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# Application of imperfect interface weight function techniques for modelling of glued structures containing cracks and small defects

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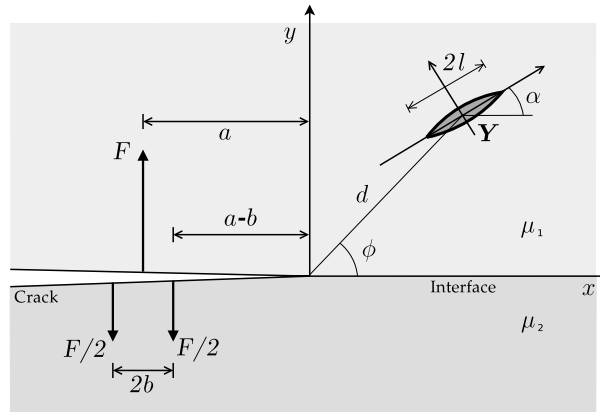
**Abstract.** We analyse a problem of anti-plane shear in a bimaterial plane containing a semi-infinite crack situated on a soft imperfect interface. The plane also contains a small thin inclusion (for instance an ellipse with high eccentricity) whose influence on the propagation of the main crack we investigate. The problem can be considered as modelling bimaterial ceramics which are joined with a thin adhesive substance. An important element of our approach is the derivation of a new weight function (a special solution to a homogeneous boundary value problem) in the imperfect interface setting. The weight function is derived using Fourier transform and Wiener-Hopf techniques and allows us to obtain an expression for an important constant  $\sigma_0$  (which may be used in a fracture criterion) that describes the leading order of tractions near the crack tip for the unperturbed problem. We present computations that demonstrate how  $\sigma_0$  varies depending on the extent of interface imperfection and contrast in material stiffness. We then perform perturbation analysis to derive an expression for the change in the leading order of tractions near the tip of the main crack induced by the presence of the small defect, whose sign can be interpreted as the inclusion's presence having an amplifying or shielding effect on the propagation of the main crack.

## 1. Introduction

We consider a problem of anti-plane shear formulated in the whole plane, with different materials occupying the regions above and below the crack line. The geometry considered contains a semi-infinite crack situated along a soft *imperfect interface*; we will formulate and solve a weight function problem in such a geometry before using the newly derived weight function to describe the tractions near the crack tip. We will then conduct perturbation analysis to evaluate the effect of a small inclusion's presence, in particular whether it encourages or shields the propagation of the main crack.

Soft imperfect interfaces model a very thin layer of adhesive between two larger bodies of material. Typically such an interface is represented in the model by transmission conditions (justified for example in [1]) which impose continuity of tractions across the interface while the jump in displacement is proportional to the traction.





**Figure 1.** Geometry of the crack, imperfect interface, and small defect in a bimaterial plane.

Weight functions are special solutions to homogeneous boundary value problems that aid in the evaluation of constants describing the behaviour of physical fields near crack tips; a weight function for the perfect interface analogue of the problem considered here has been previously constructed [5] but the presence of the imperfect interface fundamentally alters many aspects of weight functions and their application [6].

Our approach uses Betti's identity to relate physical fields to the weight function; the presence of the imperfect interface introduces new challenges here. We present computations that demonstrate how the stress at the crack tip (which is finite in the imperfect interface interfacial crack setting and may be used in a fracture criterion) varies depending on the extent of interface imperfection and the contrast in stiffness between the two materials. We also draw comparisons against the previously studied analogous perfect interface case. We then move on to consider the perturbed problem. In particular, the Betti identity again allows us to use the weight function to derive an expression for  $\Delta\sigma_0$ , the change in the leading order of tractions near the tip of the main crack induced by the presence of the small defect. The sign of  $\Delta\sigma_0$  can be interpreted as the inclusion's presence having an amplifying or shielding effect on the propagation of the main crack.

## 2. Physical problem formulation

The geometry under consideration is shown in Figure 1. A crack occupies  $\{(x, y) : x < 0, y = 0\}$  while an imperfect interface lies along the positive  $x$ -axis joining two materials with shear moduli  $\mu_1$  and  $\mu_2$  lying respectively above and below the crack. The methods described are applicable to arbitrary loadings on the crack faces, although we will concentrate on asymmetric self-balanced point loadings as shown in Figure 1. A small elliptical inclusion of shear modulus  $\mu_{in}$  is centered at a point  $\mathbf{Y}$ , making an angle  $\phi$  with the positive  $x$ -axis and oriented at an angle  $\alpha$  to the horizontal.

The anti-plane shear displacement function  $u$  satisfies the Laplace equation

$$\nabla^2 u(x, y) = 0.$$

We assume continuity of tractions and impose imperfect transmission conditions ahead of the crack, that is

$$\mu_1 \frac{\partial u_1}{\partial y} \Big|_{y=0+} = \mu_2 \frac{\partial u_2}{\partial y} \Big|_{y=0-}, \quad [[u]](x) = \kappa \mu_1 \frac{\partial u_1}{\partial y} \Big|_{y=0+}, \quad x > 0,$$

where  $\kappa$  is a parameter describing the extent of interface imperfection. On the crack faces, tractions are prescribed as follows

$$\mu_1 \frac{\partial u_1}{\partial y} \Big|_{y=0+} = p_+(x), \quad \mu_2 \frac{\partial u_2}{\partial y} \Big|_{y=0-} = p_-(x), \quad x < 0,$$

where  $p_{\pm}$  are such that the total load is self-balanced.

### 3. Weight function and Betti identity

The weight function problem is formulated similarly but with zero traction prescribed on the crack faces and no inclusion is present. Also, the geometry is modified; the crack occupies  $x > 0$  while the imperfect interface occupies  $x < 0$  (the mirror image of the physical problem setup).

The non-trivial solution to this weight function problem can be found by employing the Wiener-Hopf technique since it is a homogeneous problem of the form  $Au = 0$ . This process involves a suitable choice of factorisation of the Wiener-Hopf kernel that is obtained by taking Fourier transforms and applying boundary conditions. The factorisation results in a representation of this Wiener-Hopf kernel as the product of functions analytic in overlapping half-planes, which upon performing asymptotic analysis yields the imperfect interface weight function via Liouville's Theorem.

Betti's identity can then be applied in an imperfect interface setting; this gives a relationship between the hitherto unknown physical fields and the newly derived weight function. The presence of the imperfect interface introduces some changes in the application of Betti's identity; details of this derivation are given in [7]. For instance, the Fourier transform of the displacement jump across  $y = 0$  is analytic in the upper complex half-plane in the perfect interface case, since the displacement jump is zero across a perfect interface. This is not true for the imperfect interface and so a new version of the Betti identity is required. The identity takes the following form, where lowercase  $u$  and  $\sigma$  are the physical displacement and out-of-plane stress component, while uppercase  $U$  and  $\Sigma$  are their weight function counterparts. Bars denote Fourier transforms (with Fourier coordinate  $\xi$ ) and  $(\pm)$  superscripts represent the restriction of the preceding quantity to either the positive or negative semi-axis.

$$\overline{[U]^{(+)}}(\xi) \overline{\sigma^{(+)}}(\xi) - \overline{\Sigma}(\xi) \overline{[u]^{(-)}}(\xi) = -\overline{[U]}(\xi) \langle \bar{p} \rangle(\xi) - \langle \bar{U} \rangle(\xi) \overline{[p]}(\xi), \quad \xi \in \mathbb{R}.$$

### 4. Asymptotic behaviour near the crack tip

Full radial asymptotics near the tip of an interfacial crack sitting on an imperfect interface have been constructed in [3]; importantly, along the interface,

$$\sigma_{yz} \sim \sigma_0, \quad x \rightarrow 0^+, \quad (1)$$

where  $\sigma_0$  is a constant we aim to find. This asymptotic behaviour is markedly different to the equivalent expression for a crack sitting on a perfect interface, where a square root singularity occurs. That is, in the perfect interface case,

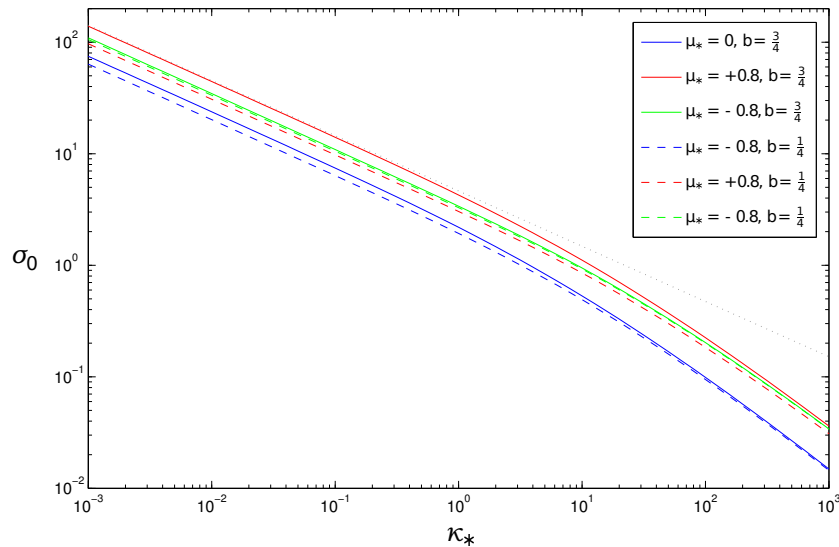
$$\sigma_{yz} = \frac{K_{III}}{\sqrt{2\pi}} x^{-1/2} + O(1), \quad x \rightarrow 0,$$

where the constant  $K_{III}$  is the mode III stress intensity factor. Stress intensity factors can be used in a simple fracture criterion; if  $K_{III}$  exceeds the fracture toughness of the cracked body's material, then the crack begins to grow [2]. There is no such singularity in the imperfect case (as is readily seen in (1)), but the constant  $\sigma_0$  plays an analogous role to  $K_{III}$  in the perfect interface setting.

The application of Betti's identity obtains an expression for the important constant  $\sigma_0$ :

$$\sigma_0 = \frac{1}{2} \sqrt{\frac{\mu_0}{\pi}} \int_{-\infty}^{\infty} \xi \left( [\bar{U}] (\xi) \langle \bar{p} \rangle (\xi) + \langle \bar{U} \rangle (\xi) [\bar{p}] (\xi) \right) d\xi.$$

Details of the derivation of this expression are given in [7]. We stress here that both the weight function  $U$  and the constant  $\mu_0$  depend heavily on the parameter of interface imperfection  $\kappa$ .



**Figure 2.** Log-log plot of  $\sigma_0$  against  $\kappa_*$  for differently contrasting materials.

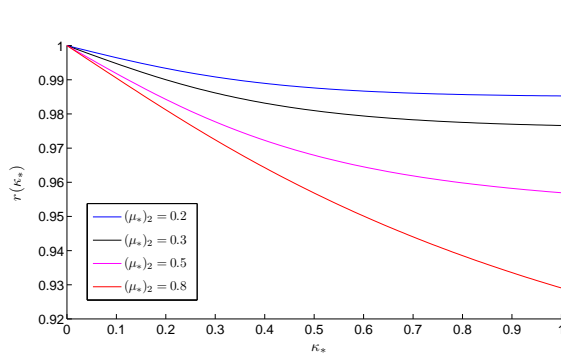
Figure 2 plots  $\sigma_0$  as a function of a dimensionless parameter of interface imperfection  $\kappa_*$  which is defined as  $\kappa_* = \kappa(\mu_1 + \mu_2)/a$ , on a log-log plot for a range of differently contrasting materials described by the dimensionless contrast parameter  $\mu_* = (\mu_1 - \mu_2)/(\mu_1 + \mu_2)$ . The plot indicates that  $\sigma_0 = O(\kappa_*^{-1/2})$  as  $\kappa_* \rightarrow 0$  (a dotted line of slope  $-\frac{1}{2}$  is included in the figure and clearly lies tangent to the curves as  $\kappa_* \rightarrow 0$ ) and  $\sigma_0 = O(\kappa_*^{-1})$  as  $\kappa_* \rightarrow \infty$ .

The problem considered is a singular perturbation problem; taking small values of  $\kappa > 0$  gives a qualitatively different solution to the perfect case in which  $\kappa = 0$  (as is seen by the different forms of crack tip asymptotics). This makes the comparison of fields near the crack tip difficult. However, given two particular pairs of materials with contrast parameters  $(\mu_*)_1$  and  $(\mu_*)_2$  say, we might expect the dimensionless ratio of stress intensity factors  $(K_{III})_1/(K_{III})_2$  from the perfect interface case and  $(\sigma_0)_1/(\sigma_0)_2$  from the imperfect interface case to be similar. The ratio

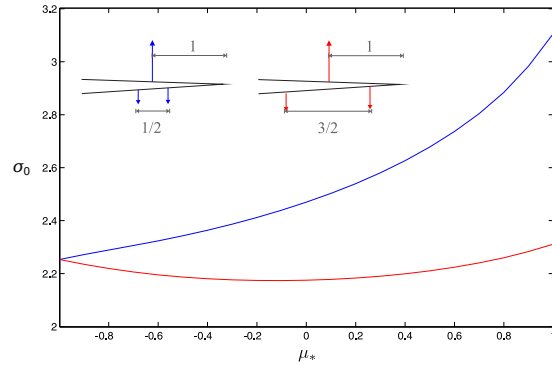
$$r(\kappa_*) = \frac{(\sigma_0)_1/(\sigma_0)_2}{(K_{III})_1/(K_{III})_2}$$

is plotted in Figure 3. From this we see that the ratio does indeed tend to 1 as  $\kappa_* \rightarrow 0$ , that is as the extent of interface imperfection decreases.

Figure 4 plots  $\sigma_0$  as a function of the dimensionless parameter  $\mu_*$  for two different configurations of point loadings. We see that the positioning of the point loadings on the lower crack face have the least effect for values of  $\mu_*$  near  $-1$ ; this is because the material occupying the lower half plane in such cases has a much higher shear modulus than the material occupying the upper half plane and so acts as an almost inelastic body. The exact positioning of the point loadings therefore has a smaller influence on the value of  $\sigma_0$ .



**Figure 3.** Plot of the ratio  $r(\kappa_*)$  with  $(\mu_*)_1 = 1$  for various values of  $(\mu_*)_2$ .



**Figure 4.** Plot of  $\sigma_0$  as a function of  $\mu_*$  for two different configurations of point loadings on the crack faces.

### 5. Perturbation analysis

We shall construct an asymptotic solution of the perturbed problem (with the inclusion present) using the method of Movchan and Movchan [4], that is the asymptotics of the solution will be taken in the form

$$u_{1,2}(\mathbf{x}, \varepsilon) = u_{1,2}^{(0)}(\mathbf{x}) + \varepsilon W^{(1)}(\boldsymbol{\xi}) + \varepsilon^2 u_{1,2}^{(1)}(\mathbf{x}) + o(\varepsilon^2), \quad \varepsilon \rightarrow 0.$$

In this expression, the leading term  $u_{1,2}^{(0)}$  corresponds to the unperturbed solution while  $\varepsilon W^{(1)}$  represents a boundary layer concentrated near the small defect that is needed to satisfy perfect transmission conditions on the boundary of the inclusion. The term  $\varepsilon^2 u_{1,2}^{(1)}$  is introduced to fulfil the boundary conditions on the crack faces and the imperfect interface which are disturbed by the boundary layer. This term, in turn, will produce perturbations of the crack tip fields and correspondingly of the constant  $\sigma_0$ .

The constant  $\sigma_0$  is expanded in the form

$$\sigma_0 = \sigma_0^{(0)} + \varepsilon^2 \Delta\sigma_0 + o(\varepsilon^2), \quad \varepsilon \rightarrow 0.$$

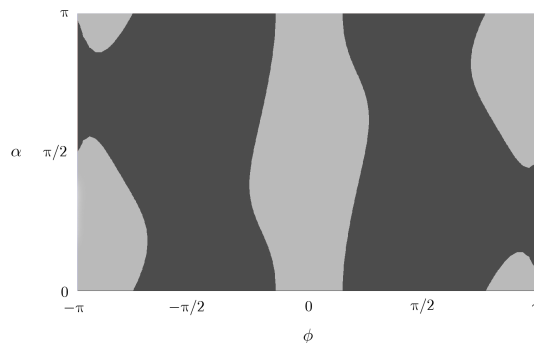
Our objective is to find the first order variation  $\Delta\sigma_0$  by employing the Betti identity corresponding to the first order perturbation  $u^{(1)}$ . The presence of “effective” tractions induced on the crack faces by the small elastic inclusion causes additional terms to be present:

$$\begin{aligned} & \llbracket \bar{U} \rrbracket^+(\xi) \langle \bar{\sigma}^{(1)+} \rangle(\xi) - \langle \bar{\Sigma} \rangle(\xi) \llbracket \bar{u}^{(1)-} \rrbracket(\xi) \\ & = -\llbracket \bar{U} \rrbracket(\xi) \bar{P}^-(\xi) - \langle \bar{U} \rangle(\xi) \bar{Q}^-(\xi) - \kappa \langle \bar{\Sigma} \rangle(\xi) \bar{P}^+(\xi) - \langle \bar{U} \rangle(\xi) \bar{Q}^+(\xi). \end{aligned}$$

Here, the functions  $\bar{P}^\pm$  and  $\bar{Q}^\pm$  are computable functions which result from the boundary layer analysis. Similar reasoning to that employed earlier for finding the unperturbed tractions near the crack tip  $\sigma_0^{(0)}$  yields an expression for  $\Delta\sigma_0$  in the form

$$\begin{aligned} \Delta\sigma_0 = & -\frac{1}{2} \sqrt{\frac{\mu_0}{\pi}} \left\{ \int_{-\infty}^{\infty} \left[ \xi \llbracket \bar{U} \rrbracket(\xi) \bar{P}^-(\xi) + \xi \langle \bar{U} \rangle(\xi) \bar{Q}^-(\xi) \right] d\xi \right. \\ & \left. + \int_{-\infty}^{\infty} \left[ \kappa \xi \langle \bar{\Sigma} \rangle^-(\xi) \bar{P}^+(\xi) + \xi \langle \bar{U} \rangle(\xi) \bar{Q}^+(\xi) \right] d\xi \right\}. \end{aligned}$$

The sign of  $\Delta\sigma_0$  is interpreted as the inclusion's presence having either a shielding effect on the propagation of the main crack if  $\Delta\sigma_0 < 0$  or an amplifying effect if  $\Delta\sigma_0 > 0$ . Figure 5 plots the sign of  $\Delta\sigma_0$  for a particular configuration of materials for varying location (characterised by the angle  $\phi$ ) and orientation  $\alpha$  of the small defect. Different analysis should be sought however when the defect lies very close to the imperfect interface (when  $\phi$  is near 0), since we assumed that the inclusion is at a finite distance from the interface between the half-planes in the boundary layer analysis.



**Figure 5.** Plot of the sign of  $\Delta\sigma_0$  for varying  $\alpha$  and  $\phi$ . The darker shaded areas are those  $(\phi, \alpha)$  for which  $\Delta\sigma_0 > 0$  while paler regions have  $\Delta\sigma_0 < 0$ .

## 6. Conclusions

The method described uses a new imperfect interface weight function to aid in the computation of the leading order of tractions near the crack tip in an imperfect interface crack problem under arbitrary loadings on the crack faces. Further, perturbation analysis obtains the next term in the asymptotics which describes the change in the tractions near the crack tip brought about by the presence of a small inclusion. The presence of the imperfect interface causes many significant differences in comparison to the analogous perfect interface problem; in particular the asymptotic behaviour of tractions near the crack tip is of a different form.

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