^6 + y_1x_2^3x_0^5\} \right\rangle \subset \mathbb{K}[y_0, y_1][x_0, x_1, x_2, x_3]$$

and the fiber over the point $[0 : 1]$ is the ideal $(x_3^3, x_3^2x_2, x_3x_2^2, x_2^3, x_3^2x_1, x_3x_2x_1, x_2^2x_1^2)_{\geqslant 8}$.

(Figure 3.5c) $x_3x_2x_1x_0^5$ and $x_2^3x_0^5$ define the deformation

$$\left\langle \{I_8\} \setminus \{x_3x_2x_1x_0^5\} \cup \{y_0\, x_3x_2x_1x_0^5 + y_1x_2^3x_0^5\}\right\rangle \subset \mathbb{K}[y_0,y_1][x_0,x_1,x_2,x_3]$$

with fiber over the point $[0:1]$ equal to $(x_3^2, x_3x_2^2, x_2^3, x_3x_2x_1^2, x_2^2x_1^2)_{\geqslant 8}$.

(Figure 3.5d) $x_2^2x_1^2x_0^4$ and $x_3x_2x_0^6$ define the deformation

$$\left\langle \{I_8\} \setminus \{x_2^2x_1^2x_0^4\} \cup \{y_0\, x_2^2x_1^2x_0^4 + y_1x_3x_2x_0^6\}\right\rangle \subset \mathbb{K}[y_0,y_1][x_0,x_1,x_2,x_3]$$

and the fiber over the point $[0:1]$ is the ideal $(x_3^2, x_3x_2, x_2^4, x_2^3x_1, x_2^2x_1^3)_{\geqslant 8}$.

(Figure 3.5e) $x_2^2x_1^2x_0^4$ and $x_2^3x_0^5$ define the deformation

$$\left\langle \{I_8\} \setminus \{x_2^2x_1^2x_0^4\} \cup \{y_0\, x_2^2x_1^2x_0^4 + y_1x_2^3x_0^5\}\right\rangle \subset \mathbb{K}[y_0,y_1][x_0,x_1,x_2,x_3]$$

with the ideal $(x_3^2, x_3x_2^2, x_2^3, x_3x_2x_1, x_2^2x_1^3)_{\geqslant 8}$ as fiber over $[0:1]$.

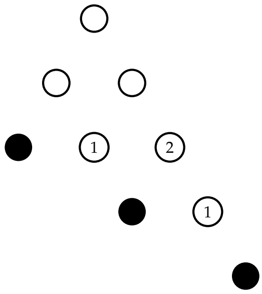
$k=1$. $\{I_8\}_{(\geqslant 1)}$ has only $x_2^2x_1^6$ as minimal element and $x_3x_1^7$ is the only maximal element of $\{I_8\}^{\mathcal{C}}_{(\geqslant 1)}$. The decreasing set of $x_2^2x_1^6$ is

$$\mathcal{F}_{x_2^2x_1^6} = \{\mathsf{id}, \mathsf{e}_1^-, 2\mathsf{e}_1^-, 3\mathsf{e}_1^-, 4\mathsf{e}_1^-\},$$

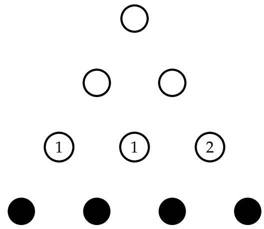
Borel-admissible w.r.t. $x_3x_1^7$. The Borel rational deformation defined by $x_2^2x_1^6$ and $x_3x_1^7$ is

$$\begin{gathered}\big\langle \{I_8\} \setminus \{x_2^2x_1^6, x_2^2x_1^5x_0, x_2^2x_1^4x_0^2, x_2^2x_1^3x_0^3, x_2^2x_1^2x_0^4\} \cup \{y_0x_2^2x_1^6 + y_1x_3x_1^7,\\ y_0x_2^2x_1^5x_0 + y_1x_3x_1^6x_0, y_0x_2^2x_1^4x_0^2 + y_1x_3x_1^5x_0^3, y_0x_2^2x_1^3x_0^3 + y_1x_3x_1^4x_0^3,\\ y_0x_2^2x_1^2x_0^4 + y_1x_3x_1^3x_0^4\}\big\rangle \subset \mathbb{K}[y_0,y_1][x_0,x_1,x_2,x_3]\end{gathered}$$

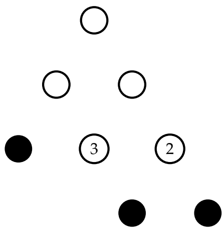
and the fiber over $[0:1]$ is the ideal $(x_3^2, x_3x_2^2, x_3x_2x_1, x_2^4, x_2^3x_1, x_3x_1^3)_{\geqslant 8}$ (Figure 3.5f).



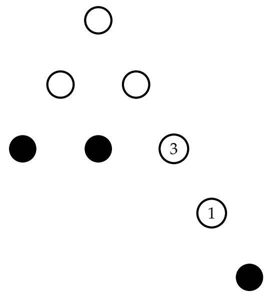
(a) $(x_3^2, x_3x_2^2, x_3x_2x_1, x_2^4, x_2^3x_1, x_2^2x_1^2)$.



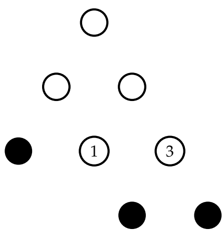
(b) $(x_3^3, x_3^2x_2, x_3x_2^2, x_2^3, x_3^2x_1, x_3x_2x_1, x_2^2x_1^2)$.



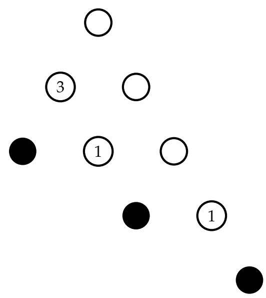
(c) $(x_3^2, x_3x_2^2, x_2^3, x_3x_2x_1^2, x_2^2x_1^2)$.



(d) $(x_3^2, x_3x_2, x_2^4, x_2^3x_1, x_2^2x_1^3)$.



(e) $(x_3^2, x_3x_2^2, x_2^3, x_3x_2x_1, x_2^2x_1^3)$.



(f) $(x_3^2, x_3x_2^2, x_3x_2x_1, x_2^4, x_2^3x_1, x_3x_1^3)$.

Figure 3.5: Green's diagrams of the Borel-fixed ideals introduced in Example 3.1.8.

Remark 3.1.9. We briefly discuss our method in relation with the Hilbert function and so with the results exposed by Mall in [61]. Since our interest is oriented toward Hilbert schemes, where the crucial aspect is the Hilbert polynomial, we usually prefer to consider as ideal I defining a subschemes with Hilbert polynomial $p(t)$ as generated in degree r, i.e. $I = I_{\geqslant r} = (I_r)$ such that the Hilbert function of $\mathbb{K}[x]/I$ is

$$HF_{\mathbb{K}[x]/I}(t) = \begin{cases} \binom{n+t}{n}, & \text{if } t < r, \\ p(t), & \text{if } t \geqslant r, \end{cases}$$

where r is the Gotzmann number of the Hilbert polynomial $p(t)$. However, it is possible to adapt the technique to work also on the Hilbert function of the quotient modules defined by the saturation of such Borel ideals. We highlight this point starting from Example 3.6 and Example 3.9 of [61]. Mall showed that the ideal $J = (x_3^2, x_3x_2, x_3x_1 - x_2^2) \subset \mathbb{K}[x_0, x_1, x_2, x_3]$ has only two possible initial ideals (varying the term ordering): $I = (x_3^2, x_3x_2, x_3x_1, x_2^3)$ and $\overline{I} = (x_3^2, x_3x_2, x_2^2)$, so that these ideals are connected through two Gröbner deformations and moreover the quotient modules defined have the same Hilbert function $HF_{\mathbb{K}[x]/I}(t) = HF_{\mathbb{K}[x]/\overline{I}}(t) = (1, 4, 7, 10, 3t + 1, \ldots)$.

In our perspective, since the Hilbert polynomial of $\mathbb{K}[x]/I$ is $p(t) = 3t + 1$ with Gotzmann number $r = 4$, we would consider the ideals $I_{\geqslant 4}$ and $\overline{I}_{\geqslant 4}$. Applying Algorithm 3.1 on the Borel set $\{\overline{I}_4\}$, we obtain the deformation

$$\begin{aligned}\overline{J} =& \big\langle \{\overline{I}_4\} \setminus \{x_2^2x_1^2, x_2^2x_1x_0, x_2^2x_0^2\} \\ & \cup \{y_0\, x_2^2x_1^2 + y_1\, x_3x_1^3, y_0\, x_2^2x_1x_0 + y_1\, x_3x_1^2x_2, y_0\, x_2^2x_0^2 + y_1\, x_3x_1x_0^2\}\big\rangle\end{aligned}$$

with fibers $I_{\geqslant 4}$ and $\overline{I}_{\geqslant 4}$. Keeping in mind that to compute the saturation of a Borel-fixed ideal it is sufficient to put the smallest variable x_0 equal to 1, $I = (I_{\geqslant 4})^{\text{sat}}$ and $\overline{I} = (\overline{I}_{\geqslant 4})^{\text{sat}}$ must have the same Hilbert function because we are swapping pairs of monomials with the same power of the variable x_0. Furthermore we can deduce also the deformation between I and $\overline{I}$: first of all we consider the saturation of the ideal $\big\langle \{\overline{I}_2\}_4 \setminus \{x_2^2x_1^2, x_2^2x_1x_0, x_2^2x_0^2\}\big\rangle = (x_3^2, x_3x_2, x_2^3)$ (the common part of I and $\overline{I}$) and then we add to the generators the binomial $y_0\, x_2^2 + y_1\, x_3x_1$

$$\overline{J}^{\text{sat}} = (x_3^2, x_3x_2, x_2^3, y_0\, x_2^2 + y_1\, x_3x_1).$$

Hence to have deformations preserving the Hilbert function of the saturation of the Borel-fixed ideals involved, the key point is to swap couples of monomials having the same power of the smallest variable (in our case x_0). This happens by construction whenever the deformation is defined by monomials x^α, x^β such that $\min x^\alpha = \min x^\beta > 0$, whereas it happens rarely if the deformation is ruled by a single couple of monomials as the following example shows. Let us consider the ideal $I_{\geqslant 4}$ and the ideal $\widetilde{I} = (x_3, x_2^4, x_2^3 x_1)_{\geqslant 4}$, fibers of the deformation

$$\left\langle \{I_4\} \setminus \left\{x_2^3 x_0\right\} \cup \left\{y_0\, x_2^3 x_0 + y_1\, x_3 x_0^3\right\}\right\rangle .$$

The monomials involved in the deformation have not the same power of x_0, indeed

$$\begin{aligned} HF_{\mathbb{K}[x]/I}(t) &= (1,4,7,10,3t+1,\dots)\,, \\ HF_{\mathbb{K}[x]/\widetilde{I}^{\text{sat}}}(t) &= (1,3,6,10,3t+1,\dots)\,. \end{aligned}$$

3.2 The connectedness of the Hilbert scheme

In this section, we will study consecutive deformations of Borel-fixed ideals and we want to introduced a way to control the "direction" toward which we move, for obtaining a technique similar to the one introduced by Peeva and Stillman [83]. The deformations introduced in [83] are affine and based on Gröbner basis tools, but anyhow the idea is to exchange monomials belonging to the ideal with monomials not belonging. The goal is to determine a sequence of deformations leading from any Borel-fixed ideal to the lexicographic ideal, so the choice of the monomials to exchange is governed by the `DegLex` term ordering.

Therefore we start by slightly modifying the strategy of Algorithm 3.1 and Algorithm 3.2, adding a term ordering σ, refinement of the Borel partial order $\leq_B$, that we will use to choose in a unique way one of the possible Borel rational deformations of an ideal. The point is to replace some monomials of the ideal with some others that are greater that them w.r.t. σ.

ORIENTEDBORELRATIONALDEGENERATION(I, σ)

Input: $I \subset \mathbb{K}[x]$, Borel-fixed ideal.

Input: σ, term ordering, refinement of the Borel partial order $\leq_B$.

Output: a saturated Borel-fixed ideal J, σ-Borel degeneration of I.

If I has not a σ-Borel degeneration, the function returns the ideal itself.

$p(t) \leftarrow$ HILBERTPOLYNOMIAL$(\mathbb{K}[x]/I)$;

$r \leftarrow$ GOTZMANNNUMBER$\big(p(t)\big)$;

for $k = 0, \ldots, n-1$ **do**

  $x^\alpha \leftarrow \min_\sigma \{I_r\}_{(\geqslant k)}$;

  $x^\beta \leftarrow \max_\sigma \{I_r\}^{\mathcal{C}}_{(\geqslant k)}$;

  if $x^\alpha <_\sigma x^\beta$ **then**

    $\mathcal{F}_\alpha \leftarrow$ DECREASINGSET$(\{I_r\}, x^\alpha)$;

    if $\mathcal{F}_\alpha$ is Borel-admissible **then**

      return $\big\langle \{I_r\} \setminus \mathcal{F}_\alpha(x^\alpha) \cup \mathcal{F}_\alpha(x^\beta) \big\rangle^{\text{sat}}$;

    end if

  end if

end for

return I;

Algorithm 3.3: How to find a Borel rational degeneration with special "direction" of a fixed term ordering. For details see Appendix B.

We remark that x^α and x^β are surely a minimal element of $\{I_r\}_{(\geqslant k)}$ and a maximal element of $\{I_r\}^{\mathcal{C}}_{(\geqslant k)}$, by Remark 2.3.4. Moreover the condition $x^\alpha <_\sigma x^\beta$ guarantees that x^β can not be obtained from x^α by an elementary move e_j^-. In fact if such a move exists, $x^\alpha >_B x^\beta$ implies $x^\alpha >_\sigma x^\beta$ for any term ordering σ.

The uniqueness of the deformation is imposed by making the algorithm returning the first deformation with the property $x^\alpha <_\sigma x^\beta$. If Algorithm 3.3 does not find any deformation, it returns the same ideal I given as input; for instance this happens if we apply the algorithm to the lexicographic ideal with the `DegLex` term ordering. Indeed $x^\gamma >_{\texttt{DegLex}} x^\delta, \forall\, x^\gamma \in \{I_m\},\ x^\delta \in \{I_m\}^{\mathcal{C}},\ \forall\, m$ by definition, thus the condition $x^\alpha <_\sigma x^\beta$ would always be false.

Definition 3.7. Given a Borel-fixed ideal I and a term ordering σ, we say that

- the ideal $J \neq I$ returned by ORIENTEDBORELRATIONALDEGENERATION(I,σ) (Algorithm 3.3) is a *σ-Borel rational degeneration* or simply a *σ-Borel degeneration* of I and we call *σ-Borel rational deformation* of I the Borel rational deformation having as fibers I and J;
- I is a *σ-endpoint* if ORIENTEDBORELRATIONALDEFORMATION(I,σ) returns the same ideal I.

After having determined a method similar to the one proposed in [83], we want to compare these two approaches, so we recall briefly the strategy and some notation of that paper. Given an ideal $I = I_{\geqslant r} \subset \mathbb{K}[x]$, Peeva and Stillman compute the monomials $x^\beta = \max_{\texttt{DegLex}}\{I_r\}^{\mathcal{C}}$ (they call it *first gap*) and $x^\alpha = \max_{\texttt{DegLex}}\{x^\delta \in \{I_r\} \mid x^\delta <_{\texttt{DegLex}} x^\beta\}$. If I is the lexicographic ideal, the set $\{x^\delta \in \{I_r\} \mid x^\delta <_{\texttt{DegLex}} x^\beta\}$ will be obviously empty. At this point they determine the set of monomials

$$T = \{x^{\overline{\alpha}}x^\gamma \in \{I_r\} \mid x^\alpha = x^{\overline{\alpha}} \cdot (\min x^\alpha)^s,\ x^{\overline{\beta}}x^\gamma \notin \{I_r\} \text{ and } \max x^\gamma <_B \min x^{\overline{\beta}}\}$$

and they fix the monomial $x^{\overline{\alpha}}$ of minimal degree and the corresponding $x^{\overline{\beta}}$ of minimal degree. Let us denote them by $x^{\widetilde{\alpha}}$ and $x^{\widetilde{\beta}}$. They finally form the ideal

$$\widetilde{I} = \Big\langle \{I_r\} \setminus \{x^{\widetilde{\alpha}}x^\gamma \in \{I_r\}\} \cup \{x^{\widetilde{\alpha}}x^\gamma - x^{\widetilde{\beta}}x^\gamma \mid x^{\widetilde{\alpha}}x^\gamma \in \{I_r\}\}\Big\rangle, \quad \max x^\gamma \leq_B \min x^{\widetilde{\beta}}.$$

The Borel ideal "DegLex-closer" to the lexicographic ideal is $\mathsf{gin}_{\texttt{DegLex}}(\mathsf{in}_{\texttt{DegLex}}(\widetilde{I}))$.

The computation of a generic initial ideal reveals an important difference between the two techniques: this choice of monomials involved in the substitution generally does not preserve the Borel condition, so that to restore it a computation of a gin could be needed, as the following example shows.

Example 3.2.1. Let us consider the ideal $I = (x_2^4, x_2^3x_1, x_2^2x_1^2, x_2x_1^3)_{\geqslant 7} \subset \mathbb{K}[x_0,x_1,x_2]$. The Hilbert polynomial is $p(t) = t+7$ with Gotzmann number 7. The first gap is $x_2^3x_0^4$ and the DegLex-greatest monomial in $\{I_7\}$, DegLex-smaller than the gap, is $x_2^2x_1^5$.

$$T = \left\{x_2^2x_1^2x_0^3, x_2^2x_1^3x_0^2, x_2^2x_1^4x_0, x_2^2x_1^5\right\}$$

so that $x^{\widetilde{\alpha}} = x_2^2 x_1^2$ and $x^{\widetilde{\beta}} = x_2^3 x_0$. We construct the ideal

$$\widetilde{I} = \Big\langle \{I_7\} \setminus \{x_2^2 x_1^2 x_0^3\} \cup \{x_2^2 x_1^2 x_0^3 - x_2^3 x_0^4\} \Big\rangle,$$

and $\mathsf{in}_{\mathtt{DegLex}}(\widetilde{I}) = (x_2^3, x_2 x_1^3)_{\geqslant 7}$ is not Borel-fixed, because $x_2 x_1^3 x_0^3 \in \mathsf{in}_{\mathtt{DegLex}}(\widetilde{I})$ and $\mathsf{e}_1^+(x_2 x_1^3 x_0^3) = x_2^2 x_1^2 x_0^3 \notin \mathsf{in}_{\mathtt{DegLex}}(\widetilde{I})$. Finally

$$\mathsf{gin}_{\mathtt{DegLex}}\big(\mathsf{in}_{\mathtt{DegLex}}(\widetilde{I})\big) = (x_2^3, x_2^2 x_1^2, x_2 x_1^4)_{\geqslant 7}$$

is again Borel-fixed.

Using Algorithm 3.3 to compute the DegLex-Borel degeneration of I, we have

$$\min_{\mathtt{DegLex}} \{I_7\} = x_2 x_1^3 x_0^3 <_{\mathtt{DegLex}} x_2^3 x_0^4 = \max_{\mathtt{DegLex}} \{I_7\}^{\mathcal{C}},$$

and since the decreasing set of $x_2 x_1^3 x_0^3$ contains only the identity, the DegLex-Borel degeneration of I is the ideal $J = \langle \{I_7\} \setminus \{x_2 x_1^3 x_0^3\} \cup \{x_2^3 x_0^4\} \rangle = (x_2^3, x_2^2 x_1^2, x_2 x_1^4)_{\geqslant 7}$.

The choice of the first gap is very similar to the choice of the maximal monomial in $\{I_r\}^{\mathcal{C}}$, because in both cases we look for a "greatest" element. But the two techniques differ in the choice of the monomial inside the ideal: we look for minimal elements whereas Peeva and Stillman consider maximal elements lower than the gap. Hence we expect that in the cases in which the monomials chosen are the same, the deformation is almost equal, in the sense that the monomial ideal we reach is the same.

Example 3.2.2. We consider the ideal $I = (x_3^2, x_3 x_2, x_2^3, x_2^2 x_1)_{\geqslant 5} \subset \mathbb{K}[x_0, x_1, x_2, x_3]$. The Hilbert polynomial is $p(t) = 3t + 2$ with Gotzmann number 5. The first gap is $x_3 x_1^4$ and the greatest monomial of the ideal smaller than it is x_2^5. Then

$$T = \left\{ x_2^2 x_1^3, x_2^2 x_1^2 x_0, x_2^2 x_1 x_0^2, x_2^3 x_1^2, x_2^3 x_1 x_0, x_2^3 x_0^2, x_2^4 x_1, x_2^4 x_0, x_2^5 \right\},$$

$x^{\widetilde{\alpha}} = x_2^2$ and $x^{\widetilde{\beta}} = x_3 x_1$, so that

$$\begin{aligned}\widetilde{I} = \Big\langle &\{I_5\} \setminus \{x_2^2 x_1^3, x_2^2 x_1^2 x_0, x_2^2 x_1 x_0^2\} \\ &\cup \left\{ x_2^2 x_1^3 - x_3 x_1^4, x_2^2 x_1^2 x_0 - x_3 x_1^3 x_0, x_2^2 x_1 x_0^2 - x_3 x_1^2 x_0^2 \right\} \Big\rangle\end{aligned}$$

and $\mathsf{in}_{\mathtt{DegLex}}(\widetilde{I}) = \mathsf{gin}_{\mathtt{DegLex}}\big(\mathsf{in}_{\mathtt{DegLex}}(\widetilde{I})\big) = (x_3^2, x_3x_2, x_3x_1^2, x_2^3)_{\geqslant 5}$.

Applying Algorithm 3.3 on I with the DegLex term ordering, we have

$$\min_{\mathtt{DegLex}}\{I_5\} = x_2^2x_1x_0^2 >_{\mathtt{DegLex}} x_2^2x_0^3 = \max_{\mathtt{DegLex}}\{I_5\}^{\mathcal{C}},$$
$$\min_{\mathtt{DegLex}}\{I_5\}_{(\geqslant 1)} = x_2^2x_1^3 <_{\mathtt{DegLex}} x_3x_1^4 = \max_{\mathtt{DegLex}}\{I_5\}^{\mathcal{C}}_{(\geqslant 1)}.$$

Then the algorithm determines $\mathcal{F}_{x_2^2x_1^3} = \{\mathrm{id}, \mathsf{e}_1^-, 2\mathsf{e}_1^-\}$ that is Borel-admissible w.r.t. $x_3x_1^4$, and swapping $\mathcal{F}_{x_2^2x_1^3}(x_2^2x_1^3) = \{x_2^2x_1^3, x_2^2x_1^2x_0, x_2^2x_1x_0^2\}$ with $\mathcal{F}_{x_2^2x_1^3}(x_3x_1^4) = \{x_3x_1^4, x_3x_1^3x_0, x_3x_1^2x_0^2\}$ we obtain again the ideal

$$J = (x_3^2, x_3x_2, x_3x_1^2, x_2^3)_{\geqslant 5} = \mathsf{in}_{\mathtt{DegLex}}(\widetilde{I}).$$

Our next goal is to prove that also with our method it is possible to reach from any point defined by a Borel-fixed ideal on the Hilbert scheme $\mathbf{Hilb}^n_{p(t)}$ some special point, by a sequence of consecutive Borel rational degenerations. Of course the lexicographic ideal defines a special point, but since we build the deformations working exclusively on the Borel sets defined by the piece of degree equal to r of the ideals (r is always the Gotzmann number of $p(t)$), we can generalize the property to the class of hilb-segment ideals (Definition 2.60).

Chosen a Hilbert scheme $\mathbf{Hilb}^n_{p(t)}$ and a term ordering σ, we want to have an overview on all σ-Borel rational degenerations among Borel-fixed ideals defining $\mathbb{K}$-rational points of $\mathbf{Hilb}^n_{p(t)}$.

Definition 3.8. Let $\mathbf{Hilb}^n_{p(t)}$ be the Hilbert scheme parametrizing subschemes of the projective space $\mathbb{P}^n$ with Hilbert polynomial $p(t)$ and let σ be any term ordering, refinement of the Borel partial order $\leq_B$. We define the *σ-degeneration graph* of $\mathbf{Hilb}^n_{p(t)}$ as the graph whose vertices correspond to all (saturated) Borel-fixed ideals defining points of $\mathbf{Hilb}^n_{p(t)}$ and whose edges represent the σ-Borel rational deformations, i.e. any edge goes from a Borel-fixed ideal to its σ-Borel degeneration. The algorithm computing the σ-degeneration graph is described in Algorithm 3.4.

Theorem 3.9. *Let $\mathbf{Hilb}^n_{p(t)}$ be a Hilbert scheme and let σ be a term ordering. If there exists a Borel-fixed ideal I defining a point of $\mathbf{Hilb}^n_{p(t)}$ which is a hilb-segment ideal w.r.t. σ, then the σ-degeneration graph of $\mathbf{Hilb}^n_{p(t)}$ is a rooted tree, with the ideal I as root.*

DegenerationGraph($\mathbf{Hilb}^n_{p(t)}, \sigma$)
Input: $\mathbf{Hilb}^n_{p(t)}$, Hilbert scheme.
Input: σ, term ordering, refinement of the Borel partial order $\leq_B$.
Output: the σ-degeneration graph (vertices, edges) of $\mathbf{Hilb}^n_{p(t)}$.
vertices $\leftarrow$ BorelGeneratorDFS($\mathbb{K}[x_0, \ldots, x_n], p(t)$);
edges $\leftarrow \emptyset$;
for all $I \in$ vertices **do**
    $J \leftarrow$ OrientedBorelRationalDeformation(I, σ);
    if $J \neq I$ **then**
        edges $\leftarrow$ edges $\cup \{(I, J)\}$;
    end if
end for
return (vertices, edges);

Algorithm 3.4: How to compute the σ-degeneration graph associated to a Hilbert scheme. For details see Appendix B.

We recall briefly what we mean by rooted tree. A *tree* is a connected graph (V, E), such that $|E| = |V| - 1$. A *rooted tree* is a tree in which a fixed vertex (the root) determines a natural orientation of the edges, "toward to" and "away from" the root.

Proof. By definition I is a σ-endpoint, because I could not have σ-Borel degeneration, so I is the natural root of the graph. To prove that the σ-degeneration graph is a rooted tree, it is sufficient to show that any other Borel ideal $J \neq I$ has a σ-Borel degeneration.

Let r be the Gotzmann number of $p(t)$. For any ideal $J \neq I$, there exists a pair of monomials (x^α, x^β) such that $\{J_r\} \ni x^\alpha <_\sigma x^\beta \in \{J_r\}^{\mathcal{C}}$. So let $0 \leqslant k < n$ be the integer such that

$$\min_\sigma \{J_r\}_{(\geqslant i)} >_\sigma \max_\sigma \{J_r\}^{\mathcal{C}}_{(\geqslant i)}, \qquad \forall\, i = 0, \ldots, k-1$$

and

$$\min_\sigma \{J_r\}_{(\geqslant k)} = x^\alpha <_\sigma x^\beta = \max_\sigma \{J_r\}^{\mathcal{C}}_{(\geqslant k)}.$$

Let $\mathcal{F}_\alpha$ be the decreasing set of x^α. If $\mathcal{F}_\alpha$ is Borel-admissible w.r.t. x^β, we finish because Algorithm 3.3 applied on J and σ does not return ideal J itself. Then let us assume that $\mathcal{F}_\alpha$ is not Borel-admissible w.r.t. x^β: we want to show that there must exist a σ-Borel degeneration of J determined by a pair of monomials $x^{\overline{\alpha}}$ and $x^{\overline{\beta}}$ such that $\min x^{\overline{\alpha}} = \min x^{\overline{\beta}} < k$, i.e. a σ-Borel degeneration of J that Algorithm 3.3 should find before examining the restriction $\{J_r\}_{(\geqslant k)}$.

The first reason for which $\mathcal{F}_\alpha$ could be not Borel-admissible w.r.t. x^β is the existence of some decreasing move in $\mathcal{F}_\alpha$ not admissible w.r.t. x^β. Let $\mathcal{G} \subset \mathcal{F}_\alpha$ be the set of the decreasing moves admissible on both monomials. Since $x^\alpha \in J$ could not belong to the hilb-segment ideal I, also every monomial in $\mathcal{F}_\alpha(x^\alpha)$ does not belong to I. To go back to I, the monomials in $\mathcal{G}(x^\alpha)$ could be replaced by the monomials in $\mathcal{G}(x^\beta)$ and the others in $\mathcal{F}_\alpha(x^\alpha) \setminus \mathcal{G}(x^\alpha)$ should to be replaced by monomials not obtained by decreasing moves from monomials in $\{J_r\}_k$, that is $\max_\sigma\{J_r\}^{\mathcal{C}}_{(\geqslant i)} >_\sigma \min_\sigma\{J_r\}_{(\geqslant i)}$, for some $i < s$.

The second reason for which $\mathcal{F}_\alpha$ could be not Borel-admissible is the existence of a move $F \in \mathcal{F}_\alpha$ such that the monomial $F(x^\beta)$ is not a k-maximal element, namely there exists e_j^+, $j \geqslant k$, such that $\mathsf{e}_j^+\big(F(x^\beta)\big) \notin \{J_r\}$. Let $i = \min F(x^\alpha) = \min F(x^\beta) < k$. $\mathsf{e}_j^+ F(x^\beta)$ can not be obtained by a monomial in $\{J_r\}^{\mathcal{C}}_{(\geqslant k)}$ applying a composition of decreasing elementary moves G, because $G(x^\delta) = \mathsf{e}_j^+ F(x^\beta)$ implies $x^\delta >_B x^\beta$ in contradiction with the hypothesis $x^\beta = \max_\sigma\{J_r\}_{(\geqslant k)}$. Since $x^\alpha <_\sigma x^\beta \Rightarrow F(x^\alpha) <_\sigma F(x^\beta)$,

$$\max_\sigma\{J_r\}_{(\geqslant i)} \geq_\sigma \mathsf{e}_j^+\big(F(x^\beta)\big) >_\sigma F(x^\beta) >_\sigma F(x^\alpha) \geq_\sigma \min_\sigma\{J_r\}_{(\geqslant k)}.$$

In both cases, there exist $i < k$ and a pair of monomials $x^{\overline{\alpha}}, x^{\overline{\beta}}$

$$x^{\overline{\beta}} = \max_\sigma\{J_r\}_{(\geqslant i)} >_\sigma \min_\sigma\{J_r\}_{(\geqslant i)} = x^{\overline{\alpha}}.$$

We compute again the decreasing set $\mathcal{F}_{\overline{\alpha}}$ and we check if it is Borel-admissible w.r.t. $x^{\overline{\beta}}$: if not we repeat the reasoning and we look for monomials involving more variables. Finally, we are sure to find a Borel-admissible set of decreasing moves because if we reach the smallest variable x_0, the decreasing set only contains the identity move. Note that it is not possible to have cycles thanks total order on the monomials, because each σ-Borel degeneration approaches to the hilb-segment ideal. $\square$

Corollary 3.10. *The Hilbert scheme* $\mathbf{Hilb}^n_{p(t)}$ *is connected.*

Proof. Let I be any ideal defining a point on $\mathbf{Hilb}^n_{p(t)}$. As usual through an affine Gröbner degeneration the point defined by I can be connected to the point defined by the Borel-fixed ideal $\mathsf{gin}(I)$.

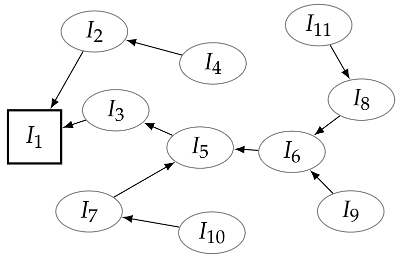
Of course on $\mathbf{Hilb}^n_{p(t)}$, there is the lexicographic point corresponding to the lexicographic ideal (Proposition 2.58), hilb-segment ideal w.r.t. `DegLex`. By Theorem 3.9, the `DegLex`-degeneration graph is a *connected* rooted tree, so the point defined by any ideal I can be connected to the lexicographic point by an initial affine degeneration and a sequence of `DegLex`-Borel rational degenerations. □

We underline that if $\mathbf{Hilb}^n_{p(t)}$ does not contain a point defined by a hilb-segment ideal w.r.t. σ, the σ-degeneration graph could be not connected, as the following example shows.

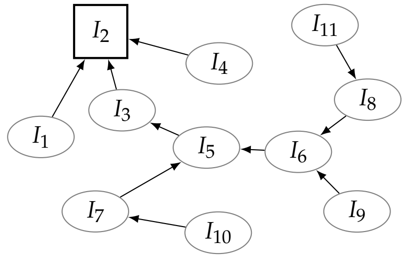
Example 3.2.3. Let us consider the Hilbert scheme $\mathbf{Hilb}^3_{6t-5}$. In $\mathbb{K}[x_0, x_1, x_2, x_3]$ there are 11 saturated Borel-fixed ideals with Hilbert polynomial $6t-5$ (whose Gotzmann number is 10) and many of them are hilb-segment ideals. In the following list of the ideals, we specify the term ordering (computed with Algorithm 2.7) for which the corresponding ideal becomes (possibly) a hilb-segment ideal:

$$
\begin{array}{ll}
I_1 = (x_3, x_2^7, x_2^6x_1^4), & \texttt{DegLex};\\
I_2 = (x_3, x_2^8, x_2^7x_1, x_2^6x_1^3), & \omega_2 = (37,6,3,1);\\
I_3 = (x_3^2, x_3x_2, x_3x_1, x_2^7, x_2^6x_1^3), & \omega_3 = (21,4,2,1);\\
I_4 = (x_3^2, x_3x_2, x_3x_1, x_2^8, x_2^7x_1, x_2^6x_1^2); & \\
I_5 = (x_3^2, x_3x_2, x_3x_1^2, x_2^7, x_2^6x_1^2), & \omega_5 = (25,5,2,1);\\
I_6 = (x_3^2, x_3x_2, x_3x_1^3, x_2^7, x_2^6x_1), & \omega_6 = (29,6,2,1);\\
I_7 = (x_3^2, x_3x_2^2, x_3x_2x_1, x_3x_1^2, x_2^7, x_2^6x_1); & \\
I_8 = (x_3^2, x_3x_2, x_3x_1^4, x_2^6), & \omega_8 = (33,7,2,1);\\
I_9 = (x_3^2, x_3x_2^2, x_3x_2x_1, x_3x_1^3, x_2^6); & \\
I_{10} = (x_3^3, x_3^2x_2, x_3x_2^2, x_3^2x_1, x_3x_2x_1, x_3x_1^2, x_2^6); & \\
I_{11} = (x_3^2, x_3x_2, x_2^5), & \omega_{11} = (33,11,2,1).
\end{array}
$$

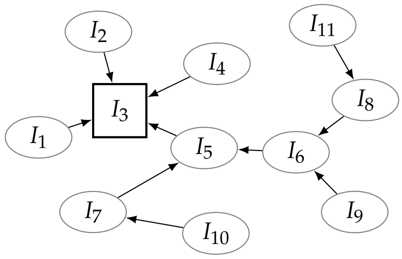
The degeneration graphs of $\mathbf{Hilb}^n_{p(t)}$ w.r.t. all the term ordering listed above turn out to be rooted trees as shown in Figures 3.6a–3.6g.



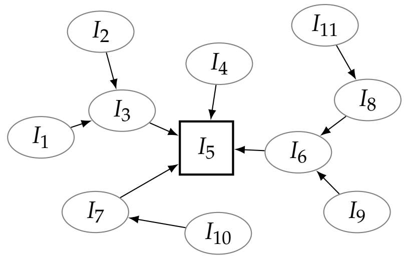
(a) The `DegLex`-deformation graph of $\mathbf{Hilb}^3_{6t-5}$.



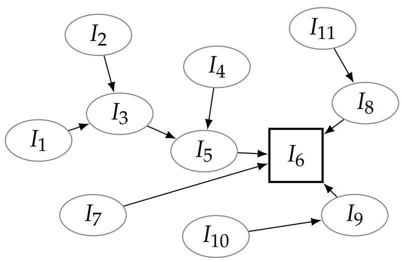
(b) The ω_2-deformation graph of $\mathbf{Hilb}^3_{6t-5}$.



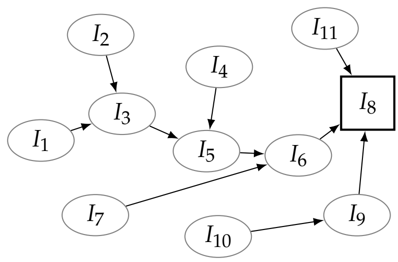
(c) The ω_3-deformation graph of $\mathbf{Hilb}^3_{6t-5}$.



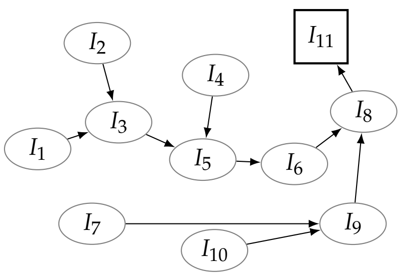
(d) The ω_5-deformation graph of $\mathbf{Hilb}^3_{6t-5}$.



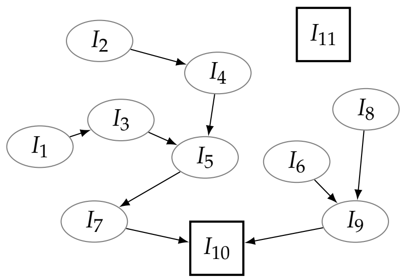
(e) The ω_6-deformation graph of $\mathbf{Hilb}^3_{6t-5}$.



(f) The ω_8-deformation graph of $\mathbf{Hilb}^3_{6t-5}$.



(g) The ω_{11}-deformation graph of $\mathbf{Hilb}^3_{6t-5}$.



(h) The `DegRevLex`-deformation graph of $\mathbf{Hilb}^3_{6t-5}$.

Figure 3.6: The graphical representation of the degeneration graphs of the Hilbert scheme $\mathbf{Hilb}^3_{6t-5}$ introduced in Example 3.2.3. The square vertices are endpoints.

The ideal generates by the greatest $\binom{3+10}{3} - p(10) = 231$ monomials in $\mathbb{K}[x_0, x_1, x_2, x_3]_{10}$ w.r.t. `DegRevLex` has constant Hilbert polynomial equal to 55, so $\mathbf{Hilb}^3_{6t-5}$ does not contain a point defined by a hilb-segment ideal w.r.t. `DegRevLex`. Applying Algorithm 3.4 on $\mathbf{Hilb}^3_{6t-5}$ and `DegRevLex`, we find that both I_{10} and I_{11} are `DegRevLex`-endpoint, so that the `DegRevLex`-degeneration graph is not connected (Figure 3.6h).

3.2.1 The special case of constant Hilbert polynomials

Let us finally consider the special case of the `DegLex`-degeneration graph of Hilbert schemes of points. Given a Borel-fixed ideal I defining a point of $\mathbf{Hilb}^n_s$, we highlight that among the minimal monomials of $\{I_s\}$ there will be surely a monomial of the type $x_1^a x_0^{s-a}$. In fact x_1^s belongs to the ideal and applying the decreasing move e_1^- repeatedly, we will find a monomial $x_1^a x_0^{s-a}$ such that $\mathsf{e}_1^-(x_1^a x_0^{s-a}) \notin \{I_r\}$. It is easy to deduce that the power a of the variable x_1 is equal to the regularity of the saturated ideal I^{sat}, because x_1^a is one of the generators of the saturation (Proposition 2.9) and it is the one of highest degree (Proposition 2.12).

Proposition 3.11. *Let $I \subset \mathbb{K}[x]$ be a Borel-fixed ideal defining a point of $\mathbf{Hilb}^n_s$ and let $\mathrm{reg}(I)$ be the regularity of its saturation I^{sat}. $s - \mathrm{reg}(I)$ consecutive `DegLex`-Borel rational degenerations lead from the point defined by I to the lexicographic point.*

Proof. For the lexicographic ideal L associated to the constant Hilbert polynomial $p(t) = s$, the order set contains the monomials

$$\{L_s\}^{\mathcal{C}} = \left\{x_0^s, x_1 x_0^{s-1}, \ldots, x_1^{s-1} x_0\right\}.$$

For the Borel-fixed ideal I defining a point of $\mathbf{Hilb}^n_s$, we can divide the order set as follows:

$$\{I_s\}^{\mathcal{C}} = \left\{x_0^s, \ldots, x_1^{\mathrm{reg}(I)-1} x_0^{s-\mathrm{reg}(I)+1}\right\} \cup \left\{x^{\gamma_1}, \ldots, x^{\gamma_{s-\mathrm{reg}(I)}}\right\},$$

where $\max x^{\gamma_i} > 1$, $\forall\, i = 1, \ldots, s - \mathrm{reg}(I)$.

The `DegLex`-Borel degeneration of I, that we can obtain applying Algorithm 3.3, is determined by the monomials

$$\min_{\texttt{DegLex}} \{I_s\} = x_1^{\mathrm{reg}(I)} x_0^{s-\mathrm{reg}(I)} \quad \text{and} \quad \max_{\texttt{DegLex}} \{I_s\}^{\mathcal{C}} = \max_{\texttt{DegLex}} \left\{x^{\gamma_1}, \ldots, x^{\gamma_{s-\mathrm{reg}(I)}}\right\} = x^{\beta},$$

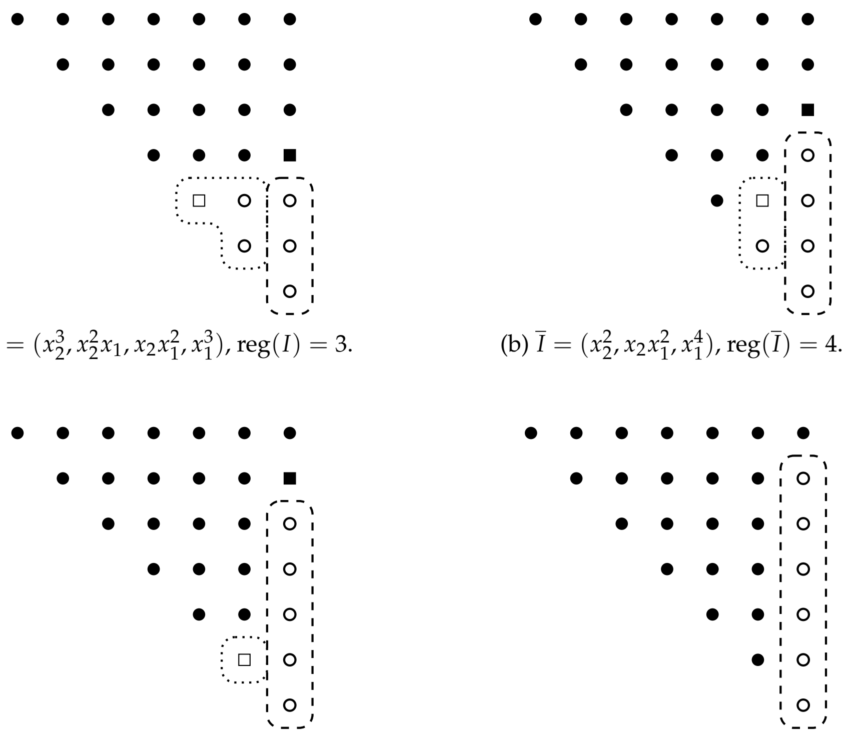
so that the DegLex-Borel degeneration of I is the ideal

$$J = \left\langle \{I_s\} \setminus \left\{ x_1^{\mathrm{reg}(I)} x_0^{s-\mathrm{reg}(I)} \right\} \cup \{x^\beta\} \right\rangle^{\mathrm{sat}}$$

whose order set in degree s is

$$\{J_s\}^{\mathcal{C}} = \left\{ x_0^s, \ldots, x_1^{\mathrm{reg}(I)-1} x_0^{s-\mathrm{reg}(I)+1}, x_1^{\mathrm{reg}(I)} x_0^{s-\mathrm{reg}(I)} \right\} \cup \Big(\{x^{\gamma_1}, \ldots, x^{\gamma_{s-\mathrm{reg}(I)}}\} \setminus \{x^\beta\} \Big)$$

and whose regularity is $\mathrm{reg}(J) = \mathrm{reg}(I) + 1$. Repeating $s - \mathrm{reg}(I)$ times this process, we will obtain an ideal with regularity equal to $\mathrm{reg}(I) + \big(s - \mathrm{reg}(I)\big) = s$, i.e. the lexicographic ideal L. $\square$



(a) $I = (x_2^3, x_2^2 x_1, x_2 x_1^2, x_1^3)$, $\mathrm{reg}(I) = 3$.

(b) $\overline{I} = (x_2^2, x_2 x_1^2, x_1^4)$, $\mathrm{reg}(\overline{I}) = 4$.

(c) $\widetilde{I} = (x_2^2, x_2 x_1, x_1^5)$, $\mathrm{reg}(\widetilde{I}) = 5$.

(d) $L = (x_2, x_1^6)$, $\mathrm{reg}(L) = 6$.

Figure 3.7: An example of the sequence of DegLex-Borel degeneration leading from the point defined by a generic Borel-fixed ideal to the lexicographic point.

Corollary 3.12. *Let* $\mathbf{Hilb}^n_s$ *be a zero-dimensional Hilbert scheme. The* `DegLex`*-degeneration graph of* $\mathbf{Hilb}^n_s$ *has height equal to* $s-a$*, where* a *is the smallest positive integer such that*

$$s \leqslant \sum_{i=0}^{a-1} \binom{n-1+i}{n-1}.$$

We recall that the height of a rooted tree is the maximal distance between a vertex and the root, where the vertices connected to the root have distance 1, the vertices connected to vertices of distance 1 have distance 2 and so on.

Proof. By Proposition 3.11, to determine the height of the `DegLex`-degeneration graph of $\mathbf{Hilb}^n_s$, we have to understand which is the lowest regularity of a Borel-fixed ideal with constant Hilbert polynomial $p(t) = s$. We saw that the regularity of such an ideal I coincides with the minimal power of the variable x_1 in a monomial of the type $x_1^a x_0^{s-a}$ belonging to $\{I_s\}$. So the question is how many monomials we can put in a order set not containing $x_1^a x_0^{s-a}$ and the answer is that we can put all the monomials with a power of the variable x_0 greater than $s-a$, i.e.

$$\underbrace{\binom{n-1+a-1}{n-1}}_{\mathbb{K}[x_1,\ldots,x_n]_{a-1}\cdot x_0^{s-a+1}} + \ldots + \underbrace{\binom{n-1+1}{n-1}}_{\mathbb{K}[x_1,\ldots,x_n]_1\cdot x_0^{s-1}} + \underbrace{\binom{n-1+0}{n-1}}_{x_0^s} = \sum_{i=0}^{a-1}\binom{n-1+i}{n-1}.$$

Finally, to obtain a Borel-fixed ideal I having x_1^a as minimal generator, namely $\mathrm{reg}(I) = a$, we need $s \leqslant \sum_{i=0}^{a-1}\binom{n-1+i}{n-1}$.

Note that the choosing monomials to put in the order set w.r.t. a decreasing order on the power of the smallest variable x_0 agrees with choosing the smallest monomial w.r.t. `DegRevLex` among those still belonging to the Borel set, hence the lowest regularity a of a Borel-fixed ideal with Hilbert polynomial $p(t) = s$ is always realized by the hilb-segment ideal w.r.t. `DegRevLex`. $\square$

Example 3.2.4. Let us consider the Hilbert scheme $\mathbf{Hilb}^3_8$. There are 12 Borel-fixed ideals, that we list again with the term ordering for which they (possibly) are hilb-segment:

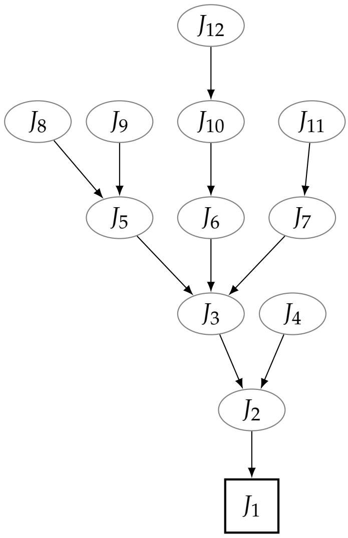
$$\begin{array}{ll}
J_1 = (x_3, x_2, x_1^8), & \texttt{DegLex};\\
J_2 = (x_3, x_2^2, x_2x_1, x_1^7), & \omega_2 = (8,7,2,1);\\
J_3 = (x_3, x_2^2, x_2x_1^2, x_1^6), & \omega_3 = (7,5,2,1);\\
J_4 = (x_3^2, x_3x_2, x_2^2, x_3x_1, x_2x_1, x_1^6), & \omega_4 = (11,10,3,1);\\
J_5 = (x_3, x_2^2, x_2x_1^3, x_1^5), & \omega_5 = (11,6,3,1);\\
J_6 = (x_3, x_2^3, x_2^2x_1, x_2x_1^2, x_1^5); & \\
J_7 = (x_3^2, x_3x_2, x_2^2, x_3x_1, x_2x_1^2, x_1^5), & \omega_7 = (5,4,2,1);\\
J_8 = (x_3, x_2^3, x_2^2x_1, x_2x_1^3, x_1^4), & \omega_8 = (9,4,3,1);\\
J_9 = (x_3^2, x_3x_2, x_2^2, x_3x_1, x_2x_1^3, x_1^4); & \\
J_{10} = (x_3^2, x_3x_2, x_3x_1, x_2^3, x_2^2x_1, x_2x_1^2, x_1^4), & \omega_{10} = (6,4,3,1);\\
J_{11} = (x_3^2, x_3x_2, x_2^2, x_3x_1^2, x_2x_1^2, x_1^4), & \omega_{11} = (6,5,3,1);\\
J_{12} = (x_3^2, x_3x_2, x_2^3, x_2^2x_1, x_3x_1^2, x_2x_1^2, x_1^3), & \texttt{DegRevLex}.
\end{array}$$

The `DegLex`-degeneration graph of $\mathbf{Hilb}^3_8$ (Figure 3.8a) has height 5, because

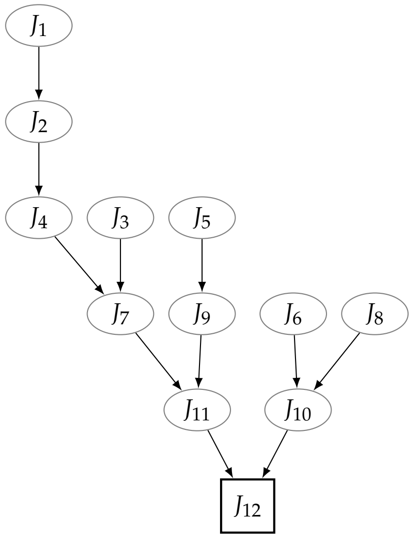
$$8 \leqslant \sum_{i=0}^{3-1} \binom{2+i}{2} = \binom{2}{2} + \binom{3}{2} + \binom{4}{2} = 10 \quad \text{and} \quad 8 > \sum_{i=0}^{2-1} \binom{2+i}{2} = 4,$$

and the `DegRevLex`-degeneration graph (Figure 3.8b) has the same height, being the hilb-segment ideal J_{12} w.r.t. `DegRevLex` the most distant vertix from the lexicographic ideal J_1.

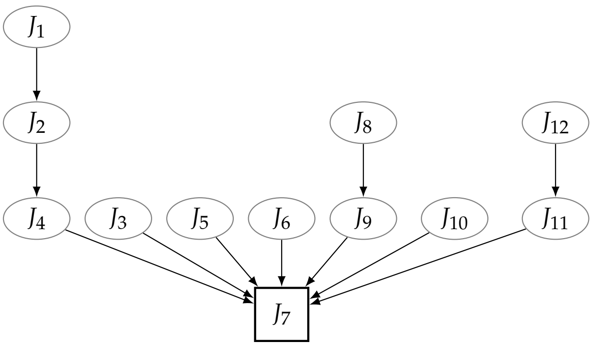
Degeneration graphs with a lower height can be found considering hilb-segment ideals with an intermediate regularity (Figure 3.8c).



(a) The `DegLex`-degeneration graph of $\mathbf{Hilb}^3_8$.



(b) The `DegRevLex`-degeneration graph of $\mathbf{Hilb}^3_8$.

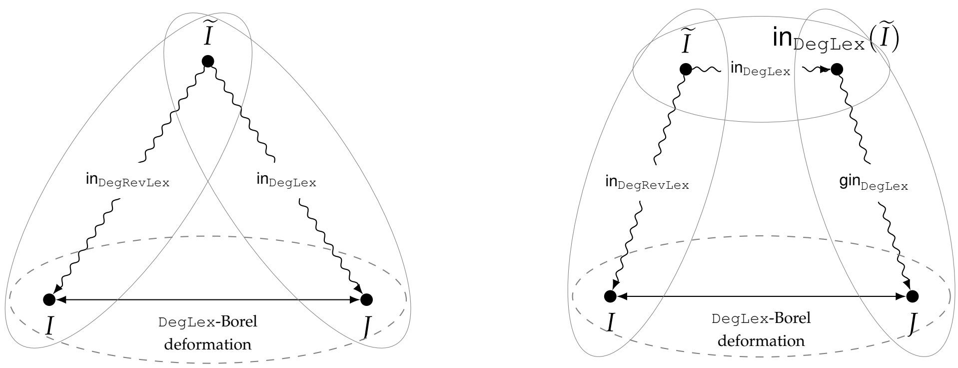


(c) The ω_7-degeneration graph of $\mathbf{Hilb}^3_8$.

Figure 3.8: The graphical representation of some degeneration graphs discussed in Example 3.2.4. The graphs are drawn as trees to highlight their height.

3.3 Borel-fixed ideals defining points lying on a same component of the Hilbert scheme

The last section of this chapter is devoted to the study of components of the Hilbert scheme, keeping in mind a question posed by Reeves in [86] "Is the subset of Borel-fixed ideals on a component enough to determine the component?". In fact the points corresponding to two ideals connected by any Borel rational deformation lie on the same component of the Hilbert scheme. We underline that the technique introduced by Peeva and Stillman in [83] is slightly different, because in order for passing from a Borel-fixed ideal to another one they use at least two affine deformations (see Figure 3.9).



(a) Example 3.2.2.   (b) Example 3.2.1.

Figure 3.9: A Borel rational deformation ensures that the two Borel ideals defining it lie on a same component (drawn with the dashed line). With Peeva-Stillman method, in the best case (Fig. 3.9a), we can deduce that the components containing two Borel-fixed ideals have non-empty intersection but in the general case (Fig. 3.9b), we could get no information.

We introduce a new graph related to Borel-fixed ideals and Hilbert scheme components, different from the incidence graph defined by Reeves in [86], but also useful to understand the intersections among components and we will call it Borel incidence graph.

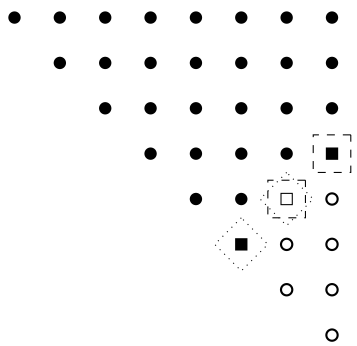
Definition 3.13. Given the Hilbert scheme $\mathbf{Hilb}^n_{p(t)}$, we define the *Borel incidence graph of* $\mathbf{Hilb}^n_{p(t)}$ as the graph whose vertices correspond to the (saturated) Borel-fixed ideals of $\mathbb{K}[x]$ with Hilbert polynomial $p(t)$ and whose edges represents Borel rational deformations.

To construct the graph we can use Algorithm 2.4 to determine the vertices and then we can apply Algorithm 3.1 on every Borel-fixed ideal to compute the edges (discarding repetitions). But we would like to add more edges to the graph, so we now discuss if it is possible to perform two Borel rational deformations of the same ideal simultaneously. Let us consider a Borel-fixed ideal $I \subset \mathbb{K}[x]$ with Hilbert polynomial $p(t)$ having two Borel degenerations: $J_1 = \big\langle \{I_r\} \setminus \mathcal{F}_{\alpha_1}(x^{\alpha_1}) \cup \mathcal{F}_{\alpha_1}(x^{\beta_1})\big\rangle$ and $J_2 = \big\langle \{I\} \setminus \mathcal{F}_{\alpha_2}(x^{\alpha_2}) \cup \mathcal{F}_{\alpha_2}(x^{\beta_2})\big\rangle$. The point is to understand under which conditions swapping at the same time $\mathcal{F}_{\alpha_1}(x^{\alpha_1})$ with $\mathcal{F}_{\alpha_1}(x^{\beta_1})$ and $\mathcal{F}_{\alpha_2}(x^{\alpha_2})$ with $\mathcal{F}_{\alpha_2}(x^{\beta_2})$ preserves the Borel property. The first problems that can arise are:

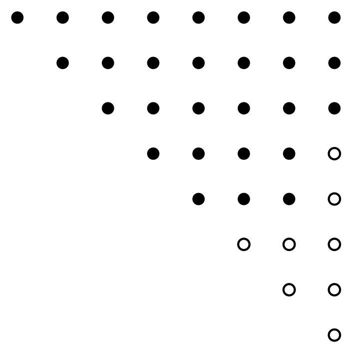
1. if the sets of monomials are not disjoint, there are some problems in the definition of the deformation. For instance if $\mathcal{F}_{\alpha_1}(x^{\alpha_1}) \cap \mathcal{F}_{\alpha_2}(x^{\alpha_2}) = \emptyset$ and $\mathcal{F}_{\alpha_1}(x^{\beta_1}) \cap \mathcal{F}_{\alpha_2}(x^{\beta_2}) \neq \emptyset$, the number of monomials we remove from the Borel set would be different from the number of monomials we add (see Figure 3.10);

2. if x^{β_2} can be obtained by a decreasing move from x^{α_1} (or viceversa x^{β_1} from x^{α_2}), performing both exchanges we would not obtain a Borel set $\mathscr{B}$, because $\mathscr{B} \ni x^{\beta_2} <_B x^{\alpha_1} \notin \mathscr{B}$ (see Figure 3.11);

3. assuming $\min x^{\alpha_1} > \min x^{\alpha_2}$, if there exists an admissible move e_j^-, $j \geqslant \min x^{\alpha_1}$, such that $\mathsf{e}_j^-(x^{\alpha_2}) <_B x^{\beta_1}$, it could happen that $\mathsf{e}_j^-(x^{\alpha_2}) \in \mathcal{F}_{\alpha_1}(x^{\beta_1})$ and again swapping both sets of monomials we would not obtain a Borel set $\mathscr{B}$ because $\mathscr{B} \ni \mathsf{e}_j^-(x^{\alpha_2}) <_B x^{\alpha_2} \notin \mathscr{B}$ (see Figure 3.11).



(a) $I = (x_2^2, x_2x_1^3, x_1^4)$, $p_I(t) = 7$.



(b) $J = (x_2^3, x_2^2x_1, x_2x_1^2, x_1^5)$, $p_J(t) = 8$.

Figure 3.10: Example of two Borel rational deformations that can not be performed simultaneously because involving not disjoint sets of monomials. Both deformations involve the monomial $x_2x_1^2x_0^4$, thus swapping the monomials $x_2^2x_0^5, x_1^5x_0^2$ with $x_2x_1^2x_0^4$ preserves the Borel property but the Hilbert polynomial changes.

Theorem 3.14. *Let $I \subset \mathbb{K}[x]$ be a Borel-fixed ideal with Hilbert polynomial $p(t)$ with Gotzmann number r. Let us assume that there are s Borel rational deformations:*

$$\mathcal{I}_k = \left\langle \{I_r\} \setminus \mathcal{F}_{\alpha_k}(x^{\alpha_k}) \cup \{y_{k\,0}F(x^{\alpha_k}) + y_{k\,1}F(x^{\beta_k}) \mid F \in \mathcal{F}_{\alpha_k}\} \right\rangle \subset \mathbb{K}[y_{k\,0}, y_{k\,1}][x],$$

$k = 1, \ldots, s$, such that

(i) $\min x^{\alpha_1} \geqslant \ldots \geqslant \min^{\alpha_s}$*;*

(ii) $\forall\, i, j$, x^{α_i} can not be obtained from x^{β_j} by applying a decreasing move;

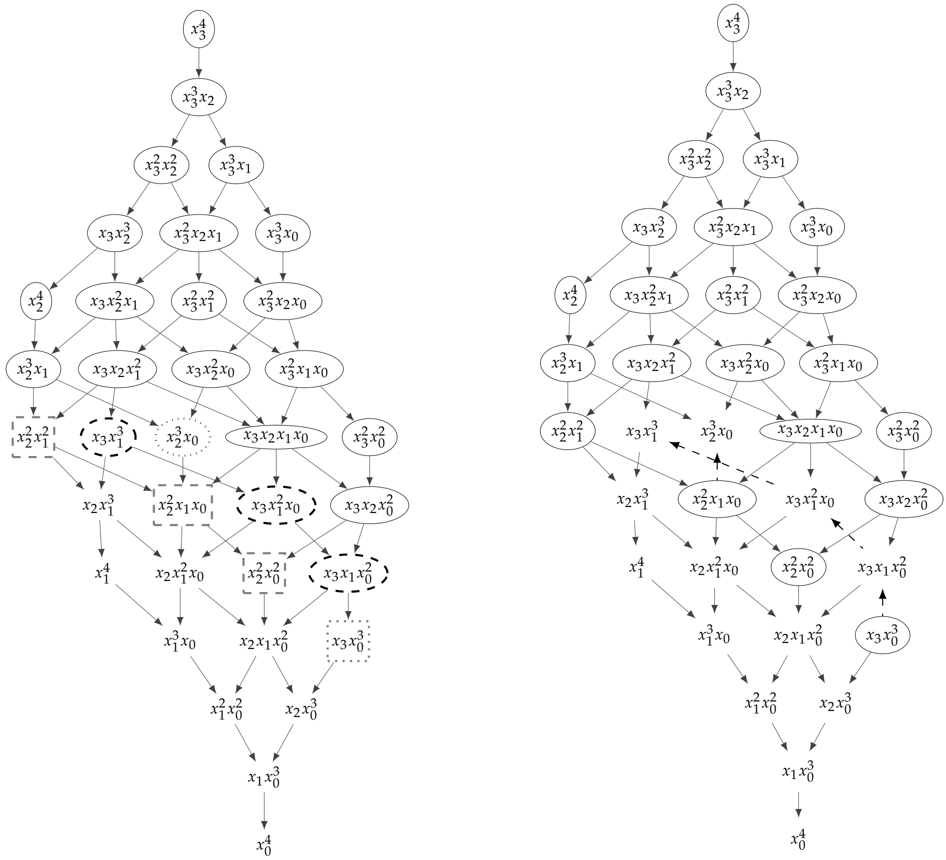
(iii) $\forall\, i, j$ such that $\min x^{\alpha_i} > \min x^{\alpha_j}$, $\mathsf{e}_h^-(x^{\alpha_j})$ cannot be obtained from x^{β_i} through a decreasing move, for all admissible e_h^-, $h \geqslant \min x^{\alpha_i}$;

(iv) $\forall\, i, j$, $\mathcal{F}_{\alpha_i}(x^{\alpha_i}) \cap \mathcal{F}_{\alpha_j}(x^{\alpha_j}) = \emptyset$ and $\mathcal{F}_{\alpha_i}(x^{\beta_i}) \cap \mathcal{F}_{\alpha_j}(x^{\beta_j}) = \emptyset$.

Then the family of subschemes of $\mathbb{P}^n$ parametrized by the ideal

$$\mathcal{I} = \left\langle \{I\} \setminus \left(\bigcup_k \mathcal{F}_{\alpha_k}(x^{\alpha_k}) \right) \cup \left(\bigcup_k \left\{ y_{k\,0}\, F(x^{\alpha_k}) + y_{k\,1}\, F(x^{\beta_k}) \;\middle|\; F \in \mathcal{F}_{\alpha_k} \right\} \right) \right\rangle \tag{3.6}$$

is flat over $(\mathbb{P}^1)^{\times s}$ and has 2^s fibers corresponding to Borel-fixed ideals.



(a) $I = (x_3^2, x_3x_2, x_3x_1, x_2^3)$.

(b) $J = (x_3^2, x_3x_2, x_2^2, x_3x_0^3)$.

Figure 3.11: First example of two Borel rational deformations that can not be performed simultaneously because not preserving the Borel property (not even the Hilbert polynomial). The deformations are defined by $\mathcal{F}_{x_3x_1^3}(x_3x_1^3) = \{x_3x_1^3, x_3x_1^2x_0, x_3x_1x_0^2\}, \mathcal{F}_{x_3x_1^3}(x_2^2x_1^2) = \{x_2^2x_1^2, x_2^2x_1x_0, x_2^2x_0^2\}$ and by $\mathcal{F}_{x_2^3x_0}(x_2^3x_0) = \{x_2^3x_0\}, \mathcal{F}_{x_2^3x_0}(x_3x_0^3) = \{x_3x_0^3\}$. The Borel property is not preserved because (2.) $x_3x_0^3 < x_3x_1^3 = 4\mathsf{e}_0^+(x_3x_0^3)$ and (3.) $\mathsf{e}_2^-(x_2^3x_0) = x_2^2x_1x_0 <_B \mathsf{e}_0^+(x_2^2x_1x_0) = x_2^2x_1^2$.

Proof. The key point is that the hypotheses ensure that there are not linear syzygies between two monomials belonging to two different groups, so assuming $y_{k\,0} \neq 0$, $\forall\ k$ and $z_k = \frac{y_{k1}}{y_{k0}}$, we can lift simultaneously the syzygies of Eliahou-Kervaire among the monomials of $\{I_r\}$ as done in the proof of Theorem 3.4 for each group of binomials $F(x^{\alpha_k}) + z_k F(x^{\beta_k})$, $\forall\ F \in \mathcal{F}_{\alpha_k}$, $\forall\ k = 1,\ldots,s$. For each localization we deduce the flatness by Proposition 3.3 and by the symmetry between the sets of monomials $\mathcal{F}_{\alpha_k}(x^{\alpha_k})$ and $\mathcal{F}_{\alpha_k}(x^{\beta_k})$, the property holds for any other choice of non-vanishing variables $y_{k\,h}$.

The Borel-fixed ideals appearing the family correspond to the points of $\left(\mathbb{P}^1\right)^{\times s}$ such that for each k $[y_{k\,0} : y_{k\,1}] = [0:1]$ or $[y_{k\,0} : y_{k\,1}] = [1:0]$, and so there are 2^s possibilities. □

Corollary 3.15. *Let $I \subset \mathbb{K}[x]$ be a Borel-fixed ideal and let $\mathcal{I}_1,\ldots,\mathcal{I}_s$ be s Borel rational deformations of I as in Theorem 3.14. For any Borel-fixed ideal J belonging to the family over $\left(\mathbb{P}^1\right)^{\times s}$, there exists a rational deformation having both I and J as fibers. Called $p(t)$ the Hilbert polynomial of I and r its Gotzmann number, the points defined by I and J on the Hilbert scheme $\mathbf{Hilb}^n_{p(t)}$ are connected by a rational curve $\mathcal{C} : \mathbb{P}^1 \to \mathbf{Hilb}^n_{p(t)}$. Considering the construction of the Hilbert scheme $\mathbf{Hilb}^n_{p(t)}$ as subscheme of the Grassmannian $\mathbf{Gr}_{\mathbb{K}}(q(r),N)$, where $N = \binom{n+r}{n}$ and $q(r) = N - p(r)$, the degree of the curve $\mathcal{C}$ via the Plücker embedding* (1.2) $\mathscr{P} : \mathbf{Gr}_{\mathbb{K}}(q(r),N) \to \mathbb{P}\big[\wedge^{q(r)} \mathbb{K}[x]_r\big]$ *is $q(r) - |\{I_r\} \cap \{J_r\}|$.*

Proof. Let $\mathcal{I}$ be the family defined in (3.6). To obtain a rational deformation having I and any other Borel-fixed ideal J of $\mathcal{I}$, it suffices to specialize the variables $y_{k\,0}, y_{k\,1}$ as follows:

- $y_{k\,0} = 1,\ y_{k\,1} = 0$, if $\mathcal{F}_{\alpha_k}(x^{\alpha_k}) \subset \{I_r\}$ and $\mathcal{F}_{\alpha_k}(x^{\alpha_k}) \subset \{J_r\}$;
- $y_{k\,0} = y_0,\ y_{k\,1} = y_1$, if $\mathcal{F}_{\alpha_k}(x^{\alpha_k}) \subset \{I_r\}$ and $\mathcal{F}_{\alpha_k}(x^{\beta_k}) \subset \{J_r\}$.

As usual, the ideal I corresponds to the fiber of the point $[1:0]$ and the ideal J to the fiber of $[0:1]$.

To determine the degree of the curve through the Plücker embedding, we can use again the matrix representation of the submodule $\mathcal{I}_r \subset (\mathbb{K}[y_0,y_1])[x]_r$ used in the proof of Theorem 3.5. In this case the columns of the first block correspond to the monomials belonging to both $\{I_r\}$ and $\{J_r\}$, the columns of the second block to the monomials in $\{I_r\} \setminus (\{I_r\} \cap \{J_r\})$ and the columns of the third block to the monomials in $\{J_r\} \setminus (\{I_r\} \cap \{J_r\})$. Thus the degree of the Plücker coordinates in the variables y_0 and y_1 is equal to $|\{I_r\} \setminus (\{I_r\} \cap \{J_r\})| = |\{I_r\}| - |\{I_r\} \cap \{J_r\}| = q(r) - |\{I_r\} \cap \{J_r\}|$. □

Definition 3.16. Let $I \subset \mathbb{K}[x]$ be a Borel-fixed ideal and let $\mathcal{I}_1, \ldots, \mathcal{I}_s$ be s Borel rational deformations of I as in Theorem 3.14. We say that $\{\mathcal{I}_1, \ldots, \mathcal{I}_s\}$ is a set of *compatible* Borel rational deformations. Given a second Borel-fixed ideal J belonging to the family of ideals $\mathcal{I}$ over $(\mathbb{P}^1)^{\times s}$ but not to the set of Borel degenerations of I computed with Algorithm 3.2, we call *composed Borel rational deformation* the deformation defined in Corollary 3.15 having both I and J as fibers and *(composed) Borel rational degeneration* of I the ideal J.

Example 3.3.1. The two Borel rational deformations

$$\begin{aligned}
\mathcal{I}_1 &= \left\langle \{I_8\} \setminus \{x_3^2x_0^6\} \cup \{y_0\, x_3^2x_0^6 + y_1 x_2^3x_0^5\} \right\rangle \\
\mathcal{I}_2 &= \left\langle \{I_8\} \setminus \mathcal{F}_{x_2^2x_1^6}(x_2^2x_1^6) \cup \left\{ y_0 F(x_2^2x_1^6) + y_1 F(x_3x_1^7) \mid F \in \mathcal{F}_{x_2^2x_1^6} \right\} \right\rangle
\end{aligned}$$

described in Example 3.1.8 are in the hypothesis of Theorem 3.14, so there is a flat family over $\mathbb{P}^1 \times \mathbb{P}^1$ defined by the ideal

$$\begin{aligned}
\mathcal{I} = \Big\langle \{I_8\} \setminus \{x_3^2x_0^6\} \cup \{y_{00}\, x_3^2x_0^6 + y_{01}\, x_2^3x_0^5\} \\
\setminus \mathcal{F}_{x_2^2x_1^6}(x_2^2x_1^6) \cup \left\{ y_{10} F(x_2^2x_1^6) + y_{11} F(x_3x_1^7) \mid F \in \mathcal{F}_{x_2^2x_1^6} \right\} \Big\rangle.
\end{aligned}$$

Let us assume $y_{00} \neq 0$, $y_{10} \neq 0$, $z_0 = \frac{y_{01}}{y_{00}}$, $z_1 = \frac{y_{11}}{y_{10}}$ and let us see how the Eliahou-Kervaire syzygies of $I_{\geqslant 8}$ lift to a set of syzygies for the family $\mathcal{I}|_{[1:z_0],[1:z_1]}$. The presentation map is

$$\begin{aligned}
\psi : \big(\mathbb{K}[z_0, z_1][x](-8)\big)^{136} &\longrightarrow \mathbb{K}[z_0, z_1][x] \\
\mathbf{f}_\gamma &\longmapsto \begin{cases} x^\gamma & \forall\, x^\gamma \in \{I_8\} \setminus \{x_3^2x_0^6\} \setminus \mathcal{F}_{x_2^2x_1^6}(x_2^2x_1^6) \\ x_3^2x_0^6 + z_0\, x_2^3x_0^5 & x^\gamma = x_3^2x_0^6 \\ F(x_2^2x_1^6) + z_1 F(x_3x_1^7), & \forall\, x^\gamma = F(x_2^2x_1^6),\ F \in \mathcal{F}_{x_2^2x_1^6}. \end{cases}
\end{aligned} \tag{3.7}$$

For the monomials in $\{I_8\} \setminus \{x_3^2x_0^6\} \setminus \mathcal{F}_{x_2^2x_1^6}(x_2^2x_1^6)$ we lift the syzygies among the same monomials in $I_8 \subset \mathbb{K}[x]$. Let us look at the binomials.

- $\mathbf{f}_{x_3^2x_0^6} \mapsto x_3^2x_0^6 + z_0\, x_2^3x_0^5$

  $(\cdot x_1)\ \ x_1\mathbf{f}_{x_3^2x_0^6} - x_0\mathbf{f}_{x_3^2x_1x_0^5} - z_0x_0\mathbf{f}_{x_2^3x_1x_0^4};$

  $(\cdot x_2)\ \ x_2\mathbf{f}_{x_3^2x_0^6} - x_0\mathbf{f}_{x_3^2x_2x_0^5} - z_0x_0\mathbf{f}_{x_2^4x_0^4};$

  $(\cdot x_3)\ \ x_3\mathbf{f}_{x_3^2x_0^6} - x_0\mathbf{f}_{x_3^3x_0^5} - z_0x_0\mathbf{f}_{x_3x_2^3x_0^4}.$

- $\mathbf{f}_{x_2^2x_1^6} \mapsto x_2^2x_1^6 + z_1\, x_3x_1^7$

  $(\cdot x_2)\ \ x_2\mathbf{f}_{x_2^2x_1^6} - x_1\mathbf{f}_{x_2^3x_1^5} - z_1x_1\mathbf{f}_{x_3x_2x_1^6};$

  $(\cdot x_3)\ \ x_3\mathbf{f}_{x_2^2x_1^6} - x_1\mathbf{f}_{x_3x_2^2x_1^5} - z_1x_1\mathbf{f}_{x_3^2x_1^6}.$

- $\mathbf{f}_{x_2^2x_1^5x_0} \mapsto x_2^2x_1^5x_0 + z_1\, x_3x_1^6x_0$

  $(\cdot x_1)\ \ x_1\mathbf{f}_{x_2^2x_1^5x_0} - x_0\mathbf{f}_{x_2^2x_1^6};$

  $(\cdot x_2)\ \ x_2\mathbf{f}_{x_2^2x_1^5x_0} - x_0\mathbf{f}_{x_2^3x_1^5} - z_1x_0\mathbf{f}_{x_3x_2x_1^6};$

  $(\cdot x_3)\ \ x_3\mathbf{f}_{x_2^2x_1^5x_0} - x_0\mathbf{f}_{x_3x_2^2x_1^5} - z_1x_0\mathbf{f}_{x_3^2x_1^6}.$

- $\mathbf{f}_{x_2^2x_1^4x_0^2} \mapsto x_2^2x_1^4x_0^2 + z_1\, x_3x_1^5x_0^2$

  $(\cdot x_1)\ \ x_1\mathbf{f}_{x_2^2x_1^4x_0^2} - x_0\mathbf{f}_{x_2^2x_1^5x_0};$

  $(\cdot x_2)\ \ x_2\mathbf{f}_{x_2^2x_1^4x_0^2} - x_0\mathbf{f}_{x_2^3x_1^4x_0} - z_1x_0\mathbf{f}_{x_3x_2x_1^5x_0};$

  $(\cdot x_3)\ \ x_3\mathbf{f}_{x_2^2x_1^4x_0^2} - x_0\mathbf{f}_{x_3x_2^2x_1^4x_0} - z_1x_0\mathbf{f}_{x_3^2x_1^5x_0}.$

- $\mathbf{f}_{x_2^2x_1^3x_0^3} \mapsto x_2^2x_1^3x_0^3 + z_1\, x_3x_1^4x_0^3$

  $(\cdot x_1)\ \ x_1\mathbf{f}_{x_2^2x_1^3x_0^3} - x_0\mathbf{f}_{x_2^2x_1^4x_0^2};$

  $(\cdot x_2)\ \ x_2\mathbf{f}_{x_2^2x_1^3x_0^3} - x_0\mathbf{f}_{x_2^3x_1^3x_0^2} - z_1x_0\mathbf{f}_{x_3x_2x_1^4x_0^2};$

  $(\cdot x_3)\ \ x_3\mathbf{f}_{x_2^2x_1^3x_0^3} - x_0\mathbf{f}_{x_3x_2^2x_1^3x_0^2} - z_1x_0\mathbf{f}_{x_3^2x_1^4x_0^2}.$

- $\mathbf{f}_{x_2^2x_1^2x_0^4} \mapsto x_2^2x_1^2x_0^4 + z_1\, x_3x_1^3x_0^4$

  $(\cdot x_1)\ \ x_1\mathbf{f}_{x_2^2x_1^2x_0^4} - x_0\mathbf{f}_{x_2^2x_1^3x_0^3};$

$$(\cdot x_2)\ \ x_2\mathbf{f}_{x_2^2x_1^2x_0^4} - x_0\mathbf{f}_{x_2^3x_1^2x_0^3} - z_1x_0\mathbf{f}_{x_3x_2x_1^3x_0^3};$$
$$(\cdot x_3)\ \ x_3\mathbf{f}_{x_2^2x_1^2x_0^4} - x_0\mathbf{f}_{x_3x_2^2x_1^2x_0^3} - z_1x_0\mathbf{f}_{x_3^2x_1^3x_0^3}.$$

Thus the Borel-fixed ideals belonging to this family are 4:

$$\begin{array}{ll} I = (x_3^2, x_3x_2^2, x_3x_2x_1, x_2^4, x_2^3x_1, x_2^2x_1^2) & \\ \left.\begin{array}{l}(x_3^3, x_3^2x_2, x_3x_2^2, x_2^3, x_3^2x_1, x_3x_2x_1, x_2^2x_1^2) \\ (x_3^2, x_3x_2^2, x_3x_2x_1, x_2^4, x_2^3x_1, x_3x_1^3)\end{array}\right\} & \text{simple Borel degenerations} \\ (x_3^3, x_3^2x_2, x_3x_2^2, x_2^3, x_3^2x_1, x_3x_2x_1, x_3x_1^3) & \text{composed Borel degeneration} \end{array}$$

and for any pair among them, the points defined by the Borel-fixed ideals on the Hilbert scheme $\mathbf{Hilb}^3_{3t+5}$ are connected by a rational curve.

$\textsc{BorelIncidenceGraph}(\mathbf{Hilb}^n_{p(t)})$
Input: $\mathbf{Hilb}^n_{p(t)}$, a Hilbert scheme.
Output: the Borel incidence graph $(\mathsf{vertices}, \mathsf{edges})$ of $\mathbf{Hilb}^n_{p(t)}$.
$r \leftarrow \textsc{GotzmannNumber}\big(p(t)\big)$;
$\mathsf{vertices} \leftarrow \textsc{BorelGeneratorDFS}\big(\mathbb{K}[x_0,\ldots,x_n], p(t)\big)$;
$\mathsf{edges} \leftarrow \emptyset$;
for all $I \in \mathsf{vertices}$ **do**
  $\mathsf{simpleDeformations} \leftarrow \textsc{BorelRationalDeformations}(I, r)$;
  for all $\{\mathcal{I}_1,\ldots,\mathcal{I}_s\} \subset \mathsf{simpleDeformations}$ **do**
    if $\{\mathcal{I}_1,\ldots,\mathcal{I}_s\}$ compatible set of Borel deformations **then**
      $J \leftarrow \big\langle \{I_r\} \setminus \big(\bigcup_k \mathcal{F}_{\alpha_k}(x^{\alpha_k})\big) \cup \big(\bigcup_k \mathcal{F}_{\alpha_k}(x^{\beta_k})\big)\big\rangle^{\mathsf{sat}}$;
      if $(I, J) \notin \mathsf{edges}$ **and** $(J, I) \notin \mathsf{edges}$ **then**
        $\mathsf{edges} \leftarrow \mathsf{edges} \cup \{(I, J)\}$;
      end if
    end if
  end for
end for
return $(\mathsf{vertices}, \mathsf{edges})$;

Algorithm 3.5: The pseudocode description of the method computing the Borel incidence graph. For details see Appendix B.

Theorem 3.14 and Corollary 3.15 introduce other Borel rational deformations between pairs of Borel-fixed ideals. Hence in the algorithm computing the Borel incidence graph (Algorithm 3.5) we add these composed Borel rational deformations to those obtained applying Algorithm 3.1 on every ideal.

Furthermore Theorem 3.14 gives us a new criterion to detect points lying on a common component of the Hilbert scheme, indeed the points defined by Borel-fixed ideals belonging to any family over $\left(\mathbb{P}^1\right)^{\times s}$ are on the same component. With the following examples we show that this criterion is a sufficient condition that does not overlap Reeves criterion [86] based on the hyperplane section.

Example 3.3.2. Let us consider the Hilbert scheme containing the rational normal curve of degree 4, that is $\mathbf{Hilb}^4_{4t+1}$. On it there are 12 points defined by Borel-fixed ideals:

$$
\begin{aligned}
J_1 &= (x_4, x_3, x_2^5, x_2^4 x_1^3),\\
J_2 &= (x_4, x_3, x_2^6, x_2^5 x_1, x_2^4 x_1^2),\\
J_3 &= (x_4, x_3^2, x_3x_2, x_3x_1, x_2^5, x_2^4 x_1^2),\\
J_4 &= (x_4, x_3^2, x_3x_2, x_3x_1^2, x_2^5, x_2^4 x_1),\\
J_5 &= (x_4^2, x_4x_3, x_3^2, x_4x_2, x_3x_2, x_4x_1, x_3x_1, x_2^5, x_2^4 x_1),\\
J_6 &= (x_4, x_3^2, x_3x_2, x_2^4, x_3x_1^3),\\
J_7 &= (x_4, x_3^2, x_3x_2^2, x_3x_2x_1, x_3x_1^2, x_2^4),\\
J_8 &= (x_4^2, x_4x_3, x_3^2, x_4x_2, x_3x_2, x_4x_1, x_3x_1^2, x_2^4),\\
J_9 &= (x_4, x_3^2, x_3x_2, x_2^4, x_2^3 x_1),\\
J_{10} &= (x_4, x_3^2, x_3x_2^2, x_2^3, x_3x_2x_1),\\
J_{11} &= (x_4^2, x_4x_3, x_3^2, x_4x_2, x_3x_2, x_4x_1, x_2^3),\\
J_{12} &= (x_4^2, x_4x_3, x_3^2, x_4x_2, x_3x_2, x_2^2).
\end{aligned}
$$

The result by Reeves [86] says that there exists a component of the Hilbert scheme containing the points defined by Borel-fixed ideals with the same hyperplane section, i.e. in the case of $\mathbf{Hilb}^4_{4t+1}$ there are components (even not different) containing:

- $J_1, J_2, J_3, J_4, J_5, J_6, J_7, J_8$ having hyperplane section (x_4, x_3, x_2^4);
- J_9, J_{10}, J_{11} having hyperplane section $(x_4, x_3^2, x_3x_2, x_2^3)$;
- J_{12}.

Computing with Algorithm 3.5 the Borel incidence graph of $\mathbf{Hilb}^4_{4t+1}$ (Figure 3.12), we find one composed deformation over $\mathbb{P}^1 \times \mathbb{P}^1$, containing the ideals J_7, J_8, J_{10}, J_{11}, so that the corresponding points lie on a same component.

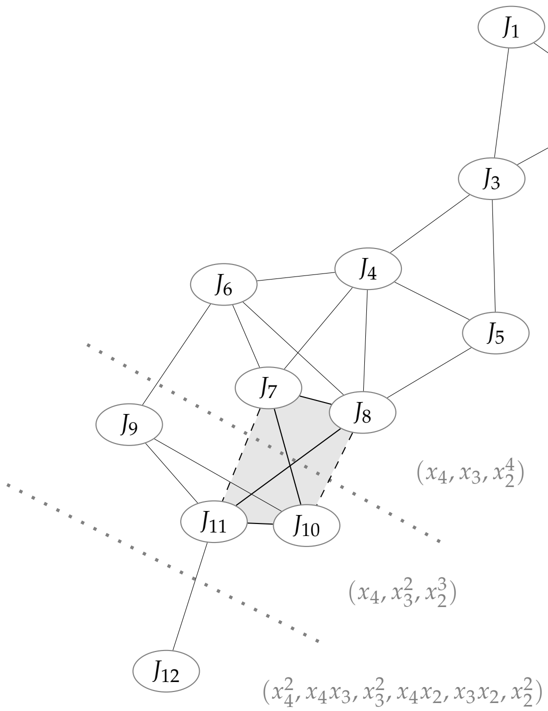


Figure 3.12: The Borel incidence graph of $\mathbf{Hilb}^4_{4t+1}$. The light gray quadrangle corresponds to the family over $\mathbb{P}^1 \times \mathbb{P}^1$ and the dashed lines correspond to composed Borel rational deformations. The dotted lines divide the ideals w.r.t. the hyperplane section.

Example 3.3.3. On the Hilbert scheme $\mathbf{Hilb}^3_{6t-3}$ containing the complete intersections $(2,3)$ in $\mathbb{P}^3$, there are 31 points defined by Borel-fixed ideals, with three different admissible hyperplane sections (of 6 points in $\mathbb{P}^2$):

(x_3, x_2^6)

$$\begin{array}{ll}
I_1 = (x_3, x_2^7, x_2^6x_1^6), & I_{14} = (x_3^2, x_3x_2, x_3x_1^4, x_2^7, x_2^6x_1^2),\\
I_2 = (x_3, x_2^8, x_2^7x_1, x_2^6x_1^5), & I_{15} = (x_3^2, x_3x_2^2, x_3x_2x_1, x_3x_1^3, x_2^7, x_2^6x_1^2),\\
I_3 = (x_3^2, x_3x_2, x_3x_1, x_2^7, x_2^6x_1^5), & I_{16} = (x_3^3, x_3^2x_2, x_3x_2^2, x_3^2x_1, x_3x_2x_1, x_3x_1^2, x_2^7, x_2^6x_1^2),\\
I_4 = (x_3, x_2^8, x_2^7x_1^2, x_2^6x_1^4), & I_{17} = (x_3^2, x_3x_2, x_3x_1^5, x_2^7, x_2^6x_1),\\
I_5 = (x_3^2, x_3x_2, x_3x_1, x_2^8, x_2^7x_1, x_2^6x_1^4), & I_{18} = (x_3^2, x_3x_2^2, x_3x_2x_1, x_3x_1^4, x_2^7, x_2^6x_1),\\
I_6 = (x_3^2, x_3x_2, x_3x_1^2, x_2^7, x_2^6x_1^4), & I_{19} = (x_3^2, x_3x_2^2, x_3x_2x_1^2, x_3x_1^3, x_2^7, x_2^6x_1),\\
I_7 = (x_3, x_2^9, x_2^8x_1, x_2^7x_1^2, x_2^6x_1^3), & I_{20} = (x_3^3, x_3^2x_2, x_3x_2^2, x_3^2x_1, x_3x_2x_1, x_3x_1^3, x_2^7, x_2^6x_1),\\
I_8 = (x_3^2, x_3x_2, x_3x_1, x_2^8, x_2^7x_1^2, x_2^6x_1^3), & I_{21} = (x_3^2, x_3x_2, x_2^6, x_3x_1^6),\\
I_9 = (x_3^2, x_3x_2, x_3x_1^2, x_2^8, x_2^7x_1, x_2^6x_1^3), & I_{22} = (x_3^2, x_3x_2^2, x_3x_2x_1, x_2^6, x_3x_1^5),\\
I_{10} = (x_3^2, x_3x_2, x_3x_1^3, x_2^7, x_2^6x_1^3), & I_{23} = (x_3^2, x_3x_2^2, x_3x_2x_1^2, x_3x_1^4, x_2^6),\\
I_{11} = (x_3^2, x_3x_2^2, x_3x_2x_1, x_3x_1^2, x_2^7, x_2^6x_1^3), & I_{24} = (x_3^3, x_3^2x_2, x_3x_2^2, x_3^2x_1, x_3x_2x_1, x_3x_1^4, x_2^6),\\
I_{12} = (x_3^2, x_3x_2, x_3x_1^3, x_2^8, x_2^7x_1, x_2^6x_1^2), & I_{25} = (x_3^2, x_3x_2^3, x_3x_2^2x_1, x_3x_2x_1^2, x_3x_1^3, x_2^6),\\
I_{13} = (x_3^2, x_3x_2^2, x_3x_2x_1, x_3x_1^2, x_2^8, x_2^7x_1, x_2^6x_1^2), & I_{26} = (x_3^3, x_3^2x_2, x_3x_2^2, x_3^2x_1, x_3x_2x_1^2, x_3x_1^3, x_2^6);
\end{array}$$

(x_3^2, x_3x_2, x_2^5)

$$\begin{array}{ll}
I_{27} = (x_3^2, x_3x_2, x_2^6, x_2^5x_1^2), & I_{29} = (x_3^2, x_3x_2^2, x_3x_2x_1^2, x_2^5),\\
I_{28} = (x_3^2, x_3x_2^2, x_3x_2x_1, x_2^6, x_2^5x_1), & I_{30} = (x_3^3, x_3^2x_2, x_3x_2^2, x_3^2x_1, x_3x_2x_1, x_2^5);
\end{array}$$

$(x_3^2, x_3x_2^2, x_2^4)$

$$I_{31} = (x_3^2, x_3x_2^2, x_2^4)$$

In this case, Algorithm 3.5 finds 5 families over $\mathbb{P}^1 \times \mathbb{P}^1$.

- Starting from I_9 and its Borel deformations

$$\begin{array}{ll}
\mathcal{I}_1 & \text{defined by} \quad x_3x_1^2x_0^9,\ x_2^6x_1^2x_0^4,\ \mathcal{F}_{x_3x_1^2x_0^9} = \{\mathrm{id}\},\\
\mathcal{I}_2 & \text{defined by} \quad x_3x_1^2x_0^9,\ x_2^7x_0^5,\ \mathcal{F}_{x_3x_1^2x_0^9} = \{\mathrm{id}\},\\
\mathcal{I}_3 & \text{defined by} \quad x_3x_2x_0^{10},\ x_2^7x_0^5,\ \mathcal{F}_{x_3x_2x_0^{10}} = \{\mathrm{id}\},\\
\mathcal{I}_4 & \text{defined by} \quad x_3x_2x_0^{10},\ x_2^6x_1^2x_0^4,\ \mathcal{F}_{x_3x_2x_0^{10}} = \{\mathrm{id}\},
\end{array}$$

we can determine 2 families over $\mathbb{P}^1 \times \mathbb{P}^1$, because both $\{\mathcal{I}_1, \mathcal{I}_3\}$ and $\{\mathcal{I}_2, \mathcal{I}_4\}$

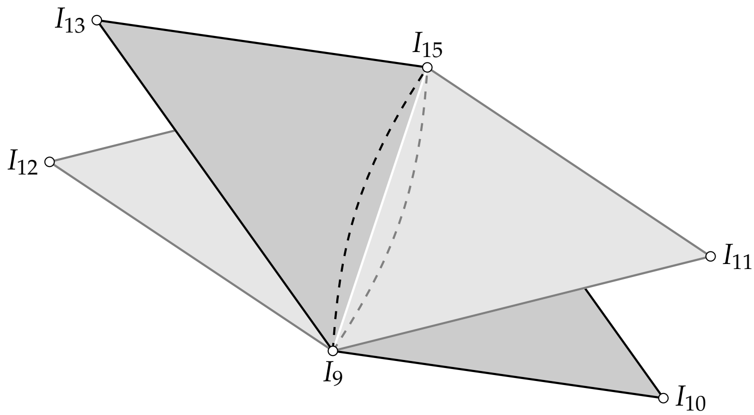


Figure 3.13: The two family over $\mathbb{P}^1 \times \mathbb{P}^1$ that give rise to two composed Borel rational deformations containing the same pairs of Borel-fixed ideals, even if they do not coincide.

are sets of compatible deformations:

$$\begin{aligned}\mathcal{I}_{1,3} = \Big\langle\, &\{(I_9)_{12}\} \setminus \{x_3x_1^2x_0^9\} \cup \{y_{00}x_3x_1^2x_0^9 + y_{01}x_2^6x_1^2x_0^4\} \\ &\setminus \{x_3x_2x_0^{10}\} \cup \{y_{10}x_3x_2x_0^{10} + y_{11}x_2^7x_0^5\}\Big\rangle,\\ \mathcal{I}_{2,4} = \Big\langle\, &\{(I_9)_{12}\} \setminus \{x_3x_1^2x_0^9\} \cup \{y_{00}x_3x_1^2x_0^9 + y_{01}x_2^7x_0^5\} \\ &\setminus \{x_3x_2x_0^{10}\} \cup \{y_{10}x_3x_2x_0^{10} + y_{11}x_2^6x_1^2x_0^4\}\Big\rangle.\end{aligned}$$

The Borel-fixed ideals belonging to $\mathcal{I}_{1,3}$ are I_9, I_{11}, I_{12} and I_{15} and the ideals belonging to $\mathcal{I}_{2,4}$ are I_9, I_{10}, I_{13} and I_{15}. Note that I_{15} turns out to be in both cases a composed Borel degeneration of I_9 but the rational deformations having I_9 and I_{15} as fibers that we can construct applying Corollary 3.15 on $\mathcal{I}_{1,3}$ and $\mathcal{I}_{2,4}$ are different (see Figure 3.13).

- Considered the ideal $I_{18} = (x_3^2, x_3x_2^2, x_3x_2x_1, x_3x_1^4, x_2^7, x_2^6x_1)$ and its Borel rational deformations

$$\begin{aligned}&\overline{\mathcal{I}}_1 \quad \text{defined by} \quad x_3x_2x_1x_0^9,\ x_2^6x_0^6,\ \mathcal{F}_{x_3x_2x_1x_0^9} = \{\mathrm{id}\},\\ &\overline{\mathcal{I}}_2 \quad \text{defined by} \quad x_3x_2x_1x_0^9,\ x_3x_1^3x_0^8,\ \mathcal{F}_{x_3x_2x_1x_0^9} = \{\mathrm{id}\},\end{aligned}$$

$$\overline{\mathcal{I}}_3 \quad \text{defined by} \quad x_3^2x_0^{10},\ x_3x_1^3x_0^8,\ \mathcal{F}_{x_3^2x_0^{10}} = \{\mathrm{id}\},$$
$$\overline{\mathcal{I}}_4 \quad \text{defined by} \quad x_3^2x_0^{10},\ x_2^6x_0^6,\ \mathcal{F}_{x_3^2x_0^{10}} = \{\mathrm{id}\},$$

we obtain again 2 families over $\mathbb{P}^1 \times \mathbb{P}^1$ by the compatibility of $\{\overline{\mathcal{I}}_1, \overline{\mathcal{I}}_3\}$ and $\{\overline{\mathcal{I}}_2, \overline{\mathcal{I}}_4\}$:

$$\begin{aligned}\overline{\mathcal{I}}_{1,3} = \Big\langle & \{(I_{18})_{12}\} \setminus \{x_3x_2x_1x_0^9\} \cup \{y_{00}x_3x_2x_1x_0^9 + y_{01}x_2^6x_0^6\} \\ & \setminus \{x_3^2x_0^{10}\} \cup \{y_{10}x_3^2x_0^{10} + y_{11}x_3x_1^3x_0^8\} \Big\rangle, \\ \overline{\mathcal{I}}_{2,4} = \Big\langle & \{(I_{18})_{12}\} \setminus \{x_3x_2x_1x_0^9\} \cup \{y_{00}x_3x_2x_1x_0^9 + y_{01}x_3x_1^3x_0^8\} \\ & \setminus \{x_3^2x_0^{10}\} \cup \{y_{10}x_3^2x_0^{10} + y_{11}x_2^6x_0^6\} \Big\rangle.\end{aligned}$$

$\overline{\mathcal{I}}_{1,3}$ contains I_{18}, I_{20}, I_{23}, I_{26}, $\overline{\mathcal{I}}_{2,4}$ contains I_{18}, I_{19}, I_{24}, I_{26}, and as before I_{26} is a composed Borel degeneration of I_{18} that can be obtained by means of 2 different composed Borel rational deformations.

- Consider the ideal $I_{23} = (x_3^2, x_3x_2^2, x_3x_2x_1^2, x_3x_1^4, x_2^6)$ and the Borel rational deformations

$$\widetilde{\mathcal{I}}_1 \quad \text{defined by} \quad x_3^2x_0^{10},\ x_3x_2x_1x_0^9,\ \mathcal{F}_{x_3^2x_0^{10}} = \{\mathrm{id}\},$$
$$\widetilde{\mathcal{I}}_2 \quad \text{defined by} \quad x_3x_1^{11},\ x_2^5x_1^7,\ \mathcal{F}_{x_3x_1^{11}} = \{\mathrm{id}, \mathsf{e}_1^-, 2\mathsf{e}_1^-, \ldots, 7\mathsf{e}_1^-\}.$$

They are compatible and give rise to the family over $\mathbb{P}^1 \times \mathbb{P}^1$

$$\begin{aligned}\mathcal{I}_1 = \Big\langle & \{(I_{23})_{12}\} \setminus \{x_3^2x_0^{10}\} \cup \{y_{00}x_3^2x_0^{10} + y_{01}x_3x_2x_1x_0^9\} \\ & \setminus \mathcal{F}_{x_3x_1^{11}}(x_3x_1^{11}) \cup \Big\{y_{10}F(x_3x_1^{11}) + y_{11}F(x_2^5x_1^7) \mid F \in \mathcal{F}_{x_3x_1^{11}}\Big\} \Big\rangle\end{aligned}$$

containing 4 Borel-fixed ideal I_{23}, I_{24}, I_{29} and I_{30}. Since one of the simple Borel deformations is defined by monomials in $\{(I_{23})_{12}\}_{(\geqslant 1)}$, the hyperplane section of the 4 ideals could not be the same, so we deduce that the corresponding four points, that we can not discuss with Reeves criterion, lie on a common component of $\mathbf{Hilb}^3_{6t-3}$.

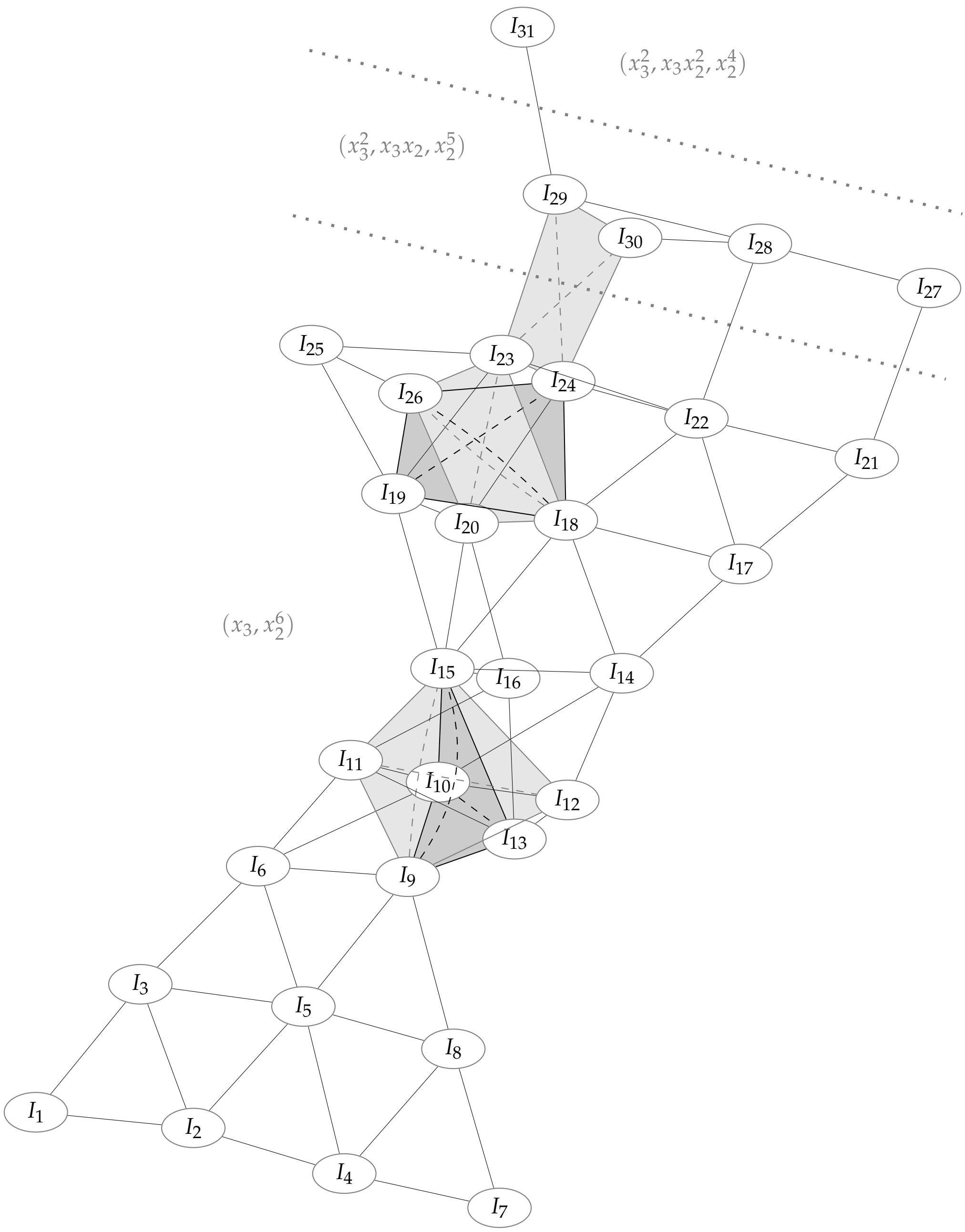


Figure 3.14: The Borel incidence graph of $\mathbf{Hilb}^3_{6t-3}$.

Chapter 4

Borel open covering of Hilbert schemes

In Chapter 1, we showed that describing explicitly a Hilbert scheme is a very hard task, even in the easiest (from a geometric point of view) cases, because the costruction of $\mathbf{Hilb}^n_{p(t)}$ as subscheme of a Grassmannian requires a huge number of variables. The first idea to reduce the complexity is to study the Hilbert scheme locally, i.e. to consider the affine open covering of the Grassmannian and to look at the intersection between the Hilbert scheme and each open subset. Indeed to study the Grassmannian $\mathbf{Gr}_{\mathbb{K}}(q,N)$ globally, we have to use the Plücker embedding (1.2) that requires $\binom{N}{q}$ (projective) variables, whereas if we want to study $\mathbf{Gr}_{\mathbb{K}}(q,N)$ locally, we can consider its open covering having open subsets described by $q(N-q)$ (affine) variables.

Since the Grassmannians considered in the study of Hilbert schemes actually parametrize ideals in a polynomial ring, the second idea is to exploit the algorithmic tools developed by the computational algebra, particularly the theory of Gröbner basis. The application of Gröbner bases to the study of Hilbert schemes was already introduced by Carrà Ferro in [19] and but our interest originates mainly in the ideas exposed in the paper [82] by Notari and Spreafico.

The results I will expose in this chapter belong to several joint papers [12, 13, 22, 58] with M. Roggero, F. Cioffi and C. Bertone.

4.1 Gröbner strata

Definition 4.1. Let us consider any term ordering σ and a monomial ideal $J \subset \mathbb{K}[x]$ (not even Borel-fixed). We define the *homogeneous tail* (in the following tail for short) of $x^\alpha \in J$ as the set of monomials:

$$\mathcal{T}_\sigma^J(x^\alpha) = \left\{ x^\beta \notin J \text{ s.t. } |\beta| = |\alpha| \text{ and } x^\beta <_\sigma x^\alpha \right\} \tag{4.1}$$

Every ideal $I \subset \mathbb{K}[x]$, having $J = (x^{\alpha_1}, \dots, x^{\alpha_s})$ as initial ideal w.r.t. σ, has a reduced Gröbner basis of $\{f_1, \dots, f_s\}$ where:

$$f_i = x^{\alpha_i} + \sum_{x^\beta \in \mathcal{T}_\sigma^J(x^{\alpha_i})} c_{\alpha_i\beta} x^\beta, \qquad c_{\alpha_i\beta} \in \mathbb{K}. \tag{4.2}$$

Thus it is very natural to parameterize the family of all the ideals I such that $\mathsf{in}_\sigma(I) = J$ by the coefficients $c_{\alpha_i\beta}$; in this way the fanily corresponds to a subset of $\mathbb{K}^{|\mathcal{T}_\sigma(J)|}$, where $\mathcal{T}_\sigma(J) = \mathcal{T}_\sigma^J(x^{\alpha_1}) \times \cdots \times \mathcal{T}_\sigma^J(x^{\alpha_s})$.

Definition 4.2. Given a monomial ideal $J = (x^{\alpha_1}, \dots, x^{\alpha_s}) \subset \mathbb{K}[x]$ and a term ordering σ, let us fix a subset T_i for each tail $\mathcal{T}_\sigma^J(x^{\alpha_i})$. Set $T = \{T_1, \dots, T_s\}$, we will denote by $\mathcal{S}t_\sigma(J, T)$ the family of all ideals I in $\mathbb{K}[x]$ such that $\mathsf{in}_\sigma(I) = J$ and whose reduced Gröbner basis $f_1, \dots, f_s$ is of the type:

$$f_i = x^{\alpha_i} + \sum_{x^\beta \in T_i} c_{\alpha_i\beta} x^\beta, \tag{4.3}$$

and we will call it *T-Gröbner stratum* of J. Moreover we will use $\mathcal{S}t_\sigma(J)$, and we will call it *Gröbner stratum* of J, whenever we consider the complete tail of every generator of J, i.e. $T_i = \mathcal{T}_\sigma^J(x^{\alpha_i})$, $\forall\, i$.

Remark 4.1.1. It will be clearer later that the term ordering affects the construction of a Gröbner stratum only because it determines which monomials can belong to the tails; indeed two different term orderings giving the same tails will lead to the same Gröbner stratum.

Every ideal I in the family $\mathcal{S}t_\sigma(J, T)$ is uniquely determined by a point in the affine space $\mathbb{A}^N$ ($N = \sum_i |T_i|$) where we fix coordinates $C_{\alpha_i\beta}$ corresponding to the coefficients $c_{\alpha_i\beta}$ that appear in (4.3). The subset of $\mathbb{A}^N$ corresponding to $\mathcal{S}t_\sigma(J,T)$ turns out to be a closed algebraic set. More precisely, we will see how it can be endowed in a very natural way with a structure of affine subscheme, possibly reducible or non reduced, that is we will see that it can be obtained as the subscheme of $\mathbb{A}^N$ defined by an ideal $\mathfrak{h}(J,T)$ in $\mathbb{K}[C]$, where C is the set of variables $C_{\alpha_i\beta}$.

Definition 4.3. We will denote by $\mathfrak{h}(J,T)$ and $\mathfrak{L}(J,T)$ the ideals in $\mathbb{K}[C]$ generated by the following procedure.

Step 1 Consider the set of polynomials $\mathcal{B} = \{F_1,\dots,F_s\}$ such that

$$F_i = x^{\alpha_i} + \sum_{x^\beta \in T_i} C_{\alpha_i\beta} x^\beta \in \mathbb{K}[C][x]. \tag{4.4}$$

Step 2 Consider a set $\mathrm{Syz}(J) = \left\{x^\gamma \mathbf{e}_i - x^\delta \mathbf{e}_j \mid x^\gamma x^{\alpha_i} - x^\delta x^{\alpha_j}\right\}$ that generates the syzygies of J.

Step 3a For every $x^\gamma \mathbf{e}_i - x^\delta \mathbf{e}_j \in \mathrm{Syz}(J)$, compute a complete reduction w.r.t. $\mathcal{B}$ of the S-polynomial $S(F_i,F_j) = x^\gamma F_i - x^\delta F_j$: $S(F_i,F_j) \xrightarrow{\mathcal{B}} R_{ij}$.

Step 3b For every $x^\gamma \mathbf{e}_i - x^\delta \mathbf{e}_j \in \mathrm{Syz}(J)$, compute a complete reduction w.r.t. J of the S-polynomial $S(F_i,F_j) = x^\gamma F_i - x^\delta F_j$: $S(F_i,F_j) \xrightarrow{J} M_{ij}$.

Step 4a Call $\mathfrak{h}(J,T)$ the ideal of $\mathbb{K}[C]$ generated by the coefficients (polynomials in $\mathbb{K}[C]$) of the reduced polynomials R_{ij} computed at **Step 3a**.

Step 4b Call $\mathfrak{L}(J,T)$ the ideal of $\mathbb{K}[C]$ generated by the coefficients of the reduced polynomials M_{ij} computed at **Step 3b**. Note that by construction the coefficients in M_{ij} are linear, so actually $\mathfrak{L}(J,T)$ turns out to be a vector subspace of the vector space $\langle C\rangle$ spanned by the variables C.

It is almost evident, that the definition of $\mathfrak{h}(J,T)$ is nothing else than Buchberger's characterization of Gröbner basis if we think to the $C_{\alpha_i\beta}$'s as constant in $\mathbb{K}$ instead of variables. In fact the variables $C_{\alpha_i\beta}$ do not appear in the leading terms w.r.t. σ of F_i and so their specialization in $\mathbb{K}$ commutes with reduction with respect to $\mathcal{B}$. Thus $(\dots,c_{\alpha_\beta},\dots)$ is a closed point in the support of $Z\big(\mathfrak{h}(J,T)\big)$ in $\mathbb{A}^N$

if and only if it corresponds to polynomials $f_1, \dots, f_s$ in $\mathbb{K}[x]$ that form a Gröbner basis. Then the support of $Z\big(\mathfrak{h}(J,T)\big)$ is uniquely defined; however a priori the ideal $\mathfrak{h}(J,T)$ could depend on the choices we perform computing it, that is (1) on the choice of the set $\mathrm{Syz}(J)$ of generators of the syzygies and (2) on the choices did during the reduction of any S-polynomial $S(F_i, F_j)$ (which in general is not uniquely determined).

Thanks again to Buchberger's criterion, we can prove that indeed $\mathfrak{h}(J,T)$ only depends on J, T and of course on σ, because it can be defined in an equivalent intrinsic way.

Proposition 4.4. *Let $J \subset \mathbb{K}[x]$ be a monomial ideal and let σ be any term ordering. Consider the set $\mathcal{B} = \{F_1, \dots, F_s\}$, $F_i \in \mathbb{K}[C][x]$ as in* (4.4) *and an ideal $\mathfrak{a}$ in $\mathbb{K}[C]$ with Gröbner basis $\mathcal{A}$. The following conditions are equivalent:*

(i) *$\mathcal{B} \cup \mathcal{A}$ is a Gröbner basis in $\mathbb{K}[C,x]$;*

(ii) *$\mathfrak{a}$ contains the coefficients (polynomials in $\mathbb{K}[C]$) of all the polynomials in the ideal $(F_1, \dots, F_s)$ that are reduced modulo J;*

(iii) *$\mathfrak{a}$ contains all the coefficients of every complete reduction of $S(F_i, F_j)$ with respect to $\mathcal{B}$ for every i, j;*

(iv) *$\mathfrak{a}$ contains all the coefficients of some (even partial) reduction with respect to $\mathcal{B}$ of $S(F_i, F_j)$ for every i, j;*

(v) *$\mathfrak{a}$ contains all the coefficients of some (even partial) reduction with respect to $\mathcal{B}$ of $S(F_i, F_j)$, for every (i, j) corresponding to a set* $\mathrm{Syz}(J)$ *of generators of the syzygies of J.*

Proof. (i) $\Rightarrow$ (ii). Let G be a polynomial in $(F_1, \dots, F_s)$ which is reduced modulo J. By hypothesis, G must be reducible to 0 through $\mathcal{B} \cup \mathcal{A}$, so that the further steps of reduction have to be performed just using $\mathcal{A}$. Any step of reduction through $\mathcal{A}$ does not change the monomials in $\mathbb{K}[x]$ but only modifies their coefficients (in $\mathbb{K}[C]$), then $G \xrightarrow{\mathcal{A}} 0$, that is every coefficient in $\mathbb{K}[C]$ of G can be reduced to 0 using $\mathcal{A}$: this shows that all the coefficients in $\mathbb{K}[x]$ of G belong to $\mathfrak{a}$.

(ii) $\Rightarrow$ *(iii)*, *(iii)* $\Rightarrow$ *(iv)* and *(iv)* $\Rightarrow$ *(v)* are obvious.

(v) $\Rightarrow$ *(i)*. We can check that $\mathcal{B} \cup \mathcal{A}$ is a Gröbner basis using the refined Buchberger criterion (see for instance [24, Theorem 9, p. 104]). If $\mathcal{A} = \{a_1, \ldots, a_r\}$, a set of generators for the syzygies of the ideal $(x^{\alpha_1}, \ldots, x^{\alpha_s}, \mathsf{in}_\sigma(a_1), \ldots, \mathsf{in}_\sigma(a_r))$ can be obtained as the union of a set of generators of the syzygies of $(x^{\alpha_1}, \ldots, x^{\alpha_s})$, a set of generators of the syzygies of $\mathsf{in}_\sigma(\mathfrak{a}) = \big(\mathsf{in}_\sigma(a_1), \ldots, \mathsf{in}_\sigma(a_r)\big)$ and the trivial syzygies $\big(x^{\alpha_i}, \mathsf{in}_\sigma(a_j)\big)$. Then:

- $S(a_i, a_j) \xrightarrow{\mathcal{B}\cup\mathcal{A}} 0$, since $\mathcal{A}$ is a Gröbner basis and $\mathcal{A} \subset \mathcal{B} \cup \mathcal{A}$;
- $S(F_i, a_j) \xrightarrow{\mathcal{B}\cup\mathcal{A}} 0$, since the leading terms of F_i and a_j are coprime and $F_i, a_j \in \mathcal{B} \cup \mathcal{A}$;
- $S(F_i, F_j) \xrightarrow{\mathcal{B}\cup\mathcal{A}} 0$ in at least one way, by hypothesis. $\square$

There are many ideals $\mathfrak{a}$ fulfilling the equivalent conditions of Proposition 4.4: for instance we can consider the irrelevant maximal ideal in $\mathbb{K}[C]$ or any ideal obtained accordingly with condition *(iv)*. Moreover, if $\mathfrak{a}$ satisfies those conditions and $\mathfrak{a}' \supset \mathfrak{a}$, then also $\mathfrak{a}'$ does, and if several ideals $\mathfrak{a}_l$ satisfy the conditions, then also their intersection $\bigcap_l \mathfrak{a}_l$ does. As a consequence of these remarks we obtain the proof of the uniqueness of the ideal $\mathfrak{h}(J, T)$ given by Definition 4.3.

Theorem 4.5. *Let J be a monomial ideal and T be the list of subsets of the tails of J as above. Then:*

(i) *$\mathfrak{h}(J, T)$ is uniquely defined; indeed $\mathfrak{h}(J, T) = \bigcap_l \mathfrak{a}_l$, $\mathfrak{a}_l$ satisfying the equivalent conditions of Proposition 4.4;*

(ii) *$\mathfrak{L}(J, T)$ is uniquely defined.*

Proof. *(i)* $\mathfrak{h}(J, T)$ is one of the ideals $\mathfrak{a}_l$, because it satisfies condition *(v)* of Proposition 4.4; on the other hand, if $\mathfrak{a}_l$ satisfies condition *(iii)* of Proposition 4.4, then clearly $\mathfrak{a} \supseteq \mathfrak{h}(J, T)$.

(ii) It suffices to observe that the generators for $\mathfrak{L}(J, T)$ are the degree 1 homogeneous components of the generators of $\mathfrak{h}(J, T)$ given in its construction (Definition 4.3). $\square$

By abuse of notation we will denote by the same symbol $\mathcal{S}t_\sigma(J,T)$ the family of ideals and the subscheme in $\mathbb{A}^N$ given by the ideal $\mathfrak{h}(J,T)$. Note that $\mathfrak{h}(J,T)$ is not always a prime ideal and so $\mathcal{S}t_\sigma(J,T)$ is not necessarily irreducible nor reduced, as the following trivial example shows.

Example 4.1.2. Let $J = (x_2^2, x_2x_1) \subset \mathbb{K}[x_0,x_1,x_2]$ and σ be any term ordering. Let us choose $T = \left\{T_{x_2^2} = \emptyset, T_{x_2x_1} = \{x_1x_0\}\right\}$ and construct the ideal of the T-Gröbner stratum $\mathcal{S}t_\sigma(J,T)$ according to Definition 4.3:

$$\begin{aligned}
&F_{x_2^2} = x_2^2, \quad F_{x_2x_1} = x_2x_1 + Cx_1x_0,\\
&S(F_{x_2^2}, F_{x_2x_1}) = x_1F_{x_2^2} - x_2F_{x_2x_1} =\\
&\qquad\qquad = -Cx_2x_1x_0 \xrightarrow{F_{x_2^2},F_{x_2x_1}} -Cx_2x_1x_0 + Cx_0F_{x_2x_1} = C^2x_1x_0^2.
\end{aligned}$$

Then $\mathfrak{h}(J,T) = (C^2)$ that is $\mathcal{S}t_\sigma(J,T)$ is a double point on the affine line $\mathbb{A}^1$.

4.1.1 Gröbner strata are homogeneous varieties

Let us denote by $\overline{\mathbb{T}}_x$ the group of the monomials in the field of fraction $\mathbb{K}(x)$ of the ring $\mathbb{K}[x]$. Any term ordering σ makes $\overline{\mathbb{T}}_x$ a totally ordered group in the obvious way.

Definition 4.6. Let us denote by ℓ the grading induced on $\mathbb{K}[C]$ by any term ordering σ on $\mathbb{K}[x]$ through the map

$$\begin{aligned}
\ell : \mathbb{K}[C] &\longrightarrow \overline{\mathbb{T}}_x,\\
C_{\alpha_i\beta} &\longmapsto \frac{x^{\alpha_i}}{x^\beta}.
\end{aligned} \tag{4.5}$$

As we will use also the usual grading over $\mathbb{Z}$ where all the variables have degree 1, we will always write explicitly the symbol ℓ when the above defined grading is concerned (so, ℓ-degree γ with $\gamma \in \mathbb{Z}^{n+1}$, ℓ-homogeneous of degree γ etc.) and we will leave the simple terms when the usual grading is involved (so, degree r with $r \in \mathbb{Z}$, homogeneous of degree r etc.).

Proposition 4.7 ([90, Lemma 2.8]). *(i) The grading ℓ is positive.*

(ii) $\mathfrak{h}(J,T)$ *is a ℓ-homogeneous ideal.*

Proof. *(i)* Let us observe that all the variables have ℓ-degree higher than that of the constant 1. Indeed

$$\ell(C_{\alpha_i\beta}) >_\ell \ell(1) \iff \frac{x^{\alpha_i}}{x^\beta} >_\sigma 1 \iff x^{\alpha_i} >_\sigma x^\beta.$$

As well known, this condition is equivalent to the positivity of the grading (see [55, Chapter 4]).

(ii) Let us consider the grading on $\mathbb{K}[C,x]$ induced by the map $\ell : \mathbb{K}[C,x] \to \overline{\mathbb{T}}_x$ sending $\ell(x_j) = x_j$ and $\ell(C_{\alpha_i\beta}) = \frac{x^{\alpha_i}}{x^\beta}$ and note that it coincides with the grading introduced in Definition 4.6 on the restriction to $\mathbb{K}[C]$. Every monomial that appears in F_i is of the type $C_{\alpha_i\beta}x^\beta$ and so its ℓ-degree is $\ell(C_{\alpha_i\beta}x^\beta) = \ell(C_{\alpha_i\beta}) \cdot \ell(x^\beta) = \frac{x^{\alpha_i}}{x^\beta}x^\beta = x^{\alpha_i}$. Thus all the polynomials F_i are ℓ-homogeneous and then also the S-polynomials $S(F_i,F_j)$ and their reductions are ℓ-homogeneous. Finally, the coefficients of any monomial x^γ (which are polynomials in $\mathbb{K}[C]$) in such reductions are ℓ-homogeneous. $\square$

Proposition 4.8 ([32, Theorem 3.2])**.** *The linear space $Z\big(\mathfrak{L}(J,T)\big)$ can be naturally identified with the Zariski tangent space to $\mathcal{S}t_\sigma(J,T)$ at the origin.*

If $C' \subset C$ is any subset of $\dim_{\mathbb{K}} Z\big(\mathfrak{L}(J,T)\big)$ variables such that $\mathfrak{L}(J,T) \oplus \langle C'\rangle = \langle C\rangle$, then $\mathfrak{h}(J,T) \cap \mathbb{K}[C']$ defines a ℓ-homogeneous subvariety in $\mathbb{A}^{|C'|}$ isomorphic to $\mathcal{S}t_\sigma(\mathfrak{h},T)$.

Proof. By definition there exist $e = |C \setminus C'|$ ℓ-homogeneous linear form $l_1,\dots,l_e$ in $\mathfrak{L}(J,T)$ such that $\{l_1,\dots,l_e\} \cup C'$ is a basis for the $\mathbb{K}$-vector space of linear forms in $\mathbb{K}[C]$. Then $\mathfrak{h}(J,T)$ has a set of ℓ-homogeneous generators of the type

$$l_1 + q_1, \dots, l_e + q_e, q_{e+1}, \dots, q_{e+s} \tag{4.6}$$

where $q_1,\dots,q_{e+s} \in \mathbb{K}[C']$ so that the inclusion

$$\mathbb{K}[C']/(q_{e+1},\dots,q_{e+s}) \hookrightarrow \mathbb{K}[C]/\mathfrak{h}(J,T) \tag{4.7}$$

is indeed an isomorphism (see also [90, Proposition 2.4]). The hypothesis $|C'| = \dim_{\mathbb{K}} \mathfrak{L}(J,T)$ ensures that $l_1,\dots,l_e$ generate $\mathfrak{L}(J,T)$ so that $q_{e+1},\dots,q_{e+s}$ belong to $(C')^2\mathbb{K}[C']$ and the tangent space at the origin of $Z((q_{e+1},\dots,q_{e+s}))$ is a linear space of dimension $|C'|$, i.e. $\mathbb{A}^{|C'|}$ itself. $\square$

We may summarize the previous result saying that $\mathcal{S}t_\sigma(J,T)$ *can be embedded in its Zariski tangent space at the origin*. This explains the following terminology.

Definition 4.9. We call *embedding dimension* of $\mathcal{S}t_\sigma(J,T)$ the dimension of the affine space $\mathbb{A}^{|C'|}$ defined in Proposition 4.8 and we will denote it by $\operatorname{ed}\mathcal{S}t_\sigma(J,T)$, i.e. $\operatorname{ed}\mathcal{S}t_\sigma(J,T) = |C'|$. The complement $C'' = C \setminus C'$ is a *maximal set of eliminable variables* for $\mathfrak{h}(J,T)$.

Corollary 4.10. *In the above notation, the following statements are equivalent:*

(i) $\mathcal{S}t_\sigma(J,T) \simeq \mathbb{A}^{\operatorname{ed}\mathcal{S}t_\sigma(J,T)}$;

(ii) $\mathcal{S}t_\sigma(J,T)$ *is smooth;*

(iii) the origin is a smooth point for $\mathcal{S}t_\sigma(J,T)$*;*

(iv) $\operatorname{ed}\mathcal{S}t_\sigma(J,T) \leqslant \dim_{\mathbb{K}} \mathcal{S}t_\sigma(J,T)$.

Note that in general a maximal set of eliminable variables (and so its complementary) is not uniquely determined. However, if $C_{\alpha_i\beta} \in \mathfrak{L}(J,T)$, then $C_{\alpha_i\beta}$ belongs to any set of eliminable variables; on the other hand, if $C_{\alpha_i\beta}$ does not appear in any element of $\mathfrak{L}(J,T)$, then $C_{\alpha_i\beta}$ does not belong to any set of eliminable variables.

There is an easy criterion that allows us to decide if a variable is eliminable or not.

Criterion 4.11. Let us consider two polynomials F_i and F_j among those defined in (4.4) such that $\mathsf{in}_\sigma(F_i) = x^{\alpha_i}$, $\mathsf{in}_\sigma(F_j) = x^{\alpha_j}$ and let $C_{\alpha_i\beta}$ be a variable appearing in the tail of F_i. Using the reduction with respect to J of a ℓ-homogeneous polynomial $x^\gamma F_i - x^\delta F_j$ we can see that:

(a) if $x^{\gamma+\beta} \notin J$ and $x^{\gamma+\beta-\delta}$ is not a monomial that appears in F_j, then $C_{\alpha_i\beta} \in \mathfrak{L}(J,T)$;

(b) if $x^{\gamma+\beta} \notin J$ and $x^{\beta'} = x^{\gamma+\beta-\delta}$ is a monomial that appears in F_j, then $C_{\alpha_i\beta} - C_{\alpha_j\beta'} \in \mathfrak{L}(J,T)$.

Moreover if $C_{\alpha_i\beta} - C_{\alpha_j\beta'} \in \mathfrak{L}(J,T)$, then every maximal set of eliminable variables must contain at least either one of them.

In most cases the number $N = |C|$ is very big and $\mathfrak{h}(J, T)$ needs a lot of generators so that finding it explicitly is a very heavy computation. On the contrary $\mathfrak{L}(J, T)$ is very fast to compute and so we can easily obtain a set of eliminable variables C''; a forgoing knowledge of C' allows a simpler computation of the ideal $\mathfrak{h}(J, T) \cap \mathbb{K}[C \setminus C'']$ that gives $\mathcal{S}t_\sigma(J, T)$ embedded in the affine space of minimal dimension $\mathbb{A}^{\mathrm{ed}\,\mathcal{S}t_\sigma(J,T)}$.

Furthermore, in many interesting cases we can greatly bring down the number of involved variables thanks to another kind of argument.

Theorem 4.12. *Let $J \subset \mathbb{K}[x]$ be a Borel-fixed saturated monomial ideal with basis G, m any integer and $\mathfrak{h}(J_{\geqslant m})$ the ideal of $\mathcal{S}t_\sigma(J_{\geqslant m})$ as in Definition 4.3.*

(i) There is a set of eliminable variables for $\mathfrak{h}(J_{\geqslant m})$ that contains all variables except at most those appearing in polynomials F_i whose leading term is either $x^\alpha \in G_{\geq m}$ or $x^\alpha x_0^{m-|\alpha|}$, where $x^\alpha \in G_{<m}$.

(ii) $\mathcal{S}t_\sigma(J_{\geqslant m-1})$ is a closed subscheme of $\mathcal{S}t_\sigma(J_{\geqslant m})$. More precisely $\mathcal{S}t_\sigma(J_{\geqslant m-1})$ is isomorphic to $\mathcal{S}t_\sigma(J_{\geqslant m}, T)$, where T contains the complete tail of a monomial in the basis of $J_{\geqslant m}$ if it is not divided by x_0, and a tail containing only monomials divided by x_0 otherwise.

(iii) If x_1 does not appear in any monomial of degree $m+1$ in the monomial basis of J, then $\mathcal{S}t_\sigma(J_{\geqslant m-1}) \simeq \mathcal{S}t_\sigma(J_{\geqslant m})$.

(iv) If x_1 appears in N monomials of degree $m+1$ in the monomial basis G of J, then $\mathrm{ed}\,\mathcal{S}t_\sigma(J_{\geqslant m}) \geqslant \mathrm{ed}\,\mathcal{S}t_\sigma(J_{\geqslant m-1}) + NM$, where M is the number of monomials of the basis of J of degree smaller than m.

(v) $\mathcal{S}t_\sigma(J_{\geqslant m-1}) \not\simeq \mathcal{S}t_\sigma(J_{\geqslant m})$ if and only if x_1 appears in monomials of degree $m+1$ in the monomial basis of J and $J_{\geqslant m-1} \neq J_{\geqslant m}$.

(vi) If s is the maximal degree of a monomial divided by x_1 in the monomial basis of J, then $\mathcal{S}t_\sigma(J_{\geqslant s-1}) \simeq \mathcal{S}t_\sigma(J_{\geqslant m})$ for every $m \geqslant s$.

Proof. (i) Let us consider any monomial x^η in the monomial basis of $J_{\geqslant m}$ which does not belong to $G_{\geqslant m}$ and such it that could be written as $x^\eta = x^\alpha x^\epsilon$ where $x^\alpha \in G_{<m}$

and x^ϵ is a monomial of degree $m-|\alpha|$, $x^\epsilon \neq x_0^{m-|\alpha|}$. Then among the polynomials F_i there are:

$$\begin{aligned} F &= x^\alpha x_0^{m-|\alpha|} + \sum C_\beta x^\beta, \\ F' &= x^{\alpha+\varepsilon} + \sum C'_\delta x^\delta. \end{aligned}$$

We have to prove that all the variables C' that appear in F' can be eliminated. The S-polynomial of F and F' is:

$$S(F,F') = x_0^p F' - x^{\varepsilon'} F = \sum C'_\delta x^\delta x_0^p - \sum C_\beta x^{\beta+\varepsilon'}.$$

No monomial $x^\delta x_0^p$ in the first summand belongs to $J_{\geqslant m}$ because $x^\delta \notin J$ and J is saturated and Borel-fixed. Thus, the linear part of the coefficient of $x^\delta x_0^p$ in the reduction of this S-polynomial will be either C'_δ or $C'_\delta - C_\beta$. Then C' is a set of eliminable variables for $J_{\geqslant m}$.

(ii) The first part of this statement is a special case of general facts proved in [41, Section 3]. We directly prove the second part (which implies the first one). Here we denote by x^α and x^γ the monomials in the basis of $J_{\geqslant m-1}$ of degree $m-1$ and $\geq m$ respectively, and we set:

$$\begin{aligned} F_\alpha &= x^\alpha + \sum C_{\alpha\beta} x^\beta, \qquad |\alpha| = m-1 \\ F_\gamma &= x^\gamma + \sum C_{\gamma\eta} x^\eta, \qquad |\gamma| \geqslant m \end{aligned}$$

where x^β varies among all monomials of degree $m-1$ in the tail of x^α and x^η among those of the same degree of x^γ in its tail. Applying the procedure described in Definition 4.3 on such set of polynomials we define $\mathcal{S}t_\sigma(J_{\geqslant m-1})$ by means of an ideal $\mathfrak{h}(J_{\geqslant m-1}) \subset \mathbb{K}[C]$.

The basis of $J_{\geqslant m}$ is made by monomials of the following three types:

- monomials x^γ of degree $\geqslant m$, that also belong to the basis of $J_{\geqslant m-1}$;
- monomials $x^\alpha x_0$ such that x^α is any monomial of degree $m-1$ in the basis of $J_{\geqslant m-1}$;
- monomials $x^\alpha x_i$ of degree m such that x^α is as above and $\min(x^\alpha) > x_i \neq x_0$.

We set:

$$\begin{aligned} \overline{F}_{\alpha 0} &= x^\alpha x_0 + \sum C_{\alpha\delta} x^\delta x_0, \\ \overline{F}_{\alpha i} &= x^\alpha x_i + \sum C'_{\alpha i \tau} x^\tau \quad |\tau| = m \quad x^\tau < x^\alpha x_i, \\ \overline{F}_\gamma &= x^\gamma + \sum C_{\gamma\eta} x^\eta. \end{aligned} \tag{4.8}$$

Note that we use the same names for some of the coefficients that appears in polynomials F and $\overline{F}$, so that $\overline{F}_{\alpha 0} = x_0 F_\alpha$ and $\overline{F}_\gamma = F_\gamma$.

Applying the procedure described in Definition 4.3 on the set of polynomials $\overline{F}$ we obtain an ideal $\mathfrak{h}' \subset \mathbb{K}[C, C']$ defining $\mathcal{S}t_\sigma(J_{\geqslant m}, T)$.

Thanks to *(i)* we know that C' is a set of eliminable variables for $\mathfrak{h}'$ and so $\mathcal{S}t_\sigma(J_{\geqslant m}, T)$ is also defined by $\overline{\mathfrak{h}} = \mathfrak{h}' \cap \mathbb{K}[C]$. The statement follows once we show that $\overline{\mathfrak{h}} = \mathfrak{h}(J_{\geqslant m-1})$.

In order to eliminate the variables C' we consider every monomial $x^\alpha x_i = \mathsf{in}(\overline{F}_{\alpha i})$ and reduce it using the polynomials F. In this way we obtain a polynomial $H_{\alpha i} \in (F)\mathbb{K}[X, C]$ such that $x^\alpha x_i + H_{\alpha i}$ is completely reduced w.r.t. J. Then also $x^\alpha x_i x_0 + H_{\alpha i} x_0 + \sum C'_{\alpha i \tau} x^\tau x_0$ (i.e. $\overline{F}_{\alpha i} x_0 + H_{\alpha i} x_0$) is reduced modulo J and moreover it belongs to $(\overline{F})\mathbb{K}[X, C, C']$ because $x_0 F \subseteq (\overline{F})\mathbb{K}[X, C, C']$. The coefficients of the monomials in the variables x belong to $\mathfrak{h}'$, because the ideal $\mathfrak{h}'$ is generated by the coefficient of monomials in x in the polynomials in $(F)\mathbb{K}[X, C, C']$ that are reduced modulo J_{m-1} or modulo J, which is the same (Proposition 4.4*(ii)* and Theorem 4.5). The coefficients of the monomials in x of $F_{\alpha i} x_0 + H_{\alpha i} x_0$ are also the coefficients of the monomials in x of $F_{\alpha i} + H_{\alpha i}$, and are precisely the set of polynomials of the type $C'_{\alpha i \tau} - \phi_{\alpha i \tau}(C)$ that allow us to eliminate the variables C'. So the elimination of C' is obtained simply putting $C'_{\alpha i \tau} = \phi_{\alpha i \tau}(C)$. In this way $F_{\alpha i}$ becomes $-H_{\alpha i}$ that belongs to $(G)\mathbb{K}[X, C]$.

The ideal $\overline{\mathfrak{h}}$, obtained from $\mathfrak{h}'$ eliminating C', can also be obtained first eliminating C' and after taking the coefficients of the monomials in x, because the procedure of eliminating C' and that of taking coefficients. So $\overline{\mathfrak{h}}$ is generated by the coefficients of monomials in x of polynomials in $(x_0 G_\alpha, -H_{\alpha i}, G_\gamma)\mathbb{K}[X, C]$ that are reduced modulo J. Hence $\overline{\mathfrak{h}} \subseteq \mathfrak{h}(J_{\geqslant m-1})$ because $(x_0 G_\alpha, -H_{\alpha i}, G_\gamma)\mathbb{K}[X, C] \subset (G)\mathbb{K}[X, C]$.

On the other hand, $x_0(G)\mathbb{K}[X, C] = (x_0 G_\alpha, x_0 G_\gamma)\mathbb{K}[X, C] \subset (x_0 G_\alpha, G_\gamma)\mathbb{K}[X, C]$. Moreover two polynomials Q and $x_0 Q$ have the same coefficients of the monomials in x and either one is reduced modulo J if and only the other is. Hence we obtain the opposite inclusion $\mathfrak{h}(J_{\geqslant m-1}) \subseteq \overline{\mathfrak{h}}$.

(iii) We use *(ii)* and prove that in the present hypothesis, $\mathcal{S}t_\sigma(J_{\geqslant m}) \simeq \mathcal{S}t_\sigma(J_{\geqslant m}, T)$, where T is defined as in *(ii)*. Following Definition 4.3, we obtain the ideal $\mathfrak{h}(J_{\geqslant m})$ of $\mathcal{S}t_\sigma(J_{\geqslant m})$ using:

$$\begin{aligned}
F''_{\alpha 0} &= x^\alpha x_0 + \sum C_{\alpha\delta} x^\delta x_0 + \sum C''_{\alpha\mu} x^\mu \quad , \quad x_0 \nmid x^\mu \\
F_{\alpha i} &= x^\alpha x_i + \sum C'_{\alpha i \tau} x^\tau \\
F_\gamma &= x^\gamma + \sum C_{\gamma\eta} x^\eta = G_\gamma.
\end{aligned}$$

Note that $F_{\alpha i}$ and F_γ are as in *(ii)*, but all the degree m monomials of the tail of $x_0 x^\alpha$ appear in $F''_{\alpha 0}$, and not only those divided by x_0.

For every monomial x^α of degree $m-1$ in the basis of J_{m-1}, let us consider the S-polynomial:

$$S(F''_{\alpha 0}, F_{\alpha 1}) = \sum C_{\alpha\delta} x^\delta x_1 x_0 + \sum C''_{\alpha\mu} x^\mu x_1 - \sum C'_{\alpha i \tau} x^\tau x_0.$$

By hypothesis no monomial appearing in it belongs to J_m. In fact $x^\mu x_1 \in J$ if and only if it is a minimal generator of J, which is excluded by hypothesis because its degree is $m+1$, or it is of the type $x^\alpha x_a$ with x^α minimal generator of J_m and $x_a = \min(x^\mu x_1) = x_1$, while $x^\mu \notin J_m$. Then $S(F''_{\alpha 0}, F_{\alpha 1})$ is already reduced with respect to J_m and so the coefficients of the monomials in x belong to $\mathfrak{h}(J_{\geqslant m})$. Especially, as both $x^\delta x_1 x_0$ and $x^\tau x_0$ are multiple of x_0, while $x^\mu x_1$ is not, the coefficient of $x^\mu x_1$ is simply $C''_{\alpha\sigma}$ so that each $C''_{\alpha\sigma}$ belongs to $\mathfrak{h}(J_{\geqslant m})$. Hence we can eliminate all the variables C'', just putting them equal to 0. In this way $F''_{\alpha 0}$ becomes $\overline{F}_{\alpha 0}$ as in (4.8) and $\mathcal{S}t_\sigma(J_{\geqslant m}) \simeq \mathcal{S}t_\sigma(J_{\geqslant m}, T)$, where T is as in *(ii)*, and we conclude because $\mathcal{S}t_\sigma(J_{\geqslant m}, T) \simeq \mathcal{S}t_\sigma(J_{\geqslant m-1})$.

(iv) By *(ii)*, we know that $\mathrm{ed}\, \mathcal{S}t_\sigma(J_{\geqslant m}) \geqslant \mathrm{ed}\, \mathcal{S}t_\sigma(J_{\geqslant m}, T) = \mathcal{S}t_\sigma(J_{\geqslant m-1})$, where the tails defined in T contain only monomials divided by x_0. Let us now consider a monomial x^α among the generators of J of degree smaller than m and a generator x^γ of degree $m+1$ divided by x_1. Computing the stratum $\mathcal{S}t_\sigma(J_{\geqslant m})$, in the tail of $x^\alpha x_0^{m-|\alpha|}$ there is the monomial $x^\beta = x^\gamma / x_1$ not belonging to T. Let us call D the

coefficient of x^β, that is

$$F = x^\alpha x_0^{m-|\alpha|} + \ldots + Dx^\beta + \ldots.$$

Thinking about the Eliahou-Kervaire syzygies of the ideal J, it is easy to see that in any S-polynomial, F is surely multiplied by a monomial x^δ, $\min x^\delta > \min x^\alpha x_0^{m-|\alpha|} = 0$. Therefore in every S-polynomial the monomial $x^\beta x^\delta = (x^\beta x_i)x^{\delta'}$ belongs to J because of the Borel-fixed hypothesis, so that it can be reduced. Finally there is no equation involving the variable D, so it is free and it cannot be eliminated. Repeating the reasoning for the M minimal generators of degree smaller than $m+1$ and for the N generators divided by x_1 of degree $m+1$, we obtain the thesis.

(v) and *(vi)* straightforward applying *(iv)* and *(iii)*. □

With the following examples, we want to underline again the not so crucial role played by term ordering in this construction (Example 4.1.3) and we want to show (Example 4.1.4 and Example 4.1.5) that the estimate of growth of the embedding dimension of the stratum introduced in Theorem 4.12*(iv)* is a lower bound.

Example 4.1.3. Let us consider the ideals $I = (x_3, x_2^2, x_2x_1)$ and $I_{\geqslant 2} = (x_3^2, x_3x_2, x_3x_1, x_3x_0, x_2^2, x_2x_1)$ in the ring $\mathbb{K}[x_0, x_1, x_2, x_3]$ and the Gröbner strata of the ideal $I_{\geqslant 2}$ according to two different term orderings: $\mathcal{S}t_{\texttt{DegLex}}(I_{\geqslant 2})$ and $\mathcal{S}t_{\texttt{DegRevLex}}(I_{\geqslant 2})$. In the first case there are 24 monomials in the complete tails, i.e. 24 new variables C, whereas in the second case they are 23, so we may guess that the family of the ideals with initial ideal $I_{\geqslant 2}$ w.r.t. `DegLex` could be different from the family of the ideals with initial ideal $I_{\geqslant 2}$ w.r.t. `DegRevLex`.

However applying Theorem 4.12, we can see that $\mathcal{S}t_{\texttt{DegLex}}(I_{\geqslant 2}) \simeq \mathcal{S}t_{\texttt{DegLex}}(I)$ and $\mathcal{S}t_{\texttt{DegRevLex}}(I_{\geqslant 2}) \simeq \mathcal{S}t_{\texttt{DegRevLex}}(I)$. Now the tails of the 3 monomials that generate I are the same w.r.t. both term orders and then (see Remark 4.1.1)

$$\mathcal{S}t_{\texttt{DegLex}}(I_{\geqslant 2}) \simeq \mathcal{S}t_{\texttt{DegLex}}(I) = \mathcal{S}t_{\texttt{DegRevLex}}(I) \simeq \mathcal{S}t_{\texttt{DegRevLex}}(I_{\geqslant 2}).$$

Example 4.1.4. Let us consider the polynomial ring $\mathbb{K}[x_0, x_1, x_2, x_3]$, the ideal $J = (x_3^2, x_3x_2, x_3x_1^4, x_2^7, x_2^6x_1^2)$ and the term ordering ω given by the matrix (2.41) with second row equal to $(23, 5, 2, 1)$. By the previous theorem we know that

$$\begin{aligned}
&\mathcal{S}t_\omega(J) \simeq \mathcal{S}t_\omega(J_{\geqslant 3}),\\
&\mathcal{S}t_\omega(J_{\geqslant 4}) \simeq \mathcal{S}t_\omega(J_{\geqslant 5}) \simeq \mathcal{S}t_\omega(J_{\geqslant 6}),\\
&\mathcal{S}t_\omega(J_{\geqslant 7}) \simeq \mathcal{S}t_\omega(J_{\geqslant m}),\ \forall\, m \geqslant 8
\end{aligned}$$

and

$$\operatorname{ed} \mathcal{S}t_\omega(J_{\geqslant 4}) \geqslant \operatorname{ed} \mathcal{S}t_\omega(J) + 2 \qquad \operatorname{ed} \mathcal{S}t_\omega(J_{\geqslant 7}) \geqslant \operatorname{ed} \mathcal{S}t_\omega(J_{\geqslant 4}) + 3$$

By an explicit computation, $\operatorname{ed} \mathcal{S}t_\omega(J) = 46$, $\operatorname{ed} \mathcal{S}t_\omega(J_{\geqslant 4}) = 50$ and $\operatorname{ed} \mathcal{S}t_\omega(J_{\geqslant 7}) = 56$.

Example 4.1.5. For any term ordering σ, there are at most two possible classes of isomorphism for the strata $\mathcal{S}t_\sigma(L_{\geqslant m})$, where $L \subset \mathbb{K}[x]$ is a saturated lexicographic ideal: $\mathcal{S}t_\sigma(L)$ and $\mathcal{S}t_\sigma(L_{\geqslant r-1})$, where r is the maximal degree of a minimal generator, in fact the variable x_1 appears (if it does) only in the generator of degree r. Called b the number of generators of degree r, applying Theorem 4.12*(iv)*, we have

$$\operatorname{ed} \mathcal{S}t_\sigma(L_{\geqslant r-1}) \geqslant \operatorname{ed} \mathcal{S}t_\sigma(L) + n - b.$$

If the monomial of maximal degree in the basis does not contain the variable x_1, we have $\mathcal{S}t_\sigma(L_{\geqslant m}) \simeq \mathcal{S}t_\sigma(L)$, $\forall\, m$.

We conclude this section with a result similar to the one stated in Theorem 4.12 that concerns only the case of Gröbner strata w.r.t. `DegRevLex`.

Proposition 4.13. *Let J be a Borel-fixed saturated ideal and let us consider the* `DegRevLex` *term ordering. Then*

$$\mathcal{S}t_{\texttt{DegRevLex}}(J) \simeq \mathcal{S}t_{\texttt{DegRevLex}}(J_{\geqslant m}), \quad \forall\, m.$$

Proof. The arguments to achieve the proof are very similar to the arguments used in the proof of Theorem 4.12. First of all let us consider the monomials

$$F_\alpha = x^\alpha + \sum C_{\alpha\beta} x^\beta$$

corresponding to the monomial basis G of J and the ideal $\mathfrak{h}(J) \subset \mathbb{K}[C]$ of the stratum $\mathcal{S}t_{\texttt{DegRevLex}}(J)$.

In order to compute $\mathcal{S}t_{\mathtt{DegRevLex}}(J_{\geqslant m})$, we have to consider again polynomials F_α as before if $x^\alpha \in G_{\geqslant m}$ and new polynomials $\overline{F}_{\alpha\varepsilon}$ such that $\mathsf{in}(\overline{F}_{\alpha\varepsilon}) = x^{\alpha+\varepsilon}$, $\forall\, x^\alpha \in G_{<m}$ and $\forall\, x^\varepsilon$ of degree $m - |\alpha|$, especially $x^\alpha x_0^{m-|\alpha|}$. Then by the definition itself of $\mathtt{DegRevLex}$, the tail of $x^\alpha x_0^{m-|\alpha|}$ contains exactly the monomials in the tail of x^α multiplied by $x_0^{m-|\alpha|}$. So we can write

$$\overline{F}_{\alpha\varepsilon} = \begin{cases} x^{\alpha+\varepsilon} + \sum E^{\varepsilon}_{\alpha\delta} x^\delta, & \forall\, x^\varepsilon \neq x_0^{m-|\alpha|}, \\ x^\alpha x_0^{m-|\alpha|} + \sum C_{\alpha\beta} x^\beta x_0^{m-|\alpha|} = x_0^{m-|\alpha|} F_\alpha, & \text{if } x^\varepsilon = x_0^{m-|\alpha|} \end{cases}$$

hence $\mathfrak{h}(J_{\geqslant m}) \subset \mathbb{K}[C, E]$ (note that in the present case variables D do not appear by construction).

By Theorem 4.12*(i)*, we know that all the variables E can be eliminated. By the same reasoning used in the proof of Theorem 4.12*(ii)*, the ideal $\overline{\mathfrak{h}} = \mathfrak{h}(J_{\geqslant m}) \cap \mathbb{K}[C]$ contains the coefficients of the monomials in x of a set of S-polynomials corresponding to a set of the S-polynomials of the monomial basis of J: so $\mathcal{S}t_{\mathtt{DegRevLex}}(J) \simeq \mathcal{S}t_{\mathtt{DegRevLex}}(J_{\geqslant m})$. □

4.2 Open subsets of the Hilbert scheme I

We now discuss the relation between Gröbner strata and Hilbert schemes, so in this section we will use again the main related notation. Given a Hilbert polynomial $p(t)$, r will denote its Gotzmann number, $N(t) = \binom{n+t}{n}$ and $q(t) = N(t) - p(t)$.

Lemma 4.14. *Let $J \subset \mathbb{K}[x]$ be any monomial ideal defining a subscheme* $\operatorname{Proj} \mathbb{K}[x]/J \subset \mathbb{P}^n$ *with Hilbert polynomial $p(t)$ and let σ be any term ordering. Then (at least set-theoretically) $\mathcal{S}t_\sigma(J) \subseteq \mathbf{Hilb}^n_{p(t)}$.*

Proof. Let I be any ideal in $\mathcal{S}t_\sigma(J)$. By hypothesis $\mathsf{in}_\sigma(I) = J$ and then I and J share the same Hilbert function. Therefore I defines a subscheme with Hilbert polynomial $p(t)$, i.e. a point of $\mathbf{Hilb}^n_{p(t)}$. □

Now we will see that with some restriction the set-theoretic inclusions are in fact algebraic maps and that for some special ideals they are indeed open injections. The crucial point is that the stratum structure (and so its injection in the

Hilbert scheme) depends on the ideal J and not on the the corresponding subscheme $Z = \operatorname{Proj} \mathbb{K}[x]/J$. For instance the stratum of the saturated lexicographic ideal L associated to the Hilbert polynomial $p(t)$ is not in general isomorphic to an open subset of $\mathbf{Hilb}^n_{p(t)}$ (see [90] and Example 4.1.5), whereas, as we will see, the stratum of its truncation $L_{\geqslant r}$ is an open subset of the Reeves-Stillman component of $\mathbf{Hilb}^n_{p(t)}$.

Let $J = J_{\geqslant r}$ be a monomial ideal with Hilbert polynomial $p(t)$ (and Gotzmann number r) and let σ be any term ordering. As seen in the previous section, every ideal I such that $\mathsf{in}_\sigma(I) = J$ has a (unique) reduced Gröbner basis $\{f_1, \dots, f_{q(r)}\}$ where f_i is as in Definition 4.2. Not every ideal generated by $q(r)$ polynomials of such a type has J as initial ideal. In order to obtain equations for $\mathcal{S}t_\sigma(J)$ we consider again the coefficients $c_{\alpha_i\beta}$ appearing in the f_i as new variables; more precisely let $C = \{C_{\alpha_i\beta},\ i = 1, \dots, q(r),\ x^\beta \notin J_r \text{ and } x^{\alpha_i} >_\sigma x^\beta\}$ be new variables and consider $q(r)$ polynomials in $\mathbb{K}[C][x]$ of the following type:

$$F_i = x^{\alpha_i} + \sum_{x^\beta \in \mathcal{T}^J_\sigma(x^{\alpha_i})} C_{\alpha_i\beta} x^\beta. \tag{4.9}$$

We obtain the ideal $\mathfrak{h}(J)$ of $\mathcal{S}t_\sigma(J)$ collecting the coefficients (polynomials in $\mathbb{K}[C]$) of the monomials in x of some complete reduction with respect to $F_1, \dots, F_{q(r)}$ of all the S-polynomials $S(F_i, F_j)$, corresponding to a set of generators for $\operatorname{Syz}(J)$ (see Theorem 4.5 and Proposition 4.4*(v)*).

Proposition 4.15. *In the above notation, let $J = J_{\geqslant r}$ be a monomial ideal with Hilbert polynomial $p(t)$ and let $\mathcal{A}$ be the $q(r)(n+1) \times N(r+1)$ matrix whose columns correspond to the monomials in $\mathbb{K}[x]_{r+1}$ and whose rows correspond to the polynomials $x_j F_i$, for all $j = 0, \dots, n$ and $i = 1, \dots, q(r)$ (and the entries are polynomials in $\mathbb{K}[C]$ corresponding to the coefficients of the monomials in x). Then the ideal $\mathfrak{h}(J)$ of the Gröbner stratum $\mathcal{S}t_\sigma(J)$ is generated by the minors of $\mathcal{A}$ of dimension $q(r+1)+1$.*

Proof. By abuse of notation we write in the same way a polynomial and the corresponding row in the matrix $\mathcal{A}$. It is quite evident by elementary arguments of linear algebra, that the ideal $\mathfrak{a} \subseteq \mathbb{K}[C]$, generated by all minors of dimension $q(r+1)+1$ does not change if we perform some row reduction on $\mathcal{A}$. Let $\mathcal{G}$ be a set of $q(r+1)$

rows whose leading terms are a basis of J_{r+1}. If $x_h F_i \notin \mathcal{G}$, then it has the same leading term than one in $\mathcal{G}$, say $x_k F_j$; we can substitute $x_h F_i$ with $x_h F_i - x_k F_j$. In this way the rows not in $\mathcal{G}$ become precisely all the S-polynomials $S(F_i, F_j)$ that have $r+1$ w.r.t. the variables x.

At the end of this sequence of row reductions, we can write the matrix as follows:

$$\left(\begin{array}{c|c} D & E \\ \hline S & L \end{array}\right) \tag{4.10}$$

where D is a $q(r+1) \times q(r+1)$ upper triangular matrix with 1's along the main diagonal, whose rows correspond to $\mathcal{G}$ and whose columns correspond to monomials in J_{r+1}.

Using rows in $\mathcal{G}$, we now perform a sequence of rows reductions on the following ones, in order to annihilate all the coefficients of monomials in J_{r+1}, that is the entries of the submatrix S: if $a(C)$ is the first non-zero entry in a row not in $\mathcal{G}$ and its column corresponds to the monomial $x^\gamma \in J_{r+1}$, we add to this row $-a(C)x_k F_j$, where $x_k F_j \in \mathcal{G}$ and $\mathsf{in}_\sigma(x_k F_j) = x^\gamma$. This is nothing else than a step of reduction with respect to $\{F_1, \dots, F_{q(r)}\}$. At the end of this second turn of rows reductions, we can write the matrix as follows:

$$\left(\begin{array}{c|c} D & E \\ \hline 0 & R \end{array}\right) \tag{4.11}$$

where the rows in $(D \mid E)$ are unchanged whereas the rows in $(0 \mid R)$ are the coefficients of the monomials in x of the complete reductions of S-polynomials of degree $r+1$ w.r.t. variables x. Then $\mathfrak{a}$ is generated by the entries of R and so $\mathfrak{a} \subset \mathfrak{h}(J)$.

We can see that this inclusion is in fact an equality taking in mind Theorem 2.20 and Proposition 4.4*(v)*: the first one says that $\mathrm{Syz}(J)$ is generated in degree $r+1$ and the second one that in this case $\mathfrak{h}(J)$ is generated by the coefficients of the monomials in x of complete reductions of the S-polynomials $S(F_i, F_j)$ of degree $r+1$ w.r.t. the variables x. $\square$

The following corollary just express in an explicit way two properties contained in the proof of Proposition 4.15.

Corollary 4.16. *In the above notation:*

- *the ideal $\mathfrak{h}(J)$ is generated by the entries of the submatrix R in* (4.11)*;*
- *the vector space $\mathfrak{L}(J)$ is generated by the entries of the submatrix L in* (4.10)*.*

Let us now look briefly at the construction of the Hilbert scheme from a local perspective, that is finding equations of the open subset $\mathcal{U}_{\Delta_\mathrm{I}} \cap \mathbf{Hilb}^n_{p(t)}$ for any open affine subset $\mathcal{U}_{\Delta_\mathrm{I}}$ of the Grassmannian $\mathbf{Gr}_{\mathbb{K}}(q(r), N(r))$ given by the non-vanishing of a Plücker coordinate Δ_I. Since the base vector space is that one spanned by the monomials of degree r in $\mathbb{K}[x]$, we can associate to any Plücker coordinate a monomial ideal J generated by $q(r)$ monomials of degree r. Therefore we will denote by $\mathcal{U}_J$ and $\mathcal{H}_J$ respectively the open subsets of $\mathbf{Gr}_{\mathbb{K}}(q(r), N(r))$ and of $\mathbf{Hilb}^n_{p(t)}$ corresponding to the the Plücker coordinate associated to the ideal J.

In a natural way $\mathcal{U}_J$ is isomorphic to the affine space $\mathbb{A}^{q(r)\left(N(r)-q(r)\right)}$. Indeed, if $J = (x^{\alpha_1}, \ldots, x^{\alpha_{q(r)}})$, every point in $\mathcal{U}_J$ is uniquely identified by the reduced, ordered set of generators $\langle g_1, \ldots, g_{q(r)}\rangle$ of the type $g_i = x^{\alpha_i} + \sum c_{\alpha_1\beta}x^\beta$, where $c_{\alpha_i\beta} \in \mathbb{K}$ and x^β is any monomial in $\mathbb{K}[x]_r \setminus J$. Then we consider on $\mathbb{A}^{q(r)\left(N(r)-q(r)\right)}$ the coordinates $C_{\alpha_i\beta}$. Note that each $C_{\alpha_i\beta}$ naturally corresponds to the Plücker coordinate associated to the ideal $J' = (x^{\alpha_1}, \ldots, x^{\alpha_{i-1}}, x^\beta, x^{\alpha_{i+1}}, \ldots, x^{\alpha_{q(r)}})$ (but of course not all the Plücker coordinates are of this type).

Now we can mimic the construction of Gröbner strata and obtain the defining ideal of $\mathcal{H}_J$ as a subscheme of $\mathbb{A}^{q(r)\left(N(r)-q(r)\right)}$. Let us consider the set of variables $\overline{C} = \{C_{\alpha_i\beta}, i = 1, \ldots, q(r),\ x^\beta \in \mathbb{K}[x]_r \setminus J\}$ and $q(r)$ polynomials $\overline{F}_1, \ldots, \overline{F}_{q(r)}$ in $\mathbb{K}[\overline{C}][x]$ of the type:

$$\overline{F}_i = x^{\alpha_i} + \sum_{\substack{|\beta|=r\\ x^\beta \notin J}} C_{\alpha_i\beta}x^\beta \tag{4.12}$$

and let $\overline{\mathcal{A}}$ be the $(n+1)q(r) \times N(r+1)$ matrix representing the polynomials $x_j\overline{F}_i$ (as the matrix $\mathcal{A}$ represents the polynomials x_jF_i in the proof of Proposition 4.15). Then consider the ideal $\overline{\mathfrak{a}}(J) \subset \mathbb{K}[\overline{C}]$ generated by the minors of $\overline{\mathcal{A}}$ of dimension $q(r+1)+1$.

Proposition 4.17. *$\overline{\mathfrak{a}}(J)$ is the ideal of $\mathcal{H}_J$ as a closed subscheme of $\mathbb{A}^{q(r)p(r)}$.*

Proof. Every ideal $I \in \mathcal{U}_J$ can be obtained from $(\overline{F}_1, \dots, \overline{F}_{q(r)})$ specializing (in a unique way) the variables $C_{\alpha_i\beta}$ to $c_{\alpha_1\beta} \in \mathbb{K}$. Obviously not all the specializations give ideals $I \in \mathcal{H}_J$, that is with Hilbert polynomial $p(t)$, because we have to ask both $\dim_{\mathbb{K}} I_r = q(r)$ and $\dim_{\mathbb{K}} I_{r+1} = q(r+1)$: thanks to Gotzmann's Persistence Theorem we know that these two necessary conditions are also sufficient.

In the open subset $\mathcal{U}_J$ the first condition always holds and the rank of every specialization of $\overline{\mathcal{A}}$ is $\geqslant q(r+1)$ by Macaulay's Estimate on the Growth of Ideals. Therefore $\mathcal{H}_J$ is given by the condition $\operatorname{rank} \overline{\mathcal{A}} \leqslant q(r+1)$. □

Let us know consider a special ordering of Plücker coordinates. We write the $q(r)$ monomials generating the ideal associated to the Plücker coordinate in decreasing order with respect to σ; then given $J_1 = (x^{\alpha_1} >_\sigma \cdots >_\sigma x^{\alpha_{q(r)}})$ and $J_2 = (x^{\gamma_1} >_\sigma \cdots >_\sigma x^{\gamma_{q(r)}})$, then $J_1 > J_2$ if $x^{\alpha_i} = x^{\gamma_i}$ for every $i = 1, \dots, s-1$ and $x^{\alpha_s} >_\sigma x^{\gamma_s}$ (i.e. a lexicographic order on the "alphabet"of the monomials of degree r ordered w.r.t. σ).

It is now easy to compare, for the same monomial ideal $J = J_{\geqslant r}$ with Hilbert polynomial $p(t)$, the Gröbner stratum $\mathcal{S}t_\sigma(J)$ and the open subset $\mathcal{H}_J$. We underline that for our purpose it will be sufficient to consider the open subsets $\mathcal{H}_J$ corresponding to monomial ideals J defining points of $\mathbf{Hilb}^n_{p(t)}$, because (scheme-theoretically) they cover $\mathbf{Hilb}^n_{p(t)}$. Indeed, if I has Hilbert polynomial $p(t)$, also $\mathsf{in}_\sigma(I)$ does and so $I \in \mathcal{H}_{\mathsf{in}_\sigma(I)}$.

Theorem 4.18. *Let $p(t)$ be any admissible Hilbert polynomial in $\mathbb{P}^n$ with Gotzmann number r and let σ be any term ordering.*

(i) If $J = J_{\geqslant r}$ is a monomial ideal with Hilbert polynomial $p(t)$, then $\mathcal{S}t_\sigma(J)$ is naturally isomorphic to the locally closed subscheme of $\mathbf{Hilb}^n_{p(t)}$ given by the conditions that the Plücker coordinate corresponding to J does not vanish and the preceding ones vanish.

(ii) For every isolated, irreducible component H of $\mathbf{Hilb}^n_{p(t)}$, there is a monomial ideal $J = J_{\geqslant r}$ with Hilbert polynomial $p(t)$ such that an irreducible component of $\operatorname{Supp} \mathcal{S}t_\sigma(J)$ *is an open subset of* $\operatorname{Supp} H$*. Then* $\operatorname{Supp} H$ *has an open subset which is a homogeneous affine variety with respect to a non-standard positive grading.*

(iii) Every smooth irreducible component H of $\mathbf{Hilb}^n_{p(t)}$ is rational. The same holds for every smooth, irreducible component of Supp $\mathbf{Hilb}^n_{p(t)}$.

Proof. (i) We obtain the two affine varieties $\mathcal{S}t_\sigma(J)$ and $\mathcal{H}_J$ in a quite similar way. The only difference comes from the definition of the set of polynomials $F_1, \ldots, F_{q(r)}$ given in (4.9), leading to equations for $\mathcal{S}t_\sigma(J)$, and the set of polynomials $\overline{F}_1, \ldots, \overline{F}_{q(r)}$ given in (4.12), leading to equations for $\mathcal{H}_J$: in $\overline{F}_i$ the sum is over all the degree r monomials $x^\beta \notin J$ whereas in F_i we also assume that $x^\beta <_\sigma \mathsf{in}_\sigma(F_i) = x^{\alpha_i}$. Therefore we can think of $\mathcal{S}t_\sigma(J)$ as the affine subscheme defined by the ideal $\overline{\mathfrak{h}}(J)$ in the ring $\mathbb{K}[\overline{C}][X]$, where $\overline{C} = \{C_{\alpha_i\beta} \mid i = 0, \ldots, q(r),\ x^\beta \in \mathbb{K}[x]_r \setminus J\}$ generated by $\mathfrak{h}(J)$ and by $\left(C_{\alpha_i\beta} \mid x^\beta >_\sigma \mathsf{in}_\sigma(F_i)\right)$, namely $\overline{\mathfrak{h}}(J) = \mathfrak{h}(J)\mathbb{K}[\overline{C}] + (\overline{C} \setminus C)$. Now we can conclude because the Plücker coordinates before that associated to J vanish if and only if all the $C_{\alpha_i\beta}$ such that $x^\beta >_\sigma \mathsf{in}_\sigma(F_i)$ vanish.

(ii) As J varies among the finite set of the monomial ideals in $\mathbf{Hilb}^n_{p(t)}$, the Gröbner strata $\mathcal{S}t_\sigma(J)$ give a set theoretical covering of $\mathbf{Hilb}^n_{p(t)}$ by locally closed subschemes. Then there is a suitable ideal J such that an irreducible component of Supp $\mathcal{S}t_\sigma(J)$ is an open subset of H. The support and the irreducible components of the support of $\mathcal{S}t_\sigma(J)$ are homogeneous (see [16, Section IV.3.3] and [32, Corollary 2.7]), having $\mathcal{S}t_\sigma(J)$ a structure of homogeneous affine scheme with respect to a non-standard positive grading ℓ.

(iii) If H is a smooth, irreducible component of either $\mathbf{Hilb}^n_{p(t)}$ or Supp $\mathbf{Hilb}^n_{p(t)}$, then it is also reduced. Thanks to the previous item we know that an open subset of H is an affine homogeneous variety with respect to a positive grading. Moreover this open subset is also smooth and so it is isomorphic to an affine space, by Corollary 4.10. □

Remark 4.2.1. Let J be a monomial ideal in $\mathbf{Hilb}^n_{p(t)}$ and let $\mathfrak{a}(J) \subset \mathbb{K}[\overline{C}]$ the ideal of $\mathcal{H}_J$. It is possible to define a grading ℓ' on $\mathbb{K}[\overline{C}]$ such that $\mathfrak{a}(J)$ becomes homogeneous, by the analogous definition: $\ell'(C_{\alpha_i\beta}) = \frac{x^{\alpha_i}}{x^\beta}$ if $C_{\alpha_i\beta}$ appears in $\overline{F}_i$ (4.12). However this grading ℓ' is not necessarily positive and so it gives less interesting consequences.

If an irreducible component H of $\mathbf{Hilb}^n_{p(t)}$ is also reduced, Theorem 4.18 insures that there is an open subset of H which has the structure of homogeneous variety with respect to a positive grading induced from that of a suitable Gröbner stratum $\mathcal{S}t_\sigma(J)$. On the other hand, in the case of a non-reduced component we only know that the support of a suitable open subset is homogeneous with respect to a positive grading, but this does not imply that the open subset itself is homogeneous.

The proof of Theorem 4.18 suggests that whenever there do not exist monomials $x^\beta >_\sigma \mathsf{in}_\sigma(F_i)$ not belonging to the ideal J, the constructions of $\mathcal{S}t_\sigma(J)$ and $\mathcal{H}_J$ are substantially equal, hence the hilb-segment ideals introduced in Definition 2.60 assume a great importance.

Corollary 4.19. *Let $p(t)$ be a Hilbert polynomial with Gotzmann number r and let $J = J_{\geqslant r}$ be a hilb-segment ideal (and so Borel-fixed) such that J_r defines a point in the Grassmannian $\mathbf{Gr}_{\mathbb{K}}(q(r), N(r))$. If $\mathbb{K}[x]/J$ has Hilbert polynomial different from $p(t)$, then the open subset $\mathcal{H}_J$ of $\mathbf{Hilb}^n_{p(t)}$ is empty.*

Proof. Called σ the term ordering realizing J as hilb-segment ideal, any point $I \in \mathcal{H}_J$ should belong to the Gröbner stratum $\mathcal{S}t_\sigma(J)$, that is it should share the same Hilbert polynomial of J, which is not $p(t)$. □

The first of the following examples highlights both that Theorem 4.18 does not hold for a monomial ideal J defining a point of $\mathbf{Gr}_{\mathbb{K}}(q(r), N(r))$ but not of $\mathbf{Hilb}^n_{p(t)}$ and that Corollary 4.19 does not hold for a monomial ideal J in $\mathbf{Gr}_{\mathbb{K}}(q(r), N(r))$ which is not a hilb-segment ideal. Moreover Example 4.2.3 presents a concrete case of empty $\mathcal{H}_J$ as discussed in the previous corollary.

Example 4.2.2. Let us consider the constant Hilbert polynomial $p(t) = 2$ in $\mathbb{P}^2$. $\mathbf{Hilb}^2_2$ is irreducible of dimension 4 (see [50]). The monomial ideal $J = (x_2^2, x_2x_1, x_1^2, x_0^2)$ is generated by 4 monomials of degree 2, but does not belong to $\mathbf{Hilb}^2_2$ because its radical is the irrelevant maximal ideal. However, $\mathcal{H}_J$ is non-empty because it contains for instance all the reduced subschemes given by couples of points $[1 : a : b]$, $[1 : a' : b'] \in \mathbb{P}^2$ such that $ab' \neq a'b$. By the way, $\mathcal{S}t_\sigma(J)$ cannot have any common point with $\mathbf{Hilb}^2_2$.

Example 4.2.3. Let us consider the Hilbert polynomial $p(t) = 4t$ and the projective space $\mathbb{P}^3$. The Gotzmann number is 6, so that the Hilbert scheme $\mathbf{Hilb}^3_{4t}$ is embedded in the Grassmannian $\mathbf{Gr}_{\mathbb{K}}(60, 84)$. The ideal J generated by the greatest 60 monomials in $\mathbb{K}[x_0, x_1, x_2, x_3]_6$ w.r.t. `DegRevLex` defines by construction a segment $\{J_r\} \subset \mathcal{P}(3,6)$ and has constant Hilbert polynomial $\overline{p}(t) = 24$. Thus J defines a point of $\mathbf{Gr}_{\mathbb{K}}(60, 84)$ not belonging to $\mathbf{Hilb}^3_{4t}$, therefore $\mathcal{H}_J$ is empty.

Corollary 4.20. *Let $p(t)$ be an admissible Hilbert polynomial in $\mathbb{P}^n$ and let H be an isolated and irreducible component of $\mathbf{Hilb}^n_{p(t)}$. If H contains a point defined by a hilb-segment ideal J w.r.t. some term ordering σ, then $\mathcal{S}t_\sigma(J)$ is an open subset of H, so that H has an open subset which is an homogeneous affine variety with respect to a non-standard positive grading.*

Proof. If J is a hilb-segment ideal, then there are no Plücker coordinates preceding that one associated to J. Thus $\mathcal{S}t_\sigma(J) \simeq \mathcal{H}_J$ (see Theorem 4.18) and so $\mathcal{H}_J$ is an affine homogeneous scheme with respect to a positive grading. □

Corollary 4.21. *Let $J \subset \mathbb{K}[x]$ be a hilb-segment ideal w.r.t. some term ordering σ defining a point of the Hilbert scheme $\mathbf{Hilb}^n_{p(t)}$ and let H be an irreducible component of $\mathbf{Hilb}^n_{p(t)}$ containing the point defined by J. If either of the following condition holds:*

1. *$\mathcal{S}t_\sigma(J)$ is an affine space,*
2. *J is a smooth point of $\mathcal{S}t_\sigma(J)$,*
3. *J is a smooth point of $\mathbf{Hilb}^n_{p(t)}$,*

then H is rational.

Proof. Straightforward consequence of the previous result and of Corollary 4.10. □

4.2.1 The Reeves and Stillman component of $\mathbf{Hilb}^n_{p(t)}$

A nice application of the results just proved concerns the component of the Hilbert scheme $\mathbf{Hilb}^n_{p(t)}$ containing the point defined by the lexicographic ideal $L \subset \mathbb{K}[x]$ associated to the Hilbert polynomial $p(t)$. This component is unique and it is usually denoted by H_{RS} and called Reeves ans Stillman component of $\mathbf{Hilb}^n_{p(t)}$ because in [87] they prove that the point of $\mathbf{Hilb}^n_{p(t)}$ corresponding to $\operatorname{Proj} \mathbb{K}[x]/L$ (the lexicographic point) is smooth.

Putting together the smoothness of the lexicographic point and Corollary 4.21, we obtain the following property.

Corollary 4.22. *The Reeves and Stillman component H_{RS} of $\mathbf{Hilb}^n_{p(t)}$ is rational.*

Reeves and Stillman get the proof by a computation of the Zariski tangent space dimension; however we are able to prove the same result applying our technique as a theoretical tool. Mimicking the notation used in [87], we denote by $L(b_0,\ldots,b_{n-1})$ the truncation of the lexicographic ideal (2.40) in degree $r = \sum_i b_i$, i.e. the hilb-segment ideal w.r.t. `DegLex`

$$L(b_0,\ldots,b_{n-1}) = \left\langle x^\alpha \;\middle|\; x^\alpha \geq_{\texttt{DegLex}} x_n^{b_{n-1}}\cdots x_1^{b_0}\right\rangle.$$

Theorem 4.23. *The Gröbner stratum $\mathcal{S}t_{\texttt{DegLex}}\big(L(b_0,\ldots,b_{n-1})\big)$ of the lexicographic ideal associated to the Hilbert polynomial $p(t)$ is isomorphic to an affine space. Therefore the component H_{RS} of $\mathbf{Hilb}^n_{p(t)}$ is rational.*

Proof. Thanks to Corollary 4.21 we can obtain the complete statement proving that the Gröbner stratum $\mathcal{S}t_{\texttt{DegLex}}(L(b_0,\ldots,b_{n-1}))$ is an affine space, that is showing that a same number is both a lower-bound for its dimension and an upper-bound for its embedding dimension; the first part corresponds to Theorem 4.1 (here in terms of initial ideals) and the second one corresponds to Theorem 3.3 of [87].

We proceed by induction on the number n of variables and on the Gotzmann number r. In order to obtain an upper-bound for the embedding dimension we look for a maximal set of eliminable variables $C'' \subset C$, using Criterion 4.11. If $\{x^{\alpha_1},\ldots,x^{\alpha_n}\}$ is the monomial basis of the saturation L of $L(b_0,\ldots,b_{n-1})$, then we can assume to order the polynomials $F_1,\ldots,F_{q(r)} \in \mathbb{K}[C][x]$ (that we use to construct $\mathcal{S}t_{\texttt{DegLex}}(L(b_0,\ldots,b_{n-1})))$ so that $\mathrm{in}_{\texttt{DegLex}}(F_1) = x^{\alpha_1}x_0^{r-|\alpha_1|} >_{\texttt{DegLex}} \cdots >_{\texttt{DegLex}} \mathrm{in}_{\texttt{DegLex}}(F_n) = x^{\alpha_n}x_0^{r-|\alpha_n|}$. So by Theorem 4.12 we can initially consider a set of eliminable variables C'', containing all the variables $C_{\alpha_j\beta}$ appearing in F_j for every $j > n$.

Step 1 The zero-dimensional case: $\mathcal{S}t_{\texttt{DegLex}}\big(L(b_0,\ldots,0)\big) \simeq \mathbb{A}^{nb_0}$.
Claim 1i. $\dim \mathcal{S}t_{\texttt{DegLex}}\big(L(b_0,\ldots,0)\big) \geqslant nb_0$.
The zero-dimensional scheme Z of b_0 general points in $\mathbb{P}^n$ has Gotzmann number b_0 and Hilbert polynomial $p(t) = b_0$. Moreover $\mathrm{in}_{\texttt{DegLex}}\big(I(Z)_{\geqslant b_0}\big) \supseteq L(b_0,\ldots,b_{n-1})$,

because for every monomial $x^\gamma \geq_{\texttt{DegLex}} x_1^{b_0}$ we can find some homogeneous polynomial of the type $x^\gamma - \sum_{j=1}^{b_0} c_j x_1^{b_0-j} x_0^j$ vanishing in the b_0 points of Z: we can find the c_j's solving a $b_0 \times b_0$ linear system with a Vandermonde associated matrix. As both $\mathsf{in}_{\texttt{DegLex}}\big(I(Z)_{\geqslant b_0}\big)$ and $L(b_0, 0, \ldots, 0)$ are generated in degree b_0, they coincide; so $I(Z)_{\geqslant r} \in \mathcal{S}t_{\texttt{DegLex}}\big(L(b_0, 0, \ldots, 0)\big)$ and we conclude since we can choose Z in a family of dimension nb_0.

Claim 1ii. $\operatorname{ed} \mathcal{S}t_{\texttt{DegLex}}\big(L(b_0, \ldots, 0)\big) \leqslant nb_0$.

The saturation of $L(b_0, 0, \ldots, 0)$ is the ideal $(x_n, x_{n-1}, \ldots, x_2, x_1^{a_0})$, which is generated by n monomials; moreover there are only b_0 monomials of degree b_0 not contained in $L(b_0, 0, \ldots, 0)$: Theorem 4.12 leads to the conclusion.

Step 2 If $\mathcal{S}t_{\texttt{DegLex}}\big(L(0, b_1, \ldots, b_{n-1})\big) \simeq \mathbb{A}^K$ then $\mathcal{S}t_{\texttt{DegLex}}(L(b_0, b_1, \ldots, b_{n-1})) \simeq \mathbb{A}^{K+nb_0}$.

Claim 2i. $\dim \mathcal{S}t_{\texttt{DegLex}}\big(L(b_0, b_1, \ldots, b_{n-1})\big) \geqslant \dim \mathcal{S}t_{\texttt{DegLex}}\big(L(0, b_1, \ldots, b_{n-1})\big) + nb_0 = K + nb_0$.

Let Y be any closed subscheme in $\mathbb{P}^n$ such that $I(Y)_{\geqslant r} \in \mathcal{S}t_{\texttt{DegLex}}\big(L(0, b_1, \ldots, b_{n-1})\big)$ and consider the set Z of b_0 points in $\mathbb{P}^n$. If we choose the b_0 points in Z general enough, then $I(Z \cup Y) = I(Z) \cdot I(Y)$. Then we conclude thanks to the previous step, as $\mathsf{in}_{\texttt{DegLex}}\big(I(Z)\big) = L(b_0, 0, \ldots, 0)$ and $L(b_0, \ldots, b_{n-1}) = L(b_1, \ldots, b_{n-1}) \cdot L(b_0, \ldots, 0)$.

Claim 2ii. $\operatorname{ed} \mathcal{S}t_{\texttt{DegLex}}\big(L(b_0, b_1, \ldots, b_{n-1})\big) \leqslant \operatorname{ed} \mathcal{S}t_{\texttt{DegLex}}\big(L(0, b_1, \ldots, b_{n-1})\big) + nb_0 = K + nb_0$.

First of all, let us consider all the polynomials F_i such that $x_0^{r-b_0} \mid \mathsf{in}_{\texttt{DegLex}}(F_i)$ and the set of variables $C_{\alpha_i\beta}$ appearing in them such that $x^\beta = x^{\beta_1} x_0^{r-b_1}$ for some monomial $x^{\beta_1} \notin L(0, b_1, \ldots, b_{n-1})$: a multiple of x^β belongs to $L(b_0, \ldots, b_{n-1})$ if and only if the corresponding multiple of x^{β_1} belongs to $L(0, b_1, \ldots, b_{n-1})$. Then $F_i = x_0^{r-b_0} F_i^{(1)} + \ldots$, where the $F_i^{(1)}$'s are the polynomials that appear in the definition of $\mathcal{S}t_{\texttt{DegLex}}\big(L(0, b_1, \ldots, b_{n-1})\big)$. Using the S-polynomials involving pairs of such polynomials we see that $\mathfrak{L}\big(L(0, b_1, \ldots, b_{n-1})\big) \subseteq \mathfrak{L}\big(L(b_0, \ldots, b_{n-1})\big)$; thus all the variables $C_{\alpha_i\beta}$ of this type are eliminable, except at most $K = \operatorname{ed} \mathcal{S}t_{\texttt{DegLex}}\big(L(0, b_1, \ldots, b_{n-1})\big)$ of them.

Moreover, for every $i \leqslant n$ there are b_0 variables $C_{\alpha_i\beta}$ such that $x^\beta \notin L(b_0, \ldots, b_{n-1})$,

$x^\beta \in L(0,\dots,b_{n-1})$: there are $x_n^{b_{n-1}} \cdots x_2^{b_1} x_1^{b_0-j} x_0^j$, $j = 1,\dots,b_0$. If we specialize to 0 all the variables of the two above considered types, the embedding dimension drops at most by ed $\mathcal{S}t_{\mathtt{DegLex}}\big(L(0,b_1,\dots,b_{n-1})\big) + nb_0 = K + nb_0$.

Now it will be sufficient to verify that all the remaining variables $C_{\alpha_i\beta}$ are eliminable, using Criterion 4.11. Assume that $x^\beta <_{\mathtt{DegLex}} x_n^{b_{n-1}} \cdots x_2^{b_1}$ and $x_0^{r-b_0} \nmid x^\beta$.

- If $i > n$, all the variables are eliminable using those appearing in $F_1,\dots,F_n$, thanks to Corollary 4.12.
- If $i < n$, using $S(F_i,F_j)$, where $\mathsf{in}_{\mathtt{DegLex}}(F_j) = x^{\alpha_i} x_1^{r-|\alpha_i|}$, we see that $C_{\alpha_i\beta} \in \mathfrak{L}\big(L(b_0,\dots,b_{n-1})\big)$.
- If $i = n$, using $S(F_n,F_{n-1}) = x_2 x_0^{b_0-1} F_n - x_1^{b_0} F_{n-1}$, we see that $C_{\alpha_n\beta} \in \mathfrak{L}\big(L(b_0,\dots,b_{n-1})\big)$ (note that by the previous item $C_{\alpha_{n-1}\beta'} \in \mathfrak{L}(L(b_0,\dots,b_{n-1})))$.

Step 3: If $L(0,a_1,\dots,a_{n-1}) \simeq \mathbb{A}^{K_1}$ then $L(0,a_1,\dots,a_d) \simeq \mathbb{A}^{K_2}$ where d is the maximal index $< n$ such that $a_d \neq 0$ (the degree of the Hilbert polynomial) and $K_2 = K_1 + (n-d)(d+1) + \binom{b_{n-1}+n}{n} - 1$ (or $K_2 = K_1 + \binom{b_{n-1}+n}{n} - 1$ if d does not exist).
Here we compare the ideal $L(0,a_1,\dots,a_{n-1})$ in $\mathbb{K}[x]$ and the ideal $L(0,a_1,\dots,a_d)$ in $\mathbb{K}[x_0,\dots,x_d]$. Observe that both $L(0,a_1,\dots,a_{n-1})^{\mathrm{sat}}$ and $L(0,a_1,\dots,a_d)^{\mathrm{sat}}$ fulfill the hypothesis of Theorem 4.12*(vi)* (see also Example 4.1.5); then it holds

$$\mathcal{S}t_{\mathtt{DegLex}}\big(L(b_0,\dots,b_{n-1})\big) \simeq \mathcal{S}t_{\mathtt{DegLex}}\left(L(0,a_1,\dots,a_{n-1})^{\mathrm{sat}}\right),$$
$$\mathcal{S}t_{\mathtt{DegLex}}\big(L(0,a_1,\dots,a_d)\big) \simeq \mathcal{S}t_{\mathtt{DegLex}}\left(L(0,a_1,\dots,a_d)^{\mathrm{sat}}\right).$$

The statement for the saturated ideals $L(0,a_1,\dots,a_{n-1})^{\mathrm{sat}}$ and $L(0,a_1,\dots,a_d)^{\mathrm{sat}}$ is proved using the same technique as above in [90, Corollary 5.5]. $\square$

I think that the main advantageous aspects of this approach for a local study of the Hilbert schemes are:

(+1) the possibility of exploiting Gröbner bases tools. Through both theoretical improvements of algorithms and the constant development of computers, this theory provides very efficient methods to study many non-trivial cases;

(+2) Theorem 4.12 allows to reduce the number of parameters required for the description of open subsets of $\mathbf{Hilb}^n_{p(t)}$, that in general is very large, being the Hilbert scheme embedded in a suitable Grassmannian.

On the other hand, there are also some critical aspects:

(−1) the ideals for which the Gröbner stratum is an open subset of $\mathbf{Hilb}^n_{p(t)}$ are hilb-segment, i.e. Borel-fixed, but as seen in Section 2.7 not every Borel-fixed ideal is a hilb-segment ideal, hence there could be components of $\mathbf{Hilb}^n_{p(t)}$ not containing hilb-segment ideals, i.e. components that can not be studied by means of Gröbner strata;

(−2) by Lemma 4.14 we know that considering "all" the monomial ideals with a fixed Hilbert polynomial $p(t)$ and the associated Gröbner strata we can cover set-theoretically the Hilbert scheme $\mathbf{Hilb}^n_{p(t)}$. Two questions that arise immediately are 1. what about the scheme structure? 2. is there a smaller set of monomial ideals sufficient to cover all the Hilbert scheme?

(−3) Theorem 4.12 says also that many of the parameters considered in the costruction of the Gröbner stratum are eliminable, so from an algorithmic point of view it would be preferable to avoid to introduce them.

Thus the first goal has been to generalize the construction of Gröbner strata to a generic Borel-fixed ideal. The main problem is that avoiding the use of a term ordering we give up some of the basic tools we used, primarily the Buchberger's algorithm and the associated noetherian reduction of polynomials. In next section we introduce a new noetherian reduction procedure which does not use any term ordering.

4.3 Cioffi and Roggero's results

In this section we recall the main results exposed in the paper by F. Cioffi and M. Roggero "Flat families by Borel-fixed ideals and a generalization of Gröbner bases" [22] adapting the notation to that used so far.

From now on, for any monomial ideal $J \subset \mathbb{K}[x]$ we will denote by G_J the set of minimal generators of J and by $\mathcal{N}(J)$ its *sous-escalier*, that it the set of monomials not belonging to J.

Lemma 4.24. *Let J be a Borel-fixed ideal in $\mathbb{K}[x]$. Then:*

(i) $x^\alpha \in J \setminus G_J \;\Rightarrow\; \dfrac{x^\alpha}{\min x^\alpha} \in J$*;*

(ii) $x^\beta \notin J$ *and* $x_i x^\beta \in J \;\Rightarrow$ *either* $x_i x^\beta \in G_J$ *or* $x_i > \min x^\beta$*.*

Proof. Both properties follow from the combinatorial characterization of Borel-fixed ideals. □

For any monomial $x^\alpha \in \mathbb{K}[x]$, we will denote by $x^{\underline{\alpha}}$ the monomial obtained from x^α with the substitution $x_0 = 1$, i.e. $\alpha = (\alpha_0, \ldots, \alpha_n) \to \overline{\alpha} = (0, \alpha_1, \ldots, \alpha_n)$. Analogously for any monomial ideal $J \subset \mathbb{K}[x]$, $\underline{J}$ will be the ideal generated by $\{x^{\underline{\alpha}} \mid x^\alpha \in G_J\}$. Note that if J is a Borel-fixed ideal $\underline{J}$ coincides with J^{sat}.

Definition 4.25. For any non-zero homogeneous polynomial $f \in \mathbb{K}[x]$, the *support* of f is the set $\operatorname{Supp} f$ of monomials that appear in f with a non-zero coefficient.

Definition 4.26 ([88])**.** A *marked polynomial* is a polynomial $f \in \mathbb{K}[x]$ together with a specified monomial of its support $\operatorname{Supp} f$ that will be called *head term* of f and denoted by $\operatorname{Ht}(f)$.

Remark 4.3.1. Although we mainly use the word "monomial", we say "head term" for coherency with the notation introduced in [88]. Anyway, there will be no possible ambiguity on the meaning of "head term of f", because we will always use marked polynomials f such that the coefficient of $\operatorname{Ht}(f)$ in f is 1.

Definition 4.27 ([22])**.** Given a monomial ideal J and an ideal I, a *J-reduced form modulo I* of a polynomial h is a polynomial h_0 such that $h - h_0 \in I$ and $\operatorname{Supp} h_0 \subseteq \mathcal{N}(J)$. A polynomial is *$J$-reduced* if its support is contained in $\mathcal{N}(J)$. If there is a unique J-reduced form modulo I of h, we call it *J-normal form modulo I* and denote it by $\mathrm{Nf}_J(h)$.

Note that every polynomial h has a unique J-reduced form modulo an ideal I if and only if $\mathcal{N}(J)$ is a $\mathbb{K}$-basis for the quotient $\mathbb{K}[x]/I$ or, equivalently, $\mathbb{K}[x] = I \oplus \langle \mathcal{N}(J) \rangle$ as a $\mathbb{K}$-vector space. If moreover I is homogeneous, the J-reduced form modulo I of a homogeneous polynomial is supposed to be homogeneous too. These facts motivate the following definitions.

Definition 4.28. A finite set G of homogeneous marked polynomials $f_\alpha = x^\alpha - \sum c_{\alpha\beta} x^\beta$, with $\mathrm{Ht}(f_\alpha) = x^\alpha$, is called *$J$-marked set* if the head terms $\mathrm{Ht}(f_\alpha)$ are pairwise different, form the monomial basis G_J of a monomial ideal J and every x^β belongs to $\mathcal{N}(J)$, i.e. $|\operatorname{Supp} f \cap J| = 1$. We call the polynomial $\mathrm{Ht}(f_\alpha) - f_\alpha$ *tail* of f_α and we denote it by $\mathcal{T}(f_\alpha)$, so that $\operatorname{Supp} \mathcal{T}(f_\alpha) \subseteq \mathcal{N}(J)$. A J-marked set G is a *J-marked basis* if $\mathcal{N}(J)$ is a basis of $\mathbb{K}[x]/(G)$ as a $\mathbb{K}$-vector space.

Definition 4.29. The family of all homogeneous ideals I such that $\mathcal{N}(J)$ is a basis of the quotient $\mathbb{K}[x]/I$ as a $\mathbb{K}$-vector space will be denoted by $\mathcal{M}f(J)$ and called *J-marked family*. If J is a Borel-fixed ideal, then $\mathcal{M}f(J)$ can be endowed with a natural structure of scheme (see [22, Section 4]) that we call *J-marked scheme*.

Remark 4.3.2. (i) The ideal (G) generated by a J-marked basis G has the same Hilbert function of J, hence $\dim_{\mathbb{K}} J_t = \dim_{\mathbb{K}} (G)_t$, by the definition of J-marked basis itself. Moreover, note that a J-marked basis is unique for the ideal that it generates, by the uniqueness of the J-normal forms modulo I of the monomials in G_J.

(ii) $\mathcal{M}f(J)$ contains every homogeneous ideal having J as initial ideal w.r.t. some term order, but it can also contain other ideals (see [22, Example 3.18]).

(iii) When J is a Borel-fixed ideal, every homogeneous polynomial has a J-reduced form modulo any ideal generated by a J-marked set G ([22, Theorem 2.2]).

Proposition 4.30. *Let J be a Borel-fixed ideal, I be a homogeneous ideal generated by a J-marked set G. The following facts are equivalent:*

(i) $I \in \mathcal{M}f(J)$

(ii) *G is a J-marked basis;*

(iii) $\dim_{\mathbb{K}} I_t = \dim_{\mathbb{K}} J_t$, *for every integer t;*

(iv) *if $h \in I$ and h is J-reduced, then $h = 0$.*

Proof. For the equivalence among the first three statements, see [22, Corollaries 2.3, 2.4, 2.5]. For the equivalence among *(i)* and *(iv)*, observe that if $I \in \mathcal{M}f(J)$, then every polynomial has a unique J-reduced form; so, the J-reduced form of a polynomial of I must to be null. Vice versa, it is enough to show that every polynomial f has a unique J-reduced form. Let $\overline{f}$ and $\widetilde{f}$ be two J-reduced form of f. Then, $\overline{f} - \widetilde{f}$ is a J-reduced polynomial of I because $f - \overline{f}$ and $f - \widetilde{f}$ belong to I by definition. We have done, because $\overline{f} - \widetilde{f}$ is null by the hypothesis. $\square$

Thinking for a moment about Hilbert schemes, we want to observe that two different ideals I_1 and $\mathfrak{b}$ of the same J-marked scheme $\mathcal{M}f(J)$ give rise to different projective schemes of the same Hilbert scheme $\mathbf{Hilb}^n_{p(t)}$. Indeed, by the uniqueness of the reduced form, there is a monomial $x^\alpha \in G_J$ such that the corresponding polynomials f^1_α and f^2_α of the J-marked bases of I_1 and I_2, respectively, are different and moreover such that $f^1_\alpha \notin I_2$, $f^2_\alpha \notin I_1$. If I_1 and I_2 defined the same projective scheme, we would have $(I_1)_r = (I_2)_r$ for some $r \gg 0$. Hence $x_0^{r-m} f^1_\alpha \in I_2$ with normal form modulo I_2 given by $x_0^{r-m} f^1_\alpha - x_0^{r-m} f^2_\alpha$, that is impossible because of Proposition 4.30*(iv)*.

Now we come back to deal with Borel-fixed ideals, exposing special properties of J-marked families in this case. So from now on J will be always considered Borel-fixed and G will be a J-marked set.

Definition 4.31. If m_J is the initial degree of J, we set $V^J_{m_J} = G_{m_J}$; so, for every term $x^\alpha \in G_J$ of degree m_J, there is a unique polynomial $g_\alpha \in V^J_{m_J}$ such that $\mathrm{Ht}(g_\alpha) = x^\alpha$. For every $m > m_J$ and for every $x^\alpha \in J_m \setminus G_J$, we set $g_\alpha = x_i g_\epsilon$, where $x_i = \min x^\alpha$

and g_ϵ is the unique polynomial of V^J_{s-1} with head term $x^\epsilon = \frac{x^\alpha}{\min x^\alpha}$, and we let $V^J_s = G_m \cup \{g_\alpha \text{ s.t. } x^\alpha \in J_m \setminus G_J\}$.

In the following we let $V^J = \cup_s V^J_s$. Moreover, $\langle V^J \rangle$ denotes the vector space generated by the polynomials in V^J and $\xrightarrow{V^J_s}$ is the reduction relation defined in the usual sense of Gröbner basis theory.

Remark 4.3.3. An ideal I belongs to $\mathcal{M}f(J)$ if and only if $I = \langle V^J \rangle$ as a vector space. Indeed, for every integer m, the number of elements in V^J_m is equal to the number of monomials in J_m.

Example 4.3.4. Let us consider the Borel-fixed ideal $J = (x_2^2, x_2x_1, x_1^3) \subset \mathbb{K}[x_0, x_1, x_2]$ and the J-marked set

$$\begin{aligned} f_{x_2^2} &= x_2^2 + 2x_1^2 - \frac{1}{3}x_1x_0, \\ f_{x_2x_1} &= x_2x_1 - x_0^2, \\ f_{x_1^3} &= x_1^3 + \frac{1}{5}x_1^2x_0 - x_1x_0^2 - 3x_0^3. \end{aligned}$$

The initial degree of J is 2, so we have:

$(m = 2)$ $J_2 = \langle x_2^2, x_2x_1 \rangle$

$$V^J_2 = \left\{ g_{x_2^2} = f_{x_2^2}, g_{x_2x_1} = f_{x_2x_1} \right\};$$

$(m = 3)$ $J_3 = \langle x_2^3, x_2^2x_1, x_2x_1^2, x_1^3, x_2^2x_0, x_2x_1x_0 \rangle$

$$\begin{aligned} V^J_3 = \{ & g_{x_2^3} = x_2 g_{x_2^2}, g_{x_2^2x_1} = x_1 g_{x_2^2}, g_{x_2x_1^2} = x_1 g_{x_2x_1}, \\ & g_{x_2^2x_0} = x_0 g_{x_2^2}, g_{x_2x_1x_0} = x_0 g_{x_2x_1} \} \cup \left\{ g_{x_1^3} = f_{x_1^3} \right\}; \end{aligned}$$

$(m = 4)$ $J_4 = \langle x_2^4, x_2^3x_1, x_2^2x_1^2, x_2x_1^3, x_1^4, x_2^3x_0, x_2^2x_1x_0, x_2x_1^2x_0, x_1^3x_0, x_2^2x_0^2, x_2x_1x_0^2 \rangle$

$$\begin{aligned} V^J_4 = \{ & g_{x_2^4} = x_2 g_{x_2^3} = x_2^2 f_{x_2^2}, g_{x_2^3x_1} = x_1 g_{x_2^3} = x_2x_1 f_{x_2^2}, g_{x_2^2x_1^2} = x_1 g_{x_2^2x_1} = x_1^2 f_{x_2^2}, \\ & g_{x_2x_1^3} = x_1 g_{x_2x_1^2} = x_1^2 f_{x_2x_1}, g_{x_1^4} = x_1 g_{x_1^3} = x_1 f_{x_1^3}, \\ & g_{x_2^3x_0} = x_0 g_{x_2^3} = x_2x_0 f_{x_2^2}, g_{x_2^2x_1x_0} = x_0 g_{x_2^2x_1} = x_1x_0 f_{x_2^2}, \\ & g_{x_2x_1^2x_0} = x_0 g_{x_2x_1^2} = x_1x_0 f_{x_2x_1}, g_{x_1^3x_0} = x_0 g_{x_1^3} = x_0 f_{x_1^3}, \\ & g_{x_2^2x_0^2} = x_0 g_{x_2^2x_0} = x_0^2 f_{x_2^2}, g_{x_2x_1x_0^2} = x_0 g_{x_2x_1x_0} = x_0^2 f_{x_2x_1} \}. \end{aligned}$$

Keeping in mind the canonical decomposition and the decomposition map of a Borel-fixed ideal introduced in Definition 2.18, we prove the following lemma.

Lemma 4.32. *Let J be a Borel-fixed ideal. If x^ϵ belongs to $\mathcal{N}(J)$ and $x^\epsilon \cdot x^\delta = x^{\epsilon+\delta}$ belongs to J for some x^δ, then $x^{\epsilon+\delta} = \langle x^\alpha | x^\eta \rangle^J$ with $x^\eta <_{\mathtt{Lex}} x^\delta$. Furthermore:*

(i) if $|\delta| = |\eta|$, then $x^\eta <_B x^\delta$; and

(ii) $x^{\underline{\eta}} <_{\mathtt{Lex}} x^{\underline{\delta}}$.

Proof. We can assume that x^δ and x^η are coprime; indeed, if this is not the case, we can divide the involved equalities of monomials by $\gcd(x^\delta, x^\eta)$. If $x^\eta = 1$, all the statements are obvious. If $x^\eta \neq 1$, then $\min x^\delta \mid x^\alpha$ because x^δ and x^η are coprime, hence $\min x^\delta \geq \min x^\alpha \geq \max x^\eta$ and so $\min x^\delta > \max x^\eta$ because they cannot coincide. This inequality implies both $x^\eta <_{\mathtt{Lex}} x^\delta$ and $x^{\underline{\eta}} <_{\mathtt{Lex}} x^{\underline{\delta}}$. Moreover, if $|\delta| = |\eta|$, this is also sufficient to conclude that $x^\eta <_B x^\delta$. $\square$

Remark 4.3.5. Observe that if $g_\beta = x^\delta f_\alpha$ belongs to V_s^J, then $x^\beta = \langle x^\alpha | x^\delta \rangle^J$.

We have already recalled that, when J is a Borel-fixed ideal, every homogeneous polynomial has a J-reduced form modulo an ideal generated by a J-marked set G (Remark 4.3.2iii). Further, a J-reduced form of a homogeneous polynomial can be constructed by a suitable reduction relation, as it is recalled by next Proposition.

Proposition 4.33 ([22, Proposition 3.6])**.** *With the above notation, every monomial $x^\beta \in J_m$ can be reduced to a J-reduced form modulo (G) in a finite number of reduction steps, using only polynomials of V_s^J. Hence, the reduction relation $\xrightarrow{V_s^J}$ is Noetherian.*

The Noetherianity of the reduction relation $\xrightarrow{V_s^J}$ provides an algorithm that reduces every homogeneous polynomial of degree m to a J-reduced form modulo (G) in a finite number of steps. We note that on one hand it is convenient to substitute the polynomials in V_s^J by their J-reduced normal forms for an efficient implementation of a reduction algorithm, but on the other hand it is convenient to use in the proofs the polynomials of V_s^J as constructed in Definition 4.31.

Using the Noetherianity of the reduction relation $\xrightarrow{V_s^J}$, we can recognize when a J-marked set is a J-marked basis by a Buchberger-like criterion [22, Theorem 3.12].

To this aim we need to pose an order on the set $W^J_m = \{x^\delta f_\alpha \text{ s.t. } f_\alpha \in G \text{ and } |\delta+\alpha| = m\}$ that becomes a set of marked polynomials by letting $\mathrm{Ht}(x^\delta f_\alpha) = x^{\delta+\alpha}$. Note that I_m is generated by W^J_m as a vector space. We set $W^J = \cup_m W^J_m$.

Remark 4.3.6. We point out that Definition 3.9 of [22] does not work well for our purpose. Hence, we introduce the following Definition 4.34 and provide a new proof of [22, Lemma 3.10] by the following Lemma 4.35.

Definition 4.34. Let $\leq$ be any order on G and $x^\delta f_\alpha$, $x^{\delta'} f_{\alpha'}$ be two elements of W^J_m. We set

$$x^\delta f_\alpha \geq_m x^{\delta'} f_{\alpha'} \Leftrightarrow x^\delta >_{\mathtt{Lex}} x^{\delta'} \text{ or } x^\delta = x^{\delta'} \text{ and } f_\alpha \geq f_{\alpha'}. \tag{4.13}$$

Lemma 4.35. *(i) For every two elements $x^\delta f_\alpha$, $x^{\delta'} f_{\alpha'}$ of W^J_m we get*

$$x^\delta f_\alpha \geq_m x^{\delta'} f_{\alpha'} \Rightarrow \forall x^\eta : \ x^{\delta+\eta} f_\alpha \geq_{m'} x^{\delta'+\eta} f_{\alpha'},$$

where $m' = |\delta+\eta+\alpha|$.

(ii) Every polynomial $g_\beta \in V^J_s$ is the minimum w.r.t. $\leq_m$ of the subset W^J_β of W^J_m containing all polynomials of W^J_m with x^β as head term.

(iii) If $x^\delta f_\alpha$ belongs to $W^J_m \setminus G_m$ and x^β belongs to $\mathrm{Supp}\, x^\delta f_\alpha \setminus \{x^\delta x^\alpha\}$ *with $g_\beta \in V^J_s$, then $x^\delta f_\alpha \succ_m g_\beta$.*

Proof. (i) This follows by the analogous property of the term order $>_{\mathtt{Lex}}$.

(ii) Let $g_\beta = x^{\delta'} f_{\alpha'}$ be the polynomial of V^J such that $x^\beta = \langle x^{\alpha'} | x^{\delta'} \rangle^J$ and $x^\delta f_\alpha$ be another polynomial of W^J_β. We can assume that x^δ and $x^{\delta'}$ are coprime; otherwise, we can divide the involved inequalities of monomials by $\gcd(x^\delta, x^{\delta'})$. By Remark 4.3.5, we have that $\max x^{\delta'} \leq \min x^{\alpha'}$ and $\max x^\delta > \min x^\alpha$. Then, we get $\max x^\delta > \max x^{\delta'}$ because $x^{\alpha'} \nmid x^\alpha$ and $x^\alpha \nmid x^{\alpha'}$. Thus, $x^\delta >_{\mathtt{Lex}} x^{\delta'}$.

(iii) If x^β belongs to G_J we are done. Otherwise, let $x^\beta = \langle x^{\alpha'} | x^{\delta'} \rangle^J$ and note that every term of $\mathrm{Supp}\, x^\delta f_\alpha$ is a multiple of x^δ, in particular $x^\beta = x^{\delta+\gamma}$ for some $x^\gamma \in \mathcal{N}(J)$. By Lemma 4.32, we get $x^{\delta'} <_{\mathtt{Lex}} x^\delta$. □

Definition 4.36. The *S-polynomial* of two elements f_α, $f_{\alpha'}$ of a J-marked set G is the polynomial $S(f_\alpha, f_{\alpha'}) = x^\gamma f_\alpha - x^{\gamma'} f_{\alpha'}$, where $x^{\gamma+\alpha} = x^{\gamma'+\alpha'} = \mathrm{lcm}(x^\alpha, x^{\alpha'})$.

Theorem 4.37 (Buchberger-like criterion)**.** *Let J be a Borel-fixed ideal and I the homogeneous ideal generated by a J-marked set G. With the above notation, the following statements are equivalent:*

(i) $I \in \mathcal{M}f(J)$*;*

(ii) $\forall f_\alpha, f_{\alpha'} \in G$*,* $S(f_\alpha, f_{\alpha'}) \xrightarrow{V_s^J} 0$*;*

(iii) $\forall f_\alpha, f_{\alpha'} \in G$*,* $S(f_\alpha, f_{\alpha'}) = x^\gamma f_\alpha - x^{\gamma'} f_{\alpha'} = \sum a_j x^{\eta_j} f_{\alpha_j}$*, with* $x^{\eta_j} <_{\texttt{Lex}} \max_{\texttt{Lex}}\{x^\gamma, x^{\gamma'}\}$ *and* $f_{\alpha_j} \in G$*.*

Proof. (i) $\Rightarrow$ *(ii)* Recall that $I \in \mathcal{M}f(J)$ if and only if G is a J-marked basis, so that every polynomial has a unique J-normal form modulo I. Since $S(f_\alpha, f_{\alpha'})$ belongs to I by construction, its J-normal form modulo I is null.

(ii) $\Rightarrow$ *(iii)* Straightforward by the definition of the reduction relation $\xrightarrow{V_s^J}$ and by Lemma 4.35*(iii)*.

(iii) $\Rightarrow$ *(i)* We want to prove that $I = \langle V^J \rangle$ or, equivalently, that $\langle V^J \rangle = \langle W^J \rangle$. It is sufficient to prove that $x^\eta \cdot V^J \subseteq \langle V^J \rangle$, for every monomial x^η. We proceed by induction on the monomials x^η, ordered according to $\texttt{Lex}$. The thesis is obviously true for $x^\eta = 1$. We then assume that the thesis holds for any monomial $x^{\eta'}$ such that $x^{\eta'} <_{\texttt{Lex}} x^\eta$.

If $|\eta| > 1$, we can consider any product $x^\eta = x^{\eta_1} \cdot x^{\eta_2}$, x^{η_1} and x^{η_2} non-constant. Since $x^{\eta_i} <_{\texttt{Lex}} x^\eta, i = 1,2$, we immediately obtain by induction

$$x^\eta \cdot V^J = x^{\eta_1} \cdot (x^{\eta_2} \cdot V^J) \subseteq x^{\eta_1} \langle V^J \rangle \subseteq \langle V^J \rangle.$$

If $|\eta| = 1$, then we need to prove that $x_i \cdot V^J \subseteq \langle V^J \rangle$. Since $x_0 V^J \subseteq V^J$, it is then sufficient to prove the thesis for $x^\eta = x_i$, assuming that the thesis holds for every $x^{\eta'} <_{\texttt{Lex}} x_i$. We consider $g_\beta = x^\delta f_\alpha \in V^J$, where $\max x^\delta \leq \min x^\alpha$. If $x_i g_\beta$ does not belong to V^J, then $\max(x_i \cdot x^\delta) > \min x^\alpha$, so $x_i > \min x^\alpha$. In particular, $x_i > \min x^\alpha \geq \max x^\delta$, so $x_i >_{\texttt{Lex}} x^\delta$: by induction, it is now sufficient to prove the thesis for $x_i f_\alpha$.

We consider an S-polynomial $S(f_\alpha, f_{\alpha'}) = x_i f_\alpha - x^\gamma f_{\alpha'}$ such that $x^\gamma <_{\texttt{Lex}} x_i$. Such S-polynomial always exists: for instance, we can consider $x_i x^\alpha = \langle x^{\alpha'} | x^{\eta'} \rangle^J$. By the hypothesis $x_i f_\alpha - x^{\eta'} f_{\alpha'} = \sum a_j x^{\eta' j} f_{\alpha_j}$ where $x^{\eta'} f_{\alpha'}, x^{\eta' j} f_{\alpha_j} \in \langle V^J \rangle$ by induction since $x^{\eta'}, x^{\eta' j}$ are lower than x_i w.r.t. `Lex` and then $x_i f_\alpha$ belongs to $\langle V^J \rangle$. □

Remark 4.3.7. In [22, Section 3] some results about syzygies of the ideal I generated by a J-marked basis are proposed, by using the order on W^J_m defined in Definition 3.9 of [22] that does not work well. Anyway, the order defined in Definition 4.34 works well also in that context of syzygies.

Definition 4.38. We call *Eliahou-Kervaire couple* of the J-marked set G any couple of polynomials f_α, f_β, $\mathrm{Ht}(f_\alpha) = x^\alpha$, $\mathrm{Ht}(f_\beta) = x^\beta$, such that

$$x_j x^\alpha = \langle x^\beta | x^\eta \rangle^J \text{ for some } x_j > \min x^\alpha.$$

We call *Eliahou-Kervaire S-polynomial* (EK-polynomial, for short) of G an S-polynomial among an Eliahou-Kervaire couple of polynomials f_α and f_β. We denote such S-polynomial by $S^{\mathrm{EK}}(f_\alpha, f_\beta)$. Observe that, thanks to the definition, an EK-polynomial is of kind

$$S^{\mathrm{EK}}(f_\alpha, f_\beta) = x_j f_\alpha - x^\eta f_\beta, \text{ for some } x_j > \min x^\alpha, \text{ with } x_j x^\alpha = \langle x^\beta | x^\eta \rangle^J.$$

Remark 4.3.8. We underline that in the proof of Theorem 4.37 the crucial point is the existence of an S-polynomial of kind $x_i f_\alpha - x^\eta f_\alpha$ with $x^\eta <_{\mathtt{Lex}} x_i$, and we use an EK-polynomial. An analogous argument will be used in the proof of Theorem 4.47 and of Theorem 4.49.

As pointed out in Remark 4.3.8, in the proof of Theorem 4.37 we just need to assume that *(iii)* holds for EK-polynomials. Hence, we have the following result.

Corollary 4.39. *With the same notation of Theorem 4.37,*

$$I \in \mathcal{M}f(J) \Leftrightarrow \text{ for every EK-polynomial, } S^{EK}(f_\alpha, f_\beta) \xrightarrow{V^J_s} 0.$$

4.4 Superminimal generators and a new Noetherian reduction

Form this section onwards, we consider a Borel-fixed ideal J obtained as truncation in degree m of a saturated ideal, that is $J = \underline{J}_{\geqslant m}$ for some integer m. In this case we have $(G_J)_{>m} \subset G_{\underline{J}}$.

Definition 4.40. The *set of superminimal generators* of J is

$$sG_J = \{x^\alpha \in G_J \mid x^{\underline{\alpha}} \in G_{\underline{J}}\}.$$

Hence, for every $x^\alpha \in sG_J$, we have an integer $t_\alpha = \alpha_0$ such that $x^\alpha = x_0^{t_\alpha} x^{\underline{\alpha}}$; more precisely,

$$t_\alpha = \begin{cases} 0, & \text{if } |\underline{\alpha}| \geq m \\ m - |\underline{\alpha}|, & \text{otherwise} \end{cases}.$$

Given a J-marked set G, the set sG of *superminimal generators* of G is

$$sG = \{f_\alpha = x^\alpha - \mathcal{T}(f_\alpha) \in G \,| x^\alpha \in sG_J\}.$$

Definition 4.41. Given a J-marked set G and two polynomials h and h_1, we say that h is in *sG-relation* with h_1 if there is a monomial $x^\gamma \in \operatorname{Supp} h \cap J$, $\operatorname{Coeff}(x^\gamma) = c$, such that x^γ is divisible by a superminimal generator x^α of J with $x^\gamma = x^\alpha \cdot x^\epsilon = \langle x^{\underline{\alpha}} | x^\eta \rangle^{\underline{J}}$ and $h_1 = h - c \cdot x^\epsilon f_\alpha$, that is h_1 is obtained by replacing in h the monomial x^γ by $x^\epsilon \cdot \mathcal{T}(f_\alpha)$. We call *superminimal reduction* the transitive closure of the above relation and denote it by $\xrightarrow{sG}$. Moreover, we say that:

- *h can be reduced to h_1* by $\xrightarrow{sG}$ if $h \xrightarrow{sG} h_1$;
- *h is reduced w.r.t. sG* if no monomial in $\operatorname{Supp} h$ is divisible by a monomial of sG_J;
- *h is strongly reduced* if no monomial in $\operatorname{Supp} h$ is divisible by a monomial of $G_{\underline{J}}$, that is h is $\underline{J}$-reduced. In other words, h is strongly reduced if for every t, $x_0^t \cdot h$ is reduced w.r.t. sG.

Remark 4.4.1. Given a polynomial f_α of a J-marked set G and any positive integer t, then $\operatorname{Supp}(x_0^t \cdot \mathcal{T}(f_\alpha)) \subseteq \mathcal{N}(J)$. Furthermore, if G is a J-marked basis, then we also have $x_0^t \cdot \mathcal{T}(f_\alpha) = \operatorname{Nf}(x_0^t \cdot x^\alpha)$ since in this case the tail of f_α is indeed the J-normal form modulo (G) of x^α (see Remark 4.3.2i). More generally, if h is a homogeneous polynomial of degree $\deg(h) \geqslant m$, then $\operatorname{Nf}(x_0^t \cdot h) = x_0^t \cdot \operatorname{Nf}(h)$.

It is interesting to notice that the subset sG of Definition 4.40 is a subset of V, but not every step of reduction by $\xrightarrow{sG}$ is also a step of reduction by $\xrightarrow{V^J_s}$, as shown in the following example.

Example 4.4.2. Consider the Borel-fixed ideal

$$J = \underline{J}_{\geqslant 2} = (x_2, x_1^2)_{\geqslant 2} = (x_2^2, x_2x_1, x_1^2, x_2x_0)$$

and let G be a J-marked set. Consider the monomial $x_2^2x_0$. The only way to reduce $x_2^2x_0$ via $\xrightarrow{V^J_3}$ leads to $x_0 \cdot \mathcal{T}(g_{x_2^2})$, where $g_{x_2^2} = f_{x_2^2} \in V^J_2$. Moreover, $x_0 \cdot \mathcal{T}(g_{x_2^2})$ is not further reducible, because all the monomials of its support belong to $\mathcal{N}(J)$. On the other hand, according to Definition 4.41, a first step of reduction of the monomial $x_2^2x_0$ via $\xrightarrow{sG}$ is $x_2^2x_0 \xrightarrow{sG} x_2 \cdot \mathcal{T}(f_{x_2x_0})$, where f is the polynomial in sG with $\mathrm{Ht}(f) = x_2x_0$. Since x_2 is a monomial of $G_{\underline{J}}$, every monomial appearing in $\mathrm{Supp}\left(x_2 \cdot \mathcal{T}(f_{x_2x_0})\right)$ belongs to $J = \underline{J}_{\geqslant 2}$, and so we will need further steps of reduction via $\xrightarrow{sG}$ to compute a polynomial reduced w.r.t. sG.

Theorem 4.42. *With the above notation:*

(i) $\xrightarrow{sG}$ *is Noetherian;*

(ii) for every homogeneous polynomial h there exist $\bar{t} = \bar{t}(h)$ and a polynomial $\overline{h}$ strongly reduced such that $x_0^{\bar{t}} \cdot h \xrightarrow{sG} \overline{h}$;

(iii) If moreover G is a J-marked basis and $\deg h \geqslant m$*, then* $\overline{h} = \mathrm{Nf}(x_0^{\bar{t}} \cdot h) = x_0^{\bar{t}} \cdot \mathrm{Nf}(h)$ *where* $\mathrm{Nf}(h)$ *is the unique J-normal form modulo (G) of h.*

Proof. (i) If $\xrightarrow{sG}$ was not Noetherian, by Lemma 4.32 and Lemma 4.35, we would be able to find infinite descending chains of monomials w.r.t. $<_{\mathtt{Lex}}$.

(ii) It is sufficient to prove the thesis for monomials x^γ in $\underline{J}$. Let $x^\gamma = \langle x^{\underline{\alpha}} | x^\eta \rangle^{\underline{J}}$. If $x^\eta = 1$, then $x^\alpha = x_0^{t_\alpha} \cdot x^{\underline{\alpha}}$ is in sG_J, f_α belongs to sG and $x_0^{t_\alpha} \cdot x^{\underline{\alpha}} \xrightarrow{sG} \mathcal{T}(f_\alpha)$, where $\mathrm{Supp}\,\mathcal{T}(f_\alpha) \subseteq \mathcal{N}(J)$. In this case $\overline{h} = \mathcal{T}(f_\alpha)$ and $\bar{t} = t_\alpha$.

If $x^\eta \neq 1$, we can assume that the thesis holds for any monomial $x^{\gamma'} = \langle x^{\underline{\beta}} | x^{\eta'} \rangle^{\underline{J}}$, such that $x^{\eta'} <_{\mathtt{Lex}} x^\eta$. We perform a first reduction $x_0^{t_\alpha} \cdot x^\gamma \xrightarrow{sG} x^\eta \cdot \mathcal{T}(f_\alpha)$. If $x^\eta \cdot \mathcal{T}(f_\alpha)$ is strongly reduced, we are done. Otherwise, we have $x^\eta \neq x_0^{|\eta|}$. For every monomial $x^{\gamma'} \in \operatorname{Supp}\left(x^\eta \cdot \mathcal{T}(f_\alpha)\right) \cap \underline{J}$ we have $x^{\gamma'} = \langle x^{\underline{\beta}'} | x^{\eta'} \rangle^{\underline{J}}$, with $x^{\eta'} <_{\texttt{Lex}} x^\eta$ by Lemma 4.32. So, we have also $x_0^t \cdot x^{\eta'} <_{\texttt{Lex}} x^\eta$, for every t. By the inductive hypothesis we can find a suitable power t of x_0 such that every monomial in $x_0^t \cdot x^\eta \cdot \mathcal{T}(f_\alpha)$ can be reduced by $\xrightarrow{sG}$ to a strongly reduced polynomial. Thus $x_0^t \cdot x^\eta \cdot \mathcal{T}(f_\alpha) \xrightarrow{sG} \overline{h}$ with $\overline{h}$ strongly reduced. In this case $\overline{t}(x^\gamma) = t_\alpha + t = t_\alpha + \overline{t}\left(x^\eta \cdot \mathcal{T}(f_\alpha)\right)$.

(ii) Since $\deg h \geqslant m$, we observe that $x_0^{\overline{t}} \cdot h - \overline{h} \in (G)$ and $\operatorname{Supp} \overline{h} \subseteq \mathcal{N}(\underline{J}) \subseteq \mathcal{N}(J)$, hence $\overline{h}$ is a J-reduced form modulo (G) of $x_0^{\overline{t}} \cdot h$. Therefore, if G is a J-marked basis, $\overline{h}$ is the unique J-normal form of $x_0^{\overline{t}} \cdot h$. Moreover, $\overline{h} = \mathrm{Nf}(x_0^{\overline{t}} \cdot h) = x_0^{\overline{t}} \cdot \mathrm{Nf}(h)$ because $\deg h \geqslant m$ (see Remark 4.4.1). □

If we consider the more general setting used in Section 4.3, we are not able to generalize the properties of the reduction $\xrightarrow{V_s^J}$ to $\xrightarrow{sG}$. Indeed, in our proofs we will often need that polynomials $f_\alpha \in G$ have the following property:

$$\forall\, x^\beta \in \operatorname{Supp} \mathcal{T}(f_\alpha),\ x^\beta \in \mathcal{N}(J_0). \tag{4.14}$$

Assume that J is Borel-fixed, with initial degree m_J, that $\underline{J}$ is its saturation and m is such that $J_{\geqslant m} = \underline{J}_{\geqslant m}$. Condition (4.14) is necessary for the properties we will prove (see Example 4.4.3) and it obviously holds for $J = \underline{J}_{\geqslant m_J}$, that is when $m_J = m$. If we work in a more general setting, in which $m > m_J$, then we need to assume that (4.14) holds for the J-marked set we consider; however, in this way, we are able to characterize only a subset of the J-marked scheme.

Theorem 4.42 has some interesting consequences.

Corollary 4.43. *Let I be an ideal generated by a J-marked set G. Then:*

$$I \in \mathcal{M}f(J) \Longleftrightarrow \forall\, h \in I, \exists\, t \text{ s.t. } x_0^t \cdot h \xrightarrow{sG} 0.$$

Proof. Let $h \in I$. If I belongs to $\mathcal{M}f(J)$, then G is a J-marked basis and the equivalent condition of Proposition 4.30*(iv)* holds. Especially $\mathrm{Nf}(h) = 0$. Moreover, by Theorem 4.42*(iii)*, we have $x_0^t \cdot h \xrightarrow{sG} x_0^t \cdot \mathrm{Nf}(h)$ for a suitable t, and we conclude.

Vice versa, we use again Proposition 4.30. For every $h \in I$ such that h is J-reduced modulo I, then h is also strongly reduced w.r.t. sG, that is $x_0^t \cdot h$ is not further reducible through $\xrightarrow{sG}$ for every t and by the hypothesis $x_0^t \cdot h = 0$. Thus $h = 0$ and we conclude. □

Corollary 4.44. *Given a set of marked polynomials* $\Gamma = \{f_\beta \mid \mathrm{Ht}(f_\beta) = x^\beta \in sG_J, \mathrm{Supp}\,(x^\beta - f_\beta) \subset \mathcal{N}(J)\}$*, there is at most one ideal* I *of* $\mathcal{M}f(J)$ *such that* Γ *is the set of superminimals of* I*.*

Proof. Suppose that I is an ideal in $\mathcal{M}f(J)$ such that Γ is its set of superminimals and let G be its J-marked basis. If f_α is a polynomial of G, then $f_\alpha = x^\alpha - \mathrm{Nf}(x^\alpha)$, where $\mathrm{Nf}(x^\alpha)$ is uniquely determined by Γ in the following way: for every $x^\alpha \in G_J \setminus sG_J$, we consider an integer t such that $x_0^t \cdot x^\alpha \xrightarrow{\Gamma} \overline{h}$ with $\overline{h}$ strongly reduced, that exists by Theorem 4.42, and set $\mathrm{Nf}(x^\alpha) = \frac{\overline{h}}{x_0^t}$. Note that if x_0^t does not divide $\overline{h}$, then such an ideal I does not exist. □

Now we will prove that the Buchberger-like criterion introduced for $\xrightarrow{V_s^J}$ can be rephrased in terms of the $\xrightarrow{sG}$ reduction, showing an analogous of Theorem 4.37 for $\xrightarrow{sG}$.

Lemma 4.45. *Let* h *be a homogeneous polynomial of degree* $s \geqslant m$*.*

$$h \in \langle V_s^J \rangle \iff x_0 \cdot h \in \langle V_{s+1}^J \rangle.$$

Proof. If $h \in \langle V_s^J \rangle$, then $x_0 \cdot h \in \langle V_{s+1} \rangle$ by definition of V.

Vice versa, assume that $x_0 \cdot h \in \langle V_{s+1} \rangle$. This is equivalent to $x_0 \cdot h \xrightarrow{V_{s+1}^J} 0$. Every monomial in $\mathrm{Supp}\,(x_0 \cdot h)$ can be written as $x_0 \cdot x^\epsilon$; observe that $x_0 \cdot x^\epsilon \notin sG_J$, because $\deg x_0 \cdot x^\epsilon > m$. Then, if $x_0 \cdot x^\epsilon$ belongs to J, we can decompose it as $x_0 \cdot x^\epsilon = \langle x^\alpha | x^\eta \rangle^J$, $x^\alpha \in G_J$ and $x^\eta \neq 1$. Since $\min x^\alpha \geq \max x^\eta$, we have that x^η is divisible by x_0. So $x^\eta = x_0 \cdot x^{\eta'}$.

Summing up, in order to reduce the monomial $x_0 \cdot x^\epsilon$ of $x_0 \cdot h$ using V^J, we use the polynomial $x_0 \cdot x^{\eta'} \cdot f_\alpha \in V^J$, $\mathrm{Ht}(f_\alpha) = x^\alpha$. If the coefficient of $x_0 \cdot x^\epsilon$ in $x_0 \cdot h$ is a, we obtain

$$x_0 \cdot h \xrightarrow{V_{s+1}^J} x_0 \cdot (h - a \cdot x^{\eta'} f_\alpha).$$

At every step of reduction, we obtain a polynomial which is divisible by x_0. In particular,

$$x_0 \cdot h \in \langle V_{s+1}^J \rangle \Rightarrow x_0 \cdot h = x_0 \cdot \sum a_i x^{\eta_i} f_{\alpha_i}, \text{ where } x_0 \cdot x^{\eta_i} f_{\alpha_i} \in V_{s+1}^J.$$

Then we have that $h = \sum a_i x^{\eta_i} f_{\alpha_i}$ and $x^{\eta_i} f_{\alpha_i} \in V_s^J$, that is $h \in \langle V_s^J \rangle$. □

Consider $f_\alpha, f_{\alpha'} \in G$, the S-polynomial $S(f_\alpha, f_{\alpha'}) = x^\gamma f_\alpha - x^{\gamma'} f_{\alpha'}$ and assume that $x^{\gamma'} <_{\texttt{Lex}} x^\gamma$. By Lemma 4.35*(iii)*, if $S(f_\alpha, f'_\alpha) \xrightarrow{V_s^J} h$, then $S(f_\alpha, f'_\alpha) - h = \sum a_j x^{\delta_j} f_{\beta_j}$ with $x^{\delta_j} f_{\beta_j} \in V_s^J$, $x^{\delta_j} <_{\texttt{Lex}} x^\gamma$. Now we show that a similar result holds for the superminimal reduction $\xrightarrow{sG}$.

Lemma 4.46. *Consider $f_\alpha, f_{\alpha'} \in G$, the S-polynomial $S(f_\alpha, f_{\alpha'}) = x^\gamma f_\alpha - x^{\gamma'} f_{\alpha'}$ and assume that $x^{\gamma'} <_{\texttt{Lex}} x^\gamma$. If $x_0^t \cdot S(f_\alpha, f'_\alpha) \xrightarrow{sG} h$, then $x_0^t \cdot S(f_\alpha, f'_\alpha) - h = \sum a_j x^{\eta_j} f_{\beta_j}$ with $f_{\beta_j} \in sG$, $x^{\eta_j} <_{\texttt{Lex}} x^\gamma$ and $x^{\underline{\eta_j}} <_{\texttt{Lex}} x^{\underline{\gamma}}$.*

Proof. Consider a monomial $x_0^t \cdot x^\gamma \cdot x^\epsilon$ in $\operatorname{Supp}\left(x_0^t \cdot x^\gamma \cdot \mathcal{T}(f_\alpha)\right) \cap J$. Such a monomial decomposes as $x_0^t \cdot x^\gamma \cdot x^\epsilon = \langle x^{\underline{\beta}} | x^\eta \rangle^{\underline{J}}$, $x^\eta <_{\texttt{Lex}} x_0^t x^\gamma$ and $x^{\underline{\eta}} <_{\texttt{Lex}} x^{\underline{\gamma}}$ by Lemma 4.32, because $x^\epsilon \in \mathcal{N}(J)$. The same holds for any further reduction and the same argument applies to monomials appearing in $\operatorname{Supp}\left(x_0^t \cdot x^{\gamma'} \cdot \mathcal{T}(f_{\alpha'})\right)$. □

If we just consider J Borel-fixed, with initial degree m_J, $\overline{J}$ its saturation and m such that $J_{\geqslant m} = \underline{J}_{\geqslant m}$, then Lemma 4.46 is no longer true when $m > m_J$: as we already pointed out we need to have $m_J = m$.

Example 4.4.3. In $\mathbb{K}[x_0, x_1, x_2, x_3]$, consider $\underline{J} = (x_3^2, x_2x_3, x_1x_3, x_2^2)$ and

$$J = (x_3^2) \cdot (x_0, x_1, x_2, x_3)_{\geqslant 2} + (x_3x_2) \cdot (x_0, x_1, x_2, x_3)_{\geqslant 2} + \\ + (x_3x_1) \cdot (x_0, x_1, x_2, x_3)_{\geqslant 2} + (x_2^2) \cdot (x_0, x_1, x_2, x_3)_{\geqslant 4}.$$

The ideal $\underline{J}$ is the saturation of J, but $J \neq \underline{J}_{\geqslant m}$ for any integer m. Consider a J-marked set G and $f_\alpha, f_\beta \in G$ such that $\operatorname{Ht}(f_\alpha) = x_3x_2x_0^2$ and $\operatorname{Ht}(f_\beta) = x_3x_1x_0^2$ and consider $x_2^4 \in \operatorname{Supp}\mathcal{T}(f_\beta)$. Then $S(f_\alpha, f_\beta) = x_1 f_\alpha - x_2 f_\beta$. If we apply Definition 4.41 and Theorem 4.42, we reduce $x_2^4 \in \operatorname{Supp} S(f_\alpha, f_\beta)$ by $\xrightarrow{sG}$, pre-multiplying by x_0^4. We get that $x_2^4x_0^4$ belongs to $\operatorname{Supp} x_0^4 S(f_\alpha, f_\beta)$ and $x_0^4x_2^4 = \langle x_2^2 | x_0^4x_2^2 \rangle^{\underline{J}}$. But $x_2^2x_0^4 >_{\texttt{Lex}} x_2$.

Theorem 4.47. *With the fixed notation, let I be the homogeneous ideal generated by a J-marked set G. The following statements are equivalent:*

(i) $I \in \mathcal{M}f(J)$;

(ii) $\forall\, f_\alpha, f_{\alpha'} \in G$, $\exists\, t$ *s.t.* $x_0^t \cdot S(f_\alpha, f_{\alpha'}) \xrightarrow{sG} 0$;

(iii) $\forall\, f_\alpha, f_{\alpha'} \in G$, $\exists\, t$ *s.t.* $x_0^t \cdot S(f_\alpha, f_{\alpha'}) = x_0^t(x^\gamma f_\alpha - x^{\gamma'} f_{\alpha'}) = \sum a_j x^{\eta_j} f_{\alpha_j}$, *with* $x^{\eta_j} <_{\texttt{Lex}} \max_{\texttt{Lex}}\{x^\gamma, x^{\gamma'}\}$ *and* $f_{\alpha_j} \in sG$.

Proof. *(i)* $\Rightarrow$ *(ii)* If $I \in \mathcal{M}f(J)$, we can apply Corollary 4.43 because any S-polynomial among elements in G belongs to I.

(i) $\Rightarrow$ *(ii)* Straightforward by Lemma 4.46.

(i) $\Rightarrow$ *(ii)* Assuming *(iii)*, we prove that $\langle V^J \rangle = \langle W^J \rangle$ by an argument analogous to that applied in the proof of Theorem 4.37. It is sufficient to prove that $x^\eta \cdot V^J \subseteq \langle V^J \rangle$, for every monomial x^η. We proceed by induction on the monomials x^η, ordered according to $\texttt{Lex}$. The thesis is obviously true for $x^\eta = 1$. We then assume that the thesis holds for any monomial $x^{\eta'}$ such that $x^{\eta'} <_{\texttt{Lex}} x^\eta$.

If $|\eta| > 1$, we can consider any product $x^\eta = x^{\eta_1} \cdot x^{\eta_2}$, x^{η_1} and x^{η_2} non-constant. Since $x^{\eta_i} <_{\texttt{Lex}} x^\eta, i = 1, 2$, we immediately obtain by induction

$$x^\eta \cdot V^J = x^{\eta_1} \cdot \left(x^{\eta_2} \cdot V^J \right) \subseteq x^{\eta_1} \langle V^J \rangle \subseteq \langle V^J \rangle.$$

If $|\eta| = 1$, then we need to prove that $x_i \cdot V^J \subseteq \langle V^J \rangle$. Since $x_0 V^J \subseteq V^J$, it is then sufficient to prove the thesis for $x^\eta = x_i$, $i \geqslant 1$, assuming that the thesis holds for every $x^{\eta'} <_{\texttt{Lex}} x_i$. We consider $g_\beta = x^\delta f_\alpha \in V^J$, where $\max x^\delta \leq \min x^\alpha$. If $x_i g_\beta$ does not belong to V^J, then $\max(x_i \cdot x^\delta) > \min x^\alpha$, so $x_i > \min x^\alpha$ because $\max x^\delta \leq \min x^\alpha$ by construction. In particular, $x_i > \min x^\alpha \geq \max x^\delta$, so $x_i >_{\texttt{Lex}} x^\delta$ and it is sufficient to prove the thesis for $x_i f_\alpha$.

We consider an S-polynomial $S(f_\alpha, f_{\alpha'}) = x_i f_\alpha - x^\gamma f_{\alpha'}$ such that $x^\gamma <_{\texttt{Lex}} x_i$. Such S-polynomial always exists: for instance, we can consider $x_i x^\alpha = \langle x^{\alpha'} | x^{\eta'} \rangle^J$.

By hypothesis there is t such that $x_0^t S(f_\alpha, f_{\alpha'}) = x_0^t(x_i f_\alpha - x^{\eta'} f_{\alpha'}) = \sum a_j x^{\eta'_j} f_{\alpha_j}$ where $x_0^t x^{\eta'}, x^{\eta'_j}$ are lower than x_i w.r.t. $\texttt{Lex}$. Then $x^{\eta'} f_{\alpha'}$, $x^{\eta'_j} f_{\alpha_j}$ belong to $\langle V^J \rangle$ by induction and by Lemma 4.45, and we conclude that $x_i f_\alpha \in \langle V^J \rangle$, by Lemma 4.45 again. $\square$

As pointed out in Remark 4.3.8 concerning the proof of Theorem 4.37, also in the proof of Theorem 4.47 it would be sufficient to assume statement *(iii)* only for EK-polynomials. We then have the following result.

Corollary 4.48. *With the same notations of Theorem 4.47,*

$$I \in \mathcal{M}f(J) \Leftrightarrow \ \forall \text{ EK-polynomial } \exists\, t \text{ s.t. } x_0^t S^{EK}(f_\alpha, f_{\alpha'}) \xrightarrow{sG} 0.$$

We have just showed that it is sufficient to work with superminimal reduction for testing if an ideal $I = (G)$ is in $\mathcal{M}f(J)$. Then, one may think that it is enough to reduce S-polynomials among elements in sG (either with $\xrightarrow{V_s^J}$ or with $\xrightarrow{sG}$). The following example clearly shows that this is not true.

Example 4.4.4. We consider the Borel-fixed ideal

$$J = \underline{J}_{\geqslant 2} = (x_3, x_2^2)_{\geqslant 2} = (x_3^2, x_3x_2, x_2^2, x_3x_1, x_3x_0) \subseteq \mathbb{K}[x_0, x_1, x_2, x_3].$$

In this case, sG_J contains only two monomials, x_3x_0 and x_2^2.

If G is any J-marked set, then $sG = \{f_{x_3x_0}, f_{x_2^2}\}$. The unique S-polynomial among superminimal elements is

$$S(f_{x_3x_0}, f_{x_2^2}) = x_2^2 f_{x_3x_0} - x_3x_0 f_{x_2^2} = x_3x_0 \cdot \mathrm{Nf}(x_2^2) - x_2^2 \cdot \mathrm{Nf}(x_3x_0).$$

Any monomial appearing in $\operatorname{Supp}\mathrm{Nf}(x_3x_0)$ is in $\mathcal{N}(J)_2 = \mathbb{K}[x_0, x_1, x_2]_2 \setminus \{x_2^2\}$. Then any monomial appearing in $\operatorname{Supp}\left(x_2^2 \cdot \mathrm{Nf}(x_3x_0)\right)$ is further reduced by $f_{x_2^2}$, obtaining by $\xrightarrow{V_s^J}$ or $\xrightarrow{sG}$

$$S(f_{x_3x_0}, f_{x_2^2}) = x_3x_0 \cdot \mathrm{Nf}(x_2^2) - \mathrm{Nf}(x_2^2) \cdot \mathrm{Nf}(x_3x_0) = \mathrm{Nf}(x_2^2) \cdot f_{x_3x_0} \to 0.$$

Nevertheless, even if the only S-polynomial among superminimal generators reduces to 0, we need to impose also other conditions in order to get a J-marked basis G.

If we consider $sG = \{f_{x_3x_0}, f_{x_2^2}\}$ with $f_{x_3x_0} = x_3x_0 + x_1^2$ and $f_{x_2^2} = x_2^2$, then for any choice of $f_{x_3^2}, f_{x_3x_2}, f_{x_3x_1}$, the S-polynomial among $f_{x_3x_1}$ and $f_{x_3x_0}$ does not reduce to 0:

$$S(f_{x_3x_1}, f_{x_3x_0}) = x_0 f_{x_3x_1} - x_1 f_{x_3x_0} = \sum_{x^{\alpha_i} \in \mathcal{N}(J)_2} a_i x^{\alpha_i} x_0 - x_1^3.$$

The monomials $x^{\alpha_i}x_0$ are in $\mathcal{N}(J)_3$ and are not further reducible (neither by $\xrightarrow{V_s^J}$ nor by $\xrightarrow{sG}$). Furthermore, x_1^3 does not appear among monomials $m_i x_0$, so it is not canceled. So, for any choice of coefficients in the tail of $f_{x_3x_1}$, we have an S-polynomial which is not reducible to 0, and so any J-marked set containing $f_{x_3x_0} = x_3x_0 + x_1^2$ is not a J-marked basis.

The previous example shows that it is not enough to impose condition *(ii)* of Theorem 4.47 to S-polynomials among elements of sG. We need to consider some other S-polynomial in order to get a J-marked basis.

Theorem 4.49. *With the fixed notations, consider the following sets of S-polynomials:*

$$L_1 = \left\{ x^{\gamma} f_{\alpha} - x^{\gamma'} f_{\alpha'} \mid x^{\alpha}, x^{\alpha'} \in sG_J \right\}, \tag{4.15}$$

$$L_2 = \left\{ x_i f_{\alpha'} - x_0 f_{\alpha} \mid x_i = \min_{j>0}\{x_j \text{ s.t. } x_j \mid x^{\alpha}\},\ |\alpha| = |\alpha'| = m \right\}. \tag{4.16}$$

Then:

$$I \in \mathcal{M}f(J) \iff \forall\, S(f_\alpha, f_{\alpha'}) \in L_1 \cup L_2,\ \exists\, t \text{ s.t. } x_0^t \cdot S(f_\alpha, f_{\alpha'}) \xrightarrow{sG} 0.$$

Proof. ($\Rightarrow$) If I belongs to $\mathcal{M}f(J)$, then it is enough to apply Theorem 4.37.

($\Leftarrow$) Vice versa, we want to prove that $\langle V^J \rangle = \langle W^J \rangle$, that is $x_i \cdot V^J \subseteq \langle V^J \rangle$ for every $i = 0, \dots, n$. We proceed by induction on the variables. By construction we have $x_0 \cdot V^J \subseteq \langle V^J \rangle$. We now assume that $(x_0, \dots, x_{i-1})V^J \subseteq \langle V^J \rangle$ and we prove that $x_i \cdot V^J \subseteq \langle V^J \rangle$. Consider $x^\delta f_\beta \in V^J$. The thesis is that $x_i \cdot x^\delta f_\beta$ is contained in $\langle V^J \rangle$. If $x_i x^\delta f_\beta$ does not belong to V^J, then $\max(x_i \cdot x^\delta) > \min x^\beta$, so $x_i > \min x^\beta$ because $\max x^\delta \leq \min x^\beta$ by construction. In particular, $x_i > \min x^\beta \geq \max x^\delta$, so that it is sufficient to prove the thesis for $x_i f_\beta$, because by induction then we have $x^\delta x_i f_\beta \in \langle V^J \rangle$.

Consider $x^\beta = \langle x^{\underline{\alpha}} | x^\eta \rangle^{\underline{J}}$. We have a first case when $x^\eta = 1$. Then $x^\beta = x^{\underline{\alpha}}$ and f_β belongs to sG. We consider $x^{\underline{\alpha}} x_i = \langle x^{\underline{\alpha}'} | x^{\eta'} \rangle^{\underline{J}}$. Observe that since $x_i > \min x^{\underline{\alpha}}$ then x_i does not divide $x^{\eta'}$ and $\max x^{\eta'} < x_i$. Consider $x^{\alpha'} = x^{\underline{\alpha}'} \cdot x_0^{t_{\alpha'}}$, so that we can take the polynomial $f_{\alpha'} \in sG$. We construct the S-polynomial between f_β and $f_{\alpha'}$

$$S(f_\beta, f_{\alpha'}) = x_0^{t_{\alpha'}} x_i f_\beta - x^{\eta'} f_{\alpha'}$$

that belongs to L_1. Thus, by the hypothesis and by Lemma 4.46, there is k such that

$$x_0^k S(f_\beta, f_{\alpha'}) = x_0^k (x_0^{t_{\alpha'}} x_i f_\beta - x^{\eta'} f_{\alpha'}) = \sum a_j x^{\eta_j} f_{\alpha_j},$$

with $x^{\eta_j} <_{\texttt{Lex}} x_i$ and $f_{\alpha_j} \in sG$. Hence we obtain that both $x^{\eta_j} f_{\alpha_j}$ and $x^{\eta'} f_{\alpha'}$ belong to $\langle V^J \rangle$ by induction on the variables, and so $x_i f_\beta$ belongs to $\langle V^J \rangle$ (by Lemma 4.45).

We have a second case when $x^\eta = x_0^t$, $t > 0$. Then, $|\beta| = m$ and f_β belongs to sG. Let $x_i x^\beta = \langle x^{\underline{\alpha}'} | x^{\eta'} \rangle^{\underline{J}}$. If $x_i > \min x^{\underline{\alpha}'}$, then $x^{\eta'}$ is not divisible by x_i and we repeat the argument above. Otherwise, $x_i \leq \min x^{\underline{\alpha}'})$ and x_i does not divide $x^{\eta'}$, so that $x_i = \min x^{\underline{\alpha}'}$ and $x^{\eta'} <_{\texttt{Lex}} x_i$. Then, we take $x^{\beta'} = \frac{x^\beta}{x_0} \cdot x_i$ that belongs to G_J because it has degree m. So, $x_i f_\beta - x_0 f_{\beta'}$ belongs to L_2 and we repeat the same reasoning above.

We now assume the thesis holds for every $f_{\beta'}$ such that $x^{\beta'} = \langle x^{\underline{\alpha}'} | x^{\eta'} \rangle^{\underline{J}}$ with $x^{\eta'} <_{\texttt{Lex}} x^\eta$. By the base of the induction, we can suppose that $x^\eta \geq_{\texttt{Lex}} x_1$; so, f_β does not belong to sG and hence it has degree m. Let $x_j = \min_{l>0}\{x_l \text{ s.t. } x_l \mid x^\beta\}$. Observe that if x_0 does not divide x^β, then $x_j = \min x^\beta$; in this case, we have $x_i > x_j$ because $x_i > \min x^\beta$. Anyway, first we suppose that $x_i \leq x_j$; so, $x_j > \min x^\beta$ and x_0 divides x^β. We consider $x^{\beta'} = \frac{x^\beta}{x_0} \cdot x_i$ and the following S-polynomial

$$S(f_\beta, f_{\beta'}) = x_i f_\beta - x_0 f_{\beta'},$$

that belongs to L_2 and we repeat the argument of the previous case.

We now assume that $x_i > x_j$ and consider $x^{\beta'} = \frac{x^\beta}{x_j} \cdot x_0 = \langle x^{\underline{\alpha}'} | x^{\eta'} \rangle^{\underline{J}}$. Observe that $x^{\eta'} <_{\texttt{Lex}} x^\eta$ because $x^{\eta'} = \frac{x^\eta}{x_j} \cdot x_0$. We consider the S-polynomial:

$$S(f_{\beta'}, f_\beta) = x_j f_{\beta'} - x_0 f_\beta$$

that belongs to L_2; so, by the hypothesis and by Lemma 4.46, there is an integer t such that

$$x_0^t S(f_{\beta'}, f_\beta) = x_0^t (x_j f_{\beta'} - x_0 f_\beta) = \sum a_l x^{\eta_l} f_{\alpha_l} \tag{4.17}$$

with $x^{\eta_l} <_{\texttt{Lex}} x_j$, $f_{\alpha_l} \in sG$. We now multiply (4.17) by x_i. We observe that $x_i f_{\alpha_l}$ belongs to $\langle V^J \rangle$, because $f_{\alpha_l} \in sG$ and by the first two cases. Also $x_i f_{\beta'}$ belongs to $\langle V^J \rangle$ because $x^{\eta'} <_{\texttt{Lex}} x^\eta$. Moreover, $x_j x_i f_{\beta'}$ belongs to $\langle V^J \rangle$ by induction on the variables. So, $x_i f_\beta$ belongs to $\langle V^J \rangle$ thanks to Lemma 4.45. □

Remark 4.4.5. Theorem 4.49 is an improvement, from the computational point of view, of Theorem 4.47. Indeed, it gives a criterion to establish if I belongs to $\mathcal{M}f(J)$ in which we compute the reduction by $\xrightarrow{sG}$ of a subset of the whole set of S-polynomials among elements in G. Actually, in the proof of Theorem 4.49 we do not need all the S-polynomials of the set L_1, but only those of type $x_0^{t_{\alpha'}} x_i f_\beta - x^{\eta'} f_{\alpha'}$. This fact can lead us to a further improvement of the efficiency of our algorithms.

4.5 Explicit construction of marked families

In this section, we define an affine scheme whose points correspond to the all the ideals belonging to the J-marked family $\mathcal{M}f(J)$, being $J = \underline{J}_{\geqslant m}$ a truncated Borel-fixed ideal as before, as done in Section 4.1 for families sharing the same initial ideal.

Definition 4.50 (Cf. with Definition 4.3)**.** Let $J = \underline{J}_{\geqslant m} \subset \mathbb{K}[x]$ be a Borel-fixed ideal.

Step 1 Consider the set of polynomials $\mathcal{G} = \{F_\alpha\}_{x^\alpha \in G_J}$

$$F_\alpha = x^\alpha + \sum_{x^\beta \in \mathcal{N}(J)_{|\alpha|}} C_{\alpha\beta} x^\beta \in \mathbb{K}[C][x] \tag{4.18}$$

where C denotes the whole set of variables $C_{\alpha\beta}$, $\forall\ x^\alpha \in G_J$, $\forall\ x^\beta \in \mathcal{N}(J)_{|\alpha|}$. Moreover we look at F_α's as marked polynomial with $\mathrm{Ht}(F^\alpha) = x^\alpha$.

Step 2 Consider the analogous of V_s^J and W_s^J denoting them by $\mathcal{V}_s^J$ and $\mathcal{W}_s^J$.

Step 3a For any pairs F_α, F'_α, compute the J-reduced forms by $\xrightarrow{\mathcal{V}_s}$ of the S-polynomial $S(F_\alpha, F'_\alpha)$: $S(F_\alpha, F'_\alpha) \xrightarrow{\mathcal{V}_s} H_{\alpha\alpha'}$.

Step 3b For any EK-pairs F_α, F'_α, compute the J-reduced forms by $\xrightarrow{\mathcal{V}_s}$ of the EK-polynomial $S^{\mathrm{EK}}(F_\alpha, F'_\alpha)$: $S^{\mathrm{EK}}(F_\alpha, F'_\alpha) \xrightarrow{\mathcal{V}_s} H^{\mathrm{EK}}_{\alpha\alpha'}$.

Step 4a Call $\mathfrak{A}_J$ the ideal in $\mathbb{K}[C]$ generated by the coefficients (polynomials of $\mathbb{K}[C]$) of the monomials in the variables x appearing in $H_{\alpha\alpha'}$.

Step 4b Call $\mathfrak{A}'_J$ the ideal in $\mathbb{K}[C]$ generated by the coefficients (polynomials of $\mathbb{K}[C]$) of the monomials in the variables x appearing in $H^{\mathrm{EK}}_{\alpha\alpha'}$.

Cioffi and Roggero in [22, Section 4] prove that the ideal $\mathfrak{A}_J$ does not depend on the reduction $\xrightarrow{\mathcal{V}_s^J}$ and defines the subscheme structure of $\mathcal{M}f(J)$ in the affine space $\mathbb{A}^{|C|}$. By definition $\mathfrak{A}'_J \subseteq \mathfrak{A}_J$. Anyway, we will prove that $\mathfrak{A}'_J$ and $\mathfrak{A}_J$ are the same ideal, although $\mathfrak{A}_J$ is defined by a set of generators bigger than the set of generators of $\mathfrak{A}'_J$. More precisely, we prove that the ideal $\mathfrak{A}'_J$ contains the coefficients of every J-reduced polynomial in $(\mathcal{G}) \subset \mathbb{K}[C][x]$.

Lemma 4.51. *(i) For every monomial $x^\beta = \langle x^\alpha | x^\delta \rangle^J \in J$, we have a formula of type*

$$x^\beta = \sum a_i x^{\gamma_i} F_{\alpha_i} + H_\beta,$$

with $a_i \in K[C]$, $x^{\gamma_i} F_{\alpha_i} \in \mathcal{V}^J$, $x^{\gamma_i} <_{\texttt{Lex}} x^\delta$ and $\operatorname{Supp} H_\beta \subset \mathcal{N}(J)$.

(ii) For every polynomial $x_i F_\alpha \in \mathcal{W}^J \setminus \mathcal{V}^J$, we have a formula of type

$$x_i F_\alpha = \sum b_j x^{\eta_j} F_{\alpha_j} + H_{i,\alpha},$$

with $b_j \in K[C]$, $x^{\eta_j} F_{\alpha_j} \in \mathcal{V}^J$, $x^{\eta_j} <_{\texttt{Lex}} x_i$, $\operatorname{Supp} H_{i,\alpha} \subset \mathcal{N}(J)$ and the coefficients appearing in $H_{i,\alpha}$ belong to $\mathfrak{A}'_J$.

Proof. Statement *(i)* follows from the existence of J-reduced forms obtained by $\xrightarrow{\mathcal{V}_s^J}$. Statement *(ii)* follows also from the definition of $\mathfrak{A}'_J$. □

Proposition 4.52. *For every polynomial $x^\delta F_\alpha \in \mathcal{W}^J \setminus \mathcal{V}^J$, we have*

$$x^\delta F_\alpha = \sum b_j x^{\eta_j} F_{\alpha_j} + H_{\delta\alpha}, \tag{4.19}$$

with $b_j \in K[C]$, $x^{\eta_j} F_{\alpha_j} \in \mathcal{V}^J$, $x^{\eta_j} <_{\texttt{Lex}} x^\delta$, $\operatorname{Supp} H_{\delta\alpha} \subset \mathcal{N}(J)$ and the coefficients appearing in $H_{\delta\alpha}$ belong to $\mathfrak{A}'_J$.

Proof. For $|\delta| = 1$ it is enough to use Lemma 4.51*(ii)*. Assume that $|\delta| > 1$ and that the thesis holds for every $x^{\delta'} <_{\texttt{Lex}} x^\delta$. Let $x_i = \min x^\delta$ and $x^{\delta'} = \frac{x^\delta}{x_i}$, so that $x^{\delta'} F_\alpha$ belongs to $\mathcal{W}^J \setminus \mathcal{V}^J$.

By the inductive hypothesis, we have $x^{\delta'} F_\alpha = \sum b'_j x^{\eta'_j} F_{\alpha_j} + H_{\delta'\alpha}$, with $x^{\eta'_j} <_{\texttt{Lex}} x^{\delta'}$. So, multiplying by x_i, we obtain $x^\delta F_\alpha = \sum b'_j x_i x^{\eta'_j} F_{\alpha_j} + x_i H_{\delta'\alpha}$ and the thesis holds for every polynomial $x_i x^{\eta'_j} F_{\alpha_j}$ that belongs to $\mathcal{W}^J \setminus \mathcal{V}^J$ because $x_i x^{\eta'_j} <_{\texttt{Lex}} x_i x^{\delta'} = x^\delta$. Then, we substitute such polynomials by formulas of type (4.19) and obtain

$$x^\delta F_\alpha = \sum b_s x^{\eta_s} F_{\alpha_s} + H' + x_i H_{\delta'\alpha}$$

where the first sum satisfies the conditions of (4.19) and H' is J-reduced with $\operatorname{Supp} H'$ contained in $\mathcal{N}(J)$ and the coefficients of H' are in $\mathfrak{A}'_J$.

Note that $x_i H_{\delta'\alpha}$ and $H_{\delta'\alpha}$ have the same coefficients belonging to $\mathfrak{A}'_J$, but we do not know if $\operatorname{Supp}(x_i H_{\delta'\alpha}) \subset \mathcal{N}(J)$. If $x^{\beta'} \in \operatorname{Supp} H_{\delta'\alpha}$ has coefficient b in

$H_{\delta'\alpha}$ and $x^\beta = x_i x^{\beta'}$ belongs to J, then we can use Lemma 4.51*(i)* obtaining $bx^\beta = \sum ba_k x^{\gamma_i} F_{\alpha_k} + bH_\beta$. Moreover, if $x^\beta = \langle x^{\alpha'} | x^\epsilon \rangle^J$, then $x^{\gamma_i} <_{\texttt{Lex}} x^\epsilon <_{\texttt{Lex}} x_i <_{\texttt{Lex}} x^\delta$ and all coefficients of H_β belong to $\mathfrak{A}'_J$ because they are divisible by b. Substituting all such monomials x^β, we obtain the thesis and $H_{\delta\alpha}$ is J-reduced with coefficients in $\mathfrak{A}'_J$, because it is the sum of J-reduced polynomials with coefficients in $\mathfrak{A}'_J$. $\square$

Corollary 4.53. *Every polynomial of $(\mathcal{G})$ can be written in a unique way as $\sum b_j x^{\eta_j} F_{\alpha_j} + H$, with $b_j \in K[C]$, $x^{\eta_j} F_{\alpha_j} \in \mathcal{V}^J$ and H J-reduced. Moreover, we obtain also that the coefficients of H belongs to $\mathfrak{A}'_J$.*

Proof. By definition, every polynomial of $(\mathcal{G})$ is a linear combination of polynomials of $\mathcal{V}^J \cap (\mathcal{W} \setminus \mathcal{V})$ with coefficients in $K[C]$ and, by Proposition 4.52, every such polynomial can be written has described in the statement. Hence, we have only to prove the uniqueness of this writing.

Let $\sum b_j x^{\eta_j} F_{\alpha_j} + H = 0$ be the difference between two writings of the same polynomial of $(\mathcal{G})$, with $b_j \neq 0$, $x^{\eta_j} F_{\alpha_j} \in \mathcal{V}$ pairwise different and H J-reduced. Let $x^{\eta_1} x^{\alpha_1}$ the maximum of the monomials w.r.t. the order for which $x^{\eta_i} x^{\alpha_i}$ is lower than $x^{\eta_j} x^{\alpha_j}$ if $x^{\eta_i} <_{\texttt{Lex}} x^{\eta_j}$ or $x^{\eta_i} = x^{\eta_j}$ and $x^{\alpha_i} < x^{\alpha_j}$, where $<$ is any order fixed on G_J. By definition of $\mathcal{V}^J$, the unique polynomial of $\mathcal{V}^J$ with head term $x^{\eta_1} x^{\alpha_1}$ is $x^{\eta_1} F_{\alpha_1}$. Moreover, the monomial $x^{\eta_1} x^{\alpha_1}$ does not appear with a non-null coefficient in any polynomial of the sum because every other monomial belongs to $\mathcal{N}(J)$ or is lower than it, by construction and by Lemma 4.32. Further, $x^{\eta_1} x^{\alpha_1}$ does not belong to $\operatorname{Supp} H$ because $\operatorname{Supp} H \subset \mathcal{N}(J)$ and $x^{\eta_1} x^{\alpha_1} \in J$. Thus, we obtain a contradiction to the fact that $b_j \neq 0$. $\square$

Corollary 4.54. *The ideal $\mathfrak{A}'_J$ contains the coefficients of every J-reduced polynomial of $(\mathcal{G})$. In particular, $\mathfrak{A}'_J = \mathfrak{A}_J$.*

Proof. Let F be a J-reduced polynomial of $(\mathcal{G})$ and let $F = \sum b_j x^{\eta_j} F_{\alpha_j} + H$ as in Corollary 4.53. Since F itself is J-reduced, also $F = 0 + F$ is a formula as described in Corollary 4.53 and we obtain that $F = H$, by the uniqueness of this formula. Hence, we have that the coefficients of F and H are the same and are in $\mathfrak{A}'_J$. The last assertion is due to the definition of $\mathfrak{A}_J$. $\square$

Remark 4.5.1. Actually, for every ideal $\widehat{\mathfrak{A}}_J \subseteq \mathfrak{A}_J \subseteq \mathbb{K}[C]$ such that condition *(ii)* of Lemma 4.51 holds, also Corollary 4.54 holds. We are then allowed to choose different sets of S-polynomials of $\mathcal{G}$ in order to obtain generators of the ideal $\mathfrak{A}_J$.

We know recall the construction of the J-marked schemes using matrices to underline the close relation between them and the open affine subsets of the Grasmmannians. By Gotzmann's Persistence Theorem, a specialization $C \to c \in \mathbb{K}^{|C|}$ transforms the J-marked set $\mathcal{G}$ in a J-marked basis G if and only if $\dim_{\mathbb{K}}(G)_s = \dim_{\mathbb{K}} J_s$, for every degree s. Thus, for each s, consider the matrix $\mathcal{A}_s$ whose columns correspond to the terms of degree s in $\mathbb{K}[x]$ and whose rows contain the coefficients of the terms in every polynomial of degree s of type $x^\delta F_\alpha$. Hence, every entry of the matrix $\mathcal{A}_s$ is 1, 0 or one of the variables C. Let $\mathfrak{a}$ be the ideal of $\mathbb{K}[C]$ generated by the minors of order $\dim_{\mathbb{K}} J_s + 1$ of $\mathcal{A}_s$, for every s.

Proposition 4.55 ([22, Lemma 4.2]). *The ideal $\mathfrak{a}$ is equal to the ideal $\mathfrak{A}$.*

Proof. Let $a_s = \dim_{\mathbb{K}} J_s$. We consider in $\mathcal{A}_s$ the $a_s \times a_s$ submatrix whose columns corresponds to the terms in J_s and whose rows are given by the polynomials $x^\beta F_\alpha$ in $\mathcal{V}_s$. Up to a permutation of rows and columns, this submatrix is upper-triangular with 1 on the main diagonal. We may also assume that it corresponds to the first a_m rows and columns in $\mathcal{A}_s$. Then the ideal $\mathfrak{a}$ is generated by the determinants of $(a_s+1) \times (a_s+1)$ submatrices containing that above considered. Moreover the Gaussian row-reduction of $\mathcal{A}_s$ with respect to the first a_m rows is nothing else than the $\mathcal{V}_s$-reduction of the S-polynomials of the special type considered defining $\mathfrak{A}$. $\square$

As the superminimal reduction uses less polynomials than $\xrightarrow{V_s^J}$, we now exploit it to embed $\mathcal{M}f(J)$ in an affine subspace of $\mathbb{A}^{|C|}$ of lower dimension.

Definition 4.56. If $\mathcal{G}$ is the set of marked polynomials given in (4.18), we will call *set of superminimal generators*, and denote it by $s\mathcal{G}$, the subset of $\mathcal{G}$

$$s\mathcal{G} = \{F_\alpha \in \mathcal{G} \mid \mathrm{Ht}(F_\alpha) = x^\alpha \in sG_J\}. \tag{4.20}$$

We will denote by $\widetilde{C} \subset C$ the set of variables appearing in the tails of the polynomials in $s\mathcal{G}$.

Note that the J-marked basis G of every $I \in \mathcal{M}f(J)$ is obtained by specializing in a suitable way the variables C in $\mathcal{G}$ and that the set of superminimal generators sG of I is obtained in the same way by $s\mathcal{G}$ through the same specialization of the variables $\widetilde{C}$.

Definition 4.57. Let $x^\alpha \in G_J$ and t be an integer such that $x_0^t \cdot x^\alpha \xrightarrow{s\mathcal{G}} H_\alpha$, with H_α strongly reduced (the integer t exists by Theorem 4.42). We can write $H_\alpha = H'_\alpha + x_0^t \cdot H''_\alpha$, where no monomial appearing in H'_α is divisible by x_0^t. We will denote by:

- $\mathfrak{B} = \left\{C_{\alpha\gamma} - \phi_{\alpha\gamma} \text{ s.t. } x^\alpha \in G_J \setminus sG_J,\ x^\gamma \in \mathcal{N}(J)_{|\alpha|}\right\}$ the set of the coefficients of $\mathcal{T}(F_\alpha) - H''_\alpha$ for every $x^\alpha \in G_J$;
- $\mathfrak{D}_1 \subset \mathbb{K}[\widetilde{C}]$ the set of the coefficients of H'_α for every $x^\alpha \in G_J \setminus sG_J$;
- $\mathfrak{D}_2$ the set of the coefficients of the strongly reduced polynomials in $(s\mathcal{G})\mathbb{K}[\widetilde{C}][x]$.

Theorem 4.58. *The J-marked scheme $\mathcal{M}f(J)$ is defined by the ideal $\widetilde{\mathfrak{A}}_J = \mathfrak{A}_J \cap \mathbb{K}[\widetilde{C}]$ as subscheme of the affine space $\mathbb{A}^{|\widetilde{C}|}$. Moreover $\mathfrak{A}_J = (\mathfrak{B} \cup \mathfrak{D}_1 \cup \mathfrak{D}_2)\mathbb{K}[C]$ and $\widetilde{\mathfrak{A}}_J = (\mathfrak{D}_1 \cup \mathfrak{D}_2)K[\widetilde{C}]$.*

Proof. For the first part it suffices to prove that $\mathfrak{A}_J$ contains $\mathfrak{B}$ and so it contains an element of the type $C_{\alpha\gamma} - \phi_{\alpha\gamma}$, for every $C_{\alpha\gamma} \in C \setminus \widetilde{C}$, where $\phi_{\alpha\gamma} \in \mathbb{K}[\widetilde{C}]$, that allows the elimination of the variables $C_{\alpha\gamma} \in C \setminus \widetilde{C}$.

It is clear by the construction in Definition 4.57 that H_α belongs to $\mathbb{K}[\widetilde{C}][x]$ and that both $x_0^t \cdot \mathcal{T}(F_\alpha)$ and H_α are strongly reduced. Thus their difference $x_0^t \cdot \mathcal{T}(F_\alpha) - H_\alpha$ is strongly reduced and moreover it belongs to $(\mathcal{G})$, because $x_0^t \cdot \mathcal{T}(F_\alpha) - H_\alpha = -x_0^t \cdot F_\alpha + (x_0^t \cdot x^\alpha - H_\alpha)$. Hence, by Corollary 4.54, its coefficients belong to $\mathfrak{A}_J$ and in particular the coefficient of $x_0^t \cdot x^\gamma$ is of the type $C_{\alpha\gamma} - \phi_{\alpha\gamma}$, with $\phi_{\alpha\gamma} \in \mathbb{K}[\widetilde{C}]$. Then $\mathfrak{A}_J \supseteq \mathfrak{B}$ and $\mathfrak{A}_J$ is generated by $\mathfrak{B} \cup \widetilde{\mathfrak{A}}_J$.

To prove the second part, it is sufficient to show that $\mathfrak{A}_J \cap \mathbb{K}[\widetilde{C}] = (\mathfrak{D}_1 \cup \mathfrak{D}_2)\mathbb{K}[\widetilde{C}]$.

($\supseteq$) Taking the coefficients in $x_0^t \cdot \mathcal{T}(F_\alpha) - H_\alpha$ of monomials that are not divisible by x_0^t, we see that $\mathfrak{A}_J$ contains the coefficients of H'_α. Then $\mathfrak{A}_J \cap \mathbb{K}[\widetilde{C}] \supseteq \mathfrak{D}_1$, because

$H'_\alpha \in \mathbb{K}[\widetilde{C}][x]$. Moreover we recall that $\mathfrak{A}_J$ is made by all the coefficients in the polynomials of $(\mathcal{G})$ that are strongly reduced. Indeed, $\mathfrak{A}_J$ is made by all the coefficients of the polynomials of $(\mathcal{G})$ that are J-reduced. But the degree of the monomials in the variables x of every polynomial in $(\mathcal{G})$ is $\geqslant m$ and then "J-reduced" is equivalent to "$\underline{J}$-reduced", that it is strongly reduced. Then $\mathfrak{A}_J \cap \mathbb{K}[\widetilde{C}] \supseteq \mathfrak{D}_2$, because $(s\mathcal{G})\mathbb{K}[\widetilde{C}][x] \subset (\mathcal{G})$.

($\subseteq$) For every polynomial $F \in \mathbb{K}[C,x]$, let us denote by F^ϕ the polynomial in $\mathbb{K}[\widetilde{C},x]$ obtained substituting every $C_{\alpha\gamma} \in C \setminus \widetilde{C}$ by $\phi_{\alpha\gamma}$. Observe that for ever $x^\alpha \in G_J$ we have $x_0^t \cdot F_\alpha^\phi = x_0^t(x^\alpha - H''_\alpha) + H'_\alpha$ and moreover $x_0^t(x^\alpha - H'') + H'_\alpha \in (s\mathcal{G})\mathbb{K}[\widetilde{C},x]$. It remains to prove that every element $w \in \mathfrak{A}_J \cap K[\widetilde{C}]$ can be obtained modulo $\mathfrak{D}_1$ as a coefficient in some strongly reduced polynomial of the ideal $(s\mathcal{G}) \subset \mathbb{K}[\widetilde{C}]$. We know that w is a coefficient in a strongly reduced polynomial $D \in (\mathcal{G})$.

If $D = \sum D_\alpha F_\alpha \in (\mathcal{G})$, then for a suitable t,

$$x_0^t \cdot D^\phi = \sum D_\alpha^\phi \cdot \left(x_0^t \cdot (x^\alpha - H''_\alpha) + H'_\alpha\right) \in (s\mathcal{G})\mathbb{K}[\widetilde{C}][x]$$

and w is still one of the coefficients of D^ϕ because it does not contain any variable in $C \setminus \widetilde{C}$ and so it remains unchanged. Moreover if D is strongly reduced, also D^ϕ is strongly reduced and so $w \in (\mathfrak{D}_1 \cup \mathfrak{D}_2)\mathbb{K}[\widetilde{C}]$. $\square$

Proposition 4.59. *Let $\mathfrak{U} \subseteq \widetilde{\mathfrak{A}}_J$ be any ideal in $\mathbb{K}[\widetilde{C}]$ such that:*

(i) for every monomial $x^\beta = \langle x^{\underline{\alpha}} | x^\delta \rangle^{\underline{J}} \in J$, there exists t such that we have a formula of type

$$x_0^t \cdot x^\beta = \sum b_i x^{\eta_i} F_{\alpha_i} + H_\beta,$$

with $a_i \in \mathbb{K}[\widetilde{C}]$, $F_{\alpha_i} \in s\mathcal{G}$, $x^{\eta_i} <_{\texttt{Lex}} x^\delta$, $x^{\underline{\eta}_j + \underline{\alpha}_j} = \langle x^{\underline{\alpha}_j} | x^{\underline{\eta}_j} \rangle^{\underline{J}}$ and $H_\beta = x_0^t \cdot H'_\beta + H''_\beta$, with $\operatorname{Supp} H_\beta \subset \mathcal{N}(J)$, x_0^t does not divide H''_β and the coefficients of H''_β belong to $\mathfrak{U}$;

(ii) for every polynomial $F_\alpha \in s\mathcal{G}$ and for every $x_i > \min x^{\underline{\alpha}}$ there exists t such that we have a formula of type

$$x_0^t x_i F_\alpha = \sum b_j x^{\eta_j} F_{\alpha_j} + H_{i,\alpha}$$

where $b_j \in \mathbb{K}[\widetilde{C}]$, $F_{\alpha_j} \in s\mathcal{G}$, $x^{\eta_j} <_{\texttt{Lex}} x_i$, $x^{\underline{\eta}_j + \underline{\alpha}_j} = \langle x^{\underline{\alpha}_j} | x^{\underline{\eta}_j} \rangle^{\underline{J}}$, $\operatorname{Supp} H_{i,\alpha} \subseteq \mathcal{N}(J)$ and the coefficients of $H_{i,\alpha}$ belongs to $\mathfrak{U}$.

Then $\mathfrak{U} = (\mathfrak{D}_1 \cup \mathfrak{D}_2)$.

Proof. Thanks to *(i)*, we immediately have that $\mathfrak{D}_1 \subseteq \mathfrak{U}$.

For the inclusion $\mathfrak{D}_2 \subseteq \mathfrak{U}$, observe that if *(i)* and *(ii)* hold for $\mathfrak{U}$, then we can use the same arguments of Proposition 4.52 and obtain that for every $F_\alpha \in s\mathcal{G}$, for every x^δ, there exists t such that

$$x_0^t x^\delta F_\alpha = \sum b_j x^{\eta_j} F_{\alpha_j} + H \tag{4.21}$$

with $b_j \in \mathbb{K}[\widetilde{C}]$, $F_{\alpha_j} \in s\mathcal{G}$, $x^{\eta_j} <_{\mathtt{Lex}} x^\delta$, $x^{\underline{\eta}_j+\underline{\alpha}_j} = \langle x^{\underline{\alpha}_j} | x^{\underline{\eta}_j} \rangle^{\underline{J}}$, Supp $H_{\delta,\alpha} \subseteq \mathcal{N}(J)$ and the coefficients of $H_{\delta,\alpha}$ belong to $\mathfrak{U}$.

We can also prove the uniqueness of such a rewriting: thanks to the uniqueness of the $\underline{J}$-canonical decomposition (Lemma 2.17), the polynomials $x^{\eta_j} F_{\alpha_j}$ that can appear in (4.21) have pairwise different head terms. So an analogous of Corollary 4.53 holds for this setting. Thanks to this uniqueness, as in Corollary 4.54, we get the non trivial inclusion of the thesis. □

Proposition 4.59 is very important from the computational point of view: indeed, different choices of sets of S-polynomials to reduce give different sets of generators for $\widetilde{\mathfrak{A}}_J$. For instance we can get a set of generators for $\widetilde{\mathfrak{A}}_J$ starting from EK-polynomials among polynomials in $\mathcal{G}$ or starting from $\mathcal{L}_1$ and $\mathcal{L}_2$ in $\mathbb{K}[C][x]$, corresponding to L_1 and L_2 as defined in Theorem 4.49. However, a good choice of the set of S-polynomials can strongly influence the efficiency of an algorithm computing equations for $\mathcal{M}f(J)$.

As seen in Section 4.1, a Gröbner stratum $\mathcal{S}t_\sigma(J)$ can be isomorphically projected in its Zariski tangent space at the origin $T_0\big(\mathcal{S}t_\sigma(J)\big)$ and moreover if the origin is a smooth point, then the stratum is isomorphic to this tangent space. In general, if we do not consider a term ordering we cannot project isomorphically $\mathcal{M}f(J)$ into $T_0\big(\mathcal{M}f(J)\big)$, but in any case, the dimension of this tangent space plays an interesting role in the following theorem.

Remark 4.5.2. Let $\mathfrak{L}(J)$ be the ideal generated in $\mathbb{K}[C]$ by the linear components of the generators of $\mathfrak{A}_J$. Then, the Zariski tangent space $T_0\big(\mathcal{M}f(J)\big)$ of the J-marked scheme at the origin can be naturally identified to the linear space of $\mathbb{A}^{|C|}$ defined as the set of zeros of $\mathfrak{L}(J)$.

We now prove the analogous of Theorem 4.12 for Gröbner strata in the case of marked families. Using at the same time several truncations of a saturated Borel-fixed ideal $\underline{J}$, we introduce the following notation:

- $s\mathcal{G}^{(m)}$ will denote the superminimal generators associated to $\underline{J}_{\geqslant m}$ and $\widetilde{C}^{(m)}$ the corresponding variables;
- $\widetilde{\mathfrak{A}}^{(m)}$ will denote the ideal defining the affine subscheme $\mathcal{M}f(\underline{J}_{\geqslant m})$ in the ring $\mathbb{K}[\widetilde{C}^{(m)}]$ (as in Theorem 4.58).

Theorem 4.60 (Cf. with Theorem 4.12)**.** *Let $\underline{J}$ be a saturated Borel-fixed ideal and let m be any integer. With the previous notations, the followings hold:*

(i) $\mathcal{M}f(\underline{J}_{\geqslant m-1})$ is a closed subscheme of $\mathcal{M}f(\underline{J}_{\geqslant m})$ cut out by a suitable linear space.

(ii) Let N be the number of monomials $x^\alpha \in G_{\underline{J}}$ of degree $m+1$ divisible by x_1 and $M = |G_{\underline{J}} \cap \mathbb{K}[x]_{\leqslant m-1}|$; then,

$$\dim_{\mathbb{K}} T_0\big(\mathcal{M}f(\underline{J}_{\geqslant m})\big) \geqslant \dim_{\mathbb{K}} T_0\big(\mathcal{M}f(\underline{J}_{\geqslant m-1})\big) + NM.$$

(iii) $\mathcal{M}f(\underline{J}_{\geqslant m-1}) \simeq \mathcal{M}f(\underline{J}_{\geqslant m})$ if and only if either $\underline{J}_{\geqslant m-1} = \underline{J}_{\geqslant m}$ or no monomial of degree $m+1$ in $G_{\underline{J}}$ is divisible by x_1.

In particular:

$$\mathcal{M}f(\underline{J}_{\rho-1}) \simeq \mathcal{M}f(\underline{J}_{\mathrm{reg}(\underline{J})-1}) \simeq \mathcal{M}f(\underline{J}_{\mathrm{reg}(\underline{J})}) \simeq \mathcal{M}f(\underline{J}_{r}), \tag{4.22}$$

where ρ is the maximal degree of monomials divisible by x_1 in $G_{\underline{J}}$ and r is the Gotzmann number of the Hilbert polynomial $p(t)$ of $\mathbb{K}[x]/\underline{J}$.

Moreover, $\mathcal{M}f(\underline{J}_{\mathrm{reg}(\underline{J})})$ can be embedded in an affine space of dimension $|G_{\underline{J}}| \cdot p\big(\mathrm{reg}(\underline{J})\big)$, and the same holds for $\mathcal{M}f(\underline{J}_{\geqslant m})$, for every $m \geqslant \mathrm{reg}(\underline{J})$.

Proof. (i) Thanks to Theorem 4.58, a marked scheme is defined by an ideal generated by polynomials of $\mathbb{K}[\widetilde{C}]$ that are constructed using only the superminimals. So, now it is enough to prove that the set of superminimals $s\mathcal{G}^{(m-1)}$ corresponds to $s\mathcal{G}^{(m)}$ modulo a subset of the variables $\widetilde{C}^{(m)}$, in the following sense.

Consider $x^\alpha \in sG_{\underline{J}_{\geqslant m-1}}$. If $|\alpha| \geqslant m$, then x^α belongs to $sG_{\underline{J}_{\geqslant m}}$ and we can identify $F_\alpha^{(m)} \in s\mathcal{G}^{(m)}$ and $F_\alpha^{(m-1)} \in s\mathcal{G}^{(m-1)}$ (and in particular the variables in their tails: $\widetilde{C}_{\alpha\gamma}^{(m)} = \widetilde{C}_{\alpha\gamma}^{(m-1)}$).

If $|\alpha| = m-1$, then we can consider the corresponding superminimal element $F_\beta^{(m)} \in s\mathcal{G}^{(m)}$, with $x^\beta = x_0 \cdot x^\alpha$. Then we identify the variable $\widetilde{C}_{\beta\delta'}^{(m)}$, which is the coefficient of a monomial in $\operatorname{Supp} F_\beta^{(m)}$ of kind $x^{\delta'} = x_0 \cdot x^\delta$, with the variable $\widetilde{C}_{\alpha\delta}^{(m-1)}$ which is the coefficient of the monomial x^δ in $\operatorname{Supp} F_\alpha^{(m-1)}$.

We repeat this identifications for all $x^\alpha \in sG_{\underline{J}_{\geqslant m-1}}$ and we denote by $\overline{C}^{(m)}$ the subset of $\widetilde{C}^{(m)}$ containing the variables non-identified with variables of $\widetilde{C}^{(m-1)}$, that is the variables appearing as coefficients of monomials not divisible by x_0 in the tails of polynomials in $s\mathcal{G}^{(m)} \setminus s\mathcal{G}^{(m-1)}$. Now, every polynomial in $s\mathcal{G}^{(m)} \mod (\overline{C}^{(m)})$ either belongs to $s\mathcal{G}^{(m-1)}$ or is a polynomials of $s\mathcal{G}^{(m-1)}$ multiplied by x_0. Thanks to Theorem 4.58, we have that

$$\widetilde{\mathfrak{A}}^{(m)} + \left(\overline{C}^{(m)}\right) \simeq \widetilde{\mathfrak{A}}^{(m-1)}.$$

(ii) We now consider $x^\gamma \in G_{\underline{J}}$, $|\gamma| = m+1$, x^γ divisible by x_1. We define $x^\beta = x^\gamma / x_1$; observe that $x^\beta \notin \underline{J}$. Furthermore, x^β is not divisible by x_0, otherwise x^γ would be too.

Then, for every $x^{\underline{\alpha}} \in G_{\underline{J}}$ with $|\underline{\alpha}| \leqslant m-1$, there is $F_\alpha = x^{\underline{\alpha}} x_0^{m-|\underline{\alpha}|} - \mathcal{T}(F_\alpha) \in s\mathcal{G}^{(m)}$ such that $x^\beta \in \operatorname{Supp} \mathcal{T}(F_\alpha)$. We focus on the coefficient $\widetilde{C}_{\alpha\beta}^{(m)}$ of x^β. Since x^β is not divisible by x_0, $\widetilde{C}_{\alpha\beta}^{(m)}$ cannot be identified with a coefficient appearing in $F_\alpha^{(m-1)} = x^{\underline{\alpha}} x_0^{m-|\underline{\alpha}|-1} - \mathcal{T}(F_\alpha^{(m-1)}) \in s\mathcal{G}^{(m-1)}$. So $\widetilde{C}_{\alpha\beta}^{(m)}$ belongs to the subset of variables $\overline{C}^{(m)}$ defined in the proof of *(i)*.

We now use the construction of $T_0\big(\mathcal{M}f(\underline{J}_{\geqslant m})\big)$ of Remark 4.5.2. If we think about syzygies of the ideal $\underline{J}_{\geqslant m}$, we can see that in any S-polynomial, $F_\alpha^{(m)}$ is multiplied by a monomial x^δ divisible by x_i, $i > 0$. In particular, $x^\delta \cdot x^\beta \in \underline{J}_{\geqslant m}$; indeed, if $x_i = x_1$ we are done by construction, otherwise we apply the Borel-fixed property because $\frac{x^\gamma}{x_i} \cdot x_1 \cdot x^\beta$ belongs to $\underline{J}$. This means that the coefficient $\widetilde{C}_{\alpha\beta}^{(m)}$ does not appear in any equation defining $T_0\big(\mathcal{M}f(\underline{J}_{\geqslant m})\big)$.

Applying this argument to the N monomials in $G_{\underline{J}}$ of degree $m+1$ which are divisible by x_1 and to the N monomials in $G_{\underline{J}}$ of degree $\leqslant m-1$, we obtain the result.

(iii) If $\underline{J}_{\geqslant m} = \underline{J}_{\geqslant m-1}$, obviously $\mathcal{M}f(\underline{J}_{\geqslant m}) = \mathcal{M}f(\underline{J}_{\geqslant m-1})$. We now assume that $\underline{J}_{\geqslant m} \neq \underline{J}_{\geqslant m-1}$ and no monomial of degree $m+1$ in the monomial basis of $\underline{J}$ is divisible by x_1; we prove that every polynomial in $s\mathcal{G}^{(m)}$ either belong to $s\mathcal{G}^{(m-1)}$ or it is the product of the "corresponding" polynomial in $s\mathcal{G}^{(m-1)}$ by x_0.

If $x^\alpha \in sG_{\underline{J}_{\geqslant m-1}}$ and $|\alpha| \geqslant m$, then $F_\alpha^{(m)} \in s\mathcal{G}^{(m)}$ and $F_\alpha^{(m-1)} \in s\mathcal{G}^{(m-1)}$ have the same shape and we can identify them letting $\widetilde{C}_{\alpha\gamma}^{(m)} = \widetilde{C}_{\alpha\gamma}^{(m-1)}$, as done in the proof of *(i)*. If $|\alpha| = m-1$, then $x^\beta = x_0 \cdot x^\alpha \in sG_{\underline{J}_{\geqslant m}}$ and all the monomials in the support of $x_0 \cdot F_\alpha^{(m-1)}$ appear in the support of $F_\beta^{(m)}$ (and we identify their coefficients as above). In the support of $F_\beta^{(m)}$ there are also some more monomials that are not divisible by x_0. We will prove now that the coefficients of these last monomials indeed belong to $\widetilde{\mathfrak{A}}^{(m)}$.

Consider the monomial $x_0 \cdot x_1 \cdot x^\alpha$. If we perform its reduction using $s\mathcal{G}^{(m)}$, the first step of reduction will lead to

$$x_0 \cdot x_1 \cdot x^\alpha \xrightarrow{s\mathcal{G}^{(m)}} x_1 \mathcal{T}(F_\beta^{(m)}).$$

Let x^γ be a monomial of $\operatorname{Supp} \mathcal{T}(F_\beta^{(m)})$. If $x_1 \cdot x^\gamma \in \underline{J}_{\geqslant m}$, then $x_1 \cdot x^\gamma = \langle x^{\underline{\alpha}'} | x^\eta \rangle^{\underline{J}}$, with $x^{\underline{\alpha}'} \in G_{\underline{J}}$ and $x^\eta <_{\mathtt{Lex}} x_1$. If $x^\eta = 1$, then $|\underline{\alpha}'| = m+1$ and $x^{\underline{\alpha}'}$ is divisible by x_1, against the hypothesis. Then $x^\eta = x_0^{\bar{t}}$, with $t > 0$, and so the monomial $x_1 \cdot x^\gamma \in \underline{J}_{\geqslant m}$ is actually divisible by x_0. If $x_1 \cdot x^\gamma \in \mathcal{N}(\underline{J}_{\geqslant m})$, then this monomial is not further reducible, so that its coefficient belongs to $\widetilde{\mathfrak{A}}^{(m)}$.

Vice versa, by absurd suppose now that $\underline{J}_{\geqslant m-1} \neq \underline{J}_{\geqslant m}$ and that there exists $x^\alpha \in G_{\underline{J}}$ divisible by x_1, $|\alpha| = m+1$. Using *(ii)*, we have that $T_0\big(\mathcal{M}f(\underline{J}_{\geqslant m-1})\big) \not\simeq T_0\big(\mathcal{M}f(\underline{J}_{\geqslant m})\big)$ because $\dim_{\mathbb{K}} T_0\big(\mathcal{M}f(\underline{J}_{\geqslant m-1})\big) < \dim_{\mathbb{K}} T_0\big(\mathcal{M}f(\underline{J}_{\geqslant m})\big)$, and so $\mathcal{M}f(\underline{J}_{\geqslant m-1}) \not\simeq \mathcal{M}f(\underline{J}_{\geqslant m})$.

For the last part of the statement, note that if ρ is the maximal degree of a monomial divisible by x_1 in the monomial basis of $\underline{J}$, for every $m \geqslant \rho$, applying iteratively *(iii)* we obtain

$$\mathcal{M}f(\underline{J}_{\geqslant \rho-1}) \simeq \mathcal{M}f(\underline{J}_{\geqslant m}).$$

If especially $m \geqslant \operatorname{reg}(\underline{J})$, the Hilbert function and the Hilbert polynomial $p(t)$ of $\mathbb{K}[x]/\underline{J}$ surely coincide, hence $\mathcal{M}f(\underline{J}_{\geqslant m})$ can be embedded in an affine space of dimension

$$|\widetilde{C}^{(m)}| = \sum_{x^\alpha \in sG_{\underline{J}_{\geqslant m}}} p(|\alpha|)$$

and in this case every monomial in $sG_{\underline{J}_{\geqslant m}}$ has degree m. □

Remark 4.5.3. By Theorem 4.60, we can embed $\mathcal{M}f(\underline{J}_{\geqslant m})$ in an affine space of dimension $|\widetilde{C}^{(m)}|$, for every $m \geqslant \rho - 1$. Anyway, we cannot always compute the dimension of this affine space using the Hilbert polynomial $p(t)$, since m may be strictly lower than the regularity of the Hilbert function. For $m < \operatorname{reg}(\underline{J})$, we have that

$$|\widetilde{C}^{(m)}| = \sum_{x^\alpha \in sG_{\underline{J}_{\geqslant m}}} |\mathcal{N}(\underline{J}_{\geqslant m})_{|\alpha|}|.$$

4.5.1 The pseudocode description of the algorithm

We will now expose the pseudocode of the algorithm for computing the equations defining the affine scheme that describes a J-marked family $\mathcal{M}f(J)$ for a truncated Borel-fixed ideal $J = \underline{J}_{\geqslant m}$, mainly based on Theorem 4.49.

MINIMALGENERATORS(J)

Input: J, a truncation of a saturated Borel-fixed ideal, i.e. $J = \underline{J}_{\geqslant m}$ for some m.

Output: the set G_J of minimal generators of J.

SUPERMINIMALGENERATORS(J)

Input: J, a truncation of a saturated Borel-fixed ideal, i.e. $J = \underline{J}_{\geqslant m}$ for some m.

Output: the set sG_J of superminimal generators of J.

REDUCE(H, nfs)

Input: H, a polynomial in $\mathbb{K}[C][x]$;

Input: nfs, a set of marked polynomials $\mathrm{Ht}(F_\alpha) - \mathcal{T}(F_\alpha)$ of $\mathbb{K}[C][x]$ such that $\mathrm{Ht}(F_\alpha) \in \mathbb{K}[x]$ and $\mathcal{T}(F_\alpha) = \mathrm{Nf}(\mathrm{Ht}(F_\alpha))$.

Output: the polynomial $\overline{H}$ computed by replacing each monomial $x^\delta \in \operatorname{Supp} H$ with $x^\eta \mathcal{T}(F_\alpha)$ with $F_\alpha \in$ nfs.

SUPERMINIMALREDUCTION$(H, s\mathcal{G})$
Input: H, a polynomial in $\mathbb{K}[C][x]$;
Input: $s\mathcal{G}$, the set of superminimal generators for some marked family.
Output: $\overline{H}$ such that there exists t for which $x_0^t H \xrightarrow{s\mathcal{G}} \overline{H}$.

SUPERMINIMALSYZYGIES(J)
Input: J, a truncation of a saturated Borel-fixed ideal, i.e. $J = \underline{J}_{\geqslant m}$ for some m.
Output: the set of syzygies between pairs of superminimal generators of J corresponding to the set L_1 of Theorem 4.49.

COEFF(H, x^β)
Input: H, a polynomial in $\mathbb{K}[C][x]$.
Input: x^β, a monomial in $\mathbb{K}[x]$.
Output: the coefficient of the monomial x^β in H (obviously 0 if $x^\beta \notin \operatorname{Supp} H$).

Algorithm 4.1: Auxiliary methods for the algorithm computing the affine scheme that describes a marked family.

MARKEDFAMILY(J)
Input: J, a truncation of a saturated Borel-fixed ideal, i.e. $J = \underline{J}_{\geqslant m}$ for some m.
Output: the ideal of the affine scheme describing $\mathcal{M}f(J)$.
$sG_J \leftarrow$ SUPERMINIMALGENERATORS(J);
$s\mathcal{G} \leftarrow \emptyset$;
for all $x^\alpha \in sG_J$ **do**
  $F_\alpha \leftarrow x^\alpha$;
  for all $x^\beta \in \mathcal{N}(J)_{|\alpha|}$ **do**
    $F_\alpha \leftarrow F_\alpha + C_{\alpha\beta} x^\beta$;
  end for
  $s\mathcal{G} \leftarrow s\mathcal{G} \cup \{F_\alpha\}$;
end for
$G \leftarrow$ MINIMALGENERATORS$(J) \setminus sG_J$;
knownNF $\leftarrow s\mathcal{G}$;

equations $\leftarrow \emptyset$;
while $G \neq \emptyset$ **do**
  $x^{\alpha} \leftarrow \min_{\texttt{DegRevLex}} G$;
  $x_i \leftarrow \min_{j>0}\{x_j \mid x^{\alpha}\}$;
  $x^{\gamma} \leftarrow \frac{x_0}{x_i} x^{\alpha}$; // This is a sygygy of the set L_2 of Theorem 4.49
  $H \leftarrow \textsc{Reduce}\big(x_i \mathrm{Nf}(x^{\gamma}), \textsf{knownNF}\big)$;
  $(Q, R) \leftarrow$ polynomials such that $H = Q \cdot x_0 + R$;
  for all $x^{\delta} \in \operatorname{Supp} R$ **do**
    equations $\leftarrow$ equations $\cup \{\textsc{Coeff}(R, x^{\delta})\}$; // We are imposing $R = 0$
  end for
  knownNF $\leftarrow$ knownNF $\cup \{x^{\alpha} - Q\}$; // Because $H = \mathrm{Nf}(x_0 x^{\gamma}) = x_0 \mathrm{Nf}(x^{\gamma})$
  $G \leftarrow G \setminus \{x^{\alpha}\}$;
end while
syzygies $\leftarrow \textsc{SuperminimalSyzygies}(J)$;
for all $x^{\gamma}\mathbf{e}_{\alpha} - x^{\gamma'}\mathbf{e}_{\alpha'} \in$ **syzygies do**
  $S(F_{\alpha}, F_{\alpha'}) \leftarrow x^{\gamma} F_{\alpha} - x^{\gamma'} F_{\alpha'}$;
  $H \leftarrow \textsc{SuperminimalReduction}\big(S(F_{\alpha}, F_{\alpha'}), s\mathcal{G}\big)$;
  for all $x^{\delta} \in \operatorname{Supp} H$ **do**
    equations $\leftarrow$ equations $\cup \{\textsc{Coeff}(H, x^{\delta})\}$;
  end for
end for
return $\big\langle \text{EQUATIONS} \big\rangle$;

Algorithm 4.2: The algorithm for computing the affine scheme that describe a marked family.

It is very useful also to have a method that computes the dimension of the tangent space at the origin of a marked family (i.e. the number of monomials in the tails of the superminimal generators) avoiding to generate the complete equations of the ideal of the family. For instance if we know a lower bound of the dimension of the tangent space (determined by geometric arguments or for any other reason) and the marked family realizes this bound, we can conclude directly that the marked family is an affine space and that there are no relations among the variables C.

TANGENTSPACEDIMENSION(J)
Input: J, a truncation of a saturated Borel-fixed ideal, i.e. $J = \underline{J}_{\geqslant m}$ for some m.
Output: the dimension of the tangent space at the origin of $\mathcal{M}f(J)$.
$sG_J \leftarrow$ SUPERMINIMALGENERATORS(J);
$N \leftarrow 0$;
for all $x^\alpha \in sG_J$ **do**
  $N \leftarrow N + \left|\mathcal{N}(J)_{|\alpha|}\right|$;
end for
return N;

Algorithm 4.3: The algorithm computing the dimension of the tangent space at the origin of a marked family.

GRÖBNERSTRATUM(J, σ)
Input: J, a truncation of a saturated Borel-fixed ideal, i.e. $J = \underline{J}_{\geqslant m}$ for some m.
Input: σ, a term ordering such that J is a gen-segment ideal w.r.t. σ.
Output: the ideal of the affine scheme describing $\mathcal{S}t_\sigma(J)$.
ideal $\leftarrow$ MARKEDFAMILY(J);
equations $\leftarrow$ generators of ideal;
for all $f \in$ equations **do**
  if $f \notin (C)^2$ **then**
    $C_{\alpha\beta} = \max_\ell \operatorname{Supp} f$;   // ℓ is the positive grading induced on C by σ (Definition 4.6)
    $\phi_{\alpha\beta} = \frac{f}{\text{COEFF}(C_{\alpha\beta}, f)} + C_{\alpha\beta}$;
    ideal $\leftarrow$ SUBSTITUTE$(C_{\alpha\beta} \leftarrow \phi_{\alpha\beta}$, ideal$)$;
    ideal $\leftarrow$ SUBSTITUTE$(C_{\alpha\beta} \leftarrow \phi_{\alpha\beta}$, equations$)$;
  end if
end for
return ideal;

Algorithm 4.4: The algorithm computing the Gröbner stratum of a gen-segment ideal.

We can start from the same algorithm also for computing the Gröbner stratum of a gen-segment ideal, indeed by Theorem 4.12*(i)* we know that also for Gröbner strata the relevant variables C are those in the tails of superminimal generators and in the case of a segment ideal J the tail of a monomial $x^\alpha \in J$ contains all the monomials of $\mathcal{N}(J)_{|\alpha|}$. After having computed the equations of the marked family, we can exploit the term ordering for further eliminated other variables $C_{\alpha\beta}$ appearing in degree 1 in any of the equation of the ideal of the scheme defining the marked family (Algorithm 4.4).

Always starting from Algorithm 4.2, we can determine the procedure computing the tangent space dimension at the origin of a Gröbner stratum, i.e. the its embedding dimension (Algorithm 4.5).

EMBEDDINGDIMENSION(J, σ)
Input: J, a truncation of a saturated Borel-fixed ideal, i.e. $J = \underline{J}_{\geqslant m}$ for some m.
Input: σ, a term ordering such that J is a gen-segment ideal w.r.t. σ.
Output: the embedding dimension of $\mathcal{S}t_\sigma(J)$.
$sG_J \leftarrow$ SUPERMINIMALGENERATORS(J);
$s\mathcal{G} \leftarrow \emptyset$;
$N \leftarrow 0$;
for all $x^\alpha \in sG_J$ **do**
    $N \leftarrow N + \left|\mathcal{N}(J)_{|\alpha|}\right|$;
    $F_\alpha \leftarrow x^\alpha$;
    for all $x^\beta \in \mathcal{N}(J)_{|\alpha|}$ **do**
        $F_\alpha \leftarrow F_\alpha + C_{\alpha\beta}x^\beta$;
    end for
    $s\mathcal{G} \leftarrow s\mathcal{G} \cup \{F_\alpha\}$;
end for
$G \leftarrow$ MINIMALGENERATORS$(J) \setminus sG_J$;
linearEquations $\leftarrow \emptyset$;

while $G \neq \emptyset$ **do**
  $x^\alpha \leftarrow \min_{\texttt{DegRevLex}} G$;
  $x_i \leftarrow \min_{j>0}\{x_j \mid x^\alpha\}$;
  $x^\gamma \leftarrow \frac{x_0}{x_i} x^\alpha$;
  $H \leftarrow \textsc{Reduce}\big(x_i \mathrm{Nf}(x^\gamma), G_J\big)$; // We delete all the monomials in J
  $(Q, R) \leftarrow$ polynomials such that $H = Q \cdot x_0 + R$;
  for all $x^\delta \in \operatorname{Supp} R$ **do**
    equations $\leftarrow$ equations $\cup \{\textsc{Coeff}(R, x^\delta)\}$;
  end for
  $G \leftarrow G \setminus \{x^\alpha\}$;
end while
syzygies $\leftarrow \textsc{SuperminimalSyzygies}(J)$;
for all $x^\gamma \mathbf{e}_\alpha - x^{\gamma'} \mathbf{e}_{\alpha'} \in$ syzygies **do**
  $S(F_\alpha, F_{\alpha'}) \leftarrow x^\gamma F_\alpha - x^{\gamma'} F_{\alpha'}$;
  $H \leftarrow \textsc{Reduce}\big(S(F_\alpha, F_{\alpha'}), G_J\big)$;
  for all $x^\delta \in \operatorname{Supp} H$ **do**
    equations $\leftarrow$ equations $\cup \{\textsc{Coeff}(H, x^\delta)\}$;
  end for
end for
return $N - \dim_{\mathbb{K}} \langle$equation$\rangle$;

Algorithm 4.5: The algorithm computing the embedding dimension of a Gröbner stratum.

Example 4.5.4. Let us consider the saturated Borel-fixed ideals in $\mathbb{K}[x_0, x_1, x_2, x_3]$ with Hilbert polynomial $p(t) = 4t$:

$$\begin{aligned} J_1 &= (x_3, x_2^5, x_2^4 x_1^2),\\ J_2 &= (x_3^2, x_3x_2, x_3x_1, x_2^5, x_2^4x_1),\\ J_3 &= (x_3^2, x_3x_2, x_3x_1^2, x_2^4),\\ J_4 &= (x_3^2, x_3x_2, x_2^3). \end{aligned}$$

They are all hilb-segment ideals and by Proposition 2.64 any truncation will be a gen-segment ideal, so that any marked family coincides with the Gröbner stratum w.r.t. the term ordering making any ideal a gen-segment ideal. We now discuss how the computational complexity of the costruction of such families of ideals decreases applying the results introduced up to this point.

J_1. This is the lexicographic ideal associated to $p(t) = 4t$. As seen in Example 4.1.5, there are only two possible marked family structures:

$$\mathcal{M}f(J_1) \simeq \mathcal{M}f\left((J_1)_{\geqslant m}\right),\ m = 2,3,4, \qquad \mathcal{M}f\left((J_1)_{\geqslant 5}\right) \simeq \mathcal{M}f\left((J_1)_{\geqslant m}\right),\ m > 5.$$

Applying Algorithm 4.3 we have that

$$\dim_{\mathbb{K}} T_0\left(\mathcal{M}f(J_1)\right) = 47 \qquad \text{and} \qquad \dim_{\mathbb{K}} T_0\left(\mathcal{M}f((J_1)_{\geqslant 5})\right) = 64.$$

Furthermore, if we use the homogeneous positive grading induced by the term ordering `DegLex`, we obtain that both $\mathcal{M}f(J_1) \simeq \mathcal{S}t_{\texttt{DegLex}}(J_1)$ and $\mathcal{M}f((J_1)_{\geqslant 5}) \simeq \mathcal{S}t_{\texttt{DegLex}}((J_1)_{\geqslant 5})$ are affine spaces and

$$\operatorname{ed} \mathcal{S}t_{\texttt{DegLex}}(J_1) = 21 \qquad \text{and} \qquad \operatorname{ed} \mathcal{S}t_{\texttt{DegLex}}((J_1)_{\geqslant 5}) = 23.$$

J_2. Again there are two possible marked scheme structure up to isomorphism

$$\begin{aligned}&\mathcal{M}f(J_2) \simeq \mathcal{M}f\left((J_2)_{\geqslant 3}\right), && \dim_{\mathbb{K}} T_0\left(\mathcal{M}f(J_2)\right) = 61,\\ &\mathcal{M}f\left((J_2)_{\geqslant 4}\right) \simeq \mathcal{M}f\left((J_2)_{\geqslant m}\right),\ \forall\, m > 4, && \dim_{\mathbb{K}} T_0\left(\mathcal{M}f((J_2)_{\geqslant 4})\right) = 88.\end{aligned}$$

By computing the Gröbner strata w.r.t. $\omega_2 = (9,3,2,1)$, we find that the variables we need to describe these families of ideals are

$$\operatorname{ed} \mathcal{S}t_{\omega_2}(J_2) = 24 \qquad \text{and} \qquad \operatorname{ed} \mathcal{S}t_{\omega_2}((J_2)_{\geqslant 4}) = 27.$$

J_3. In this case $\rho = 3$, thus the marked families of the truncations of J_3 are all isomorphic to $\mathcal{M}f(J_3) \simeq \mathcal{S}t_{\omega_3}(J_3)$, where $\omega_3 = (7,3,2,1)$:

$$\mathcal{M}f(J_3) \simeq \mathcal{M}f\left((J_3)_{\geqslant m}\right),\ \forall\, m, \quad \dim_{\mathbb{K}} T_0\left(\mathcal{M}f(J_3)\right) = 44, \quad \operatorname{ed} \mathcal{S}t_{\omega_3}(J_3) = 24.$$

J_4. In this case the ideal is ACM, so x_1 do not appear in any generator, so we can consider again the saturated ideal itself.

$$\mathcal{M}f(J_4) \simeq \mathcal{M}f\left((J_4)_{\geqslant m}\right),\ \forall\, m, \quad \dim_{\mathbb{K}} T_0\left(\mathcal{M}f(J_4)\right) = 28, \quad \operatorname{ed} \mathcal{S}t_{\omega_4}(J_4) = 16,$$

where $\omega_4 = (6,4,2,1)$.

4.6 Open subsets of the Hilbert scheme II

In this final section of the chapter we will use marked families to cover the Hilbert scheme. The results we will expose belong to the submitted paper "Borel open covering of Hilbert schemes" [13] written in collaboration with C. Bertone and M. Roggero.

We consider again the embedding of the Hilbert scheme in a suitable projective space through the Plücker embedding $\mathbf{Hilb}^n_{p(t)} \subset \mathbf{Gr}_{\mathbb{K}}(q(r), N(r)) \hookrightarrow \mathbb{P}^{\binom{N(r)}{p(r)}-1}$ and we will use the same notation introduced in Section 4.2. As we seen, each open affine subset $\mathcal{U}_J$ of the Grassmannian and the corresponding open subset $\mathcal{H}_J = \mathcal{U}_J \cap \mathbf{Hilb}^n_{p(t)}$ of the Hilbert scheme can be uniquely identified with a monomial ideal of $\mathbb{K}[x]$ generated by $q(r)$ monomials of degree r that fixes the Plücker coordinate $\Delta_J \neq 0$. We will denote this set of ideals with $\mathcal{M}^n$, with $\mathcal{B}^n$ its subset composed by Borel-fixed ideals and with $\mathcal{B}^n_{p(t)}$ the subset of Borel-fixed ideals with Hilbert polynomial $p(t)$, i.e.

$$\mathcal{B}^n_{p(t)} \subset \mathcal{B}^n \subset \mathcal{M}^n. \tag{4.23}$$

In the following results, we state some close relation between open subsets of Grassmannians and properties of the ideals that correspond to the points of $\mathbf{Gr}_{\mathbb{K}}(q(r), N(r))$.

Lemma 4.61. *Let J and I be ideals in $\mathbf{Gr}_{\mathbb{K}}(q(r), N(r))$, with $J \in \mathcal{M}^n$ and G_J its monomial basis. Then the following statements are equivalent:*

(i) $\Delta_J(I) \neq 0$;

(ii) *I_r can be represented by a matrix $\mathfrak{M}(I_r)$ of the form $\left(\begin{array}{c|c} \mathrm{Id} & R \end{array}\right)$, where the left block is the $q(r) \times q(r)$ identity matrix and corresponds to the monomials in G_J and the entries of the right block R are constants $-c_{\alpha\beta}$, where $x^\alpha \in G_J$ and $x^\beta \in \mathcal{N}(J)_r$;*

(iii) *I is generated by a J-marked set:*

$$G = \left\{ f_\alpha = x^\alpha - \sum c_{\alpha\beta} x^\beta \mid \mathrm{Ht}(f_\alpha) = x^\alpha \in G_J \right\}. \tag{4.24}$$

If the previous conditions hold and moreover J' is another monomial ideal in $\mathcal{M}^n$, then:

(iv) $\Delta_{J'}(I)/\Delta_J(I)$ *can be expressed as a polynomial in the* $c_{\alpha\beta}$*'s of degree* $|G_J \setminus (G_J \cap G_{J'})|$*;*

(v) especially, if $G_{J'} = G_J \setminus \{x^\alpha\} \cup \{x^\beta\}$*, then (up to the sign)* $\Delta_{J'}(I)/\Delta_J(I) = c_{\alpha\beta}$*;*

(vi) we can fix an isomorphism $\mathbb{A}^{p(r)q(r)} \simeq \mathcal{U}_J$ *such that the constants* $c_{\alpha\beta}$ *are the coordinates of* I *in* $\mathbb{A}^{p(r)q(r)}$*.*

Proof. *(i)* $\Rightarrow$ *(ii)* It suffices to multiply any matrix $\mathfrak{M}(I_r)$ by the inverse of its submatrix corresponding to the columns fixed by J, since its determinant is $\Delta_J(I) \neq 0$.

(ii) $\Rightarrow$ *(i)* is obvious.

(ii) $\Rightarrow$ *(iii)* The generators of I given by the rows of $\mathfrak{M}(I_r)$ are indeed a J-marked set and, vice versa, the matrix containing the coefficients of the polynomials f_α has precisely the shape required in *(ii)*.

Finally *(iv)*, *(v)* and *(vi)* are easy consequences of *(ii)*. □

As before we will denote by $\mathcal{G}$ the J-marked set:

$$\mathcal{G} = \left\{ F_\alpha = x^\alpha - \sum C_{\alpha\beta} x^\beta \mid \mathrm{Ht}(F_\alpha) = x^\alpha \in G_J, x^\beta \in \mathcal{N}(J) \right\} \tag{4.25}$$

and by $(\mathcal{G})$ the ideal generated in the ring $\mathbb{K}[C][x]$, where C is as usual the compact notation for the set of new variables $C_{\alpha\beta}$, $x^\alpha \in G_J$, $x^\beta \in \mathcal{N}(J)_r$.

Corollary 4.62. *In the hypothesis of Lemma 4.61,* $\mathcal{U}_J$ *is isomorphic to the affine space* $\mathbb{A}^{p(r)q(r)} = \operatorname{Spec} \mathbb{K}[C]$*. The (closed) points in* $\mathcal{U}_J$ *correspond to all ideals that we obtain from* $(\mathcal{G})$ *specializing the variables* $C_{\alpha\beta}$ *to* $c_{\alpha\beta} \in \mathbb{K}$*.*

Remark 4.6.1. If $J \in \mathcal{M}^n$, then the open subset $\mathcal{U}_J$ is a parameter space for the set of J-marked sets. However it is not in general isomorphic to the J-marked scheme $\mathcal{M}f(J)$ because, for instance, the Hilbert polynomial is not necessarily constant on $\mathcal{U}_J$ (see [22, Example 1.10]).

Remark 4.6.2. Let J, J' be any couple of monomial ideals in $\mathcal{M}^n$ and let $\delta = |G_J \setminus G_{J'}|$. The localization of $\Delta_{J'}$ in $\mathbb{K}[C]$, the coordinate ring of $\mathbb{A}^{p(r)q(r)} \simeq \mathcal{U}_J$, gives a polynomial of degree δ as shown in Lemma 4.61*(iv)*. In the "worst" case, if we consider J' such that $G_J \cap G_{J'} = \emptyset$, then $\delta = q(r)$.

As J varies in $\mathcal{M}^n$, the open sets $\mathcal{U}_J$ cover the Grassmannian, and so the subsets $\mathcal{H}_J = \mathcal{U}_J \cap \mathbf{Hilb}^n_{p(t)}$, $J \in \mathcal{M}^n$ give an open covering of $\mathbf{Hilb}^n_{p(t)}$ by affine subschemes. It is quite obvious that every open subset $\mathcal{U}_J$ is non-empty, because it contains the point corresponding to J itself. If J has Hilbert polynomial $p(t)$ also $\mathcal{H}_J$ is non-empty because it contains J itself, but when $\operatorname{Proj} \mathbb{K}[x]/J \notin \mathcal{H}\mathrm{ilb}^n_{p(t)}(\mathbb{K})$, it is not easy to understand general properties of $\mathcal{H}_J$ or even to decide if it is empty or not. For this reason we prefer to consider a slightly different open covering for $\mathbf{Gr}_{\mathbb{K}}(q(r), N(r))$ and $\mathbf{Hilb}^n_{p(t)}$, obtained considering only the set of Borel-fixed ideals $\mathcal{B}^n \subset \mathcal{M}^n$, that we will prove to be more convenient for our purposes.

Definition 4.63. Given the Grassmannian $\mathbf{Gr}_{\mathbb{K}}(q(r), N(r))$ and the set $\mathcal{B}^n$ of all the Borel-fixed ideals $J \subset \mathbb{K}[x]$ such that $\dim_{\mathbb{K}} J_r = q(r)$, we call *Borel region* of $\mathbf{Gr}_{\mathbb{K}}(q(r), N(r))$ the union

$$\mathcal{U} = \bigcup_{J \in \mathcal{B}^n} \mathcal{U}_J. \tag{4.26}$$

Another key role will be played by the linear group $\mathrm{GL}_{\mathbb{K}}(n+1)$ and its induced action on the Grassmannian $\mathbf{Gr}_{\mathbb{K}}(q(r), N(r))$. The action of an element of $\mathrm{GL}(n+1)$ on $\mathbb{P}^n$ corresponds to a different choice of the basis for $\mathbb{K}[x]_1$ and therefore to a different choice of the basis for $\mathbb{K}[x]_r$. So $g \in \mathrm{GL}(n+1)$ induces a linear change of Plücker coordinates in the projective space $\mathbb{P}^{\binom{N(r)}{p(r)}-1}$ in which $\mathbf{Gr}_{\mathbb{K}}(q(r), N(r))$ is embedded. Note that not all the linear changes of Plücker coordinates can be obtained by the action of some element of $\mathrm{GL}(n+1)$ on $\mathbb{P}^n$.

Lemma 4.64. *The action of* $\mathrm{GL}(n+1)$ *on the Borel region* $\mathcal{U}$ *gives an open covering of* $\mathbf{Gr}_{\mathbb{K}}(q(r), N(r))$*, that is:*

$$\bigcup_{g \in \mathrm{GL}(n+1)} g \centerdot \mathcal{U} = \bigcup_{\substack{J \in \mathcal{B}^n \\ g \in \mathrm{GL}(n+1)}} g \centerdot \mathcal{U}_J = \mathbf{Gr}_{\mathbb{K}}(q(r), N(r)). \tag{4.27}$$

Proof. Let $I \in \mathbf{Gr}_{\mathbb{K}}(q(r), N(r))$ be any ideal and let σ be any term order on the monomials of $\mathbb{K}[x]$. Due to Galligo's Theorem [33], in generic coordinates the initial ideal J' of I is Borel-fixed, and then $J = (J'_r)$ is Borel-fixed too. Moreover, by construction J is generated by $q(r)$ monomials of degree r and so $J \in \mathcal{B}^n$. Hence for a general $g \in \mathrm{GL}(n+1)$ we have $\Delta_J(g \centerdot I) \neq 0$ that is $g \centerdot I \in \mathcal{U}_J$, so that $I \in g^{-1} \centerdot \mathcal{U}_J$. $\square$

Remark 4.6.3. In the proof of Lemma 4.64 we deal with the generic initial ideal J', which may have minimal generators of degree $> r$. To avoid this problem, we consider J'_r, which is Borel-fixed and is generated by $q(r)$ monomials in degree r.

The new covering of the Grassmannian $\mathbf{Gr}_{\mathbb{K}}(q(r), N(r))$ shown in Lemma 4.64 will turn out to be more suitable to study local properties of Hilbert schemes.

Definition 4.65. Given the Grassmannian $\mathbf{Gr}_{\mathbb{K}}(q(r), N(r))$, we define the *Borel covering* of $\mathbf{Gr}_{\mathbb{K}}(q(r), N(r))$ as the family of all open subsets of the type $g \centerdot \mathcal{U}_J$ where $J \in \mathcal{B}^n$ and $g \in \mathrm{GL}(n+1)$.

What happens to the restriction to the Hilbert scheme of the Borel covering of the Grassmannian? First we investigate whether $\mathcal{H}_J = \mathcal{U}_J \cap \mathbf{Hilb}^n_{p(t)}$ is empty or not, for $J \in \mathcal{B}^n$. Of course if J belongs to $\mathbf{Hilb}^n_{p(t)}$, then $\mathcal{H}_J$ cannot be empty because it contains at least J. Moreover, since it is an open subset of $\mathbf{Hilb}^n_{p(t)}$, if it contains a point, it also contains an open subset of at least one irreducible component of $\mathbf{Hilb}^n_{p(t)}$.

Proposition 4.66. *If $J \in \mathcal{B}^n$, then:*

$$\mathcal{H}_J \neq \emptyset \quad \Longleftrightarrow \quad \operatorname{Proj} \mathbb{K}[x]/J \in \mathcal{H}\mathrm{ilb}^n_{p(t)}(\mathbb{K}). \tag{4.28}$$

As a consequence, if we define the Borel region $\mathcal{H}$ *of* $\mathbf{Hilb}^n_{p(t)}$ *as* $\mathcal{H} = \mathcal{U} \cap \mathbf{Hilb}^n_{p(t)} = \cup_{J \in B^n_{p(t)}} \mathcal{H}_J$, *we get:*

$$\mathbf{Hilb}^n_{p(t)} = \bigcup_{\substack{g \in \mathrm{GL}(n+1) \\ J \in \mathcal{B}^n_{p(t)}}} g \centerdot \mathcal{H}_J = \bigcup_{g \in \mathrm{GL}(n+1)} g \centerdot \mathcal{H}.$$

Proof. We prove only the non-trivial part ($\Rightarrow$) of the first statement. Assume that $\operatorname{Proj} \mathbb{K}[x]/J \notin \mathcal{H}\mathrm{ilb}^n_{p(t)}(\mathbb{K})$. By Gotzmann's Persistence Theorem, this is equivalent to $\dim_{\mathbb{K}} J_{r+1} > q(r+1)$. If I is any ideal in $\mathcal{U}_J$, then it has a set of generators as those given in Lemma 4.61*(iii)*, so that $\dim_{\mathbb{K}} I_{r+1} \geqslant \dim_{\mathbb{K}} J_{r+1} > q(r+1)$ (see [22, Corollary 2.3]). Hence $\operatorname{Proj} \mathbb{K}[x]/I \notin \mathcal{H}\mathrm{ilb}^n_{p(t)}(\mathbb{K})$. The other statement is a direct consequence of the first one and of Lemma 4.64. □

We point out that in Proposition 4.66, the hypothesis $J \in \mathcal{B}^n$ is necessary, as shown in the following example.

Example 4.6.4 (Cf. Example 4.2.2)**.** Let us consider again the Hilbert scheme $\mathbf{Hilb}^2_2$: in this case $r = 2$ and $q(2) = 4$. The monomial ideal $J = (x_2^2, x_2x_1, x_1^2, x_0^2)$ is generated by 4 monomials of degree 2, it is not Borel-fixed and obviously does not belong to $\mathbf{Hilb}^2_2$ because it is a primary ideal over the irrelevant maximal ideal (x_2, x_1, x_0). Nevertheless, $\mathcal{H}_J$ contains the ideal $(x_2^2 - x_2x_0, x_2x_1, x_1^2 - x_1x_0, x_0^2 - x_1x_0 - x_2x_0)$ corresponding to the set of points $\{[1 : 0 : 1], [1 : 1 : 0]\}$ and, more generally, all the ideals corresponding to pair of distinct points outside the line $x_0 = 0$ and not on the same line through $[1 : 0 : 0]$.

Corollary 4.67. *Set-theoretically we have that:*

$$\mathbf{Hilb}^n_{p(t)} \subseteq \bigcap_{\substack{g\in \mathrm{GL}(n+1)\\ J\in\mathcal{B}^n\setminus\mathcal{B}^n_{p(t)}}} g \centerdot \Pi_J \tag{4.29}$$

where Π_J is the hyperplane in $\mathbb{P}^{\binom{N(r)}{p(r)}-1}$ given by $\Delta_J = 0$.

Example 4.6.5. Let us consider the Hilbert polynomial $p(t) = 3t$ in $\mathbb{P}^3$. The closed points of $\mathbf{Hilb}^3_{3t}$ corresponds to curves in $\mathbb{P}^3$ of degree 3 and arithmetic genus 1, hence it contains all the smooth plane elliptic curves and also some singular or reducible or non-reduced curve. The Gotzmann number of $p(t) = 3t$ is $r = 3$ and so $q(3) = 11$ and $\binom{20}{11} - 1 = 167959$.

The only Borel-fixed ideal defining points on $\mathbf{Hilb}^3_{3t}$ is the lexicographic ideal:

$$L = \underline{L}_{\geqslant 3} = (x_3, x_2^3)_{\geqslant 3}.$$

The Borel region of $\mathbf{Hilb}^3_{3t}$ is then equal to the open subset $\mathcal{U}_L \cap \mathbf{Hilb}^3_{3t}$. The Grassmannian $\mathbf{Gr}_{\mathbb{K}}(11, 20)$ in which $\mathbf{Hilb}^3_{3t}$ is embedded has dimension $q(3) \cdot p(3) = 99$. Using Algorithm 2.4, we compute the complete list of Borel-fixed ideals in $\mathbf{Gr}_{\mathbb{K}}(11, 20)$ that do not belong to $\mathbf{Hilb}^3_{3t}$:

$$\begin{array}{rl}
2t+3 & J_1 = (x_3^2, x_3x_2, x_3x_1, x_2^3, x_2^2x_1)_{\geqslant 3},\\
 & J_2 = (x_3^2, x_3x_2, x_2^2, x_3x_1^2)_{\geqslant 3},\\
t+6 & J_3 = (x_3^2, x_3x_2, x_2^3, x_2^2x_1, x_3x_1^2, x_2x_1^2)_{\geqslant 3},\\
9 & J_4 = (x_3^2, x_3x_2^2, x_2^3, x_3x_2x_1, x_2^2x_1, x_3x_1^2, x_2x_1^2, x_1^3)_{\geqslant 3}
\end{array}$$

Then $\mathbf{Hilb}^3_{3t}$ as a subscheme of $\mathbf{Gr}_{\mathbb{K}}(11,20) \hookrightarrow \mathbb{P}^{167959}$ is contained set-theoretically in the intersection of the hyperplanes Π_{J_i} given by $\Delta_{J_i} = 0$, $i = 1,2,3,4$ (and in all the hyperplanes obtained from these by the action of $\mathrm{GL}(4)$).

Definition 4.68. The *Borel covering* of $\mathbf{Hilb}^n_{p(t)}$ will be the family of all the open subsets of $g \,.\, \mathcal{H}_J$ where $J \in \mathcal{B}^n_{p(t)}$ and $g \in \mathrm{GL}(n+1)$.

For any Borel ideal J in $\mathcal{B}^n_{p(t)}$, the open subset $\mathcal{H}_J = \mathcal{U}_J \cap \mathbf{Hilb}^n_{p(t)}$ will be called the *J-marked region* of $\mathbf{Hilb}^n_{p(t)}$.

The name "J-marked region" comes from Lemma 4.61 and its connection with the J-marked scheme will be clearer with Theorem 4.70.

Remark 4.6.6.

1. If the Hilbert polynomial $p(t)$ is the constant r, then every Borel ideal $J \in \mathbf{Gr}_{\mathbb{K}}(q(r), N(r))$ belongs to $\mathbf{Hilb}^n_r$ i.e. $\mathcal{B}^n = \mathcal{B}^n_{p(t)}$. Then in the zero-dimensional case the family of hyperplanes Π_J considered in Proposition 4.66 is indeed empty.

2. If $\deg p(t) = d \geqslant 1$, $\mathcal{B}^n \setminus \mathcal{B}^n_{p(t)}$ in general is not empty and its elements define subschemes of $\mathbb{P}^n$ of dimension equal to or lower than the one of the subschemes parametrized by $\mathbf{Hilb}^n_{p(t)}$. Indeed, if $I \in \mathcal{B}^n$ has Hilbert polynomial $\widetilde{p}(t) \neq p(t)$, then being $\widetilde{q}(r) = q(r)$ for Gotzmann's Persistence Theorem, $\dim_{\mathbb{K}} I_t > q(t)$ for $t \geqslant r$. Hence for $t \gg 0$, $\widetilde{q}(t) > q(t)$ and $\widetilde{p}(t) < p(t)$. So $\deg \widetilde{p}(t) \leqslant d$.

3. If $\mathcal{B}^n_{p(t)}$ contains only one ideal, then $\mathbf{Hilb}^n_{p(t)}$ is a smooth rational projective variety. Indeed, we know that $\mathcal{B}^n_{p(t)}$ contains at least the lexicographic ideal L, i.e. the ideal generated in degree r by the $q(r)$ maximal monomials w.r.t. the term order `DegLex`. In Section 4.2.1 we proved that $\mathcal{H} = \mathcal{H}_L$ is isomorphic to an affine space. By Proposition 4.66, as g varies in $\mathrm{GL}(n+1)$, the open subsets $g \,.\, \mathcal{H}$ cover $\mathbf{Hilb}^n_{p(t)}$. Thus $\mathbf{Hilb}^n_{p(t)}$ is smooth and rational as claimed.

The open subset $\mathcal{H}_J = \mathcal{U}_J \cap \mathbf{Hilb}^n_{p(t)}$ of $\mathbf{Hilb}^n_{p(t)}$ is then a closed subscheme in the affine space in $\mathbb{A}^{p(r)q(r)} \simeq \mathcal{U}_J$. Moving from Lemma 4.61 and Corollary 4.62 we can determine the scheme structure of $\mathcal{H}_J$ in $\mathbb{A}^{p(r)q(r)}$, starting from the set of J-marked polynomials $\mathcal{G}$ as in Definition 4.25.

Definition 4.69. We will denote by $\mathfrak{A}_J$ the ideal in $\mathbb{K}[C]$ defining $\mathcal{H}_J$ as an affine subscheme of $\mathbb{A}^{p(r)q(r)}$ through the isomorphism of Lemma 4.61*(vi)*.

Remark 4.6.7. We obtain every ideal $I \in \mathcal{U}_J$ specializing (in a unique way) the variables $C_{\alpha\beta}$ in $(\mathcal{G})$ to $c_{\alpha\beta} \in \mathbb{K}$, but not every specialization gives rise to an ideal I in $\mathcal{H}_J$, that is to an ideal with Hilbert polynomial $p(t)$. This last condition holds for an ideal I if and only if every polynomial has an unique J-normal form modulo I, that is if and only if every J-reduced polynomial in I vanishes. Hence, the ideal $\mathfrak{A}_J$ is made by the coefficients w.r.t. the variables x of all the polynomials $(\mathcal{G}) \subset \mathbb{K}[C][x]$ that are J-reduced.

Due to Macaulay's Estimate on the Growth of Ideals we know that if I is generated by a J-marked set, then $\dim_{\mathbb{K}} I_t \geqslant q(t)$ for every $t \geqslant r$. Moreover, by Gotzmann's Persistence Theorem, $\operatorname{Proj} \mathbb{K}[x]/I \in \mathcal{H}\mathrm{ilb}^n_{p(t)}(\mathbb{K})$ if the equality holds for $r+1$, that is if $\dim_{\mathbb{K}} I_{r+1} = q(r+1)$.

Then let $(\mathcal{G}) \subseteq \mathbb{K}[C][x]$, $\mathfrak{M}\big((\mathcal{G})_r\big)$ and $\mathfrak{M}\big((\mathcal{G})_{r+1}\big)$ be respectively the matrices whose columns correspond to the monomials in $\mathbb{K}[x]_r$ and $\mathbb{K}[x]_{r+1}$ and whose rows contain the coefficients of monomials in the polynomials F_α and $x_i F_\alpha$ respectively. Thus, a set of generators for the ideal $\mathfrak{A}_J$ is given by the minors of order $q(r+1)+1$ of the matrix $\mathfrak{M}\big((\mathcal{G})_{r+1}\big)$.

An easy computation shows that this is in general a very large set of polynomials! Indeed $\mathfrak{M}\big((\mathcal{G})_{r+1}\big)$ is a $(n+1)q(r) \times N(r)$ matrix and the number of its minors of order $q(r+1)+1$ is $\binom{(n+1)q(r)}{q(r+1)+1} \cdot \binom{N(r)}{q(r+1)+1}$ and their degree is up to $q(r+1)+1$. Looking at the special form of $\mathfrak{M}\big((\mathcal{G})_{r+1}\big)$, we will show in Theorem 4.72 that the number of minors of $\mathfrak{M}\big((\mathcal{G})_{r+1}\big)$ that are sufficient to impose the condition on the the rank can be drastically reduced and the degree of the involved determinants is bounded by $d+2$.

Example 4.6.8. Let us consider for instance the Hilbert scheme $\mathbf{Hilb}^2_2$. For every monomial ideal $J \in \mathcal{M}^n$, we have $r=2$, $q(2)=4$, $q(3)=8$, $N(3)=10$. Then $\mathfrak{M}\big((\mathcal{G}_J)_3\big)$ is a 12×10 matrix and the number of its minors of order 9 (with degree up to 9) is $\binom{3\cdot 4}{9} \cdot \binom{10}{9} = 2200$.

In order to obtain a better set of generators for $\mathfrak{A}_J$, we now prove that the open subset $\mathcal{H}_J$ for a Borel-fixed ideal J defining a point of $\mathbf{Hilb}^n_{p(t)}$ is nothing else but the J-marked scheme $\mathcal{M}f(J)$.

Theorem 4.70. *There is a scheme theoretic isomorphism:*

$$\mathcal{H}_J \simeq \mathcal{M}f(J). \tag{4.30}$$

Proof. The thesis directly follows from the two constructions of $\mathcal{M}f(J)$ and $\mathcal{H}_J$. Both constructions start from a J-marked set $\mathcal{G} \subseteq \mathbb{K}[C][x]$ (as in Definition 4.25). As shown in Proposition 4.55, we can obtain a set of generators for the ideal defining $\mathcal{M}f(J)$ imposing conditions on the rank of some matrices. In the present hypothesis, we can consider only one matrix, the one corresponding to the degree $r+1$, and impose that its rank is $\leqslant \dim_{\mathbb{K}} J_{r+1}$. This matrix turns out to be indeed $\mathfrak{M}\big((\mathcal{G})_{r+1}\big)$ and $\dim_{\mathbb{K}} J_{r+1} = q(r+1)$. Then in both cases, a set of generators is given by the minors of order $q(r+1)+1$ of the matrix $\mathfrak{M}\big((\mathcal{G})_{r+1}\big)$. $\square$

Thanks to this last result, $\mathfrak{A}_J$ is the ideal in $\mathbb{K}[C]$ defining $\mathcal{H}_J$ or equivalently $\mathcal{M}f(J)$ as an affine subscheme in $\mathbb{A}^{p(r)q(r)}$. The isomorphism between a J-marked region of $\mathbf{Hilb}^n_{p(t)}$ and the corresponding J-marked scheme allows us to embed $\mathcal{H}_J$ in affine linear spaces of "low" dimension by Theorem 4.60. We can choose linear spaces of different dimension, depending on whether we want to keep control on the degree of the equations defining the scheme structure or not.

4.6.1 Equations defining $\mathcal{H}_J$ in local Plücker coordinates

In our reasoning the matrix $\mathfrak{M}\big((\mathcal{G})_{r+1}\big)$ has a major role, therefore we now look closer at its shape. Remind that the ideal $J = J_{\geqslant r} = (J_r)$ belongs to $\mathcal{B}^n_{p(t)}$ and that we are assuming that the degree of the Hilbert polynomial $p(t)$ is equal to d.

Lemma 4.71. *Up to permutations on rows and columns, $\mathfrak{M}\big((\mathcal{G})_{r+1}\big)$ assumes the following* *simple form:*

$$\mathfrak{M}\big((\mathcal{G})_{r+1}\big) = \left(\begin{array}{cccc|ccc} \mathtt{Id}(n,\dots,d+1) & \bullet & \bullet & \bullet & \bullet & \dots & \bullet \\ 0 & \mathtt{Id}(d) & \bullet & \bullet & \bullet & \dots & \bullet \\ \vdots & \vdots & \ddots & \bullet & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \mathtt{Id}(0) & \bullet & \dots & \bullet \\ \hline \star & \bullet & \bullet & \bullet & \bullet & \dots & \bullet \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \star & \bullet & \bullet & \bullet & \bullet & \dots & \bullet \end{array}\right) \tag{4.31}$$

where

- *the columns on the left of the vertical line correspond to monomials in J_{r+1};*
- *the columns on the right of the vertical line correspond to monomials in $\mathcal{N}(J)_{r+1}$;*
- $\mathrm{Id}(n,\dots,d+1)$ *is an identity matrix of order $\binom{n-d+r}{r+1}$, corresponding to the monomials in $\mathbb{K}[x_{d+1},\dots,x_n]_{r+1}$;*
- $\mathrm{Id}(d),\dots,\mathtt{Id}(0)$ *are identity matrices of suitable dimensions $\leqslant q(r)$, corresponding to monomials in J_{r+1} with minimal variable $x_d,\dots,x_0$ respectively;*
- *"$\star$" stands for entries that are all 0, except at most one entry equal to 1 in each row;*
- *"$\bullet$" stands for entries that are either 0 or coefficients $-C_{\alpha\beta}$.*

Proof. We consider the $\mathbb{K}[C]$-module of polynomials in $(\mathcal{G})$ of degree $r+1$ with respect to the variables x and its set of generators $\{x_i F_\alpha \mid F_\alpha \in \mathcal{G}, i = 0,\dots,n\}$. We write inside the matrix $\mathfrak{M}\big((\mathcal{G})_{r+1}\big)$ the coefficients of the monomials in $\mathbb{K}[x]_{r+1}$ appearing in these polynomials $x_i F_\alpha$.

First of all we order the columns writing first the monomials in J_{r+1}, listed in decreasing order w.r.t. `DegRevLex`, and then the monomials in $\mathcal{N}(J)_{r+1}$. In this way the first monomial is x_n^{r+1}, the only one with minimal variable x_n, after this there are the monomials whose minimal variable is x_{n-1}, and so on.

The rows are ordered in a similar way. Every monomial in $x^\gamma J_{r+1}$ can be written as a product $\langle x^\alpha | x_i \rangle^J$ such that $x_i = \min x_i x^\alpha \leq \min x^\alpha$ and $x^\alpha \in J_r$ (Lemma

2.17). The first rows (those above the horizontal line in the picture) correspond to polynomials $x_i F_\alpha$ such that $x_i = \min x_i x^\alpha$ ordered w.r.t. `DegRevLex` on the initial monomials $x_i x^\alpha$. The first row corresponds to $x_n F_{x_n^r}$, after there are the rows corresponding to polynomials of the type $x_{n-1}F_\alpha$ with $x^\alpha \in \mathbb{K}[x_{n-1}, x_n]$ and so on. Below the horizontal line we list the rows corresponding to the remaining polynomials $x_i F_\alpha$ such that $\min x_i x^\alpha < x_i$.

The top left submatrix, let us call it $\mathcal{D}$, is an upper triangular matrix of order $q(r+1)$. In fact, as $J \in \mathcal{B}^n_{p(t)}$, then J_{r+1} contains $\mathbb{K}[x_{d+1}, \dots, x_n]_{r+1}$ (see Proposition 2.15) and so each monomial in J_{r+1} corresponds to one and only one column and row in $\mathcal{D}$.

Moreover in the row of $\mathcal{D}$ corresponding to a polynomial $x_i F_\alpha$ with initial monomial $x_i x^\alpha$, the entry on the main diagonal is the coefficient of $x_i x^\alpha$ in $x_i F_\alpha$, i.e. 1. If $x_i x^\beta$ is any monomial appearing in $x_i \mathcal{T}(F_\alpha)$, then either $x_i x^\beta \notin J$, hence its coefficient $-C_{\alpha\beta}$ is written on the right of the vertical line, or $x_i x^\beta \in J$, that is $x_i x^\beta = x_j x^{\alpha'}$ for some $x^{\alpha'} \in J$ and $x_j = \min x_i x^\beta < x_i$, hence its coefficient $-C_{\alpha\beta}$ is written in one of the columns corresponding to monomials with minimal variable x_j lower than x_i. Thus in $\mathcal{D}$ there are identity blocks $\mathrm{Id}(i)$ corresponding to monomials in J_{r+1} with minimal variable x_i.

Furthermore, the minimal variable in every monomial $x^\beta \in \mathcal{N}(J)_r$ is lower than or equal to x_d: hence the first block of $\mathcal{D}$ is a big identity matrix $\mathtt{Id}(n, \dots, d+1)$ of order $\binom{n-d+r}{r+1}$, corresponding to monomials in J_{r+1} with minimal variable $x_{d+1}, \dots, x_n$. The same arguments holds for the "$\star$" under the horizontal line. $\square$

We will now determine the dimension of a linear affine space in which $\mathcal{H}_J$ can be embedded and furthermore to study in which cases we can control the degree of the defining equations, bounding it using only d. As $\mathfrak{A}_J$ is the localization in the open subset $\mathcal{U}_J$ of the ideal defining the Hilbert scheme $\mathbf{Hilb}^n_{p(t)}$ in $\mathbb{K}[\Delta]$, we can derive a bound on a set of generators of $\mathfrak{A}_J$ from the known bounds for analogous global results. In Chapter 1 we showed that Iarrobino and Kleiman proved that $\mathbf{Hilb}^n_{p(t)}$ is generated in degree $q(r+1)+1$. Later on, Haiman and Sturmfels proved the Bayer's conjecture saying that $\mathbf{Hilb}^n_{p(t)}$ is generated in the far lower degree $n+1$. Unluckily, global results do not give a satisfying bound in the local case, because the global Plücker coordinate $\Delta_{J'}$, when localized in $\mathcal{U}_J$, corresponds to a polynomial in $\mathbb{K}[C]$ whose degree can vary between 1 and $q(r)$ (Remark 4.6.2).

Example 4.6.9. We consider again $\mathbf{Hilb}^3_{3t}$ as in Example 4.6.5.

- Localizing Δ_L at $\mathcal{U}_{J_1}$, we obtain a polynomial of degree 1, since the monomial basis of L is $G_{J_1} \setminus \{x_2x_1^2\} \cup \{x_3^2x_0\}$ (Lemma 4.61*(v)*);
- for $\mathcal{U}_{J_i}$, $i = 2,3,4$, we count the monomials in $G_{J_i} \setminus G_L$. We then obtain that localizing at $\mathcal{U}_{J_i}$, the Plücker coordinate Δ_L becomes a polynomial of degree i, $i = 2,3,4$ in the $C_{\alpha\gamma}$ (Lemma 4.61*(iv)*).

We now prove that the equations defining $\mathcal{H}_J$ in $\mathbb{A}^{p(r)q(r)}$, that is in the local case, are of degree $\leqslant d+2$.

Theorem 4.72. *Let $p(t)$ be an admissible Hilbert polynomial in $\mathbb{P}^n$, of degree d and Gotzmann number r. If $J \in \mathcal{B}^n_{p(t)}$, then the ideal $\mathfrak{A}_J$ defining $\mathcal{H}_J$ as a subscheme of $\mathbb{A}^{p(r)q(r)}$, that is in "local" Plücker coordinates, is generated in degree smaller than or equal to $d+2$ by $p(r+1)\cdot((n+1)q(r) - q(r+1))$ polynomials.*

We give two (equivalent) proofs that $\mathfrak{A}_J$ is generated in degree $\leqslant d+2$: the first one uses minors of the matrix $\mathfrak{M}\big((\mathcal{G})_{r+1}\big)$, and in this way we also count the number of generators; the second one uses the Buchberger-like criterion on the reduction of S-polynomials.

First proof. The ideal $\mathfrak{A}_J$ of $\mathcal{H}_J$ is generated by the minors of order $q(r+1)+1$ of $\mathfrak{M}\big((\mathcal{G})_{r+1}\big)$, that we think to be written like in Lemma 4.71. As the determinant of the top left submatrix of order $q(r+1)$ (called $\mathcal{D}$ in the proof of Lemma 4.71) is 1, we can just consider the minors of order $q(r+1)+1$ containing $\mathcal{D}$:

$$\det\left(\begin{array}{cccc|c} \mathtt{Id}(n,\ldots,d+1) & \bullet & \bullet & \bullet & \bullet \\ 0 & \mathtt{Id}(d) & \bullet & \bullet & \bullet \\ \vdots & \vdots & \ddots & \bullet & \vdots \\ 0 & 0 & 0 & \mathtt{Id}(0) & \bullet \\ \hline \star & \bullet & \bullet & \bullet & \bullet \end{array}\right). \tag{4.32}$$

We perform Gaussian reduction on the last rows. In $\star$ there is at most a non-zero element, which is 1; if necessary, we perform a first row reduction, to make it a 0. At the end of this first step of reduction, the degree of $\bullet$ in the last row remains at most 1 in $\mathbb{K}[C]$.

With the second row reduction, we obtain that the above determinant is equal to the following:

$$\det\left(\begin{array}{cccc|c} \mathtt{Id}(n,\ldots,d+1) & \bullet & \bullet & \bullet & \bullet \\ 0 & \mathtt{Id}(d) & \bullet & \bullet & \bullet \\ \vdots & \vdots & \ddots & \bullet & \vdots \\ 0 & 0 & 0 & \mathtt{Id}(0) & \bullet \\ \hline 0 & 0 & \circ_2 & \circ_2 & \circ_2 \end{array}\right).$$

where $\circ_2$ stands for polynomials in $\mathbb{K}[C]$ of degree at most 2.

Going on with Gaussian reduction, the determinant is equal to the element appearing in the last line and last column, which is a polynomial in $\mathbb{K}[C]$ of degree $\leqslant d+2$.

For the number of polynomials that generate $\mathfrak{A}_J$, we simply count the number of minors of $\mathfrak{M}\big((\mathcal{G})_{r+1}\big)$ of order $q(r+1)+1$ containing the matrix $\mathcal{D}$. $\square$

Second proof. As shown in Theorem 4.47 and Corollary 4.48, we can obtain a set of generators for $\mathfrak{A}_J$ also using a special procedure of reduction of S-polynomials of elements in $\mathcal{G}$ with a Buchberger-like criterion analogous to the one for Gröbner bases. The only difference when a term order is not defined is that reductions must be chosen in a careful way in order to have a Noetherian reduction. In the present hypothesis, we consider only EK-polynomials of degree $r+1$ with respect to the variables x, that is of the type $x_iF_\alpha - x_jF_{\alpha'}$ with $x_i > \min x^\alpha$ and $x_ix^\alpha = \langle x^{\alpha'}|x_j\rangle^J$, that correspond to a basis of the syzygies of J in degree $r+1$.

If x_ix^β is a monomial of J that appear in $x_iF_\alpha - x_jF_{\alpha'}$, then $x^\beta \in \mathcal{N}(J)_r$ and $x_ix^\beta = \langle x^\gamma|x_h\rangle^J$, i.e. $x_h = \min x_ix^\beta < x_i$ and $x^\beta \in J_r$. Then we can perform a step of reduction $\xrightarrow{\mathcal{V}^J_{r+1}}$ of x_hx^γ rewriting it by $x_h\mathcal{T}(F_\gamma)$. If some monomial of $x_hx^\beta - x_hF_\gamma$ belongs to J, then again we can reduce it using some polynomial $x_{h'}F_{\gamma'}$ with $x_{h'} < x_h$.

At every step of reduction a monomial is replaced by a sum of other monomials multiplied by one of the variables C. Then at every step of reduction the degree of the coefficients directly involved increases by 1. If $x^{\eta_0}, x^{\eta_1}, \ldots, x^{\eta_s}$ is a sequence of monomials in J_m such that $x^{\eta_{i+1}}$ appears in the tail of the reduction of x^{η_i}, then $\min x^{\eta_{i+1}} < \min x^{\eta_i}$. As the minimal variable of any monomial in $\mathcal{N}(J)_r$ is lower than or equal to x_d, the length of any such chain is at most $d+1$. Thus, the final degree of the coefficients is at most $1 + (d+1) = d+2$. $\square$

Example 4.6.10. We consider again $\mathbf{Hilb}^2_2$, already investigated in Example 4.6.8. If we consider all the minors of $\mathfrak{M}\big((\mathcal{G})_{r+1}\big)$ of order $q(r+1)+1$, we obtain a set of generators for $\mathfrak{A}_J$ of cardinality 2200. Using Theorem 4.72, we see that actually in order to define $\mathfrak{A}_J$ we just need 8 minors of degree 2.

By Theorem 4.70, we can exploit the techniques presented for J-marked schemes to embed $\mathcal{H}_J$ in linear affine spaces of lower dimension. We consider again the notation introduced for Theorem 4.60: for an ideal $J \in \mathcal{B}^n_{p(t)}$, let $\underline{J}$ be its saturation and ρ the maximal degree of a monomial divisible by x_1 in $G_{\underline{J}}$; if there are no such monomials in $G_{\underline{J}}$, we set $\rho = 0$. Moreover if $x^\alpha \in G_{\underline{J}}$, we write $x^{\overline{\alpha}}$ for the monomial $x^\alpha x_0^{m-|\alpha|} \in G_{\underline{J}_{\geqslant m}}$, if $|\alpha| < m$; otherwise $x^{\overline{\alpha}} = x^\alpha$. Finally, we will denote by $\varphi_{J,r}$ the embedding $\mathcal{H}_J \hookrightarrow \mathbb{A}^{p(r)q(r)}$ given by Theorem 4.70 and Theorem 4.72.

Theorem 4.73. *In the established setting, the followings hold:*

(i) if $m \geqslant r$, then $\mathcal{M}f(\underline{J}_{\geqslant m}) \simeq \mathcal{H}_J$;

(ii) if $m < r$, then $\mathcal{M}f(\underline{J}_{\geqslant m})$ is a closed subscheme of $\mathcal{H}_J$, (eventually equal). If we consider the embedding $\phi_{J,r}(\mathcal{H}_J) \subset \mathbb{A}^{p(r)q(r)}$, then $\mathcal{M}f(\underline{J}_{\geqslant m})$ is cut out by a suitable linear space;

(iii) $\mathcal{H}_J \simeq \mathcal{M}f(\underline{J}_{\geqslant m})$ if and only if either $\underline{J}_{\geqslant m} = J$ or $m \geqslant \rho - 1$.

In particular, if $\rho > 0$, then $\rho - 1$ is the smallest integer m such that:

$$\mathcal{H}_J \simeq \mathcal{M}f(\underline{J}_{\geqslant m}).$$

Especially, the isomorphism $\mathcal{H}_J \simeq \mathcal{M}f(\underline{J}_{\geqslant \mathrm{reg}(\underline{J})})$ induces an embedding $\phi_{J,\mathrm{reg}(\underline{J})}$ of $\mathcal{H}_J$ in an affine space of dimension $|G_{\underline{J}}| \cdot p\big(\mathrm{reg}(\underline{J})\big)$ and the isomorphism $\mathcal{H}_J \simeq \mathcal{M}f(\underline{J}_{\geqslant \rho-1})$ induces an embedding $\phi_{J,\rho-1}$ of $\mathcal{H}_J$ in an affine space of dimension

$$\sum_{x^{\overline{\alpha}} \in G_{\underline{J}_{\geqslant \rho-1}}} \Big| \mathcal{N}(\underline{J}_{\geqslant \rho-1})_{|\overline{\alpha}|} \Big|.$$

Proof. Thanks to the isomorphism $\mathcal{H}_J \simeq \mathcal{M}f(\underline{J}_{\geqslant r})$ of Theorem 4.70, the statements are straightforward consequences of Theorem 4.60. □

The embedding $\phi_{J,\rho-1}$ (or more generally $\phi_{J,m}$ with $\rho - 1 \leqslant m < \mathrm{reg}(\underline{J})$) of $\mathcal{H}_J$ in affine spaces defined in Theorem 4.73 are computationally advantageous, because in order to compute equations for $\mathcal{H}_J$ we deal with a small number of variables, namely smaller than $p(r)q(r)$; however, using these embedding we do not have any control on the degree of the equations defining $\mathcal{H}_J$.

If we can do computations for an embedding in a bigger affine space, considering $\mathcal{H}_J$ in $\mathbb{A}^{p(\mathrm{reg}(\underline{J}))q(\mathrm{reg}(\underline{J}))}$, then the equations defining $\mathcal{H}_J$ as a subscheme of $\mathbb{A}^{p(\mathrm{reg}(\underline{J}))q(\mathrm{reg}(\underline{J}))}$ are bounded, as we show in the following theorem. Furthermore we can compare computationally two open subsets of this kind.

Theorem 4.74. *Consider $J \in \mathcal{B}^n_{p(t)}$.*

(i) *$\mathcal{H}_J$ can be embedded as a closed subscheme in $\mathbb{A}^{p(m)q(m)}$ where m is any integer $\geqslant \mathrm{reg}(\underline{J})$, by an ideal generated in degree $\leqslant d+2$.*

(ii) *If $\phi_{J_i,r} : \mathcal{H}_{J_i} \to \mathbb{A}^{p(r)q(r)}$ are the embedding for the open subsets corresponding to two Borel-fixed ideals J_1 and J_2 belonging to $\mathcal{B}^n_{p(t)}$, then in general $\phi_{J_1,r}(\mathcal{H}_{J_1} \cap \mathcal{H}_{J_2}) \neq \phi_{J_2,r}(\mathcal{H}_{J_1} \cap \mathcal{H}_{J_2})$. More precisely:*

$$\phi_{J_1,r}(\mathcal{H}_{J_1} \cap \mathcal{H}_{J_2}) = \phi_{J_1,r}(\mathcal{H}_{J_1}) \setminus F_1, \quad \phi_{J_2,r}(\mathcal{H}_{J_1} \cap \mathcal{H}_{J_2}) = \phi_{J_2,r}(\mathcal{H}_{J_2}) \setminus F_2,$$

where F_1 e F_2 are hypersurfaces of the same degree $|G_{J_1} \setminus (G_{J_1} \cap G_{J_2})|$ in $\mathbb{A}^{p(r)q(r)}$.

(iii) *If we consider $\overline{m} \geqslant \max\{\mathrm{reg}(\underline{J}_1), \mathrm{reg}(\underline{J}_2)\}$ then statement (ii) holds considering the embedding $\phi_{J_i,\overline{m}} : \mathcal{H}_{J_i} \hookrightarrow \mathbb{A}^{p(\overline{m})q(\overline{m})}$.*

Proof. *(i)* Using Theorem 4.70, $\mathcal{H}_J = \mathcal{M}f(J)$. Furthermore, thanks to Theorem 4.73 , we have that $\mathcal{H}_J \simeq \mathcal{M}f(\underline{J}_{\geqslant \mathrm{reg}(\underline{J})})$. Applying Theorem 4.47, it is sufficient to consider reductions of S-polynomials in degree $\mathrm{reg}(\underline{J}) + 1$ and we conclude as in Theorem 4.72.

(ii) F_1 is defined by the equation of $\frac{\Delta_{J_2}}{\Delta_{J_1}}$ in $\mathbb{A}^{p(r)q(r)}$ and so its degree corresponds to $|G_{J_1} \setminus (G_{J_1} \cap G_{J_2})|$, by Lemma 4.61*(v)*.

(iii) is a straightforward consequence of *(i)* and *(ii)*. □

Example 4.6.11. We consider the Hilbert scheme of s points in $\mathbb{P}^n$, $\mathbf{Hilb}^n_s$. For any monomial ideal J , we have that the open subset of the Grassmannian $\mathcal{U}_J$ is isomorphic to $\mathbb{A}^{sq(s)}$, where $q(s) = \binom{n+s}{n} - s$. The saturated lexicographic ideal $\underline{L} = (x_n, \ldots, x_2, x_1^s)$ has regularity s the open subset $\mathcal{H}_{\underline{L}_{\geqslant s}} \subset \mathbf{Hilb}^n_s$, the Reeves-Stillman component, contains all the subschemes of $\mathbb{P}^n$ made up of s distinct points, so it has dimension $\geqslant ns$. Using Theorem 4.73, $\mathcal{H}_{\underline{L}_{\geqslant s}}$ is embedded in an affine space of dimension $|G_{\underline{L}}| \cdot p(s) = ns$. Then, $\mathcal{H}_{\underline{L}_{\geqslant s}} \simeq \mathbb{A}^{ns}$.

Example 4.6.12. We can now easily study some features of $\mathbf{Hilb}^3_{3t}$, that we have already investigated in Example 4.6.5. The Borel region of $\mathbf{Hilb}^3_{3t}$ is made up of one open subset only, corresponding to the lexicographic ideal $L = \underline{L}_{\geqslant 3} = (x_3, x_2^3)_{\geqslant 3}$, as already pointed out in Example 4.6.5, using Proposition 4.66 and Theorem 4.70. Since no monomial in the basis of (x_3, x_2^3) is divisible by x_1, using Theorem 4.73, we have that $\mathcal{M}f(\underline{L}) \simeq \mathcal{H}_L$ and we can embed $\mathcal{H}_L$ in $\mathbb{A}^{12}$.

Furthermore, since the monomial basis of $\underline{L}$ is made up of two coprime monomials, we have that every ideal I in $\mathcal{M}f(\underline{L})$ corresponds to the complete intersection of a plane and a cubic; we then have that $\mathcal{H}_L$ has dimension $\geqslant 12$, and so $\mathcal{H}_L \simeq \mathbb{A}^{12}$. Every point of $\mathbf{Hilb}^3_{3t}$ is, up to a change of coordinates, a point of $\mathcal{H}_L$, hence every scheme in $\mathbb{P}^3$ with Hilbert polynomial $3t$ is a $(1,3)$-complete intersection.

Example 4.6.13. Let us consider the Hilbert scheme $\mathbf{Hilb}^3_{4t}$. Continuing the computation of marked schemes started in Example 4.5.4, we obtain that the Gröbner stratum $\mathcal{S}t_{\omega_4}(J_4)$ of $J_4 = (x_3^2, x_3x_2, x_2^3)$ is isomorphic to the affine space $\mathbb{A}^{16}$, namely $\mathbf{Hilb}^3_{4t}$ has a rational component of dimension 16 corresponding to $(2,2)$-complete intersection (called Vainsencher-Avritzer component [101]). A second component is the Reeves-Stillman one, containing the lexicographic point. Being such a point smooth [87], the Gröbner stratum $\mathcal{S}t_{\texttt{DegLex}}\big((J_1)_{\geqslant 5}\big)$ has to be an affine scheme, thus isomorphic to $\mathbb{A}^{23}$ (cf. with [35]).

We know that the point defined by $J_3 = (x_3^2, x_3x_2, x_3x_1^2, x_2^4)$ lies on the intersection of these two components: indeed it belongs to the Reeves-Stillman component because it has the same hyperplane section of the lexicographic ideal (Reeves criterion) and it belongs to the Vainsencher-Avritzer component because there is a Borel rational deformation having $(J_3)_{\geqslant 6}$ and $(J_4)_{\geqslant 6}$ as fibers. By explicit computation (see Example C.2.2) we find that the ideal of the Gröbner stratum $\mathcal{S}t_{\omega_3}(J_3)$ can be decomposed in an ideal generated by a single variable (of 24), that correspond to an affine scheme of dimension 23, and an ideal defining a affine subscheme isomorphic to $\mathbb{A}^{16}$. Hence the two components of $\mathbf{Hilb}^3_{4t}$ intersect transversely.

Chapter 5

Low degree equations defining the Hilbert scheme

In this chapter, we will introduce a new set of equations defining the Hilbert scheme $\mathbf{Hilb}^n_{p(t)}$ as subscheme of the suitable Grassmannian $\mathbf{Gr}_{\mathbb{K}}(q(r), N(r))$ as already done in Chapter 1. This topic is placed here, after a long discussion on Borel-fixed ideals, because ideas behind the main result we will prove come from the construction of the Borel open covering of the Hilbert scheme discussed in Chapter 4, in particularly Section 4.6.1. We remark that it is not trivial to extend the bound on the degree of equations defining locally the Hilbert scheme to the global equations, because as seen in Lemma 4.61 global coordinates can correspond to polynomials of high degree in the local coordinates. The results of this chapter belong to the preprint "Low degree equations defining the Hilbert scheme" [17], joint paper with J. Brachat, B. Mourrain and M. Roggero.

5.1 BLMR equations

Given an admissible Hilbert polynomial $p(t)$ on $\mathbb{P}^n$ with Gotzmann number r and degree d and the associated volume polynomial $q(t) = \binom{n+t}{n} - p(t)$, we set

$$q'(t) = q(t) - \binom{n-d-1+t}{n-d-1} = \binom{n+t}{n} - \binom{n-d-1+t}{n-d-1} - p(t), \tag{5.1}$$

so that $q(t) - q'(t) = \dim_{\mathbb{K}} \mathbb{K}[x_{d+1}, \dots, x_n]_t$.

Proposition 5.1. *Let $\mathcal{U}'$ be the set of all the elements $I \in \mathbf{Gr}_{\mathbb{K}}(q(r), \mathbb{K}[x]_r)$ such that I_r has a set of generators of the type:*

$$\begin{aligned} G_I^r = & \left\{x^\alpha + f_\alpha \mid x^\alpha \in \mathbb{K}[x_{d+1}, \dots, x_n]_r \text{ and } f_\alpha \in (x_d, \dots, x_0)\right\} \\ & \cup \left\{g_j \mid g_j \in (x_d, \dots, x_0)\right\} \end{aligned} \tag{5.2}$$

where the first set of generators contains by construction $q(r) - q'(r)$ elements and the second set has $q'(r)$ polynomials, so that $\dim_{\mathbb{K}} I_r = q(r)$.

Then $\mathcal{U}'$ is a non-empty open subset in $\mathbf{Gr}_{\mathbb{K}}(q(r), \mathbb{K}[x]_r)$ and I_{r+1} has a set of generators G_I^{r+1} that can be represented by a matrix of the type:

$$\mathcal{A}_{r+1} = \left(\begin{array}{c|c} \mathtt{Id} & \bullet \quad \bullet \quad \bullet \\ \hline 0 & \mathcal{D}_1 \\ \hline 0 & \mathcal{D}_2 \end{array}\right) \tag{5.3}$$

where:

- *the columns belonging to the left part of the matrix correspond to the monomials in $\mathbb{K}[x_{d+1}, \dots, x_n]_{r+1}$ and the columns on the right to the monomials in $(x_0, \dots, x_d)_{r+1}$;*
- *the top-left submatrix* Id *is the identity matrix of order $q(r+1) - q'(r+1)$;*
- *the rows of $\mathcal{D}_1$ contain the coefficients of all the generators multiplied by a variable x_h, $h = 0, \dots, d$;*
- *the rows of $\mathcal{D}_2$ contain the coefficients of the generators g_j multiplied by a variable x_h, $h = d+1, \dots, n$ and the coefficients of the polynomials $x_{i'} f_{\alpha'} - x_i f_\alpha$ such that $x_{i'} x^{\alpha'} = x_i x^\alpha$ and $i, i' \geqslant d+1$.*

Moreover the subset $\mathcal{U} \subset \mathcal{U}'$ of all the ideals I_r such that $\operatorname{rank} \mathcal{D}_1 \geqslant q'(r+1)$ *is open and $\mathcal{U}^{\mathrm{GL}} = \{g \,.\, \mathcal{U} \mid g \in \mathrm{GL}(n+1)\}$ is an open covering of $\mathbf{Gr}_{\mathbb{K}}(q(r), \mathbb{K}[x]_r)$.*

Proof. Let us consider the canonical projection

$$\pi : \mathbb{K}[x_0, \dots, x_n]_r \longrightarrow (\mathbb{K}[x_0, \dots, x_n]/(x_0, \dots x_d))_r \simeq \mathbb{K}[x_{d+1}, \dots, x_n]_r.$$

The subset $\mathcal{U}'$ of $\mathbf{Gr}_{\mathbb{K}}(q(r),\mathbb{K}[x]_r)$ is open because $\mathcal{U}' = \pi^{-1}(\mathbb{K}[x_{d+1},\dots,x_n]_r) \cap \mathbf{Gr}_{\mathbb{K}}(q(r),\mathbb{K}[x]_r)$. Moreover $\mathcal{U}'$ is non-empty because any Borel-fixed ideal J defining a point of $\mathbf{Gr}_k(q(r),\mathbb{K}[x]_r)$ (i.e. $\dim_{\mathbb{K}} J_r = q(r)$) belongs to $\mathcal{U}'$. Indeed, $\dim_{\mathbb{K}} J_t \geqslant q(t)$, $\forall\ t \geqslant r$ (by Macaulay's Estimate on the Growth of Ideals) implies that the Hilbert polynomial of $\mathbb{K}[x_0,\dots,x_n]/J$ has degree smaller than or equal to $\deg p(t) = d$ and so $\mathbb{K}[x_{d+1},\dots,x_n]_{\geqslant r} \subseteq J$ (see Proposition 2.15). Therefore the ideal J_r has a basis G_I^r (as $\mathbb{K}$-vector space) as the one described in (5.2) so that $J_r \in \mathcal{U}'$.

For any $I_r \in \mathcal{U}'$, a set of generators of I_{r+1} is $\mathbb{K}[x]_1 \cdot G_I^r = \bigcup_i \{x_i\, G_I^r\}$. The set of generators G_I^{r+1} we are looking for can be easily obtained from $\mathbb{K}[x]_1 \cdot G_I^r$ just modifying few elements: for every monomial x^γ in $\mathbb{K}[x_{d+1},\dots,x_n]_{r+1}$ we choose only one product $x_i(x^\alpha + f_\alpha)$ such that $x^\gamma = x_i\, x^\alpha$, to be left in G_I^{r+1} (and corresponding to a row in the first block of $\mathcal{A}_{r+1}$), whereas we replace any other polynomial $x_{i'}(x^{\alpha'} + f_{\alpha'})$, such that $x^\gamma = x_{i'}x^{\alpha'}$, by $x_{i'}f_{\alpha'} - x_i f_\alpha$ (which belongs to $(x_0,\dots,x_d)$ and corresponds to a row of $\mathcal{D}_2$).

Obviously the condition $\operatorname{rank}\mathcal{D}_1 \geqslant q'(r+1)$ is an open condition and we call $\mathcal{U} \subset \mathcal{U}'$ the corresponding open subset. Again this open subset is not empty because it contains for instance all the subspaces J_r defined by a Borel-fixed ideal $J \in \mathbf{Gr}_{\mathbb{K}}(q(r),\mathbb{K}[x]_r)$.

To prove the last statement, we consider any term ordering σ, refinement of $\leq_B$ (i.e. $x_n >_\sigma \cdots >_\sigma x_0$). Then for a generic $g \in \mathrm{GL}(n+1)$, $J = (\mathrm{in}(g\,.\,I)_r)$ is Borel-fixed. Note that J belongs to $\mathbf{Gr}_{\mathbb{K}}(q(r),\mathbb{K}[x]_r)$, but if $I \notin \mathcal{H}\mathrm{ilb}^n_{p(t)}(\mathbb{K})$, J can differ from $\mathrm{in}(g\,.\,I)$. As J is Borel-fixed

$$\begin{aligned}\dim_{\mathbb{K}}(x_0J_r + \ldots + x_dJ_r) &= \dim_{\mathbb{K}} J_{r+1} \cap (x_0,\dots,x_d) = \\ &= \dim_{\mathbb{K}} J_{r+1} - \dim_{\mathbb{K}} \mathbb{K}[x_{d+1},\dots,x_n]_{r+1} \geqslant q'(r+1),\end{aligned}$$

hence

$$\begin{aligned}\dim_{\mathbb{K}}\left(x_0(g\,.\,I)_r + \ldots + x_d(g\,.\,I)_r\right) &\geqslant \dim_{\mathbb{K}}\left(x_0\mathrm{in}(g\,.\,I)_r + \ldots + x_d\mathrm{in}(g\,.\,I)_r\right) \geqslant \\ &\geqslant \dim_{\mathbb{K}}(x_0J_r + \cdots + x_dJ_r) \geqslant q'(r+1).\end{aligned}$$

Finally we can conclude that $g\,.\,I \in \mathcal{U}$ because a set of generators of the vector space $x_0(g\,.\,I)_r + \cdots + x_d(g\,.\,I)_r$ corresponds to the rows of $\mathcal{D}_1$. $\square$

Making reference to the matrix $\mathcal{A}_{r+1}$ in (5.3), let us call

$$\mathcal{D} = \begin{pmatrix} \mathcal{D}_1 \\ \mathcal{D}_2 \end{pmatrix}.$$

Corollary 5.2. *Let $I \subset \mathbb{K}[x]$ be any ideal belonging to $\mathcal{U}$. Then*

(i) $I \in \mathcal{U} \cap \mathcal{H}\mathrm{ilb}^n_{p(t)}(\mathbb{K}) \iff \operatorname{rank} \mathcal{D} = \operatorname{rank} \mathcal{D}_1 = q'(r+1)$;

(ii) $I \notin \mathcal{U} \cap \mathcal{H}\mathrm{ilb}^n_{p(t)}(\mathbb{K}) \iff \begin{array}{l} \textit{either } \operatorname{rank} \mathcal{D}_1 > q'(r+1) \\ \textit{or } \operatorname{rank} \mathcal{D}_1 = q'(r+1) < \operatorname{rank} \mathcal{D} \end{array}$.

Proof. (i) follows from the fact that $I \in \mathcal{H}\mathrm{ilb}^n_{p(t)}(\mathbb{K})$ if and only if $\dim_{\mathbb{K}} I_{r+1} = \operatorname{rank} \mathcal{A}_{r+1} = q(r+1)$ and from the special form of $\mathcal{A}_{r+1}$.

(ii) is another way to write *(i)*. ◻

We can now describe the set of equations $\mathfrak{H}$ that we will prove to describe the Hilbert scheme $\mathbf{Hilb}^n_{p(t)}$. As usual r is the Gotzmann number of $p(t)$ and let I be an element of $\mathbf{Gr}_{\mathbb{K}}(q(r), \mathbb{K}[x]_r)$. We will describe the equations of $\mathfrak{H}$ in terms of the Plücker coordinates of I via the Plücker embedding (1.2) $\mathscr{P} : \mathbf{Gr}_{\mathbb{K}}(q(r), N) \to \mathbb{P}\big[\wedge^{q(r)} \mathbb{K}[x]_r\big]$.

A first subset $\mathfrak{H}'$ of the equations is construct as follows. Let us choose in all the possible ways a set of $d+1$ elements of the type $x_i \Lambda^{(s_i)}_{\mathrm{J}_i}$ such that $i = 0, \ldots, d$ and $\sum_i s_i = q'(r+1)$. Moreover let us consider any variable x_j, $j = d+1, \ldots, n$ and any multiindex $\mathrm{K} = \{\mathrm{k}_1, \ldots, \mathrm{k}_{q(r)-1}\}$, such that all the monomials belonging to $\mathbb{K}[x_{d+1}, \ldots, x_n]_r$ are contained in the set $\{x^{\alpha(\mathrm{k}_1)}, \ldots, x^{\alpha(\mathrm{k}_{q(r)-1})}\}$ (in order for $\Lambda^{(1)}_{\mathrm{K}}$ to contain only monomials in $(x_0, \ldots, x_d)$). The first part $\mathfrak{H}'$ of the polynomials in $\mathfrak{H}$ is represented by all the coefficients of all exterior products of the type

$$\left(\bigwedge_{i=0}^{d} x_i \, \Lambda^{(s_i)}_{\mathrm{J}_i} \right) \wedge x_j \, \Lambda^{(1)}_{\mathrm{K}}. \tag{5.4}$$

The second part of the equations $\mathfrak{H}''$ is constructed as follows. We choose in all the possible ways a multiindex $\mathrm{H} = \{\mathrm{h}_1, \ldots, \mathrm{h}_{q(r)}\}$, such that again the corresponding set of monomials $\{x^{\alpha(\mathrm{h}_1)}, \ldots, x^{\alpha(\mathrm{h}_{q(r)})}\}$ of degree r contains the monomials in $\mathbb{K}[x_{d+1}, \ldots, x_n]_r$. For any $x^\gamma \in \mathbb{K}[x_{d+1}, \ldots, x_n]_{r-1}$ and any couple of variables $x_j, x_{j'} \in \{x_{d+1}, \ldots, x_n\}$, let $\overline{\mathrm{h}}$ and $\overline{\mathrm{h}}'$ be the indices such that $x_j x^\gamma = x^{\alpha(\overline{\mathrm{h}})}$ and

$x_{j'} x^\gamma = x^{\alpha(\overline{\mathrm{h}}')}$. The second part $\mathfrak{H}''$ of the polynomials in $\mathfrak{H}$ can be obtained collecting the coefficients of all the exterior products of the type

$$\left(\bigwedge_{i=0}^{d} x_i \, \Lambda^{(s_i)}_{\mathrm{J}_i} \right) \wedge \left(x_{j'} \, \Lambda^{(1)}_{\mathrm{H}_1} + x_j \, \Lambda^{(1)}_{\mathrm{H}_2} \right) \tag{5.5}$$

where $\mathrm{H}_1 = \mathrm{H} \setminus \{\overline{\mathrm{h}}\}$ and $\mathrm{H}_2 = \mathrm{H} \setminus \{\overline{\mathrm{h}}'\}$. Note that we need to sum the two vectors in order for the monomials $x_{j'} x^{\alpha(\overline{h})}$ and $x_j x^{\alpha(\overline{h}')}$ to delete, because $\Delta_{\mathrm{H}_1|\overline{h}} = -\Delta_{\mathrm{H}_2|\overline{h}'}$.

Theorem 5.3. *Let $p(t)$ be an admissible Hilbert polynomial in $\mathbb{P}^n$ of degree d and Gotzmann number r. The set $\mathcal{H}\mathrm{ilb}^n_{p(t)}(\mathbb{K})$ of $\mathbb{K}$-valued point of $\mathcal{H}\mathrm{ilb}^n_{p(t)}$ can be described by a closed subscheme of the Grassmannian $\mathbf{Gr}_{\mathbb{K}}(q(r), \mathbb{K}[x]_r)$ defined by the equations $\mathfrak{H}$ given in* (5.4), (5.5) *of degree smaller than or equal to $d+2$.*

Proof. We divide the proof in two steps.
Step 1. Firstly, we consider an ideal $I \in \mathcal{U}$ and show that every polynomial in $\mathfrak{H}$ vanishes on the Plücker coordinates $[\ldots : \Delta_{\mathrm{I}}(I) : \ldots]$ of I if and only if $I \in \mathcal{U} \cap \mathcal{H}\mathrm{ilb}^n_{p(t)}(\mathbb{K})$.

Note that the conditions given by the vanishing of the polynomials in $\mathfrak{H} = \mathfrak{H}' \cup \mathfrak{H}''$ mean that $q'(r+1)$ rows in the matrix $\mathcal{D}_1$ and one rows in the matrix $\mathcal{D}_2$ are linearly dependent (directly by the construction of the matrix $\mathcal{A}_{r+1}$ in Proposition 5.1). These conditions ensure also that $q'(r+1)+1$ rows of the matrix $\mathcal{D}_1$ are dependent, because of the well-known property of vector spaces saying that $s+1$ vectors $v_1, \ldots, v_{s+1}$, such that every subset of s elements is linearly dependent with any other vector $u \neq 0$, are dependent. The only delicate issue, that we will discuss later in Remark 5.1.1, is checking that $\mathcal{D}_2$ is not a zero matrix.

By Corollary 5.2, $I \in \mathcal{U}$ belongs to $\mathcal{H}\mathrm{ilb}^n_{p(t)}(\mathbb{K})$ if and only if the polynomials of $\mathfrak{H}$ vanish on $[\ldots : \Delta_{\mathrm{I}}(I) : \ldots]$. Note that the coefficients of the exterior products in (5.4) and (5.5) are polynomials of degree $\leqslant d+2$ (more precisely the degree is the number of non-zero s_i).

Step 2. Let I be an element of $\mathbf{Gr}_{\mathbb{K}}(q(r), \mathbb{K}[x]_r)$ and $g = (g_{i,j})$ be an element of $\mathrm{GL}(n+1)$. The Plücker coordinates $[\cdots : \Delta_{\mathrm{I}}(g \,\text{.}\, I) : \cdots]$ of $g \,\text{.}\, I \in \mathbf{Gr}_{\mathbb{K}}(q(r), \mathbb{K}[x]_r)$ are bi-homogeneous polynomials of degree 1 in the Plücker coordinates $[\cdots : \Delta_{\mathrm{I}}(I) :$

$\cdots]$ and of degree $q(r)\cdot r$ in the coefficients $g_{i,j}$ of the matrix g. So given a homogeneous polynomial P of degree $s \leqslant d+2$ in $\mathfrak{H}$, $P([\cdots : \Delta_{\mathrm{I}}(g \centerdot I) : \cdots])$ is a bi-homogeneous polynomial of degree s in $[\cdots : \Delta_{\mathrm{I}}(I) : \cdots]$ and of degree $q(r)\cdot r \cdot s$ in $g_{i,j}$. At this point we collect, and denote by C_P, the homogeneous polynomials of degree $s \leqslant d+2$ in the Plücker coordinates $[\cdots : \Delta_{\mathrm{I}}(I) : \cdots]$, that spring up as coefficients of $P([\cdots : \Delta_{\mathrm{I}}(g \centerdot I) : \cdots])$, viewed as a homogeneous polynomial of degree $q(r)\cdot r\cdot s$ in the variables $g_{i,j}$.

From Proposition 5.1 and Corollary 5.2, I belongs to $\mathcal{H}\mathrm{ilb}^n_{p(t)}(\mathbb{K})$ if and only if for a generic changes of variables $g \in \mathrm{GL}(n+1)$

$$g \centerdot I \in \mathcal{U} \cap \mathcal{H}\mathrm{ilb}^n_{p(t)}(\mathbb{K})$$

i.e.

all the homogeneous polynomials $P \in \mathfrak{H}$ vanish at $[\cdots : \Delta_{\mathrm{I}}(g \centerdot I) : \cdots]$,

or equivalently

all the coefficients C_P for $P \in \mathfrak{H}$ vanish at $[\cdots : \Delta_{\mathrm{I}}(I) : \cdots]$.

We finally proved that $I \in \mathbf{Gr}_{\mathbb{K}}(q(r), \mathbb{K}[x]_r)$ belongs to $\mathcal{H}\mathrm{ilb}^n_{p(t)}(\mathbb{K})$ if and only if $[\cdots : \Delta_{\mathrm{I}}(I) : \cdots]$ satisfies all the equations of the set

$$\bigcup_{P \in \mathfrak{H}} C_P \tag{5.6}$$

which consists of homogeneous polynomial of degree smaller than or equal to $d+2$. □

Remark 5.1.1. Note that a necessary condition in order that $\mathcal{D}_2$ is empty is that I_r has no generators belonging to G_I^r of the type g_j and now we prove that it is not possible. For the sake of simplicity, we can think about the monomial ideal obtained in the case $f_\alpha = g_j = 0$, $\forall\, \alpha$, $\forall\, j$. Such an ideal should have a Hilbert polynomial $\widetilde{p}(t)$ such that $\widetilde{p}(r) = \binom{n+r}{n} - \binom{n-d+r}{n-d}$. Let us show that $\widetilde{p}(r)$ can not be equal to $P(r)$ with P a Hilbert polynomial with Gotzmann number equal to r and of degree d.

The first point is to compute the maximal value in degree r of a Hilbert polynomial of degree d and Gotzmann number r. By the Gotzmann decomposition of

Hilbert polynomials (1.21), we know that among the Hilbert polynomials of degree d and Gotzmann number r there is

$$p(t) = \binom{t+d}{d} + \binom{t+d-1}{d} + \cdots + \binom{t+d-(r-1)}{d},$$

and that any other Hilbert polynomial has at least one binomial coefficient $\binom{t+(d-i)-j}{d-i}$ replacing $\binom{t+d-j}{d}$ $(i,j>0)$. Because of $\binom{r+d-j}{d} > \binom{r+(d-i)-j}{d-i}$, the maximal value reached is

$$P = \max\left\{P(r) \mid P(t) \text{ of degree } d \text{ and Gotzmann number } r\right\} = \sum_{i=1}^{r}\binom{d+i}{d}.$$

Finally, starting from the decomposition

$$\mathbb{K}[x_0,\ldots,x_n]_r = \bigcup_{i=0,\ldots,r} \mathbb{K}[x_0,\ldots,x_d]_i \cdot \mathbb{K}[x_{d+1},\ldots,x_n]_{r-i},$$

we have that

$$\begin{aligned}\widetilde{p}(r) &= \binom{n+r}{n} - \binom{n-d+r+1}{n-d+1} = \\ &= \sum_{i=1}^{r}\binom{d+i}{d}\binom{n-d+1+r-i}{n-d+1} > \sum_{i=1}^{r}\binom{d+i}{d} = P.\end{aligned}$$

5.2 Extension of the coefficient ring

We will now prove that the subscheme of $\mathbf{Gr}_{\mathbb{K}}(q(r), \mathbb{K}[x]_r)$ described in Theorem 5.3 representing the set $\mathcal{H}\mathrm{ilb}^n_{p(t)}(\mathbb{K})$ is indeed the Hilbert scheme we are looking for, so let us denote it by $\mathbf{Hilb}^n_{p(t)}$. We have seen in Chapter 1 that any element $Z \in \mathcal{H}\mathrm{ilb}^n_{p(t)}(A)$ defines an element of $\mathcal{G}\mathrm{r}^{N(r)}_{q(r)}(A)$, that is, being $\mathcal{G}\mathrm{r}^{N(r)}_{q(r)}$ represented by $\mathbf{Gr}_{\mathbb{K}}(q(r), N(r))$, a morphism $f_Z : \operatorname{Spec} A \to \mathbf{Gr}_{\mathbb{K}}(q(r), N(r))$. To prove that $\mathbf{Hilb}^n_{p(t)}$ represents $\mathcal{H}\mathrm{ilb}^n_{p(t)}$, we have to show that to any element $Z \in \mathcal{H}\mathrm{ilb}^n_{p(t)}(A)$ we can associate a morphism $\overline{f}_Z : \operatorname{Spec} A \to \mathbf{Hilb}^n_{p(t)}$. To accomplish this task we prove that the morphism f_Z factors through the inclusion $\mathbf{Hilb}^n_{p(t)} \hookrightarrow \mathbf{Gr}_{\mathbb{K}}(q(r), N(r))$ as subschemes of $\mathbb{P}\big[\wedge^{q(r)} \mathbb{K}[x]_r\big]$:

$$\begin{tikzcd}
\operatorname{Spec} A \arrow[rr, "f_Z"] \arrow[dr, dashed, "\overline{f}_Z"'] & \circlearrowleft & \mathbf{Gr}_{\mathbb{K}}(q(r), N(r)) \\
& \mathbf{Hilb}^n_{p(r)} \arrow[ur, hook] &
\end{tikzcd}$$

Theorem 5.4. *The Hilbert scheme $\mathbf{Hilb}^n_{p(t)}$ can be defined as subscheme of the Grassmannian $\mathbf{Gr}_{\mathbb{K}}(q(r), N(r))$ by equations of degree equal to or less than $d+2$.*

Proof. For any element of $Z \in \mathcal{H}\mathrm{ilb}^n_{p(t)}(A)$ we consider the submodule $I(Z)_r \subset A[x_0, \ldots, x_n]_r$ that belongs to $\mathcal{G}\mathrm{r}^{N(r)}_{q(r)}(A)$. To prove that the map $f_Z : \operatorname{Spec} A \to \mathbf{Gr}_{\mathbb{K}}(q(r), N(r))$ factors through $\mathbf{Hilb}^n_{p(t)}$, we now prove that an element of $I \in \mathcal{G}\mathrm{r}^{N(r)}_{q(r)}(A)$ belongs to $\mathcal{H}\mathrm{ilb}^n_{p(t)}(A)$ if and only if the equations (5.4) and (5.5) extended to the projective space $\mathbb{P}_A^{\binom{N(r)}{q(r)}-1}$ by means of the tensor product for the $\mathbb{K}$-algebra A $\mathbb{K}[x] \otimes_{\mathbb{K}} A = A[x]$ are satisfied by the Plücker coordinates of I in the Grasmmannian $\mathbf{Gr}_A(q(r), N(r)) = \mathbf{Gr}_{\mathbb{K}}(q(r), N(r)) \times_{\operatorname{Spec} \mathbb{K}} \operatorname{Spec} A$.

In Chapter 1 we discussed about the fact that the flatness is a local property, therefore we can consider A a local $\mathbb{K}$-algebra with maximal ideal $\mathfrak{m}$ and residue field $k(\mathfrak{m})$.

Step 1. Given $I_r \in \mathcal{G}\mathrm{r}^{N(r)}_{q(r)}(A)$, firstly let us prove that if equations given in Theorem 5.3 are satisfied, then I_r belongs to $\mathcal{H}\mathrm{ilb}^n_{p(t)}(A)$ (i.e. $A[x]_1 \cdot I_r$ belongs to $\mathcal{G}\mathrm{r}^{N(r+1)}_{q(r+1)}(A)$ according to Gotzmann's Persistence Theorem.

Let us consider $I_r \in \mathcal{G}\mathrm{r}^{N(r)}_{q(r)}(A)$ satisfying the extension of equations (5.4) and (5.5). Tensoring by the residue field $k(\mathfrak{m})$ and using Nakayama's Lemma, we can determine a free submodule $J_r \subset A[x]_r$ generated by $q(r)$ monomials having the Borel-fixed property, such that with a generic change of coordinates, the monomials $N(J_r) = \{x^\beta \in A[x]_r \mid x^\beta \notin J_r\}$ form a basis of $A[x]_r/I_r$ as a free A-module (see [27, Chapter 15]). Now we consider the exact sequence

$$0 \longrightarrow I_r \longrightarrow A[x]_r \longrightarrow A[x]_r/I_r \longrightarrow 0,$$

and we tensor it by the residue field $k(\mathfrak{m})$, obtaining

$$I_r \otimes_{\mathbb{K}} k(\mathfrak{m}) \longrightarrow k(\mathfrak{m})[x]_r = \mathbb{K}[x]_r \otimes_{\mathbb{K}} k(\mathfrak{m}) \longrightarrow A[x]_r/I_r \otimes_{\mathbb{K}} k(\mathfrak{m}) \longrightarrow 0.$$

Called $I_r^{k(\mathfrak{m})}$ the image of $I_r \otimes_{\mathbb{K}} k(\mathfrak{m})$ in $k(\mathfrak{m})[x]$, by the assumptions and by Theorem 5.3 we deduce that $I_r^{k(\mathfrak{m})}$ belongs to $\mathcal{H}\mathrm{ilb}^n_{p(t)}(k(\mathfrak{m}))$. Consequently, $J_r^{k(\mathfrak{m})}$ (resp. J_r) also belongs to $\mathcal{H}\mathrm{ilb}^n_{p(t)}(k(\mathfrak{m}))$ (resp. $\mathcal{H}\mathrm{ilb}^n_{p(t)}(A)$) and thus $(J_r^{k(\mathfrak{m})})$ (resp. (J_r)) defines a Borel-fixed ideal with Hilbert polynomial $p(t)$ in $k(\mathfrak{m})[x]$ (resp. in $A[x]$).

For a generic change of coordinates, as $N(J_r)$ is a basis of $A[x]_r/I_r$, I_r is also a free A-module of rank $q(r)$ with a basis of the form:

$$\left\{ x^\alpha - \sum_{x^\beta \in N(J_r)} c_{\alpha\beta} x^\beta \quad \middle| \quad x^\alpha \in J_r \right\}. \tag{5.7}$$

Therefore we can choose a system of generators for $A[x]_1 \cdot I_r$ equal to that one described with the matrix $\mathcal{A}_{r+1}$ (5.3) in Proposition 5.1. Up to a change of coordinates, the A-module $\langle \mathcal{D}_1 \rangle$ generated by the lines of $\mathcal{D}_1$ (and by definition equal to $x_0 I_r + \cdots + x_d I_r$) contains a family $\mathcal{F}$ of $q'(r+1)$ polynomials of the form:

$$\mathcal{F} = \left\{ x^\gamma - \sum_{x^\eta \in N(J_{r+1})} c_{\gamma\eta} x^\eta \quad \middle| \quad x^\gamma \in x_0 J_r + \cdots + x_d J_r \right\}. \tag{5.8}$$

Let us prove it by induction on $0 \leqslant i \leqslant d$ for $x^\alpha \in x_i J_r$. Let $x^\gamma = x_0 x^\alpha$ with x^α in J_r. Among the generators (5.7) of I_r there is

$$x^\alpha - \sum_{x^\beta \in N(J_r)} c_{\alpha\beta} x^\beta,$$

thus

$$x_0 x^\alpha - \sum_{x^\beta \in N(J_r)} c_{\alpha\beta} x_0 x^\beta$$

belongs to $x_0 I_r \subset x_0 I_r + \ldots + x_d I_r$. Moreover because of (J_r) is Borel-fixed and $(J_{r+1} : A[x]_1) = J_r$ ($J_r \in \mathcal{H}\mathrm{ilb}^n_{p(t)}(A)$), it is easy to check that $x_0 N(J_r) \subset N(J_{r+1})$. Hence the assertion is proved for $i = 0$ and let us suppose that it holds for all $0 \leqslant j < i$. Considered $x^\gamma = x_i x^\alpha$ with $x^\alpha \in J_r$, again

$$x_i x^\alpha - \sum_{x^\beta \in N(J_r)} c_{\alpha\beta} x_i x^\beta \in x_i I_r \subset x_0 I_r + \ldots + x_d I_r.$$

If $x_i x^\beta$ ($x^\beta \in N(J_r)$) does not belong to $N(J_{r+1})$, then there exists $x^\epsilon \in J_r$ such that

$$x_i x^\beta = x_j x^\epsilon$$

and $j < i$ because of the Borel-fixed property. Then, by induction, we can replace $x_i x^\beta = x_j x^\epsilon$ with an element of the A-module generated by $N(J_{r+1})$ modulo $x_0 I_r + \cdots + x_d I_r$, finally proving that the family $\mathcal{F}$ described in (5.8) belongs to $\langle \mathcal{D}_1 \rangle$ (i.e. to $x_0 I_r + \ldots + x_d I_r$).

As equations of Theorem 5.3 are satisfied, equations (5.4) and (5.5) are also satisfied for a generic change of coordinates, so that we can assume without loss of generality that there exist J_r Borel-fixed and $\mathcal{F}$ as in (5.8), such that I_r satisfies (5.4) and (5.5).

Now we want to show that equations (5.4) and (5.5) imply that $\mathcal{F}$ generates the A-module $\langle \mathcal{D}_2 \rangle$ spanned by the lines of $\mathcal{D}_2$. As a matter of fact, equations (5.4) and (5.5) imply that the exterior product between $q'(r+1)$ polynomials in $\langle \mathcal{D}_1 \rangle$ and one polynomial in $\langle \mathcal{D}_2 \rangle$ always vanishes. In particular, the exterior product between the $q'(r+1)$ polynomials that belong to $\mathcal{F}$ and any polynomial g in $\langle \mathcal{D}_2 \rangle$ is equal to zero. We deduce easily that g belongs to $\langle \mathcal{F} \rangle$ and that $\mathcal{F}$ generates $\langle \mathcal{D}_2 \rangle$.

Moreover $\mathcal{F}$ generates $\langle \mathcal{D}_1 \rangle$. With the same reasoning used in the proof of Theorem 5.3 and in Remark 5.1.1, it is easy to prove that any exterior product between $q'(r+1)+1$ polynomials in $\langle \mathcal{D}_1 \rangle$ is equal to zero. In particular, the exterior product between the $q'(r+1)$ polynomials that belong to $\mathcal{F}$ and any polynomial g in $\langle \mathcal{D}_1 \rangle$ is equal to zero. So again g belongs to $\langle \mathcal{F} \rangle$ and $\mathcal{F}$ generates $\langle \mathcal{D}_1 \rangle$ (keeping in mind that the free A-module $A[x]_r$ has a basis that contains $\mathcal{F}$).

Finally, we conclude that I_{r+1} is a free A-module with basis $\mathcal{F}$ plus the polynomials represented by the lines in the first rows of $\mathcal{A}_{r+1}$ and rewriting this family of polynomials using linear combinations of elements in $\mathcal{F}$ we can obtain a basis of the form

$$\left\{ x^\gamma - \sum_{x^\eta \in N(J_{r+1})} c_{\gamma\eta} x^\eta \;\middle|\; x^\gamma \in J_{r+1} \right\}. \tag{5.9}$$

$A[x]_{r+1}/I_{r+1}$ turns out to be an A-module with basis $N(J_{r+1})$, so $I_r \in \mathcal{H}\mathrm{ilb}^n_{p(t)}(A)$.

Step 2. Let us suppose that $I_r \in \mathcal{G}\mathrm{r}^{N(r)}_{q(r)}(A)$ belongs to $\mathcal{H}\mathrm{ilb}^n_{p(t)}(A)$ and let us prove that it satisfies the extension of equations given in Theorem 5.3. This is equivalent to prove that equations (5.4) and (5.5) are satisfied for a generic changes of coordinates. From the Generic Initial Theorem [27, Theorem 15.20] and Nakayama's

Lemma, there exists a Borel-fixed monomial ideal J with Hilbert polynomial $p(t)$, such that for a generic changes of coordinates, $N(J_r)$ and $N(J_{r+1})$ are a basis of respectively $A[x]_r/I_r$ and $A[x]_{r+1}/I_{r+1}$ as free A-modules. As mentioned in **Step 1**, we can represent I_{r+1} with the matrix $\mathcal{A}_{r+1}$ introduced in Proposition 5.1 and find a family $\mathcal{F}$ in $\langle \mathcal{D}_1 \rangle$ of the form (5.8).

As $N(J_{r+1})$ is a basis of $A[x]_{r+1}/I_{r+1}$ as a free A-module, every polynomial given by a line in $\mathcal{D}_1$ or $\mathcal{D}_2$ belongs to $\langle \mathcal{F} \rangle$. Therefore equations (5.4) and (5.5) are satisfied for a generic change of coordinates and equations $\mathfrak{H}$ of Theorem 5.3 are satisfied. □

Example 5.2.1. Let us apply Theorem 5.3 to $\mathbf{Hilb}^2_2$, already considered in Example 1.5.1, Example 1.5.4 and Example 1.5.5. Since the Hilbert polynomial is constant, to compute the first set of equations $\mathfrak{H}'$ (5.4), we have to consider the wedge product between $x_0\Lambda^{(4)}_{\emptyset}$ and the 2 elements $x_2\Lambda^{(1)}_{123}, x_1\Lambda^{(1)}_{123}$ not containing monomials in $\mathbb{K}[x_2, x_1]_2$. We obtain 12 polynomials:

- $-\Delta_{26}\Delta_{46} + \Delta_{45}\Delta_{46} + \Delta_{16}\Delta_{56}$,
- $-\Delta_{24}\Delta_{46} + \Delta_{14}\Delta_{56}$,
- $-\Delta_{24}\Delta_{45} + \Delta_{12}\Delta_{56}$,
- $-\Delta_{36}\Delta_{46} + \Delta_{26}\Delta_{56} + \Delta_{45}\Delta_{56}$,
- $-\Delta_{45}^2 - \Delta_{34}\Delta_{46} + \Delta_{24}\Delta_{56}$,
- $-\Delta_{25}\Delta_{45} + \Delta_{23}\Delta_{46}$,
- $-\Delta_{45}^2 + \Delta_{25}\Delta_{46} - \Delta_{15}\Delta_{56}$,
- $\Delta_{34}\Delta_{45} + \Delta_{23}\Delta_{46} - \Delta_{13}\Delta_{56}$,
- $\Delta_{14}\Delta_{45} - \Delta_{12}\Delta_{46}$,
- $\Delta_{35}\Delta_{46} - \Delta_{25}\Delta_{56}$,
- $\Delta_{35}\Delta_{45} - \Delta_{23}\Delta_{56}$,
- $\Delta_{15}\Delta_{45} - \Delta_{13}\Delta_{46} + \Delta_{12}\Delta_{56}$.

To compute the second set of equations $\mathfrak{H}''$ (5.5), we have to consider the coefficients of the wedge product between $x_0\Lambda^{(4)}_{\emptyset}$ and one of the following elements

$$\begin{array}{ll} x_1\Lambda^{(1)}_{234} + x_2\Lambda^{(1)}_{134}, & x_1\Lambda^{(1)}_{134} + x_2\Lambda^{(1)}_{124}, \\ x_1\Lambda^{(1)}_{235} + x_2\Lambda^{(1)}_{135}, & x_1\Lambda^{(1)}_{135} + x_2\Lambda^{(1)}_{125}, \\ x_1\Lambda^{(1)}_{236} + x_2\Lambda^{(1)}_{136}, & x_1\Lambda^{(1)}_{136} + x_2\Lambda^{(1)}_{126}. \end{array}$$

We obtain other 36 generators:

- $\Delta_{16}\Delta_{25} - \Delta_{15}\Delta_{26} - \Delta_{24}\Delta_{26} + \Delta_{14}\Delta_{36}$,
- $-\Delta_{15}\Delta_{24} - \Delta_{24}^2 + \Delta_{14}\Delta_{25} + \Delta_{14}\Delta_{34}$,
- $-\Delta_{14}\Delta_{23} + \Delta_{12}\Delta_{25}$,
- $-\Delta_{25}\Delta_{26} - \Delta_{26}\Delta_{34} + \Delta_{16}\Delta_{35} + \Delta_{24}\Delta_{36}$,
- $-\Delta_{24}\Delta_{25} + \Delta_{14}\Delta_{35}$,
- $-\Delta_{23}\Delta_{24} + \Delta_{12}\Delta_{35}$,
- $-\Delta_{24}\Delta_{46} + \Delta_{14}\Delta_{56}$,
- $\Delta_{24}\Delta_{25} - \Delta_{14}\Delta_{35}$,
- $\Delta_{15}\Delta_{23} + \Delta_{23}\Delta_{24} - \Delta_{13}\Delta_{25}$,
- $\Delta_{13}\Delta_{14} - \Delta_{12}\Delta_{15} - \Delta_{12}\Delta_{24}$,
- $\Delta_{25}^2 + \Delta_{25}\Delta_{34} - \Delta_{15}\Delta_{35} - \Delta_{24}\Delta_{35}$,
- $\Delta_{23}\Delta_{25} + \Delta_{23}\Delta_{34} - \Delta_{13}\Delta_{35}$,
- $\Delta_{13}\Delta_{24} - \Delta_{12}\Delta_{25} - \Delta_{12}\Delta_{34}$,
- $\Delta_{16}\Delta_{25} - \Delta_{15}\Delta_{26} + \Delta_{24}\Delta_{45}$,

- $-\Delta_{16}\Delta_{24}+\Delta_{14}\Delta_{26}-\Delta_{14}\Delta_{45}$,
- $\Delta_{24}^2-\Delta_{14}\Delta_{25}+\Delta_{12}\Delta_{26}$,
- $-\Delta_{26}^2+\Delta_{16}\Delta_{36}-\Delta_{34}\Delta_{46}+\Delta_{24}\Delta_{56}$,
- $-\Delta_{24}\Delta_{26}+\Delta_{14}\Delta_{36}-\Delta_{24}\Delta_{45}$,
- $-\Delta_{24}\Delta_{25}+\Delta_{24}\Delta_{34}+\Delta_{12}\Delta_{36}$,
- $\Delta_{26}^2-\Delta_{16}\Delta_{36}-\Delta_{25}\Delta_{46}+\Delta_{15}\Delta_{56}$,
- $\Delta_{24}\Delta_{26}-\Delta_{16}\Delta_{34}-\Delta_{15}\Delta_{45}$,
- $\Delta_{16}\Delta_{23}-\Delta_{15}\Delta_{25}+\Delta_{24}\Delta_{25}$,
- $-\Delta_{35}\Delta_{46}+\Delta_{25}\Delta_{56}$,
- $-\Delta_{26}\Delta_{34}+\Delta_{24}\Delta_{36}-\Delta_{25}\Delta_{45}$,
- $-\Delta_{25}^2+\Delta_{23}\Delta_{26}+\Delta_{24}\Delta_{35}$,
- $\Delta_{16}\Delta_{23}-\Delta_{13}\Delta_{26}-\Delta_{24}\Delta_{34}+\Delta_{14}\Delta_{35}$,
- $\Delta_{14}\Delta_{15}-\Delta_{12}\Delta_{16}-\Delta_{14}\Delta_{24}$,
- $\Delta_{25}\Delta_{26}-\Delta_{15}\Delta_{36}+\Delta_{34}\Delta_{45}$,
- $\Delta_{23}\Delta_{26}-\Delta_{34}^2+\Delta_{24}\Delta_{35}-\Delta_{13}\Delta_{36}$,
- $\Delta_{15}\Delta_{24}-\Delta_{12}\Delta_{26}-\Delta_{14}\Delta_{34}$,
- $-\Delta_{25}\Delta_{26}+\Delta_{16}\Delta_{35}+\Delta_{25}\Delta_{45}$,
- $-\Delta_{23}\Delta_{26}-\Delta_{25}\Delta_{34}+\Delta_{15}\Delta_{35}$,
- $\Delta_{15}^2-\Delta_{13}\Delta_{16}-\Delta_{14}\Delta_{25}+\Delta_{12}\Delta_{26}$,
- $\Delta_{26}\Delta_{35}-\Delta_{25}\Delta_{36}+\Delta_{35}\Delta_{45}$,
- $\Delta_{25}\Delta_{35}-\Delta_{34}\Delta_{35}-\Delta_{23}\Delta_{36}$,
- $\Delta_{15}\Delta_{25}-\Delta_{13}\Delta_{26}-\Delta_{14}\Delta_{35}+\Delta_{12}\Delta_{36}$.

Now we need to introduce the action of $\mathrm{GL}_{\mathbb{K}}(3)$ on $\mathbb{K}[x_0,x_1,x_2]$ and to understand how the induced action on $\mathbf{Gr}_{\mathbb{K}}(4,6)$ works. Given an element $g=(g_{ij})\in \mathrm{GL}_{\mathbb{K}}(3)$ and its action

$$\begin{pmatrix} g_{00} & g_{01} & g_{02}\\ g_{10} & g_{11} & g_{12}\\ g_{20} & g_{21} & g_{22}\end{pmatrix}\cdot\begin{pmatrix} x_0\\ x_1\\ x_2\end{pmatrix},$$

the induced action on the $\mathbb{K}$-vector space $\mathbb{K}[x_0,x_1,x_2]_2$ is represented by the matrix

$$\begin{pmatrix} g_{00}^2 & 2g_{00}g_{01} & 2g_{00}g_{02} & g_{01}^2 & 2g_{01}g_{02} & g_{02}^2\\ g_{00}g_{10} & g_{01}g_{10}+g_{00}g_{11} & g_{02}g_{10}+g_{00}g_{12} & g_{01}g_{11} & g_{02}g_{11}+g_{01}g_{12} & g_{02}g_{12}\\ g_{00}g_{20} & g_{01}g_{20}+g_{00}g_{21} & g_{02}g_{20}+g_{00}g_{22} & g_{01}g_{21} & g_{02}g_{21}+g_{01}g_{22} & g_{02}g_{22}\\ g_{10}^2 & 2g_{10}g_{11} & 2g_{10}g_{12} & g_{11}^2 & 2g_{11}g_{12} & g_{12}^2\\ g_{10}g_{20} & g_{11}g_{20}+g_{10}g_{21} & g_{12}g_{20}+g_{10}g_{22} & g_{11}g_{21} & g_{12}g_{21}+g_{11}g_{22} & g_{12}g_{22}\\ g_{20}^2 & 2g_{20}g_{21} & 2g_{20}g_{22} & g_{21}^2 & 2g_{21}g_{22} & g_{22}^2\end{pmatrix}\cdot\begin{pmatrix} x_0^2\\ x_0x_1\\ x_0x_2\\ x_1^2\\ x_1x_2\\ x_2^2\end{pmatrix}.$$

To write explicitly the action of g on the Plücker coordinates, we can consider the element $\Lambda^{(4)}_{123456}$, substitute each element x^β of the basis of $\mathbb{K}[x_0,x_1,x_2]_2$ with $g\cdot x^\beta$ obtaining the exterior product $g\cdot\Lambda^{(4)}_{123456}$ and then the action is determined by looking at the coefficients of the same element of the basis of $\wedge^4\mathbb{K}[x_0,x_1,x_2]_2$ in $\Lambda^{(4)}_{123456}$ and $g\cdot\Lambda^{(4)}_{123456}$. For instance the Plücker coordinate Δ_{25}, coefficient of $x_2^2\wedge x_1^2\wedge x_2x_0\wedge x_0^2$ in $\Lambda^{(4)}_{123456}$, will be send by the action of g to the coefficient of $x_2^2\wedge x_1^2\wedge x_2x_0\wedge x_0^2$ in $g\cdot\Lambda^{(4)}_{123456}$.

Finally, for every polynomial P contained in $\mathfrak{H}$, we have to compute the action of g, that is substituting each Δ_{ab} with $g\cdot\Delta_{ab}$, and then we have to collect all the coefficients (polynomials in the Plücker coordinates of degree 2) of $g \centerdot P$, viewed as polynomial in the variables g_{ij}. For instance collecting the coefficients of the polynomial $g \centerdot (\Delta_{35}\Delta_{46} - \Delta_{25}\Delta_{56})$, $\Delta_{35}\Delta_{46} - \Delta_{25}\Delta_{56} \in \mathfrak{H}'$, we obtain 3495 polynomials that give some of the equations defining the Hilbert scheme $\mathbf{Hilb}^2_2$.

Chapter 6

On the connectedness of Hilbert schemes of locally Cohen-Macaulay curves in $\mathbb{P}^3$

In this chapter I expose the first results obtained in collaboration with E. Schlesinger about the question of the connectedness of the Hilbert scheme of locally Cohen-Macaulay (lcm for short) curves in $\mathbb{P}^3$. Basically this is an expanded version of the paper "The Hilbert schemes of locally Cohen-Macaulay curves in $\mathbb{P}^3$ may after all be connected" [59].

This chapter wants to be a further evidence of the potential of the ideas we are proposing to study many problems of geometric nature. Indeed looking at locally Cohen-Macaulay curves in $\mathbb{P}^3$ with a computational and combinatorial eye, we prove quite easily the connectedness of the Hilbert scheme of (locally Cohen-Macaulay) curves of degree 4 and genus -3, considered one of the best candidates to be a non-connected Hilbert scheme.

Throughout this chapter, unless otherwise specified we will say Hilbert scheme meaning Hilbert scheme of locally Cohen-Macaulay curves.

6.1 Introduction to the problem

By the term *curve* we will mean a one dimensional subscheme without isolated or embedded zero dimensional components; in the literature such an object is usually called a *locally Cohen-Macaulay* curve. One of the major tool in the study of Hilbert schemes of codimension two subschemes of a fixed projective scheme is the liaison theory. We refer to [46] and [71] for a modern treatment of this topic. Let us recall the main results of biliaison theory in the context of curves in $\mathbb{P}^3$. We say that a curve D is obtained from a curve C by *elementary biliaison* of *height* h if D is linearly equivalent to $C + hH$ on a surface S that contains both C and D and has H as its plane section [44] (see also [66, Chapter III]). For example, an effective divisor D of bidegree (a, b) on a smooth quadric surface $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ is obtained by an elementary biliaison of nonnegative height from a curve C which is the disjoint union of $d = |a - b|$ lines on the quadric. Biliaison is the equivalence relation generated by elementary biliaisons. Rao [85] proved that there is an invariant that distinguishes biliaison equivalence classes: the finite length graded module $M_C = \bigoplus_{n\in\mathbb{Z}} H^1(\mathbb{P}^3, \mathcal{I}_C(n))$, which is now commonly referred to as the Rao (or Hartshorne-Rao) module of C; two curves are in the same biliaison class if and only if their Rao modules are isomorphic up to a twist. The structure of a biliaison class is well understood. A curve C in a biliaison class is said to be *minimal* if for every other curve D of the class $M_D \simeq M_C(-h)$ with $h \geqslant 0$; this means the Rao module of D is obtained shifting the Rao module of C to the right. The main result of the theory is:

(i) Every biliaison class contains minimal curves; the family of the minimal curves of the class is irreducible, and any two minimal curves of the class can be joined by a finite number of elementary biliaisons of height zero.

(ii) If D is a non minimal curve, then D is obtained from a minimal curve C of the class by a finite sequence of elementary biliaisons of positive height. This is known as the *Lazarsfeld-Rao property*.

Note that from 6.1 it follows that a curve is minimal in its biliaison class if and only if it has minimal degree among curves of the class. This result has a long history. Lazarsfeld and Rao [56] proved the Lazarsfeld Rao property under a cohomological condition on C that guarantees C is minimal; they only considered *trivial* elementary biliaisons ($C_i = C_{i-1} + nH$, no linear equivalence allowed), but at the end of the process they needed a deformation with constant cohomology. The existence of minimal curves and the Lazarsfeld-Rao property were proven independently in [66] and [5]. Strano [100] showed the deformation at the end of the Lazarsfeld-Rao process is not needed if one allows linear equivalence in the definition of elementary biliaison. The version of the theorem we have given is due to Hartshorne [46, Theorem 3.4], where the precise conditions on the ambient projective scheme are determined for the Lazarsfeld-Rao property to hold for biliaison classes of codimension 2 subschemes.

We would like to stress the fact that for this theory to work it is necessary to consider locally Cohen-Macaulay curves: even if one starts with a smooth irreducible curve, the minimal curve of the class may fail to be reduced or irreducible; it may even not be generically a local complete intersection (as it was assumed by Rao in [85]). So let $\mathbf{H}_{d,g}$ denote the Hilbert scheme parametrizing (locally Cohen-Macaulay) curves of degree d and arithmetic genus g in $\mathbb{P}^3$; it is an open subscheme of the full Hilbert scheme $\mathbf{Hilb}^3_{dt+1-g}$. In [67] and [68] Martin-Deschamps and Perrin have shown that, whenever nonempty, $\mathbf{H}_{d,g}$ has a component $E_{d,g}$ whose closed points correspond to curves that have maximal cohomology, in the sense that

$$\dim H^i(\mathbb{P}^3, \mathcal{I}_C(n)) \leqslant \dim H^i(\mathbb{P}^3, \mathcal{I}_E(n)) \tag{6.1}$$

for every $C \in \mathbf{H}_{d,g}$, $E \in E_{d,g}$, $n \in \mathbb{Z}$ and $i = 0, 1, 2$. Note that by (6.1) there is no obstruction coming from the semicontinuity of cohomology to specializing any curve in $\mathbf{H}_{d,g}$ to an extremal curve. This remark raised the question whether every curve can be specialized to an extremal curve. This is known to be false for curves that are not generically of embedding dimension two [81], but it is open for, say, smooth curves. A weaker version of this question, proposed by Hartshorne in [45] and [47], is whether $\mathbf{H}_{d,g}$ is connected, that is, if every curve in $\mathbf{H}_{d,g}$ belongs to the connected component containing the extremal curves. It is an interesting problem because the Hilbert scheme of smooth curves is not connected [43], while the full Hilbert scheme is connected (as seen in Chapter 3), but through schemes with zero dimensional components.

Let us review what is known about the problem of connectedness of $\mathbf{H}_{d,g}$. No example of a nonconnected $\mathbf{H}_{d,g}$ has been found so far. The Hilbert scheme $\mathbf{H}_{d,g}$ is connected when $g \geqslant \binom{d-3}{2} - 1$ (see [1, 48, 91]) and when $d \leqslant 4$ (see [79, 81]; this is non trivial, because when g is very negative the Hilbert scheme has a large number of irreducible components). Hartshorne [45] has shown that smooth irreducible nonspecial curves and arithmetically Cohen-Macaulay (ACM) curves are in the connected component containing the extremal curves, and so are Koszul curves by [84]. Building on the results of [45], E. Schlesinger has shown [92] that, if a minimal curve C can be connected to an extremal curve by flat families lying on surfaces of degree s, where s is the least degree of a surface containing C, then *every* curve in the biliaison class of C is in the connected component of extremal curves in its Hilbert scheme.

By [44] and [49], any curve contained in a singular quadric surface, including a double plane, is in the connected component of extremal curves [49]. On the other hand, the case of curves on a smooth quadric surface has been open so far. By biliaison it is enough to deal with divisors of bidegree $(d, 0)$. The case $d \leqslant 2$ is then trivial, and Nollet [79] has shown that is possible to specialize a divisor of bidegree $(3, 0)$ to an extremal curve, but it has been an open question whether one could specialize four (or more) disjoint lines on a smooth quadric to an extremal curve of the same genus; consequently, the Hilbert scheme of curves of degree $\mathbf{H}_{10,12}$, which has an irreducible component whose general member is a divisor of bidegree $(7, 3)$ on a smooth quadric surface, was proposed as a candidate for an example of a nonconnected $\mathbf{H}_{d,g}$: see [45, Ex. 4.2], [47, Section 4], and the open problems list of the 2010 AIM workshop on *Components of Hilbert Schemes* available at `aimpl.org/hilbertschemes`.

6.2 Extremal curves

In this section we establish notation and terminology and review some known results that we will need later. We work over an algebraically closed field $\mathbb{K}$ of arbitrary characteristic. We denote with the symbol $\mathcal{I}_X$ the ideal sheaf of a subscheme $X \subset \mathbb{P}^3$. Given a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^3$, we define $h^i(\mathcal{F}) = \dim H^i(\mathbb{P}^3, \mathcal{F})$ and

$H^i_*(\mathcal{F}) = \bigoplus_{n\in\mathbb{Z}} H^i(\mathbb{P}^3, \mathcal{F}(n))$. We write $\mathbb{K}[x] = \mathbb{K}[x_0, x_1, x_2, x_3]$ for the homogeneous coordinate ring $H^0_*(\mathcal{O}_{\mathbb{P}^3})$ of $\mathbb{P}^3$.

Definition 6.1. A *curve* in $\mathbb{P}^3$ (or more precisely a locally Cohen-Macaulay curve) is a one dimensional subscheme $C \subset \mathbb{P}^3$ without zero dimensional associated points; this means that all irreducible components of C have dimension 1, and that C has no embedded points.

We denote by $\mathbf{H}_{d,g}$ the Hilbert scheme parametrizing curves of degree d and arithmetic genus g in $\mathbb{P}^3$ [66]. This is an open subscheme of the Hilbert scheme $\mathbf{Hilb}^3_{dt+1-g}$ parametrizing all one dimensional subschemes of $\mathbb{P}^3$ with Hilbert polynomial $dt + 1 - g$.

Definition 6.2. We say that a curve $E \subset \mathbb{P}^3$ of degree d and genus g is *extremal* if

$$h^i(\mathcal{I}_C(n)) \leqslant h^i(\mathbb{P}^3, \mathcal{I}_E(n)) \tag{6.2}$$

for $i = 0, 1, 2$, for every curve C of degree d and genus g, and for every $n \in \mathbb{Z}$,

Thus a curve is extremal if it has the largest possible cohomology. One knows that $\mathbf{H}_{d,g}$ is nonempty if and only if either $g = \binom{d-1}{2}$, in which case it is irreducible and consists of plane curves, or $g \leqslant \binom{d-2}{2}$. Whenever nonempty, $\mathbf{H}_{d,g}$ contains extremal curves [67]; in fact, the extremal curves form an irreducible $E_{d,g}$ of $\mathbf{H}_{d,g}$ [68].

Remark 6.2.1. Our definition of extremal curves is equivalent to the one given by Hartshorne [45, 47]. Martin-Deschamps and Perrin did not include ACM curves with maximal cohomology among extremal curves; the difference comes up only when $g = \binom{d-1}{2}$ and $g = \binom{d-2}{2}$. In [67] the functions $h^i(\mathcal{I}_E(n))$ are computed explicitly for an extremal curve E, and the bounds (6.2) are proven, under the assumption that the field $\mathbb{K}$ has characteristic zero; this assumption on the characteristic is not necessary [80].

Martin-Deschamps and Perrin also compute the Rao module $M_E = H^1_*(\mathcal{I}_E)$ of an extremal curve:

Theorem 6.3 ([67, 68])**.** *Let (d, g) be two integers satisfying $d \geqslant 2$ and $g \leqslant \binom{d-2}{2} - 1$. A curve E of degree d and genus g is extremal if and only if*

$$M_E \simeq R/(l_1, l_2, F, G)(b)$$

where (l_1, l_2, F, G) is a regular sequence, $\deg l_1 = \deg l_2 = 1$, $\deg F = \binom{d-2}{2} - g$, $\deg G = \binom{d-1}{2} - g$, and $b = \deg F - 1$.

The following proposition describes extremal curves supported on a line. It is a special case of [68, Proposition 0.6]; we state it in the form needed later in the chapter.

Proposition 6.4. *Let (d, g) be a pair of integers satisfying $g \leqslant \binom{d-2}{2} - 1$. Let F and G be two forms of degrees $\deg F = \binom{d-2}{2} - g$ and $\deg G = \binom{d-1}{2} - g$ in $\mathbb{K}[x_1, x_0]$ with no common zeros. The surface S of equation $x_3 G - x_2^{d-1} F = 0$ is irreducible and generically smooth along the line L of equations $x_3 = x_2 = 0$. It therefore contains a unique curve E of degree d supported on L. The curve E is extremal of degree d and genus g, and its Rao module is*

$$M_E \simeq \mathbb{K}[x_0, x_1, x_2, x_3]/(x_3, x_2, F, G)(b) \simeq \mathbb{K}[x_0, x_1]/(F, G)(b)$$

where $b = \deg(F) + 1$. The homogeneous ideal of E is generated by $x_3^2, x_3 x_2, x_2^d$ and $x_3 G - x_2^{d-1} F$.

Proof. The surface S is irreducible because F and G have no common zeros, and it is smooth at points of L where G is different from zero. Therefore the ideal of L in the local ring $\mathcal{O}_{S,\xi}$ of the generic point ξ of L is generated by one function t, and the ideal of a curve of degree d supported on L must be $t^d \mathcal{O}_{S,\xi}$. Since a locally Cohen-Macaulay curve supported on L is determined by its ideal at the generic point of L, we see that there is a unique curve $D_m \subset S$ supported on L of degree m for every $m \geqslant 1$. For $m = d - 1$, the curve $P = D_{d-1}$ is the planar multiple structure of equations $x_3 = x_2^{d-1} = 0$. We note that $\mathcal{I}_P \otimes \mathcal{O}_L \simeq \mathcal{O}_L(-1) \oplus \mathcal{O}_L(1-d)$ where the two generators are the images of x_3 and x_2^{d-1}. The two forms F and G define a surjective map $\mathcal{O}_L(-1) \oplus \mathcal{O}_L(1-d) \to \mathcal{O}_L(b)$; composing this with the natural map

$\mathcal{I}_P \to \mathcal{I}_P \otimes \mathcal{O}_L$ we obtain a surjection $\phi : \mathcal{I}_P \to \mathcal{O}_L(b)$. We let E be the subscheme of $\mathbb{P}^3$ whose ideal sheaf is the kernel of ϕ. By construction we have an exact sequence

$$0 \longrightarrow \mathcal{I}_E \longrightarrow \mathcal{I}_P \longrightarrow \mathcal{O}_L(b) \longrightarrow 0. \tag{6.3}$$

This sequence shows that E is a (locally Cohen-Macaulay) curve of degree d and genus g, and that its homogeneous ideal is generated by x_3^2,x_3x_2,x_2^d and $x_3G - x_2^{d-1}F$. Therefore $E = D_d$ is the unique curve of degree d contained in S and supported on L. Finally, the long exact cohomology sequence associated to (6.3) shows that the Rao module of E is

$$M_E = \mathbb{K}[x_0, x_1]/(F, G)(b) = \mathbb{K}[x_0, x_1, x_2, x_3]/(x_3, x_2, F, G)(b).$$

Hence E is an extremal curve. $\square$

Extremal curves seem to have a special role on the study of the connectedness of $\mathbf{H}_{d,g}$ as the lexicographic ideal for the connectedness of the full Hilbert scheme $\mathbf{Hilb}^n_{p(t)}$. The lexicographic ideal is so important for two reasons:

1. every Hilbert scheme $\mathbf{Hilb}^n_{p(t)}$ contains a point defined by the lexicographic ideal associated to $p(t)$;

2. being the lexicographic ideal a segment ideal w.r.t. `DegLex`, this term ordering fixes the "direction" to follow to define flat deformations and specializations of ideals in order to approach the lexicographic point (polarizations for Hartshorne [42], Gröbner degenerations for Peeva and Stillman [83] and Borel degenerations in Chapter 3).

Each Hilbert scheme of locally Cohen-Macaulay curves $\mathbf{H}_{d,g}$ contains extremal curves [66], therefore they are a good candidate to play the analogous special role as the lexicographic ideal. The point we want to discuss is how to detect the "direction" that allows to approaching them, basically applying Gröbner degenerations. This direction can not be given by a term ordering, mainly for two reasons:

- there is not a unique extremal curve, but we have to consider a wide class of curves, precisely a component of $\mathbf{H}_{d,g}$;

- the ideal defining an extremal curve is not a monomial ideal.

Thus the idea is to consider a weight order ω such that

1. it does not distinguish x_1 and x_0;

2. the generator $x_3G - x_2^{d-1}F$ is ω-homogeneous.

The first consequence is that the order defined by ω on the monomials of $\mathbb{K}[x]$ is not a total order, but the Gröbner machinery still works. Indeed given a polynomial $P(x_0, x_1, x_2, x_3)$, the *initial form* $\mathsf{in}_\omega(P)$ of P with respect to ω is the sum of all the terms $c_\alpha x^\alpha$ in P for which the scalar product

$$\omega \cdot \alpha = \omega_3\alpha_3 + \omega_2\alpha_2 + \omega_1\alpha_1 + \omega_0\alpha_0$$

is maximal. The initial ideal $\mathsf{in}_\omega(I)$ of an ideal I is the ideal generated by the initial forms $\mathsf{in}_\omega(P)$ as P varies in I. Also in this case, there is a flat family over the affine line $\mathbb{A}^1 = \operatorname{Spec} \mathbb{K}[t]$ whose fibers over $t \neq 0$ are isomorphic to $\operatorname{Proj} \mathbb{K}[x]/I$, while the special fiber over zero is the subscheme of $\mathbb{P}^3$ defined by the $\mathsf{in}_\omega(I)$: see for example [9] and [27, Theorem 15.17]. Roughly, this family is defined letting the one dimensional torus act on $\mathbb{P}^3$ by $t \centerdot [x_3 : x_2 : x_1 : x_0] = [t^{\omega_3}x : t^{\omega_3}x_2 : t^{\omega_1}x_1 : t^{\omega_0}x_0]$ and taking the limit as t goes to zero, so that set theoretically we are projecting C onto the line L of equations $x_3 = x_2 = 0$, but what is interesting is the scheme theoretic structure of the limit.

Let us suppose

$$x_3 >_\omega x_2 >_\omega x_1 =_\omega x_0 \tag{6.4}$$

and $\deg_\omega x_1 = \deg_\omega x_0 = 1$, i.e. $\omega = (\omega_3, \omega_2, 1, 1)$. The order induced by the hypothesis (6.4) on the monomials of a fixed degree can be represented as a planar graph in similar way to that one used to represent the Borel partial order $\leq_B$.

Definition 6.5. We call $\Omega(m)$ the graph defined as follows:

- the vertices correspond to the classes of polynomials $x_3^{\alpha_3}x_2^{\alpha_2}F$, with F in $\mathbb{K}[x_0, x_1]$ such that $\alpha_3 + \alpha_2 + \deg F = m$;

- the edges correspond to the two classes of transformations $\underline{\mathsf{e}}_3^- = \frac{x_2}{x_3}$ and $\underline{\mathsf{e}}_2^- = \frac{l}{x_2}$ with $l \in \mathbb{K}[x_0, x_1]_1$.

Proposition 6.6. *The graph $\Omega(m)$ is isomorphic to the graph representing the poset $\mathcal{P}(2,m)$.*

Proof. Assumed that the monomials of $\mathcal{P}(2,m)$ belong to the ring $\mathbb{K}[x_1,x_2,x_3]$, it suffices to consider the map $\psi : \Omega(m) \to \mathcal{P}(2,m)$

$$\begin{array}{ccll} x_3^{\alpha_3}x_2^{\alpha_2}F & \longmapsto & x_3^{\alpha_3}x_2^{\alpha_2}x_1^{\deg F}, & F \in \mathbb{K}[x_1,x_0] \\ \underline{\mathsf{e}}_i^- & \longmapsto & \mathsf{e}_i^-, & i = 2,3. \end{array}$$

□

Definition 6.7. Let $I \subset \mathbb{K}[x]$ be an ideal defining a locally Cohen-Macaulay curve C in $\mathbf{H}_{d,g}$, $d \geqslant 3$ and let r be the maximal degree of a generator of I. We will say that I is a *lcm-segment* ideal if there exists a weight order $\omega = (\omega_3, \omega_2, 1, 1)$ such that

(i) every polynomial in the basis of $\langle I_r \rangle$ as $\mathbb{K}$-vector space is ω-homogeneous;

(ii) the image through ψ of the set of all monomials involved in a basis of $\langle I_r \rangle$ is a segment of $\mathcal{P}(2,r)$ w.r.t. $\overline{\omega} = (\omega_3, \omega_2, 1)$.

Proposition 6.8. *The ideal $I_E \subset \mathbb{K}[x_0,x_1,x_2,x_3]$ defining any extremal curve $E \in \mathbf{H}_{d,g}$ supported on a line is a lcm-segment ideal w.r.t. $\omega = (d,2,1,1)$.*

Proof. By Proposition 6.4, we know that the ideal of an extremal curve in of degree d and genus g supported on a line is

$$I_E = (x_3^2, x_3x_2, x_2^d, x_3G - x_2^{d-1}F) \quad \text{with} \quad \deg(x_3G - x_2^{d-1}F) = \binom{d-1}{2} - g + 1.$$

Set $r = \binom{d-1}{2} - g + 1$, the vector space $\langle (I_E)_r \rangle$ as only one non-monomial generator: $x_3G - x_2^{d-1}F$. So we need a weight order ω such that

$$\deg_\omega x_3G = \omega_3 + \binom{d-1}{2} - g = (d-1)\omega_2 + \binom{d-2}{2} - g = \deg_\omega x_2^{d-1}F$$

i.e. $\omega_3 = (d-1)\omega_2 - \binom{d-2}{1} = (d-1)\omega_2 - d + 2$.

Then the image through ψ of monomials involved in the basis of $\langle (I_E)_r \rangle$ corresponds to the Borel set $\mathscr{B} = \{(x_3^2, x_3x_2, x_2^d)_r\} \cup \{x_3x_1^{r-1}, x_2^{d-1}x_1^{r-d+1}\} \subset \mathcal{P}(2,r)$ (see Figure 6.1), so that

$$\mathscr{N} = \mathcal{P}(2,r) \setminus \mathscr{B} = \left\{x_2^{d-2}x_1^{r-d+2}, \ldots, x_1^r\right\}.$$

The two monomials of $\mathscr{B}$ images of the generator $x_3G - x_2^{d-1}F$ are minimal elements and $\mathscr{N}$ has as unique maximal monomial $x_2^{d-2}x_1^{r-d+2}$, hence in order for $\mathscr{B}$ to be a segment w.r.t. $\omega = (\omega_3, \omega_2, \omega_1)$ we need

$$\begin{cases} \omega_3 > \omega_2 \\ \omega_2 > \omega_1 \\ \omega_3 + (r-1)\omega_1 > (d-2)\omega_2 + (r-d+2)\omega_1 \end{cases}$$

Finally adding the hypothesis $\omega_1 = 1$ and the requirement of homogeneity for $x_3G - x_2^{d-1}F$, we have to solve the system

$$\begin{cases} \omega_3 > \omega_2 \\ \omega_2 > 1 \\ \omega_3 + (r-1) > (d-2)\omega_2 + (r-d+2) \\ \omega_3 = (d-1)\omega_2 - d + 2 \end{cases}$$

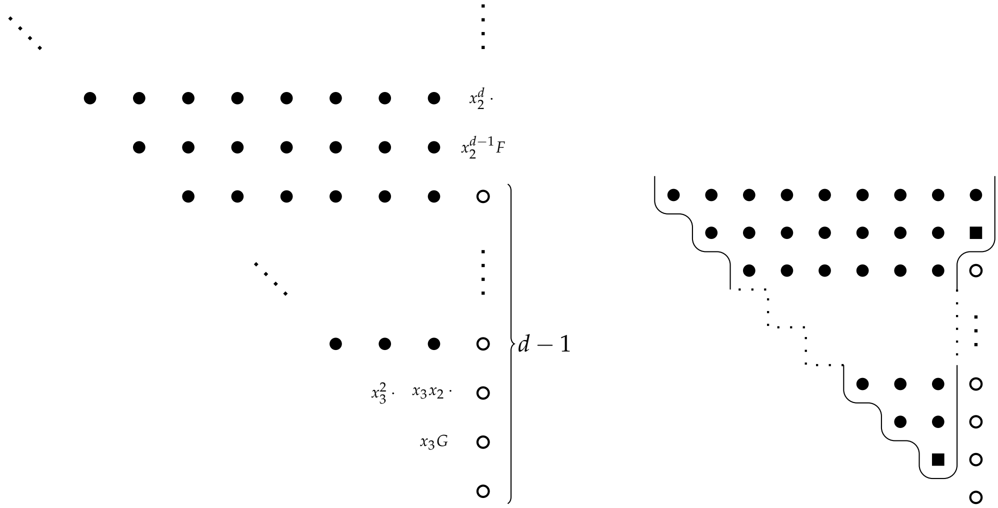
Replacing ω_3 in the third inequality using the equality we obtain $\omega_2 > 1$, so that a solution of the system is

$$\begin{cases} \omega_3 > \omega_2 \\ \omega_2 > 1 \\ \omega_3 = (d-1)\omega_2 - d + 2 \end{cases} \quad \Rightarrow \quad \begin{cases} \omega_2 = 2 \\ \omega_3 = 2(d-1) - d + 2 = d > 2 \end{cases} \qquad \square$$

Remark 6.2.2. The weight order making the ideal of an extremal curve $E \in \mathbf{H}_{d,g}$ a lcm-segment ideal does not depend on the genus of the curve, but only on the degree.

6.3 $\mathbf{H}_{4,-3}$ is connected

In this section we construct a flat family of curves whose general member is a disjoint union of lines on a smooth quadric surface Q and whose special member is an



(a) The ideal of an extremal curve drawn in the graph $\Omega(r)$.

(b) The Borel set in $\mathcal{P}(2,r)$ associated to an extremal curve.

Figure 6.1: The ideal defining an extremal curve represented in $\Omega(r)$ and the associated Borel set in $\mathcal{P}(2,r)$.

extremal multiple line. This specialization is obtained considering the initial ideal with respect to the weight vector $\omega = (d,2,1,1)$. In case $d = 3$, such a specialization was constructed by Nollet [79] for a triple structure $3L$ on the line L on Q (without using the language of weight vectors and initial ideals). For $d \geqslant 4$, if we begin with the d-uple structure dL on the quadric, we obtain a limit with embedded points. We now show that, if we take a sufficiently general divisor of bidegree $(d,0)$, for any $d \geqslant 3$, then we obtain an extremal curve as a limit.

Theorem 6.9. *Let Q be the quadric surface of equation*

$$q(x_0,x_1,x_2,x_3) = x_3(x_3+x_0) - x_2x_1 = 0. \tag{6.5}$$

For every $a \in \mathbb{K}$ let $L_a \subset Q$ denote the line of equations $x_3 - ax_1 = x_2 - a(ax_1+x_0) = 0$. Given $d \geqslant 3$ and $a_1,\dots,a_d \in \mathbb{K}$, consider the divisor

$$C = L_{a_1} + \cdots + L_{a_d} \tag{6.6}$$

on Q. If the sums $a_i + a_j$ for $1 \leqslant i < j \leqslant d$ are all distinct, then there is a flat family of pairs $\mathcal{C} \subset \mathcal{Q} \to \mathbb{A}^1$, whose fiber over 1 is (C, Q), whose fiber over $t \neq 0$ consists of d disjoint lines on a smooth quadric surface, and whose fiber over 0 is an extremal curve in the double plane of equation $x_3^2 = 0$.

Proof. Let C_0 denote the subscheme defined by (the saturation of) $\mathsf{in}_\omega(I_C)$, where I_C denotes the homogeneous ideal of C. By Gröbner bases theory, there is a flat specialization from C to C_0; since $\mathsf{in}_\omega(q) = x_3^2$, the smooth quadric Q specializes to the double plane $x_3^2 = 0$ as C specializes to C_0. We let $l_i = x_3 - a_i x_1$ and $m_i = x_2 - a_i(a_i x_1 + x_0)$ denote the given equations for the line L_{a_i}. Since I_C contains the product of the ideals of the lines L_{a_i},

$$x_2^d = \mathsf{in}_\omega(m_1 \cdot \ldots \cdot m_d) \in I_{C_0}.$$

Therefore C_0 is contained in the complete intersection $x_3^2 = x_2^d = 0$ and so it is supported on the line L. We will show that I_C contains a polynomial $A(x_0, x_1, x_2, x_3)$ such that $\mathsf{in}_\omega(A) = x_3 G - x_2^{d-1} F$ where F and G are homogeneous forms in $\mathbb{K}[x_0, x_1]$, $\deg G = \binom{d}{2}$, $\deg F = \binom{d-1}{2} + 1$, and F and G have no common zero in $\mathbb{P}^1 = \operatorname{Proj} \mathbb{K}[x_0, x_1]$. It follows that C_0 is contained in the surface S of equation $x_3 G - x_2^{d-1} F$. By flatness, the Hilbert polynomial of C_0 coincides with that of C, so C_0 is a one dimensional subscheme of $\mathbb{P}^3$ of degree d and genus $1 - d$. Let E the largest Cohen-Macaulay curve contained in C_0: it is the curve of degree d obtained from C_0 throwing away its embedded points. By Proposition 6.4 E is the unique curve of degree d contained in S and supported on the line L; it is an extremal curve of degree d and genus $1 - d$. Since $E \subset C_0$ and the two schemes have the same Hilbert polynomial, we conclude $E = C_0$. Thus the limit is an extremal curve, and the statement is proven.

To conclude the proof, we need to find $A \in I_C$ with $\mathsf{in}_\omega(A) = x_3 G - x_2^{d-1} F$. In principle, this is a Gröbner basis calculation, but luckily we can bypass such a calculation because we were able to find, with the help of some computation performed with *Macaulay2* [37], a determinantal formula for A (note that, while I_C is generated by forms of degree $\leqslant d$, the degree of A is much larger than d). Let $A = x_3 G - x_1 B$

denote the determinant

$$\begin{vmatrix} l_1 & m_1 & m_1^2 & \dots & m_1^{d-1} \\ l_2 & m_2 & m_2^2 & \dots & m_2^{d-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_d & m_d & m_d^2 & \dots & m_d^{d-1} \end{vmatrix}.$$

Since $l_i = x_3 - a_i\, x_1$, by linearity

$$A = x_3 \begin{vmatrix} 1 & m_1 & m_1^2 & \dots & m_1^{d-1} \\ 1 & m_2 & m_2^2 & \dots & m_2^{d-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & m_d & m_d^2 & \dots & m_d^{d-1} \end{vmatrix} - x_1 \begin{vmatrix} a_1 & m_1 & m_1^2 & \dots & m_1^{d-1} \\ a_2 & m_2 & m_2^2 & \dots & m_2^{d-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_d & m_d & m_d^2 & \dots & m_d^{d-1} \end{vmatrix}. \tag{6.7}$$

The linear forms l_i and m_i are the equations of the line L_{a_i}, hence the polynomial A belongs to the ideal of C. In the expansion $A = xG - zB$ the polynomial G is a Vandermonde determinant, i.e.

$$G = \prod_{1\leqslant i<j\leqslant d} (m_j - m_i) = \prod_{1\leqslant i<j\leqslant d} (a_i - a_j)\big((a_i + a_j)x_1 + x_0\big) \tag{6.8}$$

We note that G is nonzero: the hypothesis that the sums $a_i + a_j$ be all distinct implies that $a_i - a_j \neq 0$ for every $i < j$. Furthermore, the zeros of G in $\mathbb{P}^1 = \operatorname{Proj} \mathbb{K}[x_0, x_1]$ are the points $[1 : -a_i - a_j]$.

The polynomial B is

$$\begin{aligned} B = \begin{vmatrix} a_1 & m_1 & m_1^2 & \dots & m_1^{d-1} \\ a_2 & m_2 & m_2^2 & \dots & m_2^{d-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_d & m_d & m_d^2 & \dots & m_d^{d-1} \end{vmatrix} = \\ = \sum_{j=1}^{d} (-1)^{j-1} a_j\, m_1 \cdots \widehat{m_j} \cdots m_d \prod_{\substack{1\leqslant h<k\leqslant d \\ h\neq j, k\neq j}} (m_k - m_h). \end{aligned} \tag{6.9}$$

Since $m_k - m_h = (a_h - a_k)\big((a_h + a_k)x_1 + x_0\big)$ and the initial term of m_j is x_2, the initial term of B is the polynomial

$$\sum_{j=1}^{d} (-1)^{j-1} a_j\, x_2^{d-1} \prod_{\substack{1\leqslant h<k\leqslant d \\ h\neq j, k\neq j}} (a_h - a_k)\big((a_h + a_k)x_1 + x_0\big) = x_2^{d-1} P \tag{6.10}$$

provided $P = P(x_0, x_1) \neq 0$. We will prove not only that P is not zero, but also that it has no zero in common with G, that is, it does not vanish at the points $[1 : -a_i - a_j]$. By symmetry, it is enough to show that $P(1, -a_1 - a_2) \neq 0$. For this, we write P as a determinant: if we set $p_i = a_i^2 x_1 + a_i x_0$, then

$$P = \sum_{j=1}^{d} (-1)^{j-1} a_j \prod_{\substack{1 \leqslant h < k \leqslant d \\ h \neq j, k \neq j}} (p_k - p_h) = \begin{vmatrix} a_1 & 1 & p_1 & \dots & p_1^{d-2} \\ a_2 & 1 & p_2 & \dots & p_2^{d-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_d & 1 & p_d & \dots & p_d^{d-2} \end{vmatrix} \tag{6.11}$$

Now note that $p_1(1, -a_1 - a_2) = p_2(1, -a_1 - a_2) = -a_1 a_2$ and $p_i(1, -a_1 - a_2) = a_i^2 - (a_1 + a_2) a_i$ for $j \geqslant 3$. For simplicity we write p_i in place of $p_i(1, -a_1 - a_2)$. Then

$$\begin{aligned} P(1, -a_1 - a_2) &= \begin{vmatrix} a_1 & 1 & (-a_1 a_2) & \dots & (-a_1 a_2)^{d-2} \\ a_2 & 1 & (-a_1 a_2) & \dots & (-a_1 a_2)^{d-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_d & 1 & p_d & \dots & p_d^{d-2} \end{vmatrix} = \\ &= (a_1 - a_2) \begin{vmatrix} 1 & (-a_1 a_2) & \dots & (-a_1 a_2)^{d-2} \\ 1 & p_3 & \dots & p_3^{d-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & p_d & \dots & p_d^{d-2} \end{vmatrix} = \\ &= (a_1 - a_2) \left(\prod_{j=3}^{d} (p_j + a_1 a_2) \right) \left(\prod_{3 \leqslant h < k \leqslant d} (p_k - p_h) \right) = \\ &= (a_1 - a_2) \left(\prod_{j=3}^{d} (a_j - a_1)(a_j - a_2) \right) \left(\prod_{3 \leqslant h < k \leqslant d} (a_k - a_h)(a_h + a_k - a_1 - a_2) \right). \end{aligned} \tag{6.12}$$

This shows $P(1, -a_1 - a_2) \neq 0$ because of the assumption that sums $a_i + a_j$ be all distinct.

To finish, we let $F = x_1 P$. Then $\mathsf{in}_\omega(A) = x_3 G - x_2^{d-1} F$, and F and G are homogeneous forms in $\mathbb{K}[x_0, x_1]$, $\deg G = \binom{d}{2}$, $\deg F = \binom{d-1}{2} + 1$, and F and G have no common zero in $\mathbb{P}^1$. $\square$

We would like to make some remarks on the polynomial P. It is divisible by x_1^{d-2}. Indeed, P is a form in x_1 and x_0 of degree $\binom{d-1}{2}$ with coefficients in $\mathbb{K}[a_1, \ldots, a_d]$. It is divisible by the Vandermonde determinant $V(a_1, \ldots, a_d)$ because it is antisymmetric in the a_i's. Furthermore, the coefficient of $x_1^{\alpha_1} x_0^{\alpha_0}$ in P is an antisymmetric polynomial of degree $2\alpha_1 + \alpha_0 + 1$ in the a_i's: in order for it to be nonzero, it is necessary that $2\alpha_1 + \alpha_1 + 1 \geqslant \deg V = \binom{d}{2}$. Since $\alpha_1 + \alpha_0 = \binom{d-1}{2}$, we deduce $\alpha_1 \geqslant d-2$. This means that P is divisible by x_1^{d-2}, and the coefficient of $x_1^{d-2} x_0^{\binom{d-1}{2}}$ is $V(a_1, \ldots, a_d)$ times a constant $-c_d$ that depends on d but not on the a_i's. To compute c_d, we eliminate the highest power of x_0 in each column by subtracting the correct linear combination of the previous columns: we obtain $c_2 = 1$ and

$$c_d = \sum_{k=1}^{d-2} (-1)^{k+1} \binom{d-1-k}{k} c_{d-k}, \qquad d \geqslant 3. \tag{6.13}$$

The first few values are $c_3 = 1$, $c_4 = 2$, $c_5 = 5$, $c_6 = 14$, $c_7 = 42$, and at the end of the chapter we will prove that c_d coincides with the $(d-2)$-th Catalan number.

Remark 6.3.1. We can give a geometric interpretation of the condition that the sums $a_i + a_j$ be all distinct. In the family constructed in the proof of the theorem, the union of the two lines L_{a_i} and L_{a_j} specializes to the planar double line $x_3 = x_2^2 = 0$ plus the embedded point $x_3 = x_2 = (a_i + a_j)x_1 + x_0 = 0$. Thus the condition means that these embedded points are all distinct.

Example 6.3.2. If the condition that the $a_i + a_j$ be all distinct is not satisfied, we expect the limit to acquire embedded points. The reason is that in this case the proof of Theorem 6.9 shows that the polynomials F and G have a common zero, and so $x_3 G - x_2^{d-1} F$ is no longer irreducible. For a specific example, we take $d = 4$ and $a_1 = 0$, $a_2 = 1$, $a_3 = 2$ and $a_4 = 3$ (in characteristic $\neq 2, 3$), so that $a_1 + a_4 = a_2 + a_3 = 3$. In this case,

$$\begin{aligned}\frac{x_3 G - x_2^3 F}{12} &= x_3(x_1 + x_0)(x_1 + 2x_0)(x_1 + 3x_0)^2(x_1 + 4x_0)(x_1 + 5x_0) \\ &\quad - 2x_2^3 x_1^3 (3x_1 + x_0) = (3x_1 + x_0)(x_3 G_1 - x_2^3 F_1).\end{aligned}$$

The initial ideal of I_C is

$$\begin{aligned}\mathsf{in}_{(4,2,1,1)}(I_C) = \big(&x_3^2, 6x_3x_2x_1^2 + 2x_3x_2x_1x_0, x_2^4, x_3x_2^2x_0, 6x_3x_2^2x_1, \\ &6x_3x_2x_1x_0^2 + 2x_3x_2x_0^3, x_3G_1 - x_2^3F_1, 6x_3x_2^3\big)\end{aligned}$$

The saturation of this ideal is

$$\left(x_3^2, 3x_3x_2x_1 + x_3x_2x_0, x_3x_2^2, x_2^4, x_3G_1 - x_2^3F_1\right),$$

therefore the limit C_0 consists of the unique 4 structure supported on L contained in the surface $x_3G_1 - x_2^3F_1$, which by Proposition 6.4 is an extremal curve of genus -2, plus an embedded point (of equation $3x_1 + x_0 = 0$ on L).

Corollary 6.10. *Let C be an effective divisor of bidegree $(d, 0)$ on a smooth quadric surface. Then every curve in the biliaison class of C is in the connected component of the extremal curves in its Hilbert scheme. In particular, this holds for every curve on a smooth quadric surface.*

Proof. The case $0 \leqslant d \leqslant 2$ has been proven by Hartshorne [45]. When $d \geqslant 3$, the statement follows from Theorem 6.9 and [92, Theorem 2.3] because C is a minimal curve. □

Catalan numbers

Definition 6.11. Let us define for any $n > k \geqslant 2$

$$\left\langle {n \atop k} \right\rangle = \begin{cases} 1, & \text{if } k = 2 \\ \sum\limits_{i=k+1}^{d} \left\langle {i \atop k-1} \right\rangle, & \text{otherwise.} \end{cases} \tag{6.14}$$

Proposition 6.12. *The number $\langle {d \atop d-1} \rangle$ coincides with the Catalan number $C(d-3)$.*

Proof. The definition of $\langle {n \atop k} \rangle$ is the same as the construction of the Catalan numbers through the generalized Pascal triangle (see [99, Example 3.5.5]). □

We recall a binomial identity that will be very useful in the following. For any n, m

$$\sum_{i=0}^{m} (-1)^i \binom{n}{i} = (-1)^m \binom{n-1}{m} \tag{6.15}$$

and equivalently for $n > 0$

$$\sum_{i=1}^{m} (-1)^{i+1} \binom{n}{i} = 1 + (-1)^{m+1} \binom{n-1}{m}. \tag{6.16}$$

	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$	$n=11$
$k=2$	1	1	1	1	1	1	1	1	1
$k=3$		1	2	3	4	5	6	7	8
$k=4$			2	5	9	14	20	27	35
$k=5$				5	14	28	48	75	110
$k=6$					14	42	90	165	275
$k=7$						42	132	297	572
$k=8$							132	429	1001
$k=9$								429	1430
$k=10$									1430

Table 6.1: First values of $\left\langle {n \atop k} \right\rangle$.

Lemma 6.13. *For the coefficients c_d described in* (6.13)*,*

$$c_d = \sum_{h=0}^{a-2} \left\langle {d-h \atop a-h} \right\rangle c_{2+h} + \sum_{k=1}^{d-a-1} (-1)^{k+1} \binom{d-k-a}{k} c_{d-k}, \quad \forall\, d > a \geqslant 2. \tag{6.17}$$

Proof. We proceed by induction on a. Let us consider any c_d with $d > 2$. The sum

$$\sum_{i=3}^{d} c_i = \sum_{i=3}^{d} \left(\sum_{k=1}^{i-2} (-1)^k \binom{i-k-1}{k} c_{i-k} \right).$$

can be rewritten as

$$\sum_{i=3}^{d} c_i = \sum_{k=1}^{d-2} \left(\sum_{i=1}^{k} (-1)^{i+1} \binom{d-k-1}{i} \right) c_{d-k}$$

as suggested by the following diagram

$$\begin{array}{ll}
c_d & \binom{d-2}{1}c_{d-1} - \binom{d-3}{2}c_{d-2} + \binom{d-4}{3}c_{d-3} - \binom{d-5}{4}c_{d-4} + \ldots + \binom{1}{d-2}c_2 \\
+\, c_{d-1} & \qquad\qquad\quad\; + \binom{d-3}{1}c_{d-2} - \binom{d-4}{2}c_{d-3} + \binom{d-5}{3}c_{d-4} + \ldots + \binom{1}{d-3}c_2 \\
+\, c_{d-2} & \qquad\qquad\qquad\qquad\qquad\quad + \binom{d-4}{1}c_{d-3} - \binom{d-5}{2}c_{d-4} + \ldots + \binom{1}{d-4}c_2 \\
\vdots & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \vdots \\
+\, c_3 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \binom{1}{1}c_2
\end{array}$$

Since $d-k-1$ is greater that 0 for each k, we can apply the binomial identity (6.16), obtaining

$$\sum_{i=3}^{d} c_i = \sum_{k=1}^{d-2}\left(1+(-1)^{k+1}\binom{d-k-2}{k}\right)c_{d-k} = \\ = \sum_{i=2}^{d-1} c_i + \sum_{k=1}^{d-2}(-1)^{k+1}\binom{d-k-2}{k}c_{d-k}.$$

The binomial coefficient corresponding to $k=d-2$ surely vanishes and $1=\left\langle{d \atop 2}\right\rangle$ so we proved

$$c_d = \left\langle{d \atop 2}\right\rangle c_2 + \sum_{k=1}^{d-3}(-1)^{k+1}\binom{d-k-2}{k}c_{d-k}.$$

Let us now suppose that the statement is true for $a-1$, that is for every $d>a-1$

$$c_d = \sum_{h=0}^{a-3}\left\langle{d-h \atop a-1-h}\right\rangle c_{2+h} + \sum_{k=1}^{d-(a-1)-1}(-1)^{k+1}\binom{d-k-(a-1)}{k}c_{d-k}.$$

With the same reasoning applied before we compute

$$\sum_{i=a+1}^{d} c_i = \sum_{i=a+1}^{d}\left(\sum_{h=0}^{a-3}\left\langle{d-h \atop a-1-h}\right\rangle c_{2+h}\right) + \\ + \sum_{i=a+1}^{d}\left(\sum_{k=1}^{d-(a-1)-1}(-1)^{k+1}\binom{d-k-(a-1)}{k}c_{d-k}\right).$$

For the first part of the sum, we can change the order of the summations obtaining

$$\sum_{i=a+1}^{d}\left(\sum_{h=0}^{a-3}\left\langle{d-h \atop a-1-h}\right\rangle c_{2+h}\right) = \sum_{h=0}^{a-3}\left(\sum_{i=a+1}^{d}\left\langle{i-h \atop a-1-h}\right\rangle\right)c_{2+h} = \\ = \sum_{h=0}^{a-3}\left(\sum_{j=(a-h)+1}^{d-h}\left\langle{j \atop (a-h)-1}\right\rangle\right)c_{2+h} = \sum_{h=0}^{a-3}\left\langle{d-h \atop a-h}\right\rangle c_{2+h}.$$

For the second part of the sum, we change the order and we use again the binomial identity (6.16), since $d-k-(a-1)>0$, $\forall\, k$:

$$\begin{aligned}
&\sum_{i=a+1}^{d}\left(\sum_{k=1}^{d-(a-1)-1}(-1)^{k+1}\binom{d-k-(a-1)}{k}c_{d-k}\right)=\\
&=\sum_{k=1}^{d-(a-1)-1}\left(\sum_{i=1}^{k}(-1)^{i+1}\binom{d-k-(a-1)}{i}\right)c_{d-k}=\\
&=\sum_{k=1}^{d-(a-1)-1}\left(1+(-1)^{k+1}\binom{d-k-a}{k}\right)c_{d-k}=\\
&=\sum_{i=a}^{d-1}c_i+\sum_{k=1}^{d-(a-1)-1}(-1)^{k+1}\binom{d-k-a}{k}c_{d-k}.
\end{aligned}$$

For $k=d-(a-1)-1$, the binomial coefficient vanishes and $1=\left\langle{d-(a-2) \atop a-(a-2)}\right\rangle$ so

$$\begin{aligned}
c_d&=\sum_{h=0}^{a-3}\left\langle{d-h \atop a-h}\right\rangle c_{2+h}+c_a+\sum_{k=1}^{d-a-1}(-1)^{k+1}\binom{d-k-a}{k}c_{d-k}=\\
&=\sum_{h=0}^{a-2}\left\langle{d-h \atop a-h}\right\rangle c_{2+h}+\sum_{k=1}^{d-a-1}(-1)^{k+1}\binom{d-k-a}{k}c_{d-k}.
\end{aligned}$$ □

Proposition 6.14.

$$c_d=\left\langle{d+1 \atop d}\right\rangle=C(d-2). \tag{6.18}$$

Proof. We prove the equality by induction on d. Obviously $c_2=1=\left\langle{3 \atop 2}\right\rangle=C(0)$ and let us assume the statement true for any c_k, $2\leqslant k<d$. Considering the equality (6.17) with $a=d-1$ the second sum is empty, so that

$$c_d=\sum_{h=0}^{d-3}\left\langle{d-h \atop d-h-1}\right\rangle c_{2+h}.$$

By the inductive hypothesis

$$\sum_{h=0}^{d-3}\left\langle{d-h \atop d-h-1}\right\rangle c_{2+h}=\sum_{h=0}^{d-3}C(d-h-3)C(2+h-2)=\sum_{h=0}^{D}C(D-h)C(h).$$

This is the recursive definition of Catalan numbers (see [36, Section 7.5]) so

$$\sum_{h=0}^{D}C(D-h)C(h)=C(D+1)\quad\Rightarrow\quad c_d=C(d-2)=\left\langle{d+1 \atop d}\right\rangle.$$ □

Corollary 6.15.

$$c_d = \frac{1}{d-1}\binom{2d-4}{d-2}. \tag{6.19}$$

Corollary 6.16 (A new recurrence relation for Catalan numbers)**.**

$$C(0) = 1, \qquad C(n) = \sum_{k=1}^{n} (-1)^{k+1}\binom{n-k+1}{k} C(n-k). \tag{6.20}$$

Appendix A

The *Macaulay2* package `HilbertSchemesEquations`

This chapter is supposed to be a handbook for the *Macaulay2* [37] package `HilbertSchemeEquations.m2`. We will introduce and explain the main functions of the package through the complete computation of the examples introduced in Chapter 1, that is computing various sets of equations for the Hilbert scheme $\mathbf{Hilb}^2_2$.

A.1 Basic features

Firstly we need a method for computing the Gotzmann number of a Hilbert polynomial. `HilbertSchemesEquations.m2` provides two methods:

- `gotzmannDecomposition`, taking as input a single variable polynomial $p(t)$ and returning the Gotzmann decomposition of the polynomial as list of pairs $\{\ldots,(a_i,b_i),\ldots\}$ such that $p(t) = \sum_i \binom{t+a_i}{b_i}$:

```
gotzmannDecomposition = method(TypicalValue => List)
--  INPUT: p, polynomial (one variable).
-- OUTPUT: a list of pairs {...(a_i,b_i)...} containing the
--         Gotzmann decomposition of p.
--  ERROR: if numgens(ring(p)) > 1.
--         if coefficientRing(ring(p)) =!= ZZ and
--            coefficientRing(ring(p)) =!= QQ.
```

```
--        if p is not an admissible Hilbert polynomial.
```

- `gotzmannNumber`, taking as input a single variable polynomial $p(t)$ and returning the Gotzmann number:

```
gotzmannNumber = method(TypicalValue => ZZ)
--  INPUT: p, polynomial (one variable).
-- OUTPUT: the Gotzmann number of p.
--  ERROR: if numgens(ring(p)) > 1.
--         if coefficientRing(ring(p)) =!= ZZ and
--            coefficientRing(ring(p)) =!= QQ.
--         if p is not an admissible Hilbert polynomial.
```

Example A.1.1. We test these two methods on the polynomials $p(t) = 4t+1$ and $q(t) = \frac{1}{3}t^2 - \frac{2}{3}t + 1$.

```
Macaulay2, version 1.4
i1 : loadPackage "HilbertSchemesEquations";
i2 : R = QQ[t];
i3 : p = 4*t+1;
i4 : q = (t^2-2*t+3)/3;
i5 : gotzmannNumber p
o5 = 7
i6 : gotzmannDecomposition p
o6 = {(1, 1), (0, 1), (-1, 1), (-2, 1), (-4, 0), (-5, 0), (-6, 0)}
o6 : List
i7 : gotzmannDecomposition q
stdio:7:1:(3): error: argument 1: not admissible Hilbert polynomial
```

Hence $q(t)$ is not admissible (as expected) and $p(t)$ can be decomposed as

$$\binom{t+1}{1} + \binom{t}{1} + \binom{t-1}{1} + \binom{t-2}{1} + \binom{t-4}{0} + \binom{t-5}{0} + \binom{t-6}{0}.$$

To compute the Plücker relations defining the Grassmannian, we will use the function `Grassmannian(ZZ,ZZ)` provided by *Macaulay2*. We remark that to compute the ideal defining $\mathbf{Gr}_{\mathbb{K}}(q,N) \subset \mathbb{P}_{\mathbb{K}}^{\binom{N}{q}-1}$ we need to call `Grassmannian(`$q-1$`,`$N-1$`)`.

The package `HilbertSchemesEquations` gives methods to compute the generic generators of a subspace (and of its exterior powers) parametrized by $\mathbf{Gr}_{\mathbb{K}}(q,N)$ according to Definition 1.9.

- `genericSubspaceGen`. This method requires 3 arguments:

1. the exterior algebra generated by the basis $\{\underline{v}_1, \ldots, \underline{v}_N\}$ of the base vector space $V \simeq \mathbb{K}^N$, with coefficient in the ring of the Grassmannian $\mathbf{Gr}_{\mathbb{K}}(q, N)$;
2. the dimension q of the subspaces parametrized by $\mathbf{Gr}_{\mathbb{K}}(q, N)$;
3. a multiindex J;

and it returns the element $\Lambda_{\mathrm{J}}^{(q-|\mathrm{J}|)}$ associated to $\mathbf{Gr}_{\mathbb{K}}(q, N)$.

```
genericSubspaceGen = method(TypicalValue => RingElement)
--  INPUT: A, an exterior algebra with coefficientRing(A)
--            corresponding to a ring defining a grassmannian.
--            gens(A) has to be a basis of the base vector
--            space of the grassmannian.
--         q, the dimension of the subspaces parametrized by
--            the grassmannian.
--         J, a set of indices.
-- OUTPUT: the generator Lambda^{(q-#J)}_J.
--  ERROR: if q < 1 or q > numgens(A).
--         if numgens(coefficientRing(A)) != binomial(numgens(A),q).
--         if #J >= q,
--         if min(J) < 0 or max(J) > numgens(A)-1.
```

- `genericSetOfSubspaceGens`. Also this method requires 3 arguments: the first 2 are the same required by `genericSubspaceGen` and the third is the order s of the exterior power. It returns the set of generators $\Gamma^{(s)}$ associated to $\mathbf{Gr}_{\mathbb{K}}(q, N)$.

```
genericSetOfSubspaceGens = method(TypicalValue => List)
--  INPUT: A, an exterior algebra with coefficientRing(A)
--            corresponding to a ring defining a grassmannian.
--            gens(A) has to be a basis of the base vector
--            space of the grassmannian.
--         q, the dimension of the subspaces parametrized by
--            the grassmannian.
--         s, the order of the exterior power.
-- OUTPUT: list containing the set of generators Gamma^{(s)}.
--  ERROR: if q < 1 or q > numgens(A).
--         if numgens(coefficientRing(A)) != binomial(numgens(A),q).
--         if s < 1 or s > q.
```

Example A.1.2. We introduce these two new methods applying them to the case of $\mathbf{Gr}_{\mathbb{K}}(4, 6)$ discussed in Example 1.2.3 and Example 1.2.4.

```
Macaulay2, version 1.4
i1 : loadPackage "HilbertSchemesEquations";
i2 : grass = Grassmannian (3,5,CoefficientRing=>QQ);
o2 : Ideal of QQ[p       , p       , p       , p       , p       , p       ,
                  0,1,2,3   0,1,2,4   0,1,3,4   0,2,3,4   1,2,3,4   0,1,2,5
     ------------------------------------------------------------------------
     p       , p       , p       , p       , p       , p       , p       ,
      0,1,3,5   0,2,3,5   1,2,3,5   0,1,4,5   0,2,4,5   1,2,4,5   0,3,4,5
     ------------------------------------------------------------------------
     p       , p       ]
      1,3,4,5   2,3,4,5
i3 : pluckerRelations = first entries gens grass;
i4 : #pluckerRelations
o4 = 15
i5 : for i from 0 to #pluckerRelations-1 do (print(pluckerRelations#i););
     p       p       - p       p       + p       p
      1,2,4,5 0,3,4,5   0,2,4,5 1,3,4,5   0,1,4,5 2,3,4,5
     p       p       - p       p       + p       p
      1,2,3,5 0,3,4,5   0,2,3,5 1,3,4,5   0,1,3,5 2,3,4,5
     p       p       - p       p       + p       p
      1,2,3,4 0,3,4,5   0,2,3,4 1,3,4,5   0,1,3,4 2,3,4,5
     p       p       - p       p       + p       p
      1,2,3,5 0,2,4,5   0,2,3,5 1,2,4,5   0,1,2,5 2,3,4,5
     p       p       - p       p       + p       p
      1,2,3,4 0,2,4,5   0,2,3,4 1,2,4,5   0,1,2,4 2,3,4,5
     p       p       - p       p       + p       p
      1,2,3,5 0,1,4,5   0,1,3,5 1,2,4,5   0,1,2,5 1,3,4,5
     p       p       - p       p       + p       p
      0,2,3,5 0,1,4,5   0,1,3,5 0,2,4,5   0,1,2,5 0,3,4,5
     p       p       - p       p       + p       p
      1,2,3,4 0,1,4,5   0,1,3,4 1,2,4,5   0,1,2,4 1,3,4,5
     p       p       - p       p       + p       p
      0,2,3,4 0,1,4,5   0,1,3,4 0,2,4,5   0,1,2,4 0,3,4,5
     p       p       - p       p       + p       p
      1,2,3,4 0,2,3,5   0,2,3,4 1,2,3,5   0,1,2,3 2,3,4,5
     p       p       - p       p       + p       p
      1,2,3,4 0,1,3,5   0,1,3,4 1,2,3,5   0,1,2,3 1,3,4,5
     p       p       - p       p       + p       p
      0,2,3,4 0,1,3,5   0,1,3,4 0,2,3,5   0,1,2,3 0,3,4,5
     p       p       - p       p       + p       p
      1,2,3,4 0,1,2,5   0,1,2,4 1,2,3,5   0,1,2,3 1,2,4,5
     p       p       - p       p       + p       p
      0,2,3,4 0,1,2,5   0,1,2,4 0,2,3,5   0,1,2,3 0,2,4,5
     p       p       - p       p       + p       p
      0,1,3,4 0,1,2,5   0,1,2,4 0,1,3,5   0,1,2,3 0,1,4,5
```

Then we introduce the exterior algebra generated by the basis $\{\underline{v}_0, \ldots, \underline{v}_5\}$ of the base vector space V and we compute the sets of generators of any exterior power of a subspace parametrized by $\mathbf{Gr}_{\mathbb{K}}(4,6)$.

```
i6 : G = ring(grass)/grass;
i7 : A = G[v_0..v_5,SkewCommutative=>true];
i8 : genericSubspaceGen (A,4,{0,4,5})
```

```
o8 = p       v  + p       v  + p       v
      0,1,4,5 1    0,2,4,5 2    0,3,4,5 3
o8 : A
i9 : Gamma1 = genericSetOfSubspaceGens (A,4,1);
i10 : #Gamma1
o10 = 20
i11 : genericSubspaceGen (A,4,{1,3})
o11 = - p       v v  + p       v v  - p       v v  + p       v v  - p       v v
         0,1,2,3 0 2    0,1,3,4 0 4    1,2,3,4 2 4    0,1,3,5 0 5    1,2,3,5 2 5
      ---------------------------------------------------------------------------
      + p       v v
         1,3,4,5 4 5
o11 : A
i12 : Gamma2 = genericSetOfSubspaceGens (A,4,2);
i13 : #Gamma2
o13 = 15
i14 : genericSubspaceGen (A,4,{2})
o14 = p       v v v  + p       v v v  - p       v v v  - p       v v v  +
       0,1,2,3 0 1 3    0,1,2,4 0 1 4    0,2,3,4 0 3 4    1,2,3,4 1 3 4
      ---------------------------------------------------------------------------
      p       v v v  - p       v v v  - p       v v v  - p       v v v  -
       0,1,2,5 0 1 5    0,2,3,5 0 3 5    1,2,3,5 1 3 5    0,2,4,5 0 4 5
      ---------------------------------------------------------------------------
      p       v v v  + p       v v v
       1,2,4,5 1 4 5    2,3,4,5 3 4 5
o14 : A
i15 : Gamma3 = genericSetOfSubspaceGens (A,4,3);
i16 : #Gamma3
o16 = 6
i17 : genericSubspaceGen (A,4,{})
o17 = p       v v v v  + p       v v v v  + p       v v v v  + p       v v v v
       0,1,2,3 0 1 2 3    0,1,2,4 0 1 2 4    0,1,3,4 0 1 3 4    0,2,3,4 0 2 3 4
      ---------------------------------------------------------------------------
      + p       v v v v  + p       v v v v  + p       v v v v  +
         1,2,3,4 1 2 3 4    0,1,2,5 0 1 2 5    0,1,3,5 0 1 3 5
      ---------------------------------------------------------------------------
      p       v v v v  + p       v v v v  + p       v v v v  + p       v v v v
       0,2,3,5 0 2 3 5    1,2,3,5 1 2 3 5    0,1,4,5 0 1 4 5    0,2,4,5 0 2 4 5
      ---------------------------------------------------------------------------
      + p       v v v v  + p       v v v v  + p       v v v v  +
         1,2,4,5 1 2 4 5    0,3,4,5 0 3 4 5    1,3,4,5 1 3 4 5
      ---------------------------------------------------------------------------
      p       v v v v
       2,3,4,5 2 3 4 5
o17 : A
```

A.2 Hilbert scheme equations

Gotzmann equations

The package `HilbertSchemesEquations` provides the function `GotzmannHilbEquations`:

```
GotzmannHilbEquations = method(TypicalValue => List,
          Options => {PluckerRelations => true,SingleGrassmannian => true})
--  INPUT: p, admissible Hilbert polynomial.
--         n, dimension of the projective space.
-- OUTPUT: a list containing:
--           - #0 the ring in which the Hilbert scheme is embedded;
--           - #1 the ideal defining the Hilbert scheme.
-- OPTION: PluckerRelations (Boolean), default value true.
--            If true, the ideal of Plucker relations is computed.
--            If false, the Plucker coordinates are considered
--            without relations among them.
--         SingleGrassmannian (Boolean), default value true.
--            If true, the ideal is embedded in a single
--            grassmannian. If false, the ideal is embedded
--            in the product of two grassmannians.
--  ERROR: if numgens(ring(p)) > 1.
--         if coefficientRing(ring(p)) != ZZ and
--            coefficientRing(ring(p)) != QQ
--         if p is not admissible.
--         if n < 1.
--         if first degree (p) >= n.
```

This method requires two inputs: a Hilbert polynomial $p(t)$ and a dimension n of a projective space $\mathbb{P}^n_{\mathbb{K}}$. It returns a sequence with two elements: the first is the ring in which the ideal of the Hilbert scheme is computed and the second is just the ideal.

There are two options:

- `PluckerRelations`, since computing the ideal of the Plücker relations is in general a hard task, it is possible to tell the function to ignore Plücker relations;

- `SingleGrassmannian`, with this option it is possible to choose the embedding of the Hilbert scheme: if in the single Grassmannian $\mathbf{Gr}_{\mathbb{K}}(q(r), N(r))$ or in the product $\mathbf{Gr}_{\mathbb{K}}(q(r), N(r)) \times \mathbf{Gr}_{\mathbb{K}}(p(r+1), N(r+1))$. To compute the image of the Hilbert scheme $\mathbf{Hilb}^n_{p(t)} \subset \mathbf{Gr}_{\mathbb{K}}(q(r), N(r)) \times \mathbf{Gr}_{\mathbb{K}}(p(r+1), N(r+1))$ by the projection π on the first factor, we use the projective elimination theory (see [24, Chapter 8 Section 5]). Given the map $\pi : \operatorname{Proj} \mathbb{K}[\Delta] \times \operatorname{Proj} \mathbb{K}[\nabla] \to \operatorname{Proj} \mathbb{K}[\Delta]$ and the ideal $\mathcal{I}_{\mathcal{H}}$ defining $\mathbf{Hilb}^n_{p(t)} \subset \operatorname{Proj} \mathbb{K}[\Delta] \times \operatorname{Proj} \mathbb{K}[\nabla]$, the ideal $\widehat{\mathcal{I}}_{\mathcal{H}}$ defining $\pi(\mathbf{Hilb}^n_{p(t)})$ can be computed applying the standard elimination algorithm on the affine open subset of the product of projective spaces. Denoted by U_I the open subset of $\operatorname{Proj}\mathbb{K}[\Delta]$ where $\Delta_\mathrm{I} \neq 0$ and by V_J the open subset of $\operatorname{Proj}\mathbb{K}[\nabla]$ where $\nabla_\mathrm{J} \neq 0$, we dehomogenize the ideal $\mathcal{I}_{\mathcal{H}}$, then we eliminate the variables ∇ and finally we homogenize the ideal obtained with the variable Δ_I. Repeating this procedure for any pair I, J we recover the ideal $\widetilde{\mathcal{I}}_{\mathcal{H}}$.

To recap, let us consider the Hilbert polynomial $p(t)$ with Gotzmann number r and the projective space $\mathbb{P}^n_{\mathbb{K}}$. Moreover let $\mathbf{Gr}_{\mathbb{K}}(q(r), N(r)) \subset \mathbb{P}_{\mathbb{K}}^{\binom{N(r)}{q(r)}-1} = \operatorname{Proj}\mathbb{K}[\Delta_\mathrm{I}]$ be defined by the ideal $\mathcal{Q}_1$ and $\mathbf{Gr}_{\mathbb{K}}(p(r), N(r+1)) \subset \mathbb{P}_{\mathbb{K}}^{\binom{N(r+1)}{p(r+1)}-1} = \operatorname{Proj}\mathbb{K}[\nabla_\mathrm{J}]$ defined by the ideal $\mathcal{Q}_2$. Finally let $\mathcal{B}_{\mathcal{H}}$ be the ideal generated by the bilinear equations introduced in the proof of Theorem 1.24.

Calling the method `GotzmannHilbEquations` on the pairs $(p(t), n)$ and varying the option, there are the four possibilities illustrated in the following table:

		`SingleGrassmannian`	
		`false`	**`true`**
`PluckerRelations`	**`true`**	$\mathcal{I}_{\mathcal{H}} = (\mathcal{B}_{\mathcal{H}}, \mathcal{Q}_1, \mathcal{Q}_2)$ $\mathbf{Hilb}^n_{p(t)} \simeq \operatorname{Proj}\frac{\mathbb{K}[\Delta,\nabla]}{\mathcal{I}_{\mathcal{H}}}$	$\widehat{\mathcal{I}}_{\mathcal{H}} \subset \mathbb{K}[\Delta]$ $\mathbf{Hilb}^n_{p(t)} \simeq \operatorname{Proj}\frac{\mathbb{K}[\Delta]}{\widehat{\mathcal{I}}_{\mathcal{H}}}$
	`false`	$\mathcal{B}_{\mathcal{H}} \subset \mathbb{K}[\Delta, \nabla]$	$\widehat{\mathcal{B}}_{\mathcal{H}} \subset \mathbb{K}[\Delta]$

Example A.2.1. We compute entirely Gotzmann equations of the Hilbert scheme $\mathbf{Hilb}^2_2$, completing Example 1.5.1.

```
Macaulay2, version 1.4
i1 : loadPackage "HilbertSchemesEquations";
i2 : R = QQ[t];
i3 : time Hilb = GotzmannHilbEquations (2_R,2);
     -- 600 bilinear equations
     -- used 10943.3 seconds
i4 : gensHilb = first entries gens (Hilb#1);
```

```
i5 : #gensHilb
o5 = 30376
i6 : gbHilb = first entries gens gb (Hilb#1);
i7 : #gbHilb
o7 = 50
i8 : hilbertPolynomial (Hilb#1,Projective=>false)
     7 4   15 3   45 2   15
o8 = -i  + --i  + --i  + --i + 1
     8      4      8      4
o8 : QQ[i]
```

The computation is very long because to compute the projection of the ideal given by the bilinear equations on a single Grassmannian, the method has to eliminate the variables ∇ in each open subset $U_I \times V_J$ of the open covering of $\mathbb{P}^{14} \times \mathbb{P}^{44}$. Each elimination correspond to a computation of a Gröbner basis and there are $15 \cdot 45 = 675$ possible open subsets.

If we want to embed $\mathbf{Hilb}^2_2$ in the product $\mathbf{Gr}_{\mathbb{K}}(4,6) \times \mathbf{Gr}_{\mathbb{K}}(2,10)$, we have to switch to `false` the option `SingleGrassmannian` and the computation turns out to be very quick.

```
i9 : time Hilb = GotzmannHilbEquations (2_P,2,SingleGrassmannian=>false);
     -- 600 bilinear equations
     -- used 0.542746 seconds
i10 : gensHilb = first entries gens (Hilb#1);
i11 : #gensHilb
o11 = 735
i12 : gbHilb = first entries gens gb (Hilb#1);
i13 : #gbHilb
o13 = 1992
i14 : hilbertPolynomial (Hilb#1,Projective=>false)
           1      16       1     15       739    14      239   13
o14 = -----------i   + ---------i   + ----------i   + --------i   +
      14631321600      182891520      3657830400      52254720
      ---------------------------------------------------------------------------
         74153   12     41759  11    24703913  10    3192863 9    3190527833 8
      ----------i   + --------i   + ----------i   + --------i  + -----------i  +
      1045094400      52254720      3657830400      73156608     14631321600
      ---------------------------------------------------------------------------
       8866763 7   674339683 6   30564427 5   1668104321 4   7824563 3
      --------i  + ---------i  + --------i  + ----------i  + -------i  +
      10450944     261273600     3265920      101606400      508032
      ---------------------------------------------------------------------------
      6898529 2   5477
      -------i  + ----i + 1
       705600     1260
o14 : QQ[i]
```

Iarrobino-Kleiman equations

The package `HilbertSchemesEquations` provides also the code to compute Iarrobino-Kleiman global equations for the Hilbert scheme, even if this method is totally unusable because the huge number of product to be computed as shown in Example 1.5.4.

```
IKHilbEquations = method(TypicalValue => List,
                        Options => {PluckerRelations => true})
--  INPUT: p, admissible Hilbert polynomial.
--         n, dimension of the projective space.
-- OUTPUT: a list containing:
--           - #0 the ring in which the Hilbert scheme is embedded;
--           - #1 the ideal defining the Hilbert scheme.
-- OPTION: PluckerRelations (Boolean), default value true.
--            If true, the ideal of Plucker relations is computed.
--            If false, the Plucker coordinates are considered
--            without relations among them.
--  ERROR: if numgens(ring(p)) > 1.
--         if coefficientRing(ring(p)) != ZZ and
--            coefficientRing(ring(p)) != QQ
--         if p is not admissible.
--         if n < 1.
--         if first degree (p) >= n.
```

Bayer-Haiman-Sturmfels equations

The method implementing the strategy introduced in the proof of Theorem 1.26 for computing the equations of $\mathbf{Hilb}^n_{p(t)}$ is called `BHSHilbEquations`.

```
BHSHilbEquations = method(TypicalValue => List,
                         Options => {PluckerRelations => true})
--  INPUT: p, admissible Hilbert polynomial.
--         n, dimension of the projective space.
-- OUTPUT: a list containing:
--           - #0 the ring in which the Hilbert scheme is embedded;
--           - #1 the ideal defining the Hilbert scheme.
-- OPTION: PluckerRelations (Boolean), default value true.
--            If true, the ideal of Plucker relations is computed.
--            If false, the Plucker coordinates are considered
--            without relations among them.
--  ERROR: if numgens(ring(p)) > 1.
--         if coefficientRing(ring(p)) != ZZ and
--            coefficientRing(ring(p)) != QQ
--         if p is not admissible.
--         if n < 1.
--         if first degree (p) >= n.
```

This function requires as input the same objects as the previous methods and it also has the option `PluckerRelation` to choose if considering the Plücker relations or not. If `PluckerRelations => true` then the relations are used during the computation of the exterior products, in order to reduce any coefficient in the Plücker coordinates.

Example A.2.2. We test the method `BHSHilbEquations` on the Hilbert scheme $\mathbf{Hilb}^2_2$, completing Example 1.5.5.

```
Macaulay2, version 1.4
i1 : loadPackage "HilbertSchemesEquations";
i2 : R = QQ[t];
i3 : time Hilb = BHSHilbEquations (2_R,2);
     -- 577 exterior products
     -- used 1.67346 seconds
i4 : gensHilb = first entries gens (Hilb#1);
i5 : #gensHilb
o5 = 3976
i6 : gbHilb = first entries gens gb (Hilb#1);
i7 : #gbHilb
o7 = 272
i8 : hilbertPolynomial (Hilb#1,Projective=>false)
     7 4   15 3   45 2   15
o8 = -i  + --i  + --i  + --i + 1
     8      4      8      4
o8 : QQ[i]
i9 : time Hilb = BHSHilbEquations (2_R,2,PluckerRelations=>false);
     -- 617 exterior products
     -- used 1.01678 seconds
i10 : gensHilb = first entries gens (Hilb#1);
i11 : #gensHilb
o11 = 1642
```

We underline that by this computation the Hilbert scheme $\mathbf{Hilb}^2_2$ embedded in $\mathbb{P}^{14}$ has dimension 4 and degree 21, i.e. $\frac{7}{8} = \frac{21}{4!}$, confirming what stated by Haiman and Sturmfels in [41, page 756].

Appendix B

The H(ilbert) S(cheme) C(omputation) java library

B.1 The description of the library

The HSC library contains several packages that we now briefly describe.

HSC.math It contains the implementations of the rational numbers and some basic mathematical operations and functions.

HSC.hilbpoly It contains the implementations of Hilbert polynomials and some basic operations on them. For instance given a Hilbert polynomial $p(t)$, it is possible to compute $\Delta p(t)$, its Gotzmann number and Gotzmann decomposition and the saturated lexicographic ideal associated to it.

HSC.monomials It contains the implementations of monomials, monomial ideals and term orderings. All the basic operation on these objects are made available (elementary moves and evaluation on maximal and minimal variable, regularity of a Borel-fixed ideal, etc.).

HSC.borelfixed This package contains the implementation of the most important algorithms introduce in the thesis to work on Borel-fixed ideals. Let us look closer at its classes:

`PosetGraph` contains the implementation of the poset of monomials of a fixed degree with the Borel partial order $\leq_B$ represented as planar graph;

`BorelGenerator` contains the algorithm for computing all the saturated Borel-fixed with chosen number of variables and Hilbert polynomial;

`BorelInequalitiesSystem` contains the algorithms determining whenever a Borel-fixed ideal is a gen/reg/hilb-segment ideal (and computing the relative term ordering);

`ConnectingPath` contains all the algorithms about Borel rational deformations of Borel-fixed ideals.

`HSC.inequality` It contains the implementations of the linear inequalities with integer coefficients needed for the simplex algorithm used in the computation of segment ideals.

`HSC.utilities` It contains methods to manage the input/output.

B.2 Borel-fixed ideals and segment ideals

Algorithm 2.4, computing all the saturated Borel-fixed ideals with fixed number of variables and Hilbert polynomial, can be tested by the applet `Borel Generator` (Figure B.1) available at `www.personalweb.unito.it/paolo.lella/HSC/borelGenerator.html`. There are two field to fill:

Projective space requires the dimension of the projective space;

Hilbert polynomial requires the Hilbert polynomial as list of coefficients enclosed in square brackets. Any rational coefficient has to be enclosed in round brackets. As example

$$\begin{aligned} p(t) = 4t &\qquad \rightsquigarrow \qquad [0,4], \\ p(t) = \frac{3}{2}t^2 + \frac{5}{2}t - 1 &\qquad \rightsquigarrow \qquad [-1,(5/2),(3/2)]. \end{aligned}$$

In the output window (Figure B.2), there will be the list of all Borel-fixed saturated ideals, with first element always the lexicographic ideal.

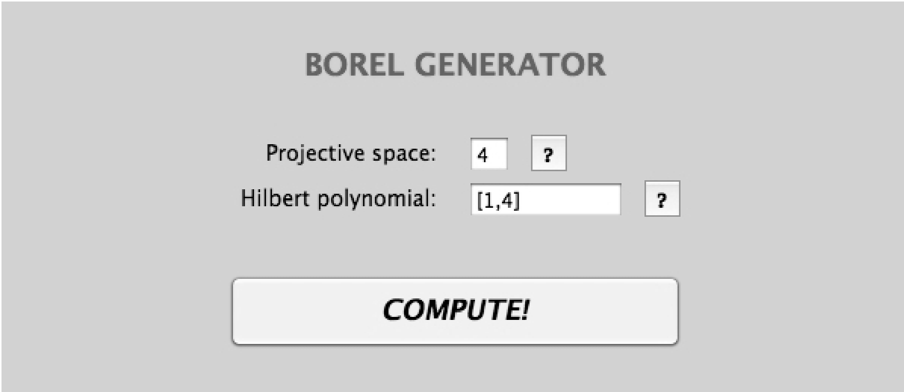


Figure B.1: The applet `Borel Generator`.

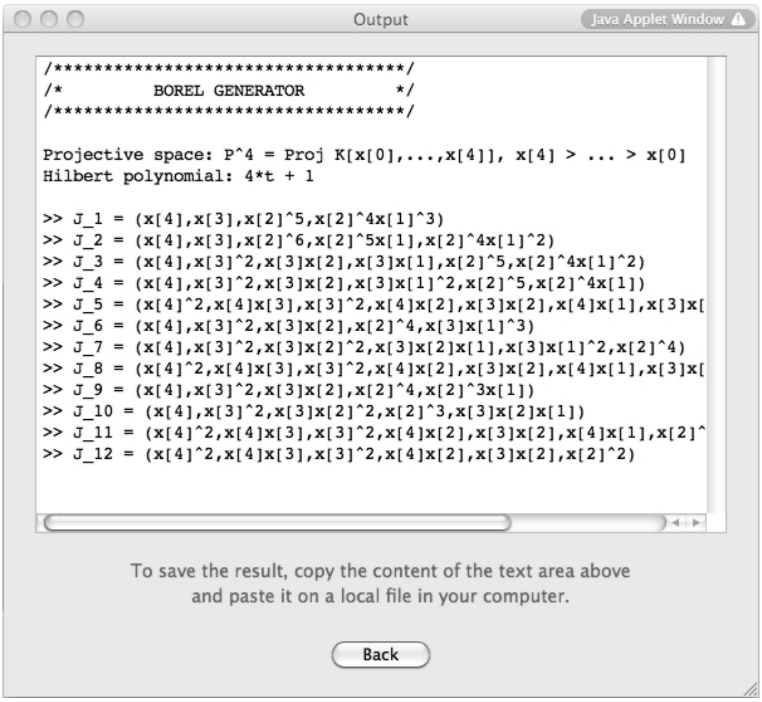


Figure B.2: The output window of the applet `Borel Generator`.

The applet `Segment Ideals` (Figure B.3) makes available the algorithms to determine whenever a Borel-fixed ideals is hilb/reg/gen-segment ideal (Algorithm 2.7 and Algorithm 2.8). It can be found at `www.personalweb.unito.it/paolo.lella/HSC/segment.html` requires three arguments as input:

Projective space needs a positive integer declaring the dimension of the projective space (i.e. n means that the polynomial ring used will be $\mathbb{K}[x_0, \ldots, x_n]$);

Borel-fixed ideal needs a string describing a Borel-fixed ideal, with the same syntax used in the output window of the applet `Borel Generator`;

Truncation degree needs a positive integer, which will be used to compute if the ideal truncated in such degree is a gen-segment ideal.

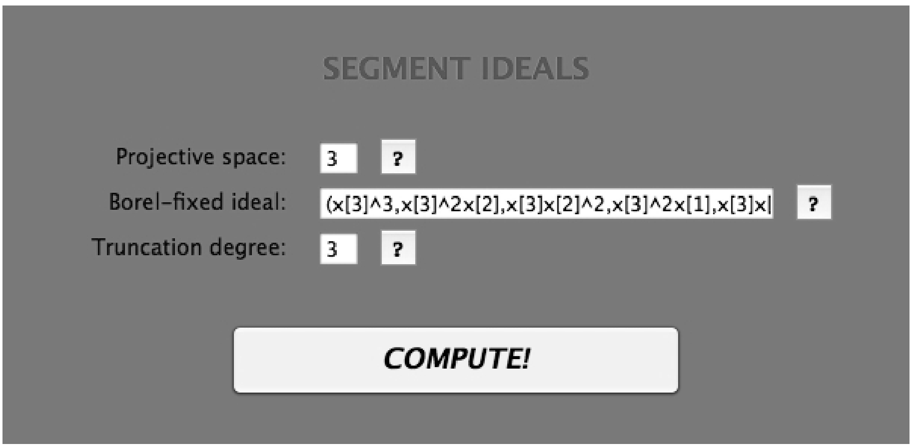


Figure B.3: The applet `Segment Ideals`.

In Figure B.4, there is the output window of this applet, with the results of the computation on the ideal discussed in Example 2.7.4. If the ideal is some segment ideal, the applet specifies the term ordering with the first solution (that one with smallest coefficients) of the system of constraints given by the symplex algorithm.

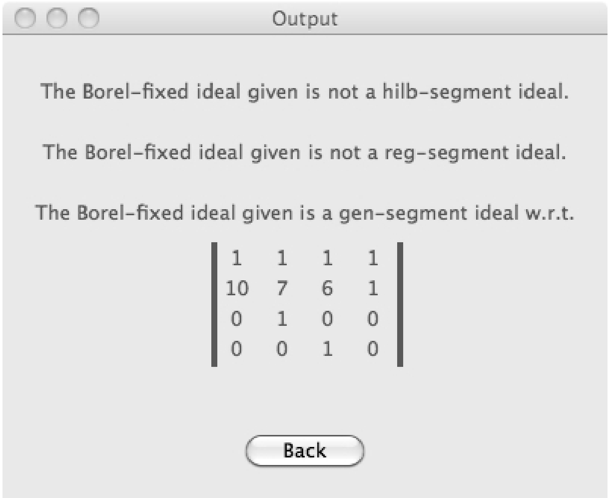


Figure B.4: The output window of the applet `Segment Ideals`.

B.3 Borel rational deformations

The applet computing all the Borel rational deformations of a Borel-fixed ideal (Algorithm 3.1) is called `Borel Rational Deformations` (Figure B.5) and it can be found at `www.personalweb.unito.it/paolo.lella/HSC/deformations.html`.

It requires two arguments:

Projective space is the dimension of the projective space (as before).

Borel-fixed ideal needs the string of the Borel-fixed ideals that we want to deform, again with the systax used in the applet `Borel Generator`.

The algorithm computes both simple and composed Borel rational deformations, considering the Borel set defined by the homogeneous piece of the ideal of degree equal to the Gotzmann number of its Hilbert polynomial, so that all the possible Borel rational curves on the Hilbert scheme passing through the given ideal are determined.

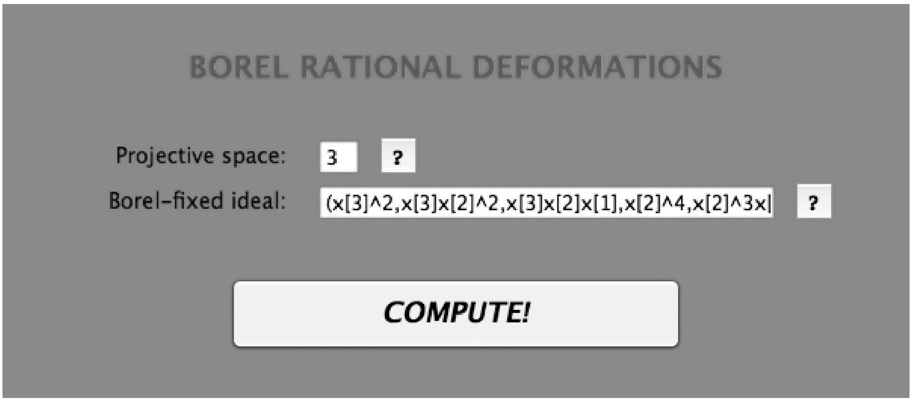


Figure B.5: The applet `Borel Rational Deformations`.

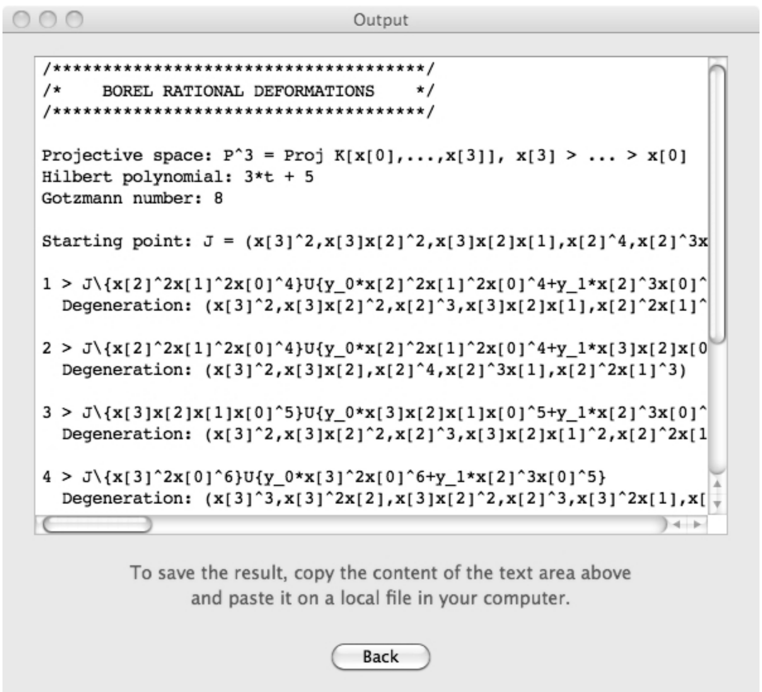


Figure B.6: The output window of the applet `Borel Rational Deformations`.

To have a global glance of Borel rational curves on a Hilbert scheme, we can use the applet `Borel Incidence Graph` (Figure B.7) that computes simple and composed Borel rational deformations of every Borel-fixed ideal with number of variables and Hilbert polynomial fixed (Algorithm 3.5). It requires the same two arguments of the applet `Borel Generator` (Projective space and Hilbert polynomial) and it is available at `www.personalweb.unito.it/paolo.lella/HSC/borelIncidenceGraph.html`

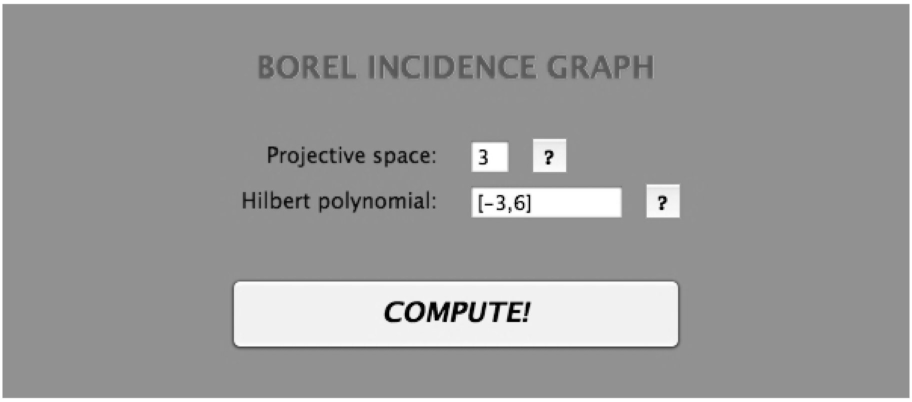


Figure B.7: The applet `Borel Incidence Graph`.

The output window (Figure B.8) is divided in two part. The upper part contains the details about the deformations: for each Borel-fixed ideals there is the list of all the simple and composed Borel rational deformations involving it. In the lower part of the window there is the description of the graph (vertices and edges) described with the language of the free software `Graphviz`. Copying the code and pasting it in a file with `.dot` extension and compiling with the `neato` processor it is possible to get the picture of the Borel incidence graph (see for instance Figure 3.12 and Figure 3.14).

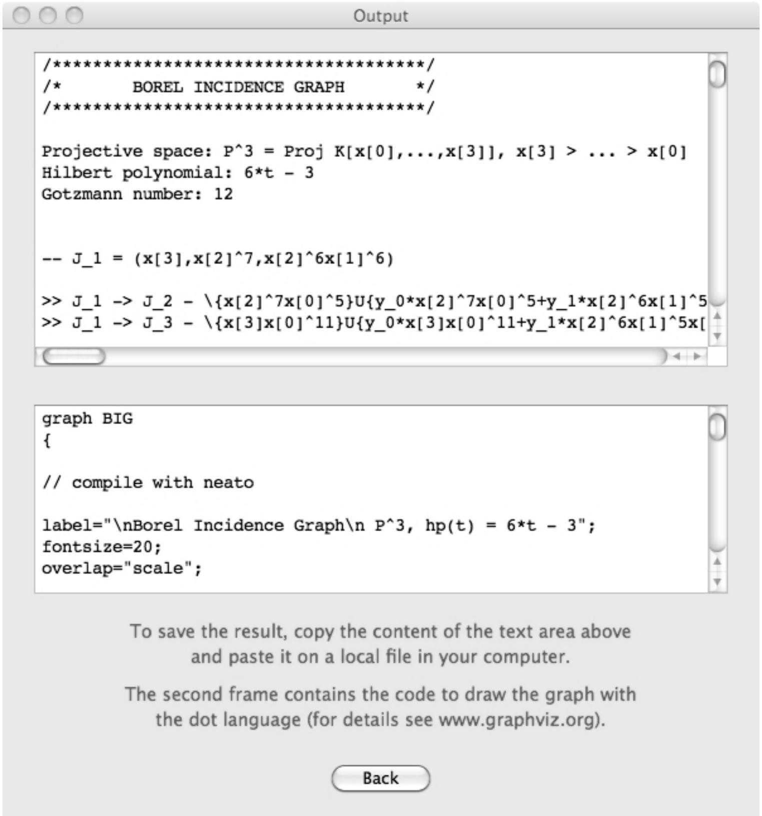


Figure B.8: The output window of the applet `Borel Incidence Graph`.

B.4 σ-Borel degenerations

The algorithms dealing with oriented Borel rational deformations are made available in the applets `Oriented Borel Rational Degeneration` (Figure B.9) and `Degeneration Graph` (Figure B.12).

`Oriented Borel Rational Degeneration` (Algorithm 3.3) is available at `www.personalweb.unito.it/paolo.lella/HSC/TOdeformation.html` and requires three arguments: Projective space and Borel-fixed ideal are as in the applet `Borel Rational Deformations`. Moreover a term ordering is required: the default term order is `DegLex`. To change it, we need to click on the button

"Change": in the dialog window that opens (Figure B.10) there are two options, Graded and Reverse, and the fields to fill with a sequence of rational coefficients determining the term ordering. The `DegLex` term ordering corresponds to Graded: yes, Reverse: no, Weights: 1,0,. . . ,0, whereas `DegRevLex` corresponds to Graded: yes, Reverse: yes, Weights: 0,. . . ,0,-1. To fix the term ordering described in (2.41), it suffices to choose Graded: yes, Reverse: no and to insert the vector $(\omega_n, \ldots, \omega_0)$.

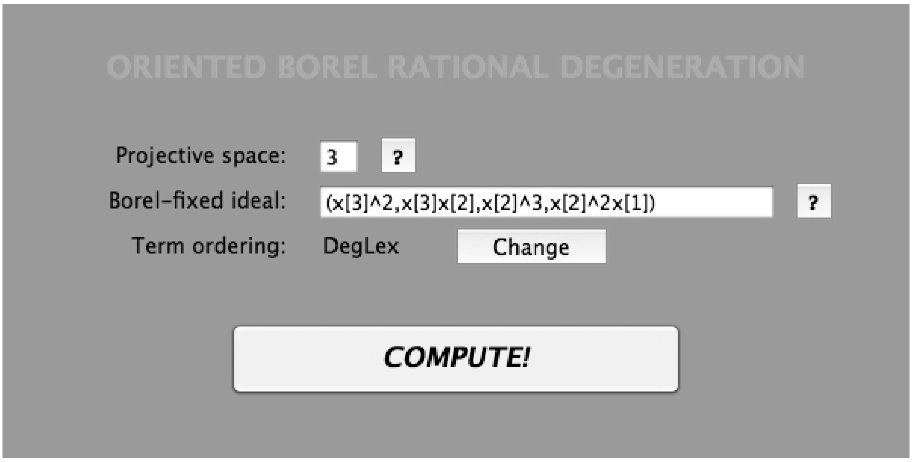


Figure B.9: The applet `Oriented Borel Rational Degeneration`.

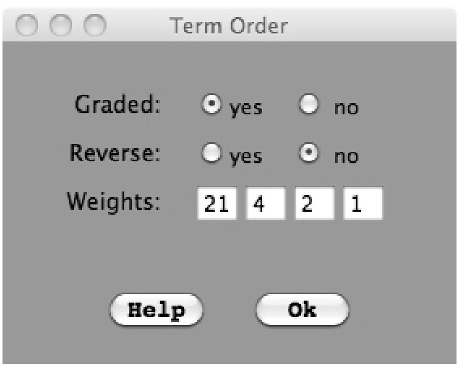


Figure B.10: The dialog window for changing the term ordering.

In the output window (Figure B.11), there is the description of the degeneration computed in degree equal to the Gotzmann number of the Hilbert polynomial of the ideal, with the monomials exchanged specified and the Borel-fixed ideal obtained.

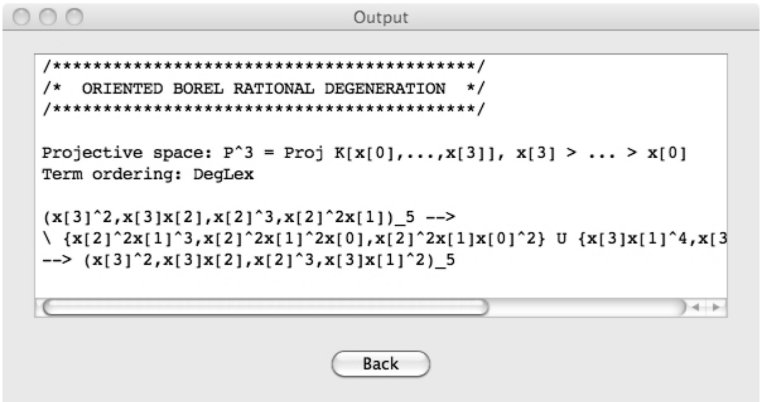


Figure B.11: The output window of the applet `Oriented Borel Rational Degeneration`.

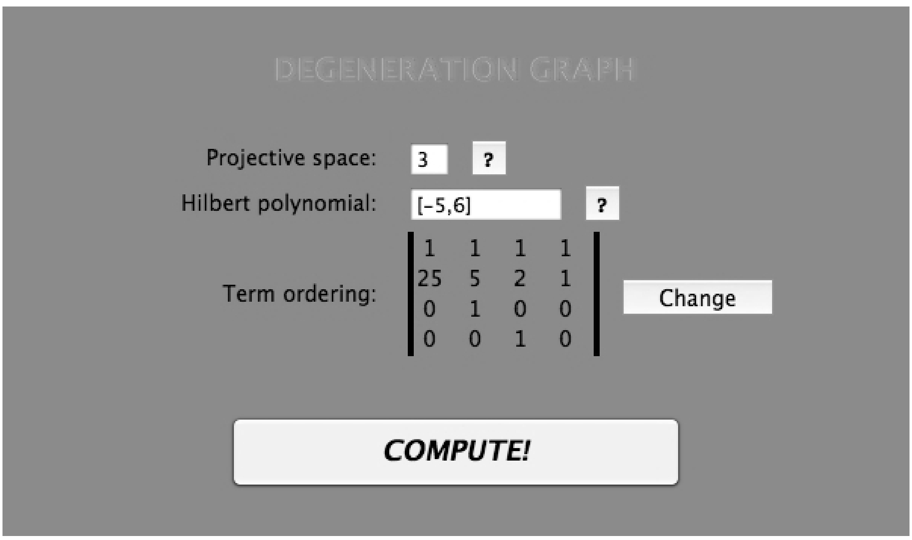


Figure B.12: The applet `Degeneration Graph`.

The applet `Degeneration Graph` (available at `www.personalweb.unito.it/paolo.lella/HSC/deformationGraph.html`) realizes Algorithm 3.4 and needs three arguments: Projective space, Hilbert polynomial and Term ordering. As for the applet computing the Borel incidence graph, its output window (Figure B.13) splits in two parts: the upper one contains the explicit description of the Borel-fixed ideals defining points on the chosen Hilbert polynomial with the relative Borel degeneration prescribed by the fixed term ordering, while the lower one contains the description of the direct graph again with the `Graphviz` code. Copying the code and pasting it in a file with `.dot` extension and compiling with the `dot` processor it is possible to get the picture of the forest representing the degeneration graph (see for instance Figure 3.6 and Figure 3.8).

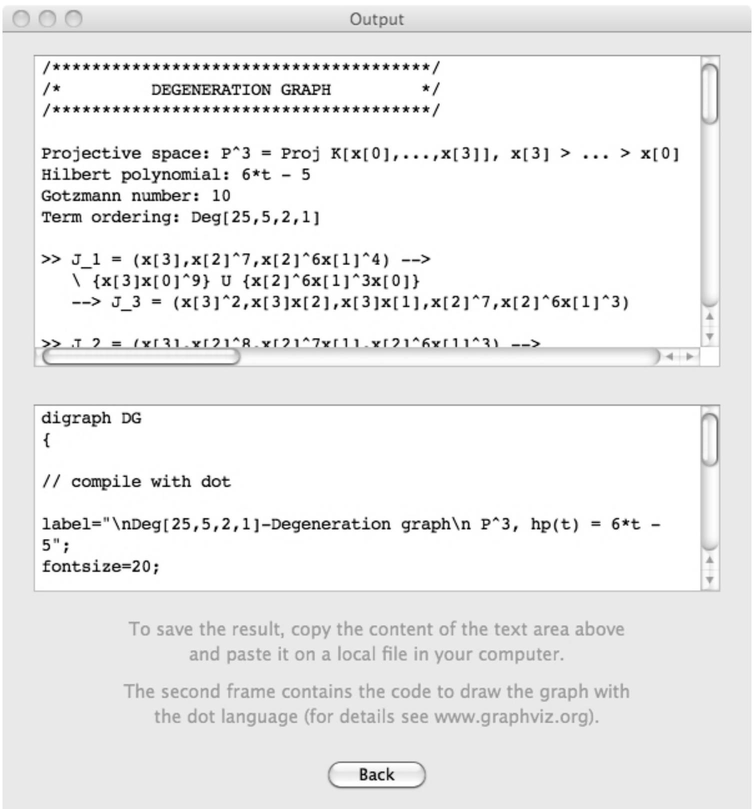


Figure B.13: The output window of the applet `Degeneration Graph`.

Appendix C

The *Macaulay2* package `MarkedSchemes`

This chapter is supposed to be a handbook for the *Macaulay2* [37] package `MarkedSchemes.m2`. We will introduce and explain the main functions of the package that allow to compute affine schemes associated to marked families and Gröbner strata.

C.1 Basic features

Firstly there are some basic methods to manipulate monomials (in the context of Borel-fixed ideals).

- `moveUp` and `moveDown` implement the elementary moves and require two arguments: a monomial and the index of the elementary move.

```
moveUP = method(TypicalValue => RingElement)
--  INPUT: m monomial
--         i variable index
-- OUTPUT: the monomial m*(ring(m)_(i-1)/ring(m)_i);
--  ERROR: if m is not a monomial
--         if m is a constant
--         if i < 1 or i > numgens(ring(m))-1
--         if the i-th variable does not divide m

moveDOWN = method(TypicalValue => RingElement)
```

```
--  INPUT: m monomial
--         i variable index
-- OUTPUT: the monomial m*(ring(m)_(i+1)/ring(m)_i);
--  ERROR: if m is not a monomial
--         if m is a constant
--         if i < 0 or i >= numgens(ring(m))-1
--         if the i-th variable does not divide m
```

- `minimum` and `maximum` require as argument a monomial and return the index of the minimum/maximum variable dividing the monomial.

```
minimum = method(TypicalValue => ZZ)
--  INPUT: m monomial
-- OUTPUT: the index of the smallest variable dividing m
--  ERROR: if m is not a monomial
--         if m is a constant

maximum = method(TypicalValue => ZZ)
--  INPUT: m monomial
-- OUTPUT: the index of the greatest variable dividing m
--  ERROR: if m is not a monomial
--         if m is a constant
```

- `canonicalDecomposition` is a method for computing the canonical decomposition of a monomial w.r.t. a Borel-fixed ideal containing it. It returns a sequence with two entries, the first one is the unique generator of the ideal giving the decomposition.

```
canonicalDecomposition = method(TypicalValue => Sequence)
--  INPUT: m monomial
--         J Borel-fixed ideal
-- OUTPUT: a sequence of two elements that represent the
--           canonical decomposition of m over J
--  ERROR: if m is not a monomial
--         if m is constant
--         if J is not a monomial ideal
--         if J is not a Borel-fixed ideal
--         if m does not belong to J
```

Example C.1.1. With this example, we want to point out that *Macaulay2* does not permit to work with decreasing indexed variables, so we will have to consider $x_0 > \ldots > x_n$.

```
Macaulay2, version 1.4
i1 : loadPackage "MarkedSchemes";
```

```
i2 : R := QQ[x_0..x_3];
i3 : m = x_0*x_1
o3 = x x
      0 1
o3 : QQ[x , x , x , x ]
         0   1   2   3
i4 : mUP = moveUP(m,1)
      2
o4 = x
      0
o4 : QQ[x , x , x , x ]
         0   1   2   3
i5 : mDOWN = moveDOWN(m,1)
o5 = x x
      0 2
o5 : QQ[x , x , x , x ]
         0   1   2   3
i6 : minimum mDoWN
o6 = 2
i7 : maximum m
o7 = 0
i8 : J = ideal(mUP,m,mDoWN,x_1^5,x_1^4*x_2)
              2                5   4
o8 = ideal (x , x x , x x , x , x x )
              0   0 1   0 2   1   1 2
o8 : Ideal of QQ[x , x , x , x ]
                   0   1   2   3
i9 : f = x_0*x_1^5*x_2
         5
o9 = x x x
       0 1 2
o9 : QQ[x , x , x , x ]
          0   1   2   3
i10 : canonicalDecomposition (f,J)
              4
o10 = (x x , x x )
        0 1   1 2
o10 : Sequence
```

C.2 Marked families and Gröbner strata

The basic method implementing Algorithm 4.2 is called `markedScheme`.

```
markedScheme = method(TypicalValue => Sequence,
                      Options => {MonomialOrder => GRevLex,
                                  Segment => false,
                                  DescribeFamily => false,
                                  EliminateParameters => false});
--   INPUT: J Borel-fixed saturated ideal
--          s degree in which J has to be truncated
-- OPTIONS: Segment, if true there exists a term ordering for which the
--                   ideal is a gen-segment ideal.
```

```
--          MonomialOrder, if the option Segment is true, the option
--                         MonomialOrder contains the term ordering for
--                         which the ideal is a gen-segment ideal.
--          DescribeFamily, if true the function returns the complete
--                          list of polynomial generators with coefficients
--                          in the parameters.
--          EliminateParameters, if true and Segment is true some
--                               parameters willbe eliminated.
--  OUTPUT: sequence containting
--             - #0 the ring of parameters
--             - #1 the ideal of the marked scheme
--             - #2 the ring of the parameters and the starting variables
--             - #3 the polynomial generators of the family
--                  (if opts.DescribeFamily all the polynomials,
--                  otherwise only the superminimal generators)
--   ERROR: if J is not a monomial ideal
--          if J is not a Borel-fixed ideal
--          if s < 0
-- WARNING: if J is not saturated, the ideal will be saturated.
```

With the default choices for the options of the method, the scheme of the J-marked family is computed avoiding any term ordering. If we are interested in the open subset of the Hilbert scheme defined by J, there is the method `openSubsetHilb` that computes the maximal degree ρ of a monomial generator of J divided by the last but one variable and then calls `markedScheme` with arguments J and $\rho - 1$.

Example C.2.1. Let us consider the ideal $(x_0^2, x_0x_1^2, x_1^5) \subset \mathbb{K}[x_0, x_1, x_2]$, $x_0 > x_1 > x_2$ and its truncation in degree 3.

```
Macaulay2, version 1.4
i1 : loadPackage "MarkedSchemes";
i2 : R := QQ[x_0..x_2];
i3 : J = ideal(x_0^2,x_0*x_1^2,x_1^5)
             2     2   5
o3 = ideal (x , x x , x )
             0   0 1   1
o3 : Ideal of QQ[x , x , x ]
                  0   1   2
i4 : Mf = markedScheme(J,2);
     -- Computation of normal forms in progress...
     -- Completed in .073413
     -- Computation of S-polynomials in progress...
     -- Completed in .106108
i5 : Mf#0
o5 = QQ[p , p , p , p , p , p , p , p , p , p , p  , p  , p  , p  , p  , p  ,
          0   1   2   3   4   5   6   7   8   9   10    11    12    13    14    15
     ---------------------------------------------------------------------------
```

```
        p  , p  ]
         16   17
o5 : PolynomialRing
i6 : Mf#1
                   2            2                                        2
o6 = ideal (- p p    + p   - p , p p    - p p  + p p  - p p   + 2p p   - p , p p
               4 10    10     3   9 10    4 8    3 9    2 10     8 10    1   9 10
     ----------------------------------------------------------------------------
                                                         2
     + p p  + p p   + p , p p p   - p p  + p p  - p p  + p  + p p  + p p   - p ,
        8 9    7 10    6   8 9 10    4 6    3 7    2 8    8    1 9    6 10    0
     ----------------------------------------------------------------------------

     p p p   + p p  + p , p p p   - p p  - p p  + p p  + p p  + p p  + p p  ,
      7 9 10    7 8    5   6 9 10    4 5    2 6    1 7    6 8    0 9    5 10
     ----------------------------------------------------------------------------
                                   2                            3
     p p p   - p p  + p p  + p p , - p p   - p p  - p p   - p , - p p   +
      5 9 10    2 5    0 7    5 8     9 10    8 9    7 10    6     9 10
     ----------------------------------------------------------------------------
      2            2                         2
     p p  p   - p p  - 2p p p   + p p  p   - p  p   + p p p   + p p  p   - p p
      9 10 17    8 9      7 9 10    4 10 15    10 15    8 9 17    7 10 17    7 8
     ----------------------------------------------------------------------------
                                     4    3           2
     - p p  + p p   + p p   - p , - p  + p p   - 3p p  + p p p   - 2p p  p   +
        6 9    3 15    6 17    5     9    9 17     7 9    4 9 15     9 10 15
     ----------------------------------------------------------------------------
      2                  2
     p p   + 2p p p   - p  + p p   - p  p   + p p   + p p   - p p   + p p   +
      9 16     7 9 17    7    4 13    10 13    9 14    2 15    8 15    7 16
     ----------------------------------------------------------------------------
              3      2                      2
     p  , - p p  + p p p   - 2p p p  - p p  - p p  p   + p p p   - p p  p   +
      12     8 9    8 9 17     7 8 9    6 9    9 10 14    4 8 15    8 10 15
     ----------------------------------------------------------------------------

     p p p   + p p p   + p p p   - p p  - p p  - p  p   + p p   + p p   + p p
      8 9 16    7 8 17    6 9 17    6 7    5 9    10 12    3 13    1 15    6 16
     ----------------------------------------------------------------------------
                   3      2        2
     + p p  , - p p  + p p p   - 2p p  - p p  p   + p p p   - p p  p   + p p p
        5 17     7 9    7 9 17     7 9    9 10 13    4 7 15    7 10 15    7 9 16
     ----------------------------------------------------------------------------
        2                                          3      2                    2
     + p p   + p p   - p p   + p p   + p  , - p p  + p p p   - 2p p p  - p p  -
        7 17    2 13    8 13    7 14    11     6 9    6 9 17     6 7 9    5 9
     ----------------------------------------------------------------------------

     p p  p   + p p p   - p p  p   + p p p   + p p p   + p p p   - p p  - p  p
      9 10 12    4 6 15    6 10 15    6 9 16    6 7 17    5 9 17    5 7    10 11
     ----------------------------------------------------------------------------
                                                  3      2
     - p p   + p p   + p p   + p p   + p p  , - p p  + p p p   - 2p p p  -
        8 12    1 13    6 14    0 15    5 16     5 9    5 9 17     5 7 9
     ----------------------------------------------------------------------------
```

```
     p p  p   + p p p  - p p  p   + p p p  + p p p  - p p  + p p  + p p  )
      9 10 11    4 5 15    5 10 15    5 9 16    5 7 17    8 11    0 13    5 14

o6 : Ideal of QQ[p , p , p , p , p , p , p , p , p , p , p  , p  , p  , p  ,
                  0   1   2   3   4   5   6   7   8   9   10    11    12    13
     --------------------------------------------------------------------------
                 p  , p  , p  , p  ]
                  14    15    16    17
i7 : Mf#2

o7 = QQ[p , p , p , p , p , p , p , p , p , p , p  , p  , p  , p  , p  , p  ,
         0   1   2   3   4   5   6   7   8   9   10    11    12    13    14    15
         p  , p  ][x , x , x ]
          16    17   0   1   2
o7 : PolynomialRing
i8 : Mf#3
       2               2                         2     2       3
o8 = {x  - p x x  - p x  - p x x  - p x x  - p x , x x  - p  x  - p x x x  -
       0    4 0 1    3 1    2 0 2    1 1 2    0 2   0 1    10 1    9 0 1 2
     --------------------------------------------------------------------------
        2         2         2       3   5       4         3 2          3
     p x x  - p x x  - p x x  - p x , x  - p  x x  - p  x x  - p  x x x  -
      8 1 2    7 0 2    6 1 2    5 2   1    17 1 2    16 1 2    15 0 1 2
     --------------------------------------------------------------------------
         2 3         4         4       5
     p  x x  - p  x x  - p  x x  - p  x }
      14 1 2    13 0 2    12 1 2    11 2
o8 : List
```

Since the highest degree of a generator divisible by x_1 is 5, the optimal degree to compute the open subset of $\mathbf{Hilb}^2_7$ defined by J is 4. Note that there are more parameters because there are more monomials in the tails of the superminimal generators.

```
i9 : HJ = openSubsetHilb (J);
     -- Computation of normal forms in progress...
     -- Completed in .363012
     -- Computation of S-polynomials in progress...
     -- Completed in .152566
i10 : HJ#0
o10 = QQ[p , p , p , p , p , p , p , p , p , p , p  , p  , p  , p  , p  , p  ,
          0   1   2   3   4   5   6   7   8   9   10    11    12    13    14    15
      --------------------------------------------------------------------------
          p  , p  , p  , p  , p  ]
           16    17    18    19    20
o10 : PolynomialRing
i11 : HJ#3
            4      3      2 2          2       2 2        3        3      4
o11 = {- p x  - p x x  + x x  - p x x x  - p x x  - p x x  - p x x  - p x , -
          6 1    5 1 2    0 2    4 0 1 2    3 1 2    2 0 2    1 1 2    0 2
      --------------------------------------------------------------------------
          4      2         3            2        2 2        3        3      4
      p  x  + x x x  - p  x x  - p  x x x  - p  x x  - p x x  - p x x  - p x ,
       13 1    0 1 2    12 1 2    11 0 1 2    10 1 2    9 0 2    8 1 2    7 2
      --------------------------------------------------------------------------
```

```
      5       4         3 2          3        2 3        4         4        5
     x  - p  x x  - p  x x  - p  x x x  - p  x x  - p  x x  - p  x x  - p  x }
      1    20 1 2    19 1 2    18 0 1 2    17 1 2    16 0 2    15 1 2    14 2
o11 : List
```

There are also the methods computing the dimension of the tangent space at the origin.

```
EDmarkedScheme = method(TypicalValue => ZZ,
                        Options => {MonomialOrder => GRevLex,
                        Segment => false});
--   INPUT: J Borel-fixed saturated ideal
--          s degree in which J has to be truncated
-- OPTIONS: Segment, if true there exists a term ordering for which the
--                   ideal is a gen-segment ideal.
--          MonomialOrder, if the option Segment is true, the option
--                         MonomialOrder contains the term ordering for
--                         which the ideal is a gen-segment ideal.
--  OUTPUT: the number of parameters necessary to describe the marked family
--   ERROR: if J is not a monomial ideal
--          if J is not a Borel-fixed ideal
--          if s < 0
-- WARNING: if J is not saturated, the ideal will be saturated.

EDopenSubsetHilb = method(TypicalValue => ZZ,
                          Options => {MonomialOrder => GRevLex,
                          Segment => false});
--   INPUT: J Borel-fixed saturated ideal
-- OPTIONS: Segment, if true there exists a term ordering for which the
--                   ideal is a gen-segment ideal.
--          MonomialOrder, if the option Segment is true, the option
--                         MonomialOrder contains the term ordering for
--                         which the ideal is a gen-segment ideal.
--  OUTPUT: the number of parameters necessary to describe the marked scheme
--   ERROR: if J is not a monomial ideal
--          if J is not a Borel-fixed ideal
--          if s < 0
-- WARNING: if J is not saturated, the ideal will be saturated.
```

Considering Example C.2.1, we have that

```
i12 : EDmarkedScheme(J,3)
o12 = 19
i13 : EDopenSubsetHilb(J)
o13 = 21
```

To compute the Gröbner stratum of a gen-segment ideal, i.e. to add to the marked family the structure of homogeneous variety w.r.t. a positive grading, we have to use the options putting `Segment = true`, giving the term ordering for which the ideal is a gen-segment ideal and requiring the elimination of the parameters (`EliminateParameters = true`).

Example C.2.2. Let us consider the Borel-fixed ideal $J = (x_0^2, x_0x_1, x_0x_2^2, x_1^4)$ in $\mathbb{K}[x_0, x_1, x_2, x_3]$ $(x_0 > x_1 > x_2 > x_3)$, that defines a point on the Hilbert scheme $\mathbf{Hilb}^3_{4t}$ (cf. Example 4.5.4).

```
Macaulay2, version 1.4
i1 : loadPackage "MarkedSchemes";
i2 : R := QQ[x_0..x_3];
i3 : J = ideal(x_0^2,x_0*x_1,x_0*x_2^2,x_1^4)
             2           2   4
o3 = ideal (x , x x , x x , x )
             0   0 1   0 2   1
o3 : Ideal of QQ[x , x , x , x ]
                  0   1   2   3
i4 : EDopenSubsetHilb (J)
o4 = 44
i5 : EDopenSubsetHilb (J,Segment=>true,MonomialOrder=>{Weights=>{7,3,2,1}})
o5 = 24
i6 : St = openSubsetHilb (J,Segment=>true,
                          MonomialOrder=>{Weights=>{7,3,2,1}},
                          EliminateParameters=>true);
     -- Computation of normal forms in progress...
     -- Completed in .116961
     -- Computation of S-polynomials in progress...
     -- Completed in .395245
     -- Elimination of parameters in progress...
     -- Completed in 5.42916
i7 : A := St#0
o7 = QQ[p  , p  , p  , p  , p  , p  , p  , p  , p  , p  , p  , p  , p  , p  ,
         12   13   16   17   19   20   23   24   25   26   28   30   31   32
     --------------------------------------------------------------------------
        p  , p  , p  , p  , p  , p  , p  , p  , p  , p  ]
         34   35   36   37   38   39   40   41   42   43
o7 : PolynomialRing
i8 : numgens A
o8 = 24
i9 : idealSt = St#1;
i10 : gensIdealSt := first entries gens idealSt;
i11 : #gensIdealSt
o11 = 40
i12 : (factor(gensIdealSt#0))#0
o12 = p
       39
i13 : (factor(gensIdealSt#1))#0
o13 = p
       39
i14 : (factor(gensIdealSt#2))#0
o14 = p
       39
i15 : dec1 := ideal(p_39);
```

```
i16 : RScomponent := Spec (A/dec1);
i17 : dim RScomponent
o17 = 23
i18 : dec2 := saturate (idealSt,p_39);
i19 : VAcomponent := Spec (A/dec2);
i20 : dim VAcomponent
o20 = 16
```

Catalog of Hilbert schemes

The list of all Hilbert schemes discussed in the examples with a brief description of the geometric objects parametrized.

Hilbert scheme	Geometric objects parametrized	pages
$\mathbf{Hilb}^2_2$	2 points in the projective plane $\mathbb{P}^2$	32, 35, 38, 175, 219, 221, 227, 241, 271, 274
$\mathbf{Hilb}^3_{3t+1}$	curves of degree 3 and genus 0 in the projective space $\mathbb{P}^3$ (containing rational normal curves of degree 3)	120
$\mathbf{Hilb}^3_{6t-5}$	curves of degree 6 and genus 6 in the projective space $\mathbb{P}^3$	134
$\mathbf{Hilb}^3_8$	8 points in the projective space $\mathbb{P}^3$ (first example of Hilbert scheme of points with reducible components)	138
$\mathbf{Hilb}^4_{4t+1}$	curves of degree 4 and genus 0 in the projective space $\mathbb{P}^4$ (containing rational normal curves of degree 4)	149
$\mathbf{Hilb}^3_{6t-3}$	curves of degree 6 and genus 4 in the projective space $\mathbb{P}^3$ (containing $(2,3)$-complete intersections)	151
$\mathbf{Hilb}^3_{4t}$	curves of degree 4 and genus 1 in the projective space $\mathbb{P}^3$ (containing $(2,2)$-complete intersections)	175, 229, 294

Catalog of Hilbert schemes

Hilbert scheme	Geometric objects parametrized	pages
$\mathbf{Hilb}^3_{3t}$	curves of degree 3 and genus 1 in the projective space $\mathbb{P}^3$ (containing $(1,3)$-complete intersections)	219, 225, 229
$\mathbf{H}_{4,-3}$	locally Cohen-Macaulay curves of degree 4 and genus -3 in the projective space $\mathbb{P}^3$ (containing disjoint unions of 4 lines on a smooth quadric surface)	254
$\mathbf{Hilb}^2_7$	7 points in the projective plane $\mathbb{P}^2$	292

Symbols and Notation

Symbol	Typical usage or definition	page
$\mathbb{K}$	algebraically closed field of characteristic 0	1
$\mathbb{K}[x]$	compact notation for the polynomial ring $\mathbb{K}[x_0,\dots,x_n]$	1
$(\text{schemes})_{\mathbb{K}}$	the category of schemes over $\mathbb{K}$	7
(sets)	the category of sets	7
h_X	the functor of points of the scheme X	7
$\mathrm{Ob}_{\mathcal{C}}$	the set of objects of the category $\mathcal{C}$	7
$\mathrm{Mor}_{\mathcal{C}}$	the set of morphisms between objects of the category $\mathcal{C}$	7
$\mathrm{Fun}(\mathcal{C},\mathcal{C}')$	the category of functors between the categories $\mathcal{C}$ and $\mathcal{C}'$	9
$(\text{schemes})_{\mathbb{K}}^{\mathrm{Aff}}$	the category of affine schemes over $\mathbb{K}$	10
$(\mathbb{K}\text{-algebras})$	the category of $\mathbb{K}$-algebras	10
$\mathscr{P}$	Plücker embedding of the Grassmannian	11
$\wedge^q V$	exterior algebra of the vector space V of order q	11
$\mathbb{K}[\Delta]$	compact notation for the polynomial ring of Plücker coordinates $\mathbb{K}[\dots,\Delta_{\mathrm{I}},\dots]$	11
$\mathcal{G}\mathrm{r}_q^N$	Grassmann functor	22
$Z(I)$	the Zariski closed set defined by the ideal I	22
$\mathcal{H}\mathrm{ilb}^n$	Hilbert functor	25
$\mathcal{H}\mathrm{ilb}^n_{p(t)}$	the subfunctor of the Hilbert functor associated to the Hilbert polynomial $p(t)$	26
$\mathbf{Hilb}^n_{p(t)}$	the Hilbert scheme, representing the functor $\mathcal{H}\mathrm{ilb}^n_{p(t)}$	26
$I(X)$	the saturated ideal defining the scheme X	28

Symbol	Typical usage or definition	page
$\mathrm{GL}_{\mathbb{K}}(n+1)$	the group of invertible matrices	41
$\mathsf{in}_\sigma(I)$	the initial ideal of the ideal I with respect to the term order σ	41
$\mathsf{gin}_\sigma(I)$	the generic initial ideal of the ideal I with respect to the term order σ	42
$\mathrm{B}^{\mathrm{GL}}_{\mathbb{K}}(n+1)$	the Borel subgroup of $\mathrm{GL}_{\mathbb{K}}(n+1)$ of the upper triangular matrices	43
$\mathrm{T}^{\mathrm{GL}}_{\mathbb{K}}(n+1)$	the torus subgroup of $\mathrm{GL}_{\mathbb{K}}(n+1)$ of the diagonal matrices	43
I^{sat}	the saturation of the ideal I	45
$\min x^\alpha$	the smallest variable (or its index) dividing x^α	47
$\max x^\alpha$	the greatest variable (or its index) dividing x^α	47
$\langle x^{\alpha_i} \vert x^\gamma \rangle^I$	the canonical decomposition of the monomial $x^\beta = x^{\alpha_i} x^\gamma \in I$, namely x^α_i is a minimal generator of the Borel-fixed ideal I and $\min x^{\alpha_1} \geq \max x^\gamma$	48
∂_I	the decomposition function of the Borel-fixed ideal I. For each monomial $x^\beta \in I$, ∂_I maps x^β to the minimal generator x^{α_i} of I appearing in its canonical decomposition, i.e. $x^\beta = \langle x^{\alpha_i} \vert x^\gamma \rangle^I$	48
$\mathrm{Syz}(I)$	the module of syzygies of the ideal I (also a set of generators of the module)	49
$\Delta^i p(t)$	the i-th difference of the polynomial $p(t)$: $\Delta^i p(t) = \Delta^{i-1} p(t) - \Delta^{i-1} p(t-1)$	50
$\Sigma p(t)$	the inverse operation of Δ on polynomials, namely $\Delta(\Sigma p)(t) = p(t)$ and $\Sigma(\Delta p)(t) = p(t) - c,\ c \geqslant 0$	51
e_i^+	the element $\frac{x_{i+1}}{x_i}$ in the field of fractions $\mathbb{K}(x_0,\ldots,x_n)$	52
e_j^-	the element $\frac{x_{j-1}}{x_j}$ in the field of fractions $\mathbb{K}(x_0,\ldots,x_n)$	52
$\leq_B$	the Borel partial order	53
$\mathcal{P}(n,m)$	the poset of monomials in $\mathbb{K}[x_0,\ldots,x_n]_m$ w.r.t. the Borel order	54

Symbol	Typical usage or definition	page
$\mathscr{B}$	Borel set	54
$\{I_m\}$	the Borel set defined by the monomials in I_m	54
$\mathscr{N}$	order set, the complement of a Borel set	54
$S_{(\geqslant i)}$	the subset of S containing the monomials with minimum variable greater than or equal to x_i	55
$\mathcal{B}^n_{p(t)}$	the set of saturated Borel-fixed ideals of $\mathbb{K}[x_0,\ldots,x_n]$ with Hilbert polynomial $p(t)$	82
$\mathcal{N}^n_{p(t)}$	the number of saturated Borel-fixed ideals of $\mathbb{K}[x_0,\ldots,x_n]$ with Hilbert polynomial $p(t)$	82
$\max\operatorname{var}\mathscr{N}$	the greatest variable (or its index) dividing a monomial belonging to $\mathscr{N}$	84
$\mathcal{N}^\bullet_{p(t)}$	the sequence of the number of saturated Borel-fixed ideals with Hilbert polynomial $p(t)$ in polynomials ring with increasing number of variables	85
$\Delta\mathcal{N}^i_s$	the difference between the number of saturated Borel-fixed ideals with Hilbert polynomial s in the polynomial rings $\mathbb{K}[x_0,\ldots,x_{s-i}]$ and $\mathbb{K}[x_0,\ldots,x_{s-i-1}]$	90
$\Delta\mathcal{N}^\bullet_s$	the sequence of the differences $\Delta\mathcal{N}^i_s$ for i varying from 1 to $s-2$	90
$\Delta\mathcal{N}^i$	the constant value of $\Delta\mathcal{N}^i_s$ for $s \gg 0$	91
$\Delta\mathcal{N}^\bullet$	the sequence of integers $\Delta\mathcal{N}^i$ for increasing value of i	91
`Lex`	the lexicographic term order (not graded)	99
`DegLex`	the degree lexicographic term order	99
`DegRevLex`	the degree reverse lexicographic term order	99
$\mathcal{T}^J_\sigma(x^\alpha)$	the set of monomials of the same degree of x^α not belonging to the ideal J, smaller than x^α w.r.t. the term ordering σ	156
$\mathcal{S}t_\sigma(J)$	the family of homogeneous ideals having J as initial ideal w.r.t. the term ordering σ (also the affine scheme describing the family)	156

Symbol	Typical usage or definition	page
$S(f_i, f_j)$	the S-polynomial between f_i and f_j: i.e. $\frac{\mathrm{in}_\sigma(f_j)}{\gcd(\mathrm{in}_\sigma(f_i),\mathrm{in}_\sigma(f_j))} f_i - \frac{\mathrm{in}_\sigma(f_i)}{\gcd(\mathrm{in}_\sigma(f_i),\mathrm{in}_\sigma(f_j))} f_j$ if a term order σ is consider, $\frac{\mathrm{Ht}(f_j)}{\gcd(\mathrm{Ht}(f_i),\mathrm{Ht}(f_j))} f_i - \frac{\mathrm{Ht}(f_i)}{\gcd(\mathrm{Ht}(f_i),\mathrm{Ht}(f_j))} f_j$ if f_i and f_j are marked polynomials	157
$\overline{\mathbb{T}}_x$	the multiplicative group of monomials in the fiels of fractions $\mathbb{K}(x)$	160
$\mathrm{ed}\,\mathcal{S}t_\sigma(J, T)$	the embedding dimension of the Gröbner stratum $\mathcal{S}t_\sigma(J, T)$	162
$\mathrm{Supp}\, X$	the support of the subscheme X	173
$\mathcal{N}(I)$	the sous-escalier of the ideal I, that is the set of monomials not belonging to I	181
$\underline{J}$	the saturation of a Borel-fixed ideal J	181
$\mathrm{Supp}\, f$	the support of a polynomial $f \in \mathbb{K}[x]$, i.e. the set of monomials appearing in f with non-zero coefficient	181
$\mathrm{Ht}(f)$	the specified monomial in $\mathrm{Supp}\, f$ that makes f a marked polynomial	181
$\mathcal{T}(f)$	the tail of a marked polynomial f, i.e. $\mathcal{T}(f) = f - \mathrm{Ht}(f)$	182
$\mathcal{M}f(J)$	the family of ideals I such that the sous-escalier $\mathcal{N}(J)$ of a Borel-fixed ideal J is a basis as $\mathbb{K}$-vector space of $\mathbb{K}[x]/I$	182
lcm	least common multiple	186
$\mathcal{B}^n$	the set of (saturated) Borel-fixed ideals of $\mathbb{K}[x_0, \ldots, x_n]$	215
$\mathcal{M}^n$	the set of (saturated) monomial ideals of $\mathbb{K}[x_0, \ldots, x_n]$	215
$\mathbf{H}_{d,g}$	the Hilbert scheme of locally Cohen-Macaulay curves in $\mathbb{P}^3$ of degree d and genus g (contained in $\mathbf{Hilb}^3_{dt+1-g}$)	247

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