Inconsistency evaluation in preference relations: a characterization based on metrics induced by a norm

Michele Fedrizzi[1], Nino Civolani[2], Andrew Critch[3]

[1] *Department of Industrial Engineering, University of Trento, Italy*
e-mail: michele.fedrizzi@unitn.it
[2] *Department of Economics and Management,University of Trento, Italy*
e-mail: nino.civolani@unitn.it
[3] *Department of Mathematics, University of California, Berkeley, CA 94720, United States*
e-mail: critch@math.berkeley.edu

Assume that an $n \times n$ pairwise comparison matrix $\mathbf{A}$ is represented by a point in the vector space $\mathbb{R}^{n\times n}$ of $n \times n$ real matrices. Then, an inconsistency index of $\mathbf{A}$ can be defined as a distance between $\mathbf{A}$ and the set of consistent matrices, i.e. the minimum distance between $\mathbf{A}$ and a consistent matrix. This approach is already known, but the notion of distance has been proved to be too general to conveniently characterize inconsistency, so that it can lead to unsatisfactory inconsistency indices [3]. In this paper it is proved that a suitable characterization of inconsistency indices is obtained by restricting distances to those induced by norms, under the assumption of additive representation of preferences. Then, an inconsistency index is a seminorm in the linear space $\mathcal{L}$ of skew–symmetric matrices and from this main result several other properties are derived. In particular, the linear space $\mathcal{L}$ can be partitioned into equivalence classes, where each class is an affine subspace. All matrices in an equivalence class share the same value of the inconsistency index, since all of them have the same norm–induced distance from the linear subspace of consistent matrices. Some results due respectively to Barzilai [1] and to Crawford & Williams [2] are extended in a more general framework. It is also proved that norm–based inconsistency indices satisfy a set of five characterizing properties previously introduced, as well as an upper bound property for group preference aggregation.

Key words: Inconsistency index, pairwise comparison matrix , norm, distance.

References:

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