FINITE MORPHIC p-GROUPS

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Abstract. According to Li, Nicholson and Zan, a group G is said to be morphic if, for every pair N_1, N_2 of normal subgroups, each of the conditions $G/N_1 \cong N_2$ and $G/N_2 \cong N_1$ implies the other. Finite, homocyclic p-groups are morphic, and so is the nonabelian group of order p^3 and exponent p, for p an odd prime. It follows from results of An, Ding and Zhan on self dual groups that these are the only examples of finite, morphic p-groups. In this paper we obtain the same result under a weaker hypotesis.

1. Introduction

A module M over a ring is said to be *morphic* if whenever a submodule N is a homomorphic image of M, so that there is an epimorphism $\varphi : M \to N$, then $M/N \cong \ker(\varphi)$. In other words, if N_1 and N_2 are submodules of M, then $M/N_1 \cong N_2$ if and only if $M/N_2 \cong N_1$. This condition, introduced by G. Ehrlich in [Ehr76], has been investigated by W.K. Nicholson and E. Sánchez Campos in [NSC04, NSC05], and by J. Chen, Y. Li and Y. Zhou in [CLZ06]. An extension to groups of results of Ehrlich was obtained by Li and Nicholson in [LN10].

The following analog for groups of the definition of Ehrlich was given by Li, Nicholson and L. Zan in [LNZ10].

Definition 1.1. A group G is said to be *morphic* if, whenever N is a normal subgroup of G, such that there is an epimorphism $\varphi : G \to N$, then $G/N \cong \ker(\varphi)$.

In other words, G is morphic if for every pair N_1 and N_2 of normal subgroups of G, each of the conditions $G/N_1 \cong N_2$ and $G/N_2 \cong N_1$ implies the other.

In [LNZ10] finite, nilpotent groups were considered, and it was shown that such a group is morphic if and only if each of its Sylow subgroups is morphic. This leads to the study of finite, morphic p-groups.

Clearly finite, homocyclic p-groups are morphic, and so is the nonabelian group of order p^3 and exponent p, for p an odd prime. F. Aliniaeifard, Li and Nicholson conjectured that these are the only examples:

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Conjecture 1.2 ([ALN13, Conjecture 3.5]). *The only finite, morphic p-groups are the abelian, homocyclic p-groups, and the nonabelian group of order p^3 and exponent p, for p an odd prime.*

They were able to prove their conjecture for two-generated groups:

Theorem 1.3 ([ALN13, Theorem 2.10]). *Let G be a finite, morphic p-group. If G can be generated by two elements, then*

(1) *either G is abelian and homocyclic,*
(2) *or p is odd, and G is isomorphic to the nonabelian group of order p^3 and exponent p.*

Aliniaeifard, Li and Nicholson also proved that a finite, morphic p-group satisfies the following property:

Theorem 1.4 ([ALN13, Proposition 2.1]). *Let G be a finite, morphic p-group.*

(1) *Every subgroup of G is a homomorphic image of G.*
(2) *Every homomorphic image of G is isomorphic to a normal subgroup of G.*

The latter result shows that a finite, morphic p-group belongs to the class of *self dual* groups, as introduced by A. E. Spencer [Spe72], and studied more recently by L. An, J. Ding and Q. Zhang[ADZ11]. A group is said to be self dual if the isomorphism classes of its subgroups and of its quotient groups coincide.

Now An, Ding and Zhang prove the following

Theorem 1.5 ([ADZ11, Corollary 7.2]). *Let G be a finite, self dual p-group. Then*

(1) *either G is abelian, or*
(2) *p is odd, and G is the direct product of the nonabelian group of order p^3 and exponent p, by an elementary abelian group.*

We observe here that a proof of Conjecture 1.2 can be rather straightforwardly obtained as a corollary of this result.

In this paper we study a class of finite p-groups that properly includes that of morphic p-groups, and is not contained in the class of self dual p-groups:

Definition 1.6. A finite p-group G is said to be *elementary abelian morphic*, or *ea*-morphic for short, if, whenever N is a normal subgroup of G, such that either N is elementary abelian, or G/N is elementary abelian, then there is an epimorphism $\varphi : G \to N$, and $G/N \cong \ker(\varphi)$.

Our main result is:

Theorem 1.7. *Let G be a finite, nonabelian* ea*-morphic p-group. Then G is 2-generated.*

Since finite, morphic p-groups are *ea*-morphic by Theorem 1.4, this result, together with Theorem 1.3, yields another proof of Conjecture 1.2.

In Section 2 we introduce a linear algebra setting, and use a counting argument to obtain a crucial estimate for the elementary abelian quotients of the derived subgroup. This estimate is closely related to the one in [ADZ11, Lemma 5.3], but our treatment is self contained, and elementary, as it avoids the appeal made in [ADZ11, Theorem 5.2] to a substantial result in the text of N. Blackburn and B. Huppert [HB82, Theorem 9.8]. In Section 3 we complete the proof of Theorem 1.7, by comparing two series of normal subgroups on the top and on the bottom of the group.

Our notation is mainly standard. If H is a subgroup of the group K, or H is a subspace of the vector space K, we write $H \lessdot K$ to indicate that H is maximal in K.

We write $\mathbf{F}_p$ for the field with p elements, p a prime.

2. Morphic triples

We record the following immediate consequence of the second isomorphism theorem, which we will use repeatedly.

Lemma 2.1. *If N is a normal subgroup of the group G, then*

$$(G/N)' \cong \frac{G'}{G' \cap N}.$$

Let $G \neq \{ 1 \}$ be a finite, *ea*-morphic p-group, with minimal number of generators d, that is, $|G/\Phi(G)| = p^d$.

Let N be a normal subgroup of G of order p. Then for every maximal subgroup M of G one has $G/M \cong N$, and thus, according to Definition 1.6, also $G/N \cong M$. In particular we obtain

Lemma 2.2 ([LNZ10, Theorem 37 (1)])**.** *In a finite, morphic p-group, all maximal subgroups are isomorphic.*

From now on, assume G to be nonabelian, and take $N \leq G'$. Since $M \cong G/N$, Lemma 2.1 yields

$$|M'| = |G'/N| = \frac{|G'|}{p},$$

so that

$$|G'/M'| = p.$$

Consider the characteristic subgroup K of G defined by

$$K = \bigcap \{ M' : M \lessdot G \}.$$

G'/K is isomorphic to a subgroup of $\prod_{M \lessdot G} G'/M'$, and thus it is elementary abelian.

Consider the map

$$\beta : G/\Phi(G) \times G/\Phi(G) \to G'/K$$
$$(a\Phi(G), b\Phi(G)) \mapsto [a, b]K.$$

β is well defined, as we have

Lemma 2.3. $[G, \Phi(G)] \leq K$. *In particular,* $\Phi(G)' \leq K$.

Proof. $(G/M')' = G'/M'$ has order p. This yields first $[G, G'] \le M'$ for all $M \lessdot G$, so that $[G, G'] \le K$. Also, if $a, b \in G$, then for all $M \lessdot G$ we have

$$[a, b^p] \equiv [a, b]^p \equiv 1 \pmod{M'},$$

as we have just seen that $[b, [a, b]] \in M'$. Therefore also $[G, G^p] \le K$. □

This also yields that β is bilinear, as

$$[ab, c] = [a, c] \cdot [[a, c], b] \cdot [b, c] \equiv [a, c] \cdot [b, c] \pmod{K}.$$

The $\mathbf{F}_p$-vector spaces $V = G/\Phi(G)$ and $W = G'/K$, with the map β, thus satisfy the following definition.

Definition 2.4. Let V, W be vector spaces over $\mathbf{F}_p$, and

$$\beta : V \times V \to W$$
$$(v_1, v_2) \mapsto [v_1, v_2]$$

be an alternating bilinear map. If U_1, U_2 are subspaces of V, write $[U_1, U_2]$ for the linear span of $\beta(U_1, U_2)$, and shorten $[U_1, U_1]$ to U_1'.

(V, W, β) is said to be a *morphic triple* if the following conditions hold.

(1) $V' = W$.
(2) For every $U \lessdot V$ one has $U' \lessdot W$.
(3) $\bigcap \{ U' : U \lessdot V \} = \{ 0 \}$.

Considering morphic triples alone is not sufficient to prove Theorem 1.7, as one can construct examples of morphic triples that are not associated to finite, morphic p-groups.

Proposition 2.5. *Let (V, W, β) be a morphic triple.*

Let $U \lessdot V$. Then there exist a unique $T = \mathcal{T}(U) \lessdot U$ which satisfies the following property.

For $S \lessdot V$, the following are equivalent.

(1) *$U' = S'$, and*
(2) *$S \ge T$.*

Proof. Let $a \in V \setminus U$. Consider the linear map τ

$$U \to W \to W/U'$$

given by $x \mapsto [a, x] + U'$. Since $W = V' = \langle a, U \rangle' = [a, U] + U'$, this is a surjective linear map, with $\dim(W/U') = 1$. Thus

$$T = \mathcal{T}(U) = \ker(\tau) = \{ x \in U : [a, x] \in U' \}$$

is a maximal subspace of U. By definition, $[a, T] \le U'$.

T is easily seen to be independent of the choice of $a \in V \setminus U$. Thus if $T \lessdot S \lessdot V$, and $S \ne U$, we may assume $S = \langle a, T \rangle$. It follows that

$$S' = [a, T] + T' \le U' + T' \le U'$$

and thus $S' = U'$, as they are both maximal subspaces of W.

Suppose conversely that $U \neq S \lessdot V$, and $S' = U'$. Thus $S \cap U \lessdot U$, and if $S = \langle a, S \cap U \rangle$, then $a \notin U$, and we have $[a, S \cap U] \leq S' = U'$, so that $S \cap U = \mathcal{T}(U)$. □

Corollary 2.6. *For each $U \lessdot V$, the set*

$$\{ S \lessdot V : S' = U' \}$$

has $p+1$ elements, namely the subspaces S such that $\mathcal{T}(U) \lessdot S \lessdot V$.

The set $\mathcal{M}$ of maximal subspaces of W has

$$1 + p + \cdots + p^{e-1},$$

elements, where $e = \dim(W)$. Corollary 2.6 implies that the subset

$$\mathcal{Z} = \{ Z \lessdot W : Z = U' \text{ for some } U \lessdot V \}$$

of $\mathcal{M}$ has $(1 + p + \cdots + p^{d-1})/(1+p)$ elements. Thus d is even, and

$$\frac{1 + p + \cdots + p^{d-1}}{1+p} = 1 + p^2 + \cdots + p^{d-2}.$$

Clearly if $e < d-1$ we have

$$|\mathcal{M}| = 1 + p + \cdots + p^{e-1} < 1 + p^2 + \cdots + p^{d-2} = |\mathcal{Z}|,$$

a contradiction. We have obtained

Corollary 2.7. *In a morphic triple (V, W, β), we have*

$$\dim(W) \geq \dim(V) - 1.$$

In particular, in a finite, nonabelian, ea*-morphic p-group G with minimal number of generators d we have*

$$|G'/K| \geq p^{d-1}.$$

3. Proofs

We are now ready to prove Theorem 1.7.

Let G be a finite, nonabelian, *ea*-morphic group, with minimum number of generators $d > 2$. We want to derive a contradiction.

By Definition 1.6, there is an epimorphism $G \to \Phi(G)$, and if E is its kernel, so that $G/E \cong \Phi(G)$, we also have $G/\Phi(G) \cong E$. Since $|G/\Phi(G)| = p^d$, we have that E is an elementary abelian, normal subgroup of G, of order p^d. Let

$$p^t = |E \cap G'|,$$

so that $t \leq d$. Since $G/E \cong \Phi(G)$, Lemma 2.1 yields

$$|\Phi(G)'| = \left| \frac{G'}{E \cap G'} \right| = \frac{|G'|}{p^t}.$$

In view of Lemma 2.3 and Corollary 2.7, we obtain

$$|E \cap G'| = \left| \frac{G'}{\Phi(G)'} \right| = p^t, \qquad \text{for } t \in \{ d, d-1 \}.$$

Let thus L be an elementary abelian, normal subgroup of order p^{d-1} contained in G' (we take L to be $E \cap G'$ if this has order p^{d-1}, otherwise if $E \leq G'$ we take L to be a maximal subgroup of E which is normal in G), and let

$$\{ 1 \} = L_0 \lessdot L_1 \lessdot L_2 \lessdot \ldots \lessdot L_{d-1} = L \leq G',$$

be a series of subgroups, each normal in G. In particular, $|L_i| = p^i$ for each i.

Consider an arbitrary series

$$\Phi(G) = S_0 \lessdot S_1 \lessdot S_2 \lessdot \ldots \lessdot S_{d-1} \lessdot S_d = G.$$

(We will make a more precise choice of S_2 later.) In particular, each S_i is normal in G, and $|G/S_i| = p^{d-i}$ for each i. Clearly for $i \geq 1$ one has $G/S_i \cong L_{d-i}$ as both groups are elementary abelian, of order p^{d-i}. By Definition 1.1, we also have

$$G/L_{d-i} \cong S_i, \qquad \text{for all } i \geq 1.$$

For $i \geq 1$ we thus have from Lemma 2.1

$$|S_i'| = |G'/L_{d-i}| = \frac{|G'|}{p^{d-i}},$$

that is,

$$|G'/S_i'| = p^{d-i}. \tag{3.1}$$

In particular, $|G'/S_1'| = p^{d-1}$. Since $S_1 = \langle a, \Phi(G) \rangle$ for some $a \in G$, Lemma 2.3 implies $S_1' \leq [\langle a \rangle, \Phi(G)]\, \Phi(G)' \leq K$, so that Corollary 2.7 yields $|G'/K| = p^{d-1}$ and $S_1' = K$. It follows that $S_i' \geq S_1' = K$ for each $i \geq 1$. We obtain from (3.1)

$$|S_i'/K| = |(G'/K)/(G'/S_i')| = p^{i-1}$$

for each $i \geq 1$.

In particular, $|S_2'/K| = p$ and $|S_3'/K| = p^2$. (This is the only point where we use the assumption $d > 2$.) Since $S_3 = \langle a, b, c, \Phi(G) \rangle$ for some $a, b, c \in G$, Lemma 2.3 yields $S_3' \leq \langle [a, b], [a, c], [b, c], K \rangle$. Since every element of the exterior square of a vector space of dimension 3 is a decomposable tensor, we may choose a, b, c so that $[a, b] \in K$. Choosing $S_2 = \langle a, b, \Phi(G) \rangle$, Lemma 2.3 yields $S_2' \leq K$, a final contradiction.

Finally, we derive a proof of Conjecture 1.2.

If G is a finite, abelian, morphic p-group, it is not difficult to see from Lemma 2.2 that G must be homocyclic.

In view of Theorem 1.3, let thus assume that G is nonabelian. Since morphic p-groups are *ea*-morphic, by Theorem 1.7 we conclude that G is 2-generated, and by Theorem 1.4 we have the result.

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