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Toroidal black holes and T-duality

Massimiliano Rinaldi

Department of Mathematical Physics, University College Dublin, Belfield, Dublin 4, Ireland

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Abstract

We consider the toroidal black holes that arise as a generalization of the $AdS_5 \times S^5$ solution of type IIB supergravity. The symmetries of the horizon space allow T-duality transformations that can be exploited to generate new inequivalent black hole solutions of both type IIB and type IIA supergravity, with non-trivial dilaton, B-field, and RR-forms. We examine the asymptotic structure and thermodynamical properties of these solutions.

1. Introduction

In its simplest form, target space duality claims that a bosonic closed string moving on a circle of radius R is physically equivalent to one moving on a circle of radius $2\alpha'\hbar/R$, where α' is the string tension; for a review, see [1]. This symmetry was then extended to many compact directions, including the internal degrees of freedom typical of heterotic string theory by Narain et al. [2,3]. For time-dependent backgrounds [4–6], the analogue of T-duality is the scale factor duality that relates a physically expanding and a physically contracting universe. More generally, if the background fields of a low energy effective action in string theory are independent of D coordinates, then this action shows an $O(D,D)$ symmetry. The symmetry group is replaced by $SO(D-1,1)$ if one of the coordinates is time-like. Hence, given a solution of the equations of motion, one may generate a new one by means of an $O(D,D)$ transformation. However, only the action of the subgroup $O(D)\otimes O(D)/O(D)$ of $O(D,D)$ generates physically inequivalent solutions [7–10]. T-duality is a particular case of an $O(D)\otimes O(D)$ transformation and if the D coordinates are compact, the old and the new solutions represent the same conformal field theory [11].

The aim of the present work is to investigate the symmetries of toroidal black holes. These belong to a wider class of black hole solutions to Einstein's equation with a negative cosmological constant, called topological black holes [12–21]. They generalize the ordinary asymptotically flat Schwarzschild black hole in D dimensions to black holes that are locally asymptotically anti-de Sitter such that the topology of the horizon may be elliptic, toroidal, or hyperbolic. The toroidal case clearly shows translational invariance along the compact coordinates of the $(D-2)$-dimensional horizon space. The metric is then independent of these $(D-2)$ coordinates, and the effective action must exhibit an $O(D-2)$ symmetry group. Moreover, since the metric is static, the action is also

E-mail address: massimiliano.rinaldi@ucd.ie (M. Rinaldi).

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invariant under an $SO(D-2,1)$ group of transformations as well. We will investigate the inequivalent solutions obtained by the action of the relevant subgroups on the toroidal black hole metric. In particular, we will focus on the 5-dimensional toroidal black hole that can replace the AdS_5 sector of the $AdS_5 \times S^5$ solution of type IIB supergravity [22]. Therefore, we must also study the action of the group on the anti-self-dual 5-form that implements the model.

By performing a few preliminary computations, following [7–10], we find that the action of the $SO(3,1) \otimes SO(3,1)/SO(3,1)$ group always leads to metrics with naked singularities. Hence, we will focus on the T-duality transformations only, that belong to the $O(3) \otimes O(3)$ invariance group. In general, T-duality is a map between type IIB and type IIA solutions, and we will explicitly see this through the transformations of the spectrum of RR-forms. In strict analogy with the duality between the uncharged and charged black strings [23], we will first boost the toroidal black hole and then apply T-dualities in order to generate an axion charge that is dual to the momentum. Also, we will check that the entropy of the black hole is invariant under T-duality [24]. Finally, we will see that the local asymptotic structure of the black hole changes from anti-de Sitter to warped products of compactified Minkowski space–times and hyperbolic spaces.

2. Toroidal black holes

Toroidal black holes belong to a larger class of topological black holes that arise as solutions of the Einstein equations with a negative cosmological constant [12–19]. They are Einstein space–times which are locally asymptotically anti-de Sitter. The horizon may have elliptic, toroidal, or hyperbolic topology as opposed to asymptotically flat black holes, for which only spherical topology is possible. The toroidal case has the D-dimensional metric

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}\,dr^2 + \frac{r^2}{l^2}\delta_{\alpha\beta}\,dx^\alpha\,dx^\beta, \tag{1}$$

where

$$f(r) = \frac{r^2}{l^2} - \frac{2M}{r^{D-3}}. \tag{2}$$

Here, M is the mass parameter, l is the AdS radius, and x^α ($\alpha = 1,\ldots,D-2$) are the compact coordinates parameterizing the torus horizon. The horizon is located at $r_+^{D-1} = 2Ml^2$. For $D = 5$, this metric may replace the AdS_5 sector of the $AdS_5 \times S^5$ solution of type IIB supergravity [22], see also [25,26]. Then, the 10-dimensional metric is

$$\begin{aligned} ds^2 &= -f(r)\,dt^2 + f(r)^{-1}\,dr^2 \\ &\quad + \frac{r^2}{l^2}\left[dx_1^2 + dx_2^2 + dx_3^2\right] + l^2\,d\Omega_5^2. \end{aligned} \tag{3}$$

This solution must be supplemented with an anti-self-dual 5-form

$$\begin{aligned} F_{\mu_1\ldots\mu_5} &= +\frac{4}{l}\varepsilon_{\mu_1\ldots\mu_5}, \\ F_{m_1\ldots m_5} &= -\frac{4}{l}\varepsilon_{m_1\ldots m_5}, \end{aligned} \tag{4}$$

in order to satisfy the fundamental equation [22]

$$R_{MN} = \frac{1}{6\cdot 4^2} F_M^{\;A_2A_3A_4A_5} F_{NA_2A_3A_4A_5}. \tag{5}$$

With these conventions, $\varepsilon_{01\ldots 89} = -\sqrt{-\det G}$, $\mu = t, r, x_1, x_2, x_3$, and the m's are the indices of the coordinates on the 5-sphere S^5. Upper case Latin letters label all the coordinates. The Hodge operator $*$ applied to a p-form is defined such that $^{**}F^{(p)} = (-1)^{p+1}F^{(p)}$.

It is clear that the metric (3) is static and translationally invariant along the space-like compact directions x_1, x_2, and x_3. Thus, we can generate new inequivalent solutions by means of an $O(3) \otimes O(3)/O(3)$ or an $SO(3,1) \otimes SO(3,1)/SO(3,1)$ group of transformations. As mentioned above, in the latter case we always obtain metrics with naked singularities, and we will focus on T-duality transformations that belong to the more general $O(3) \otimes O(3)$ group. Since x_1, x_2, and x_3 are compact, all the new solutions that we obtain represent the same conformal theory [11].

3. Application of T-duality

The T-duality transformations for the Neveu–Schwarz sector of an effective theory are well known. Given a metric G_{MN}, a B-field B_{MN}, and dilaton ϕ, independent of the coordinate x, the transformation rules are [27,28]

$$\begin{aligned}
&\widetilde{G}_{xx} = 1/G_{xx}, \qquad \widetilde{G}_{xN} = B_{xN}/G_{xx}, \\
&\widetilde{G}_{MN} = G_{MN} - (G_{xM}G_{xN} - B_{xM}B_{xN})/G_{xx}, \\
&\widetilde{B}_{xM} = G_{xM}/G_{xx}, \\
&\widetilde{B}_{MN} = B_{MN} - 2G_{x[M}B_{N]x}/G_{xx}, \\
&2\tilde{\phi} = 2\phi - \ln G_{xx},
\end{aligned} \tag{6}$$

where M and N run over all the coordinates except x. Here, $T_{[MN]} = (1/2!)(T_{MN} - T_{NM})$. The type IIA/IIB T-duality transformation rules for the RR-form fields were derived more recently in [29,30], and put in a general form by Hassan in [31] (see also [32] for the $SO(D,D)$ transformation rules). According to [31], the components of an RR-form F which is independent of x are related to the components of its T-dual $\widetilde{F}$ by

$$\begin{aligned}
&\widetilde{F}^{(n)}_{xM_2\ldots M_n} = F^{(n-1)}_{M_2\ldots M_n} - (n-1)G^{xx}G_{x[M_2}F^{(n-1)}_{xM_3\ldots M_n]}, \\
&\widetilde{F}^{(n)}_{M_1\ldots M_n} = F^{(n+1)}_{xM_1\ldots M_n} - nB_{x[M_1}\widetilde{F}^{(n)}_{xM_2\ldots M_n]}.
\end{aligned} \tag{7}$$

In this formalism, the index M runs over all the coordinates except x, the anti-symmetrization symbols on the right-hand sides do not involve the index x, and the upper index of the form indicates its rank. Following closely the method explained in [23] (see also [33, 34]), we first perform a boost by the change of coordinates $x_1 \to x_1 \cosh\alpha + t \sinh\alpha$, $t \to t\cosh\alpha + x_1 \sinh\alpha$. Then, we apply a T-duality on the background fields along x_1 using (6). The resulting metric, dilaton field, and B-field take the form

$$\begin{aligned}
&ds^2 = -\frac{r^2}{l^2}h(r)g(r)^{-1}\,dt^2 + \frac{l^2}{r^2}h(r)^{-1}\,dr^2 \\
&\qquad + \frac{l^2}{r^2}g(r)^{-1}\,dx_1^2 + \frac{r^2}{l^2}\big[dx_2^2 + dx_3^2\big] + l^2\,d\Omega_5^2, \\
&B_{tx_1} = -\frac{q}{r^4}g(r)^{-1}, \\
&e^{-2\phi} = \frac{r^2}{l^2}g(r),
\end{aligned} \tag{8}$$

where

$$\begin{aligned}
&h(r) = 1 - \frac{m}{r^4} + \frac{q^2}{mr^4}, \\
&g(r) = 1 + \frac{q^2}{mr^4}.
\end{aligned} \tag{9}$$

The axion charge is given by $q = 2Ml^2\cosh\alpha\sinh\alpha$, and the rescaled mass is $m = 2Ml^2\cosh^2\alpha$. In analogy with the charged black string found in [23], the horizon is unique and its location is unchanged: $r_+^4 = (m^2 - q^2)/m = 2Ml^2$. Also, in the extremal limit $M \to 0$, $\alpha \to \infty$ (such that $m = q$) one can check that the horizon disappears and the metric is no longer singular. However, as opposed to [23], in the limit $\alpha \to 0$ we do not recover the original metric, because of the term r^2/l^2 that multiplies the horizon submetric.

In the large r limit, the metric of the black hole reduces to

$$\begin{aligned}
&ds^2 \approx \frac{r^2}{l^2}\big[-dt^2 + dx_2^2 + dx_3^2\big] \\
&\qquad + \frac{l^2}{r^2}\big[dr^2 + dx_1^2\big]
\end{aligned} \tag{10}$$

which looks singular as $r \to \infty$. However, it can be checked that the scalar curvature of the black hole tends to $G_{\mu\nu}R^{\mu\nu} = -8/l^2$ and all the scalar curvature polynomials are finite (and constant) in the limit. This is not surprising since the topology of the space–time for large r is a warped product of a 3-dimensional Minkowski space–time, compactified along x_2 and x_3, and a 2-dimensional hyperbolic space compactified along x_1.

In general, T-duality is a map between type IIA and type IIB supergravity solutions. Hence, we expect that the anti-self-dual 5-form is mapped to forms with even rank. This can be readily verified by means of the transformation rules (7), and we find

$$\begin{aligned}
&F^{(4)}_{\mu_1\ldots\mu_4} = \frac{4r}{l^2}\sqrt{g(r)}\,\varepsilon_{\mu_1\ldots\mu_4}, \quad \mu \neq x_1, \\
&F^{(6)}_{A_1\ldots A_6} = -\frac{4r}{l^2}\sqrt{g(r)}\,\varepsilon_{A_1\ldots A_6}, \quad A_i \neq t, r, x_2, x_3,
\end{aligned} \tag{11}$$

where ε denotes the volume form of the corresponding subspace. It can be verified that $F^{(4)} = {}^*F^{(6)}$ and $F^{(6)} = -{}^*F^{(4)}$. Thus, we have obtained a non-trivial solution for type IIA supergravity with a black hole sector, B-field, dilaton, and RR-forms.

The above solution is still translationally invariant along x_1, x_2, and x_3. For example, we could apply another boost and a T-duality along x_2 to generate an axion charge independent of q. For simplicity, we restrict our attention to T-duality only. First, a T-duality transformation along x_2 yields

$$\begin{aligned} ds^2 &= -\frac{r^2}{l^2}h(r)g(r)^{-1}\,dt^2 + \frac{l^2}{r^2}h(r)^{-1}\,dr^2 \\ &\quad + \frac{l^2}{r^2}\big[g(r)^{-1}\,dx_1^2 + dx_2^2\big] + \frac{r^2}{l^2}\,dx_3^2 + l^2\,d\Omega_5^2, \\ B_{tx_1} &= -\frac{q}{r^4}g(r)^{-1}, \\ e^{-2\phi} &= \frac{r^4}{l^4}g(r). \end{aligned} \tag{12}$$

By computing the forms, we find a 3-form and a 7-form

$$\begin{aligned} F^{(3)}_{\mu_1\mu_2\mu_3} &= \frac{4r^2}{l^3}\sqrt{g(r)}\,\varepsilon_{\mu_1\mu_2\mu_3}, \quad \mu \neq x_1, x_2, \\ F^{(7)}_{A_1\ldots A_7} &= -\frac{4r^2}{l^3}\sqrt{g(r)}\,\varepsilon_{A_1\ldots A_7}, \quad A_i \neq t, r, x_3. \end{aligned} \tag{13}$$

It can be verified that $F^{(3)} = {}^*F^{(7)}$, $F^{(7)} = {}^*F^{(3)}$, and we see that this is again a solution for type IIB supergravity. The horizon location is unaltered and, in the extremal limit $m = q$, the singularity disappears. In the large r limit, the black hole metric has the topology of a warped product of 2-dimensional Minkowski space–time, with x_3 compact, and a 3-dimensional hyperbolic space compactified along x_1 and x_2. Again, there are no singularities as $r \to \infty$ and the scalar curvature tends to a constant.

As a final step, we apply a T-duality along x_3 to the solution above. The metric, the B-field, and the dilaton are given by

$$\begin{aligned} ds^2 &= -\frac{r^2}{l^2}h(r)g(r)^{-1}\,dt^2 + \frac{l^2}{r^2}h(r)^{-1}\,dr^2 \\ &\quad + \frac{l^2}{r^2}\big[g(r)^{-1}\,dx_1^2 + dx_2^2 + dx_3^2\big] + l^2\,d\Omega_5^2, \\ B_{tx_1} &= -\frac{q}{r^4}g(r)^{-1}, \\ e^{-2\phi} &= \frac{r^6}{l^6}g(r). \end{aligned} \tag{14}$$

The previous forms are mapped into a 2-form and an 8-form

$$\begin{aligned} F^{(2)}_{\mu_1\mu_2} &= \frac{4r^3}{l^4}\sqrt{g(r)}\,\varepsilon_{\mu_1\mu_2}, \quad \mu \neq x_1, x_2, x_3, \\ F^{(8)}_{A_1\ldots A_8} &= -\frac{4r^3}{l^4}\sqrt{g(r)}\,\varepsilon_{A_1\ldots A_8}, \quad A_i \neq t, r, \end{aligned} \tag{15}$$

that satisfy the type IIA duality condition $F^{(2)} = {}^*F^{(8)}$ and $F^{(8)} = -{}^*F^{(2)}$. All the previous characteristics are unchanged, i.e., the horizon location and the absence of curvature singularities as $r \to \infty$. Also, the $r = 0$ curvature singularity disappears in the extremal limit $m = q$. In particular, we see that in this limit the black hole metric reduces to a warped product of time and a 4-dimensional hyperbolic space compactified along x_1, x_2 and x_3, with a scalar curvature $G_{\mu\nu}R^{\mu\nu} \to -8/l^2$.

According to the general formalism, these duality transformations generate solutions of type IIA or type IIB supergravity. In particular, we explicitly checked that the fields (14), (15) are a non-trivial solution of the type IIA supergravity. Indeed, these are solutions for the equations of motion [35]

$$\begin{aligned} 0 &= R_{AB} + 2\nabla_A\nabla_B\phi - \frac{1}{4}H_A^{CD}H_{BCD} \\ &\quad + \frac{1}{4\cdot 2!}e^{2\phi}F_{CD}F^{CD}G_{AB} - \frac{1}{2}e^{2\phi}F_A^C F_{BC}, \\ 0 &= \nabla_M\big(e^{-2\phi}H^M_{PQ}\big), \\ 0 &= 4\nabla^2\phi - 4\big(\nabla\phi\big)^2 + R - \frac{1}{12}H_{ABC}H^{ABC}, \end{aligned} \tag{16}$$

where $H_{MNP} = 3!\partial_{[M}B_{NP]}$. These equations are obtained by variation of the effective type IIA action

$$\begin{aligned} S &= \frac{1}{2\kappa^2}\int d^{10}x\,\sqrt{-G} \\ &\quad \times \left\{e^{-2\phi}\left[R + 4\nabla^2\phi - \frac{1}{12}H_{MNP}H^{MNP}\right] \right. \\ &\quad \left. - \frac{1}{4}F_{MN}F^{MN}\right\}, \end{aligned} \tag{17}$$

with vanishing 4-form.

In general, if an asymptotically flat low energy string theory solution has a horizon and at least one space-like symmetry, then the dual solution also has a horizon, with the same area (in Einstein frame), and the same temperature [24]. The horizon area is invariant in our case as well. After the boost of the metric (3), the area is $A = (2\pi r_+/l)^3\sqrt{g(r_+)}$. The 5-dimensional black hole metric in Einstein frame $ds^2_{(\mathrm{E})}$ is related to the one in string frame $ds^2_{(\mathrm{ST})}$ through $ds^2_{(\mathrm{E})} = \exp(-4\phi/3)\,ds^2_{(\mathrm{ST})}$. In particular, let H_{ij} ($i = 1, 2, 3$) be the horizon metric of (14) in Einstein frame.

Then

$$H_{ij}\,dx^i\,dx^j = \frac{r^2}{l^2} g(r)^{-1/3}\,dx_1^2 + \frac{r^2}{l^2} g(r)^{2/3}\big(dx_2^2 + dx_3^2\big). \tag{18}$$

The area is then $(2\pi)^3\sqrt{\det H(r_+)}$ which coincides with A. Hence, the black hole and its dual have the same entropy. It is straightforward to check that this is true for the metrics (8) and (12) as well.

4. Conclusions

The application of T-duality transformations to the toroidal black hole generates new black hole solutions in the context of type IIA and type IIB supergravity with non-trivial dilaton, B-field, and RR-forms. The analogy with the duality between the charged and uncharged black string is strict, and all the new metrics have a unique horizon with unchanged location and entropy. We also analyzed the asymptotic structure and showed that it takes the form of a warped product of compactified hyperbolic spaces and Minkowski space–times. As opposed to the original toroidal black hole, the new metrics are not Einstein space–times, and the Ricci scalar is constant only in the large r limit.

In the original solution (3) and (4), the effect of the 5-form is to produce an effective cosmological constant. One might wonder whether a similar mechanism is valid for the dualized solutions. Namely, one would like to check if, for example, the solution (14) can be generated by replacing the RR-forms (15) by a constant term in the action (17). Preliminary computations (with $q = 0$), indicate that this mechanism does not work. Further investigations may be worthy. Finally, we note that topological black holes with hyperbolic topology [19] show translational invariance as well and it would be interesting to apply T-duality transformations, in analogy with the toroidal case.

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