Rigidity and cardinality of moduli space in Real Algebraic Geometry

Riccardo Ghiloni

Department of Mathematics, University of Trento, 38050 Povo, Italy
ghiloni@science.unitn.it

Abstract

Let R be a real closed field and let X be an affine algebraic variety over R. We say that X is universally map rigid (UMR for short) if, for each irreducible affine algebraic variety Y over R, the set of nonconstant rational maps from Y to X is finite. A bijective map $\varphi : \widetilde{X} \longrightarrow X$ from an affine algebraic variety over R to X is called weak change of the algebraic structure of X if it is regular and φ^{-1} is a Nash map preserving nonsingular points. We prove that, when $\dim(X) \geq 1$, there exists a set $\{\varphi_t : \widetilde{X}_t \longrightarrow X\}_{t \in R}$ of weak changes of the algebraic structure of X such that each $\widetilde{X}_t$ is UMR and, for each $t, s \in R$ with $t \neq s$, $\widetilde{X}_t$ and $\widetilde{X}_s$ are birationally nonisomorphic. As an immediate consequence, we solve the problem about the cardinality of the moduli space of birationally nonisomorphic affine real algebraic structures on a topological space, on an affine real Nash manifold and, when R is the field of real numbers, on a smooth manifold. The answer to this problem was already known in the case of compact smooth manifolds.

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1 The theorems

The purpose of this paper is to prove a theorem of rigidity for affine algebraic varieties over a real closed field, which implies basic facts about the cardinality of the set of distinct affine real algebraic structures on a topological space or on a manifold.

Let R be a fixed real closed field. By real algebraic variety, we mean an affine algebraic variety over R. Algebraic varieties and regular maps between them are understood in the sense of Serre [20, 8]. The concept of rational map can be defined in the standard way. Unless otherwise indicated, all real algebraic varieties are equipped with the strong topology induced by the ordering structure on R. We will use standard notions from Real Nash Geometry also (see [8]).

Let us introduce the notions of universal map rigidity and of weak change of the algebraic structure of a real algebraic variety.

Definition 1.1 *A real algebraic variety X is said to be universally map rigid, or* UMR *for short, if, for each irreducible real algebraic variety Y, the set of nonconstant rational maps from Y to X is finite.*

This definition is very restrictive. For example, if a real algebraic variety is UMR, then its group of birational automorphisms is finite. For this reason, all rational real algebraic varieties of positive dimension are not UMR. Significant examples of UMR real algebraic varieties are the irreducible real algebraic curves of genus ≥ 2 (see Lemma 2.3 and Section 3).

Recall that a map $\psi : X \longrightarrow M$ between real algebraic varieties is said to be a biregular embedding if $\psi(X)$ is an algebraic subvariety of M and the restriction of ψ from X to $\psi(X)$ is a biregular isomorphism.

Definition 1.2 *Let X and $\widetilde{X}$ be real algebraic varieties. We say that a map $\varphi : \widetilde{X} \longrightarrow X$ is a weak change of the algebraic structure of X if there exist nonsingular real algebraic varieties M and $\widetilde{M}$, biregular embeddings $\psi : X \longrightarrow M$ and $\widetilde{\psi} : \widetilde{X} \longrightarrow \widetilde{M}$ and a bijective map $\Phi : \widetilde{M} \longrightarrow M$ such that:*

(1) *Φ is regular and, when M and $\widetilde{M}$ are equipped with their natural structures of affine real Nash manifold, Φ^{-1} is a Nash map,*

(2) *the following diagram commutes*

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{\psi}} & \widetilde{M} \\ \varphi \downarrow & & \downarrow \Phi \\ X & \xrightarrow{\psi} & M. \end{array}$$

Observe that, if $\varphi : \widetilde{X} \longrightarrow X$ has the above properties, then it is bijective and regular, and φ^{-1} is a Nash map which sends nonsingular points into nonsingular points. In particular, if X is nonsingular, then $\widetilde{X}$ is nonsingular also.

We have the following theorem of rigidity.

Theorem 1.3 *Given a real algebraic variety X, there is a weak change $\varphi : \widetilde{X} \longrightarrow X$ of its algebraic structure such that $\widetilde{X}$ is* UMR.

Let X and Y be real algebraic varieties. By the symbol $X \not\mapsto Y$, we mean that every rational map from X to Y is Zariski locally constant or, equivalently, that every rational map from an irreducible component of X to Y is constant.

Definition 1.4 *We say that X and Y are algebraically unfriendly if both conditions $X \not\mapsto Y$ and $Y \not\mapsto X$ hold.*

Observe that if X and Y are algebraically unfriendly, then they are birationally nonisomorphic also.

Let X be a fixed real algebraic variety of positive dimension.

Applying Theorem 1.3 to $X \times R$, we obtain the main result of this paper:

Theorem 1.5 *There exists a set $\{\varphi_t : \widetilde{X}_t \longrightarrow X\}_{t \in R}$ of weak changes of the algebraic structure of X such that each $\widetilde{X}_t$ is* UMR *and, for each $t, s \in R$ with $t \neq s$, $\widetilde{X}_t$ and $\widetilde{X}_s$ are algebraically unfriendly.*

This result allows us to compute the cardinality of the moduli space of birationally nonisomorphic affine real algebraic structures on a topological space, on an affine real Nash manifold and on a smooth manifold.

Corollary 1.6 *Let T be a topological space. Suppose T is infinite and admits an affine real algebraic structure. Then the set of birationally nonisomorphic* UMR *affine real algebraic structures on T is equipollent to R.*

When $R = \mathbb{R}$, examples of topological spaces admitting an affine real algebraic structure can be found in [1, 2, 3, 4, 5, 7, 17]. In [10], it is proved that every affine real Nash manifold is Nash isomorphic to a nonsingular real algebraic variety so we have:

Corollary 1.7 *Given a Nash submanifold N of R^n of positive dimension, the set of birationally nonisomorphic* UMR *nonsingular real algebraic varieties that are Nash isomorphic to N is equipollent to R.*

Suppose now $R = \mathbb{R}$. Recall that a smooth manifold admits a structure of nonsingular real algebraic variety if and only if it is diffeomorphic to the interior of a compact smooth manifold with (possibly empty) boundary (see [1, 22]).

Corollary 1.8 *Let M be a smooth manifold. Suppose M has positive dimension and is diffeomorphic to the interior of a compact smooth manifold with* (*possibly empty*) *boundary. Then the set of birationally nonisomorphic* UMR *nonsingular real algebraic varieties that are diffeomorphic to M has the power of continuum.*

The previous result was already known when M is compact and the UMR condition is dropped [6, 9].

Remark. Since the set of real algebraic varieties is equipollent to R, the three previous corollaries remain true if we replace "birationally" with "biregularly" and/or we omit the UMR condition.

The theorems presented above were announced in [12, Section 3]. Further results concerning rigidity–type properties of regular and rational maps between real algebraic varieties can be found in [14].

Our proofs are based on elementary arguments. We are greatly indebted to Marco Forti and János Kollár for their help in improving the original version of Theorem 1.5. We thank also A. Tognoli, R. Benedetti, E. Ballico, S. Baratella and M. Andreatta for several useful discussions.

2 A real de Franchis theorem and a technique of Whitney

The main results of this section are Lemma 2.3 and Lemma 2.8. The first is a real version of the classical finiteness theorem of de Franchis [11], which follows immediately from Theorem 1.4 of [13] and Lemma 3.1 of [14]. The second concerns the existence of vector bases of R^n having good properties with respect to a given algebraic subset of R^n. The proof of this result is available adapting to the real algebraic situation the arguments used by Whitney in Section 10, Chapter 7 of [23]. The present section is, however, self–contained.

Let C be the algebraic closure of the fixed real closed field R, which coincides with $R[\sqrt{-1}\,] = R[x]/(x^2+1)$. Equip each projective space $\mathbb{P}^n(C)$ with its natural structure of algebraic variety over C, indicate by $\sigma_n : \mathbb{P}^n(C) \longrightarrow \mathbb{P}^n(C)$ the complex conjugation map and identify $\mathbb{P}^n(R)$ with the fixed point set of σ_n. A subset S of $\mathbb{P}^n(C)$ is said to be defined over R if it is σ_n–invariant and its real part $S(R)$ is defined as the intersection $S \cap \mathbb{P}^n(R)$. By real algebraic curve and complex algebraic curve (defined over R), we mean respectively a 1–dimensional irreducible real algebraic variety and a 1–dimensional nonsingular irreducible Zariski closed algebraic subvariety of some $\mathbb{P}^n(C)$ (defined over R). For each real algebraic curve D, there is a unique (up to biregular isomorphism) complex algebraic curve D_C defined over R such that $D_C(R)$, viewed in the natural way as a real algebraic variety, is birationally isomorphic to D. Such a curve D_C is called nonsingular projective complexification of D. The genus $g(D)$ of D is defined to be the genus $g(D_C)$ of D_C.

Let X be a real algebraic variety. We indicate by $\mathrm{Nonsing}(X)$ the set of nonsingular points of X of maximum dimension, i.e., of dimension $\dim(X)$. If an algebraic subvariety D of X is a real algebraic curve, then we say that D is a real algebraic curve of X (recall that, according to Serre's definition, algebraic subvarieties are assumed to be Zariski locally closed). For each integer k and for each point p of X, we denote by $\mathcal{C}_X(k,p)$ the set of real algebraic curves of X of genus k and containing p.

Definition 2.1 *Let X be an algebraic subset of R^n of dimension r. First, suppose $r < n$. We define the complete intersection degree* $\mathrm{cideg}(X, R^n)$ *of X in R^n as the minimum integer c such that there are a point $p \in \mathrm{Nonsing}(X)$ and polynomials $P_1, \ldots, P_{n-r}$ in $R[x_1, \ldots, x_n]$ vanishing on X with independent gradients at p and $c = \prod_{i=1}^{n-r} \deg(P_i)$. If $r = n$, then we consider* $\mathrm{cideg}(X, R^n)$ *equal to* 1.

Lemma 2.2 *Let X be an irreducible algebraic subset of R^n and let Z be a non–empty Zariski open subset of X. Define $c := \mathrm{cideg}(X, R^n)$ and $e := (c-1)(c-2)/2$. Then there are $k \in \{0, 1, \ldots, e\}$ and $p \in Z$ such that $\bigcup_{D \in \mathcal{C}_X(k,p)} D$ is Zariski dense in X.*

Proof. Let $r := \dim(X)$. If $r = n$, then the lemma is evident. Suppose $r < n$. By definition of c, there are polynomials $P_1, \ldots, P_{n-r}$ vanishing on X with independent gradients at some point of $\mathrm{Nonsing}(X)$ such that $c = \prod_{i=1}^{n-r} \deg(P_i)$. For each $i \in \{1, \ldots, n-r\}$, define $X_i := P_i^{-1}(0)$. Fix a point $p \in \mathrm{Nonsing}(X) \cap Z$ such that the gradients of $P_1, \ldots, P_{n-r}$ at p are independent. Identify R^n with a subset of $\mathbb{P}^n(R)$ by the affine chart which maps $(x_1, \ldots, x_n)$ into $[x_1, \ldots, x_n, 1]$. Consider X and each X_i as subsets of $\mathbb{P}^n(R)$ and hence as subsets of $\mathbb{P}^n(C)$. Let $\mathcal{L}$ be the set of $(n-r+1)$–dimensional linear subspaces of $\mathbb{P}^n(R)$ containing p and transverse to $\mathrm{Nonsing}(X)$ at p. For each $L \in \mathcal{L}$, denote by D_L the irreducible component of $L \cap X$ containing p. Observe that each D_L is a real algebraic curve of X. Fix $L \in \mathcal{L}$ and denote by $L_C, D_{L,C}, X_{1,C}, \ldots, X_{n-r,C}$ the Zariski closures of $L, D_L, X_1, \ldots, X_{n-r}$ in $\mathbb{P}^n(C)$ respectively. Remark that $D_{L,C}$ is an irreducible component of $L_C \cap \bigcap_{i=1}^{n-r} X_{i,C}$, $p \in \mathrm{Nonsing}(L_C) \cap \bigcap_{i=1}^{n-r} \mathrm{Nonsing}(X_{i,C})$ and $\mathrm{Nonsing}(L_C)$ and $\{\mathrm{Nonsing}(X_{i,C})\}_{i=1}^{n-r}$ are in general position in $\mathbb{P}^n(C)$ locally at p. By Bezout's theorem, it follows that $\deg(D_{L,C}) \leq \prod_{i=1}^{n-r} \deg(X_{i,C}) \leq c$. Applying the Castelnuovo Bound Theorem, we obtain that the geometric genus of $D_{L,C}$, which is equal to $g(D_L)$, is less than or equal to e. On the other hand, the Implicit Function Theorem for Nash maps ensures that $\bigcup_{L \in \mathcal{L}} D_L$ contains a neighborhood of p in $\mathrm{Nonsing}(X)$ so it is Zariski dense in X. Since X is irreducible and $\bigcup_{L \in \mathcal{L}} D_L \subset \bigcup_{k=0}^{e} \bigcup_{D \in \mathcal{C}_X(k,p)} D$, there is $k \in \{0, 1, \ldots, e\}$ such that $\bigcup_{D \in \mathcal{C}_X(k,p)} D$ is Zariski dense in X.

Let X and Y be real algebraic varieties. Suppose X irreducible. We indicate by $\mathcal{R}atio^*(X,Y)$ (resp. $\mathcal{R}^*(X,Y)$) the set of nonconstant rational (resp. regular) maps from X to Y.

Let D and E be real algebraic curves, let $f \in \mathcal{R}^*(D,E)$ and let D_C and E_C be the nonsingular projective complexifications of D and E respectively. Define $D_f := \mathrm{Nonsing}(D) \cap f^{-1}(\mathrm{Nonsing}(E))$ and $f' : D_f \longrightarrow \mathrm{Nonsing}(E)$ as the restriction of f from D_f to $\mathrm{Nonsing}(E)$. Identify D_f and $\mathrm{Nonsing}(E)$ with Zariski open subsets of $D_C(R)$ and $E_C(R)$ respectively. By Zariski's Main Theorem, we know that there is a unique complex regular map $f_C : D_C \longrightarrow E_C$ which extends f'. This map is called complexification of f.

Let A and B be complex algebraic curves. Suppose $g(B) \geq 2$. The finiteness theorem of de Franchis asserts that the set $\mathcal{R}^*_C(A,B)$ of nonconstant complex regular maps from A to B is finite. In [18], H. Martens improved this result showing first the existence of an upper bound for $\sharp\mathcal{R}^*_C(A,B)$ depending only on $g(A)$. Let $\mathbb{N}$ be the set of non–negative integers. We denote by $\mathcal{M} : \mathbb{N} \times (\mathbb{N} \setminus \{0,1\}) \longrightarrow \mathbb{N}$ the function which maps (a,b) into the maximum integer $k \in \mathbb{N}$ such that there are complex algebraic curves A and B with $g(A) = a$, $g(B) = b$ and $\sharp\mathcal{R}^*_C(A,B) = k$. By Hurwitz's formula, it follows that $\mathcal{M}(a,b) = 0$ if $a < b$ so we can define the function $\mathcal{M}_*$ as the smallest function $f : \mathbb{N} \times (\mathbb{N} \setminus \{0,1\}) \longrightarrow \mathbb{N}$ such that $\mathcal{M} \leq f$ and $f(a,b+1) \leq f(a,b) \leq f(a+1,b)$ for each $(a,b) \in \mathbb{N} \times (\mathbb{N} \setminus \{0,1\})$. An explicit upper bound for $\mathcal{M}_*$ can be found in [21].

Lemma 2.3 *Let X be an irreducible algebraic subset of R^n and let D be a real algebraic curve with $g(D) \geq 2$. Then $\mathcal{R}atio^*(X,D)$ is finite. Moreover, if $c := \mathrm{cideg}(X,R^n)$ and $e := (c-1)(c-2)/2$, then we have:*

$$\sharp\mathcal{R}atio^*(X,D) \leq \mathcal{M}_*(e, g(D)).$$

Proof. Let $h := \mathcal{M}_*(e,g(D))$. Suppose $\sharp\mathcal{R}atio^*(X,D) > h$. Then, there are a non–empty Zariski open subset Z of X and maps $f_0, \ldots, f_h$ in $\mathcal{R}^*(Z,D)$ which represent distinct elements of $\mathcal{R}atio^*(X,D)$. By Lemma 2.2, there are $k \in \{0,1,\ldots,e\}$ and $p \in Z$ such that $\bigcup_{E \in \mathcal{C}_X(k,p)} E$ is Zariski dense in X. Define

$$\Delta := \textstyle\bigcup_{i=0}^h f_i^{-1}(f_i(p)) \cup \bigcup_{i \neq j}\{x \in Z \,|\, f_i(x) = f_j(x)\}.$$

Since Δ is a proper Zariski closed subset of Z, there is $E \in \mathcal{C}_X(k,p)$ such that $E \not\subset \Delta$. Let $E' := E \cap Z$. Observe that $g(E') = g(E) = k$. By definition of Δ, we have that the maps $f_0|_{E'}, \ldots, f_h|_{E'}$ are $h+1$ distinct elements of $\mathcal{R}^*(E',D)$. In particular, their complexifications are $h+1$ distinct elements of $\mathcal{R}^*_C(E'_C, D_C)$. This is impossible. In fact, by the de Franchis–Martens theorem, $\sharp\mathcal{R}^*_C(E'_C, D_C) \leq \mathcal{M}(k, g(D)) \leq h$.

By Nash set, we mean a Nash subset of an open semi–algebraic subset of some R^n (see Definition 8.6.1 of [8]). Let $V \subset R^n$ be a Nash set. A point p of V is Nash nonsingular of dimension d if there is an open semi–algebraic neighborhood P of p in R^n such that $V \cap P$ is a Nash submanifold of R^n of dimension d. Indicate by V^* the set of Nash nonsingular points of V of maximum dimension. Let $W \subset R^m$ be another Nash set. A map $f : V \longrightarrow W$ is a Nash map if there exists an open semi–algebraic neighborhood U of V in R^n and an extension $F : U \longrightarrow R^m$ of f from U to R^m, which is Nash, i.e., semi–algebraic of class C^∞.

The next result is quite known. However, for completeness, we give a simple proof. First, we fix a convenction: the dimension of the empty set is equal to -1.

Lemma 2.4 *Let V and W be Nash sets and let $f : V \longrightarrow W$ be a Nash map. For each integer k, the sets $S_k(f) := \{x \in V \,|\, \dim(f^{-1}(f(x))) \geq k\}$ and $T_k(f) := \{y \in W \,|\, \dim(f^{-1}(y)) \geq k\}$ are semi–algebraic.*

Proof. Fix an integer k. We may suppose k non–negative. The set $S_k(f)$ is the inverse image of $T_k(f)$ under f so it suffices to prove that $T_k(f)$ is a semi–algebraic subset of W. Let us proceed by induction on $\nu := \dim(V)$. The case $\nu = 0$ is evident. Let $\nu \geq 1$. Indicate by $V_1, \ldots, V_q$ the Nash irreducible components of V. Since $T_k(f) = \bigcup_{i=1}^q T_k(f|_{V_i})$, we may suppose that V is Nash irreducible. Moreover, replacing W with the smallest Nash subset of W containing $f(V)$, we may suppose that $\dim(f(V)) = \dim(W)$ also. Let $\overline{V}$ be the smallest Nash subset of W containing $V \setminus V^*$. By Sard's theorem, there is a Nash subset Y of W such that $W \setminus Y \subset W^*$, $\dim(f^{-1}(Y)) < \nu$ and the restriction g of f from $V \setminus (\overline{V} \cup f^{-1}(Y))$ to $W \setminus Y$ is a submersion. In particular, we have that $T_k(g) = f(V \setminus \overline{V}) \setminus Y$ if $k \leq \nu - \dim(W)$ and $T_k(g) = \emptyset$ if $k > \nu - \dim(W)$. In any case, $T_k(g)$ is semi–algebraic. Since $\dim(\overline{V} \cup f^{-1}(Y)) < \nu$ and $T_k(f) = T_k(g) \cup T_k(f|_{\overline{V} \cup f^{-1}(Y)})$, the lemma follows by induction.

Let $\mathbb{G}_{n,k}(R)$ be the grassmannian of k–dimensional vector subspaces of R^n. For each $L \in \mathbb{G}_{n,k}(R)$ and for each $x \in R^n$, we indicate by $x + L$ the affine k–plane of R^n defined as $\{x + v \in R^n \,|\, v \in L\}$. As is usual, $\mathbb{G}_{n,1}(R)$ is denoted by $\mathbb{P}^{n-1}(R)$. For each $v \in R^n \setminus \{0\}$, we use the symbol $[v]$ to indicate the vector line of R^n generated by v, viewed as an element of $\mathbb{P}^{n-1}(R)$. Given a family $\{S_1, \ldots, S_m\}$ of subsets of R^n, we denote by $\mathrm{Span}(S_1, \ldots, S_m)$ the smallest vector subspace of R^n containing $\bigcup_{i=1}^m S_i$.

In the remainder of this section, X will be a fixed algebraic subset of R^n of dimension r.

Definition 2.5 *Let $k \in \{0, 1, \ldots, n-r\}$ and let $L \in \mathbb{G}_{n,k}(R)$. We say that L is good for X in R^n if $\sup_{x \in R^n} \sharp((x + L) \cap X)$ is finite. We denote by $W_k(X, R^n)$ the set of elements of $\mathbb{G}_{n,k}(R)$ good for X in R^n.*

Lemma 2.6 *For each $k \in \{0, 1, \ldots, n-r\}$, $W_k(X, R^n)$ is a dense semi–algebraic subset of $\mathbb{G}_{n,k}(R)$.*

Proof. *Step I.* Let us show that each $W_k(X, R^n)$ is semi–algebraic. Fix $k \in \{0, 1, \ldots, n-r\}$. For each $L \in \mathbb{G}_{n,k}(R)$, let $L^\perp$ be the orthogonal of L in R^n and let $\pi_L : R^n \longrightarrow L^\perp$ be the orthogonal projection of R^n onto $L^\perp$. Define $V := \{(x, L) \in R^n \times \mathbb{G}_{n,k}(R) \,|\, x \in L^\perp\}$ and $\rho : X \times \mathbb{G}_{n,k}(R) \longrightarrow V$ by $\rho(x, L) := (\pi_L(x), L)$. Observe that V is Zariski closed in $R^n \times \mathbb{G}_{n,k}(R)$, ρ is a regular map and, for each $(x, L) \in X \times \mathbb{G}_{n,k}(R)$, $\rho^{-1}(\rho(x, L)) = ((x + L) \cap X) \times \{L\}$. Let S_k^* be the set of points $(x, L) \in X \times \mathbb{G}_{n,k}(R)$ such that the dimension of $\rho^{-1}(\rho(x, L))$ is positive and let $\pi_{n,k} : R^n \times \mathbb{G}_{n,k}(R) \longrightarrow \mathbb{G}_{n,k}(R)$ be the natural projection. We have that S_k^* (and hence $\pi_{n,k}(S_k^*)$) is semi–algebraic. On the other hand, using Milnor's theorem concerning upper bounds for the sum of Betti numbers of a real algebraic set [8, Proposition 11.5.4], it follows easily that $S_k(X, R^n)$ is equal to $\mathbb{G}_{n,k}(R) \setminus \pi_{n,k}(S_k^*)$ and hence it is semi–algebraic.

Step II. We will prove that each $W_k(X,R^n)$ is dense in $\mathbb{G}_{n,k}(R)$. First, let us consider the case $k=1$ (hence $n>r$). If $r=0$, then it is evident. Let $r\geq 1$ (hence $n\geq 2$). We know that $W_1(X,R^n)=\mathbb{P}^{n-1}(R)\setminus\pi_{n,1}(S_1^*)$ so it suffices to show that, as a semi–algebraic set, $\pi_{n,1}(S_1^*)$ has dimension $<n-1$. Suppose this is not true. Then there are a non–empty Nash submanifold M of $X\times\mathbb{P}^{n-1}(R)$ contained in S_1^* and an open subset N of $\mathbb{P}^{n-1}(R)$ contained in $\pi_{n,1}(S_1^*)$ such that the restriction $\pi':M\longrightarrow N$ of $\pi_{n,1}$ from M to N is a Nash isomorphism. Let $\phi_n:R^{n-1}\longrightarrow\mathbb{P}^{n-1}(R)$ be the affine chart of $\mathbb{P}^{n-1}(R)$ which maps $w=(w_1,\ldots,w_{n-1})$ into $[\overline{w}]$ where $\overline{w}:=(w_1,\ldots,w_{n-1},1)\in R^n$, let U be a non-empty open semi–algebraic subset of R^{n-1} such that $\phi_n(U)\subset N$ and let $G:U\longrightarrow R^n$ be the unique Nash map such that $(\pi')^{-1}([\overline{w}])=(G(w),[\overline{w}])$ for each $w\in U$. Define the Nash map $\psi:U\times R\longrightarrow R^n$ by $\psi(w,t):=G(w)+t\overline{w}$ and the Nash function $D_\psi:U\times R\longrightarrow R$ as the determinant of the jacobian matrix of ψ. Observe that, being $(G(w),[\overline{w}])\in S_1^*$ for each $w\in U$, $\psi(U\times R)\subset X$. Fix $w\in U$. By simple considerations, we see that $D_\psi(w,t)$ is a monic polynomial in $R[t]$ of degree $n-1$ so, for some t, $D_\psi(w,t)\neq 0$. In particular, it follows that $\psi(U\times R)$ (and hence X) has dimension n, which contradicts our assumption. Let us complete the proof proceeding by induction on $n\geq r$. If $n=r$, the density is evident. Let $n>r$. The case $k=0$ is also evident so we may suppose that $k\geq 1$. Fix $L_0\in\mathbb{G}_{n,k}(R)$. Let ν_0 be an element of $\mathbb{P}^{n-1}(R)$ contained in L_0. By the case $k=1$, we can choose $\nu\in W_1(X,R^n)$ so close to ν_0 that $\nu^\perp\not\supset L_0$ and $L':=\mathrm{Span}(\nu,L_0\cap\nu^\perp)$ is arbitrarily close to L_0. Let $\pi_\nu:R^n\longrightarrow\nu^\perp$ be the orthogonal projection of R^n onto $\nu^\perp$ and let X_ν be the Zariski closure of $\pi_\nu(X)$ in $\nu^\perp$. Observe that, being $\nu\in W_1(X,R^n)$, $\dim(X_\nu)=r$. By induction, there is a $(k-1)$–dimensional vector subspace T of $\nu^\perp$ good for X_ν in $\nu^\perp$ so close to $L_0\cap\nu^\perp$ that $L:=\mathrm{Span}(\nu,T)$ is arbitrarily close to L' (and hence to L_0). Define $a,b\in\mathbb{N}$ as follows: $a:=\sup_{x\in R^n}\sharp((x+\nu)\cap X)$ and $b:=\sup_{x\in\nu^\perp}\sharp((x+T)\cap X_\nu)$. Fix $x\in R^n$. We have that $(\pi_\nu(x)+T)\cap X_\nu=\{p_1,\ldots,p_m\}$ where $m\leq b$. Since $(x+L)\cap X=\bigcup_{j=1}^m((p_j+\nu)\cap X)$, $\sharp((x+L)\cap X)\leq ab$. It follows that $L\in W_k(X,R^n)$ and the proof is complete.

Let $(R^n)^n_*$ be the set of n–uples $(v_1,\ldots,v_n)$ in $(R^n)^n$ such that $\{v_1,\ldots,v_n\}$ is a base of R^n. Equip $(R^n)^n_*$ with the relative topology induced by $(R^n)^n$. Define an equivalence relation on $(R^n)^n_*$ as follows: $(v_1,\ldots,v_n)\sim(v'_1,\ldots,v'_n)$ if and only if the sets $\{[v_1],\ldots,[v_n]\}$ and $\{[v'_1],\ldots,[v'_n]\}$ coincide. Indicate by $\mathcal{B}(R^n)$ the quotient topological space $(R^n)^n_*/\sim$. Observe that $\mathcal{B}(R^n)$ can be identified, in a natural way, with a dense open subset of $(\mathbb{P}^{n-1}(R))^n_{\mathrm{sym}}$. We call an element $\{\nu_1,\ldots,\nu_n\}$ of $\mathcal{B}(R^n)$ geometric base of R^n and each vector line ν_i axis of such base. Let B be a geometric base of R^n and let $k\in\{0,1,\ldots,n\}$. A coordinate k–plane of B is a k–dimensional vector subspace of R^n generated by k axes of B. The unique coordinate 0–plane of B is $\{0\}$.

Definition 2.7 *We say that a geometric base B of R^n is good for X if, for each $k\in\{0,1,\ldots,n-r\}$, all coordinate k–planes of B are good for X in R^n.*

Lemma 2.8 *Let $\mathcal{H}(X,R^n)$ be the set of elements of $\mathcal{B}(R^n)$ good for X. Then the interior of $\mathcal{H}(X,R^n)$ in $\mathcal{B}(R^n)$ is dense in $\mathcal{B}(R^n)$. In particular, $\mathcal{H}(X,R^n)$ is non–empty.*

Proof. The case $n=r$ is evident. Suppose $n>r$. Let $\sigma:(R^n)^n_*\longrightarrow\mathcal{B}(R^n)$ be the natural projection. For each $k\in\{1,\ldots,n-r\}$, let H_k be the interior of $W_k(X,R^n)$ in $\mathbb{G}_{n,k}(R)$. By Lemma 2.6, it follows that each H_k is a dense (semi–algebraic) subset of $\mathbb{G}_{n,k}(R)$. For each $\chi \subset \{1,\ldots,n\}$ with $\chi \neq \emptyset$, let $\varphi_\chi : (R^n)^n_* \longrightarrow \mathbb{G}_{n,\sharp\chi}(R)$ be the map which sends $(v_1,\ldots,v_n)$ into $\mathrm{Span}(\{v_i\}_{i\in\chi})$. Define the map

$$\varphi : (R^n)^n_* \longrightarrow \prod_{\chi\subset\{1,\ldots,n\},1\leq\sharp\chi\leq n-r} \mathbb{G}_{n,\sharp\chi}(R)$$

by $\varphi := \prod_{\chi\subset\{1,\ldots,n\},1\leq\sharp\chi\leq n-r} \varphi_\chi$ and the set H by

$$H := \varphi^{-1}\left(\prod_{\chi\subset\{1,\ldots,n\},1\leq\sharp\chi\leq n-r} H_{\sharp\chi}\right).$$

Since each map φ_χ is continuous and open, we have that H is a dense open subset of $(R^n)^n_*$. On the other hand, $H = \sigma^{-1}(\sigma(H))$ so $\sigma(H)$ is a dense open subset of $\mathcal{B}(R^n)$. Since $\sigma(H)$ is contained in $\mathcal{H}(X,R^n)$, we are done.

3 Proofs of the theorems

We need some preparations.

Let $E_1,\ldots,E_n$ be real algebraic curves, let T be the product variety $\prod_{i=1}^n E_i$ and let Y be a s–dimensional algebraic subvariety of T. For each $\chi \subset \{1,\ldots,n\}$, define $T_\chi := \prod_{i\in\chi} E_i$ (where $T_\emptyset$ is considered equal to a point) and indicate by $\pi_\chi : T \longrightarrow T_\chi$ the natural projection. We say that Y is in good position into T if, for each $\chi \subset \{1,\ldots,n\}$ with $\sharp\chi \geq s$, $G_\chi(Y,T) := \sup_{p\in T_\chi} \sharp(Y \cap \pi_\chi^{-1}(p))$ is finite. If Y has this property, then, for each $k \in \{0,1,\ldots,n-s\}$, we define the integer $G_k(Y,T) := \sum_{\chi\subset\{1,\ldots,n\},\sharp\chi=n-k} G_\chi(Y,T)$.

Let X be a r–dimensional algebraic subset of R^n and let $k \in \{0,1,\ldots,n-r\}$. Given a k–plane L good for X in R^n, we define the integer $N_L(X,R^n) := \sup_{x\in R^n} \sharp((x+L)\cap X)$. Let B be a geometric base of R^n good for X. Indicate by $B(k)$ the set of coordinate k–planes of B and define $N_{B(k)}(X,R^n) := \sum_{L\in B(k)} N_L(X,R^n)$. Observe that, if B is the geometric base of R^n induced by the canonical base, then B is good for X if and only if X is in good position into R^n. Moreover, in this situation, $G_k(X,R^n) = N_{B(k)}(X,R^n)$ for each $k \in \{0,1,\ldots,n-r\}$.

Let d be an odd positive integer. Define the nonsingular real algebraic curve D_d as $\{(x,y) \in R^2 \,|\, y^d = 1 + x^{2d}\}$ and the regular map $\psi_d : D_d \longrightarrow R$ by $\psi_d(x,y) := x$. Since D_d is the graph of the Nash function on R which maps x into $\sqrt[d]{1+x^{2d}}$, we have that ψ_d is a Nash isomorphism. Let us show that $g(D_d) \geq (d-1)(d-2)/2$. Let $F_d := \{(x,y) \in R^2 \,|\, x^d + y^d = 1\}$ and let $P_d : D_d \longrightarrow F_d$ be the polynomial map defined by $P_d(x,y) := (-x^2, y)$. Evidently, the complexification $P_{d,C} : D_{d,C} \longrightarrow F_{d,C}$ of P_d is nonconstant so, by Hurwitz's formula, it follows that $g(D_d) \geq g(F_d) = (d-1)(d-2)/2$ as desired.

Proof of Theorem 1.3. Step I. Without loss of generality, we may suppose that X is a r–dimensional algebraic subset of R^n with $r \geq 1$ and $n \geq 2r-1$. By Lemma 2.8, there is a geometric base B of R^n good for X. Up to a linear change of coordinates of R^n, we may suppose that B coincides with the geometric base of R^n induced by the canonical base. Let d be an odd positive integer such that $h := (d-1)(d-2)/2 \geq 2$ and let D_d and $\psi_d : D_d \longrightarrow R$ be as above. Define T as the product variety D_d^n, $\psi_d^n : T \longrightarrow R^n$ as the n^{th}–power of ψ_d, $\widetilde{X} := (\psi_d^n)^{-1}(X)$ and $\varphi : \widetilde{X} \longrightarrow X$ as the restriction of ψ_d^n from $\widetilde{X}$ to X. It is immediate to see that φ is a weak change of the algebraic structure of X and $\widetilde{X}$ is in good position into T. Moreover, for each $k \in \{0, 1, \ldots, n-r\}$, $G_k(\widetilde{X}, T) = N_{B(k)}(X, R^n)$.

Step II. We will show that, for each irreducible algebraic subset Y of some R^m, it holds:

$$\sharp \mathcal{R}atio^*(Y, \widetilde{X}) \leq \textstyle\sum_{k=0}^{r-1} N_{B(k)}(X, R^n) \cdot \mathcal{M}_*(e_Y, h)^{n-k} \tag{1}$$

where, setting $c_Y := \mathrm{cideg}(Y, R^m)$, $e_Y := (c_Y - 1)(c_Y - 2)/2$. In particular, $\widetilde{X}$ will be UMR. Fix such a Y. By Lemma 2.3, we know that $M := \sharp \mathcal{R}atio^*(Y, D_d) \leq \mathcal{M}_*(e_Y, h)$ (recall that $g(D_d) \geq h$). If $M = 0$, then $\mathcal{R}atio^*(Y, T) = \emptyset$ so $\mathcal{R}atio^*(Y, \widetilde{X}) = \emptyset$ also and (1) is true. Suppose $M \geq 1$. Let Z be a non–empty Zariski open subset of Y and let $g_1, \ldots, g_M \in \mathcal{R}^*(Z, D_d)$ such that the rational maps from Y to D_d represented by the pairs $\{(Z, g_i)\}_{i=1}^M$ are exactly the elements of $\mathcal{R}atio^*(Y, D_d)$. Observe that $\mathcal{R}^*(Z, D_d) = \{g_1, \ldots, g_M\}$. Define:

$$\mathcal{R}^* := \{f \in \mathcal{R}^*(Z, T) \,|\, f(Z) \subset \widetilde{X}\}.$$

Identify $\mathcal{R}atio^*(Y, \widetilde{X})$ with $\mathcal{R}^*$ in the natural way. For each $i \in \{1, \ldots, n\}$, indicate by $\pi_i : T \longrightarrow D_d$ the natural projection of T onto its i^{th}–coordinate space. For each $\chi \subset \{1, \ldots, n\}$ with $\chi \neq \emptyset$, let $F(\chi)$ be the set of functions from χ to $\{1, \ldots, M\}$, let $\chi' := \{1, \ldots, n\} \setminus \chi$ and define $T_{\chi'}$ and $\pi_{\chi'} : T \longrightarrow T_{\chi'}$ as above. Moreover, for each $\chi \subset \{1, \ldots, n\}$ with $\chi \neq \emptyset$ and for $\xi \in F(\chi)$, define:

$$\mathcal{R}^*_{\chi,\xi} := \{f \in \mathcal{R}^* \,|\, \pi_i \circ f = g_{\xi(i)} \text{ for each } i \in \chi,\ \pi_{\chi'} \circ f \text{ is constant}\}.$$

Evidently, the family of $\mathcal{R}^*_{\chi,\xi}$'s is a partition of $\mathcal{R}^*$. Fix $\chi \subset \{1, \ldots, n\}$ with $\chi \neq \emptyset$ and $\xi \in F(\chi)$. First, suppose $\sharp\chi \leq n - r$. Let us show that $\mathcal{R}^*_{\chi,\xi} = \emptyset$. Suppose this is not true. Then there would exist $f \in \mathcal{R}^*(Z, T)$ and $p \in T_{\chi'}$ such that $f(Z) \subset \widetilde{X} \cap \pi_{\chi'}^{-1}(p)$. Since $\widetilde{X}$ is in good position into T, f would be constant which contradicts our assumption. Suppose now $\sharp\chi > n - r$ and $\mathcal{R}^*_{\chi,\xi} \neq \emptyset$. Fix $z \in Z$. Define $\Psi_z : \mathcal{R}^*_{\chi,\xi} \longrightarrow \widetilde{X}$ by $\Psi_z(f) := f(z)$ and $z_{\chi,\xi}$ as the point of T_χ such that $\{z_{\chi,\xi}\} = \pi_\chi(f(Z))$ for some (and hence every) element f of $\mathcal{R}^*_{\chi,\xi}$. Evidently, Ψ_z is injective and $\Psi_z(\mathcal{R}^*_{\chi,\xi}) \subset \widetilde{X} \cap \pi_\chi^{-1}(z_{\chi,\xi})$. On the other hand, $\sharp\chi \geq n - r + 1 \geq r$ so, by the good position of $\widetilde{X}$ into T, we infer that $\mathcal{R}^*_{\chi,\xi}$ is finite and $\sharp\mathcal{R}^*_{\chi,\xi} \leq G_\chi(\widetilde{X}, T)$. In particular, $\mathcal{R}^*$ is finite and it holds:

$$\begin{aligned}
\sharp\mathcal{R}^* &= \textstyle\sum_{\chi \subset \{1,\ldots,n\},\, \sharp\chi > n-r} G_\chi(\widetilde{X}, T) \cdot \sharp F(\chi) = \\
&= \textstyle\sum_{k=0}^{r-1} G_k(\widetilde{X}, T) \cdot M^{n-k} = \\
&= \textstyle\sum_{k=0}^{r-1} N_{B(k)}(X, R^n) \cdot M^{n-k} \leq \\
&\leq \textstyle\sum_{k=0}^{r-1} N_{B(k)}(X, R^n) \cdot \mathcal{M}_*(e_Y, h)^{n-k}.
\end{aligned}$$

The next lemma is the semi–algebraic version of a result suggested to us by János Kollár. It will be used in the proof of Theorem 3.3 below.

Lemma 3.1 *Let M be a Nash submanifold of R^n, let ν be an integer and let $f : M \longrightarrow R$ be a Nash function such that, for each $s \in f(M)$, $f^{-1}(s)$ is a semi–algebraically connected Nash submanifolds of R^n of dimension ν. For* *each $s \in f(M)$, denote by X_s the Zariski closure of $f^{-1}(s)$ in R^n. Let Y be a real algebraic variety. Define $\mathcal{K}_f(Y)$ as the set of points s of $f(M)$ such that there exists a nonconstant rational map from X_s to Y. Then $\mathcal{K}_f(Y)$ is a countable union of semi–algebraic subsets of R.*

Proof. Replacing M with the graph of f, we may suppose that M is a Nash submanifold of $R^n \times R$ and f coincides with the restriction to M of the natural projection of $R^n \times R$ onto R. Moreover, without loss of generality, we may suppose that Y in an algebraic subset of R^m also. For each $d \in \mathbb{N}$, define $N_d := \binom{n+d}{d}$. Identify each polynomial q in $R[x_1, \dots, x_n]$ of degree $\leq d$ with the point of R^{N_d}, whose coordinates are the coefficients of q. In this way, if $\bar{p} = (p_1, \dots, p_m)$ is a m–uple of polynomials in $R[x_1, \dots, x_n]$ of degree $\leq d$, then $\bar{p}$ correspondes with a point of R^{mN_d}. Let $q \in R^{N_d}$, let $\bar{p} \in R^{mN_d}$ and let A be a subset of R^m. Denote by $(\bar{p}/q)^{-1}(A)$ the empty set if $q = 0$ or the set $\{x \in R^n \setminus q^{-1}(0) \,|\, \bar{p}(x)/q(x) \in A\}$ if $q \neq 0$. For each $s \in f(M)$, define $M_s := f^{-1}(s)$ and, for each $d \in \mathbb{N}$, define:

$R^{(d)} := R \times R^{mN_d} \times R^{N_d}$,

$K_d := \{(s, \bar{p}, q) \in R^{(d)} \,|\, \emptyset \neq M_s \setminus q^{-1}(0) \subset (\bar{p}/q)^{-1}(Y)\}$,

$G_d := \{(s, \bar{p}, q, y) \in R^{(d)} \times R^m \,|\, \emptyset \neq M_s \setminus q^{-1}(0) \subset (\bar{p}/q)^{-1}(y)\}$

and denote by $\pi_d : R^{(d)} \times R^m \longrightarrow R^{(d)}$ and $\rho_d : R^{(d)} \longrightarrow R$ the natural projections. It is immediate to see that $\mathcal{K}_f(Y) = \bigcup_{d \in \mathbb{N}} \rho_d(K_d \setminus \pi_d(G_d))$ so it suffices to prove that, for each $d \in \mathbb{N}$, both K_d and G_d are semi–algebraic subsets of $R^{(d)}$ and of $R^{(d)} \times R^m$ respectively. Let g be the polynomial in $R[y] = R[y_1, \dots, y_m]$ such that $g^{-1}(0) = Y$. Write g as follows: $g(y) := \sum_{j=0}^{e} g_j(y)$ where e is the degree of g and each g_j is a homogeneous polynomial in $R[y]$ of degree j. Define the polynomial $G \in R[y, t] = R[y_1, \dots, y_m, t]$ by $G(y, t) := \sum_{j=0}^{e} t^{e-j} g_j(y)$ and observe that $G(y, t) = t^e g(\frac{y}{t})$ over $R^m \times (R \setminus \{0\})$. Define: the Nash set $M^{(d)} \subset R^n \times R^{(d)}$ by

$$M^{(d)} := M \times R^{mN_d} \times R^{N_d},$$

the algebraic subset Q_d of $R^n \times R^{(d)}$ by

$$Q_d := \{(x, s, \bar{p}, q) \in R^n \times R^{(d)} \,|\, q(x) = 0\},$$

the polynomial function $\xi_d : R^n \times R^{(d)} \longrightarrow R$ and the algebraic subset S_d of $R^n \times R^{(d)}$ by

$$\xi_d(x, s, \bar{p}, q) := G(\bar{p}(x), q(x)) \;\text{ and }\; S_d := \xi_d^{-1}(0),$$

the Nash map $\Pi_d : (M^{(d)} \cap S_d) \setminus Q_d \longrightarrow R^{(d)}$ as the restriction to $(M^{(d)} \cap S_d) \setminus Q_d$ of the natural projection of $R^n \times R^{(d)}$ onto $R^{(d)}$ and the Nash map $\eta_d : M^{(d)} \setminus Q_d \longrightarrow R^{(d)} \times R^m$ by

$$\eta_d(x, s, \bar{p}, q) := (s, \bar{p}, q, \bar{p}(x)/q(x)).$$

For each $(s, \bar{p}, q) \in R^{(d)}$ and for each $y \in R^m$, we have:

$$\Pi_d^{-1}(s, \bar{p}, q) = (M_s \cap (\bar{p}/q)^{-1}(Y)) \times \{(s, \bar{p}, q)\}$$

and

$$\eta_d^{-1}(s, \bar{p}, q, y) = (M_s \cap (\bar{p}/q)^{-1}(y)) \times \{(s, \bar{p}, q)\}.$$

In particular, it holds: $K_d = \{(s,\bar{p},q) \in R^{(d)} \mid \dim(\Pi_d^{-1}(s,\bar{p},q)) \geq \nu\}$ and $G_d = \{(s,\bar{p},q,y) \in R^{(d)} \times R^m \mid \dim(\eta_d^{-1}(s,\bar{p},q,y)) \geq \nu\}$. By Lemma 2.4, K_d and G_d are semi–algebraic as desired.

Let X be a real algebraic variety. Denote by $A(X)$ the set of Nash nonsingular points of X. Since $A(X)$ is semi–algebraic, we know that it has finitely many semi–algebraically connected components. Let $M_1,\ldots,M_a$ be such semi–algebraically connected components and, for each $i \in \{1,\ldots,a\}$, let $\overline{M}_i$ be the Zariski closure of M_i in X. Observe that the set $\{\overline{M}_1,\ldots,\overline{M}_a\}$ contains the set of irreducible components of X. Let Y be a real algebraic variety. By the symbol $X \rightsquigarrow Y$, we mean that, for some $i \in \{1,\ldots,a\}$, there is a nonconstant rational map from $\overline{M}_i$ to Y.

Definition 3.2 *We say that X and Y are algebraically enemy if both conditions $X \rightsquigarrow Y$ and $Y \rightsquigarrow X$ do not hold.*

Observe that if X and Y are algebraically enemy, then they are algebraically unfriendly also.

The following result extends Theorem 1.5.

Theorem 3.3 *Given a real algebraic variety X of positive dimension, there exists a set $\{\varphi_t : \widetilde{X}_t \longrightarrow X\}_{t\in R}$ of weak changes of the algebraic structure of X such that each $\widetilde{X}_t$ is* UMR *and, for each $t,s \in R$ with $t \neq s$, $\widetilde{X}_t$ and $\widetilde{X}_s$ are algebraically enemy.*

Proof. We organize the proof into four steps.

Step I. We begin as in *Step I* of the proof of Theorem 1.3. We may suppose that X is a r–dimensional algebraic subset of R^n with $n \geq 2r$. Identify X with $X\times\{0\} \subset R^n \times R = R^{n+1}$ and R^{n+1} with $R^{n+1}\times\{0\} \subset R^{n+1}\times R^{n+1} = R^{2n+2}$. Let $c := \mathrm{cideg}(X,R^{n+1})$. Applying Lemma 2.8 to $X \times R \subset R^{n+1}$, we find a linear automorphism $\Psi : R^{n+1} \longrightarrow R^{n+1}$ such that the geometric base B of R^{n+1} induced by the canonical base is good for $X^* := \Psi(X)$. For each $t \in R$, define $X_t := \Psi(X \times \{t\})$. Fix an odd positive integer d such that $h := (d-1)(d-2)/2 \geq 2$. Define: $D_d \subset R^2$ and $\psi_d : D_d \longrightarrow R$ as above, $\psi_d^{n+1} : D_d^{n+1} \longrightarrow R^{n+1}$ as the $(n+1)^{\mathrm{th}}$–power of ψ_d, $\widetilde{X}^* := (\psi_d^{n+1})^{-1}(X^*)$ and, for each $t \in R$, $\widetilde{X}_t := (\psi_d^{n+1})^{-1}(X_t) \subset R^{2n+2}$, $\widetilde{\varphi}_t : \widetilde{X}_t \longrightarrow X_t$ as the restriction of ψ_d^{n+1} from $\widetilde{X}_t$ to X_t and $\varphi_t : \widetilde{X}_t \longrightarrow X$ as the composition $\pi\circ\Psi^{-1}\circ\widetilde{\varphi}_t$ where $\pi : X \times R \longrightarrow X$ is the natural projection. Observe that each φ_t is a weak change of the algebraic structure of X and $\widetilde{X}^*$ is in good position into D_d^{n+1}. By hypothesis $n+1 \geq 2(r+1)-1$ so, applying *Step II* of the proof of Theorem 1.3 to $\widetilde{X}^*$, we obtain that, for each irreducible algebraic subset Y of some R^m, it holds:

$$\sharp\mathcal{R}atio^*(Y,\widetilde{X}^*) \leq \textstyle\sum_{k=0}^{r} N_{B(k)}(X^*,R^{n+1})\cdot\mathcal{M}_*(e_Y,h)^{n+1-k} \tag{2}$$

where, setting $c_Y := \mathrm{cideg}(Y,R^m)$, $e_Y := (c_Y-1)(c_Y-2)/2$. In particular, $\widetilde{X}^*$ is UMR and hence each $\widetilde{X}_t$ is.

Step II. Let $M_1,\ldots,M_a$ be the semi–algebraically connected components of $A(X)$ and let $\overline{M}_1,\ldots,\overline{M}_a$ be their Zariski closures in X. Let $\gamma := \max_{i\in\{1,\ldots,a\}} \mathrm{cideg}(\overline{M}_i,R^{n+1})$. For each $t \in R$ and for each $i \in \{1,\ldots,a\}$, define $M_{i,t}^* := \varphi_t^{-1}(M_i)$ and $\overline{M}_{i,t}^*$ as the Zariski closure of $M^*_{i,t}$ in $\widetilde{X}_t$. Observe that, being φ_t a Nash isomorphism, $M^*_{1,t},\ldots,M^*_{a,t}$ are the semi–algebraically connected components of $A(\widetilde{X}_t)$ for each $t\in R$. We will show that:

$$\max_{i\in\{1,\ldots,a\}}\operatorname{cideg}(\overline{M}^*_{i,t},R^{2n+2})\leq(2d)^{n+1}\gamma \tag{3}$$

for each $t\in R$. Fix $t\in R$. Let $\pi_t:X_t\longrightarrow X$ be the biregular isomorphism defined by $\pi_t:=\pi\circ\Psi^{-1}|_{X_t}$. For each $i\in\{1,\ldots,a\}$, define $M_{i,t}:=\pi_t^{-1}(M_i)$ and $\overline{M}_{i,t}$ as the Zariski closure of $M_{i,t}$ in X_t. Observe that X and X_t are affinely equivalent so $\operatorname{cideg}(\overline{M}_i,R^{n+1})=\operatorname{cideg}(\overline{M}_{i,t},R^{n+1})$ for each $i\in\{1,\ldots,a\}$. In particular, $\gamma=\max_{i\in\{1,\ldots,a\}}\operatorname{cideg}(\overline{M}_{i,t},R^{n+1})$. Fix $i\in\{1,\ldots,a\}$. Since $\widetilde{\varphi}_t(M^*_{i,t})=M_{i,t}$, there is a non–empty open subset U of $\operatorname{Nonsing}(\overline{M}_{i,t})$ such that $(\widetilde{\varphi}_t)^{-1}(U)$ is a non–empty open subset of $\operatorname{Nonsing}(\overline{M}^*_{i,t})$. Let $\rho:R^{2n+2}\longrightarrow R^{n+1}$ be the natural projection of $R^{n+1}\times R^{n+1}$ onto its first coordinate space. Since $(\widetilde{\varphi}_t)^{-1}(U)=D^{n+1}_d\cap\rho^{-1}(U)$ and $\overline{M}^*_{i,t}$ is irreducible, it is easy to verify that $\operatorname{cideg}(\overline{M}^*_{i,t},R^{2n+2})\leq(2d)^{n+1}\operatorname{cideg}(\overline{M}_{i,t},R^{n+1})$. In particular, we have (3).

Step III. For each $t\in R$, we define:
$A_t:=\{s\in R\setminus\{t\}\,|\,\widetilde{X}_t\rightsquigarrow\widetilde{X}_s\}$,
$B_t:=\{s\in R\setminus\{t\}\,|\,\widetilde{X}_s\rightsquigarrow\widetilde{X}_t\}$,
$C_t:=R\setminus(A_t\cup B_t)=\{t\}\cup\{s\in R\,|\,\widetilde{X}_t$ and $\widetilde{X}_s$ are alg. enemy$\}$.
Moreover, we define:
$e:=\frac{1}{2}\big((2d)^{n+1}\gamma-1\big)\big((2d)^{n+1}\gamma-2\big)$,
$b:=\sum_{k=0}^{r}N_{B(k)}(X^*,R^{n+1})\cdot\mathcal{M}_*(e,h)^{n+1-k}$,
$\ell:=ab$.

From (2) and (3), it follows immediately that:

$$\sup_{t\in R}\sharp A_t<\ell. \tag{4}$$

Let $t_1,\ldots,t_\ell$ be distinct elements of R. We have:

$$\textstyle\bigcap_{i=1}^{\ell}B_{t_i}\text{ is empty.} \tag{5}$$

Otherwise, fixed t in $\bigcap_{i=1}^{\ell}B_{t_i}$, A_t would contain $\{t_1,\ldots,t_\ell\}$ and (4) would be contradicted. In particular, it holds:

$$\text{the set }R\setminus\textstyle\bigcup_{i=1}^{\ell}C_{t_i}\text{ is finite.} \tag{6}$$

In fact, $R\setminus\bigcup_{i=1}^{\ell}C_{t_i}=\bigcap_{i=1}^{\ell}(A_{t_i}\cup B_{t_i})\subset\bigcup_{i=1}^{\ell}A_{t_i}$ so, from (4), it follows that $\sharp(R\setminus\bigcup_{i=1}^{\ell}C_{t_i})<\ell^2$.

We will show that there is a non–empty open (bounded or unbounded) interval I of R such that

$$I\cap B_t\text{ is countable for each }t\in I. \tag{7}$$

Fix $t\in R$. For each $i\in\{1,\ldots,a\}$, define: the Nash submanifold $M^{(i)}$ of R^{2n+2} by $M^{(i)}:=(\psi^{n+1}_d)^{-1}(\Psi(M_i\times R))$, the Nash function $f_i:M^{(i)}\longrightarrow R$ as the restriction to $M^{(i)}$ of the composition map $\pi\circ\Psi^{-1}\circ\psi^{n+1}_d$ and $\mathcal{K}_i(\widetilde{X}_t)$ as the set of points s of R such that there is a nonconstant rational map from the Zariski closure of $f_i^{-1}(s)=M^*_{i,t}$ in R^{2n+2} (which is $\overline{M}^*_{i,s}$) to $\widetilde{X}_t$. Applying Lemma 3.1 to each f_i, we obtain that $\mathcal{K}_i(\widetilde{X}_t)$ is a countable union of semi–algebraic subsets of R for each $i \in \{1, \ldots, a\}$. Since $B_t = \bigcup_{i=1}^{a} (\mathcal{K}_i(\widetilde{X}_t) \setminus \{t\})$, we have that B_t is a countable union of semi–algebraic subsets of R also. Indicate by $\mathrm{Int}(B_t)$ the interior of B_t in R and define $N_t := B_t \setminus \mathrm{Int}(B_t)$. Since a semi–algebraic subset of R is a finite union of points and open intervals, it follows that N_t is countable. Thanks to this property of N_t, it suffices to prove the existence of a non–empty open interval I of R such that $I \cap \mathrm{Int}(B_t) = \emptyset$ for each $t \in I$. Suppose this is not true. Then, by a simple inductive argument, it would be possible to find a ℓ–uple $(t_1, \ldots, t_\ell)$ of distinct points of R such that $t_i \in \bigcap_{k=1}^{i} \mathrm{Int}(B_{t_k})$ for each $i \in \{1, \ldots, \ell\}$. In particular, $\bigcap_{k=1}^{\ell} B_{t_k}$ would contain the point t_ℓ, which is impossible by (5). This proves (7). Let I be an interval of R, which satisfies (7). Since I is equipollent to R, replacing R with I and A_t, B_t and C_t with $I \cap A_t$, $I \cap B_t$ and $I \cap C_t$ respectively, we may suppose that

$$\text{the set } R \setminus C_t = A_t \cup B_t \text{ is countable for each } t \in R. \tag{8}$$

Given a subset S of R and an integer $k \in \mathbb{N}$, we say that S has the property $\mathcal{P}(k)$ if $\sharp S = \sharp R$ and the set $S \setminus \bigcup_{i=1}^{k} C_{t_i}$ is finite for each k–uple $(t_1, \ldots, t_k)$ of distinct elements of S. Let L be the set of positive integers k such that there is a subset S of R having the property $\mathcal{P}(k)$. Thanks to (6), we know that $\ell \in L$ so L is non–empty. Let ℓ^* be the minimum of L and let R^* be a subset of R with the property $\mathcal{P}(\ell^*)$. Let us show that, for each $t \in R^*$, $\sharp(R^* \setminus C_t) < \sharp R^*$. If $\ell^* = 1$, this is obvious because each set $R^* \setminus C_t$ is finite. Let $\ell^* \geq 2$. If there would exist $t \in R^*$ with $\sharp(R^* \setminus C_t) = \sharp R^*$, then $R^* \setminus C_t$ would satisfy $\mathcal{P}(\ell^* - 1)$ contradicting the minimality of l^*. In this way, replacing R with R^* and each C_t with $C_t \cap R^*$, we may suppose that:

$$\sharp(R \setminus C_t) < \sharp R \text{ for each } t \in R. \tag{9}$$

Step IV. We will prove the existence of a subset H of R equipollent to R such that, for each $t, s \in H$ with $t \neq s$, $s \in C_t$ or, equivalently, $H \subset \bigcap_{t \in H} C_t$ (observe that $s \in C_t$ implies $t \in C_s$). Evidently, the set $\{\varphi_t : \widetilde{X}_t \longrightarrow X\}_{t \in H}$ will satisfy the properties required in Theorem 1.5. Let $\mathcal{F}$ be the family of subsets S of R such that $S \subset \bigcap_{t \in S} C_t$, equipped with the partial ordering induced by the inclusion. Observe that $\mathcal{F}$ is non–empty because $\{t\} \in \mathcal{F}$ for each $t \in R$. Moreover, if $\{S_j\}_{j \in J}$ is a chain in $\mathcal{F}$, then $\bigcup_{j \in J} S_j$ is an upper bound in $\mathcal{F}$ for that chain. In this way, by Zorn's lemma, $\mathcal{F}$ has a maximal element H. Remark that $H = \bigcap_{t \in H} C_t$. Otherwise, there would exist $s \in (\bigcap_{t \in H} C_t) \setminus H$ and $H \cup \{s\}$ would be an element of $\mathcal{F}$ contradicting the maximality of H. Since $R \setminus H = \bigcup_{t \in H} (R \setminus C_t)$, using (8) when R is uncountable and (9) when R is countable, we obtain that H is equipollent to R. The proof is complete.

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