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OF TRENTO – Italy

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ASYMPTOTIC BEHAVIOR OF THIN ELASTIC INTERPHASES

April, 2012

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ASYMPTOTIC BEHAVIOR OF THIN ELASTIC INTERPHASES

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2012

UNIVERSITY OF TRENTO
Engineering of Civil and Mechanical Structural Systems XXIV cycle

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Final Examination: April 5, 2012

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ACKNOWLEDGEMENTS

The present thesis contains the work I have performed in the last three years during my Ph.D. training under the supervision of Prof. Davide Bigoni and Prof. Francesco-Serra Cassano at the University of Trento, Italy.

I bring my deepest gratitude to both my tutors for their continuous encouragement and precious scientific support. I am indebted to Prof. Gennady S. Mishuris and Prof. Antonio Gaudiello for their close instruction with both the problems discussed, without whom this work wouldn't have been concluded.

Trento, April 2012

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SUMMARY

The asymptotic behavior of a linearly elastic composite material that contains a thin interphase of thickness ε is described and analyzed by means of two complementary methods: the asymptotic expansions method and the study of the weak form using variational methods on Sobolev spaces. We recover the solution of the system of linearized elasticity in the two dimensional vectorial case and we find limit transmission conditions.

The same steps are followed for harmonic oscillations of the elasticity system, and different solutions are found for concentrated mass densities. The cases in which the elastic coefficients depend on ε, for soft as well as stiff materials are considered. An approximated solution is found for harmonic oscillations of the elasticity system and limit transmission conditions are derived.

Considering a bounded rectangular composite domain, with an ε-dependent subdomain, we describe the weak formulation of the linearized system of elasticity. In the case of constant elastic coefficients, we estimate the bounds of the strain tensor and so, the energetic functional in the rescaled domain. We perform a variational formulation of the system of linearized elasticity and find estimates for the energetic functional of the system.

CONTENTS

1 Linear Elasticity **13**
- 1.1 Derivation of linear theory of elasticity . . . 13
- 1.2 Linear elastostatics . . . 16
- 1.3 Principle of Minimum Potential Energy . . . 18
- 1.4 Elastodynamics . . . 19

2 Sobolev spaces **21**
- 2.1 Weak derivatives . . . 21
- 2.2 The L^p spaces . . . 22
- 2.3 Definition of $W^{m,p}(\Omega)$ . . . 23
  - 2.3.1 The space $W^{1,2}(\Omega)$ . . . 23
- 2.4 Definition of $W_0^{m,p}(\Omega)$ . . . 25
  - 2.4.1 The space $W_0^{1,2}(\Omega)$ . . . 25
- 2.5 Sobolev imbedding theorem . . . 26
  - 2.5.1 The case $\Omega = \mathbb{R}^N$ . . . 26
  - 2.5.2 The case $\Omega \subset \mathbb{R}^N$ . . . 28
- 2.6 Compactness. Rellich-Kondrachov theorem . . . 28
- 2.7 Poincaré inequality . . . 29
- 2.8 Generalized Poincaré inequality . . . 30
- 2.9 Friedrichs inequality . . . 31
- 2.10 Traces . . . 31

3 Existence and uniqueness **33**
  - 3.0.1 Lax-Milgram Theory . . . 35

4 Estimates. Korn inequalities **41**
- 4.1 First Korn inequality . . . 41
- 4.2 Second Korn inequality . . . 42

4.3 Third Korn inequality 44

5 Asymptotic expansions **45**
5.1 Elastostatics 45
5.1.1 Limit Transmission conditions 64
5.2 Harmonic oscillations 68
5.2.1 Limit Transmission Conditions 75

6 Variational method **79**
6.1 Description of Problem 79
6.2 Rescaling 83
6.3 Estimates 87

A Function spaces **91**

B Hölder inequality **93**

C Strain tensor in polar coordinates **95**

Notation

(x,y) .... generic position of a point in $\mathbb{R}^2$ described in Cartesian coordinates
X,Y .... rescaling variables
(r,θ) .... generic position of a point in $\mathbb{R}^2$ described in polar coordinates
R .... fixed value of radius
ρ .... rescaling variable
$\varepsilon \in (0,1)$ .... small parameter
$\Psi_\alpha; \alpha \in \{b,c,a\}$ .... rescaling mappings
$\Omega^\varepsilon \subset \mathbb{R}^2$ .... domain that contains an ε-dependent subdomain
$\Omega \subset \mathbb{R}^2$ .... the rescaled counterpart of the above defined Ω
$\mathbf{U} : \Omega^\varepsilon \to \mathbb{R}^2$ .... displacement on the ε-dependent domain
$\mathbf{u} : \Omega \to \mathbb{R}^2$ .... displacement on the associated rescaled domain
$\boldsymbol{div}$ .... divergence operator
$\mathbf{tr}$ .... trace operator
$\triangle$ .... Laplace operator
∇ .... gradient operator
$\mathbf{F}$ .... deformation gradient
$\mathbf{u}$ .... displacement vector field
t .... time parameter
$\dot{\mathbf{u}} = \dfrac{\partial \mathbf{u}}{\partial t}$ .... velocity of displacement vector field
$\ddot{\mathbf{u}} = \dfrac{\partial^2 \mathbf{u}}{\partial t^2}$ .... acceleration of displacement vector field
$\tilde{\mathbf{s}} = \left[\tilde{\mathbf{u}}, \tilde{\mathbf{E}}, \tilde{\mathbf{S}}\right]$ .... kinematically admissible state
$\mathscr{S}_1; \mathscr{S}_2$ .... boundaries
On $\mathscr{S}_1$ we consider displacement boundary conditions, whereas on $\mathscr{S}_2$ we consider traction boundary conditions.
M, M^* .... mass density
$\mathbf{S}$ .... stress tensor
$\mathbf{e}$ .... strain tensor on Ω^ε
L .... elasticity operator

$\mathbf{e}_r, \mathbf{e}_\theta$ .... base unit vectors in polar coordinates

$\mathbf{F} \in L^2(\Omega^\varepsilon)$ .... force field applied on Ω^ε

$\mathbf{f} \in L^2(\Omega)$ .... force field applied on the rescaled domain Ω

$\mathbf{n} := (n_1, n_2)$ or $\mathbf{n} := (n_r, n_\theta)$ .... normal vector field

$\mathbb{C}; \mathbb{C}^\alpha, \alpha \in \{b, c, a\}$ .... elasticity tensor

$C_{rr}, C_{r,\theta}, C_{\theta\theta}; C^*_{rr}, C^*_{r,\theta}, C^*_{\theta\theta}$ ... components of elasticity tensor in polar coordinates

$\lambda, \mu, \lambda_b, \mu_b, l_c, \mu_c$ .... Lamé constants

$\mathscr{U}$ .... strain energy

$\mathscr{V}_\varepsilon$ .... space of admissible displacements on Ω^ε

ω .... 1-periodic cell

$L^p(\Omega)$ .... space of p integrable Lebesgue functions

$W^{m,p}(\Omega)$ .... Sobolev space

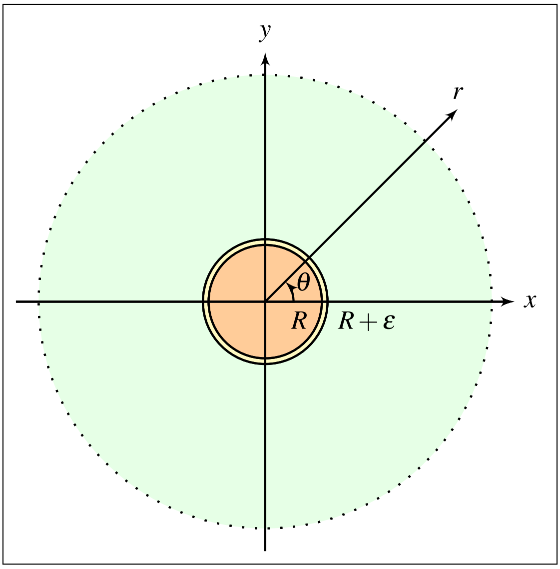
$\|.\|$ .... norm

$\rightarrow$ .... strong convergence

$\rightharpoonup$ .... weak convergence

Introduction

This work brings a twofold approach in the study of the asymptotic behavior of elastic anisotropic composite materials. The main interest is to describe and analyze the asymptotic behavior of an elastic composite material when one of its components is assumed to be dependent on a small parameter. Not only the geometry, but also some of the physical variables used in the mathematical modeling of such elastic systems are considered dependent on the same parameter. More, if the component which is dependent on the parameter happens to separate completely other components of the same composite, we call it interphase and its asymptotic behavior, both geometric and density wise plays a crucial role in the limit description of the system, when the parameter tends to zero.



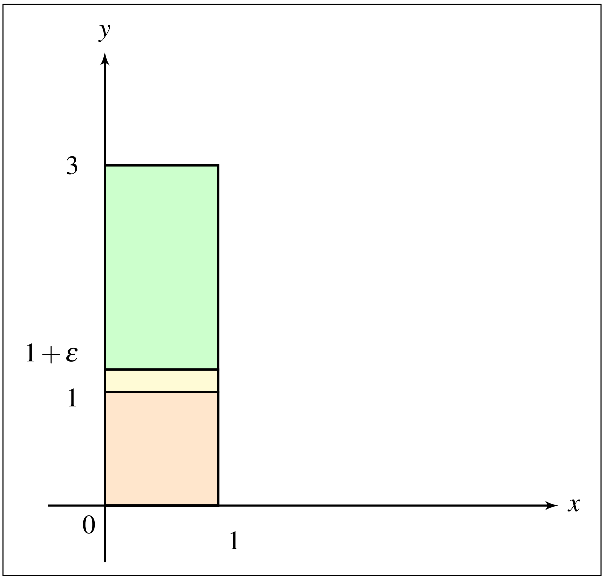
An approximated solution is found for the harmonic oscillations of the linearized system of elasticity by means of asymptotic expansions method in particular cases of concentrated mass densities. In the following, we will call an interphase, a thin domain of thickness ε and an interface the interphase's limit when $\varepsilon \to 0$. Transmission conditions on the boundaries of the interphase are imposed both where the continuity of displacement and tractions are concerned and limit transmission conditions are obtained in the case when the parameter tends to zero. In the second part of the thesis, a qualitative study is being performed in order to verify and match the results found by the method of asymptotic expansions. In the second part, a bounded, curvature free geometry is imposed for the system of linearized elasticity.

Estimates are being presented and a limit formulation gives bounds for the solution of the system. In the first chapter we introduce some basic theory on the linearized system of elasticity and its strong form and the passing to the weak form. This approach is used mostly in engineering. In the second chapter we give the mathematical background of functional analysis defining notions as weak derivative, function spaces like $L^p(\Omega)$, and the well known Sobolev type spaces. In this chapter we also present some of the most important theorems in the study of Sobolev spaces: the Sobolev imbedding theorem and it's compact version - the Rellich-Kondrachov theorem. The Sobolev imbedding theorem is presented for the case of bounded as well as unbounded domains. Inequalities like that of Poincaré and Friedrichs are presented, since they are of great importance in estimating the elastic energy.

In the third chapter we present existence conditions for the solution of the linearized system of elasticity. A case of uniqueness for Dirichlet boundary conditions imposed on a three dimensional bounded elastic body is presented. Korn type inequalities are crucial for estimating the bounds of the strain tensor. In the fourth chapter we introduce the Korn inequalities for bounded domains with functions in the spaces $W^{1,2}(\Omega)$ and $W_0^{1,2}(\Omega)$. For estimating traces of the functions we present several lemmas throughout the first four chapters pointing how we can find bounds in small neighborhoods of a point or on parts of a boundary. The fifth chapter is concerned with presenting two problems that we like to solve in detail by means of asymptotic expansions. We solve the system of linearized elasticity for statics as well as harmonic oscillations with mixed boundary conditions in the form of continuity of displacement and continuity of tractions through the two boundaries of the thin interphase.

When imposing the elasticity operator to this domain with different anisotropy for each component and considering continuity of displacement and tractions on the boundary layers, we compute an approximated solution in the thin interphase and we write limit transmission conditions that reflect the direct dependence of the traction components with respect to the small parameter ε. The limit transmission conditions describe the asymptotic behavior of the system when the small parameter ε tends to zero. We underline that the problems solved are multiple in the sense that several variables are ε dependent. When performing the limit we can observe the dependence in the behavior of the system when the elasticity tensor changes with ε, representing a more soft or stiff material that occupies the interphase. The critical cases of very soft or very stiff materials are sometimes called in the literature explosion of coefficients (see Attouch [Att]).

In the case of harmonic oscillations, the system shows a similar behavior as in the static case unless we considered concentrated mass densities. When considering concentrated mass densities per unit volume, a solution can be recovered in an explicit way that has direct dependence on all the parameters of the system. Moreover, different ε-dependent scalings of the mass densities recover the same solution for different types of materials considered. As in the statical case, the limit transmission conditions depend on ε. Moreover, in this case, we observe an explicit dependence of the transmission conditions with respect to parameters like the frequency ω and the mass density M.
The last chapter is devoted to describing and analyzing the asymptotic behavior of a bounded composite material that contains a thin component of thickness ε written in weak form. A general case for body forces applied on the domain is considered.



We estimate the rescaled energetic functional a priori estimates in Sobolev spaces. The method of asymptotic expansions and the study of the linearized elasticity system in weak form are complementary. The second approach brings a weak formulation of the limit problem which reflects the limit boundary value problem while the first one can find an approximated solution.

In the last part of this introduction we will refer some literature for the various connections of the problems treated. We start by recommending some books on nonlinear solid mechanics and continuum mechanics. Since the theory needed in this study is that of linear elasticity we will not present here any theoretical background on nonlinear elasticity. But we still like to present a theoretical introduction to how to linearize the system of elasticity in the first chapter. Since the notions used there are of particular detail from nonlinear theory, we refer to the books of Antman [An], Bigoni [B], Berdichevsky [Ber], Gurtin [Gu], Holzapfel [H], Liu [L], Truesdell and Noll [TN].

Where linear elasticity is concerned we recommend the books of Gurtin [Gu], Ogden [O]. For asymptotic expansions solutions we refer to Oleinik [Ol] and Ladyzhenskaya [Lad]. Solutions of the elasticity system under harmonic oscillations for different composites can be found in works like Sanchez-Palencia [SPZ]. For references on functional analysis and partial differential equations and Sobolev spaces we recommend the books of Brezis [Bre], Adams [Ad], Miranda [Mir]. Some of the books and papers in which the same type of problems are being treated by means on Γ-convergence are Acerbi and Buttazzo [AB1], [AB2],Braides [Bra], [DM], Serra-Cassano [SC]. For etimates of Korn type and elasticity treated in the weak form, we refer to Ciarlet [Cia1], [Cia2], Cioră-nescu [Cio] and Oleinik [Ol]. Important articles studying asymptotic behavior of elastic structures, thin elastic structures, as well as multi-domains containing an ε-dependent components are those of Bigoni et al. [B1], [B2], [B3], Mishouris et al. [M], Gaudiello [G1], [G2], [G3], [G4] and Freddi [Fr1].

CHAPTER

ONE

LINEAR ELASTICITY

1.1. Derivation of linear theory of elasticity

We start by presenting a theoretical background on linear elasticity by bringing a point of view from the nonlinear theory towards the linearized equations of elasticity. We deduce the linearized theory of elasticity when the gradient of displacement ∇u is small. Considering the constitutive equation

$$\mathbf{S} = \hat{\mathbf{S}}(\mathbf{F}) \tag{1.1}$$

for the Piola-Kirchhoff stress, we start by linearizing this equation.

In order to describe the behavior of this equation when

$$\mathbf{H} = \nabla u \to 0, \tag{1.2}$$

we consider $\hat{\mathbf{S}}(\mathbf{F})$ as a function of $\mathbf{H}$ using the relation

$$\mathbf{F} = \mathbf{I} + \mathbf{H}. \tag{1.3}$$

Theorem 1 (*Asymptotic form of constitutive relation*) *An important step is to assume that the residual stress vanishes. Then*

$$\hat{\mathbf{S}}(\,\mathbf{F}\,) \;=\; \mathbb{C}[\,\mathbf{E}\,] + \mathscr{O}(\,\mathbf{H}\,) \tag{1.4}$$

as $\mathbf{H} \;\to\; 0$*, where* $\mathbb{C}$ *is the elasticity tensor and*

$$\mathbf{E} \;=\; \frac{1}{2}\left(\,\mathbf{H} + \mathbf{H}^T\,\right) \tag{1.5}$$

is the infinitesimal strain.

Proof

Since the residual stress vanishes, we conclude that

$$\begin{aligned}
\hat{\mathbf{S}}(\,\mathbf{F}\,) \;&=\; \hat{\mathbf{S}}(\,\mathbf{I}\,) + D\hat{\mathbf{S}}(\,\mathbf{I}\,)[\,\mathbf{H}\,] + \mathscr{O}(\,\mathbf{H}\,) && (1.6)\\
&=\; \mathbf{C}[\,\mathbf{H}\,] + \mathscr{O}(\,\mathbf{H}\,) && (1.7)\\
&=\; \mathbf{C}[\,\mathbf{E}\,] + \mathscr{O}(\,\mathbb{H}\,). && (1.8)
\end{aligned}$$

Using equation (1.4) we can write the asymptotic form of the constitutive equation as

$$\mathbf{S} \;=\; \mathbf{C}[\,\mathbf{E}\,] + \mathscr{O}(\,\nabla u\,). \tag{1.9}$$

If the residual stress in the reference configuration vanishes, then to within terms of $\mathscr{O}(\,\nabla u\,)$ *as* $\nabla u \;\to\; 0$ *the stress* $\mathbf{S}$ *is a linear function of the infinitesimal strain* $\mathbf{E}$*. Since* $\mathbf{C}$ *has symmetric values, to within the same error* $\mathbf{S}$ *is symmetric.* ∎

The linear theory of elasticity is based on the stress-strain law when the order $\mathscr{O}$ is neglected, the strain displacement relation and the equation of motion:

$$\begin{cases}
\mathbf{S} \;=\; \mathbf{C}[\,\mathbf{E}\,] \\[2ex]
\mathbf{E} \;=\; \dfrac{1}{2}\Big(\,\nabla u \;+\; (\,\nabla u\,)^T\,\Big) \\[2ex]
\boldsymbol{div}\,\mathbf{S} \;+\; \mathbf{b}_0 \;=\; \rho\ddot{\mathbf{u}}
\end{cases} \tag{1.10}$$

These equations are expressed in terms of displacement

$$\mathbf{u}(\,\mathbf{p},t\,) \;=\; \mathbf{x}(\,\mathbf{p},t\,) \;-\; \mathbf{p} \tag{1.11}$$

rather than the motion $\mathbf{x}$. It is important to emphasize that the formal derivation of the linearized constitutive equation (1.10), was based on the following two assumptions.

(a) The residual stress in the reference configuration vanishes.

(b) The displacement gradient is small.

Observation 1 $\mathbf{E} \;=\; \mathbf{S} \;=\; \mathbf{0}$ *is an infinitesimal rigid displacement.*

Given $\mathbf{C},\ \rho_0,\ \mathbf{b}_0$, (1.10) is a linear system of partial differential equations for the fields $\mathbf{u},\ \mathbf{E},\ \mathbf{S}$.
When the body is isotropic, (1.10) can be replaced by

$$\mathbf{S} = 2\mu\mathbf{E} + \lambda(\mathbf{trE})\mathbf{I}. \tag{1.12}$$

If the body is homogeneous, ρ_o, μ, λ are constants. If we assume that the body is homogeneous and isotropic.

$$\boldsymbol{div}\,\left(\,\nabla\mathbf{u} \;+\; (\,\nabla\mathbf{u}\,)^T\,\right) \;=\; \triangle\mathbf{u} \;+\; \nabla\,\boldsymbol{div}\,\mathbf{u} \tag{1.13}$$

and

$$\mathbf{trE} \;=\; \boldsymbol{div}\,\mathbf{u}, \tag{1.14}$$

the second and third equation in (1.10) and (1.12) are easily combined to give the displacement equation of motion

$$\mu\,\triangle\mathbf{u} \;+\; (\,\mu+\lambda\,)\nabla\,\boldsymbol{div}\,\mathbf{u} \;+\; \mathbf{b}_0 \;=\; \rho_0\ddot{\mathbf{u}}. \tag{1.15}$$

In the statical theory, $\ddot{\mathbf{u}} \;=\; \mathbf{0}$ and we have the displacement equation of equilibrium

$$\lambda\triangle\mathbf{u} \;+\; (\,\lambda+\mu\,)\nabla\,\boldsymbol{div}\,\mathbf{u} \;+\; \mathbf{b}_0 \;=\; \mathbf{0}. \tag{1.16}$$

1.2. Linear elastostatics

The system of field equations for the statical behavior of an elastic body, in the framework of the linear theory, consists of the strain-displacement relation

$$\mathbf{E} = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T \right), \tag{1.17}$$

the stress-strain relation

$$\mathbf{S} = \mathbf{C}[\mathbf{E}], \tag{1.18}$$

and the equation of equilibrium

$$\boldsymbol{div}\,\mathbf{S} + \mathbf{b} = \mathbf{0}, \tag{1.19}$$

where $\mathbf{b} \equiv \mathbf{b}_0$. The elasticity tensor $\mathbf{C}$, which is a linear mapping of tensors into symmetric tensors, will generally depend on the position $\mathbf{p}$ in Ω ; writing $\mathbf{C}_\mathbf{p}$ to emphasize this dependence, we assume henceforth that $\mathbf{C}_\mathbf{p}$ is a smooth function of $\mathbf{p}$ on Ω .

A list $[\mathbf{u}, \mathbf{E}, \mathbf{S}]$ of fields which are smooth on Ω and satisfy (1.1),(1.4), (1.5), for a given body force $\mathbf{b}$ will be called an elastic state corresponding to $\mathbf{b}$. By (1.1) and (1.4), the properties of $\mathbf{C}, \mathbf{E}, \mathbf{S}$ are symmetric.

Assuming that Ω is bounded, we have the following theorem:

Theorem 2 ***(Theorem of Work and Energy)***

Let $[\mathbf{u}, \mathbf{E}, \mathbf{S}]$ *be an elastic state corresponding to the body force* $\mathbf{b}$. *Then*

$$\int_{\partial\Omega} (\mathbf{S}(\mathbf{u}))\cdot\mathbf{n}\,dA + \int_{\Omega} \mathbf{b}\cdot\mathbf{u}\,dV = 2\,\mathscr{U}\{\mathbf{E}\} \tag{1.20}$$

where

$$\mathscr{U} = \frac{1}{2}\int_{\Omega} \mathbf{E}\cdot\mathbf{C}[\mathbf{E}]\,dV \tag{1.21}$$

is the strain energy.

The proof of this theorem is the immediate consequence of the following lemma:

Lemma 1 *Let* $\mathbf{S}$ *be a smooth symmetric tensor field on* $\mathbf{B}$, *let* $\tilde{\mathbf{u}}$ *be a smooth vector field on* Ω, *and let*

$$\boldsymbol{div}\, \mathbf{S} + \mathbf{b} = \mathbf{0}, \tag{1.22}$$

$$\tilde{\mathbf{E}} = \frac{1}{2}\Big(\nabla\tilde{\mathbf{u}} + (\nabla\tilde{\mathbf{u}})^T \Big). \tag{1.23}$$

Then,

$$\int_{\partial\Omega} (\mathbf{S}(\tilde{\mathbf{u}}))\cdot\mathbf{n}\, dA + \int_{\Omega} \mathbf{b}\cdot\tilde{\mathbf{u}}\, dV = \int_{\Omega} \mathbf{S}\cdot\tilde{\mathbf{E}}\, dV \tag{1.24}$$

Proof

By the symmetry of $\mathbf{S}$ *and the divergence theorem,*

$$\int_{\partial\Omega} (\mathbf{S}(\tilde{\mathbf{u}}))\cdot\mathbf{n}\, dA = \int_{\partial\Omega} (\mathbf{S}(\tilde{\mathbf{u}}))\cdot\mathbf{n}\, dA \tag{1.25}$$

$$= \int_{\Omega} \boldsymbol{div}\, (\mathbf{S}(\tilde{\mathbf{u}}))\, dV \tag{1.26}$$

$$= \int_{\Omega} (\tilde{\mathbf{u}}\cdot \boldsymbol{div}\,\mathbf{S} + \mathbf{S}\cdot\nabla\tilde{\mathbf{u}})\, dV \tag{1.27}$$

$$\mathbf{S}\cdot\nabla\tilde{\mathbf{u}} = \mathbf{S}\cdot\left\{\frac{1}{2}\Big(\nabla\tilde{\mathbf{u}} + (\nabla\tilde{\mathbf{u}})^T\Big)\right\} = \mathbf{S}\cdot\tilde{\mathbf{E}} \tag{1.28}$$

∎

These equations imply (1.24).

We can interpret the left side of (1.20) as the work done by the external forces; (1.20) asserts that this work is equal to twice the strain energy. When $\mathbf{C}$ is definite positive, $\mathscr{U}\{\mathbf{E}\} \geq \mathbf{0}$, and the work is nonnegative.

1.3. Principle of Minimum Potential Energy

Theorem 3 *Assume that* $\mathbf{C}$ *is symmetric and positive definite. Let* $\mathfrak{s} = [\mathbf{u}, \mathbf{E}, \mathbf{S}]$ *be a solution of the mixed problem. Then*

$$\Phi\{\,\mathfrak{s}\,\} \;\leq\; \Phi\{\,\tilde{\mathfrak{s}}\,\} \tag{1.29}$$

for every kinematically admissible state $\tilde{\mathfrak{s}} = \left[\,\tilde{\mathbf{u}},\, \tilde{\mathbf{E}},\, \tilde{\mathbf{S}}\,\right]$ *and equality holds if* $\tilde{\mathbf{u}} = \mathbf{u} + \mathbf{w}$ *with* $\mathbf{w}$ *an infinitesimal rigid displacement of* Ω.

Proof

Let

$$\mathbf{w} \,=\, \tilde{\mathbf{u}} \,-\, \mathbf{u};\; \bar{\mathbf{E}} \,=\, \tilde{\mathbf{E}} \,-\, \mathbf{E}. \tag{1.30}$$

Then, since $\mathfrak{s}$ *and* $\tilde{\mathfrak{s}}$ *is kinematically admissible,*

$$\bar{\mathbf{E}} \,=\, \frac{1}{2}\left(\,\nabla\mathbf{w} \,+\, (\,\mathbf{w}\,)^T\,\right), \tag{1.31}$$

$$\mathbf{w} \,=\, \mathbf{0} \text{ on } \mathscr{S}_1. \tag{1.32}$$

Further, since $\mathbf{C}$ *is symmetric and* $\mathbf{S} \,=\, \mathbf{C}[\,\mathbf{E}\,]$,

$$\begin{aligned}
\tilde{\mathbf{E}}\cdot\mathbf{C}\left[\,\tilde{\mathbf{E}}\,\right] \;&=\; \mathbf{E}\cdot\mathbf{C}[\,\mathbf{E}\,] + \bar{\mathbf{E}}\cdot\mathbf{C}\left[\,\bar{\mathbf{E}}\,\right] + \mathbf{E}\cdot\mathbf{C}\left[\,\bar{\mathbf{E}}\,\right] + \bar{\mathbf{E}}\cdot\mathbf{C}[\,\mathbf{E}\,] && (1.33)\\
&=\; \mathbf{E}\cdot\mathbf{C}[\,\mathbf{E}\,] + \bar{\mathbf{E}}\cdot\mathbf{C}\left[\,\bar{\mathbf{E}}\,\right] + 2\mathbf{S}\cdot\bar{\mathbf{E}} && (1.34)
\end{aligned}$$

hence

$$\mathscr{U}\left\{\,\tilde{\mathbf{E}}\,\right\} - \mathscr{U}\{\,\mathbf{E}\,\} \;=\; \mathscr{U}\left\{\,\bar{\mathbf{E}}\,\right\} + \int_\Omega \mathbf{S}\cdot\bar{\mathbf{E}}\,dV. \tag{1.35}$$

Because $\mathfrak{s}$ *is a solution, we conclude from (1.32), and (1.24) with* $\tilde{\mathbf{u}}$ *and* $\tilde{\mathbf{E}}$ *replaced by* $\mathbf{w}$ *and* $\bar{\mathbf{E}}$ *that*

$$\int_{\Omega} \mathbf{S} \cdot \bar{\mathbf{E}}\, dV \;=\; \int_{\partial\Omega} \mathbf{Sn} \cdot \mathbf{w}\, dA + \int_{\Omega} \mathbf{b} \cdot \mathbf{w}\, dV \;=\; \int_{\mathscr{S}_2} \hat{\mathbf{s}} \cdot \mathbf{w}\, dA + \int_{\Omega} \mathbf{b} \cdot \mathbf{w}\, dV. \tag{1.36}$$

In view of the last two relations,

$$\Phi\{\, \tilde{\mathfrak{s}} \,\} - \Phi\{\, \mathfrak{s} \,\} \;=\; \mathscr{U}\left\{\, \bar{\mathbf{E}} \,\right\}. \tag{1.37}$$

Thus, since $\mathbf{C}$ *is positive definite,*

$$\Phi\{\, \mathfrak{s} \,\} \;\leq\; \Phi\{\, \tilde{\mathfrak{s}} \,\} \tag{1.38}$$

and

$$\Phi\{\, \mathfrak{s} \,\} \;=\; \Phi\{\, \tilde{\mathfrak{s}} \,\} \textit{ only when } \bar{\mathbf{E}} \;=\; \mathbf{0}; \tag{1.39}$$

that is, only when $\mathbf{w} \;=\; \bar{\mathbf{u}} - \mathbf{u}$ *is an infinitesimal rigid displacement.* ∎

The principle of minimum potential energy asserts that the difference between the strain energy and the work done by the body force and prescribed surface traction assumes a smaller value for the solution of the mixed problem than for any other kinematically admissible state.

1.4. Elastodynamics

Given the equations described before:

$$\begin{cases} \mathbf{E} \;=\; \dfrac{1}{2}\Big(\, \nabla\mathbf{u} + (\, \nabla\mathbf{u} \,)^T \,\Big) \\[2ex] \mathbf{S} \;=\; \mathbf{C}[\, \mathbf{E} \,] \\[2ex] \textit{\textbf{div}}\, \mathbf{S} + \mathbf{b} \;=\; \rho\ddot{\mathbf{u}} \end{cases} \tag{1.40}$$

we assume that Ω is bounded and ρ is continuous. Let $[\, \mathbf{u}, \mathbf{E}, \mathbf{S} \,]$ be a list of fields on $\Omega \times [\, 0, \infty \,)$ with $\mathbf{u}$ of class C^2 and $\mathbf{E}$ and $\mathbf{S}$ smooth, and suppose that the equations hold with $\mathbf{b}$ given body force field on $\Omega \times [\, 0, \infty \,)$.

$[\,\mathbf{u},\,\mathbf{E},\,\mathbf{S}\,]$ is called an elastic process corresponding to $\mathbf{b}$. Since $\mathbf{E}$ is time dependent, the strain energy

$$\mathscr{U}\{\,\mathbf{E}\,\} \;=\; \frac{1}{2}\int_{\Omega} \mathbf{E}\cdot\mathbf{C}[\,\mathbf{E}\,]\,dV \tag{1.41}$$

depends on time. If $\mathbf{C}$ is symmetric

$$\frac{1}{2}\left(\,\mathbf{E}\cdot\mathbf{C}[\,\mathbf{E}\,]\,\right)^{\cdot} \;=\; \frac{1}{2}\left(\dot{\mathbf{E}}\cdot\mathbf{C}[\,\mathbf{E}\,] + \mathbf{E}\cdot\mathbf{C}[\,\dot{\mathbf{E}}\,]\right) \;=\; \dot{\mathbf{E}}\cdot\mathbf{C}[\,\mathbf{E}\,] \;=\; \mathbf{S}\cdot\dot{\mathbf{E}} \tag{1.42}$$

Thus,

$$\left(\,\mathscr{U}\{\,\mathbf{E}\,\}\,\right)^{\cdot} \;=\; \int_{\Omega} \mathbf{S}\cdot\dot{\mathscr{E}}\,dV \tag{1.43}$$

so that the rate of change of strain energy is equal to the stress power.

Theorem 4 ***(Theorem of power and energy)***

Assume that $\mathbf{C}$ *is symmetric. Let* $[\,\mathbf{u},\,\mathbf{E},\,\mathbf{S}\,]$ *be an elastic process corresponding to the body force* $\mathbf{b}$. *Then*

$$\int_{\partial\Omega} \mathbf{S}\mathbf{n}\cdot\dot{\mathbf{u}}\,dA + \int_{\Omega} \mathbf{b}\cdot\dot{\mathbf{u}}\,dV \;=\; \left(\,\mathscr{U}\{\,\mathbf{E}\,\} + \mathscr{K}\{\,\dot{\mathbf{u}}\,\}\,\right) \tag{1.44}$$

where

$$\mathscr{K}\{\dot{\mathbf{u}}\} \;=\; \frac{1}{2}\int_{\Omega} \rho\dot{\mathbf{u}}^2\,dV \tag{1.45}$$

is the kinetic energy.

CHAPTER

TWO

SOBOLEV SPACES

2.1. Weak derivatives

In order to understand the concept of weak formulation of an equilibrium equation, we must first bring to attention basic notions like weak derivatives. For particular definitions of some of the notations where functional analysis is concerned we invite to the references indicated in the introduction as well as in the bibliography provided. We will define first the concept of a function being the weak derivative of another function.

Let $u \in L^1_{loc}(\Omega)$. If there exists $v_\alpha \in L^1_{loc}(\Omega)$ such that $T_{v_\alpha} = D^\alpha T_u$ in $\mathscr{D}'(\Omega)$, then it is unique up to sets of measure zero and is called weak or distributional derivative of u and denoted by $D^\alpha u$.

$D^\alpha u = v_\alpha$ in the weak sense, provided $v_\alpha \in L^1_{loc}(\Omega)$ satisfies:

$$\int_\Omega u(x)\, D^\alpha\, \phi(x)\, dx = (-1)^{|\alpha|} \int_\Omega v_\alpha(x)\, \phi(x)\, dx \tag{2.1}$$

for every $\phi \in \mathscr{D}(\Omega)$.

If u is sufficiently smooth to have a continuous partial derivative D^α in the usual sense, the D^α is also a weak partial derivative of u. D^α may exist in the weak sense without existing in the classical sense.

2.2. The L^p spaces

In order to introduce the definition of p-integrable function spaces, we will first bring a few basic definitions from topology.

Definition 1 *Let* $(\Omega, \mathscr{M}, \mu)$ *denote a measure space, i.e.,* Ω *is a set and* $\mathscr{M}$ *is a* σ *-algebra in* Ω*, i.e.,* $\mathscr{M}$ *is a collection of subsets of* Ω *such that:*

1. $\varnothing \in \mathscr{M}$
2. $A \in \mathscr{M} \Rightarrow A^c \in \mathscr{M}$
3. $\bigcup_{n=1}^{\infty} A_n \in \mathscr{M}$ *for* $A_n \in \mathscr{M}, \forall n.$

μ *is a measure. The members of* $\mathscr{M}$ *are called measurable sets. We can write either* $|A|$ *or* $\mu(A)$.

Definition 2 Ω *is* σ *-finite, i.e.* $\exists$ *a countable family* (Ω_n) *in* $\mathscr{M}$ *such that* $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$ *and* $\mu(\Omega_n) \leq \infty, \forall n.$

Definition 3 ***(The*** L^p ***spaces)***
Let $p \in \mathbb{R}$ *with* $1 < p < \infty$*; we set*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R};\ f \text{ is measurable and } |f|^p \in L^1(\Omega) \right\} \tag{2.2}$$

with

$$\|f\|_{L^p(\Omega)} = \|f\|_p = \left[\int_\Omega |f(x)|^p \, d\mu \right]^{1/p} \tag{2.3}$$

Observation 2 $\|.\|_p$ *is a norm.*

2.3. Definition of $W^{m,p}(\Omega)$

Definition 4 ***(Sobolev norm)*** *Let us consider m a positive integer and $1 \leq p < \infty$. Then we can define a functional $\|.\|_{m,p}$ as:*

$$\|u\|_{m,p} = \left(\sum_{0 \leq |\alpha| \leq m} \|D^{\alpha} u\|_p^p \right)^{1/p} \quad if\ 1 \leq p < \infty \tag{2.4}$$

$$\|u\|_{m,\infty} = \max_{0 \leq |\alpha| \leq m} \|D^{\alpha} u\|_{\infty} \tag{2.5}$$

for any function u for which the right side makes sense, $\|.\|_p$ being the norm in $L^p(\Omega)$.

Definition 5 ***(The spaces*** *$H^{m,p}(\Omega)$)For any positive integer m and $1 \leq p \leq \infty$ we define the vector space:*

$$H^{m,p}(\Omega) = \text{ the completion of } \left\{ u \in C^m(\Omega) |\ \|u\|_{m,p} < \infty \right\} \tag{2.6}$$

with respect to the norm $\|.\|_{m,p}$.

Definition 6 ***(The Sobolev spaces*** *$W^{m,p}(\Omega)$)For any positive integer m and $1 \leq p \leq \infty$ we define the vector spaces on which $\|.\|_{m,p}$ is a norm:*

$$W^{m,p}(\Omega) = \{ u \in L^p(\Omega) | D^{\alpha} u \in L^p(\Omega)\ for\ 0 \leq |\alpha| \leq m \} \tag{2.7}$$

where D^{α} is the weak partial derivative of u.

Theorem 5 *(Meyers-Serrin, 1964)*
Let Ω be an open set in $\mathbb{R}^N$. Then:$H^{m,p}(\Omega) \equiv W^{m,p}(\Omega)$.

2.3. The space $W^{1,2}(\Omega)$

Definition 7 *($W^{1,p}(\Omega)$)*
Let $\Omega \subset \mathbb{R}^n$ be an open set and let $p \in \mathbb{R}$ with $1 \leq p \leq \infty$. The Sobolev space $W^{1,p}(\Omega)$ is defined by

$$W^{1,p}(\Omega) = \left\{ \begin{array}{c|l} u \in L^p(\Omega) & \exists\, g_1, ..., g_n \in L^p(\Omega) \text{ such that} \\ \multicolumn{2}{c}{\int\limits_{\Omega} u \dfrac{\partial \varphi}{\partial x_i} = -\int\limits_{\Omega} g_i \varphi,\ \forall\, \varphi \in C_c^{\infty}(\Omega),\ \forall\, i = 1, 2, ..., n} \end{array} \right\} \tag{2.8}$$

Also,

$$\|u\|_{W^{1,p}} \;=\; \|u\|_p \;+\; \sum_{i=1}^{n} \left\| \frac{\partial u}{\partial x_i} \right\|_p \tag{2.9}$$

For $p = 2$ in the last definition, we get $W^{1,2}(\Omega)$ or $H^1(\Omega)$.

Proposition 1 *Equipped with the scalar product*

$$\langle u,v\rangle = \int_\Omega (u(x)v(x) + \nabla u(x)\cdot\nabla v(x))\,dx \tag{2.10}$$

and with the norm

$$\|u\|_{W^{1,2}(\Omega)} \;=\; \left(\int_\Omega (\; |u(x)|^2 \;+\; |\nabla u(x)|^2 \;)\, dx \right)^{1/2} \tag{2.11}$$

the Sobolev space $W^{1,2}(\Omega)$ is a Hilbert space.

Proof

It is obvious that (2.10) is a scalar product in $W^{1,2}(\Omega)$. It therefore remains to show that $W^{1,2}(\Omega)$ is complete for the associated norm. Let $(u_n)_{n\geq 1}$ be a Cauchy sequence in $L^2(\Omega)$. As $L^2(\Omega)$ is complete, there exist limits u and w_i such that u_n converges to u and $\dfrac{\partial u_n}{\partial x_i}$ converges to w_i in $L^2(\Omega)$. Now, by definition of the weak derivative of u_n , for every function $\phi \in C_c^\infty\Omega$, we have

$$\int_\Omega u_n(x)\frac{\partial \phi}{\partial x_i}(x)\, dx \;=\; -\int_\Omega \frac{\partial u_n}{\partial x_i}(x)\phi(x)\, dx. \tag{2.12}$$

Passing to the limit for $n \to +\infty$, we obtain

$$\int_\Omega u(x)\frac{\partial \phi}{\partial x_i}(x)\, dx \;=\; -\int_\Omega w_i(x)\phi(x)\, dx, \tag{2.13}$$

which proves that u is differentiable in the weak sense and that w_i is the i^{th} weak partial derivative of u , $\dfrac{\partial u}{\partial x_i}$. Therefore, u belongs to $W^{1,2}(\Omega)$ and $(u_n)_{n\geq 1}$ converges to u in $W^{1,2}(\Omega)$. ∎

It is very important in practice to know if regular functions are dense in the Sobolev space $W^{1,2}(\Omega)$. This partly justifies the idea of a Sobolev space which occurs very simply as the set of regular functions completed by the limits of sequences of regular functions in the energy norm $\|u\|_{W^{1,2}(\Omega)}$. This allows us to prove several properties easily by establishing them first for regular functions, then by using a density argument.

2.4. Definition of $W_0^{m,p}(\Omega)$

Definition 8 $W_0^{m,p}(\Omega)$ = *the closure of* $C_0^\infty(\Omega)$ *in the space* $W^{m,p}(\Omega)$

For any m, the chain of imbeddings

$$W_0^{m,p}(\Omega) \to W^{m,p}(\Omega) \to L^p(\Omega) \tag{2.14}$$

holds.

2.4. The space $W_0^{1,2}(\Omega)$

Let us now define another Sobolev space which is a subspace of $W^{1,2}(\Omega)$ and which will be very useful for problems with Dirichlet boundary conditions.

Definition 9 *Let* $C_c^\infty(\Omega)$ *be the space of functions of class* C^∞ *with compact support in* Ω. *The Sobolev space* $W_0^{1,2}(\Omega)$ *is defined as the closure of* $C_c^\infty(\Omega)$ *in* $W^{1,2}(\Omega)$.

$W_0^{1,2}(\Omega)$ is in fact the subspace of $W^{1,2}(\Omega)$ composed of functions which are zero on the boundary $\partial\Omega$ since this is the case for functions of $C_c^\infty(\bar{\Omega})$. In general, $W_0^{1,2}(\Omega)$ is strictly smaller than $W^{1,2}(\Omega)$ since $C_c^\infty(\Omega)$ is a strict subspace of $C_c^\infty(\bar{\Omega})$. An important exception is the case where $\Omega = \mathbb{R}^N$: in effect, in this case $\bar{\Omega} = \mathbb{R}^N = \Omega$ shows that $C_c^\infty(\mathbb{R}^N)$ is dense in $W^{1,2}(\mathbb{R}^N)$, therefore we have $W_0^{1,2}(\mathbb{R}^N) = W^{1,2}(\mathbb{R}^N)$. This exception is easily understood as the whole space $\mathbb{R}^N$ does not have a boundary.

Proposition 2 *Equipped with the scalar product of* $W^{1,2}(\Omega)$, *the Sobolev space* $W_0^{1,2}(\Omega)$ *is a Hilbert space.*

Proof *By definition* $W_0^{1,2}(\Omega)$ *is a closed subspace of* $W^{1,2}(\Omega)$ *(which is a Hilbert space), therefore it is also a Hilbert space.* ∎

An essential result for the applications of the fifth chapter is the following inequality.

2.5. Sobolev imbedding theorem

In this section, we will present the well known imbedding theorem of Sobolev. The proofs will not be presented here since they are very long and can be found in the books of Brezis [Bre], Adams [Ad] or Evans [E].

Observation 3 *If Ω has dimension 1, then $W^{1,p}(\Omega) \subset L^{\infty}(\Omega)$ with continuous injection, for all $1 \leq p \leq \infty$. In dimension $N = 2$, this inclusion is true only for $p > N$. When $p \leq N$ one can construct functions in $W^{1,p}$ that do not belong to $L^{\infty}(\Omega)$.*

Nevertheless, an important result, essentially due to Sobolev, asserts that if $1 \leq p < N$ then $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$ with continuous injection, for some $p^* \in (p, +\infty)$. This result is often called the Sobolev imbedding theorem. We will show the two cases: when $\Omega = \mathbb{R}^N$ and the case $\Omega \subset \mathbb{R}^N$.

2.5. The case $\Omega = \mathbb{R}^N$

Theorem 6 ***(Sobolev, Gagliardo, Nirenberg)*** *Let $1 \leq p \leq \infty$. Then*

$$W^{1,p}(\mathbb{R}^N) \subset L^{p^*}(\mathbb{R}^N), \tag{2.15}$$

where p^* is given by $\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{N}$, and there exists a constant $C = C(p,N)$ such that

$$\|u\|_{p^*} \leq C\|\nabla u\|_p, \ \forall\, u \in W^{1,p}(\mathbb{R}^N). \tag{2.16}$$

Lemma 2 *Let $N \geq 2$ and let $f_1, f_2, f_3, ..., f_N \in L^{N-1}(\mathbb{R}^{N-1})$. For $x \in \mathbb{R}^N$ and $i \leq p \leq N$ set*

$$\tilde{x}_i = (\, x_1, x_2, ..., x_{i-1}, x_{i+1}, ..., x_N \,) \in \mathbb{R}^{N-1}, \tag{2.17}$$

i.e. x_i is omitted from the list. Then the function

$$f(x) = f_1(\tilde{x}_1) f_2(\tilde{x}_2) ... f_N(\tilde{x}_N), \ x \in \mathbb{R}^N, \tag{2.18}$$

belongs to $L^1(\mathbb{R}^N)$ and

$$\|f\|_{L^1(\mathbb{R}^N)} \leq \prod_{i=1}^{N} \|f_i\|_{L^{N-1}(\mathbb{R}^{N-1})}. \tag{2.19}$$

Corollary 1 *Let* $1 \leq p < N$ *. Then*

$$W^{1,p}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N), \forall\, q \in [p, p^*] \tag{2.20}$$

with continuous injection.

Corollary 2 *(***Limit case** $p = N$ *)*
We have

$$W^{1,p}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N), \forall\, q \in [N, +\infty). \tag{2.21}$$

Theorem 7 ***(Morrey)*** *Let* $p > N$*. Then*

$$W^{1,p}(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N) \tag{2.22}$$

with continuous injection.
Furthermore, for all $u \in W^{1,p}(\mathbb{R}^N)$ *we have*

$$|u(x) - u(y)| \leq C|x-y|^\alpha \|\nabla u\|_p \text{ a.e. } x, y \in \mathbb{R}^N \tag{2.23}$$

where $\alpha = 1 - (N/p)$ *and C is a constant (depending only on p and N).*

Corollary 3 *Let* $m \geq 1$ *be an integer and let* $p \in [1, +\infty)$ *. We have for* $\dfrac{1}{q} = \dfrac{1}{p} - \dfrac{m}{N}$ *:*

$$\begin{cases} W^{m,p}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N), \text{ if } \dfrac{1}{p} - \dfrac{m}{N} > 0 \\ W^{m,p}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N), \forall\, q \in [p, +\infty), \text{ if } \dfrac{1}{p} - \dfrac{m}{N} = 0 \\ W^{m,p}(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N), \text{ if } \dfrac{1}{p} - \dfrac{m}{N} < 0 \end{cases} \tag{2.24}$$

and all these injections are continuous. Moreover, if $m - (N/p) > 0$ *is not an integer set*

$$k = [\, m - (N/p)\,] \text{ and } \theta = m - (N/p) - k\, (0 < \theta < 1). \tag{2.25}$$

We have, for all $u \in W^{m,p}(\mathbb{R})^N$,

$$\|D^{\alpha}u\|_{L^{\infty}(\mathbb{R}^N)} \leq C\,\|u\|_{W^{m,p}(\mathbb{R}^N)},\ \forall\,\alpha \text{ with } |\alpha| \leq k \tag{2.26}$$

and

$$|\,D^{\alpha}u(x) - D^{\alpha}u(y)\,| \leq C\,\|u\|_{W^{m,p}(\mathbb{R}^N)}|x-y|^{\theta} \text{ a.e. } x,y \in \mathbb{R}^N,\ \forall\,\alpha \text{ with } |\alpha| = k. \tag{2.27}$$

In particular, $W^{m,p}(\mathbb{R}^N) \subset C^k(\mathbb{R}^N)$.

2.5. The case $\Omega \subset \mathbb{R}^N$

We suppose that Ω is an open set of class C^1 with bounded boundary or else $\Omega = \mathbb{R}^N_+$.

Corollary 4 *Let* $1 \leq p \leq \infty$. *We have for* $\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{N}$:

$$\begin{cases} W^{1,p}(\Omega) \subset L^{p^*}(\Omega), \text{ if } p < N \\ W^{1,p}(\Omega) \subset L^q(\Omega),\ \forall\,q \in [p,+\infty), \text{ if } p = N \\ W^{1,p}(\Omega) \subset L^{\infty}(\Omega), \text{ if } p > N \end{cases} \tag{2.28}$$

and all these injections are continuous. Moreover, if $p > N$ *we have, for all* $u \in W^{1,p}(\Omega)$,

$$|\,u(x) - u(y)\,| \leq C\,\|\,u\,\|_{W^{1,p}}|\,x-y\,|^{\alpha} \text{ a.e. } x,y \in \Omega \tag{2.29}$$

with $\alpha = 1 - (\,N/p\,)$ *and C depends only one* Ω, p *and* N. *In particular,*

$$W^{1,p}(\Omega) \subset C(\bar{\Omega}) \tag{2.30}$$

2.6. Compactness. Rellich-Kondrachov theorem

Theorem 8 ***(Rellich-Kondrachov)*** *Suppose that* Ω *is bounded and of class* C^1. *Then we have the following compact injections, for* $\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{N}$:

$$\begin{cases} W^{1,p}(\Omega) \subset L^q(\Omega), \forall\, q \in [1, p^*),\ \text{if}\ p < N \\ W^{1,p}(\Omega) \subset L^q(\Omega), \forall\, q \in [p, +\infty),\ \text{if}\ p = N \\ W^{1,p}(\Omega) \subset C(\bar{\Omega}),\ \text{if}\ p > N. \end{cases} \tag{2.31}$$

In particular, $W^{1,p} \subset L^p$ with compact injection for all p and all N.

2.7. Poincaré inequality

Theorem 9 ***(Poincaré inequality)*** *Let Ω be a bounded, connected, open subset of $\mathbb{R}^n$ with a C^1 boundary $\partial\Omega$. Assume $1 \leq p \leq \infty$. Then there exists a constant, depending only on n, p and Ω such that*

$$\| u - (u)_\Omega \|_{L^p(\Omega)} \leq C \| Du \|_{L^p(\Omega)} \tag{2.32}$$

for each function $u \in W^{1,p}(\Omega)$.

Proof

We argue by contradiction. If the stated estimate is false, then

$$\exists, \text{ for each integer } k = 1, 2, \ldots \text{ a function } u_k \in W^{1,p}(\Omega) \tag{2.33}$$

satisfying

$$\| u_k - (u_k)_\Omega \| > k \|Du_k\|_{L^p(\Omega)}. \tag{2.34}$$

We renormalize by defining

$$v_k := \frac{u_k - (u_k)_\Omega}{\| u_k - (u_k)_\Omega \|_{L^p(\Omega)}},\ k = 1, 2, \ldots \tag{2.35}$$

Then

$$(v_k)_\Omega = 0, \| v_k \|_{L^p(\Omega)} = 1 \tag{2.36}$$

and (2.34) implies $\|Dv_k\|_{L^p(\Omega)} < \dfrac{1}{k}, k = 1,2,...$

In particular, the functions $\{v_k\}_{k=1}^{\infty}$ *are bounded in* $W^{1,p}(\Omega)$.
From the Rellich-Kondrachov theorem we have

$$\exists \, \{ \, v_{k_j} \, \}_{j=1}^{\infty} \, \subset \, \{ \, v_k \, \}_{k=1}^{\infty} \tag{2.37}$$

and a function $v \, \in \, L^p(\Omega)$ *such that*

$$v_{k_j} \, \rightarrow \, v \text{ in } L^p(\Omega). \tag{2.38}$$

From (2.35) it follows that

$$(v)_{\Omega} \, = \, 0, \| \, v \, \|_{L^p(\Omega)} \, = \, 1. \tag{2.39}$$

For each $i \, = \, 1,2,...,n$ *and* $\phi \, \in \, C_c^{\infty}(\Omega)$ *that*

$$\int\limits_{\Omega} v\phi_{x_i} \, dx \, = \, \lim_{k_j \to \infty} \int\limits_{\Omega} v_{k_j}\phi_{x_i} \, dx \, = \, - \lim_{k_j \to \infty} \int\limits_{\Omega} v_{k_j,x_i}\phi \, dx \, = \, 0 \tag{2.40}$$

Consequently, $v \, \in \, W^{1,p}(\Omega)$ *, with* $Dv \, = \, 0$ *a.e..*

Thus v *is constant, since* Ω *is connected. However, this conclusion is at variance with (2.39): since v is constant and* $(v)_{\Omega} \, = \, 0$*, we must have* $v \, \equiv \, 0$ *; in which case* $\|v\|_{L^p(\Omega)} \, = \, 0$*. This contradiction establishes estimate (2.32).* ∎

2.8. Generalized Poincaré inequality

Theorem 10 ***(Poincaré inequality on a ball)*** *Assume* $1 \leq \, p \, \leq \infty$*. Then there exists a constant C, depending only on n and p such that*

$$\| \, u \, - \, (u)_{x,r} \, \|_{L^p(B(x,r))} \, \leq \, Cr\| \, Du \, \|_{L^p(B(x,r))} \tag{2.41}$$

for each ball $B(x,r) \, \subset \, \mathbb{R}^n$ *and each function* $u \, \in \, W^{1,p}(B^0(x,r))$.

Proof *The case* $\Omega = B^0(0,1)$ *follows from the Poincaré inequality. In general, if* $u \in W^{1,p}(B^0(x,r))$ *we have*

$$v(y) := u(x + ry), \; y \in B(0,1). \tag{2.42}$$

Then $v \in W^{1,p}(B^0(0,1))$ *and we have*

$$\|v - (v)_{0,1}\|_{L^p(B(0,1))} \leq C\|Dv\|_{L^p(B(0,1))}. \tag{2.43}$$

Changing variables, we can recover estimate (2.41). ∎

2.9. Friedrichs inequality

Theorem 11 ***Friedrichs Inequality*** *Let* Ω *be a bounded Lipschitz domain and let* γ *be a subset of its boundary* $\partial\Omega$. *Suppose that* γ *has a positive Lebesgue measure on* $\partial\Omega$. *Then for any* $\varphi \in W^{1,2}(\Omega,\gamma)$ *the inequality*

$$\| \varphi \|_{L^2(\Omega)} \leq \| \nabla\varphi \|_{L^2(\Omega)} \tag{2.44}$$

holds with a constant C independent of φ . *If* $\gamma = \partial\Omega$ *then the inequality holds for any bounded domain* Ω *and any* $\varphi \in W_0^{1,2}(\Omega)$.

$W^{m,2}(\Omega,\gamma)$ *, for* $m > 0$ *is the completion with respect to the norm of the subspace of* $C^\infty(\bar{\Omega})$ *formed by all functions vanishing in a neighborhood of* γ .

2.10. Traces

Theorem 12 ***(Trace)*** *Let* Ω *be an open bounded regular set of class* C^1 *, or* $\Omega = \mathbb{R}^N_+$. *We define the trace mapping*

$$\begin{cases} W^{1,2}(\Omega) \cap C(\bar{\Omega}) \to L^2(\partial\Omega) \cap C(\bar{\partial\Omega}) \\ v \to \gamma_0(v) = v\big|_{\partial\Omega} \end{cases} \tag{2.45}$$

This mapping γ_0 is extended by continuity to a continuous linear mapping of $W^{1,2}(\Omega)$ into $L^2(\partial\Omega)$, again called γ_0. In particular, there exists a constant $C > 0$ such that, for every function $v \in W^{1,2}(\Omega)$, we have

$$\| v \|_{L^2(\partial\Omega)} \leq C\| v \|_{W^{1,2}(\Omega)}. \tag{2.46}$$

Proof

Let us proof the result for the half-space $\Omega = \mathbb{R}^N_+ = x \in \mathbb{R}^N, x_N > 0$. Let $v \in C_c^\infty(\bar{\mathbb{R}}^N_+)$. With the notation $x = (x', x_N)$, we have

$$|v(x',0)|^2 = -2\int_0^\infty v(x',x_N)\frac{\partial v}{\partial x_N}(x',x_N)\, dx_N, \tag{2.47}$$

and, by using the inequality $2ab \leq a^2 + b^2$,

$$|v(x',0)|^2 \leq \int_0^\infty \left(| v(x',x_N) |^2 + \left| \frac{\partial v}{\partial x_N}(x',x_N) \right|^2 \right) dx_N. \tag{2.48}$$

By integration in x', we deduce

$$\int_{\mathbb{R}^{N-1}} | v(x',0) |^2 \leq \int_{\mathbb{R}^N_+} \left(| v(x) |^2 + \left| \frac{\partial v}{\partial x_N}(x) \right|^2 \right) dx \tag{2.49}$$

that is, $\| v \|_{L^2(\partial\mathbb{R}^N_+)} \leq \| v \|_{W^{1,2}(\mathbb{R}^N_+)}$. By the density of $C_c^\infty(\bar{\mathbb{R}}^N_+)$ in $W^{1,2}(\mathbb{R}^N_+)$, we therefore obtain the result.

For an open bounded regular set of class C^1, we use and argument involving local coordinates on the boundary which allows us to reduce it to the case of $\Omega = \mathbb{R}^N_+$. ∎

CHAPTER

THREE

EXISTENCE AND UNIQUENESS

We will consider the elasticity system with mixed boundary conditions:

$$\begin{cases} \textbf{\textit{div}}\ (2\mu e(u)) + \lambda\ \textbf{\textit{tr}}\ (e(u))\mathbf{I} = f,\ \textit{in}\ \Omega \\ u = 0,\ \textit{on}\ \partial\Omega_D \\ \sigma n = g,\ \textit{on}\ \partial\Omega_N \end{cases} \tag{3.1}$$

Theorem 13 ***(Weak Solution)*** *Let Ω be a regular open bounded connected set of class C^1 of $\mathbb{R}^N$. Let $f \in L^2(\Omega)^N$ and $g \in L^2(\partial\Omega_N)^N$. We define the space*

$$V = \left\{ v \in W^{1,2}(\Omega)^N \text{ such that } v = 0 \text{ on } \partial\Omega_D \right\}. \tag{3.2}$$

There exists a unique (weak) solution $u \in V$ of (3.1) which depends linearly and continuously on the data f and g.
Proof

The space V, defined by (3.2), contains the Dirichlet boundary condition on $\partial\Omega_D$ and is a Hilbert space as a closed subspace of $W^{1,2}(\Omega)^N$. We then obtain the variational formulation: find $u \in V$ such that

$$\int_\Omega 2\mu e(u)\cdot e(v)\,dx + \int_\Omega \lambda\ \textbf{\textit{div}}\ u\ \textbf{\textit{div}}\ v\,dx = \int_\Omega f\cdot v\,dx + \int_{\partial\Omega_N} g\cdot v\,ds,\ \forall\, v \in V. \tag{3.3}$$

To be able to apply the Lax-Milgram theorem to the variational formulation of the elasticity system, the only delicate hypothesis to be verified is once again the coercivity of the bilinear form. In other words, we must show that there exists a constant $C > 0$ such that, for every function $v \in V$, we have

$$\| v \|_{W^{1,2}(\Omega)} \leq C\|e(v)\|_{L^2(\Omega)}. \tag{3.4}$$

First we note that $\| e(v) \|_{L^2(\Omega)} = 0$ implies that $v = 0$. Suppose therefore that $\| e(v) \|_{L^2(\Omega)} = 0$. We check that, if $M \neq 0$, then the points x, solutions of $b + Mx = 0$, form a line in $\mathbb{R}^3$ and a point in $\mathbb{R}^2$. Now $v(x) = 0$ on $\partial\Omega_D$, which has nonzero surface measure, therefore $M = 0$ and $b = 0$. If (3.3) is false, then there exists a sequence $v_n \in V$ such that

$$\| v_n \|_{W^{1,2}(\Omega)} = 1 > n \| e(v_n) \|_{L^2(\Omega)}. \tag{3.5}$$

In particular, the sequence $e(v_n)$ tends to zero in $L^2(\Omega)^{N^2}$. On the other hand, as v_n is bounded in $W^{1,2}(\Omega)^N$, by application of the Rellich theorem, there exists a subsequence v_n which converges in $L^2(\Omega)^N$. Korn's inequality implies that

$$\| v_{n'} - v_{p'} \|^2_{W^{1,2}(\Omega)} \leq C\| v_{n'} - v_{p'} \|^2_{L^2(\Omega)} + \| e(v_{n'}) - e(v_{p'}) \|^2_{L^2(\Omega)} \tag{3.6}$$

from which we deduce that the sequence $v_{n'}$ is Cauchy in $W^{1,2}(\Omega)^N$, and therefore converges to a limit v_∞ which satisfies $\| e(v_\infty) \|_{L^2(\Omega)} = 0$. As this is a norm we deduce that the limit is zero, $v_\infty = 0$, which is a contradiction with the fact that

$$\| v_{n'} \|_{W^{1,2}(\Omega)} = 1. \tag{3.7}$$

The mapping $(f,g) \to u$ is linear. To show that it is continuous from $L^2(\Omega)^N \times L^2(\partial\Omega)^N$ into $W^{1,2}(\Omega)^N$, we take $v = u$ in the variational formulation (3.3). By using the coercivity of the bilinear form and by bounding the linear form above, we obtain the energy estimate

$$C\| u \|^2_{W^{1,2}(\Omega)} \leq \| f \|_{L^2(\Omega)}\| u \|_{L^2(\Omega)} + \| g \|_{L^2(\partial\Omega_N)}\| u \|_{L^2(\partial\Omega)}. \tag{3.8}$$

Thanks to the Poincaré inequality and to the trace theorem, we can bound the term on the right of (3.8) above by $C\Big(\| f \|_{L^2(\Omega)} + \| g \|_{L^2(\partial\Omega_N)} \Big)\| u \|_{W^{1,2}(\Omega)}$, which proves the continuity. ∎

The variational formulation is nothing other than the principle of virtual work in mechanics. Following this analogy, the space V is the space of kinematically admissible displacements v, and the space of symmetric tensors $\mathbf{S} \in L^2(\Omega)^{N^2}$, such that $-\,\boldsymbol{div}\ \boldsymbol{\sigma} = f$ in Ω and $\mathbf{S}n = g$ on $\partial\Omega_N$ is that of tensors of statically admissible stress tensors. For the Laplacian, the solution of the variational formulation (3.3) attains the minimum of a mechanical energy defined for $v \in V$ by

$$J(v) = \frac{1}{2}\int_{\Omega} \left(2\mu|e(v)|^2 + \lambda|\ \boldsymbol{div}\ v\,|^2 \right) dx - \int_{\Omega} f \cdot v\, dx - \int_{\partial\Omega_N} g \cdot v\, ds \tag{3.9}$$

In mechanical terms, $J(v)$ is the sum of the energy of deformation

$$\frac{1}{2}\int_{\Omega} \left(2\mu|e(v)|^2 + \lambda|\ \boldsymbol{div}\ v\,|^2 \right) dx \tag{3.10}$$

and of the potential energy of exterior forces (or work of exterior forces up to a given sign)

$$-\int_{\Omega} f \cdot v\, dx - \int_{\partial\Omega_N} g \cdot v\, ds \tag{3.11}$$

Remark 1 *When the Lamé coefficients are constant and the boundary conditions are Dirichlet and homogeneous, the elasticity equations may be rearranged to the Lamé system:*

$$\begin{cases} -\mu\Delta u - (\,\mu + \lambda\,)\nabla(\,\boldsymbol{div}\ u) = f, \text{ in } \Omega \\ u = 0, \text{ on } \partial\Omega \end{cases} \tag{3.12}$$

3.0. Lax-Milgram Theory

We describe an abstract theory to obtain the existence and the uniqueness of the solution of a variational formulation in a Hilbert space. We denote by V a real Hilbert space with scalar product $<,>$ and norm $\|\cdot\|$. We consider a variational formulation of the type:

$$\textit{find } u \in V, \textit{ such that } a(\,u,v\,) = L(\,v\,), \textit{ for every } v \in V. \tag{3.13}$$

The hypotheses on a and L are:

(i) *Continuous Linear Operator*

$L(\cdot)$ is a continuous linear form on V, that is, $v \rightarrow L(v)$ is linear from V to $\mathbb{R}$ and there $\exists\, C > 0$ such that

$$|L(v)| \leq C\|v\|, \text{ for all } v \in V; \tag{3.14}$$

(ii) *Bilinear Form*

$a(\cdot,\cdot)$ is a bilinear form on V, that is, $w \rightarrow a(w,v)$ is a linear form from V into $\mathbb{R}$ for all $v \in V$; and $v \rightarrow a(w,v)$ is a linear form from V into $\mathbb{R}$ for all $w \in V$;

(iii) *Continuous Bilinear Form*

$a(\cdot,\cdot)$ is continuous, that is, $\exists\, M > 0$ such that

$$|a(w,v)| \leq M\|w\|\|v\|, \text{ for all } w,v \in V; \tag{3.15}$$

(iv) *Coercive (or Elliptic) Bilinear Form*

$a(\cdot,\cdot)$ is coercive (or elliptic), that is, $\exists\, \upsilon > 0$ such that

$$a(v,v) \geq \upsilon\,\|v\|^2, \text{ for all} v \in V \tag{3.16}$$

Theorem 14 ***(Riesz representation theorem)***

Let V be a real Hilbert space, and let $V^{'}$ be its dual. For every continuous linear form $L \in V^{'}$ there exists a unique $y \in V$ such that

$$L(x) = \langle y,x \rangle, \forall\, x \in V. \tag{3.17}$$

Further, we have

$$\|L\|_{V'} = \|y\| \tag{3.18}$$

Proof *Let $M = KerL$. This is a closed subspace of V since L is continuous. If $M = V$, then L is identically zero and we have $y = 0$. If $M \neq V$, then there exists $z \in V \setminus M$. Let z_M be its projection over M. As z does not belong to M, $z - z_M$ is nonzero and, by the theorem of projection over a convex set, is orthogonal to every element of M. Finally, let*

$$z_0 = \frac{z - z_M}{\|z - z_M\|}. \tag{3.19}$$

Every vector $x \in V$ can be written

$$x = w + \lambda z_0 \; with \; \lambda = \frac{L(x)}{\| L(z_0) \|}. \tag{3.20}$$

We see easily that $L(w) = 0$, therefore $w \in M$. This proves that $V = Vect(z_0) \oplus M$.By definition of z_M and of z_0, we have $\langle w, z_0 \rangle = 0$, which implies

$$L(x) = \langle x, z_0 \rangle L(z_0), \tag{3.21}$$

from where then we have the result with $y = L(z_0)z_0$ (the uniqueness is obvious). On the other hand, we have

$$\|y\| = |L(z_0)|, \tag{3.22}$$

and

$$\|L\|_{V'} = \sup_{x \in V, x \neq 0} \frac{|L(s)|}{\|x\|} = L(z_0) \sup_{x \in V, x \neq 0} \frac{\langle x, z_0 \rangle}{\|x\|}. \tag{3.23}$$

The maximum in the last term of this equality is attained by $x = z_0$, which implies that

$$\|L\|_{V'} = \|y\|. \tag{3.24}$$

∎

Theorem 15 ***(Lax - Milgram)***
Let V be a real Hilbert space, $L(\cdot)$ a continuous linear form on V, $a(\cdot,\cdot)$ a continuous coercive bilinear form on V. Then the variational formulation

$$\textit{find } u \in V, \textit{ such that } a(u,v) = L(v), \textit{ for every } v \in V \tag{3.25}$$

has a unique solution. Further, this solution depends continuously on the linear form L.

Proof *For all $w \in V$, the mapping $v \rightarrow a(w,v)$ is a continuous linear form on V:consequently, Riesz representation theorem implies that there exists an element of V, denoted $A(w)$, such that*

$$a(w,v) = \langle A(w), v \rangle, \textit{ for all } v \in V. \tag{3.26}$$

Moreover, the bilinearity of $a(w,v)$ *obviously implies the linearity of the mapping* $w \to A(w)$. *Further, by taking* $v = A(w)$ *, the continuity of* $a(w,v)$ *shows that*

$$\|A(w)\|^2 = a(w,A(w)) \leq M\|w\|\|A(w)\|; \tag{3.27}$$

that is, $\|A(w)\| \leq M\|w\|$ *and therefore* $w \to A(w)$ *is continuous.Another application of the Riesz representation theorem implies that there exists an element of* V *, denoted by* f *, such that* $\|f\|_V = \|L\|_{V'}$ *and*

$$L(v) = \langle f,v \rangle, \textit{for all } v \in V. \tag{3.28}$$

Finally, the variational problem

$$\textit{find } u \in V, \textit{ such that } a(u,v) = L(v), \textit{for every } v \in V \tag{3.29}$$

is equivalent to:

$$\textit{find } u \in V, \textit{ such that } A(u) = f. \tag{3.30}$$

To prove the theorem we must therefore show that the operator A *is bijective from* V *to* V *(which implies the existence and the uniqueness of* u *) and that its inverse is continuous(which proves the continuous dependence of* u *with respect to* L *).The coercivity of* $a(w,v)$ *shows that*

$$\upsilon\|w\|^2 \leq a(w,w) = \langle A(w),w \rangle \leq \|A(w)\|\|w\|, \tag{3.31}$$

which gives

$$\upsilon\|w\| \leq \|A(w)\|, \textit{for allw} \in V, \tag{3.32}$$

that is, A *is injective.To show that* A *is injective, that is,* $Im(A) = V$ *(which is not obvious if* V *is infinite dimensional),it is enough to show that* $Im(A)$ *is closed in V and that* $Im(A)^\perp = 0$. *Indeed, in this case we see that* $V = 0^\perp = (Im(A)^\perp)^\perp = Im\bar{(A)} = Im(A)$, *which proves that* A *is surjective. Let* $A(w_n)$ *be a sequence in* $Im(A)$ *which converges to* b *in* V *. By virtue of*

$$\upsilon\|w\| \leq \|A(w)\|, \textit{for all } w \in V, \tag{3.33}$$

we have

$$\upsilon\|w_n - w_p\| \leq \|A(w_n) - A(w_p)\| \tag{3.34}$$

which tends to zero as n *and* p *tend to infinity. Therefore,* w_n *is a Cauchy sequence in the Hilbert space* V *,that is, it converges to a limit* $w \in V$.

Then, by continuity of A *we deduce that* $A(w_n)$ *converges to* $A(w) = b$ *,that is,* $b \in Im(A)$ *and* $Im(A)$ *is therefore closed. On the other hand, let* $v \in Im(A)^{\perp}$ *; the coercivity of* $a(w,v)$ *implies that*

$$\upsilon\|v\|^2 \leq a(v,v) = \langle A(v),v \rangle = 0, \tag{3.35}$$

that is, $v = 0$ *and* $Im(A)^{\perp} = 0$*, which proves that* A *is bijective.Let* A^{-1} *be its inverse: the inequality*

$$\upsilon\|w\| \leq \|A(w)\|, \textit{for all} w \in V, \tag{3.36}$$

with $w = A^{-1}(v)$ *proves* A^{-1} *is continuous, therefore the solution* u *depends on* f . ∎

CHAPTER

FOUR

ESTIMATES. KORN INEQUALITIES

4.1. First Korn inequality

Theorem 16 *Let* Ω *be a bounded domain of* $\mathbb{R}^N$ *. Then every vector valued function* $u \in W_0^{1,2}(\Omega)$ *satisfies the inequality*

$$\|\nabla u\|_{L^2(\Omega)}^2 \leq 2\|e(u)\|_{L^2(\Omega)}^2 \tag{4.1}$$

Proof *Since* $C_0^\infty(\Omega)$ *is dense in* $W_0^{1,2}(\Omega)$ *, it is sufficient to prove (4.1) for functions* $C_0^\infty(\Omega)$*. By virtue of the Green formula we get*

$$\begin{aligned} \int_\Omega |e(u)|^2\, dx &= \int_\Omega \left(\frac{1}{2}\frac{\partial u_i}{\partial x_h}\frac{\partial u_i}{\partial x_h} + \frac{1}{2}\frac{\partial u_i}{\partial x_h}\frac{\partial u_h}{\partial x_i}\right) dx && (4.2)\\ &= \int_\Omega \frac{1}{2}|\nabla u|^2\, dx - \int_\Omega \frac{1}{2}\frac{\partial^2 u_i}{\partial x_h \partial x_i} u_h\, dx && (4.3)\\ &= \frac{1}{2}\int_\Omega |\nabla u|^2\, dx + \frac{1}{2}\int_\Omega \frac{\partial u_i}{\partial x_i}\frac{\partial u_h}{\partial x_h}\, dx && (4.4) \end{aligned}$$

for any $u \in C_0^\infty(\Omega)$ *. Therefore (4.1) is valid for* u *since the second integral in the right hand side of the last equality is non-negative.* ∎

4.2. Second Korn inequality

For proving this second Korn inequality we will need some preliminary lemmas that we will give here without proofs. The proofs can be found in the book of Oleinik [Ol].

Lemma 3 *Let* $v \in C^\infty(\Omega) \cap L^2(\Omega)$, $\rho^2 \triangle v \in L^2(\Omega)$. *Then* $\rho \nabla v \in L^2(\Omega)$ *and the estimate*

$$\|\rho \nabla v\|_{L^2(\Omega)} \leq C \Big(\|v\|_{L^2(\Omega)} + \|\rho^2 \triangle v\|_{L^2(\Omega)} \Big) \tag{4.5}$$

holds with a constant C independent of v.

Lemma 4 *Let* $w \in C^\infty \in C^\infty(\Omega) \cap L^2(\Omega)$, $\rho \dfrac{\partial^2 w}{\partial x_i \partial x_j} \in L^2(\Omega)$.
Then $w \in W^{1,2}(\Omega)$ *and*

$$\|\nabla w\|_{L^2(\Omega)} \leq C \left[\|w\|_{L^2(\Omega)} + \sum_{i,j=1}^{n} \|\rho \frac{\partial^2 w}{\partial x_i \partial x_j}\|_{L^2(\Omega)} \right] \tag{4.6}$$

where the constant C does not depend on w.

Theorem 17 ***(The second Korn inequality)*** *Let* Ω *be a bounded Lipschitz domain. Then each vector valued function* $u \in W^{1,2}(\Omega)$ *satisfies the inequality*

$$\|u\|_{H^2(\Omega)} \leq C \Big(\|u\|_{L^2(\Omega)} + \|e(u)\|_{L^2(\Omega)} \Big), \; u = (u_1, u_2, \ldots, u_n) \in W^{1,2}(\Omega) \tag{4.7}$$

with a constant C depending only on Ω.

Proof

We can restrict ourselves to the case of $u \in C^\infty(\bar{\Omega})$. *By the definition of the matrix* $e(u)$ *we have*

$$\frac{\partial^2 u_i}{\partial x_j^2} = 2 \frac{\partial}{\partial x_j} e_{ij}(u) - \frac{\partial}{\partial x_i} e_{jj}(u), \tag{4.8}$$

with no summation over i and j. Consider the following equations

$$\Delta v_i = \sum_{j=1}^{n} \left(2 \frac{\partial}{\partial x_j} e_{ij}(u) - \frac{\partial}{\partial x_i} e_{jj}(u) \right) = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} F_j^i. \tag{4.9}$$

Set $F_j^i = 0$ *outside* Ω, $i,j = 1,2,...,n$. *Let* $v_i \in W_0^{1,2}(\overset{\circ}{\Omega}$ *be a solution of the equation (4.9) in a smooth domain* $\overset{\circ}{\Omega}$ *such that* $\bar{\Omega} \subset \overset{\circ}{\Omega}$. *According to the well-known a priori estimate we have*

$$\|v_i\|_{W^{1,2}(\overset{\circ}{\Omega})} \leq C_1 \sum_{j=1}^{n} \|F_j^i\|_{L^2(\Omega)} \leq C_2 \|e(u)\|_{L^2(\Omega)}. \tag{4.10}$$

This inequality can be easily obtained by virtue of the Friedrichs inequality and the integral identity for solutions of the Dirichlet problem for equation (4.9).
Set $v = (v_1, v_2, ..., v_n)^*$, $w = u - v$.
Then

$$\Delta(e_{ij}(w)) = 0 \; in \; \Omega, \; e_{ij}(w) \in C^\infty(\Omega), \; i,j = 1,...,n \tag{4.11}$$

Due to (4.10) we get

$$\|e(w)\|_{L^2(\Omega)} \leq \|e(u)\|_{L^2(\Omega)} + \|e(v)\|_{L^2(\Omega)} \leq C\|e(u)\|_{L^2(\Omega)}, \tag{4.12}$$

where the constant C_3 *does not depend on* u. *Therefore we find that*

$$\| \rho \nabla e_{ij}(w) \|_{L^2(\Omega)} \leq C_4 \|e_{ij}(w)\|_{L^2(\Omega)} \leq C_5 \|e(u)\|_{L^2(\Omega)}. \tag{4.13}$$

We can see that

$$\frac{\partial^2 w_i}{\partial x_p \partial x_l} = \frac{\partial}{\partial x_p} e_{il}(w) + \frac{\partial}{\partial x_l} e_{ip} - \frac{\partial}{\partial x_i} e_{lp}(w). \tag{4.14}$$

Therefore, (4.13) yields the inequality

$$\left\| \rho \sum_{i,j=1}^{n} \left| \frac{\partial^2 w}{\partial x_i \partial x_j} \right| \right\| \leq C_6 \|e(u)\|_{L^2(\Omega)}. \tag{4.15}$$

So we establish

$$\|\nabla w\|_{L^2(\Omega)} \leq C_7 \left[\left\| \rho \sum_{i,j=1}^{n} \left| \frac{\partial^2 w}{\partial x_i \partial x_j} \right| \right\|_{L^2(\Omega)} + \|w\|_{L^2(\Omega)} \right] \tag{4.16}$$

$$\leq C_8 \left(\|e(u)\|_{L^2(\Omega)} + \|w\|_{L^2(\Omega)} \right). \tag{4.17}$$

Since $w = u - v$ *the above estimate implies*

$$\|\nabla u\|_{L^2(\Omega)} \leq C_9 \Big(\|e(u)\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} + \|v\|_{W^{1,2}(\Omega)} \Big) . \tag{4.18}$$

Therefore, owing to (4.10) we find that (4.7) is satisfied. ∎

4.3. Third Korn inequality

Theorem 18 ***(Korn inequality for 1-periodic vector valued functions)***
Let ω *be an unbounded domain in a 1-periodic structure and let* $\omega \cap Q$ *a domain with a Lipschitz boundary. Then for any* $v \in \hat{W}^{1,2}(\omega)$ *such that*

$$\int\limits_{\omega\cap Q} v\, dx = 0, Q = (0,1)^N \tag{4.19}$$

the inequality

$$\|v\|_{W^{1,2}(\omega\cap Q)} \leq C\|e(v)\|_{L^2(\omega\cap Q)} \tag{4.20}$$

holds with a constant C independent of v.
Proof

Denote by V the linear space consisting of all restrictions to $\omega \cap Q$ *of vector valued functions in* $\hat{W}^{1,2}(\omega)$ *satisfying the conditions (4.19). It is easy to see that V is a closed subspace of* $W^{1,2}(\omega \cap Q)$ *and that any rigid displacement 1-periodic in c is a constant vector. Therefore if* $v \in V \cap \mathbb{R}$ *then by virtue of (4.19) we have* $v \equiv 0$ *.* ∎

CHAPTER

FIVE

ASYMPTOTIC EXPANSIONS

5.1. Elastostatics

Let Ω^{ε} be an unbounded domain written in polar coordinates as:

$$
\begin{aligned}
\Omega^{\varepsilon} &= \{(r,\theta) : r > 0, 0 < \theta < 2\pi\} && (5.1) \\
\Omega^{\varepsilon} &= \Omega^{b} \cup \Omega^{c}_{\varepsilon} \cup \Omega^{a}_{\varepsilon} && (5.2) \\
\Omega^{c}_{\varepsilon} &= \{(r,\theta) : 1 < r < R+\varepsilon, 0 < \theta < 2\pi\} && (5.3) \\
\Omega^{c}_{\varepsilon} &= \{(r,\theta) : 1 < r < R+\varepsilon, 0 < \theta < 2\pi\} && (5.4) \\
\Omega^{a}_{\varepsilon} &= \{(r,\theta) : r > R+\varepsilon\} && (5.5)
\end{aligned}
$$

Consider also, the following boundaries :

$$
\begin{aligned}
\gamma^{-} &= \{(r,\theta) : r = 1, 0 < \theta < 2\pi\} && (5.6) \\
\gamma^{+} &= \{(r,\theta) : r = R+\varepsilon, 0 < \theta < 2\pi\} && (5.7)
\end{aligned}
$$

We define the displacement field $\mathbf{U}$ on Ω^{ε} in the following way:

$$\mathbf{U}(r,\theta) \equiv \begin{cases} \mathbf{U^b}, \text{ if } 0 < r < R; \\ \mathbf{U^c}, \text{ if } 1 < r < R+\varepsilon; \\ \mathbf{U^a}, \text{ if } r > R+\varepsilon. \end{cases} \tag{5.8}$$

All displacement fields $\mathbf{U}^b, \mathbf{U}^c, \mathbf{U}^a$ are vector fields, defined as:

$$\mathbf{U}^b : \Omega^b \to \mathbf{R}^2 \tag{5.9}$$
$$\mathbf{U}^b = \left(U_r^b, U_\theta^b\right) \tag{5.10}$$
$$\mathbf{U}^c : \Omega_\varepsilon^c \to \mathbf{R}^2 \tag{5.11}$$
$$\mathbf{U}^c = (U_r^c, U_\theta^c) \tag{5.12}$$
$$\mathbf{U}^a : \Omega_\varepsilon^a \to \mathbf{R}^2 \tag{5.13}$$
$$\mathbf{U}^a = (U_r^a, U_\theta^a) \tag{5.14}$$

and

$$U_\alpha^b : \mathbb{R} \to \mathbb{R}; \alpha \in \{r,\theta\} \tag{5.15}$$
$$U_\alpha^c : \mathbb{R} \to \mathbb{R}; \alpha \in \{r,\theta\} \tag{5.16}$$
$$U_\alpha^a : \mathbb{R} \to \mathbb{R}; \alpha \in \{r,\theta\} \tag{5.17}$$
$$\tag{5.18}$$

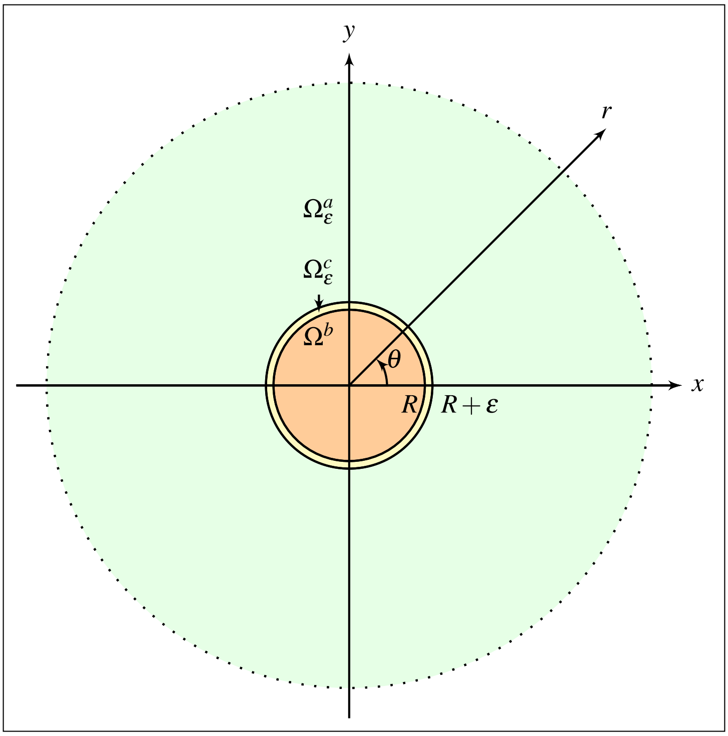


Figure 5.1: *Thin Cylindrical Interphase*

On Ω^ε, we impose the equations of the linearized system of elasticity in the strong form with transmission conditions on the inner and outer boundaries of Ω_ε^c in the form of continuity of displacement and continuity of tractions. For now, no forces are being considered.

$$\begin{cases} -\,\boldsymbol{div}\,\mathbf{S}(\mathbf{U}) = \mathbf{0}, \text{ on } \Omega^\varepsilon; \\ \mathbf{U}^b(R,\theta) = \mathbf{U}^c(R,\theta); \\ \mathbf{U}^c(R+\varepsilon,\theta) = \mathbf{U}^a(R+\varepsilon,\theta); \\ \left(\mathbf{S}(U^b)\cdot\mathbf{n}\right)(R,\theta) = \left(\mathbf{S}(U^c)\cdot\mathbf{n}\right)(R,\theta); \\ \left(\mathbf{S}(U^c)\cdot\mathbf{n}\right)(R+\varepsilon,\theta) = \left(\mathbf{S}(U^a)\cdot\mathbf{n}\right)(R+\varepsilon,\theta). \end{cases} \tag{5.19}$$

$$\mathbf{E} := \frac{1}{2}\left(\nabla\mathbf{U} + (\nabla\mathbf{U})^T\right) \tag{5.20}$$

In polar coordinates, the strain tensor takes the form:

$$\mathbf{E} = \frac{\partial U_r}{\partial r}\mathbf{e}_r \otimes \mathbf{e}_r + \frac{1}{2}\left(\frac{1}{r}\frac{\partial U_r}{\partial \theta} + \frac{\partial U_\theta}{\partial r} - \frac{1}{r}U_\theta\right)(\mathbf{e}_r \otimes \mathbf{e}_\theta + \mathbf{e}_\theta \otimes \mathbf{e}_r) + \frac{1}{r}\left(\frac{\partial U_\theta}{\partial \theta} + U_r\right)\mathbf{e}_\theta \otimes \mathbf{e}_\theta \tag{5.21}$$

so, the components of the strain tensor can be written as:

$$\begin{cases} E_{rr} = \dfrac{\partial U_r}{\partial r}; \\ E_{\theta\theta} = \dfrac{1}{r}\left(\dfrac{\partial U_\theta}{\partial \theta} + U_r\right); \\ E_{r\theta} = E_{\theta r} = \dfrac{1}{2}\left(\dfrac{1}{r}\dfrac{\partial U_r}{\partial \theta} + \dfrac{\partial U_\theta}{\partial r} - \dfrac{1}{r}U_\theta\right). \end{cases} \tag{5.22}$$

The constitutive equation for the stress tensor, for a general anisotropic material is:

$$\{\mathbf{S}\}_{ij} := \mathscr{C}_{ijhk}\{\mathbf{E}\}_{hk} \tag{5.23}$$

where $\mathbb{C}$ is a fourth order tensor called the elasticity tensor. For an isotropic elastic material the constitutive law is knows as Hooke's law and it is written as:

$$\mathbf{S} = \lambda\,(\mathbf{tr}\,(\mathbf{E}))\,\mathbf{I} + 2\mu\mathbf{E} \tag{5.24}$$

$$\mathbf{S} = \begin{bmatrix} \lambda E_{rr} + \lambda E_{\theta\theta} + 2\mu E_{rr} & 2\mu E_{r\theta} \\ 2\mu E_{r\theta} & \lambda E_{rr} + \lambda E_{\theta\theta} + 2\mu E_{\theta\theta} \end{bmatrix} \tag{5.25}$$

So, the components of the stress tensor are:

$$\begin{cases} S_{rr} = (\lambda + 2\mu)\dfrac{\partial U_r}{\partial r} + \lambda\dfrac{1}{r}\dfrac{\partial U_\theta}{\partial \theta} + \lambda\dfrac{1}{r}U_r; \\ S_{\theta\theta} = \lambda\dfrac{\partial U_r}{\partial r} + (\lambda + 2\mu)\dfrac{1}{r}\dfrac{\partial U_\theta}{\partial \theta} + (\lambda + 2\mu)\dfrac{1}{r}U_r; \\ S_{r\theta} = \mu\dfrac{1}{r}\dfrac{\partial U_r}{\partial \theta} + \mu\dfrac{\partial U_\theta}{\partial r} - \mu\dfrac{1}{r}U_\theta. \end{cases} \tag{5.26}$$

We want to study the asymptotic behavior of a cylindrical interphase, occupied by an orthotropic material of elastic coefficients $C_{rr}, C_{r\theta}, C_{\theta\theta}, G_{\theta r}$ included in an infinite isotropic elastic material and surrounding an isotropic elastic material. Next, we characterize the materials that occupy these domains:

The circular inclusion Ω^b

Constitutive equations

$$\mathbf{S} = \lambda_b \left(\mathbf{tr}\left(\mathbf{E}\right)\right)\mathbf{I} + 2\mu_b \mathbf{E} \tag{5.27}$$

$$\mathbf{S} = \begin{bmatrix} \lambda_b E_{rr} + \lambda_b E_{\theta\theta} + 2\mu_b E_{rr} & 2\mu_b E_{r\theta} \\ 2\mu_b E_{r\theta} & \lambda_b E_{rr} + \lambda_b E_{\theta\theta} + 2\mu_b E_{\theta\theta} \end{bmatrix} \tag{5.28}$$

$$\begin{cases} S_{rr} = (\lambda_b + 2\mu_b)\dfrac{\partial U_r^b}{\partial r} + \lambda_b \dfrac{1}{r}\dfrac{\partial U_\theta^b}{\partial \theta} + \lambda_b \dfrac{1}{r} U_r^b; \\ S_{\theta\theta} = \lambda_b \dfrac{\partial U_r^b}{\partial r} + (\lambda_b + 2\mu_b)\dfrac{1}{r}\dfrac{\partial U_\theta^b}{\partial \theta} + (\lambda_b + 2\mu_b)\dfrac{1}{r} U_r^b; \\ S_{r\theta} = \mu_b \dfrac{1}{r}\dfrac{\partial U_r^b}{\partial \theta} + \mu_b \dfrac{\partial U_\theta^b}{\partial r} - \mu_b \dfrac{1}{r} U_\theta^b. \end{cases} \tag{5.29}$$

The equilibrium equations in statics for this material follow the law:

$$\boldsymbol{div}\ \left(\mathbf{S}\left(\mathbf{U}^b\right)\right) = \boldsymbol{div}\ \left(\mathbf{S}\left(\mathbf{U}^b\right)\right)_r \mathbf{e}_r + \boldsymbol{div}\ \left(\mathbf{S}\left(\mathbf{U}^b\right)\right)_\theta \mathbf{e}_\theta = \mathbf{0} \tag{5.30}$$

where

$$\begin{cases} \boldsymbol{div}\ \left[\mathbf{S}\left(\mathbf{U}^b\right)\right]_r = \dfrac{\partial\left[\mathbf{S}_{rr}\left(\mathbf{U}^b\right)\right]}{\partial r} + \dfrac{1}{r}\dfrac{\partial\left[\mathbf{S}_{r\theta}\left(\mathbf{U}^b\right)\right]}{\partial \theta} + \dfrac{\mathbf{S}_{rr}\left(\mathbf{U}^b\right) - \mathbf{S}_{\theta\theta}\left(\mathbf{U}^b\right)}{r} \\ \boldsymbol{div}\ \left[\mathbf{S}\left(\mathbf{U}^b\right)\right]_\theta = \dfrac{\partial\left[\mathbf{S}_{r\theta}\left(\mathbf{U}^b\right)\right]}{\partial r} + \dfrac{1}{r}\dfrac{\partial\left[\mathbf{S}_{\theta\theta}\left(\mathbf{U}^b\right)\right]}{\partial \theta} + \dfrac{2}{r}\mathbf{S}_{r\theta}\left(\mathbf{U}^b\right) \end{cases} \tag{5.31}$$

Using the last relations, equilibrium equations for material Ω^b become

Equilibrium equations

$$\begin{cases} (\lambda_b + 2\mu_b)\dfrac{\partial^2 U_r^b}{\partial r^2} + \mu_b \dfrac{1}{r^2}\dfrac{\partial^2 U_r^b}{\partial \theta^2} - (\lambda_b + 3\mu_b)\dfrac{1}{r^2}\dfrac{\partial U_\theta^b}{\partial \theta} + (\lambda_b + 2\mu_b)\dfrac{1}{r}\dfrac{\partial U_r^b}{\partial r} \\ -(\lambda_b + 2\mu_b)\dfrac{1}{r^2}U_r^b + (\lambda_b + \mu_b)\dfrac{1}{r}\dfrac{\partial^2 U_\theta^b}{\partial r \partial \theta} = 0 \\ \mu_b \dfrac{\partial^2 U_\theta^b}{\partial r^2} + (\lambda_b + 2\mu_b)\dfrac{1}{r^2}\dfrac{\partial^2 U_\theta^b}{\partial \theta^2} + (\lambda_b + 3\mu_b)\dfrac{1}{r^2}\dfrac{\partial U_r^b}{\partial \theta} + \mu_b \dfrac{1}{r}\dfrac{\partial U_\theta^b}{\partial r} \\ -\mu_b \dfrac{1}{r^2}U_\theta^b + (\lambda_b + \mu_b)\dfrac{1}{r}\dfrac{\partial^2 U_r^b}{\partial \theta \partial r} = 0 \end{cases} \tag{5.32}$$

For Ω^b, the stiffness matrix $\mathscr{C}$ is given by

$$\mathbb{C}^b = \begin{bmatrix} \lambda_b + 2\mu_b & \lambda_b & 0 \\ \lambda_b & \lambda_b + 2\mu_b & 0 \\ 0 & 0 & 2\mu_b \end{bmatrix} \tag{5.33}$$

Infinite media Ω^a_ε

Constitutive Equations

We will denote the set of points occupied by this material by Ω^a_ε. It is an isotropic material of Lamé modulus λ_a and shear modulus μ_a. The prescribed displacement for this material will be denoted by $\mathbf{U}^a$, which, in polar coordinates can be represented as $\left(U^a_r, U^a_\theta\right)$.

The constitutive equations for this material follow Hooke's law for isotropic materials.

$$\mathbf{S} = \lambda_a \left(\mathbf{trE}\right)\mathbf{I} + 2\mu_a \mathbf{E} \tag{5.34}$$

$$\mathbf{S} = \begin{bmatrix} \lambda_a E_{rr} + \lambda_a E_{\theta\theta} + 2\mu_a E_{rr} & 2\mu_a E_{r\theta} \\ 2\mu_a E_{r\theta} & \lambda_a E_{rr} + \lambda_a E_{\theta\theta} + 2\mu_a E_{\theta\theta} \end{bmatrix} \tag{5.35}$$

$$\begin{cases} S_{rr} = (\lambda_a + 2\mu_a)\dfrac{\partial U^a_r}{\partial r} + \lambda_a \dfrac{1}{r}\dfrac{\partial U^a_\theta}{\partial \theta} + \lambda_a \dfrac{1}{r} U^a_r; \\ S_{\theta\theta} = \lambda_a \dfrac{\partial U^a_r}{\partial r} + (\lambda_a + 2\mu_a)\dfrac{1}{r}\dfrac{\partial U^a_\theta}{\partial \theta} + (\lambda_a + 2\mu_a)\dfrac{1}{r} U^a_r; \\ S_{r\theta} = \mu_a \dfrac{1}{r}\dfrac{\partial U^a_r}{\partial \theta} + \mu_a \dfrac{\partial U^a_\theta}{\partial r} - \mu_a \dfrac{1}{r} U^a_\theta. \end{cases} \tag{5.36}$$

Equilibrium equations

The equilibrium equations are:

$$\boldsymbol{div}\ \left[\mathbf{S}\left(\mathbf{U}^b\right)\right] = \boldsymbol{div}\ \left[\mathbf{S}\left(\mathbf{U}^a\right)\right]_r \mathbf{e}_r + \boldsymbol{div}\ \left[\mathbf{S}\left(\mathbf{U}^a\right)\right]_\theta \mathbf{e}_\theta = \mathbf{0} \tag{5.37}$$

where

$$
\begin{cases}
\boldsymbol{div}\ [\mathbf{S}(\mathbf{U}^a)]_r = \dfrac{\partial\,[\mathbf{S}_{rr}(\mathbf{U}^a)]}{\partial r} + \dfrac{1}{r}\dfrac{\partial\,[\mathbf{S}_{r\theta}(\mathbf{U}^a)]}{\partial\theta} + \dfrac{\mathbf{S}_{rr}(\mathbf{U}^a) - \mathbf{S}_{\theta\theta}(\mathbf{U}^a)}{r} \\
\boldsymbol{div}\ [\mathbf{S}(\mathbf{U}^a)]_\theta = \dfrac{\partial\,[\mathbf{S}_{r\theta}(\mathbf{U}^a)]}{\partial r} + \dfrac{1}{r}\dfrac{\partial\,[\mathbf{S}_{\theta\theta}(\mathbf{U}^a)]}{\partial\theta} + \dfrac{2}{r}\mathbf{S}_{r\theta}(\mathbf{U}^a)
\end{cases}
\tag{5.38}
$$

Using the last relations, equilibrium equations for material Ω^a are

$$
\begin{cases}
(\lambda_a + 2\mu_a)\dfrac{\partial^2 U_r^a}{\partial r^2} + \mu_a \dfrac{1}{r^2}\dfrac{\partial^2 U_r^a}{\partial\theta^2} - (\lambda_a + 3\mu_a)\dfrac{1}{r^2}\dfrac{\partial U_\theta^a}{\partial\theta} \\
+(\lambda_a + 2\mu_a)\dfrac{1}{r}\dfrac{\partial U_r^a}{\partial r} - (\lambda_a + 2\mu_a)\dfrac{1}{r^2}U_r^a + (\lambda_a + \mu_a)\dfrac{1}{r}\dfrac{\partial^2 U_\theta^a}{\partial r\partial\theta} = 0 \\
\mu_a \dfrac{\partial^2 U_\theta^a}{\partial r^2} + (\lambda_a + 2\mu_a)\dfrac{1}{r^2}\dfrac{\partial^2 U_\theta^a}{\partial\theta^2} + (\lambda_a + 3\mu_a)\dfrac{1}{r^2}\dfrac{\partial U_r^a}{\partial\theta} \\
+\mu_a \dfrac{1}{r}\dfrac{\partial U_\theta^a}{\partial r} - \mu_a \dfrac{1}{r^2}U_\theta^a + (\lambda_a + \mu_a)\dfrac{1}{r}\dfrac{\partial^2 U_r^a}{\partial\theta\partial r} = 0
\end{cases}
\tag{5.39}
$$

For Ω^a, the stiffness matrix is given by

$$
\mathbb{C}^a = \begin{bmatrix} \lambda_a + 2\mu_a & \lambda_a & 0 \\ \lambda_a & \lambda_a + 2\mu_a & 0 \\ 0 & 0 & 2\mu_a \end{bmatrix}
\tag{5.40}
$$

Cylindrical Interphase Ω^c

Constitutive equations

The constitutive equations of this material which is orthotropic will be written in what follows. The prescribed displacement in Ω^a will be denoted by U^c which, in polar coordinates can be written as $\left(U_r^c, U_\theta^c\right)$ The interphase is an orthotropic material characterized by elastic coefficients $C_{rr}, C_{r\theta}, C_{\theta\theta}, G_{\theta r}$. So, the stiffness matrix C for this material is

$$
\mathbb{C}^c = \begin{bmatrix} C_{rr} & C_{r\theta} & 0 \\ C_{r\theta} & C_{\theta\theta} & 0 \\ 0 & 0 & 2G_{\theta r} \end{bmatrix}
\tag{5.41}
$$

Writing the stress and strain tensors in Voigt notation we will have

$$\mathbf{S} = \begin{bmatrix} S_{rr} & S_{r\theta} \\ S_{r\theta} & S_{\theta\theta} \end{bmatrix} \mapsto \begin{bmatrix} S_{rr} \\ S_{\theta\theta} \\ \sqrt{2}S_{r\theta} \end{bmatrix} \tag{5.42}$$

$$\mathbf{E} = \begin{bmatrix} E_{rr} & E_{r\theta} \\ E_{r\theta} & E_{\theta\theta} \end{bmatrix} \mapsto \begin{bmatrix} E_{rr} \\ E_{\theta\theta} \\ \sqrt{2}E_{r\theta} \end{bmatrix} \tag{5.43}$$

This way we can write the constitutive equations of material Ω^c by the constitutive law

$$\mathbf{S} = C_{\Omega^c}\mathbf{E} \tag{5.44}$$

So, in matrix form, using the Voigt notation, we write

$$\begin{bmatrix} S_{rr} \\ S_{\theta\theta} \\ \sqrt{2}S_{r\theta} \end{bmatrix} = \begin{bmatrix} C_{rr} & C_{r\theta} & 0 \\ C_{r\theta} & C_{\theta\theta} & 0 \\ 0 & 0 & 2G_{\theta r} \end{bmatrix} \begin{bmatrix} E_{rr} \\ E_{\theta\theta} \\ \sqrt{2}E_{r\theta} \end{bmatrix} \tag{5.45}$$

$$\begin{cases} S_{rr} = C_{rr}E_{rr} + C_{r\theta}E_{\theta\theta}; \\ S_{\theta\theta} = C_{r\theta}E_{rr} + C_{\theta\theta}E_{\theta\theta}; \\ S_{r\theta} = 2G_{\theta r}E_{r\theta}. \end{cases} \tag{5.46}$$

Using the components of the strain in terms of displacement, we can rewrite the constitutive equations of the second material:

$$\begin{cases} S_{rr} = C_{rr}\dfrac{\partial U_r^c}{\partial r} + C_{r\theta}\dfrac{1}{r}\left(\dfrac{\partial U_\theta^c}{\partial \theta} + U_r^c\right); \\ S_{\theta\theta} = C_{r\theta}\dfrac{\partial U_r^c}{\partial r} + C_{\theta\theta}\dfrac{1}{r}\left(\dfrac{\partial U_\theta^c}{\partial \theta} + U_r^c\right); \\ S_{r\theta} = G_{\theta r}\left(\dfrac{1}{r}\dfrac{\partial U_r^c}{\partial \theta} + \dfrac{\partial U_\theta^c}{\partial r} - \dfrac{1}{r}U_\theta^c\right). \end{cases} \tag{5.47}$$

In component form the constitutive equations of the elastic material on Ω^c, are:

$$
\begin{cases}
S_{rr} = C_{rr}\dfrac{\partial U_r^c}{\partial r} + C_{r\theta}\dfrac{1}{r}\dfrac{\partial U_\theta^c}{\partial \theta} + C_{r\theta}\dfrac{1}{r}U_r^c; \\
S_{\theta\theta} = C_{r\theta}\dfrac{\partial U_r^c}{\partial r} + C_{\theta\theta}\dfrac{1}{r}\dfrac{\partial U_\theta^c}{\partial \theta} + C_{\theta\theta}\dfrac{1}{r}U_r^c; \\
S_{r\theta} = G_{\theta r}\dfrac{1}{r}\dfrac{\partial U_r^c}{\partial \theta} + G_{\theta r}\dfrac{\partial U_\theta^c}{\partial r} - G_{\theta r}\dfrac{1}{r}U_\theta^c.
\end{cases}
\tag{5.48}
$$

Equilibrium equations

Using these last relations and replacing them in the equilibrium equations for an elastic material, when no forces are applied, we get:

$$
\begin{cases}
C_{rr}\dfrac{\partial^2 U_r^c}{\partial r^2} - \dfrac{1}{r}C_{r\theta}\dfrac{\partial U_\theta^c}{\partial \theta} + \dfrac{1}{r}C_{r\theta}\dfrac{\partial^2 U_\theta^c}{\partial r \partial \theta} - \dfrac{1}{r^2}C_{r\theta}U_r^c \\
+\dfrac{1}{r}C_{r\theta}\dfrac{\partial U_r^c}{\partial r} + \dfrac{1}{r}G_{\theta r}\dfrac{\partial^2 U_r^c}{\partial \theta^2} + G_{\theta r}\dfrac{\partial^2 U_\theta^c}{\partial r \partial \theta} - \dfrac{1}{r}G_{\theta r}\dfrac{\partial U_\theta^c}{\partial \theta} \\
+\dfrac{1}{r}(C_{rr} - C_{r\theta})\dfrac{\partial U_r^c}{\partial r} + \dfrac{1}{r^2}(C_{r\theta} - C_{\theta\theta})\dfrac{\partial U_\theta^c}{\partial \theta} + \dfrac{1}{r^2}(C_{r\theta} - C_{\theta\theta})U_r^c = 0 \\
-\dfrac{1}{r^2}G_{\theta r}\dfrac{\partial U_r^c}{\partial \theta} + \dfrac{1}{r}G_{\theta r}\dfrac{\partial^2 U_r^c}{\partial r \partial \theta} + G_{\theta r}\dfrac{\partial^2 U_\theta^c}{\partial r^2} + \dfrac{1}{r^2}G_{\theta r}U_\theta^c \\
-\dfrac{1}{r}G_{\theta r}\dfrac{\partial U_\theta^c}{\partial r} + \dfrac{1}{r}C_{r\theta}\dfrac{\partial^2 U_r^c}{\partial r \partial \theta} + \dfrac{1}{r^2}C_{\theta\theta}\dfrac{\partial^2 U_\theta^c}{\partial \theta^2} + \dfrac{1}{r^2}C_{\theta\theta}\dfrac{\partial U_r^c}{\partial \theta} \\
+\dfrac{2}{r}G_{\theta r}\dfrac{\partial U_r^c}{\partial \theta} + 2G_{\theta r}\dfrac{\partial U_\theta^c}{\partial r} - \dfrac{2}{r}G_{\theta r}U_\theta^c = 0
\end{cases}
\tag{5.49}
$$

In order to perform an asymptotic study of this system, we need to rescale the thin interphase to a domain of normal dimensions. By that we mean that we need to move the parameter ε from the geometry of the system into the elasticity operator.

Defining the *fast* variable $\rho = \dfrac{r - R}{\varepsilon}$ then we will have to change from $U(r, \theta)$ to $U(\rho, \theta)$, where R is the radius of the circular inclusion and ε is the thickness of the interphase.

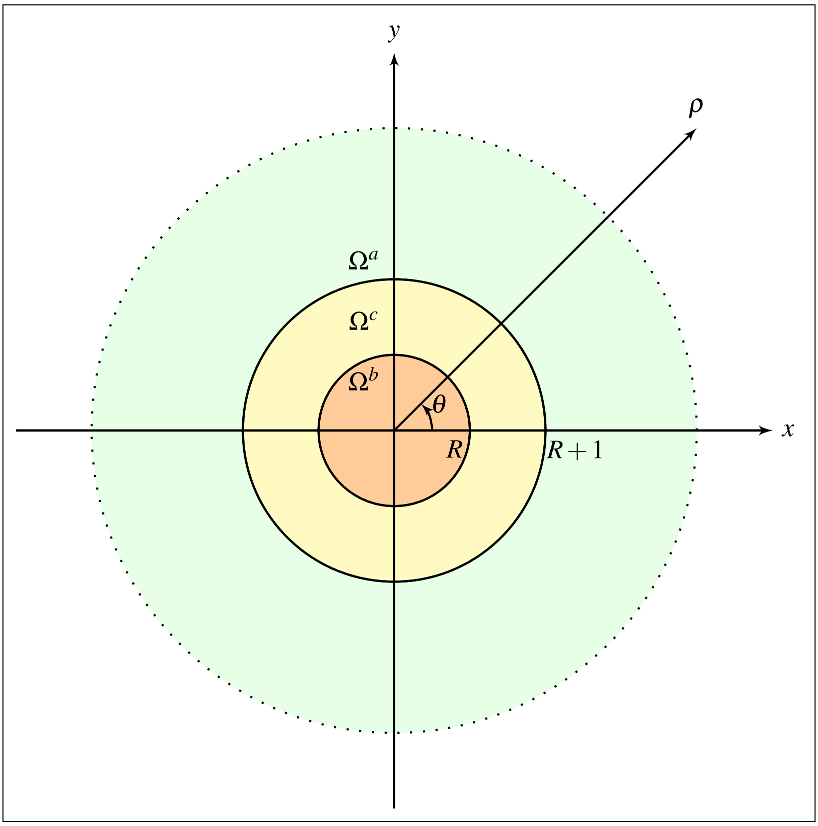


Figure 5.2: *Rescaled Cylindrical Interphase*

$$\begin{cases} \rho = \dfrac{r-R}{\varepsilon} \Rightarrow r = (\rho\varepsilon + R) \\ \dfrac{1}{r} = \dfrac{1}{(\rho\varepsilon + R)} \end{cases} \tag{5.50}$$

By the chain rule, we will have the transformations:

$$\frac{\partial^2 U_r^c}{\partial r^2} = \frac{\partial}{\partial \rho}(\frac{\partial U_r^c}{\partial \rho}\frac{\partial \rho}{\partial r})\frac{\partial \rho}{\partial r} + \frac{\partial}{\partial \theta}(\frac{\partial U_r^c}{\partial \rho}\frac{\partial \rho}{\partial r})\frac{\partial \theta}{\partial r} + \frac{\partial}{\partial \rho}(\frac{\partial U_r^c}{\partial \theta}\frac{\partial \theta}{\partial r})\frac{\partial \rho}{\partial r} + \frac{\partial}{\partial \theta}(\frac{\partial U_r^c}{\partial \theta}\frac{\partial \theta}{\partial r})\frac{\partial \theta}{\partial r}$$
$$= \frac{1}{\varepsilon^2}\frac{\partial^2 U_r^c}{\partial \rho^2}$$
$$\frac{\partial^2 U_r^c}{\partial \theta^2} = \frac{\partial}{\partial \rho}(\frac{\partial U_r^c}{\partial \rho}\frac{\partial \rho}{\partial \theta})\frac{\partial \rho}{\partial \theta} + \frac{\partial}{\partial \theta}(\frac{\partial U_r^c}{\partial \rho}\frac{\partial \rho}{\partial \theta})\frac{\partial \theta}{\partial \theta} + \frac{\partial}{\partial \rho}(\frac{\partial U_r^c}{\partial \theta}\frac{\partial \theta}{\partial \theta})\frac{\partial \rho}{\partial \theta} + \frac{\partial}{\partial \theta}(\frac{\partial U_r^c}{\partial \theta}\frac{\partial \theta}{\partial \theta})\frac{\partial \theta}{\partial \theta}$$

$$= \frac{\partial^2 U_r^c}{\partial \theta^2}$$

$$\frac{\partial^2 U_\theta^c}{\partial r \partial \theta} = \frac{\partial}{\partial \rho}(\frac{\partial U_\theta^c}{\partial \rho}\frac{\partial \rho}{\partial r})\frac{\partial \rho}{\partial \theta} + \frac{\partial}{\partial \theta}(\frac{\partial U_\theta^c}{\partial \rho}\frac{\partial \rho}{\partial r})\frac{\partial \theta}{\partial \theta} + \frac{\partial}{\partial \rho}(\frac{\partial U_\theta^c}{\partial \theta}\frac{\partial \theta}{\partial r})\frac{\partial \rho}{\partial \theta} + \frac{\partial}{\partial \theta}(\frac{\partial U_\theta^c}{\partial \theta}\frac{\partial \theta}{\partial r})\frac{\partial \theta}{\partial \theta}$$

$$= \frac{1}{\varepsilon}\frac{\partial^2 U_\theta^c}{\partial \theta^2}$$

$$\frac{\partial U_r^c}{\partial r} = \frac{\partial U_r^c}{\partial \rho}\frac{\partial \rho}{\partial r} + \frac{\partial U_r^c}{\partial \theta}\frac{\partial \theta}{\partial r} = \frac{1}{\varepsilon}\frac{\partial U_r^c}{\partial \rho}$$

$$\frac{\partial U_\theta^c}{\partial \theta} = \frac{\partial U_\theta^c}{\partial \rho}\frac{\partial \rho}{\partial \theta} + \frac{\partial U_\theta^c}{\partial \theta}\frac{\partial \theta}{\partial \theta} = \frac{\partial U_\theta^c}{\partial \theta}$$

$$\frac{\partial^2 U_\theta^c}{\partial r^2} = \frac{\partial}{\partial \rho}(\frac{\partial U_\theta^c}{\partial \rho}\frac{\partial \rho}{\partial r})\frac{\partial \rho}{\partial r} + \frac{\partial}{\partial \theta}(\frac{\partial U_\theta^c}{\partial \rho}\frac{\partial \rho}{\partial r})\frac{\partial \theta}{\partial r} + \frac{\partial}{\partial \rho}(\frac{\partial U_\theta^c}{\partial \theta}\frac{\partial \theta}{\partial r})\frac{\partial \rho}{\partial r} + \frac{\partial}{\partial \theta}(\frac{\partial U_\theta^c}{\partial \theta}\frac{\partial \theta}{\partial r})\frac{\partial \theta}{\partial r}$$

$$= \frac{1}{\varepsilon^2}\frac{\partial^2 U_\theta^c}{\partial \rho^2}$$

$$\frac{\partial^2 U_\theta^c}{\partial \theta^2} = \frac{\partial}{\partial \rho}(\frac{\partial U_\theta^c}{\partial \rho}\frac{\partial \rho}{\partial \theta})\frac{\partial \rho}{\partial \theta} + \frac{\partial}{\partial \theta}(\frac{\partial U_\theta^c}{\partial \rho}\frac{\partial \rho}{\partial \theta})\frac{\partial \theta}{\partial \theta} + \frac{\partial}{\partial \rho}(\frac{\partial U_\theta^c}{\partial \theta}\frac{\partial \theta}{\partial \theta})\frac{\partial \rho}{\partial \theta} + \frac{\partial}{\partial \theta}(\frac{\partial U_\theta^c}{\partial \theta}\frac{\partial \theta}{\partial \theta})\frac{\partial \theta}{\partial \theta}$$

$$= \frac{\partial^2 U_\theta^c}{\partial \theta^2}$$

$$\frac{\partial^2 U_r^c}{\partial r \partial \theta} = \frac{\partial}{\partial \theta}(\frac{\partial U_r^c}{\partial r}) = \frac{\partial}{\partial \theta}(\frac{1}{\varepsilon}\frac{\partial U_r^c}{\partial \rho}) = \frac{\partial}{\partial \rho}(\frac{1}{\varepsilon}\frac{\partial U_r^c}{\partial \rho})\frac{\partial \rho}{\partial \theta} + \frac{\partial}{\partial \theta}(\frac{1}{\varepsilon}\frac{\partial U_r^c}{\partial \rho})\frac{\partial \theta}{\partial \theta}$$

$$= \frac{1}{\varepsilon}\frac{\partial^2 U_r^c}{\partial \rho \partial \theta}$$

$$\frac{\partial^2 U_r^c}{\partial r \partial \theta} = \frac{\partial}{\partial \rho}(\frac{\partial U_r^c}{\partial \rho}\frac{\partial \rho}{\partial r})\frac{\partial \rho}{\partial \theta} + \frac{\partial}{\partial \theta}(\frac{\partial U_r^c}{\partial \rho}\frac{\partial \rho}{\partial r})\frac{\partial \theta}{\partial \theta} + \frac{\partial}{\partial \rho}(\frac{\partial U_r^c}{\partial \theta}\frac{\partial \theta}{\partial r})\frac{\partial \rho}{\partial \theta} + \frac{\partial}{\partial \theta}(\frac{\partial U_r^c}{\partial \theta}\frac{\partial \theta}{\partial r})\frac{\partial \theta}{\partial \theta}$$

$$= \frac{1}{\varepsilon}\frac{\partial^2 U_r^c}{\partial \rho \partial \theta}$$

$$\frac{\partial U_\theta^c}{\partial r} = \frac{\partial U_\theta^c}{\partial \rho}\frac{\partial \rho}{\partial r} + \frac{\partial U_\theta^c}{\partial \theta}\frac{\partial \theta}{\partial r} = \frac{1}{\varepsilon}\frac{\partial U_\theta^c}{\partial \rho}$$

$$\frac{\partial U_r^c}{\partial \theta} = \frac{\partial U_r^c}{\partial \rho}\frac{\partial \rho}{\partial \theta} + \frac{\partial U_r^c}{\partial \theta}\frac{\partial \theta}{\partial \theta} = \frac{\partial U_r^c}{\partial \theta}$$

With the change of coordinate, we will also introduce the asymptotic expansions:

$$U_r^c = \sum_{n=0}^{\infty} \varepsilon^n (U_r^c)_n$$

$$U_\theta^c = \sum_{n=0}^{\infty} \varepsilon^n (U_\theta^c)_n$$

$$\frac{\partial^2 U_r^c}{\partial \rho^2} = \frac{\partial^2}{\partial \rho^2} U_r^c = \frac{\partial^2}{\partial \rho^2} \left(\sum_{n=0}^{\infty} \varepsilon^n (U_r^c)_n \right) = \sum_{n=0}^{\infty} \left(\varepsilon^n \frac{\partial^2}{\partial \rho^2} (U_r^c)_n \right) = \sum_{n=0}^{\infty} \left(\varepsilon^n \frac{\partial^2 (U_r^c)_n}{\partial \rho^2} \right)$$

$$\frac{\partial^2 U_r^c}{\partial \theta^2} = \frac{\partial^2}{\partial \theta^2} U_r^c = \frac{\partial^2}{\partial \theta^2} \left(\sum_{n=0}^{\infty} \varepsilon^n (U_r^c)_n \right) = \sum_{n=0}^{\infty} \left(\varepsilon^n \frac{\partial^2}{\partial \theta^2} (U_r^c)_n \right) = \sum_{n=0}^{\infty} \left(\varepsilon^n \frac{\partial^2 (U_r^c)_n}{\partial \theta^2} \right)$$

$$\frac{\partial U_r^c}{\partial \rho} = \frac{\partial}{\partial \rho} U_r^c = \frac{\partial}{\partial \rho} \left(\sum_{n=0}^{\infty} \varepsilon^n (U_r^c)_n \right) = \sum_{n=0}^{\infty} \left(\varepsilon^n \frac{\partial}{\partial \rho} (U_r^c)_n \right) = \sum_{n=0}^{\infty} \left(\varepsilon^n \frac{\partial (U_r^c)_n}{\partial \rho} \right)$$

$$\frac{\partial U_r^c}{\partial \theta} = \frac{\partial}{\partial \theta} U_r^c = \frac{\partial}{\partial \theta} \left(\sum_{n=0}^{\infty} \varepsilon^n (U_r^c)_n \right) = \sum_{n=0}^{\infty} \left(\varepsilon^n \frac{\partial}{\partial \theta} (U_r^c)_n \right) = \sum_{n=0}^{\infty} \left(\varepsilon^n \frac{\partial (U_r^c)_n}{\partial \theta} \right)$$

$$\frac{\partial U_\theta^c}{\partial \rho} = \frac{\partial}{\partial \rho} U_\theta^c = \frac{\partial}{\partial \rho} \left(\sum_{n=0}^{\infty} \varepsilon^n (U_\theta^c)_n \right) = \sum_{n=0}^{\infty} \left(\varepsilon^n \frac{\partial}{\partial \rho} (U_\theta^c)_n \right) = \sum_{n=0}^{\infty} \left(\varepsilon^n \frac{\partial (U_\theta^c)_n}{\partial \rho} \right)$$

$$\frac{\partial U_\theta^c}{\partial \theta} = \frac{\partial}{\partial \theta} U_\theta^c = \frac{\partial}{\partial \theta} \left(\sum_{n=0}^{\infty} \varepsilon^n (U_\theta^c)_n \right) = \sum_{n=0}^{\infty} \left(\varepsilon^n \frac{\partial}{\partial \theta} (U_\theta^c)_n \right) = \sum_{n=0}^{\infty} \left(\varepsilon^n \frac{\partial (U_\theta^c)_n}{\partial \theta} \right)$$

$$\frac{\partial^2 U_\theta^c}{\partial \rho^2} = \frac{\partial^2}{\partial \rho^2} U_\theta^c = \frac{\partial^2}{\partial \rho^2} \left(\sum_{n=0}^{\infty} \varepsilon^n (U_\theta^c)_n \right) = \sum_{n=0}^{\infty} \left(\varepsilon^n \frac{\partial^2}{\partial \rho^2} (U_\theta^c)_n \right) = \sum_{n=0}^{\infty} \left(\varepsilon^n \frac{\partial^2 (U_\theta^c)_n}{\partial \rho^2} \right)$$

$$\frac{\partial^2 U_\theta^c}{\partial \theta^2} = \frac{\partial^2}{\partial \theta^2} U_\theta^c = \frac{\partial^2}{\partial \theta^2} \left(\sum_{n=0}^{\infty} \varepsilon^n (U_\theta^c)_n \right) = \sum_{n=0}^{\infty} \left(\varepsilon^n \frac{\partial^2}{\partial \theta^2} (U_\theta^c)_n \right) = \sum_{n=0}^{\infty} \left(\varepsilon^n \frac{\partial^2 (U_\theta^c)_n}{\partial \theta^2} \right)$$

$$\frac{\partial^2 U_r^c}{\partial \rho \partial \theta} = \frac{\partial^2}{\partial \rho \partial \theta} U_r^c = \frac{\partial^2}{\partial \rho \partial \theta} \left(\sum_{n=0}^{\infty} \varepsilon^n (U_r^c)_n \right) = \sum_{n=0}^{\infty} \left(\varepsilon^n \frac{\partial^2}{\partial \rho \partial \theta} (U_r^c)_n \right) = \sum_{n=0}^{\infty} \left(\varepsilon^n \frac{\partial^2 (U_r^c)_n}{\partial \rho \partial \theta} \right)$$

$$\frac{\partial^2 U_\theta^c}{\partial \rho \partial \theta} = \frac{\partial^2}{\partial \rho \partial \theta} U_\theta^c = \frac{\partial^2}{\partial \rho \partial \theta} \left(\sum_{n=0}^{\infty} \varepsilon^n (U_\theta^c)_n \right) = \sum_{n=0}^{\infty} \left(\varepsilon^n \frac{\partial^2}{\partial \rho \partial \theta} (U_\theta^c)_n \right) = \sum_{n=0}^{\infty} \left(\varepsilon^n \frac{\partial^2 (U_\theta^c)_n}{\partial \rho \partial \theta} \right)$$

$$\frac{1}{(\rho\varepsilon + R)} = \frac{1}{R} \frac{1}{\left(\frac{\rho\varepsilon}{R} + 1 \right)} = \frac{1}{R} \sum_{n=0}^{\infty} (-1)^n \left(\frac{\rho\varepsilon}{R} \right)^n = \frac{1}{R} \sum_{n=0}^{\infty} (-1)^n \varepsilon^n \left(\frac{\rho}{R} \right)^n$$

$$\frac{1}{(\rho\varepsilon + R)^2} = \frac{1}{R^2} \frac{1}{\left(\frac{\rho\varepsilon}{R} + 1 \right)^2} = \frac{1}{R^2} \sum_{n=0}^{\infty} (-1)^n (n+1) \left(\frac{\rho}{R} \right)^n \varepsilon^n$$

Taking in account the change of variable and the asymptotic expansions, the system can be written as:

$$\begin{cases}\frac{1}{\varepsilon^2}C_{rr}\frac{\partial^2 U_r^c}{\partial\rho^2}+\frac{1}{\varepsilon}C_{r\theta}\frac{1}{(\rho\varepsilon+R)}\frac{\partial^2 U_\theta^c}{\partial\rho\partial\theta}+\frac{1}{\varepsilon}C_{r\theta}\frac{1}{(\rho\varepsilon+R)}\frac{\partial U_r^c}{\partial\rho}+\frac{1}{\varepsilon}G_{\theta r}\frac{\partial^2 U_\theta^c}{\partial\rho\partial\theta}+\\ \frac{1}{\varepsilon}(C_{rr}-C_{r\theta})\frac{1}{(\rho\varepsilon+R)}\frac{\partial U_r^c}{\partial\rho}-C_{r\theta}\frac{1}{(\rho\varepsilon+R)}\frac{\partial U_\theta^c}{\partial\theta}-C_{r\theta}\frac{1}{(\rho\varepsilon+R)^2}U_r^c+G_{\theta r}\frac{1}{(\rho\varepsilon+R)}\frac{\partial^2 U_r^c}{\partial\theta^2}\\ -G_{\theta r}\frac{1}{(\rho\varepsilon+R)}\frac{\partial U_\theta^c}{\partial\theta}+(C_{r\theta}-C_{\theta\theta})\frac{1}{(\rho\varepsilon+R)^2}\frac{\partial U_\theta^c}{\partial\theta}+(C_{r\theta}-C_{\theta\theta})\frac{1}{(\rho\varepsilon+R)^2}U_r^c=0\\ \frac{1}{\varepsilon^2}G_{\theta r}\frac{\partial^2 U_\theta^c}{\partial\rho^2}+\frac{1}{\varepsilon}G_{\theta r}\frac{1}{(\rho\varepsilon+R)}\frac{\partial^2 U_r^c}{\partial\rho\partial\theta}-\frac{1}{\varepsilon}G_{\theta r}\frac{1}{(\rho\varepsilon+R)}\frac{\partial U_\theta^c}{\partial\rho}+\frac{1}{\varepsilon}C_{r\theta}\frac{1}{(\rho\varepsilon+R)}\frac{\partial^2 U_r^c}{\partial\rho\partial\theta}+\\ \frac{1}{\varepsilon}2G_{\theta r}\frac{\partial U_\theta^c}{\partial\rho}-G_{\theta r}\frac{1}{(\rho\varepsilon+R)^2}\frac{\partial U_r^c}{\partial\theta}+G_{\theta r}\frac{1}{(\rho\varepsilon+R)^2}U_\theta^c+C_{\theta\theta}\frac{1}{(\rho\varepsilon+R)^2}\frac{\partial^2 U_\theta^c}{\partial\theta^2}\\ +C_{\theta\theta}\frac{1}{(\rho\varepsilon+R)^2}\frac{\partial U_r^c}{\partial\theta}+G_{\theta r}\frac{2}{(\rho\varepsilon+R)}\frac{\partial U_r^c}{\partial\theta}-G_{\theta r}\frac{2}{(\rho\varepsilon+R)}U_\theta^c=0\end{cases} \tag{5.51}$$

$$\begin{cases}\frac{1}{\varepsilon^2}C_{rr}\frac{\partial^2 U_r^c}{\partial\rho^2}+\frac{1}{\varepsilon}[C_{r\theta}\frac{1}{(\rho\varepsilon+R)}\frac{\partial^2 U_\theta^c}{\partial\rho\partial\theta}+C_{r\theta}\frac{1}{(\rho\varepsilon+R)}\frac{\partial U_r^c}{\partial\rho}\\ +G_{\theta r}\frac{\partial^2 U_\theta^c}{\partial\rho\partial\theta}+(C_{rr}-C_{r\theta})\frac{1}{(\rho\varepsilon+R)}\frac{\partial U_r^c}{\partial\rho}]+[-C_{r\theta}\frac{1}{(\rho\varepsilon+R)}\frac{\partial U_\theta^c}{\partial\theta}\\ -C_{r\theta}\frac{1}{(\rho\varepsilon+R)^2}U_r^c+G_{\theta r}\frac{1}{(\rho\varepsilon+R)}\frac{\partial^2 U_r^c}{\partial\theta^2}-G_{\theta r}\frac{1}{(\rho\varepsilon+R)}\frac{\partial U_\theta^c}{\partial\theta}\\ +(C_{r\theta}-C_{\theta\theta})\frac{1}{(\rho\varepsilon+R)^2}\frac{\partial U_\theta^c}{\partial\theta}+(C_{r\theta}-C_{\theta\theta})\frac{1}{(\rho\varepsilon+R)^2}U_r^c]=0|\varepsilon^2\\ \frac{1}{\varepsilon^2}G_{\theta r}\frac{\partial^2 U_\theta^c}{\partial\rho^2}+\frac{1}{\varepsilon}[G_{\theta r}\frac{1}{(\rho\varepsilon+R)}\frac{\partial^2 U_r^c}{\partial\rho\partial\theta}-G_{\theta r}\frac{1}{(\rho\varepsilon+R)}\frac{\partial U_\theta^c}{\partial\rho}\\ +C_{r\theta}\frac{1}{(\rho\varepsilon+R)}\frac{\partial^2 U_r^c}{\partial\rho\partial\theta}+2G_{\theta r}\frac{\partial U_\theta^c}{\partial\rho}]+[-G_{\theta r}\frac{1}{(\rho\varepsilon+R)^2}\frac{\partial U_r^c}{\partial\theta}\\ +G_{\theta r}\frac{1}{(\rho\varepsilon+R)^2}U_\theta^c+C_{\theta\theta}\frac{1}{(\rho\varepsilon+R)^2}\frac{\partial^2 U_\theta^c}{\partial\theta^2}+C_{\theta\theta}\frac{1}{(\rho\varepsilon+R)^2}\frac{\partial U_r^c}{\partial\theta}\\ +G_{\theta r}\frac{2}{(\rho\varepsilon+R)}\frac{\partial U_r^c}{\partial\theta}-G_{\theta r}\frac{2}{(\rho\varepsilon+R)}U_\theta^c]=0|\varepsilon^2\end{cases} \tag{5.52}$$

$$\begin{cases} C_{rr}\dfrac{\partial^2 U_r^c}{\partial\rho^2}+\varepsilon[C_{r\theta}\dfrac{1}{(\rho\varepsilon+R)}\dfrac{\partial^2 U_\theta^c}{\partial\rho\partial\theta}+C_{r\theta}\dfrac{1}{(\rho\varepsilon+R)}\dfrac{\partial U_r^c}{\partial\rho}+G_{\theta r}\dfrac{\partial^2 U_\theta^c}{\partial\rho\partial\theta} \\ +(C_{rr}-C_{r\theta})\dfrac{1}{(\rho\varepsilon+R)}\dfrac{\partial U_r^c}{\partial\rho}]+\varepsilon^2[-C_{r\theta}\dfrac{1}{(\rho\varepsilon+R)}\dfrac{\partial U_\theta^c}{\partial\theta}-C_{r\theta}\dfrac{1}{(\rho\varepsilon+R)^2}U_r^c \\ +G_{\theta r}\dfrac{1}{(\rho\varepsilon+R)}\dfrac{\partial^2 U_r^c}{\partial\theta^2}-G_{\theta r}\dfrac{1}{(\rho\varepsilon+R)}\dfrac{\partial U_\theta^c}{\partial\theta}+(C_{r\theta}-C_{\theta\theta})\dfrac{1}{(\rho\varepsilon+R)^2}\dfrac{\partial U_\theta^c}{\partial\theta} \\ +(C_{r\theta}-C_{\theta\theta})\dfrac{1}{(\rho\varepsilon+R)^2}U_r^c]=0 \\ G_{\theta r}\dfrac{\partial^2 U_\theta^c}{\partial\rho^2}+\varepsilon[G_{\theta r}\dfrac{1}{(\rho\varepsilon+R)}\dfrac{\partial^2 U_r^c}{\partial\rho\partial\theta}-G_{\theta r}\dfrac{1}{(\rho\varepsilon+R)}\dfrac{\partial U_\theta^c}{\partial\rho}+C_{r\theta}\dfrac{1}{(\rho\varepsilon+R)}\dfrac{\partial^2 U_r^c}{\partial\rho\partial\theta} \\ +2G_{\theta r}\dfrac{\partial U_\theta^c}{\partial\rho}]+\varepsilon^2[-G_{\theta r}\dfrac{1}{(\rho\varepsilon+R)^2}\dfrac{\partial U_r^c}{\partial\theta}+G_{\theta r}\dfrac{1}{(\rho\varepsilon+R)^2}U_\theta^c+C_{\theta\theta}\dfrac{1}{(\rho\varepsilon+R)^2}\dfrac{\partial^2 U_\theta^c}{\partial\theta^2} \\ +C_{\theta\theta}\dfrac{1}{(\rho\varepsilon+R)^2}\dfrac{\partial U_r^c}{\partial\theta}+G_{\theta r}\dfrac{2}{(\rho\varepsilon+R)}\dfrac{\partial U_r^c}{\partial\theta}-G_{\theta r}\dfrac{2}{(\rho\varepsilon+R)}U_\theta^c]=0 \end{cases} \tag{5.53}$$

$$\frac{1}{(\rho\varepsilon+R)}=\frac{1}{R}\frac{1}{\left(\frac{\rho\varepsilon}{R}+1\right)}=\frac{1}{R}\sum_{n=0}^{\infty}(-1)^n\left(\frac{\rho\varepsilon}{R}\right)^n=\frac{1}{R}\sum_{n=0}^{\infty}\varepsilon^n\left(-\frac{\rho}{R}\right)^n \tag{5.54}$$

and

$$U_r^c=\sum_{n=0}^{\infty}\varepsilon^n\,(U_r^c)_n \tag{5.55}$$

then, using the Cauchy product for formal power series, we get

$$\frac{1}{(\rho\varepsilon+R)}U_r^c=\frac{1}{R}\sum_{n=0}^{\infty}c_n\varepsilon^n \tag{5.56}$$

where

$$c_n=\sum_{k=0}^{n}\left(-\frac{\rho\varepsilon}{R}\right)^k(U_r^c)_{n-k} \tag{5.57}$$

With the same formula, having

$$\frac{1}{(\rho\varepsilon+R)}=\sum_{n=0}^{\infty}\varepsilon^n\left(-\frac{\rho}{R}\right)^n \tag{5.58}$$

and

$$\frac{\partial^2 U_\theta^c}{\partial\rho\partial\theta} = \sum_{n=0}^{\infty} \varepsilon^n \frac{\partial^2 (U_\theta^c)_n}{\partial\rho\partial\theta} \tag{5.59}$$

we get

$$\frac{1}{(\rho\varepsilon + R)} \frac{\partial^2 U_\theta^c}{\partial\rho\partial\theta} = \sum_{n=0}^{\infty} a_n \varepsilon^n \tag{5.60}$$

where

$$a_n = \sum_{k=0}^{n} \left(-\frac{\rho}{R}\right)^k \frac{\partial^2 \left(U_\theta^c\right)_{n-k}}{\partial\rho\partial\theta} \tag{5.61}$$

$$\begin{cases} C_{rr} \sum\limits_{n=0}^{\infty} \left(\varepsilon^k \dfrac{\partial^2 (U_r^c)_k}{\partial\rho^2} \right) + \varepsilon [C_{r\theta} \dfrac{1}{R} \sum\limits_{n=0}^{\infty} A_n \varepsilon^n + C_{r\theta} \dfrac{1}{R} \sum\limits_{n=0}^{\infty} B_n \varepsilon^n \\ + G_{\theta r} \sum\limits_{n=0}^{\infty} \varepsilon^n \dfrac{\partial^2 (U_\theta^c)_n}{\partial\rho\partial\theta} + (C_{rr} - C_{r\theta}) \sum\limits_{n=0}^{\infty} C_n \varepsilon^n] \\ + \varepsilon^2 [-C_{r\theta} \dfrac{1}{R} \sum\limits_{n=0}^{\infty} D_n \varepsilon^n - C_{r\theta} \dfrac{1}{R^2} \sum\limits_{n=0}^{\infty} E_n \varepsilon^n + G_{\theta r} \dfrac{1}{R} \sum\limits_{n=0}^{\infty} F_n \varepsilon^n \\ - G_{\theta r} \dfrac{1}{R} \sum\limits_{n=0}^{\infty} G_n \varepsilon^n + (C_{r\theta} - C_{\theta\theta}) \dfrac{1}{R^2} \sum\limits_{n=0}^{\infty} H_n \varepsilon^n + (C_{r\theta} - C_{\theta\theta}) \dfrac{1}{R^2} \sum\limits_{n=0}^{\infty} I_n \varepsilon^n] = 0 \\ G_{\theta r} \sum\limits_{k=0}^{\infty} \left(\varepsilon^k \dfrac{\partial^2 (U_\theta^c)_k}{\partial\rho^2} \right) + \varepsilon [G_{\theta r} \dfrac{1}{R} \sum\limits_{n=0}^{\infty} J_n \varepsilon^n - G_{\theta r} \dfrac{1}{R} \sum\limits_{n=0}^{\infty} K_n \varepsilon^n \\ + C_{r\theta} \dfrac{1}{R} \sum\limits_{n=0}^{\infty} L_n \varepsilon^n + 2 G_{\theta r} \sum\limits_{n=0}^{\infty} \varepsilon^n \dfrac{\partial (U_\theta^c)_n}{\partial\rho}] + \varepsilon^2 [-G_{\theta r} \dfrac{1}{R^2} \sum\limits_{n=0}^{\infty} M_n \varepsilon^n \\ + G_{\theta r} \dfrac{1}{R^2} \sum\limits_{n=0}^{\infty} N_n \varepsilon^n + C_{\theta\theta} \dfrac{1}{R^2} \sum\limits_{n=0}^{\infty} O_n \varepsilon^n \\ + C_{\theta\theta} \dfrac{1}{R^2} \sum\limits_{n=0}^{\infty} P_n \varepsilon^n + G_{\theta r} \dfrac{1}{R} \sum\limits_{n=0}^{\infty} R_n \varepsilon^n - G_{\theta r} \dfrac{1}{R} \sum\limits_{n=0}^{\infty} S_n \varepsilon^n] = 0 \end{cases} \tag{5.62}$$

Written in more detail, which is necessary for separating ε-dependent components, we get:

$$
\begin{cases}
C_{rr}[\left(\varepsilon^0 \dfrac{\partial^2 (U_r^c)_0}{\partial \rho^2}\right) + \left(\varepsilon^1 \dfrac{\partial^2 (U_r^c)_1}{\partial \rho^2}\right) + ...] + C_{r\theta}\dfrac{1}{R}[A_0\varepsilon^1 + A_1\varepsilon^2 + ...] \\
+C_{r\theta}\dfrac{1}{R}[B_0\varepsilon^1 + B_1\varepsilon^2 + ...] + G_{\theta r}[\varepsilon^1 \dfrac{\partial^2 (U_\theta^c)_0}{\partial \rho \partial \theta} + \varepsilon^2 \dfrac{\partial^2 (U_\theta^c)_1}{\partial \rho \partial \theta} + ...] \\
+(C_{rr} - C_{r\theta})[C_0\varepsilon^1 + C_1\varepsilon^2 + ...] - C_{r\theta}\dfrac{1}{R}[D_0\varepsilon^2 + D_1\varepsilon^3 + ...] - C_{r\theta}\dfrac{1}{R^2}[E_0\varepsilon^2 + ...] \\
+G_{\theta r}\dfrac{1}{R}[F_0\varepsilon^2 + F_1\varepsilon^3 + ...] - G_{\theta r}\dfrac{1}{R}[G_0\varepsilon^2 + G_1\varepsilon^3 + ...] \\
+(C_{r\theta} - C_{\theta\theta})\dfrac{1}{R^2}[H_0\varepsilon^2 + H_1\varepsilon^3 + ...] + (C_{r\theta} - C_{\theta\theta})\dfrac{1}{R^2}[I_0\varepsilon^2 + I_1\varepsilon^3 + ...] = 0 \\
G_{\theta r}[\left(\varepsilon^0 \dfrac{\partial^2 (U_\theta^c)_0}{\partial \rho^2}\right) + \left(\varepsilon^1 \dfrac{\partial^2 (U_\theta^c)_1}{\partial \rho^2}\right) + \left(\varepsilon^2 \dfrac{\partial^2 (U_\theta^c)_2}{\partial \rho^2}\right) + ...] \\
+[G_{\theta r}\dfrac{1}{R}[J_0\varepsilon^1 + J_1\varepsilon^2 + ...] - G_{\theta r}\dfrac{1}{R}[K_0\varepsilon^1 + K_1\varepsilon^2 + ...] + C_{r\theta}\dfrac{1}{R}[L_0\varepsilon^1 + L_1\varepsilon^2 + ...] \\
+2G_{\theta r}[\varepsilon^1 \dfrac{\partial (U_\theta^c)_0}{\partial \rho} + \varepsilon^2 \dfrac{\partial (U_\theta^c)_n}{\partial \rho} + ...] + [-G_{\theta r}\dfrac{1}{R^2}[M_0\varepsilon^2 + ...] + G_{\theta r}\dfrac{1}{R^2}[N_0\varepsilon^2 \\
+N_1\varepsilon^3 + ...] + C_{\theta\theta}\dfrac{1}{R^2}[O_0\varepsilon^2 + O_1\varepsilon^3 + ...] + C_{\theta\theta}\dfrac{1}{R^2}[P_0\varepsilon^2 + P_1\varepsilon^3 + ...] \\
+G_{\theta r}\dfrac{1}{R}[R_0\varepsilon^2 + R_1\varepsilon^3 + ...] - G_{\theta r}\dfrac{1}{R}[S_0\varepsilon^2 + ...] = 0
\end{cases}
\tag{5.63}
$$

Well known is, that even if divergent, asymptotic expansions give good approximation at a local level. This is also why, we perform a study of the solution in the thin domain. Nontheless, a higher order approximation can be performed, but the output would be less explicit.

$$
\begin{cases}
\varepsilon^0 \left(C_{rr}\dfrac{\partial^2 (U_r^c)_0}{\partial \rho^2}\right) + \varepsilon^1 \left(C_{rr}\dfrac{\partial^2 (U_r^c)_1}{\partial \rho^2} + C_{r\theta}\dfrac{1}{R}A_0 + C_{r\theta}\dfrac{1}{R}B_0 + ...\right) \\
+\varepsilon^2 \left(C_{rr}\dfrac{\partial^2 (U_r^c)_2}{\partial \rho^2} + G_{\theta r}\dfrac{1}{R}A_1 - G_{\theta r}\dfrac{1}{R}B_1 + ...\right) + \varepsilon^3 \left(C_{rr}\dfrac{\partial^2 (U_r^c)_3}{\partial \rho^2} + ...\right) = 0 \\
\varepsilon^0 \left(G_{\theta r}\dfrac{\partial^2 (U_r^c)_0}{\partial \rho^2}\right) + \varepsilon^1 \left(G_{\theta r}\dfrac{\partial^2 (U_r^c)_1}{\partial \rho^2} + G_{\theta r}\dfrac{1}{R}J_0 + G_{\theta r}\dfrac{1}{R}K_0 + ...\right) \\
+\varepsilon^2 \left(G_{\theta r}\dfrac{\partial^2 (U_r^c)_2}{\partial \rho^2} + G_{\theta r}\dfrac{1}{R}J_1 - G_{\theta r}\dfrac{1}{R}K_1 + ...\right) + \varepsilon^3 \left(G_{\theta r}\dfrac{\partial^2 (U_r^c)_3}{\partial \rho^2} + ...\right) = 0
\end{cases}
\tag{5.64}
$$

Here, we choose to approximate the functions in the first term, for a clearer output of the approximated solution. The system thought, at this point, reduces to the form:

$$\begin{cases} \varepsilon^0 \left(C_{rr} \dfrac{\partial^2 (U_r^c)_0}{\partial \rho^2} \right) = 0 \\ \varepsilon^0 \left(G_{\theta r} \dfrac{\partial^2 (U_r^c)_0}{\partial \rho^2} \right) = 0 \end{cases} \tag{5.65}$$

which obviously gives

$$\begin{cases} C_{rr} \dfrac{\partial^2 (U_r^c)_0}{\partial \rho^2} = 0 \\ G_{\theta r} \dfrac{\partial^2 (U_r^c)_0}{\partial \rho^2} = 0 \end{cases} \tag{5.66}$$

and so, the solution can be written in an approximated form as:

$$\begin{cases} (U_r^c)_0 = A(\theta)\rho + B(\theta) \\ \\ (U_\theta^c)_0 = A(\theta)\rho + B(\theta) \end{cases} \tag{5.67}$$

Stiff Material

Considering on the thin subdomain, an elastic material with elastic properties being ε-dependent, we can obtain two critical cases for the elasticity of the system: that of the material occupying the thin domain either as extremely soft or extremely thin. This type of study is sometimes called in the literature explosion of coefficients.

The goal is to describe in a formal way the asymptotic behavior of a system of linear elasticity when one of the components reaches an extremal high or low elasticity. The first one considers extremely stiff materials whereas the second one take in account very soft materials. We intend to show in these two critical cases, the direct dependence on our small parameter in the asymptotic behavior of the system. We show that the tractions which are imposed continuous in our boundary value problem, to have different limit behaviors.

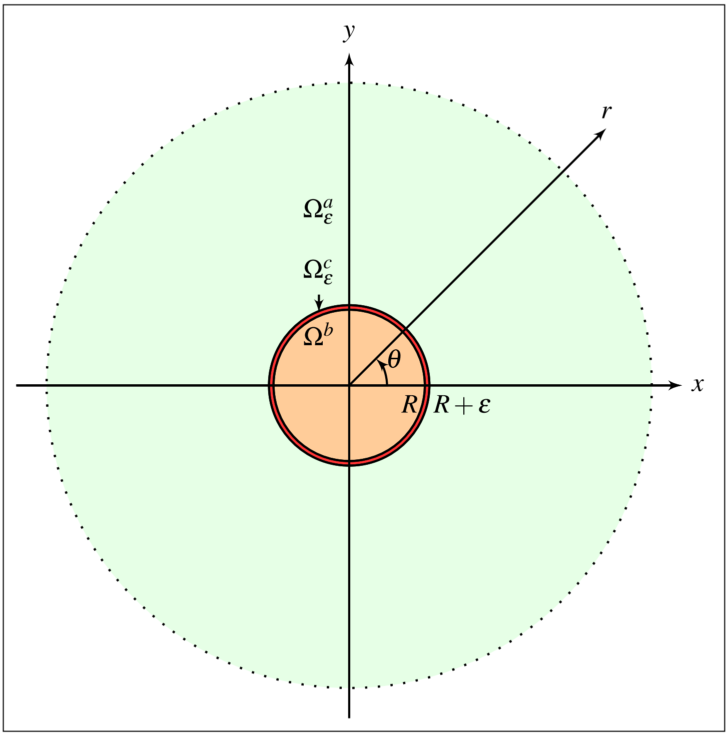


Figure 5.3: *Thin Stiff Cylindrical Interphase*

Considering that the second material becomes stiffer than the inclusion and infinite media, we will consider the proper rescaling of the elastic coefficients, as

$$\begin{cases} C_{rr} = \dfrac{1}{\varepsilon} C^*_{rr} \\ C_{r\theta} = \dfrac{1}{\varepsilon} C^*_{r\theta} \\ C_{\theta\theta} = \dfrac{1}{\varepsilon} C^*_{\theta\theta} \\ G_{\theta r} = \dfrac{1}{\varepsilon} G^*_{\theta r} \end{cases} \tag{5.68}$$

we get the solution

$$\begin{cases} \dfrac{\partial^2 (U_r^c)_0}{\partial \rho^2} = 0 \\ \dfrac{\partial^2 (U_\theta^c)_0}{\partial \rho^2} = 0 \end{cases} \tag{5.69}$$

and so,

$$\begin{cases} (U_r^c)_0 = A(\theta)\rho + B(\theta) \\ (U_\theta^c)_0 = A(\theta)\rho + B(\theta) \end{cases} \tag{5.70}$$

Soft Material

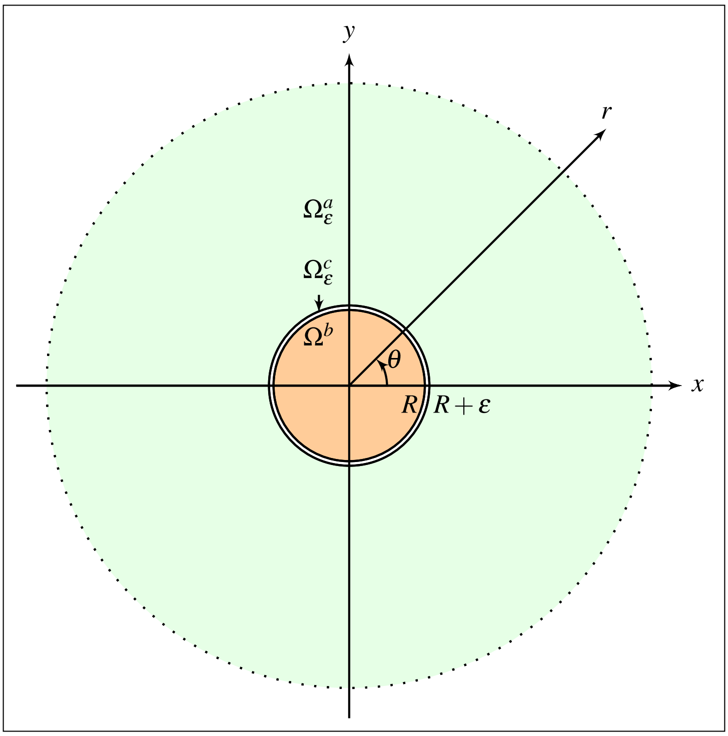


Figure 5.4: *Thin Soft Cylindrical Interphase*

Considering that the second matrial becomes softer than the inclusion and infinite media, we will consider the proper rescaling of the elastic coefficients, as

$$\begin{cases} C_{rr} = \varepsilon C^*_{rr} \\ C_{r\theta} = \varepsilon C^*_{r\theta} \\ C_{\theta\theta} = \varepsilon C^*_{\theta\theta} \\ G_{\theta r} = \varepsilon G^*_{\theta r} \end{cases} \tag{5.71}$$

In the leading term, this system can be written as:

$$\begin{cases} C^*_{rr}\dfrac{\partial^2 (U^c_r)_0}{\partial \rho^2} = 0 \\ G^*_{\theta r}\dfrac{\partial^2 (U^c_\theta)_0}{\partial \rho^2} = 0 \end{cases} \tag{5.72}$$

And so, a solution from this, can be written as:

$$\begin{cases} \dfrac{\partial^2 (U^c_r)_0}{\partial \rho^2} = 0 \\ \dfrac{\partial^2 (U^c_\theta)_0}{\partial \rho^2} = 0 \end{cases} \tag{5.73}$$

and so, the solution we recover in the statics case is:

$$\begin{cases} (U^c_r)_0 = A(\theta)\rho + B(\theta) \\ (U^c_\theta)_0 = A(\theta)\rho + B(\theta) \end{cases} \tag{5.74}$$

5.1. Limit Transmission conditions

$$\mathbf{S}(\mathbf{U}^c)\cdot\mathbf{n} = \frac{1}{\varepsilon}\begin{bmatrix} C_{rr}\dfrac{\partial (U^c_r)_0}{\partial \rho} n_r \\ G_{\theta r}\dfrac{\partial (U^c_\theta)_0}{\partial \rho} n_\theta \end{bmatrix} = \frac{1}{\varepsilon}\begin{bmatrix} C_{rr}A(\theta)\, n_r \\ G_{\theta r}C(\theta)\, n_\theta \end{bmatrix} \tag{5.75}$$

The stress-strain relation, in the limit can be written in terms of definition of the derivative:

$$\begin{cases} \dfrac{\varepsilon}{C_{rr}}\mathbf{S}_{rr}\left((U_r^c)_0\right) = \dfrac{\partial (U_r^c)_0}{\partial r} = \lim\limits_{\varepsilon\to 0}\dfrac{(U_r^c)_0|_{r=R+\varepsilon} - (U_r^c)_0|_{r=R}}{\varepsilon} \\ \dfrac{\varepsilon}{G_{\theta r}}\mathbf{S}_{r\theta}\left((U_\theta^c)_0\right) = \dfrac{\partial (U_\theta^c)_0}{\partial r} = \lim\limits_{\varepsilon\to 0}\dfrac{(U_\theta^c)_0|_{r=R+\varepsilon} - (U_\theta^c)_0|_{r=R}}{\varepsilon} \end{cases} \tag{5.76}$$

$$\begin{cases} \dfrac{\varepsilon}{C_{rr}}\mathbf{S}_{rr}\left((U_r^c)_0\right) = (U_r^a)_0|_{\rho=1} - (U_r^b)_0|_{\rho=0} \\ \dfrac{\varepsilon}{G_{\theta r}}\mathbf{S}_{r\theta}\left((U_\theta^c)_0\right) = (U_\theta^a)_0|_{\rho=1} - (U_\theta^b)_0|_{\rho=0} \end{cases} \tag{5.77}$$

$$\begin{cases} (U_r^c)_0|_{\rho=0} = (U_r^c)_0|_{\rho=0} + \varepsilon\dfrac{\partial (U_r^c)_0}{\partial r}|_{\rho=0} + \mathscr{O}\left(\varepsilon^2\right) \\ (U_\theta^c)_0|_{\rho=0} = (U_\theta^c)_0|_{\rho=0} + \varepsilon\dfrac{\partial (U_\theta^c)_0}{\partial r}|_{\rho=0} + \mathscr{O}\left(\varepsilon^2\right) \end{cases} \tag{5.78}$$

Taking only the first term in this expansion we have

$$\begin{cases} \dfrac{\varepsilon}{C_{rr}}\mathbf{S}_{rr}\left((U_r^b)_0|_{\rho=0}\right) = (U_r^a)_0|_{\rho=0} - (U_r^b)_0|_{\rho=0} = [\![(U_r^c)_0]\!] \\ \dfrac{\varepsilon}{G_{\theta r}}\mathbf{S}_{r\theta}\left((U_\theta^b)_0|_{\rho=0}\right) = (U_\theta^a)_0|_{\rho=0} - (U_\theta^b)_0|_{\rho=0} = [\![(U_\theta^c)_0]\!] \end{cases} \tag{5.79}$$

Stiff Material

$$\mathbf{S}(\mathbf{U}^c)\cdot\mathbf{n} = \frac{1}{\varepsilon}\begin{bmatrix} C_{rr}\dfrac{\partial (U_r^c)_0}{\partial \rho}n_r \\ G_{\theta r}\dfrac{\partial (U_\theta^c)_0}{\partial \rho}n_\theta \end{bmatrix} = \frac{1}{\varepsilon}\begin{bmatrix} \dfrac{1}{\varepsilon}C_{rr}^*A(\theta)\,n_r \\ \dfrac{1}{\varepsilon}G_{\theta r}^*C(\theta)\,n_\theta \end{bmatrix} = \frac{1}{\varepsilon^2}\begin{bmatrix} C_{rr}^*A(\theta)\,n_r \\ G_{\theta r}^*C(\theta)\,n_\theta \end{bmatrix} \tag{5.80}$$

The stress-strain relation, in the limit can be written in terms of definition of the derivative:

$$\begin{cases} \dfrac{1}{\varepsilon^2}C_{rr}^*\mathbf{S}_{rr}\left((U_r^c)_0\right) = \dfrac{\partial (U_r^c)_0}{\partial r} = \lim\limits_{\varepsilon\to 0}\dfrac{((U_r^c)_0)_{r=R+\varepsilon} - ((U_r^c)_0)_{r=R}}{\varepsilon} \\ \dfrac{1}{\varepsilon^2}G_{\theta r}^*\mathbf{S}_{r\theta}\left((U_\theta^c)_0\right) = \dfrac{\partial (U_\theta^c)_0}{\partial r} = \lim\limits_{\varepsilon\to 0}\dfrac{\left((U_\theta^c)_0\right)_{r=R+\varepsilon} - \left((U_\theta^c)_0\right)_{r=R}}{\varepsilon} \end{cases} \tag{5.81}$$

$$
\begin{cases}
\dfrac{\varepsilon^2}{C^*_{rr}}\mathbf{S}_{rr}\left((U^c_r)_0\right) = (U^a_r)_0|_{\rho=1} - (U^b_r)_0|_{\rho=0} \\
\dfrac{\varepsilon^2}{G^*_{\theta r}}\mathbf{S}_{r\theta}\left((U^c_\theta)_0\right) = (U^a_\theta)_0|_{\rho=1} - (U^b_\theta)_0|_{\rho=0}
\end{cases} \tag{5.82}
$$

$$
\begin{cases}
(U^c_r)_0|_{\rho=0} = (U^c_r)_0|_{\rho=0} + \varepsilon\dfrac{\partial (U^c_r)_0}{\partial r}|_{\rho=0} + \mathscr{O}\left(\varepsilon^2\right) \\
(U^c_\theta)_0|_{\rho=0} = (U^c_\theta)_0|_{\rho=0} + \varepsilon\dfrac{\partial (U^c_\theta)_0}{\partial r}|_{\rho=0} + \mathscr{O}\left(\varepsilon^2\right)
\end{cases} \tag{5.83}
$$

Taking only the first term in this expansion we have

$$
\begin{cases}
\dfrac{\varepsilon^2}{C^*_{rr}}\mathbf{S}_{rr}\left((U^b_r)_0|_{\rho=0}\right) = (U^a_r)_0|_{\rho=0} - (U^b_r)_0|_{\rho=0} = [\![(U^c_r)_0]\!] \\
\dfrac{\varepsilon^2}{G^*_{\theta r}}\mathbf{S}_{r\theta}\left((U^b_\theta)_0|_{\rho=0}\right) = (U^a_\theta)_0|_{\rho=0} - (U^b_\theta)_0|_{\rho=0} = [\![(U^c_\theta)_0]\!]
\end{cases} \tag{5.84}
$$

Soft Material

$$
\mathbf{S}\left(\mathbf{U}^c\right)\cdot\mathbf{n} = \frac{1}{\varepsilon}\begin{bmatrix} C_{rr}\dfrac{\partial (U^c_r)_0}{\partial \rho}n_r \\ G_{\theta r}\dfrac{\partial (U^c_\theta)_0}{\partial \rho}n_\theta \end{bmatrix} = \frac{1}{\varepsilon}\begin{bmatrix} \varepsilon C^*_{rr}A\left(\theta\right)n_r \\ \varepsilon G^*_{\theta r}C\left(\theta\right)n_\theta \end{bmatrix} = \begin{bmatrix} C^*_{rr}A\left(\theta\right)n_r \\ G^*_{\theta r}C\left(\theta\right)n_\theta \end{bmatrix} \tag{5.85}
$$

The stress-strain relation, in the limit can be written in terms of definition of the derivative:

$$
\begin{cases}
\dfrac{1}{C^*_{rr}}\mathbf{S}_{rr}\left((U^c_r)_0\right) = \dfrac{\partial (U^c_r)_0}{\partial r} = \lim\limits_{\varepsilon\to 0}\dfrac{(U^c_r)_0|_{r=R+\varepsilon} - (U^c_r)_0|_{r=R}}{\varepsilon} \\
\dfrac{1}{G^*_{\theta r}}\mathbf{S}_{r\theta}\left((U^c_\theta)_0\right) = \dfrac{\partial (U^c_\theta)_0}{\partial r} = \lim\limits_{\varepsilon\to 0}\dfrac{(U^c_\theta)_0|_{r=R+\varepsilon} - (U^c_\theta)_0|_{r=R}}{\varepsilon}
\end{cases} \tag{5.86}
$$

$$
\begin{cases}
\dfrac{1}{C^*_{rr}}\mathbf{S}_{rr}\left((U^c_r)_0\right) = (U^a_r)_0|_{\rho=1} - (U^b_r)_0|_{\rho=0} \\
\dfrac{1}{G^*_{\theta r}}\mathbf{S}_{r\theta}\left((U^c_\theta)_0\right) = (U^a_\theta)_0|_{\rho=1} - (U^b_\theta)_0|_{\rho=0}
\end{cases} \tag{5.87}
$$

$$\begin{cases} (U_r^c)_0|_{\rho=0} = (U_r^c)_0|_{\rho=0} + \varepsilon \dfrac{\partial (U_r^c)_0}{\partial r}|_{\rho=0} + \mathscr{O}\left(\varepsilon^2\right) \\ (U_\theta^c)_0|_{\rho=0} = (U_\theta^c)_0|_{\rho=0} + \varepsilon \dfrac{\partial (U_\theta^c)_0}{\partial r}|_{\rho=0} + \mathscr{O}\left(\varepsilon^2\right) \end{cases} \tag{5.88}$$

Taking only the first term in this expansion we have

$$\begin{cases} \dfrac{1}{C_{rr}^*}\mathbf{S}_{rr}\left((U_r^b)_0|_{\rho=0}\right) = (U_r^a)_0|_{\rho=0} - (U_r^b)_0|_{\rho=0} = [\![(U_r^c)_0]\!] \\ \dfrac{1}{G_{\theta r}^*}\mathbf{S}_{r\theta}\left((U_\theta^b)_0|_{\rho=0}\right) = (U_\theta^a)_0|_{\rho=0} - (U_\theta^b)_0|_{\rho=0} = [\![(U_\theta^c)_0]\!] \end{cases} \tag{5.89}$$

New Elastic Coefficients

The elastic coefficients in the new formulation for the stiffer, respectively, for the softer material considered, can be written in dependence to their initial counterparts.

Stiff Material

$$\begin{cases} C_{rr}^* = \varepsilon C_{rr} \\ C_{r\theta}^* = \varepsilon C_{r\theta} \\ C_{\theta\theta}^* = \varepsilon C_{\theta\theta} \\ G_{\theta r}^* = \varepsilon G_{\theta r} \end{cases} \tag{5.90}$$

Soft Material

$$\begin{cases} C_{rr}^* = \dfrac{1}{\varepsilon} C_{rr} \\ C_{r\theta}^* = \dfrac{1}{\varepsilon} C_{r\theta} \\ C_{\theta\theta}^* = \dfrac{1}{\varepsilon} C_{\theta\theta} \\ G_{\theta r}^* = \dfrac{1}{\varepsilon} G_{\theta r} \end{cases} \tag{5.91}$$

5.2. Harmonic oscillations

Starting from the wave equation

$$\boldsymbol{div}\ \mathbf{S}(\mathbf{U}(r,\theta,t)) \quad = \quad M\frac{\partial^2 \mathbf{U}(r,\theta,t)}{\partial t^2} \tag{5.92}$$

and applying a Fourier transform as:

$$\mathbf{U}(r,\theta,t) \quad = \quad e^{-i\omega t}\bar{\mathbf{U}}(r,\theta) \tag{5.93}$$

where we denoted by M the mass density of the material and ω represents the frequency.

We denoted the mass density by M for simple practical reasons,since ρ has already been used as the rescaling variable.

$$\boldsymbol{div}\ \mathbf{S}(\bar{\mathbf{U}}(r,\theta)) \quad = \quad -M\omega^2\bar{\mathbf{U}}(r,\theta) \tag{5.94}$$

This last system represents the system of linearized elasticity in the frequency domain and which can be written in vectorial form, on the cylindrical interphase as:

$$\begin{cases} C_{rr}\dfrac{\partial^2 \bar{U}_r^c}{\partial r^2} - \dfrac{1}{r}C_{r\theta}\dfrac{\partial \bar{U}_\theta^c}{\partial \theta} + \dfrac{1}{r}C_{r\theta}\dfrac{\partial^2 \bar{U}_\theta^c}{\partial r\partial \theta} - \dfrac{1}{r^2}C_{r\theta}\bar{U}_r^c + \dfrac{1}{r}C_{r\theta}\dfrac{\partial \bar{U}_r^c}{\partial r} \\ +\dfrac{1}{r}G_{\theta r}\dfrac{\partial^2 \bar{U}_r^c}{\partial \theta^2} + G_{\theta r}\dfrac{\partial^2 \bar{U}_\theta^c}{\partial r\partial \theta} - \dfrac{1}{r}G_{\theta r}\dfrac{\partial \bar{U}_\theta^c}{\partial \theta} + \dfrac{1}{r}(C_{rr}-C_{r\theta})\dfrac{\partial \bar{U}_r^c}{\partial r} \\ +\dfrac{1}{r^2}(C_{r\theta}-C_{\theta\theta})\dfrac{\partial \bar{U}_\theta^c}{\partial \theta} + \dfrac{1}{r^2}(C_{r\theta}-C_{\theta\theta})\bar{U}_r^c = -M\omega^2\bar{U}_r^c \\ -\dfrac{1}{r^2}G_{\theta r}\dfrac{\partial \bar{U}_r^c}{\partial \theta} + \dfrac{1}{r}G_{\theta r}\dfrac{\partial^2 \bar{U}_r^c}{\partial r\partial \theta} + G_{\theta r}\dfrac{\partial^2 \bar{U}_\theta^c}{\partial r^2} + \dfrac{1}{r^2}G_{\theta r}\bar{U}_\theta^c - \dfrac{1}{r}G_{\theta r}\dfrac{\partial \bar{U}_\theta^c}{\partial r} \\ +\dfrac{1}{r}C_{r\theta}\dfrac{\partial^2 \bar{U}_r^c}{\partial r\partial \theta} + \dfrac{1}{r^2}C_{\theta\theta}\dfrac{\partial^2 \bar{U}_\theta^c}{\partial \theta^2} + \dfrac{1}{r^2}C_{\theta\theta}\dfrac{\partial \bar{U}_r^c}{\partial \theta} + \dfrac{2}{r}G_{\theta r}\dfrac{\partial \bar{U}_r^c}{\partial \theta} \\ +2G_{\theta r}\dfrac{\partial \bar{U}_\theta^c}{\partial r} - \dfrac{2}{r}G_{\theta r}\bar{U}_\theta^c = -M\omega^2\bar{U}_\theta^c \end{cases} \tag{5.95}$$

As in the statical case, for the *fast* variable $\rho = \dfrac{r-R}{\varepsilon}$ then we will have to change from $u(r,\theta)$ to $u(\rho,\theta)$, where R is the radius of the circular inclusion and ε is the thickness of the interphase.

$$\begin{cases} \rho = \dfrac{r-R}{\varepsilon} \Rightarrow r = (\rho\varepsilon + R) \\ \dfrac{1}{r} = \dfrac{1}{(\rho\varepsilon + R)} \end{cases} \tag{5.96}$$

For the change of variable, the system becomes:

$$\begin{cases} \dfrac{1}{\varepsilon^2}C_{rr}\dfrac{\partial^2 \bar{U}_r^c}{\partial \rho^2} + \dfrac{1}{\varepsilon}C_{r\theta}\dfrac{1}{(\rho\varepsilon+R)}\dfrac{\partial^2 \bar{U}_\theta^c}{\partial\rho\partial\theta} + \dfrac{1}{\varepsilon}C_{r\theta}\dfrac{1}{(\rho\varepsilon+R)}\dfrac{\partial \bar{U}_r^c}{\partial\rho} \\ +\dfrac{1}{\varepsilon}G_{\theta r}\dfrac{\partial^2 \bar{U}_\theta^c}{\partial\rho\partial\theta} + \dfrac{1}{\varepsilon}(C_{rr} - C_{r\theta})\dfrac{1}{(\rho\varepsilon+R)}\dfrac{\partial \bar{U}_r^c}{\partial\rho} - C_{r\theta}\dfrac{1}{(\rho\varepsilon+R)}\dfrac{\partial \bar{U}_\theta^c}{\partial\theta} \\ -C_{r\theta}\dfrac{1}{(\rho\varepsilon+R)^2}\bar{U}_r^c + G_{\theta r}\dfrac{1}{(\rho\varepsilon+R)}\dfrac{\partial^2 \bar{U}_r^c}{\partial\theta^2} - G_{\theta r}\dfrac{1}{(\rho\varepsilon+R)}\dfrac{\partial \bar{U}_\theta^c}{\partial\theta} \\ +(C_{r\theta} - C_{\theta\theta})\dfrac{1}{(\rho\varepsilon+R)^2}\dfrac{\partial \bar{U}_\theta^c}{\partial\theta} + (C_{r\theta} - C_{\theta\theta})\dfrac{1}{(\rho\varepsilon+R)^2}\bar{U}_r^c = -M\omega^2\bar{U}_r^c \\ \dfrac{1}{\varepsilon^2}G_{\theta r}\dfrac{\partial^2 \bar{U}_\theta^c}{\partial\rho^2} + \dfrac{1}{\varepsilon}G_{\theta r}\dfrac{1}{(\rho\varepsilon+R)}\dfrac{\partial^2 \bar{U}_r^c}{\partial\rho\partial\theta} - \dfrac{1}{\varepsilon}G_{\theta r}\dfrac{1}{(\rho\varepsilon+R)}\dfrac{\partial \bar{U}_\theta^c}{\partial\rho} \\ +\dfrac{1}{\varepsilon}C_{r\theta}\dfrac{1}{(\rho\varepsilon+R)}\dfrac{\partial^2 \bar{U}_r^c}{\partial\rho\partial\theta} + \dfrac{1}{\varepsilon}2G_{\theta r}\dfrac{\partial \bar{U}_\theta^c}{\partial\rho} - G_{\theta r}\dfrac{1}{(\rho\varepsilon+R)^2}\dfrac{\partial \bar{U}_r^c}{\partial\theta} \\ +G_{\theta r}\dfrac{1}{(\rho\varepsilon+R)^2}\bar{U}_\theta^c + C_{\theta\theta}\dfrac{1}{(\rho\varepsilon+R)^2}\dfrac{\partial^2 \bar{U}_\theta^c}{\partial\theta^2} + C_{\theta\theta}\dfrac{1}{(\rho\varepsilon+R)^2}\dfrac{\partial \bar{U}_r^c}{\partial\theta} \\ +G_{\theta r}\dfrac{2}{(\rho\varepsilon+R)}\dfrac{\partial \bar{U}_r^c}{\partial\theta} - G_{\theta r}\dfrac{2}{(\rho\varepsilon+R)}\bar{U}_\theta^c = -M\omega^2\bar{U}_\theta^c \end{cases} \tag{5.97}$$

In the following, going through the same steps as we did in the statical case, only taking in account the left-hand side component of the system, we arrive at the form:

$$
\begin{cases}
\varepsilon^0 \left[C_{rr} \dfrac{\partial^2 (\bar{U}_r^c)_0}{\partial \rho^2} \right] + \varepsilon^1 \left[C_{rr} \dfrac{\partial^2 (\bar{U}_r^c)_1}{\partial \rho^2} + C_{r\theta} \dfrac{1}{R} A_0 + C_{r\theta} \dfrac{1}{R} B_0 + ... \right] \\
+\varepsilon^2 \left[C_{rr} \dfrac{\partial^2 (\bar{U}_r^c)_2}{\partial \rho^2} + G_{\theta r} \dfrac{1}{R} A_1 - G_{\theta r} \dfrac{1}{R} B_1 + ... \right] + \varepsilon^3 \left[C_{rr} \dfrac{\partial^2 (\bar{U}_r^c)_3}{\partial \rho^2} + ... \right] \\
= -M\varepsilon^2 \omega^2 \left[\varepsilon^0 (\bar{U}_r^c)_0 + \varepsilon^1 (\bar{U}_r^c)_1 + \varepsilon^2 (\bar{U}_r^c)_2 + \varepsilon^3 (\bar{U}_r^c)_3 + ... \right] \\
\varepsilon^0 \left[G_{\theta r} \dfrac{\partial^2 (\bar{U}_r^c)_0}{\partial \rho^2} \right] + \varepsilon^1 \left[G_{\theta r} \dfrac{\partial^2 (\bar{U}_r^c)_1}{\partial \rho^2} + G_{\theta r} \dfrac{1}{R} J_0 + G_{\theta r} \dfrac{1}{R} K_0 + ... \right] \\
+\varepsilon^2 \left[G_{\theta r} \dfrac{\partial^2 (\bar{U}_r^c)_2}{\partial \rho^2} + G_{\theta r} \dfrac{1}{R} J_1 - G_{\theta r} \dfrac{1}{R} K_1 + ... \right] + \varepsilon^3 \left[G_{\theta r} \dfrac{\partial^2 (\bar{U}_r^c)_3}{\partial \rho^2} + ... \right] \\
= -M\varepsilon^2 \omega^2 \left[\varepsilon^0 (\bar{U}_\theta^c)_0 + \varepsilon^1 (\bar{U}_\theta^c)_1 + \varepsilon^2 (\bar{U}_\theta^c)_2 + \varepsilon^3 (\bar{U}_\theta^c)_3 + ... \right]
\end{cases} \tag{5.98}
$$

In the leading term, the system can be written as:

$$
\begin{cases}
\varepsilon^0 \left[C_{rr} \dfrac{\partial^2 (\bar{U}_r^c)_0}{\partial \rho^2} \right] = -M\varepsilon^2 \omega^2 \left[\varepsilon^0 (\bar{U}_r^c)_0 \right] \\
\varepsilon^0 \left[G_{\theta r} \dfrac{\partial^2 (\bar{U}_r^c)_0}{\partial \rho^2} \right] = -M\varepsilon^2 \omega^2 \left[\varepsilon^0 (\bar{U}_\theta^c)_0 \right]
\end{cases} \tag{5.99}
$$

$$
\begin{cases}
C_{rr} \dfrac{\partial^2 (\bar{U}_r^c)_0}{\partial \rho^2} = -M\varepsilon^2 \omega^2 (\bar{U}_r^c)_0 \\
G_{\theta r} \dfrac{\partial^2 (\bar{U}_r^c)_0}{\partial \rho^2} = -M\varepsilon^2 \omega^2 (\bar{U}_\theta^c)_0
\end{cases} \tag{5.100}
$$

$$
\begin{cases}
\dfrac{\partial^2 (\bar{U}_r^c)_0}{\partial \rho^2} = 0 \\
\dfrac{\partial^2 (\bar{U}_\theta^c)_0}{\partial \rho^2} = 0
\end{cases} \tag{5.101}
$$

and so, the solution is, like in the static case

$$
\begin{cases}
(\bar{U}_r^c)_0 = A(\theta)\rho + B(\theta) \\
(\bar{U}_\theta^c)_0 = A(\theta)\rho + B(\theta)
\end{cases} \tag{5.102}
$$

For the rescaling of the mass density as

$$\begin{cases} M = \dfrac{1}{\varepsilon^2} M^* \\ M = \dfrac{1}{\varepsilon^2} M^* \end{cases} \tag{5.103}$$

$$\begin{cases} C_{rr} \dfrac{\partial^2 (\bar{U}_r^c)_0}{\partial \rho^2} = -M^* \omega^2 (\bar{U}_r^c)_0 \\ G_{\theta r} \dfrac{\partial^2 (\bar{U}_\theta^c)_0}{\partial \rho^2} = -M^* \omega^2 (\bar{U}_\theta^c)_0 \end{cases} \tag{5.104}$$

We get the solution

$$\begin{cases} (\bar{U}_r^c)_0 = C_1 \cos \dfrac{\rho \omega M^*}{C_{rr}} + C_2 \sin \dfrac{\rho \omega M^*}{C_{rr}} \\ (\bar{U}_\theta^c)_0 = C_3 \cos \dfrac{\rho \omega M^*}{G_{\theta r}} + C_4 \sin \dfrac{\rho \omega M^*}{G_{\theta r}} \end{cases} \tag{5.105}$$

Stiff Material

Considering that the second material becomes stiffer than the inclusion and infinite media, we will consider the proper rescaling of the elastic coefficients, as

$$\begin{cases} C_{rr} = \dfrac{1}{\varepsilon} C_{rr}^* \\ C_{r\theta} = \dfrac{1}{\varepsilon} C_{r\theta}^* \\ C_{\theta\theta} = \dfrac{1}{\varepsilon} C_{\theta\theta}^* \\ G_{\theta r} = \dfrac{1}{\varepsilon} G_{\theta r}^* \end{cases} \tag{5.106}$$

In the leading term, this system can be written as:

$$\begin{cases} C_{rr}^* \dfrac{\partial^2 (\bar{U}_r^c)_0}{\partial \rho^2} = -\varepsilon^3 M \omega^2 (\bar{U}_r^c)_0 \\ G_{\theta r}^* \dfrac{\partial^2 (\bar{U}_\theta^c)_0}{\partial \rho^2} = -\varepsilon^3 M \omega^2 (\bar{U}_\theta^c)_0 \end{cases} \tag{5.107}$$

And so, a solution from this, can be written as:

$$\begin{cases} \dfrac{\partial^2 (\bar{U}_r^c)_0}{\partial \rho^2} = 0 \\ \dfrac{\partial^2 (\bar{U}_\theta^c)_0}{\partial \rho^2} = 0 \end{cases} \tag{5.108}$$

and so,

$$\begin{cases} (\bar{U}_r^c)_0 = A(\theta)\rho + B(\theta) \\ (\bar{U}_\theta^c)_0 = A(\theta)\rho + B(\theta) \end{cases} \tag{5.109}$$

For the rescaling of the mass density as

$$\begin{cases} M = \dfrac{1}{\varepsilon^3} M^* \\ M = \dfrac{1}{\varepsilon^3} M^* \end{cases} \tag{5.110}$$

$$\begin{cases} C_{rr}(\dfrac{\partial^2 (\bar{U}_r^c)_0}{\partial \rho^2}) = -M^* \omega^2 (\bar{U}_r^c)_0 \\ G_{\theta r}(\dfrac{\partial^2 (\bar{U}_\theta^c)_0}{\partial \rho^2}) = -M^* \omega^2 (\bar{U}_\theta^c)_0 \end{cases} \tag{5.111}$$

So, we get the solution

$$\begin{cases} (\bar{U}_r^c)_0 = C_1 \cos \dfrac{\rho \omega M^*}{C_{rr}} + C_2 \sin \dfrac{\rho \omega M^*}{C_{rr}} \\ (\bar{U}_\theta^c)_0 = C_3 \cos \dfrac{\rho \omega M^*}{G_{\theta r}} + C_4 \sin \dfrac{\rho \omega M^*}{G_{\theta r}} \end{cases} \tag{5.112}$$

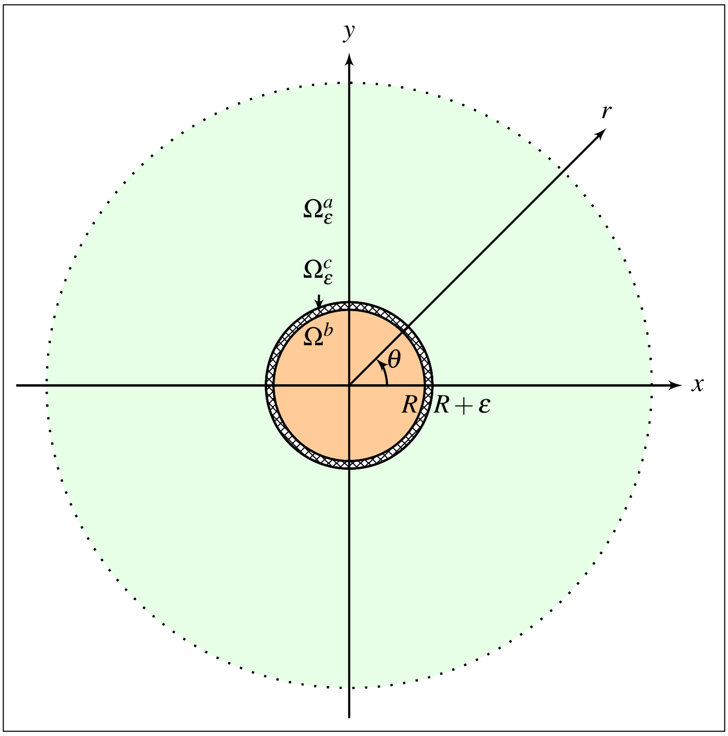


Figure 5.5: *Cylindrical Interphase. High Mass Density*

Soft Material

Considering that the second material becomes softer than the inclusion and infinite media, we will consider the proper rescaling of the elastic coefficients, as

$$\begin{cases} C_{rr} = \varepsilon C^*_{rr} \\ C_{r\theta} = \varepsilon C^*_{r\theta} \\ C_{\theta\theta} = \varepsilon C^*_{\theta\theta} \\ G_{\theta r} = \varepsilon G^*_{\theta r} \end{cases} \tag{5.113}$$

we get

$$\begin{cases} C^*_{rr}\dfrac{\partial^2(\bar{U}^c_r)_0}{\partial\rho^2} = -\varepsilon M\omega^2(\bar{U}^c_r)_0 \\ G^*_{\theta r}\dfrac{\partial^2(\bar{U}^c_\theta)_0}{\partial\rho^2} = -\varepsilon M\omega^2(\bar{U}^c_\theta)_0 \end{cases} \tag{5.114}$$

And so, a solution from this, can be written as:

$$\begin{cases} \dfrac{\partial^2(\bar{U}^c_r)_0}{\partial\rho^2} = 0 \\ \dfrac{\partial^2(\bar{U}^c_\theta)_0}{\partial\rho^2} = 0 \end{cases} \tag{5.115}$$

and so,

$$\begin{cases} (\bar{U}^c_r)_0 = A(\theta)\rho + B(\theta) \\ (\bar{U}^c_\theta)_0 = A(\theta)\rho + B(\theta) \end{cases} \tag{5.116}$$

For the rescaling of the mass density as

$$\begin{cases} M = \dfrac{1}{\varepsilon}M^* \\ M = \dfrac{1}{\varepsilon}M^* \end{cases} \tag{5.117}$$

$$\begin{cases} C^*_{rr}\dfrac{\partial^2(\bar{U}^c_r)_0}{\partial\rho^2} = -M^*\omega^2(\bar{U}^c_r)_0 \\ G^*_{\theta r}\dfrac{\partial^2(\bar{U}^c_\theta)_0}{\partial\rho^2} = -M^*\omega^2(\bar{U}^c_\theta)_0 \end{cases} \tag{5.118}$$

we get the solution

$$\begin{cases} (\bar{U}^c_r)_0 = C_1\cos\dfrac{\rho\,\omega M^*}{C_{rr}} + C_2\sin\dfrac{\rho\,\omega M^*}{C_{rr}} \\ (\bar{U}^c_\theta)_0 = C_3\cos\dfrac{\rho\,\omega M^*}{G_{\theta r}} + C_4\sin\dfrac{\rho\,\omega M^*}{G_{\theta r}} \end{cases} \tag{5.119}$$

5.2. Limit Transmission Conditions

$$\mathbf{S}\left(\bar{\mathbf{U}}^c\right)\cdot\mathbf{n}=\frac{1}{\varepsilon}\begin{bmatrix} S_{rr}\left((\bar{U}_r^c)_0\right)n_r \\ S_{\theta r}\left((\bar{U}_r^c)_0\right)n_\theta \end{bmatrix} \tag{5.120}$$

The stress-strain relation, in the limit can be written in terms of definition of the derivative:

$$\begin{cases} S_{rr}=\dfrac{C_{rr}}{\varepsilon}\dfrac{\partial\,(\bar{U}_r^c)_0}{\partial\rho}+C_{r\theta}\dfrac{1}{\rho\varepsilon+1}\,(\bar{U}_r^c)_0 \\ S_{r\theta}=\dfrac{G_{\theta r}}{\varepsilon}\dfrac{\partial\left(\bar{U}_\theta^c\right)_0}{\partial\rho}-G_{\theta r}\dfrac{1}{\rho\varepsilon+1}\,(\bar{U}_r^c)_0 \end{cases} \tag{5.121}$$

$$\frac{\partial(\bar{U}_r^c)_0}{\partial\rho} = \frac{M^*\omega}{C_{rr}}\left\{-C_2\sin\frac{M^*\omega\rho}{C_{rr}}+C_1\cos\frac{M^*\omega\rho}{C_{rr}}\right\} \tag{5.122}$$

$$\frac{\partial(\bar{U}_\theta^c)_0}{\partial\rho} = \frac{M^*\omega}{G_{\theta r}}\left\{-C_4\sin\frac{M^*\omega\rho}{G_{\theta r}}+C_3\cos\frac{M^*\omega\rho}{G_{\theta r}}\right\} \tag{5.123}$$

So,

$$\begin{aligned} S_{rr} &= \frac{C_{rr}}{\varepsilon}\left\{\frac{M^*\omega}{C_{rr}}\left\{-C_2\sin\frac{M^*\omega\rho}{C_{rr}}+C_1\cos\frac{M^*\omega\rho}{C_{rr}}\right\}\right\} \\ &+ \frac{C_{r\theta}}{\rho\varepsilon+1}\left\{C_1\cos\frac{\rho\omega M^*}{C_{rr}}+C_2\sin\frac{\rho\omega M^*}{C_{rr}}\right\} \end{aligned} \tag{5.124}$$

$$\begin{aligned} S_{r\theta} &= \frac{G_{\theta r}}{\varepsilon}\left\{\frac{M^*\omega}{G_{\theta r}}\left\{-C_4\sin\frac{M^*\omega\rho}{G_{\theta r}}+C_3\cos\frac{M^*\omega\rho}{G_{\theta r}}\right\}\right\} \\ &- \frac{G_{\theta r}}{\rho\varepsilon+1}\left\{C_3\cos\frac{\rho\omega M^*}{G_{\theta r}}+C_4\sin\frac{\rho\omega M^*}{G_{\theta r}}\right\} \end{aligned} \tag{5.125}$$

and

$$
\begin{aligned}
S_{rr} &= \frac{M^* \omega}{\varepsilon} \left\{ -C_2 \sin \frac{M^* \omega \rho}{C_{rr}} + C_1 \cos \frac{M^* \omega \rho}{C_{rr}} \right\} \\
&+ \frac{C_{r\theta}}{\rho \varepsilon + 1} \left\{ C_1 \cos \frac{\rho \omega M^*}{C_{rr}} + C_2 \sin \frac{\rho \omega M^*}{C_{rr}} \right\}
\end{aligned} \tag{5.126}
$$

$$
\begin{aligned}
S_{r\theta} &= \frac{M^* \omega}{\varepsilon} \left\{ -C_4 \sin \frac{M^* \omega \rho}{G_{\theta r}} + C_3 \cos \frac{M^* \omega \rho}{G_{\theta r}} \right\} \\
&- \frac{G_{\theta r}}{\rho \varepsilon + 1} \left\{ C_3 \cos \frac{\rho \omega M^*}{G_{\theta r}} + C_4 \sin \frac{\rho \omega M^*}{G_{\theta r}} \right\}
\end{aligned} \tag{5.127}
$$

When $\varepsilon \to 0$ we get

$$
\begin{aligned}
S_{rr} &= C_{r\theta} \left\{ C_1 \cos \frac{\rho \omega M^*}{C_{rr}} + C_2 \sin \frac{\rho \omega M^*}{C_{rr}} \right\} \\
S_{r\theta} &= G_{\theta r} \left\{ C_3 \cos \frac{\rho \omega M^*}{G_{\theta r}} + C_4 \sin \frac{\rho \omega M^*}{G_{\theta r}} \right\}
\end{aligned}
$$

Stiff Material

$$
\mathbf{S}\left(\bar{\mathbf{U}}^c\right) \cdot \mathbf{n} = \frac{1}{\varepsilon} \begin{bmatrix} S_{rr}\left((\bar{U}_r^c)_0\right) n_r \\ S_{\theta r}\left((\bar{U}_r^c)_0\right) n_\theta \end{bmatrix} \tag{5.128}
$$

for

$$
\begin{cases}
C_{rr} = \dfrac{1}{\varepsilon} C_{rr}^* \\
C_{r\theta} = \dfrac{1}{\varepsilon} C_{r\theta}^* \\
C_{\theta\theta} = \dfrac{1}{\varepsilon} C_{\theta\theta}^* \\
G_{\theta r} = \dfrac{1}{\varepsilon} G_{\theta r}^*
\end{cases} \tag{5.129}
$$

we get

$$
\begin{aligned}
S_{rr} &= \frac{M^*\omega}{\varepsilon}\left\{-C_2\sin\frac{\varepsilon M^*\omega\rho}{C^*_{rr}}+C_1\cos\frac{\varepsilon M^*\omega\rho}{C^*_{rr}}\right\} \\
&+ \frac{C^*_{r\theta}}{\rho\varepsilon+1}\left\{C_1\cos\frac{\varepsilon\rho\omega M^*}{C^*_{rr}}+C_2\sin\frac{\varepsilon\rho\omega M^*}{C^*_{rr}}\right\}
\end{aligned}
\tag{5.130}
$$

$$
\begin{aligned}
S_{r\theta} &= \frac{M^*\omega}{\varepsilon}\left\{-C_4\sin\frac{\varepsilon M^*\omega\rho}{G^*_{\theta r}}+C_3\cos\frac{\varepsilon M^*\omega\rho}{G^*_{\theta r}}\right\} \\
&- \frac{G_{\theta r}}{\rho\varepsilon+1}\left\{C_3\cos\frac{\varepsilon\rho\omega M^*}{G^*_{\theta r}}+C_4\sin\frac{\varepsilon\rho\omega M^*}{G^*_{\theta r}}\right\}
\end{aligned}
\tag{5.131}
$$

Soft Material

$$
\mathbf{S}\left(\bar{\mathbf{U}}^c\right)\cdot\mathbf{n}=\frac{1}{\varepsilon}\begin{bmatrix} S_{rr}\left((\bar{U}^c_r)_0\right)n_r \\ S_{\theta r}\left((\bar{U}^c_r)_0\right)n_\theta \end{bmatrix} \tag{5.132}
$$

for

$$
\begin{cases}
C_{rr}=\varepsilon C^*_{rr} \\
C_{r\theta}=\varepsilon C^*_{r\theta} \\
C_{\theta\theta}=\varepsilon C^*_{\theta\theta} \\
G_{\theta r}=\varepsilon G^*_{\theta r}
\end{cases} \tag{5.133}
$$

we get

$$
\begin{aligned}
S_{rr} &= \frac{M^*\omega}{\varepsilon}\left\{-C_2\sin\frac{M^*\omega\rho}{\varepsilon C^*_{rr}}+C_1\cos\frac{M^*\omega\rho}{\varepsilon C^*_{rr}}\right\} \\
&+ \frac{C^*_{r\theta}}{\rho\varepsilon+1}\left\{C_1\cos\frac{\rho\omega M^*}{\varepsilon C^*_{rr}}+C_2\sin\frac{\rho\omega M^*}{\varepsilon C^*_{rr}}\right\}
\end{aligned}
\tag{5.134}
$$

$$
\begin{aligned}
S_{r\theta} &= \frac{M^*\omega}{\varepsilon}\left\{-C_4\sin\frac{M^*\omega\rho}{\varepsilon G^*_{\theta r}}+C_3\cos\frac{M^*\omega\rho}{\varepsilon G^*_{\theta r}}\right\} \\
&- \frac{\varepsilon G^*_{\theta r}}{\rho\varepsilon+1}\left\{C_3\cos\frac{\rho\omega M^*}{\varepsilon G^*_{\theta r}}+C_4\sin\frac{\rho\omega M^*}{\varepsilon G^*_{\theta r}}\right\}
\end{aligned}
\tag{5.135}
$$

New Elastic Coefficients

The dependence of limit transmission conditions with respect to new elastic coefficients considered as imposing a stiffer or softer material in the thin interphase, is written in an explicit way and also the new elastic coefficients, like in the static case, can be written in terms of the initially imposed coefficients.

Stiff Material

$$\begin{cases} C^*_{rr} = \varepsilon C_{rr} \\ C^*_{r\theta} = \varepsilon C_{r\theta} \\ C^*_{\theta\theta} = \varepsilon C_{\theta\theta} \\ G^*_{\theta r} = \varepsilon G_{\theta r} \end{cases} \tag{5.136}$$

Soft Material

$$\begin{cases} C^*_{rr} = \dfrac{1}{\varepsilon} C_{rr} \\ C^*_{r\theta} = \dfrac{1}{\varepsilon} C_{r\theta} \\ C^*_{\theta\theta} = \dfrac{1}{\varepsilon} C_{\theta\theta} \\ G^*_{\theta r} = \dfrac{1}{\varepsilon} G_{\theta r} \end{cases} \tag{5.137}$$

CHAPTER

SIX

VARIATIONAL METHOD

6.1. Description of Problem

We consider the linearized system of elasticity in $\mathbb{R}^2$ on a bounded open domain composed of three subdomains, of which one is dependent of a small parameter ε. We will denote the domain with Ω^ε and define it like:

$$\begin{aligned}
\Omega^\varepsilon &= \Omega^b \cup \Omega^c_\varepsilon \cup \Omega^a_\varepsilon \qquad (6.1)\\
\Omega^b &= \left\{(X,Y) \in \mathbb{R}^2 | 0 < X < 1, 0 < Y < 1\right\}\\
\Omega^c_\varepsilon &= \left\{(X,Y) \in \mathbb{R}^2 | 0 < X < 1, 1 < Y < 1+\varepsilon\right\}\\
\Omega^a_\varepsilon &= \left\{(X,Y) \in \mathbb{R}^2 | 0 < X < 1, 1+\varepsilon < Y < 3\right\}
\end{aligned}$$

In this domain, we solve the weak form of the following boundary value problem:

$$\begin{cases} -\boldsymbol{div}\, \mathbf{S}(\mathbf{U}^{\alpha}) = \mathbf{F}^{\alpha},\ on\ \Omega_{\varepsilon}^{\alpha}, \alpha \in \{b,c,a\} \\ \mathbf{U}^{b}(X,0) = \mathbf{0} \\ \mathbf{U}^{\alpha}(0,Y) = \mathbf{U}^{\alpha}(1,Y); \alpha \in \{b,c,a\} \\ \mathbf{U}^{a}(X,3) = \mathbf{0} \\ \textit{Transmission conditions in thin domain} \\ \mathbf{U}^{b}(X,1) = \mathbf{U}^{c}(X,1) \\ \mathbf{U}^{c}(X,1+\varepsilon) = \mathbf{U}^{a}(X,1+\varepsilon) \\ \big(\mathbf{S}(\mathbf{U}^{b})\cdot\mathbf{n}\big)(X,1) = \left(\mathbf{S}(\mathbf{U}^{c})\cdot\mathbf{n}\right)(X,1) \\ \left(\mathbf{S}(\mathbf{U}^{c})\cdot\mathbf{n}\right)(X,1+\varepsilon) = \left(\mathbf{S}(\mathbf{U}^{a})\cdot\mathbf{n}\right)(X,1+\varepsilon). \end{cases} \tag{6.2}$$

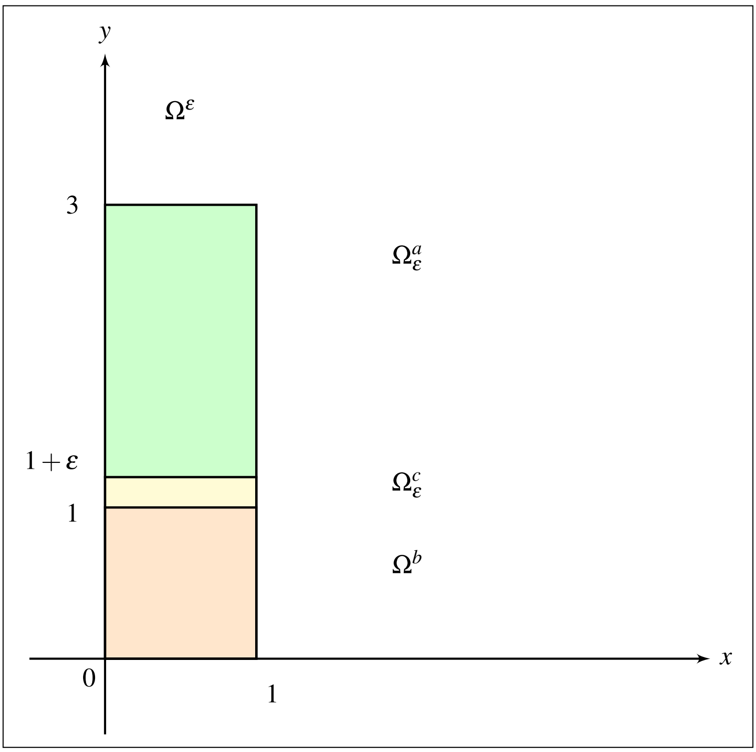


Figure 6.1: *Thin Rectangular Interphase*

The displacement vector field in the 'thin' domain splits depending on the subdomain like:

$$
\begin{aligned}
\mathbf{U}^b &: \Omega^b \subset \mathbb{R}^2 \to \mathbb{R}^2 && (6.3)\\
\mathbf{U}^b &= (U_1^b, U_2^b) \\
\mathbf{U}^c &: \Omega_\varepsilon^c \subset \mathbb{R}^2 \to \mathbb{R}^2 && (6.4)\\
\mathbf{U}^c &= (U_1^c, U_2^c) \\
\mathbf{U}^a &: \Omega_\varepsilon^a \subset \mathbb{R}^2 \to \mathbb{R}^2 && (6.5)\\
\mathbf{U}^a &= (U_1^a, U_2^a)
\end{aligned}
$$

Also, the strain tensor is defined separately for each subdomain.

$$
\begin{cases}
\mathbf{e}(\mathbf{U}^b) = \begin{bmatrix} \dfrac{\partial U_1^b}{\partial X} & \dfrac{1}{2}\left(\dfrac{\partial U_1^b}{\partial Y} + \dfrac{\partial U_2^b}{\partial X}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial U_1^b}{\partial Y} + \dfrac{\partial U_2^b}{\partial X}\right) & \dfrac{\partial U_2^b}{\partial Y} \end{bmatrix} \\
\mathbf{e}(\mathbf{U}^b) \in (L^2(\Omega^b))^{2\times 2}
\end{cases} \tag{6.6}
$$

$$
\begin{cases}
\mathbf{e}(\mathbf{U}^c) = \begin{bmatrix} \dfrac{\partial U_1^c}{\partial X} & \dfrac{1}{2}\left(\dfrac{\partial U_1^c}{\partial Y} + \dfrac{\partial U_2^c}{\partial X}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial U_1^c}{\partial Y} + \dfrac{\partial U_2^c}{\partial X}\right) & \dfrac{\partial U_2^c}{\partial Y} \end{bmatrix} \\
\mathbf{e}(\mathbf{U}^c) \in (L^2(\Omega_\varepsilon^c))^{2\times 2}
\end{cases} \tag{6.7}
$$

$$
\begin{cases}
\mathbf{e}(\mathbf{U}^a) = \begin{bmatrix} \dfrac{\partial U_1^a}{\partial X} & \dfrac{1}{2}\left(\dfrac{\partial U_1^a}{\partial Y} + \dfrac{\partial U_2^a}{\partial X}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial U_1^a}{\partial Y} + \dfrac{\partial U_2^a}{\partial X}\right) & \dfrac{\partial U_2^a}{\partial Y} \end{bmatrix} \\
\mathbf{e}(\mathbf{U}^a) \in (L^2(\Omega_\varepsilon^a))^{2\times 2}
\end{cases} \tag{6.8}
$$

So, in the domain Ω^ε we solve the weak form of the elasticity system:

$$\begin{aligned}
&\textit{Find } \mathbf{U} \in \mathscr{V}_\varepsilon \textit{ such that} \\
&\int\limits_{\Omega^\varepsilon} [\mathbb{C}\mathbf{e}(\mathbf{U}), \mathbf{e}(\mathbf{V})]\, dXdY = \int\limits_{\Omega^\varepsilon} \mathbf{F} \cdot \mathbf{V} dXdY \\
&\textit{for any } \mathbf{V} \in \mathscr{V}_\varepsilon
\end{aligned} \tag{6.9}$$

where

$$\mathscr{V}_\varepsilon = \left\{ \begin{array}{l} \mathbf{U} := (\mathbf{U}^b, \mathbf{U}^c, \mathbf{U}^a) \in (H^1(\Omega^b))^2 \times (H^1(\Omega^c_\varepsilon))^2 \times (H^1(\Omega^a_\varepsilon))^2 \textit{ such that} \\ \begin{cases} \mathbf{U}^b(X,0) = \mathbf{0}, \mathbf{U}^\alpha(0,Y) = \mathbf{U}^\alpha(1,Y); \alpha \in \{b,c,a\} \\ \mathbf{U}^a(X,3) = \mathbf{0}, \mathbf{U}^b(X,1) = \mathrm{U}^c(X,1), \mathbf{U}^c(x,1+\varepsilon) = \mathbf{U}^a(X,1+\varepsilon) \end{cases} \end{array} \right\} \tag{6.10}$$

We consider $\mathbb{C}$ the 4^{th} order elasticity tensor with constant coefficients satisfying the symmetry and coercivity conditions:

$$\exists C > 0, \forall \xi \in \mathbb{R}^{2\times 2}_s, [\mathbb{C}\xi, \xi] \geq C|\xi|^2 \tag{6.11}$$

$$(\mathbb{C}\xi)_{ij} \quad = \quad \sum_{hk} \mathscr{C}_{ijhk}\xi_{hk} \tag{6.12}$$

The scalar product $[.,.]$ in $\mathbb{R}^{2\times 2}$ is defined by:

$$[\zeta, \xi] \quad = \quad \sum_{ij} \zeta_{ij}\xi_{ij} \tag{6.13}$$

We also consider the forces

$$\mathbf{F} \equiv \begin{cases} \mathbf{F}^b \in L^2(\Omega^b)^2, \; \textit{if}\,(X,Y) \in \Omega^b \\ \mathbf{F}^c \in L^2(\Omega^c_\varepsilon)^2, \; \textit{if}\,(X,Y) \in \Omega^c_\varepsilon \\ \mathbf{F}^a \in L^2(\Omega^a_\varepsilon)^2, \; \textit{if}\,(X,Y) \in \Omega^a_\varepsilon \end{cases} \tag{6.14}$$

The weak form of this problem can be written in more detail as:

$$
\begin{aligned}
&\textit{Find } \mathbf{U} = (U^b, U^c, U^a) \in \mathscr{V}_\varepsilon \textit{ such that} \\
&\int_{\Omega^b} \left[\mathbb{C}\mathbf{e}(\mathbf{U}^b), \mathbf{e}(\mathbf{V}^b)\right] dXdY + \int_{\Omega^c_\varepsilon} \left[\mathbb{C}\mathbf{e}(\mathbf{U}^c), \mathbf{e}(\mathbf{V}^c)\right] dXdY + \int_{\Omega^a_\varepsilon} \left[\mathbb{C}\mathbf{e}(\mathbf{U}^a), \mathbf{e}(\mathbf{V}^a)\right] dXdY = \\
&= \int_{\Omega^b} \mathbf{F}^b \cdot \mathbf{V}^b dXdY + \int_{\Omega^c_\varepsilon} \mathbf{F}^c \cdot \mathbf{V}^c dXdY + \int_{\Omega^a_\varepsilon} \mathbf{F}^a \cdot \mathbf{V}^a dXdY \\
&\textit{for any } \mathbf{V} = (V^b, V^c, V^a) \in \mathscr{V}_\varepsilon
\end{aligned}
$$

6.2. Rescaling

In the following, we will rescale the domain, the displacement field and the strains in order to be able to write the problem in a fixed domain independent of ε. However, by rescaling the problem we basically move the small parameter from the geometry to the system of elasticity, step that allows us to understand how the mechanics of the system is dependent of the small parameter.

$$
\begin{aligned}
\Omega &= \Omega^b \cup \Omega^c \cup \Omega^a && (6.15)\\
\Omega^b &= \left\{(x,y) \in \mathbb{R}^2 | 0 < x < 1, 0 < y < 1\right\} \\
\Omega^c &= \left\{(x,y) \in \mathbb{R}^2 | 0 < x < 1, 1 < y < 2\right\} \\
\Omega^a &= \left\{(x,y) \in \mathbb{R}^2 | 0 < x < 1, 2 < y < 3\right\}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{u}^b &: \Omega^b \subset \mathbb{R}^2 \to \mathbb{R}^2 && (6.16)\\
\mathbf{u}^b &:= (u_1^b, u_2^b) \\
\mathbf{u}^c &: \Omega^c \subset \mathbb{R}^2 \to \mathbb{R}^2 && (6.17)\\
\mathbf{u}^c &:= (u_1^c, u_2^c) \\
\mathbf{u}^a &: \Omega^a \subset \mathbb{R}^2 \to \mathbb{R}^2 && (6.18)\\
\mathbf{u}^a &:= (u_1^a, u_2^a) \\
& && (6.19)
\end{aligned}
$$

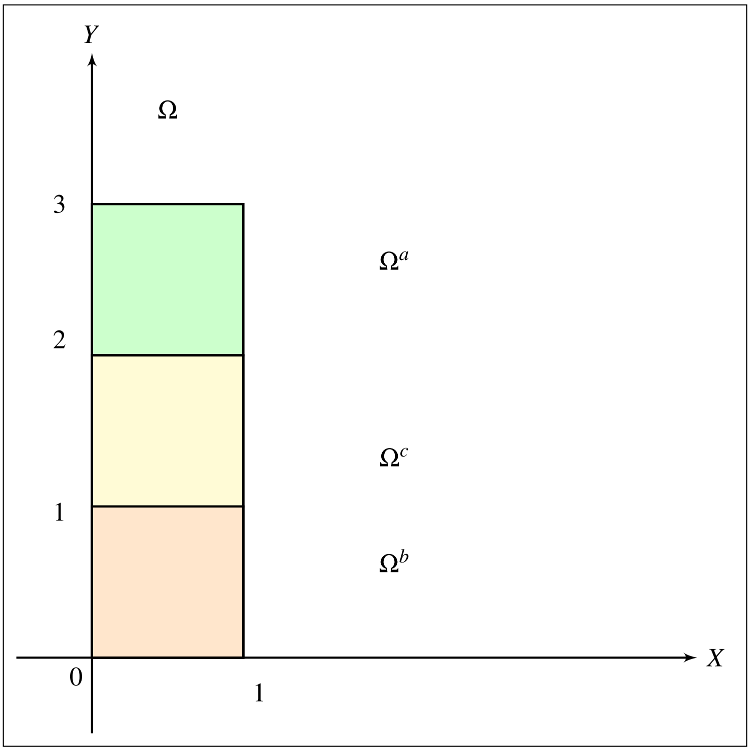


Figure 6.2: *Rescaled Rectangular Interphase*

At this point we can see the dependence of the displacement and strain vector fields in the fixed domain through the parameter ε of the displacement and strain in the domain that contains an ε-dependent subdomain.

Using the dipslacement rescaling

$$
\begin{aligned}
u_1^b(x,y) &= U_1^b(x,y) && (6.20)\\
u_2^b(x,y) &= U_2^b(x,y) && (6.21)\\
x &= X\\
y &= Y\\
u_1^c(x,y) &= \sqrt{\varepsilon}U_1^c(x,(y-1)\varepsilon+1) && (6.22)\\
u_2^c(x,y) &= \varepsilon\sqrt{\varepsilon}U_2^c(x,(y-1)\varepsilon+1) && (6.23)\\
x &= X\\
(y-1)\varepsilon+1 &= Y\\
u_1^a(x,y) &= U_1^a(x,(y-2)(2-\varepsilon)+1+\varepsilon) && (6.24)\\
u_2^a(x,y) &= (2-\varepsilon)U_2^a(x,(y-2)(2-\varepsilon)+1+\varepsilon) && (6.25)\\
x &= X\\
(y-2)(2-\varepsilon)+1+\varepsilon &= Y
\end{aligned}
$$

with the change of variable

$$
\begin{aligned}
\frac{\partial U_1^b}{\partial X} &= \frac{\partial u_1^b}{\partial x} && (6.26)\\
\frac{\partial U_1^b}{\partial Y} &= \frac{\partial u_1^b}{\partial y} && (6.27)\\
\frac{\partial U_2^b}{\partial X} &= \frac{\partial u_2^b}{\partial x} && (6.28)\\
\frac{\partial U_2^b}{\partial Y} &= \frac{\partial u_2^b}{\partial y} && (6.29)
\end{aligned}
$$

$$\frac{\partial U_1^c}{\partial X} = \frac{1}{\sqrt{\varepsilon}}\frac{\partial u_1^c}{\partial x} \tag{6.30}$$

$$\frac{\partial U_1^c}{\partial Y} = \frac{1}{\varepsilon\sqrt{\varepsilon}}\frac{\partial u_1^c}{\partial y} \tag{6.31}$$

$$\frac{\partial U_2^c}{\partial X} = \frac{1}{\varepsilon\sqrt{\varepsilon}}\frac{\partial u_2^c}{\partial x} \tag{6.32}$$

$$\frac{\partial U_2^c}{\partial Y} = \frac{1}{\varepsilon^2\sqrt{\varepsilon}}\frac{\partial u_2^c}{\partial y} \tag{6.33}$$

$$\frac{\partial U_1^a}{\partial X} = \frac{\partial u_1^a}{\partial x} \tag{6.34}$$

$$\frac{\partial U_1^a}{\partial Y} = \frac{1}{(2-\varepsilon)}\frac{\partial u_1^a}{\partial y} \tag{6.35}$$

$$\frac{\partial U_2^a}{\partial X} = \frac{1}{(2-\varepsilon)}\frac{\partial u_2^a}{\partial x} \tag{6.36}$$

$$\frac{\partial U_2^a}{\partial Y} = \frac{1}{(2-\varepsilon)^2}\frac{\partial u_2^a}{\partial y} \tag{6.37}$$

we get the rescaling of the strain:

$$\mathbf{e}_\varepsilon(\mathbf{u}^b) = \begin{bmatrix} e_{11}(\mathbf{u}^b) & e_{12}(\mathbf{u}^b) \\ e_{12}(\mathbf{u}^b) & e_{22}(\mathbf{u}^b) \end{bmatrix} \tag{6.38}$$

$$\mathbf{e}_\varepsilon(\mathbf{u}^c) = \begin{bmatrix} \frac{1}{\sqrt{\varepsilon}}e_{11}(\mathbf{u}^c) & \frac{1}{\varepsilon\sqrt{\varepsilon}}e_{12}(\mathbf{u}^c) \\ \frac{1}{\varepsilon\sqrt{\varepsilon}}e_{12}(\mathbf{u}^c) & \frac{1}{\varepsilon^2\sqrt{\varepsilon}}e_{22}(\mathbf{u}^c) \end{bmatrix} \tag{6.39}$$

$$\mathbf{e}_\varepsilon(\mathbf{u}^a) = \begin{bmatrix} e_{11}(\mathbf{u}^a) & \frac{1}{(2-\varepsilon)}e_{12}(\mathbf{u}^a) \\ \frac{1}{(2-\varepsilon)}e_{12}(\mathbf{u}^a) & \frac{1}{(2-\varepsilon)^2}e_{22}(\mathbf{u}^a) \end{bmatrix} \tag{6.40}$$

Also, when moving the transmission conditions onto the normal domain, we get:

$$\mathbf{U}^b(X,1) = \mathbf{U}^c(X,1) \Rightarrow \begin{cases} \sqrt{\varepsilon}u_1^b(x,1) = u_1^c(x,1) \\ \varepsilon\sqrt{\varepsilon}u_2^b(x,1) = u_2^c(x,1) \end{cases} \tag{6.41}$$

$$\mathbf{U}^c(X,1+\varepsilon) = \mathbf{U}^a(X,1+\varepsilon) \Rightarrow \begin{cases} u_1^c(x,2) = \sqrt{\varepsilon}u_1^a(x,2) \\ u_2^c(x,2) = \dfrac{\varepsilon\sqrt{\varepsilon}}{2-\varepsilon}u_2^a(x,2) \end{cases} \tag{6.42}$$

For the rescaling of the forces:

$$\begin{aligned} f_1^b(x,y) &= F_1^b(X,Y) \\ f_2^b(x,y) &= F_2^b(X,Y) \\ f_1^c(x,y) &= F_1^c(X,Y) \\ \varepsilon f_2^c(x,y) &= F_2^c(X,Y) \\ f_1^a(x,y) &= F_1^a(X,Y) \\ (2-\varepsilon)f_2^a(x,y) &= F_2^a(X,Y) \end{aligned}$$

we get:

$$\begin{aligned} &\int\limits_{\Omega^b} \mathbb{C}|e(\mathbf{u}^b)|^2 dxdy + \int\limits_{\Omega^c} \mathbb{C}|e(\mathbf{u}^c)|^2 dxdy + (2-\varepsilon)\int\limits_{\Omega^a} \mathbb{C}|e(\mathbf{u}^c)|^2 dxdy \\ = &\int\limits_{\Omega^b} \mathbf{f}^b\mathbf{u}^b dxdy + \sqrt{\varepsilon}\int\limits_{\Omega^c} \mathbf{f}^c\mathbf{u}^c dxdy + (2-\varepsilon)\int\limits_{\Omega^a} \mathbf{f}^a\mathbf{u}^a dxdy \end{aligned} \tag{6.43}$$

6.3. Estimates

Theorem 19 (*Korn Inequality*)

$$\|v\|_{H^1(\Omega)} \leq C\left(\|v\|_{L^2(\Omega)} + \|e(v)\|_{L^2(\Omega)}\right), \forall v \in H^1(\Omega)$$

Corollary 5 *Let Ω be an open set and Γ a part of its boundary. Then*

$$\exists C > 0 \quad \textit{such that} \quad \|v\|_{H^1(\Omega)} \leq C\left(\|e(v)\|_{L^2(\Omega)} + \|v|_\Gamma\|_{L^2(\Gamma)}\right), \forall v \in H^1(\Omega)$$

Observation 4 *If $e(u) = 0$ on Ω and $u = 0$ on a part of $\partial\Omega$ then $u = 0$ on all Ω.*

Corollary 6 *Let Ω be a rectangle and Γ a part of its boundary. Then*

$$\exists C > 0 \quad : \quad \|v\|_{H^1(\Omega)} \leq C\left(\|e(v)\|_{L^2(\Omega)} + \|v\big|_{\Gamma}\|_{L^2(\Gamma)}\right), \forall v \in H^1(\Omega)$$

Proof *Suppose $\exists\{v_n\}_{n\in\mathbb{N}} \subseteq H^1(\Omega)$:*

$$1 \quad = \quad \|v_n\|_{H^1(\Omega)} > u\left(\|e(v_n)\|_{L^2(\Omega)} + \|v_n\big|_{\Gamma}\|_{L^2(\Omega)}\right), \forall n \in \mathbb{N} \tag{6.44}$$

For a subsequence

$$\begin{aligned} v_n &\rightharpoonup v \in H^1(\Omega), v_n \to \in L^2(\Omega), e(v_n) \to 0 \in L^2(\Omega) \\ v_n\big|_{\Gamma} &\to 0 \in L^2(\Gamma) \Rightarrow e(v) = 0, v\big|_{\Gamma} = 0 \Rightarrow v = 0. \end{aligned}$$

Moreover,

$$\begin{aligned} \|v_n - v_m\|_{H^1(\Omega)} &\overset{\text{Korn Inequality}}{\leq} C\left(\|v_n - v_m\|_{L^2(\Omega)} + \|e(v_n) - e(v_m)\|_{L^2(\Omega)}\right) \\ &\Rightarrow \{v_n\}_{n\in\mathbb{N}} \textit{ is Cauchy in } H^1(\Omega) \\ &\Rightarrow v_n \to v \in H^1(\Omega) \\ &\Rightarrow \|v\| = 1. \textit{ False.} \\ &\Rightarrow v = 0 \end{aligned}$$

∎

In the following we will estimate each of the energetic integrals on $\Omega^b, \Omega^c, \Omega^a$.

$$\int_{\Omega^a} |\mathbf{e}_\varepsilon(\mathbf{u}^a_\varepsilon)|^2 \quad \geq \quad \int_{\Omega^a} |\mathbf{e}(\mathbf{u}^a)|^2 \overset{\text{Korn Inequality}}{\geq} C\|\mathbf{u}^a\|^2_{H^1(\Omega^a)} \tag{6.45}$$

$$\begin{aligned} \|\mathbf{u}^c\|^2_{H^1(\Omega^c)} &\overset{\text{Corrolary}}{\leq} C\left(\|\mathbf{e}(\mathbf{u}^c)\|^2_{L^2(\Omega^c)} + \|\mathbf{u}^c\|^2_{L^2(\Gamma_{y=2})}\right) \\ &\leq C\left(\|\mathbf{e}^c(\mathbf{u}^c)\|^2_{L^2(\Omega^c)} + \|\mathbf{u}^c\|^2_{L^2(\Gamma_{y=2})}\right) \\ &\overset{\text{Transmission conditions}}{\leq} C\left(\|\mathbf{e}^c(\mathbf{u}^c)\|^2_{L^2(\Omega^c)} + \varepsilon\|u^a_1\|^2_{L^2(\Gamma_{y=2})} + \varepsilon^3\|u^a_2\|^2_{L^2(\Gamma_{y=2})}\right) \\ &\overset{\text{from(6.45)}}{\leq} C\left(\|\mathbf{e}^c_\varepsilon(\mathbf{u}^c)\|^2_{L^2(\Omega^c)} + \|\mathbf{e}^c(\mathbf{u}^a)\|^2_{L^2(\Omega^a)}\right) \end{aligned} \tag{6.46}$$

$$\int_{\Omega^b} |\mathbf{e}^b_\varepsilon(\mathbf{u}^b)|^2 \quad = \quad \int_{\Omega^b} |\mathbf{e}^b(\mathbf{u}^b)|^2 \overset{\textit{Korn Inequality}}{\geq} C\|\mathbf{u}^b\|^2_{H^1(\Omega^b)} \tag{6.47}$$

Choosing **u** as test function in the variational formulation and using the estimates (6.45), (6.46), (6.47)we obtain that the displacement is bounded in $H^1(\Omega^b), H^1(\Omega^c), H^1(\Omega^a)$.

$$\|\mathbf{u}^a\|_{H^1(\Omega^a)} \leq C \tag{6.48}$$

$$\|\mathbf{u}^c\|_{H^1(\Omega^c)} \leq C \tag{6.49}$$

$$\|\mathbf{u}^b\|_{H^1(\Omega^b)} \leq C \tag{6.50}$$

APPENDIX

A

FUNCTION SPACES

$C_c(\Omega)$ = the space of continuous functions with compact support in Ω

$C^k(\Omega)$ = the space of k times continuously differentiable functions on $\Omega, k \geq 0$

$C^\infty(\Omega) \ = \ \bigcap\limits_{k\geq 0} C^k(\Omega)$

$C^k(\bar{\Omega})$ = functions in $C^k(\Omega)$ such that for every multi-index α with $|\alpha| \leq k$, the function $x \mapsto D^\alpha u(x)$ admits a continuous extension to $\bar{\Omega}$

$C^\infty(\bar{\Omega}) \ = \ \bigcap\limits_{k\geq 0} C^k(\bar{\Omega})$

$$C^{0,\alpha}(\bar{\Omega}) \ = \ \left\{ u \in C(\bar{\Omega}) | \underset{\substack{x,y\in\Omega \\ x\neq y}}{sup} \frac{|u(x)-u(y)|}{|x-y|^\alpha} < \infty \right\} \text{ with } 0 < \alpha < 1$$

$C^{k,\alpha}(\bar{\Omega}) \ = \ \left\{ u \in C^k(\Omega) | D^j u \in C^{0,\alpha}(\bar{\Omega}) \forall j, |j| \leq k \right\}$

$L^p(\Omega) \ = \ \left\{ u : \Omega \to \mathbb{R} | u \text{ is measurable and } \int\limits_\Omega |u|^p < \infty \right\}, 1 \leq p < \infty$

$L^\infty(\Omega) = \{u : \Omega \to \mathbb{R} | u$ is measurable and $|u(x)| \leq C$ a.e. in Ω for some constant C$\}$

$W^{m,p}(\Omega), W^{1,p}(\Omega), W_0^{1,p}(\Omega), W^{1,2}(\Omega), W_0^{1,2}(\Omega), H^m(\Omega)$ = Sobolev spaces

APPENDIX

B

HÖLDER INEQUALITY

Theorem 20 ***(Hölder inequality)***

Assume that $f \in L^p$ and $g \in L^{p'}$ with $1 \leq p \leq \infty$. Then $fg \in L^1$ and

$$\int |fg| \leq \|f\|_p \|g\|_{p'}. \tag{B.1}$$

Proof *The conclusion is obvious if $p = 1$ or $p = \infty$; therefore we assume that $1 \leq p \leq \infty$. We recall Young's inequality:*

$$ab \leq \frac{1}{p} a^p + \frac{1}{p'} b^{p'}, \forall a \geq 0, \forall b \geq 0. \tag{B.2}$$

$$\boldsymbol{log} \left(\frac{1}{p} a^p + \frac{1}{p'} b^{p'} \right) \geq \frac{1}{p} \boldsymbol{log} a^p + \frac{1}{p'} \boldsymbol{log} b^{p'} = \boldsymbol{log} ab. \tag{B.3}$$

We have

$$|f(x)g(x)| \leq \frac{1}{p} |f(x)|^p + \frac{1}{p'} |g(x)|^{p'}, a.e. x \in \Omega. \tag{B.4}$$

It follows that $fg \in L^1$ and

$$\int |fg| \leq \frac{1}{p} \|f\|_p^p + \frac{1}{p'} \|g\|_{p'}^{p'}. \tag{B.5}$$

Replacing f by $\lambda f (\lambda > 0)$ in (B.5), yields

$$\int |fg| \leq \frac{\lambda^{p-1}}{p} \|f\|_p^p + \frac{1}{\lambda p'} \|g\|_{p'}^{p'}. \tag{B.6}$$

Choosing $\lambda = \|f\|_p^{-1} \|g\|_p^{p'/p}$ (so as to minimize the right-hand side in (B.6), we obtain (B.1). ∎

APPENDIX

C

STRAIN TENSOR IN POLAR COORDINATES

In this small appendix we like to present the transformation of the strain tensor from Cartesian to polar coordinates: Given the strain tensor, by definition as the symmetric part of the gradient, in Cartesian coordinates, $e(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$ we compute first the gradient in polar coordinates:

$$
\begin{aligned}
\nabla u &= \vec{\nabla} \otimes \vec{u} && \text{(C.1)}\\
&= \left(\hat{r}\partial_r + \frac{1}{r}\partial_\theta \hat{\theta}\right) \otimes \left(u_r \hat{r} + u_\theta \hat{\theta}\right) + \hat{r}\left(u_{\theta,r} \otimes \hat{\theta} + u_\theta \otimes \hat{\theta}_{,r}\right)\\
&+ \hat{\theta}\left(\frac{1}{r}u_{r,\theta} \otimes \hat{r} + \frac{1}{r}u_r \otimes \hat{r}_{,\theta}\right)\hat{\theta}\left(\frac{1}{r}u_{\theta,\theta} \otimes \hat{\theta} + u_\theta \otimes \frac{1}{r}\hat{\theta}_{,\theta}\right)\\
&= \hat{r}\left(u_{r,r} \otimes \hat{r}\right) + \hat{r}\left(u_{\theta,r} \otimes \hat{\theta}\right) + \hat{\theta}\left(\frac{1}{r}u_{r,\theta} \otimes \hat{r} + \frac{1}{r}u_r \otimes \hat{\theta}\right)\\
&+ \hat{\theta}\left(\frac{1}{r}u_{\theta,\theta} \otimes \hat{\theta} - u_\theta \otimes \frac{1}{r}\hat{r}\right)\\
&= u_{r,r}\hat{r} \otimes \hat{r} + u_{\theta,r}\hat{r} \otimes \hat{\theta} + \frac{1}{r}u_{r,\theta}\hat{\theta} \otimes \hat{r} + \frac{1}{r}u_r\hat{\theta} \otimes \hat{\theta} + \frac{1}{r}u_{\theta,\theta}\hat{\theta} \otimes \hat{\theta}\\
&- \frac{1}{r}u_\theta \hat{\theta} \otimes \hat{r}\\
&= u_{r,r}\hat{r} \otimes \hat{r} + u_{r,r}\hat{r} \otimes \hat{\theta} + \frac{1}{r}\left(u_{r,\theta} - u_\theta\right)\hat{\theta} \otimes \hat{r} + \frac{1}{r}\left(u_r + u_{\theta,\theta}\right)\hat{\theta} \otimes \hat{\theta} && \text{(C.2)}
\end{aligned}
$$

where

$$
\begin{cases}
\hat{r} := \dfrac{\vec{r}}{|\vec{r}|} \\
|\vec{r}| = 1 \\
\hat{\theta} := \dfrac{\vec{\theta}}{|\vec{\theta}|} \\
|\vec{\theta}| = 1
\end{cases}
\tag{C.3}
$$

are the unit base vectors for polar coordinates.

$$
\begin{cases}
\vec{u} = u_r \hat{r} + u_\theta \hat{\theta} \\
\vec{\nabla} = \partial_r \hat{r} + \dfrac{1}{r} \partial_\theta \hat{\theta} \\
\vec{r} = (\cos\theta, \sin\theta) \\
\vec{\theta} = (-\sin\theta, \cos\theta)
\end{cases}
\tag{C.4}
$$

Also, we know that:

$$
\begin{cases}
\dfrac{\partial \vec{r}}{\partial r} = 0 \\
\dfrac{\partial \vec{\theta}}{\partial r} = 0 \\
\dfrac{\partial \vec{r}}{\partial \theta} = \vec{\theta} \\
\dfrac{\partial \vec{\theta}}{\partial \theta} = -\vec{r}
\end{cases}
\tag{C.5}
$$

$$
\nabla u(r,\theta) = \begin{bmatrix} u_{r,r} & u_{\theta,r} \\ \dfrac{1}{r}\left(u_{r,\theta} - u_\theta\right) & \dfrac{1}{r}\left(u_{\theta,\theta} + u_r\right) \end{bmatrix}
\tag{C.6}
$$

$$
\left(\nabla u(r,\theta)\right)^T = \begin{bmatrix} u_{r,r} & \dfrac{1}{r}\left(u_{r,\theta} - u_\theta\right) \\ u_{\theta,r} & \dfrac{1}{r}\left(u_{\theta,\theta} + u_r\right) \end{bmatrix}
\tag{C.7}
$$

$$e(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right) = \begin{bmatrix} u_{r,r} & \frac{1}{2}\left(\frac{1}{r}u_{r,\theta} + u_{\theta,r} - \frac{1}{r}u_\theta\right) \\ \frac{1}{2}\left(\frac{1}{r}u_{r,\theta} + u_{\theta,r} - \frac{1}{r}u_\theta\right) & \frac{1}{r}\left(u_{\theta,\theta} + u_r\right) \end{bmatrix}$$

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