A NOTE ON SOME NAIVE ESTIMATES OF THE FACTORIAL

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Abstract. In this very short note we make a remark on a rough and naive but easy way to estimate from below the factorial. This note arise from an example from the classical book [1].

1. Introduction

In order to estimate $m!$ in a rough but very easy way it is possible argue as follows:

(1) **Upper estimate**: by Arithmetic-Geometric means inequality it is

$$\sqrt[m]{1 \cdot 2 \cdot \cdots \cdot m} < \frac{1 + 2 + \cdots m}{m}.$$

hence

$$\sqrt[m]{m!} < \frac{m\,(m+1)}{2m} = \frac{(m+1)}{2}.$$

Therefore

$$m! < 2^{-m}\,(m+1)^m\,.$$

(2) **Lower estimate**: since

$$m! = m \cdot (m-1) \cdots 2 \cdot 1$$
$$m! = 1 \cdot 2 \cdots (m-1) \cdot m$$

it is

$$(m!)^2 = (m \cdot 1)\,[(m-1) \cdot 2] \cdot \cdots [3 \cdot (m-1)]\,[2 \cdot (m-1)]\,(1 \cdot m) > m^m$$

Therefore

$$m! > \sqrt{m^m}.$$

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2. The improved estimate

We can improve the latter estimate in a very easy way as follows.

(1) If m is even then $m = 2n$ and

$$(2n)! = (2n) \cdot (2n-1) \cdot \cdots \cdot 1$$
$$(2n)! = 1 \cdot 2 \cdots (2n-1) \cdot (2n)$$

so that

$$[(2n)!]^2 = (2n)^2 [(2n-1) \cdot 2]^2 \cdots [(n+1)\, n]^2$$

but

$$[(n+1)\, n] > \cdots [(2n-2) \cdot 3] > [(2n-1) \cdot 2]$$

thus

$$(2n)! > (2n)\, 2^{n-1} (2n-1)^{n-1}$$

(2) If m is odd we have $m = 2n+1$ and

$$(2n+1)! = (2n+1)\,(2n)!$$

hence

$$(2n+1)! > (2n+1)\,(2n)\, 2^{n-1} (2n-1)^{n-1}$$

3. Conclusion

Since

$$\lim_{n\to\infty} \frac{\sqrt{(2n)^{2n}}}{(2n)\, 2^{n-1} (2n-1)^{n-1}} = 0$$

and

$$\lim_{n\to\infty} \frac{\sqrt{(2n+1)^{2n+1}}}{(2n+1)\,(2n)\, 2^{n-1} (2n-1)^{n-1}} = 0$$

our estimates are a bit better.

References

[1] H. Dörrie "Unendliche Reihen" Verlag von R. Oldenburg, 1951.

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