



UNIVERSITÀ DEGLI STUDI DI TRENTO

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Doctoral School in Mathematics, 23rd cycle

On problems in homogenization and
two-scale convergence

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April 2012

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Chapter 1

Preface

Looking back on the path that led me to writing this thesis, one cannot say that it was always a straight one. Having been to Trento for the first time of my life in September 2007, it took me just a couple of days to decide to enrol in the doctoral school of the Università di Trento. Since I had already started a research project at the Technische Universität München earlier that year, the original idea of both my supervisors, Prof. Augusto Visintin and Prof. Martin Brokate, was to do a joint PhD between the Università di Trento and the Technische Universität München, and that I stay half of my time in Italy and the other half in Germany. Yet, once more it turned out that life was non-deterministic. In the end, I spent the entire time from December 2007 to October 2010 in Trento, I wrote one PhD thesis which I defended in December 2009 at the Technische Universität München, and I wrote *this* PhD thesis which I shall defend at the Università di Trento within this month. Over the years that have passed since 2007, just like any other PhD student, I had to struggle with the inevitable ups and downs, with the joy and the frustration that comes along with science and mathematics, with the uncertainty and the concern that some error, some ε might have escaped my attention, and all the other things that make a PhD student's life what it is. For instance, when I came to Trento, I only spoke two languages (besides German, of course) – English and also some Mathematics, but with a strong analysis accent. Unfortunately, I had to find out that both are unsuitable to lead a life outside the faculty building. To make a long story short, I found myself working on one PhD thesis, yet another PhD thesis, and all that while I was also trying to get a grip on the Italian language (di cui mi sono ‘disperatamente’ innamorato alla fine). After having submitted my thesis in München in summer 2009 and passed my Italian language exams not long after, for the first time since I started doing my own research I could finally concentrate on one single project only. During this intense and exciting period (not free of setbacks, of course) I actually developed most of the ideas that should lay the basis for this thesis. At this point I would like to emphasize that it was always my supervisor, Prof. Augusto Visintin, who left me all the freedom to explore these ideas while guiding me with his valuable advice and his strong support in establishing scientific contacts and bonds. In particular, as a result of this support I could join Prof. Giuseppe Geymonat and Prof. Françoise Krasucki

in Montpellier – whom I am also very grateful to – and started working on interface problems. This year or so of complete scientific freedom ended in October 2010 when I chose to follow my desire to find out what mathematics is actually good for outside the academic world, *i.e.* when I joined an industrial research and development department. From then on I found myself again exposed to double workload, knowing that there was still some mathematical research to do for the second thesis while leading the challenging life of an industrial research scientist. This ‘double existence’ proved once more to be both ambitious and exhausting (time-consuming as it was), and maybe even cost me some of my hair. Moreover, sometimes I simply could not afford the time to develop things and theorems until perfection like mathematicians usually do, but rather showed the feasibility of ideas and concepts by means of illustrative examples. To conclude, in retrospect the years since 2007 and the research work I was able to conduct taught me many aspects of life that go well beyond the beloved homogenization theory and Γ -convergence: stress and pressure; patience, persistence and passion; confidence; creativity; a feeling for misplaced alliterations and a particular sense of humor; a new language. And last but not least friendships for life. For this I am deeply indebted to all those who made this possible.

Acknowledgments.

Foremost I would like to thank my supervisors, Prof. Augusto Visintin (for the present thesis) and Prof. Martin Brokate (for the thesis I defended at the Technische Universität München) for their guidance, their support and their patience. I am also very grateful to Prof. Micol Amar for reading and evaluating this thesis, and for being part of the doctoral committee. (To all members of the committee I humbly ask forgiveness for all the terribly long sentences found throughout the thesis. Their use is not bad intent, it’s just that a leopard can’t change his spots.) Special thanks go to my collaborator and former colleague Stefan Neukamm for having endured endless but enlightening discussions on homogenization and Γ -convergence, and for the article we wrote together. Most notably however, for good and lasting friendship. Also, I would like to thank Prof. Giuseppe ‘Pippo’ Geymonat and Prof. Françoise Krasucki for their seemingly endless patience (I always finish what I began, even if it takes some time). I am also grateful to Prof. Valli and the essential support I always received from both the doctoral school and the university, as I am grateful to Miriam Stettermayer for her help with the bureaucracy that confused me so often. And finally, thanks that go beyond words to my family and my friends for their unconditional trust and support.

Philipp Emanuel Stelzig, April 2012.

Chapter 2

Methods for periodic homogenization: Yet another introduction

In this chapter I would like to give the reader yet another, but also a somewhat particular introduction into the contemporary theory of periodic homogenization. That is, an introduction aimed at the needs of a graduate student who got attracted by the beauty of the subject and would like to enter the field for his PhD, but who – for whatever reason – cannot expect to be actively taught homogenization theory. Thus in effect has to learn it from what he finds in the literature. The simple reason for this unusual kind of introduction is that I, having come into contact with the matter four and a half years ago, found myself in exactly the same situation and had to realize that this is not an easy task at all. Therefore, the introduction of this thesis shall – besides laying the foundations for the mathematics to follow in Chapters 3 and 4 – ease the access to the mathematical theory of periodic homogenization rather than being a concise or even complete treatment of the current knowledge. In the mathematical literature there already exist a couple of monographs and also a vast number of research articles on periodic homogenization that are accessible for graduate students with a good knowledge of weak solutions for partial differential equations and the basics of functional analysis. However, all of these contributions share one common drawback that the reader is likely to have encountered regularly in his own life as a mathematician. To my opinion, one could illustrate this drawback as follows. Writing a PhD thesis is like crossing a desert: in order to succeed it is essential to choose the right vehicle so one won't get stuck in the sand. In this situation, one would most probably turn to a well-sorted car dealer and ask for information on the most promising vehicles for the journey. Though if the car dealer were a mathematician, one would be given brochures appraising say ten different types of vehicles, with every brochure being 300 pages thick, explaining chassis, suspension, motor, gearbox etc. in every possible detail and maybe that the performance of the vehicle becomes arbitrarily bad in the quick sand limit. Whereas no word will be lost on how fast the vehicle goes, how much fuel it consumes or what it actually looks like. Of

course, after having read and *understood* the entire brochure it is an easy task to infer all those things, but by then global warming will have expanded the desert's diameter by a factor of two. This might sound exaggerated to the reader, but indeed mathematical literature is generally not good at giving quick but nonetheless comprehensive overviews over certain topics and the methods involved. But since this is, at least to my opinion, exactly what a student entering the field would want, the present introduction shall be an attempt to provide such a comprehensive overview. The result is undoubtedly another 'personalized introduction' to homogenization, thus in principle not dissimilar to the recent monograph Tartar [2009] (which is why I chose to use the first person, *i.e.* speak for myself rather than using the more common 'we'). Needless to say that it can by no standards be compared to the both unusual and inspiring work of Luc Tartar whom one might with good conscience call a grandmaster of homogenization. Here, I focus on the Italian and French schools of homogenization from the 1990s and 2000s, for the simple reason that I am most familiar with these schools and moreover because the remainder of the thesis will contribute to the concepts developed there.

To make a long story short, in this introduction I provide a selection of what I deem the 'most appealing' methods and theorems in the field of periodic homogenization. Besides sketching the basic intuition of the respective methods in terms of the 'classical homogenization problem', I also tried to include the major and to my opinion most accessible literature references and some brief remarks on the scientific context as well. Therefore, this introduction should be considered a first 'fingerpost' for students entering the field of periodic homogenization and shall moreover familiarize with the by now mostly standardized notation used in homogenization theory. Experts on the other hand will find only little new insight in this chapter – besides some of my personal experiences with the didactics of periodic homogenization, and some new ideas on the generalization of a well-known homogenization method (see Subsection 2.5.2). However, much of the notation used in the upcoming parts of the thesis will be introduced here.

Section 2.1 briefly illustrates the usefulness of mathematical analysis for multiscale modeling in mathematical physics and indicates some recent interesting developments. Particular attention is given to multiscale methods for problems in continuum physics involving periodic microstructures, *i.e.* to the concept of periodic homogenization. In the following sections I then present a selection of methods and theorems from different homogenization schools and show how each of these methods are used to solve the by now 'classical homogenization problem'. I chose to present these methods in chronological order not only for reasons of 'historical correctness', but for the fact that in this fashion one can step-by-step trace the ideas and intuition that led from the early heuristics of the 1960s to today's abstract formalisms in the theory of periodic homogenization. (Also because I strongly oppose today's 'definition-theorem-proof'-style didactics in which mathematics is mainly taught to students, including myself.) The start is marked by the basic modeling principles and methods of the late 1960s and 1970s. I will only present the asymptotic expansion method and the Γ -convergence approach, while Spagnolo's G -convergence (cf. Spagnolo [1969]) and Tartar's H -convergence (cf. Tartar [1984]) are not subject of this introduction. A paragraph on the major results

on periodic homogenization from the 1980s (*i.e.* the passage to nonconvex homogenization) is then followed by a presentation of the 1990s' fundamental contributions to the topic (*i.e.* two-scale convergence and two-scale compactness). The introduction concludes with the most recent insights and concepts developed in the course of the last decade (periodic unfolding).

2.1 The many reasons for multiscale modeling and homogenization

In physics (as is the case in chemistry, materials science, mechanical, electrical, civil and other engineering disciplines) once a certain effect is observed for the first time, scientists develop theories to explain the effect and employ mathematical models to describe and predict it quantitatively. It is the way of things that the pursuing research on the effect requires the refinement of such theories or even calls for completely new explanations as scientists gain more and more insight into the mechanisms behind the effect. Often, an evolution of theories is triggered by the discovery of smaller, formerly undetected *length scales* (or *time scales*) of a physical system. As a consequence, also the corresponding mathematical models have to be adapted to capture the newly discovered scales. While being closer to the actual physical nature, a mathematical model for a physical system that resolves smaller scales is usually more complicated and sometimes even virtually impossible to solve. (The theory of solid mechanics illustrates this evolutionary process very well. With the 18th century having seen the development of continuum theories for *structural* mechanics, scientists of the 19th century turned onto materials and established *constitutive* theories which were still continuous. Just to discover in the late 19th and early 20th century that the supposedly continuous materials are actually discrete on even smaller length scales – constituted by grains, crystals, molecules, atoms, elementary particles or maybe even smaller things carrying mass. Unfortunately, even the simplest mathematical models for solid materials that resolve length scales on which the continuum hypothesis no longer holds true are useless for practical purposes. This is imply due to the sheer number of particles and interactions involved.) The resulting ‘tradeoff dilemma’ between accuracy and complexity leads to a rather simple but nonetheless crucial question.

How to simplify a mathematical model that resolves small length scales in a way that the simplification

is no more or little more complicated to solve than a coarse scale model, but

still carries information about the small scale?

(For instance, from [Bakhvalov and Panasenko, 1989, p. xxxiii] I learned that already in 1929 Reuss [1929] tried to compute elasticity parameters for a polycrystalline material by taking adequate ‘averages’ of the material’s small scales. More precisely, the single

crystals' stresses and deformations. Thus, Reuss could use classical mechanics to predict the polycrystalline material's elastic response, while its single crystals would only enter the calculation of the elasticity parameters.) In multiscale physics the existence of mathematical models corresponding to different length scales also gives rise another important issue, namely the question of consistency. Most mathematical models for physical systems are checked by experiment, thus valid for at least the circumstances under which the experiment was carried out. Suppose there were two experimentally validated mathematical models that resolve different length scales of one and the same physical system. In this case, one would have to make a choice. Therefore, the natural question is:

*How to relate mathematical models that describe the same physical system
but on different length scales?*

(Again, solid mechanics proves to be a good 'showcase' for the issue. For instance, in structural mechanics there are various theories explaining the mechanics of beams, *i.e.* elastic continua of cylindrical shape with small cross-sectional diameter. Here, one must name the classical Euler-Bernoulli theory, Timoshenko's theory or the Cosserat theory, each of which neglects the small scales, *i.e.* the cross-sectional dimensions. Instead, these theories consider a beam a one-dimensional body deforming in space; see also [Antman, 2005, Chapter 8]. But if one treated a beam as a three-dimensional continuous body by incorporating its small cross-sectional length scales, one could also describe it by means of three-dimensional elasticity. The same reasoning also holds for strings, arcs, membranes, plates and shells. Unfortunately, the equations of three-dimensional elasticity are somewhat nasty partial differential equations (especially for large deformations, see [Ciarlet, 1988, Chapter 2]) involving a time and three space variables. Whereas beam theories result to partial differential equations with one time and one space variable only, and in the static case even reduce to ordinary differential equations. In other words, beam theories are often much simpler to solve. Still, even three-dimensional elasticity is a simplification in that it does not take into account the *discrete* nature of the material that constitutes the beam. Clearly, one could also model a beam by viewing its slender three-dimensional shape as filled by atoms which interact through certain force potentials. Yet, the sheer number of atoms involved would render such a discrete model useless for actual computations.)

Of general interest, especially in the engineering disciplines, are *microstructured* physical systems featuring *two* distinguished and *continuous* length scales. The so-called 'macroscale' – in my personal 'definition' the length scale on which the system interacts with its environment – and the 'microscale'. The latter is defined by some recurring property where the distance of recurrence is much smaller than the dimensions defining the macroscale. The resulting microscale pattern is referred to as the physical system's *microstructure*. (Again solid mechanics provides numerous examples for this type of physical systems. In materials science, many composite materials have a distinguished 'two-scale' nature, most notably fibre-reinforced composites like glass-reinforced plastic or carbon-fibre composites, but also laminates. Porous media like *e.g.* soil or ceramics show this property as well. In structural mechanics one could name

brick walls, multi-strand ropes and cables or perforated sheet metal. Also, woven structures have a distinguished two-scale nature where yarns are interlaced to form fabric or clothes.) A major difficulty in establishing mathematical models for this type of physical systems is that experimental data is often available for macroscale quantities only, but not for the microscale. The reason is that macroscopic quantities can be measured more easily. Microscale quantities on the other hand are sometimes not even accessible for measurements, at least not without ‘destroying’ the physical system, *e.g.* for material samples. This is why early mathematical models for microstructured physical systems are sometimes restricted to macroscale quantities whereas microscale parameters do not *explicitly* enter the model. However, if microscale quantities are also accessible, more elaborate mathematical models can be constructed that explicitly incorporate microscale parameters. (As an example one might think of *e.g.* the thermal conductivity matrix for a fibre-reinforced composite. Obviously, one could perform a series of experiments to identify its entries – which is most likely not going to be an easy task due to the strong thermal anisotropy of many fibre-composites. On the other hand, the thermal conductivity of the composite’s single constituents – often homogeneous – is far easier to measure. Therefore, if it were known which parts of the composite are occupied by which constituent, one could simply write down the common heat equation with the conductivity matrix’s entries varying according to the constituents’ spatial distribution. Yet, how to find out about the exact spatial distribution of the constituents without destroying the material for samples?) A special case of paramount importance among microstructured physical systems is formed by those having a *periodic microstructure*. At this point, many mathematicians tend to name fibre-reinforced composites as the prototypical case (as did I just the line before) or leave it completely to the reader to imagine a relevant example and then dive into mathematical analysis. Often, no word is lost neither on the periodicity assumption itself, nor on its limitations. Of course, this issue is quite difficult to deal with in general and therefore I will not even try; instead, I would like to point out a personal comment on the periodicity assumption.

1. For continuous physical systems with periodic microstructure, ‘periodic’ often implies ‘man made’ and even then periodicity is by no means perfect. Although one might find a periodic *reference configuration*. Thus, one should only employ a periodicity assumption if there is good reason to support it and otherwise indicate its limitations explicitly.
2. The main and maybe only reason for periodic media or structures is that they are easy and cheap to mass produce: one can repeat the same production steps over and over again. (Philosophically, one might say that the production process translates periodicity in time into periodicity in space.)
3. Periodic media and structures (such as fibre-reinforced composites, perforated sheet metal or fabric) often come in fairly simple, mostly *thin* geometries. Again the reasons are rather practical: creating a periodic microstructure inside a bulky geometry is delicate because one cannot mechanically reach into it. Whereas for thin geometries the microstructure stays close to the surface. Another aspect is

that periodic media and structures are in many cases delivered as *semi-finished materials*, thus the final geometry is only decided afterwards. Also, the more complicated a geometry, the bigger is its surface and the more often a periodic microstructure would touch it. This however may be undesirable, in particular for fibre-reinforced composites: due to their tiny bending stiffness, fibres are only good under tension loads. Cutting the fibres of a fibre-reinforced composite, *e.g.* when bringing it to its final shape, would greatly reduce the composite's resistance to tension.

Over the last four decades mathematicians have developed many strategies to answer the crucial questions on how to relate mathematical models that resolve different length scales of one and the same physical system. In principle, they all follow the same idea. That is, for a physical system with two distinguished length scales one identifies the parameter $0 < \varepsilon \ll 1$ describing the size of the small length scale relative to the coarse scale, lets it tend to zero and studies the asymptotics of the mathematical model. For obvious reasons, the quantity ε is called the *(micro)scale parameter*. (Of course, the same methodology can be applied for mathematical models involving any finite number of scales.) In case the mathematical model 'converges' to a limit in some adequate topology, one would end up with a mathematical model for the physical system that no longer shows the small scale but the coarse scale only. Although there is a priori no guarantee that any information about the small scale is transferred into the limit model, in this fashion one can at least ensure that the resulting coarse scale model is asymptotically consistent with the original model resolving both the coarse and the small scale. If there already were another coarse scale mathematical model, one could now compare this and the coarse scale model obtained by the limit process. It is clear that the asymptotics of the model resolving both scales might very well depend on a specific 'notion of convergence', *i.e.* on the topology. (However, one could then pose an inverse question: which topology would I have to choose for the asymptotics of the mathematical model resolving both the coarse and the small scale, in order to obtain the coarse scale model already known.)

This methodology has been applied to a wide variety of multiscale problems (especially in materials science, solid mechanics and structural mechanics). Most recently, the approach was used to perform so-called *discrete-to-continuous limits* where one starts with a set of discrete particles that are arranged in a regular, periodic lattice and interact by some force potential. Identifying the microscale parameter ε as the distance of neighboring particles, a number of works has shown that the sum of the particles' interaction potentials converges in a certain sense to the stored energy of a continuous elastic material occupying the same domain as the discrete particles. Hence, to some extent mathematics managed to close the theoretical gap between competing discrete and continuous models for elastic materials. See *e.g.* the works of Andrea Braides and co-authors Braides [2000]; Braides and Gelli [2006], but also Friesecke and James [2000] and more recently Schmidt [2006, 2008, 2009]. Another prominent field of research where the passage from small scale models to coarse scale models has led to fundamental insights is structural mechanics. More precisely, the theory of elastic,

one- or two-dimensional continua deforming in space such as strings, rods, (curved) beams or membranes, plates and shells. Starting with the equations of (linear or nonlinear) static elasticity for the respective three-dimensional *slender* bodies, mathematicians identify the cross-sectional diameter (for strings, rods, (curved) beams) or respectively the thickness (membranes, plates, shells) as the small length scale. By letting the small scale parameter tend to zero one can often show convergence, in an appropriate sense of course, to models of one-dimensional or two-dimensional elastic structures. To my knowledge, this approach was *formally* already employed in the late 1970s, see [Ciarlet and Destuynder \[1979\]](#); [Ciarlet \[1997\]](#), but it was not until [Acerbi et al. \[1991\]](#) and [Le Dret and Raoult \[1995\]](#) that such passage from three-dimensional slender elastic bodies to low-dimensional theories could be made rigorous. The same strategy led to most impressive results that were published in a series of scholarly articles by Gero Friesecke, Richard D. James and Stefan Müller, see [Friesecke et al. \[2002, 2006\]](#) where the authors managed to rigorously justify plate equations that are widely used in engineering mechanics. For the same development in shell theories cf. [Friesecke et al. \[2003\]](#); [Lewicka et al. \[2010\]](#). Similar results on beams and curved beams are mainly due to Maria Giovanna Mora and Stefan Müller, see [Mora and Müller \[2003, 2004\]](#) but also the interesting recent contribution [Davoli \[2011\]](#).

From now on this introduction shall be concerned with physical systems having two continuous length scales where the small scale is defined by a recurring property of spatial dimension ε , called the system's microstructure. In this situation, passing to the limit $\varepsilon \rightarrow 0$ in the associated mathematical model is referred to as *homogenization*. The wording is more or less self-explaining: the limit model has no microstructure any more since it was eliminated by letting its 'size' ε tend to zero. Thus, it describes a simpler, *homogeneous* physical system. The scientific literature distinguishes between stochastic and deterministic microstructures and therefore also between stochastic homogenization and deterministic homogenization. (An example for stochastic microstructures is once more given by a special class of fibre-reinforced composites where *short fibres* are randomly embedded into a matrix material; for instance, a close look to modern carbon fibre-reinforced ceramic brakes reveals this kind of microstructure. Also foams have a stochastic microstructure.) Important references for stochastic homogenization are *e.g.* [Bensoussan et al. \[1978\]](#) or [Jikov et al. \[1994\]](#). Deterministic homogenization on the other hand is mostly concerned with periodic homogenization (see the previous examples for materials with microstructures). Nevertheless, there are research contributions that study certain examples of non-periodic deterministic microstructures; see most notably the works of Gabriel Nguetseng [Nguetseng \[2003, 2004b\]](#) and [Nguetseng \[2004a\]](#), but also [Braides et al. \[2009\]](#). In the mathematical literature, the homogenization of physical systems with a periodic microstructure goes under the name of *periodic homogenization*. In the remainder of this introduction I am going to focus on some selected approaches and methods for periodic homogenization that I found either very inspiring or helpful for the research I conducted. Before doing so however I would like to state a remark on what I said so far.

Remark 2.1. It is my personal 'definition' to restrict homogenization to microstructured

physical systems where both the macro- and the microscale are *continuous*. This is to keep the field of homogenization well separated from others, in particular discrete-to-continuous limits. Otherwise, also the limit of vanishing atom-to-atom distance ε in a model for a crystalline solid where atoms are arranged periodically along the $\varepsilon\mathbb{Z}^3$ -grid (like *e.g.* in Braides and Gelli [2006]) would be called homogenization – although this situation is prototypical for discrete-to-continuous limits. In fact, several mathematicians use the term homogenization also for problems in which the underlying physical system has a discrete microscale, *e.g.* [Hornung, 1997, Section 1.2.2] or Braides and Gelli [2005]. However, since this use of language appears somewhat inconsistent to me, I prefer to apply the term homogenization exclusively to microstructured physical systems with both continuous macroscale and continuous microscale. (An interesting recent progress in discrete-to-continuous limits is Yeung et al. [2009] which shows that one can very well pass from discrete crystalline systems to continuous solids *without* a priori assuming the crystal's atoms to be arranged in some kind of periodic microstructure. Hence, homogenization and discrete-to-continuous limits address in general different physical and mathematical models and should therefore not be confused.)

2.2 The beginnings: 60s and 70s

The first major steps in the mathematical analysis of physical systems with periodic microstructure were done in the late 1960s (I prefer to avoid the term ‘periodic media’ used in the mathematical literature because it could lead to the misbelief that the methods involved apply to materials science only). Also, both the notation and the basic modeling principles have not changed since then. In fact, like in many other branches of mathematics there has evolved a kind of standard notation which I am going to use throughout this thesis.

2.2.1 Basic modeling principles and notation

Assume that the microstructured physical system under consideration exists in $N \in \mathbb{N}$ space dimensions, mostly $N \in \{2, 3\}$, and exhibits a periodic microstructure whose length scale is quantified by a small parameter $0 < \varepsilon \ll 1$. The modeling of a periodic microstructure relies on the notion of a ‘periodicity cell’ of fixed size, usually denoted Y . Here, one assumes that the microstructure can be viewed as the disjoint union of translated ε -homotheties εY of the periodicity cell; see also Figure 2.1. To this end, one has to ask that Y has the *paving property*, *i.e.* that one can indeed cover the whole space \mathbb{R}^N with the disjoint union of translated copies of the periodicity cell Y . In symbols, $\mathbb{R}^N = \bigcup_{j \in \mathbb{N}} (t_j + Y)$ for a family of \mathbb{R}^N -vectors $\{t_j : j \in \mathbb{N}\}$. The microstructure is then tiled by the ε -homothety of this union, *i.e.* by $\bigcup_{j \in \mathbb{N}} \varepsilon(t_j + Y)$. Now, since also the *constitutive properties* of the physical system's microstructure repeat periodically, it suffices to describe them on the periodicity cell by some *constitutive function* $A : Y \rightarrow \mathbb{R}^M$. Thanks to the paving property of Y , one may assume the function A to be extended to the entire \mathbb{R}^N by periodicity. For an arbitrary $y \in \mathbb{R}^N$ there is a unique translated

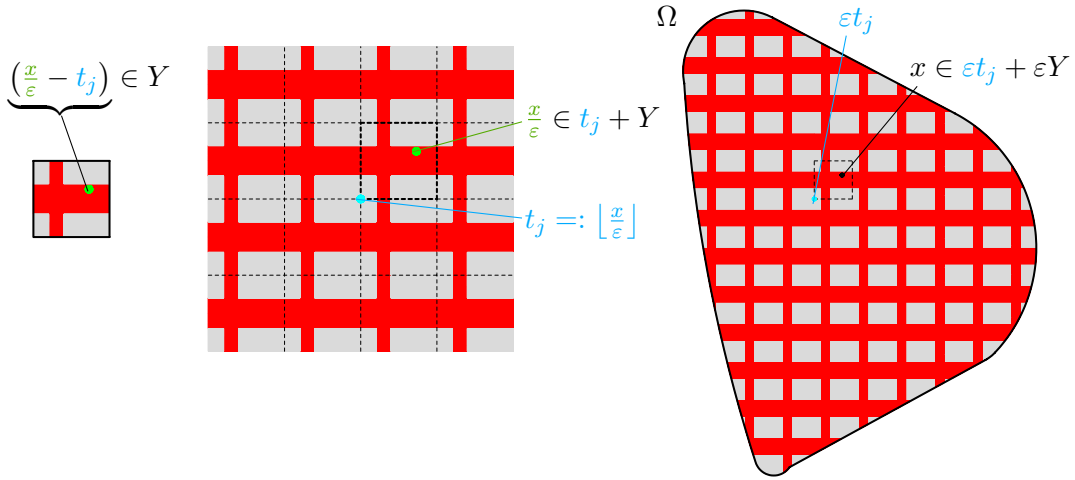


Figure 2.1: Illustration of the periodicity cell concept

periodicity cell $t_j + Y$ containing y , hence one defines $A(y) := A(y - t_j)$. (For such a microstructure tiling it is convenient to define the function $\lfloor \cdot \rfloor : \mathbb{R}^N \rightarrow \mathbb{R}^N$ that maps $y \in \mathbb{R}^N$ to the unique translation vector t_j such that $y \in t_j + Y$. In other words, $\lfloor y \rfloor$ tells in which translation of the periodicity cell the vector $y \in \mathbb{R}^N$ lies.) The constitutive properties in any material point x of the microstructure can then be easily determined by evaluating $A_\varepsilon(x) := A\left(\frac{x}{\varepsilon}\right)$ (again see Figure 2.1). The choice of the periodicity cell depends of course on the specific microstructure under consideration. In the literature, the most commonly used periodicity cell is the unit cube $[0, 1)^N$, which I am also going to use in the remainder of this introduction.

Remark 2.2. There are good reasons for choosing the periodicity cell to be the unit cube, probably the most relevant being simplicity. However, one can easily think of periodic microstructures with periodicity cells different from the cube, *e.g.* honeycomb-like microstructures where Y becomes a regular hexagon. Hexagonal periodicity cells are most suitable to describe the microstructure in cross-sections of large wire ropes that are made from wires of identical diameter (see Figure 2.2, left); here, each wire cross-section can be viewed as the incircle of a regular hexagon, which then make up the wire's cross-section if arranged like in a honeycomb. Another situation where it appears reasonable to take a periodicity cell different from the unit cube is when the microstructure shows periodically recurring voids – the literature speaks of perforated media – that would touch the periodicity cell's boundary. However, in some cases one might simply avoid this problem by choosing a suitable periodicity cell, like *e.g.* in Figure 2.2 (right); cf. also the recent preprint Cioranescu et al. [2011].

Remark 2.3. Interestingly, the mathematical literature on homogenization methods is – to my current knowledge – almost exclusively concerned with periodic microstructures that are covered by *translations* of a periodicity cell (let me call them ‘translatory periodic microstructures’ for the time being). But there are indeed materials with periodic microstructure where the periodicity does not stem from translations, but *e.g.* from *ro-*

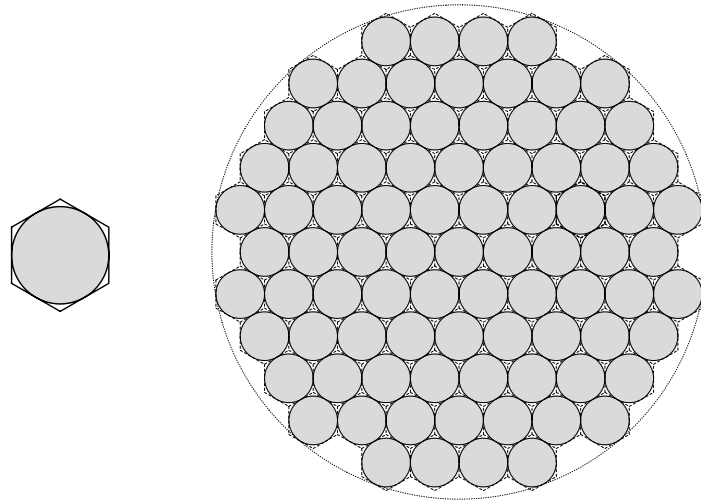


Figure 2.2: Hexagonal periodicity cells in cross sections of wire rope

tations of a periodicity cell; see the *functionally graded annular disk* depicted in Figure 2.3. Some of these situations may be resolved by changing to an appropriate coordinate

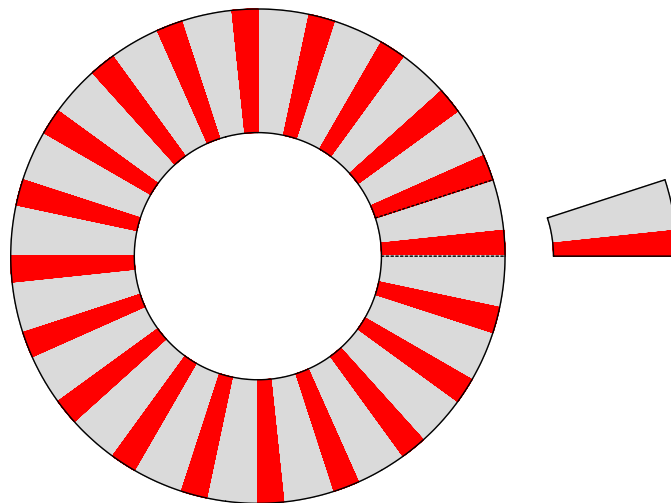


Figure 2.3: A functionally graded annular disk

system where rotations result to translations of an angle variable. For the above example polar coordinates would do the trick. More generally, one might think of materials with a kind of ‘generalized periodic microstructure’, where periodicity is understood in a sense that there is a coordinate transformation relating this generalized periodic microstructure to a common translatory periodic microstructure. In the case of elastic solids with this kind of microstructure, applying such a coordinate transformation – which is nothing but an elastic deformation of the solid – would result in a *reference*

configuration with translatory periodic microstructure. However, this reference configuration would in general be no *natural state*, *i.e.* it would be *pre-stressed* (see [Ciarlet, 1988, Chapters 2, 3 and 4] for elasticity related terms). There is already some work on prestressed microstructured materials, see Parnell [2007]. Nonetheless, to my opinion there is still much work to do in this field; as concerns the example shown in Figure 2.3, I will come back to it in Subsection 2.5.2 in a more abstract context.

In continuum physics, quantities $u(x) \in \mathbb{R}^K$ that describe the physical state of a system in some material point x (it would be better to say in a ‘volume element’ dx around x) are usually related to the constitutive properties in x (or better dx) by partial differential equations. A both physically and mathematically intuitive example is the static heat equation: in this case, the physical system reduces to a (for simplicity solid) body occupying some bounded, smooth domain $\Omega \subseteq \mathbb{R}^N$, $N \in \{2, 3\}$. The constitutive properties, which in general vary within the body Ω , would be given by a thermal conductivity matrix $\hat{A} : \Omega \rightarrow \mathbb{R}^{N \times N}$. Assume furthermore to be given an internal heat source $f : \Omega \rightarrow \mathbb{R}$ (*e.g.* heat losses of some technical process) and constant temperature on the body’s boundary (*e.g.* due to the body being imbedded in a much larger body with far higher thermal conductivity; in appropriate units equal to 0). Then, the equation for the equilibrium temperature $u : \Omega \rightarrow \mathbb{R}$ becomes

$$\begin{cases} -\operatorname{div}(\hat{A}(x)\nabla u(x)) &= f(x) & \text{for all } x \in \Omega, \\ u(x) &= 0 & \text{for all } x \in \partial\Omega. \end{cases}$$

Now, if the body had a periodic microstructure with cubic periodicity cell $Y = [0, 1)^N$, a constitutive function $A : Y \rightarrow \mathbb{R}^{N \times N}$ that tells us how the thermal conductivity varies in the periodicity cell, and a microscale parameter $0 < \varepsilon \ll 1$, then the thermal conductivity in a material point $x \in \Omega$ is – according to what has been said before – $A\left(\frac{x}{\varepsilon}\right)$. Thus, the heat equation for the resulting equilibrium temperature $u_\varepsilon : \Omega \rightarrow \mathbb{R}$ reads

$$\begin{cases} -\operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon(x)\right) &= f(x) & \text{for all } x \in \Omega \\ u_\varepsilon &= 0 & \text{for all } x \in \partial\Omega. \end{cases} \quad (\text{CHP})$$

Here, it is reasonable to assume that

$$A(y) \text{ is a symmetric matrix for all } y \in Y \quad (\text{A1})$$

and moreover

$$\begin{cases} A \text{ is strongly elliptic, } i.e. \\ \frac{1}{C}|v|^2 \leq A(y)v \cdot v \leq C|v|^2 \\ \text{for all } v \in \mathbb{R}^N, y \in Y \text{ and some constant } C > 0. \end{cases} \quad (\text{A2})$$

The problem (CHP) is a linear elliptic partial differential equation and if the domain Ω were reasonably simple, then any decent finite element solver would produce a pretty good approximation for the solution u_ε . Problem is that to capture the oscillations of

the thermal conductivity $A\left(\frac{x}{\varepsilon}\right)$ the underlying finite element mesh has to be *at least* as fine as the linear size of the periodicity cell, *i.e.* ε . Hence, the number of mesh nodes is of the same order as the number of periodicity cells in Ω , *i.e.* of order $\frac{1}{\varepsilon^N}$. Plainly speaking, as ε becomes smaller computation times rise quite rapidly. On the other hand, in applications it is desirable to regard a microstructured material like *e.g.* the solid body Ω not as a complicated geometric arrangement of the microstructure's constituents, but as one *homogeneous* material only, maybe at the cost of strong material anisotropy. The reason is that the behavior of single microstructure constituents is often completely out of interest (sometimes they are not even accessible to measurements). It is the macroscopic behavior that counts (see the explanations at the beginning of this section). Hence, if I were an experimental materials scientist, I would ask my laboratory assistant to take a material sample from the solid body Ω , lie to him about the microstructure but make him believe that the body's material is *homogeneous* and anisotropic, and then simply wait until he comes back with the measured entries of the *homogenized* thermal conductivity matrix $A_{\text{Hom}} \in \mathbb{R}^{N \times N}$. Having still the same heat source f in the body Ω and the same boundary conditions on $\partial\Omega$, the resulting temperature distribution in the solid body $u_{\text{Hom}} : \Omega \rightarrow \mathbb{R}$ would certainly satisfy

$$\begin{cases} -\operatorname{div}\left(A_{\text{Hom}}(x)\nabla u_{\text{Hom}}(x)\right) & = f(x) & \text{for all } x \in \Omega \\ u_{\text{Hom}}(x) & = 0 & \text{for all } x \in \partial\Omega. \end{cases} \quad (2.1)$$

Assuming that the laboratory assistant's measurements are correct, one has to ask the crucial question: How does this problem relate to the problem (CHP) which resolved the actual microstructure? Is the solution u_ε of (CHP) 'close' to the homogenized solution u_{Hom} of (2.1) for small enough microscale parameters ε ? Can one compute the homogenized thermal conductivity matrix A_{Hom} from the thermal conductivity $A\left(\frac{\cdot}{\varepsilon}\right)$ of the microstructured body Ω to avoid costly and time-consuming measurements? In fact, homogenization theory gives positive answers to these questions. More precisely, it establishes convergence of the solutions u_ε of the problem with microstructure (CHP) to the solution u_{Hom} of the homogenized problem (2.1), and also relates the homogenized constitutive matrix A_{Hom} to the original constitutive function A defined on the periodicity cell. Indeed, homogenization theory provides not only one but a whole variety of different methods for this task which allow to study more general mathematical models than the elementary heat equation (CHP). For instance, mathematical models for solid mechanics, fluid mechanics, electromagnetism, biophysics, materials science,...

At this point I would like to make the reader aware of the paramount importance of (CHP) for homogenization theory. In the mathematical literature, it is simply referred to as the *classical homogenization problem*. In fact, (CHP) has evolved as a kind of 'benchmark problem' for new methods in periodic homogenization (although the mathematical literature does not use this term explicitly, as far as I know). Which is interesting because benchmark problems are rather uncommon in mathematical analysis, other than in numerical analysis or scientific computing. Whenever a new method for periodic homogenization emerges, one can judge its 'performance' by the 'ease' it allows to solve the classical homogenization problem. Of course, this is far a less rigorous criterion than say convergence rates or computation times for benchmark problems on standard

hardware in scientific computing (unless someone invents a standardized graduate student). Nonetheless, it allows researchers in the field to compare the ‘usability’ of certain methods. For instance, the less problem-specific or homogenization-specific knowledge is required for a certain method, the better; the stronger the topology in which convergence to the homogenized solution is shown, the better; and so on... Therefore, all the methods for periodic homogenization I chose to present in this introduction will address the classical homogenization problem (CHP). Moreover, I will point out drawbacks and advantages of certain methods of periodic homogenization when compared to others.

2.2.2 Asymptotic Expansions

The most traditional method in homogenization theory is the so-called method of *asymptotic expansions* which dates way back to the 1960s. Its basic feature is to postulate that the solution u_ε to a problem with periodic microstructure – like (CHP) – is for every microscale parameter ε a series of the form

$$u_\varepsilon(x) = u_0(x) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \dots \quad (2.2)$$

with functions $u_0 : \Omega \rightarrow \mathbb{R}$ and $u_i : \Omega \times Y \rightarrow \mathbb{R}$, $i \in \mathbb{N}$, where the latter are Y -periodic in their second arguments. This *ansatz* reflects the notion that u_ε stays close to some macroscopic part (likewise the solution of the homogenized problem, like (2.1)) while the microstructure causes small ‘fluctuations’ $\varepsilon u_1\left(\cdot, \frac{\cdot}{\varepsilon}\right) + \varepsilon^2 u_2\left(\cdot, \frac{\cdot}{\varepsilon}\right) + \dots$ around the homogeneous part. With the microstructure being ε -periodic, it is natural to assume that the fluctuations are of the same length scale. As it is natural to assume that these fluctuations vary over the whole domain Ω (e.g. due to external data that varies over the domain, just like the heating f in (CHP)). Indeed, many mathematical monographs on homogenization theory outline the method of asymptotic expansions (like Bensoussan et al. [1978]; Sánchez-Palencia [1980]; Cioranescu and Donato [1999] or Bakhvalov and Panasenko [1989]), but often little emphasis is given to the motivation of the *ansatz* (2.2) itself, which though is fundamental to the method. Here, the reader might find [Sánchez-Palencia, 1980, Chapter 5] a very inspiring introduction to the topic. (Generally, in homogenization theory one cannot speak of the method of asymptotic expansions without giving credit to Evariste Sanchez-Palencia and his numerous important contributions to the topic.) The basic idea of the method of asymptotic expansions is now to insert *ansatz* (2.2) into the equations (CHP) governing the problem with microstructure and to formally apply the differential operator to every component of the series. Sorting the result according to powers of the microscale parameter ε and equating the coefficients on both sides of the equation sign leads to a cascade of equations for the functions u_0 and u_ℓ , $\ell \in \mathbb{N}$. Finally, the homogenized problem (2.1) is obtained by letting the microscale parameter ε tend to 0 in every single of the resulting sequence of equations. (It should be noticed that the last step renders the use of asymptotic expansions in homogenization theory far more delicate than say formal Taylor expansions or Fourier series that are often used to analytically compute solutions to particular ordinary or partial differential equations.)

Before applying the method of asymptotic expansions to the classical homogenization problem (CHP) and deriving the relations between (CHP) and the homogenized problem (2.1) in a ‘personalized’ fashion, I would like to briefly state the method’s principal advantages and drawbacks.

Due to being formal in nature, the method of asymptotic expansion is a simple but universal method to approach various kinds of *linear* problems showing a periodic microstructure. It can basically be applied without any knowledge about the actual solution (existence, a priori estimates, compactness, regularity, . . .) and therefore allows to infer homogenization results more easily and also faster. In addition, the asymptotic expansion method gives the user a good ‘feeling’ about how the single terms $u_\ell(\cdot, \frac{\cdot}{\varepsilon})$ and their *scaling* ε^ℓ , $\ell \in \mathbb{N}$, contribute to the solution of the microstructured problem u_ε . Whereas rigorously justifying an asymptotic expansion of a solution to a problem with periodic microstructure is usually very hard and problem-specific (in contrast to the universality of the method itself). This is why asymptotic expansions are rarely used to obtain strict convergence of the microstructured problem to the homogenized problem. Moreover, while no knowledge about specific properties of the solution to the microstructured problem is required, the user often needs to have a deep understanding of the problem itself in order to guess the correct form of the asymptotic expansion (*e.g.* in case the microstructure shows more than two length scales). Another drawback is that this method usually requires long and cumbersome formal calculations which makes the method quite error-prone. Finally, the asymptotic expansion method is quite unsuitable for problems involving nonlinear differential operators (but not necessarily for nonlinear problems in general, see *e.g.* [Sánchez-Palencia, 1980, Chapter 6] or Chacha and Sanchez-Palencia [1992]). Nevertheless, although more modern homogenization methods are available for problems with periodic microstructure (see the remainder of the introduction), asymptotic expansion methods are still popular today, both in the mathematical literature (*e.g.* Geymonat et al. [2010]; Rohan and Lukeš [2010]) and the engineering literature (*e.g.* Marigo and Pideri [2011]).

In the case of the classical homogenization problem (CHP) with solution $u_\varepsilon \in W_0^{1,2}(\Omega)$, the asymptotic expansion method with ansatz (2.2) for the solution is as follows. First, the gradient $\nabla u_\varepsilon(x)$ of the equilibrium temperature in the microstructured body is formally given through

$$\begin{aligned} \nabla u_\varepsilon(x) &= \nabla u_0(x) + \varepsilon \left(\nabla_x u_1 \left(x, \frac{x}{\varepsilon} \right) + \frac{1}{\varepsilon} \nabla_y u_1 \left(x, \frac{x}{\varepsilon} \right) \right) \\ &\quad + \varepsilon^2 \left(\nabla_x u_2 \left(x, \frac{x}{\varepsilon} \right) + \frac{1}{\varepsilon} \nabla_y u_2 \left(x, \frac{x}{\varepsilon} \right) \right) \\ &\quad + \dots \\ &= \nabla u_0(x) + \nabla_y u_1 \left(x, \frac{x}{\varepsilon} \right) \\ &\quad + \sum_{\ell=1}^{\infty} \varepsilon^\ell \left(\nabla_x u_\ell \left(x, \frac{x}{\varepsilon} \right) + \nabla_y u_{\ell+1} \left(x, \frac{x}{\varepsilon} \right) \right) \end{aligned} \quad (2.3)$$

where $\nabla_x u_\ell$ denotes the gradient of $u_\ell : \Omega \times Y \rightarrow \mathbb{R}$ w.r.t. the first argument and $\nabla_y u_\ell$ the gradient of u_ℓ w.r.t. the second argument. Here it should be noticed that it is in gen-

eral not possible to formally apply the divergence operator in **(CHP)** to $A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon(x)$, for the simple reason that the constitutive function, *i.e.* the thermal conductivity matrix $A : Y \rightarrow \mathbb{R}^{N \times N}$, is often not differentiable (but say piecewise constant, if the microstructure of the body is made up from two different constituents). In this case I prefer to resort to the weak form of the classical homogenization problem **(CHP)**, *i.e.*

$$\int_{\Omega} A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon(x) \cdot \nabla \psi(x) \, dx = \int_{\Omega} f(x)\psi(x) \, dx \quad \forall \psi \in C_c^\infty(\Omega). \quad (2.4)$$

At this point it is essential to notice the role of $\psi \in C_c^\infty(\Omega)$ which has to act as a testfunction for the solutions to *all* ε -problems. In particular, the testfunctions ψ have to be such that they can capture (or ‘sample’) any fine properties of the solutions u_ε , most notably their ε -oscillations. This is why it appears natural to choose testfunctions that also oscillate ε -periodically, *i.e.*

$$\psi(x) := \varphi\left(x, \frac{x}{\varepsilon}\right) \quad \text{for } \varphi \in C_c^\infty(\Omega; C_{\text{per}}^\infty(Y)),$$

and therefore

$$\nabla \psi(x) = \nabla_x \varphi\left(x, \frac{x}{\varepsilon}\right) + \frac{1}{\varepsilon} \nabla_y \varphi\left(x, \frac{x}{\varepsilon}\right).$$

Here, $C_{\text{per}}^\infty(Y)$ is the space of all smooth functions on the cubic periodicity cell $Y = [0, 1]^N$ that have identical trace on opposite faces of Y . Then the weak form of **(CHP)** reads as

$$\begin{aligned} & \int_{\Omega} A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon(x) \cdot \nabla_x \varphi\left(x, \frac{x}{\varepsilon}\right) + \frac{1}{\varepsilon} A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon(x) \cdot \nabla_y \varphi\left(x, \frac{x}{\varepsilon}\right) \, dx \\ &= \int_{\Omega} f(x)\varphi\left(x, \frac{x}{\varepsilon}\right) \, dx \quad \forall \varphi \in C_c^\infty(\Omega; C_{\text{per}}^\infty(Y)). \end{aligned} \quad (2.5)$$

Inserting (2.3) and sorting by powers of the microscale parameter ε yields

$$\begin{aligned}
& \varepsilon^0 \int_{\Omega} f(x) \varphi \left(x, \frac{x}{\varepsilon} \right) dx \\
&= \varepsilon^{-1} \left(\int_{\Omega} A \left(\frac{x}{\varepsilon} \right) (\nabla u_0(x) + \nabla_y u_1 \left(x, \frac{x}{\varepsilon} \right)) \cdot \nabla_y \varphi \left(x, \frac{x}{\varepsilon} \right) dx \right) \\
&+ \varepsilon^0 \left(\int_{\Omega} A \left(\frac{x}{\varepsilon} \right) (\nabla u_0(x) + \nabla_y u_1 \left(x, \frac{x}{\varepsilon} \right)) \cdot \nabla_x \varphi \left(x, \frac{x}{\varepsilon} \right) \right. \\
&\quad \left. + A \left(\frac{x}{\varepsilon} \right) (\nabla_x u_1 \left(x, \frac{x}{\varepsilon} \right) + \nabla_y u_2 \left(x, \frac{x}{\varepsilon} \right)) \cdot \nabla_y \varphi \left(x, \frac{x}{\varepsilon} \right) dx \right) \\
&+ \varepsilon^1 \left(\int_{\Omega} A \left(\frac{x}{\varepsilon} \right) (\nabla_x u_1 \left(x, \frac{x}{\varepsilon} \right) + \nabla_y u_2 \left(x, \frac{x}{\varepsilon} \right)) \cdot \nabla_x \varphi \left(x, \frac{x}{\varepsilon} \right) \right. \\
&\quad \left. + A \left(\frac{x}{\varepsilon} \right) (\nabla_x u_2 \left(x, \frac{x}{\varepsilon} \right) + \nabla_y u_3 \left(x, \frac{x}{\varepsilon} \right)) \cdot \nabla_y \varphi \left(x, \frac{x}{\varepsilon} \right) dx \right) \\
&+ \dots
\end{aligned}$$

By equating powers of the microscale parameter ε on both sides of '=' one obtains the following cascade of equations: For the coefficient of ε^{-1} one has

$$0 = \int_{\Omega} A \left(\frac{x}{\varepsilon} \right) (\nabla u_0(x) + \nabla_y u_1 \left(x, \frac{x}{\varepsilon} \right)) \cdot \nabla_y \varphi \left(x, \frac{x}{\varepsilon} \right) dx, \quad (2.6)$$

for the coefficient of ε^0

$$\begin{aligned}
\int_{\Omega} f(x) \varphi \left(x, \frac{x}{\varepsilon} \right) dx &= \int_{\Omega} A \left(\frac{x}{\varepsilon} \right) (\nabla u_0(x) + \nabla_y u_1 \left(x, \frac{x}{\varepsilon} \right)) \cdot \nabla_x \varphi \left(x, \frac{x}{\varepsilon} \right) \\
&\quad + A \left(\frac{x}{\varepsilon} \right) (\nabla_x u_1 \left(x, \frac{x}{\varepsilon} \right) + \nabla_y u_2 \left(x, \frac{x}{\varepsilon} \right)) \cdot \nabla_y \varphi \left(x, \frac{x}{\varepsilon} \right) dx, \quad (2.7)
\end{aligned}$$

and for the coefficient of ε^ℓ , $\ell \in \mathbb{N}$,

$$\begin{aligned}
0 &= \int_{\Omega} A \left(\frac{x}{\varepsilon} \right) (\nabla_x u_\ell \left(x, \frac{x}{\varepsilon} \right) + \nabla_y u_{\ell+1} \left(x, \frac{x}{\varepsilon} \right)) \cdot \nabla_x \varphi \left(x, \frac{x}{\varepsilon} \right) \\
&\quad + A \left(\frac{x}{\varepsilon} \right) (\nabla_x u_{\ell+1} \left(x, \frac{x}{\varepsilon} \right) + \nabla_y u_{\ell+2} \left(x, \frac{x}{\varepsilon} \right)) \cdot \nabla_y \varphi \left(x, \frac{x}{\varepsilon} \right) dx, \quad (2.8)
\end{aligned}$$

where $\varphi \in C_c^\infty(\Omega; C_{\text{per}}^\infty(Y))$. Now, since the value of the microscale parameter ε may become arbitrarily small one might wonder whether the integrals in (2.6), (2.7) and (2.8) converge. This is indeed the case and can be regarded as the *fundamental lemma* of periodic homogenization. The version I state here is a simple corollary from [Visintin, 2006, Lemma 1.1]; another proof is e.g. [Cioranescu and Donato, 1999, Theorem 2.6].

Lemma 2.1. *Let Ω be an open and bounded subset of \mathbb{R}^N satisfying $\mathbb{1}_\Omega(\cdot + \eta_k(\cdot)) \rightarrow \mathbb{1}_\Omega$ pointwise a.e. in \mathbb{R}^N for every sequence $(\eta_k)_k$ in $L^\infty(\mathbb{R}^N)$ that vanishes uniformly. Moreover, let Y be a bounded and measurable subset of \mathbb{R}^N that has the paving property, and let $\{t_j : j \in \mathbb{N}\}$ be a set of \mathbb{R}^N -vectors such that $\mathbb{R}^N = \bigcup_{j \in \mathbb{N}} (t_j + Y)$. Let $f : \Omega \times Y \rightarrow \mathbb{R}$ be bounded, continuous in its first variable and in its second variable be extended to the whole \mathbb{R}^N by Y -periodicity, i.e. $f(x, y) := f(x, y - \lfloor y \rfloor)$. Herein, $\lfloor \cdot \rfloor : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is the function that maps any $y \in \mathbb{R}^N$ to the unique t_j such that $y \in t_j + Y$. Then*

(i) *we have the identity*

$$\int_{\Omega} f\left(x, \frac{x}{\varepsilon}\right) dx = \int_{\mathbb{R}^N} \int_Y \mathbb{1}_\Omega\left(\varepsilon \left\lfloor \frac{x}{\varepsilon} \right\rfloor + \varepsilon y\right) f\left(\varepsilon \left\lfloor \frac{x}{\varepsilon} \right\rfloor + \varepsilon y, y\right) dy dx,$$

(ii) *there holds the convergence*

$$\int_{\Omega} f\left(x, \frac{x}{\varepsilon}\right) dx \rightarrow \int_{\Omega} \int_Y f(x, y) dy dx \quad \text{as } \varepsilon \rightarrow 0.$$

Remark 2.4. For the common choice $Y = [0, 1]^N$ or any other periodicity cell of mass 1, the integral mean $\int_Y \cdots dy$ obviously coincides with the unnormalized integral $\int_Y \cdots dy$. However, since all of the thesis' contents remain valid for other choices of the periodicity cell Y , in particular such of mass different from 1, I always explicitly write the integral mean.

Due the lemma's fundamental role in periodic homogenization I will shortly repeat its proof.

Proof. The basic idea behind the proof is to write the integral $\int_{\Omega} f\left(x, \frac{x}{\varepsilon}\right) dx$ as an integral over the whole space \mathbb{R}^N , then pave the \mathbb{R}^N with ε -homotheties of the periodicity cell Y , i.e. $\mathbb{R}^N = \bigcup_{j \in \mathbb{N}} \varepsilon(t_j + Y)$, and sum up the integrals over the respective tiles (see also Figure 2.1):

$$\begin{aligned} \int_{\Omega} f\left(x, \frac{x}{\varepsilon}\right) dx &= \int_{\mathbb{R}^N} \mathbb{1}_\Omega(x) f\left(x, \frac{x}{\varepsilon}\right) dx = \sum_{j \in \mathbb{N}} \int_{\varepsilon(t_j + Y)} \mathbb{1}_\Omega(x) f\left(x, \frac{x}{\varepsilon}\right) dx \\ &= \sum_{j \in \mathbb{N}} \varepsilon^N \int_Y \mathbb{1}_\Omega(\varepsilon(t_j + y)) f(\varepsilon(t_j + y), t_j + y) dy \\ &= \sum_{j \in \mathbb{N}} \varepsilon^N \int_Y \mathbb{1}_\Omega(\varepsilon t_j + \varepsilon y) f(\varepsilon t_j + \varepsilon y, y) dy \end{aligned} \quad (2.9)$$

where we performed the change of variables $y := \frac{x - t_j}{\varepsilon}$ for $x \in \varepsilon(t_j + Y)$ and inferred $f(\varepsilon(t_j + y), t_j + y) = f(\varepsilon(t_j + y), y)$ from the Y -periodicity of f in its second argument. Here it is crucial to notice that for all $x \in \varepsilon(t_j + Y)$ one has the identity $\left\lfloor \frac{x}{\varepsilon} \right\rfloor = t_j$.

Moreover, it is $\varepsilon^N = \frac{1}{\text{vol} Y} \int_{\varepsilon(t_j+Y)} dy$, thus

$$\begin{aligned} & \varepsilon^N \int_Y \mathbf{1}_\Omega(\varepsilon t_j + \varepsilon y) f(\varepsilon t_j + \varepsilon y, y) dx \\ &= \frac{1}{\text{vol} Y} \int_{\varepsilon(t_j+Y)} \int_Y \mathbf{1}_\Omega(\varepsilon t_j + \varepsilon y) f(\varepsilon t_j + \varepsilon y, y) dy dx \\ &= \int_{\varepsilon(t_j+Y)} \int_Y \mathbf{1}_\Omega\left(\varepsilon \left\lfloor \frac{x}{\varepsilon} \right\rfloor + \varepsilon y\right) f\left(\varepsilon \left\lfloor \frac{x}{\varepsilon} \right\rfloor + \varepsilon y, y\right) dy dx. \end{aligned} \quad (2.10)$$

Equations (2.9) and (2.10) finally yield the first assertion of the lemma. The second assertion follows from the first one and Lebesgue's theorem of dominated convergence. To this end, one realizes that for all $x \in \mathbb{R}^N$ and $y \in Y$

$$\left| \left(\varepsilon \left\lfloor \frac{x}{\varepsilon} \right\rfloor + \varepsilon y\right) - x \right| \leq \varepsilon \left| \left\lfloor \frac{x}{\varepsilon} \right\rfloor - \frac{x}{\varepsilon} \right| + \varepsilon |y| \leq \varepsilon 2 \sup_{\tilde{y} \in Y} |\tilde{y}|, \quad (2.11)$$

hence $\mathbf{1}_\Omega\left(\varepsilon \left\lfloor \frac{x}{\varepsilon} \right\rfloor + \varepsilon y\right) f\left(\varepsilon \left\lfloor \frac{x}{\varepsilon} \right\rfloor + \varepsilon y, y\right) \rightarrow \mathbf{1}_\Omega(x) f(x, y)$ for a.e. $(x, y) \in \mathbb{R}^N \times Y$ by the assumptions on Ω and the continuity of f in its first argument. \square

Thus, with the help of Lemma 2.1, one can for small microscale parameter ε approximate (2.6) with

$$0 = \int_\Omega \int_Y A(y) (\nabla u_0(x) + \nabla_y u_1(x, y)) \cdot \nabla_y \varphi(x, y) dy dx, \quad (2.12)$$

moreover (2.7) with

$$\begin{aligned} \int_\Omega \int_Y f(x) \varphi(x, y) dy dx &= \int_\Omega \int_Y A(y) (\nabla u_0(x) + \nabla_y u_1(x, y)) \cdot \nabla_x \varphi(x, y) \\ &\quad + A(y) (\nabla_x u_1(x, y) + \nabla_y u_2(x, y)) \cdot \nabla_y \varphi(x, y) dy dx, \end{aligned} \quad (2.13)$$

and (2.8) with ($\ell \in \mathbb{N}$)

$$\begin{aligned} 0 &= \int_\Omega \int_Y A(y) (\nabla_x u_\ell(x, y) + \nabla_y u_{\ell+1}(x, y)) \cdot \nabla_x \varphi(x, y) \\ &\quad + A(y) (\nabla_x u_{\ell+1}(x, y) + \nabla_y u_{\ell+2}(x, y)) \cdot \nabla_y \varphi(x, y) dy dx, \end{aligned} \quad (2.14)$$

which are to be individually satisfied by $u_0 : \Omega \rightarrow \mathbb{R}$, $u_\ell : \Omega \times Y \rightarrow \mathbb{R}$, $\ell \in \mathbb{N}$, and all $\varphi \in C_c^\infty(\Omega; C_{\text{per}}^\infty(Y))$. The final challenge in applying the method of asymptotic expansions is now to infer from the above limiting relations (2.12), (2.13) and (2.14) the equations that the macroscopic part u_0 of the *ansatz* for u_ε satisfies on the domain Ω . In other words, the *homogenized equations* for the problem with microstructure (CHP). Plainly speaking this comes down to choosing the right testfunctions and indeed this is where experience with homogenization theory is most needed. First, if one chooses the

testfunction $\varphi \in C_c^\infty(\Omega; C_{\text{per}}^\infty(Y))$ in (2.13) as independent of the second argument, *i.e.* with slight abuse of notation $\varphi(x, y) = \varphi(x)$ in $\Omega \times Y$, then one would obtain

$$\int_{\Omega} f(x) \varphi(x) dx = \int_{\Omega} \int_Y A(y) (\nabla u_0(x) + \nabla_y u_1(x, y)) dy \cdot \nabla \varphi(x) dx \quad \forall \varphi \in C_c^\infty(\Omega). \quad (2.15)$$

This equation is already the weak form of an elliptic partial differential equation for u_0 on the domain Ω , and still contains information about the actual microstructure through the constitutive function $A : Y \rightarrow \mathbb{R}^{N \times N}$. Now, all that remains to do is to determine the function u_1 ; to this end, in (2.12) one takes the testfunction as $\varphi(x, y) = \rho(x) \psi(y)$ for $\rho \in C_c^\infty(\Omega)$ and $\psi \in C_{\text{per}}^\infty(Y)$. By the arbitrariness of $\rho \in C_c^\infty(\Omega)$ this yields in a.e. $x \in \Omega$

$$0 = \int_Y A(y) (\nabla u_0(x) + \nabla_y u_1(x, y)) \cdot \nabla_y \psi(y) dy \quad \forall \psi \in C_{\text{per}}^\infty(Y). \quad (2.16)$$

Upon noticing the linearity of the solution to this equation in $\nabla u_0(x)$, it is convenient to define $w_i \in W_{\text{per}}^{1,2}(Y)$, $i = 1, \dots, N$, as the unique solutions of

$$0 = \int_Y A(y) (e_i + \nabla_y w_i(y)) \cdot \nabla_y \psi(y) dy \quad \forall \psi \in C_{\text{per}}^\infty(Y) \quad (2.17)$$

(uniqueness up to an additive constant; existence by the assumptions (A1), (A2) and the Lax-Milgram Lemma). Hence, the function u_1 can be uniquely expressed as (again up to an additive constant)

$$u_1(x, y) = \nabla u_0(x) \cdot \sum_{i=1}^N w_i(y) e_i = \begin{bmatrix} w_1(y) \\ \vdots \\ w_N(y) \end{bmatrix} \cdot \nabla u_0(x), \quad (2.18)$$

thus in particular

$$\nabla_y u_1(x, y) = \left[\nabla_y w_1(y) \mid \cdots \mid \nabla_y w_N(y) \right] \nabla u_0(x).$$

Inserting this into (2.15) leads to

$$\begin{aligned} & \int_{\Omega} f(x) \varphi(x) dx \\ &= \int_{\Omega} \left(\int_Y A(y) \left(I + \left[\nabla_y w_1(y) \mid \cdots \mid \nabla_y w_N(y) \right] \right) dy \right) \nabla u_0(x) \cdot \nabla \varphi(x) dx \\ &= \int_{\Omega} A_{\text{Hom}} \nabla u_0(x) \cdot \nabla \varphi(x) dx \end{aligned} \quad (2.19)$$

for all $\varphi \in C_c^\infty(\Omega)$, where

$$A_{\text{Hom}} = \int_Y A(y) \left(I + \left[\nabla_y w_1(y) \mid \cdots \mid \nabla_y w_N(y) \right] \right) dy. \quad (2.20)$$

Obviously, (2.19) is the weak form of a linear elliptic partial differential equation for the macroscopic part u_0 of the *ansatz* (2.2). More precisely, it is the weak form of (2.1) which I motivated earlier in this section by means of physical reasoning. Speaking once more in terms of heat conduction, (2.19) is nothing but the equation for the equilibrium temperature in the body Ω , whose microstructure is now so small that it can be regarded as being made from one homogeneous material with constant thermal conductivity matrix A_{Hom} . Moreover, the equilibrium temperature in the homogenized body Ω is indeed the macroscopic part u_0 in the assumed expansion of the solution u_ε to the problem with microstructure. And thanks to (2.20) one now has a mathematical expression of how to relate the homogenized thermal conductivity matrix A_{Hom} to the constitutive properties of the body with microstructure given through $A(\frac{\cdot}{\varepsilon})$. Hence, there is no more need to ask a laboratory assistant to determine A_{Hom} experimentally. Instead one can simply *compute* its entries by means of (2.20) and solving (2.17) (*e.g.* numerically). In other words, homogenization theory is a way to avoid solving *one* complicated problem with microstructure (like (CHP)). Yet, it comes at the cost of solving *two* simpler problems. One to compute the homogenized constitutive properties A_{Hom} (by (2.17) and (2.20)) which are inferred exclusively from the microstructure, *i.e.* the periodicity cell Y and how the constitutive properties vary over the periodicity cell by $A : Y \rightarrow \mathbb{R}^{N \times N}$. And another one to determine the solution u_{Hom} to the homogenized problem, *i.e.* (2.1) or equivalently (2.19). Plainly speaking, the homogenization process leads – as expected – to a *separation of scales*. Due to the importance of the above homogenization results to the upcoming parts of the introduction and homogenization theory in general, at the cost of some redundancy I will state it once again as a self-contained definition.

Definition 2.2. Let $Y = [0, 1]^N$ be the N -dimensional unit cube and the constitutive function $A : Y \rightarrow \mathbb{R}^{N \times N}$ be such that it satisfies the assumptions (A1) and (A2). Moreover, let $w_i \in W_{\text{per}}^{1,2}(Y)$, $i = 1, \dots, N$ be the (up to an additive constant) unique solutions of

$$0 = \int_Y A(y)(e_i + \nabla_y w_i(y)) \cdot \nabla_y \psi(y) \, dy \quad \forall \psi \in C_{\text{per}}^\infty(Y).$$

Then, the *homogenized constitutive matrix* $A_{\text{Hom}} \in \mathbb{R}^{N \times N}$ is defined through

$$A_{\text{Hom}} = \int_Y A(y) \left(I + \left[\nabla_y w_1(y) \mid \dots \mid \nabla_y w_N(y) \right] \right) dy.$$

Still, although one now has a way to calculate the homogenized thermal conductivity matrix A_{Hom} , it is a priori not clear whether it also has the natural properties of a conductivity matrix. Or, more mathematically speaking, whether A_{Hom} is ‘nice enough’ to ensure the existence of a solution to the homogenized problem (2.1). The answer is found in the following statement.

Proposition 2.3. Let Y , $A : Y \rightarrow \mathbb{R}^{N \times N}$, $A_{\text{Hom}} \in \mathbb{R}^{N \times N}$ and $w_i \in W_{\text{per}}^{1,2}(Y)$ be given as in Definition 2.2. Then

(i) A_{Hom} is symmetric and strongly elliptic, i.e.

$$\frac{1}{C}|v|^2 \leq A_{\text{Hom}}v \cdot v \leq C|v|^2$$

for some constant $C > 0$.

(ii) A_{Hom} can be written as

$$A_{\text{Hom},ij} = \int_Y A(y)(e_j + \nabla_y w_j(y)) \cdot (e_i + \nabla_y w_i(y)) \, dy, \\ i, j \in \{1, \dots, N\}.$$

This is again one of the fundamental results of periodic homogenization, which is why I chose to provide a proof (as found in e.g. [Bensoussan et al., 1978, Chapter 1, Remark 2.6], [Sánchez-Palencia, 1980, Chapter 5, Section 3] or [Cioranescu and Donato, 1999, Proposition 6.9]).

Proof. Obviously, the second assertion and the symmetry of A by (A1) imply the symmetry of A_{Hom} . To prove the second statement, one first writes for $i, j \in \{1, \dots, N\}$

$$\begin{aligned} & \int_Y A(y)(e_j + \nabla_y w_j(y)) \cdot (e_i + \nabla_y w_i(y)) \, dy \\ &= \int_Y A(y)e_j \cdot e_i \, dy \\ &+ \int_Y A(y)e_j \cdot \nabla_y w_i(y) \, dy + \int_Y A(y)\nabla_y w_j(y) \cdot e_i \, dy \\ &+ \int_Y A(y)\nabla_y w_j(y) \cdot \nabla_y w_i(y) \, dy. \end{aligned}$$

However, testing the equation (2.17) defining w_j with w_i yields

$$0 = \int_Y A(y)e_j \cdot \nabla_y w_i(y) \, dy + \int_Y A(y)\nabla_y w_j(y) \cdot \nabla_y w_i(y) \, dy,$$

thus

$$\begin{aligned} & \int_Y A(y)(e_j + \nabla_y w_j(y)) \cdot (e_i + \nabla_y w_i(y)) \, dy \\ &= \int_Y A(y)e_j \cdot e_i \, dy + \int_Y A(y)\nabla_y w_j(y) \cdot e_i \, dy \\ &= \left(\int_Y A(y) \left(I + \left[\nabla_y w_1(y) \mid \cdots \mid \nabla_y w_N(y) \right] \right) \, dy e_j \right) \cdot e_i \\ &= A_{\text{Hom},ij} \end{aligned}$$

and the second assertion follows. It remains to prove the ellipticity of A_{Hom} . Since A_{Hom} is a constant matrix, it is enough to show that A_{Hom} is positive definite, i.e.

$A_{\text{Hom}} v \cdot v > 0$ for all $v \in \mathbb{R}^N \setminus \{0\}$. For such v observe that by the second statement of the proposition

$$\begin{aligned} A_{\text{Hom}} v \cdot v &= \sum_{i,j=1}^N v_i A_{\text{Hom},ij} v_j \\ &= \int_Y A(y) \left(\sum_{j=1}^N v_j (e_j + \nabla_y w_j(y)) \right) \cdot \left(\sum_{i=1}^N v_i (e_i + \nabla_y w_i(y)) \right) dy. \end{aligned}$$

An easy calculation shows that $\sum_{i=1}^N v_i (e_i + \nabla_y w_i)$ is nothing but the gradient of the function $\zeta_v \in W^{1,2}(Y)$, $\zeta_v(y) := \sum_{i=1}^N v_i (y_i + w_i(y))$, and by the ellipticity (A2) of A one infers

$$A_{\text{Hom}} v \cdot v \geq \frac{1}{C} \int_Y |\nabla_y \zeta_v(y)|^2 dy.$$

If the right hand side were zero, then ζ_v would result as a constant function, hence $\zeta_v \in W_{\text{per}}^{1,2}(Y)$. Since $W_{\text{per}}^{1,2}(Y)$ is obviously a vector space, one would further obtain $Y \ni y \mapsto v \cdot y = \zeta_v - \sum_{i=1}^N v_i w_i \in W_{\text{per}}^{1,2}(Y)$. This however is only possible if $v = 0$, which contradicts the assumption $v \in \mathbb{R}^N \setminus \{0\}$. Thus A_{Hom} is positive definit and the proof is finished. \square

With the ellipticity of A_{Hom} at hand, the Lax-Milgram Lemma immediately yields the existence of a unique solution to the homogenized equation (2.1), more precisely to its weak form (2.19). Thus, the problem defining the macroscopic part u_0 of the *ansatz* (2.2), i.e. the equilibrium temperature u_{Hom} in the homogenized body Ω , is well-posed.

Remark 2.5. I would like to advice the reader that homogenization does not always conserve the mathematical ‘nature’ of a problem with microstructure, although this is the case for the classical homogenization problem (CHP) and its homogenized counterpart (2.1). In fact, the homogenization process may lead to ‘strange phenomena’ like e.g. in [Cioranescu et al., 2011, Theorem 5.13], where the authors observe entirely new boundary conditions in the homogenized equations caused by periodically recurring microscopic quantities *inside* the microstructured domain.

Although the derivation of the homogenized equations (2.1) and the homogenized constitutive properties (see Definition 2.2) has been completely formal, in the case of the classical homogenization problem (CHP) one can indeed justify the *ansatz* (2.2). For this one might turn to e.g. [Bensoussan et al., 1978, Chapter 1, Section 2.4] or [Cioranescu and Donato, 1999, Section 7.2]. However, the justification heavily relies on regularity theory for solutions of elliptic partial differential equations and therefore requires additional smoothness assumptions on the domain and the data. In order to obtain strict convergence of the solutions u_ε for the problem with microstructure (CHP) to the solution u_{Hom} of the homogenized problem (2.1), other methods for periodic homogenization appear to be more suitable (see the upcoming sections of this introduction).

Before turning to other methods in the theory of periodic homogenization, I would like to state a critical remark regarding the presence, or better, the absence of asymptotic expansions in lectures on periodic homogenization for students of mathematics. In fact, many lecture notes or books on periodic homogenization I have come across in the past four and a half years hardly teach the method of asymptotic expansions but merely include it for seemingly historic reasons. An argument which is often heard among mathematicians is that its formal nature would make it an engineer's method rather than a mathematician's method and that homogenization theory is about proving convergence to a homogenization limit. As concerns proving convergence of a problem with microstructure to a homogenization limit, there are indeed more suitable methods available than asymptotic expansions. However, I tend to disagree with the opinion that homogenization theory is about proving convergence – it's about deriving the homogenization limit, whereas proving convergence is about showing that the obtained homogenization limit is correct. In fact, the steps leading to the identification of the homogenization limit for the classical homogenization problem exposed in this section are *self-contained* and cover little more than 5 pages. Whereas proving convergence in a self-contained fashion would require considerably more efforts (even when using a very generous definition of the term 'self-contained'). While there is definitely a need to prove convergence, to my opinion it is nonetheless very useful to employ a fast and simple method beforehand to derive the homogenization limit, no matter how formal or 'quick and dirty' the method may be. After all, this is why the asymptotic expansion method is still popular today, even in the mathematical literature (see the examples I gave previously). Another advantage of asymptotic expansions in periodic homogenization is that it is *constructive* and allows for an explicit calculation of the homogenization limit rather than producing abstract existence results (*e.g.* from compactness arguments). To conclude, I would strongly advocate a more prominent role of the asymptotic expansion method in courses on periodic homogenization for students of mathematics, as well as examples of its use in contemporary research.

2.2.3 Γ -convergence

The late 1960s and 1970s have witnessed various approaches of paramount importance to periodic homogenization. G -convergence and H -convergence, for instance, interpret the classical homogenization problem (CHP) as a sequence of linear differential operators and exploit methods related to operator convergence in linear functional analysis to pass to the homogenization limit. For detailed information on these two methods, see the references Spagnolo [1969] or Dal Maso [1993] for G -convergence, as well as Tartar [1984] and Tartar [2009] for H -convergence. However, it was not until Paolo Marcellini and his seminal work Marcellini [1978] that advantage was taken from a particular property of the classical homogenization problem. More precisely, from the fact that the classical homogenization problem is nothing but the Euler-Lagrange equation

for the minimizer of a quadratic functional: by definition

$$u_\varepsilon \in W_0^{1,2}(\Omega) \text{ solves (CHP)} \iff \int_{\Omega} A\left(\frac{x}{\varepsilon}\right) \nabla u_\varepsilon(x) \cdot \nabla \psi(x) \, dx = \int_{\Omega} f(x) \psi(x) \, dx \quad \forall \psi \in C_c^\infty(\Omega)$$

and by density for all $\psi \in W_0^{1,2}(\Omega)$. But this is indeed the Euler-Lagrange equation for the unique minimizer u_ε of the functional

$$\mathcal{E}_\varepsilon : W_0^{1,2}(\Omega) \rightarrow \mathbb{R},$$

$$\mathcal{E}_\varepsilon(v) := \frac{1}{2} \int_{\Omega} A\left(\frac{x}{\varepsilon}\right) \nabla v(x) \cdot \nabla v(x) \, dx - \int_{\Omega} f(x)v(x) \, dx \quad (2.21)$$

(uniqueness by the strong ellipticity (A1) of the constitutive function A). Physically speaking, \mathcal{E}_ε can be interpreted as the *dissipation potential* (see e.g. [Lemaitre and Chaboche, 1990, Chapter 2.5]) for the heat conduction in the microstructured body Ω , and the minimizer of the dissipation potential is actually the body's equilibrium temperature u_ε . Hence, as Marcellini realized the classical homogenization problem (CHP) can for varying values of the microscale parameter ε be viewed as sequence of functionals $(\mathcal{E}_\varepsilon)_\varepsilon$ indexed by ε . More specifically, as a sequence of *minimization problems* or *variational problems*. By physical intuition, as well as by e.g. the method of asymptotic expansions one already knows that there is a homogeneous approximation (2.1) for the classical homogenization problem (CHP) and that the corresponding solutions u_ε – i.e. the minimizers of \mathcal{E}_ε – converge in a certain sense to the solution u_{Hom} of the homogenized problem (2.1). However, the homogenized equation (2.1) is again nothing but the Euler-Lagrange equation of the *homogenized functional*

$$\mathcal{E}_{\text{Hom}} : W_0^{1,2}(\Omega) \rightarrow \mathbb{R},$$

$$\mathcal{E}_{\text{Hom}}(v) := \frac{1}{2} \int_{\Omega} A_{\text{Hom}} \nabla v(x) \cdot \nabla v(x) \, dx - \int_{\Omega} f(x)v(x) \, dx \quad (2.22)$$

where A_{Hom} is the homogenized constitutive matrix as given in Definition 2.2. In particular, u_{Hom} can therefore be viewed as the unique minimizer of \mathcal{E}_{Hom} (uniqueness by the ellipticity of A_{Hom} , see Proposition 2.3). Consequently, given convergence of the solutions u_ε to (CHP) to the solution of u_{Hom} of the homogenized problem (2.1), one can express this like

$$\arg \min_{v \in W_0^{1,2}(\Omega)} \mathcal{E}_\varepsilon(v) = u_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} u_{\text{Hom}} = \arg \min_{v \in W_0^{1,2}(\Omega)} \mathcal{E}_{\text{Hom}}(v) \quad (2.23)$$

Now, one may ask whether there is a notion of convergence for sequences of functionals like $(\mathcal{E}_\varepsilon)_\varepsilon$ to some limit functional \mathcal{E}_{Hom} that ensures convergence of the minimizers $(u_\varepsilon)_\varepsilon$ to a minimizer of the limit functional, here denoted u_{Hom} . Indeed, this is what Γ -convergence as introduced by Ennio De Giorgi in the 1970s is all about (see De Giorgi and Franzoni [1975]). Roughly speaking, Γ -convergence defines a convergence for ‘energy functionals’ posed over a common topological vector space such that under suitable

coercivity and compactness assumptions on the functionals and the underlying vector space every cluster point of the corresponding sequence of minimizers is a minimizer of the Γ -limit functional. Moreover, the associated sequence of minima also converges to the minimum of the Γ -limit. (The term ‘energy functional’ is commonly used in the context of Γ -convergence. In applications of Γ -convergence to continuum physics, the functionals encountered often quantify the energy ‘stored’ in a certain configuration of the underlying physical system. Compare *e.g.* Γ -convergence approaches to problems of elasticity, like those in Friesecke et al. [2002] or Le Dret and Raoult [1995], where suitable functionals capture the stored elastic energy of the underlying system.) This is why Paolo Marcellini in Marcellini [1978] employed methods closely related to Γ -convergence (although the term ‘ Γ -convergence’ is never stated in that article). Yet, he did not only derive the homogenized equations for the classical *linear* homogenization problem, but for a far wider class of *nonlinear* homogenization problems. This was due to the major advantage that Γ -convergence requires no or only minor assumptions on the particular form of the functionals \mathcal{E}_ε . In principle, any homogenization problem that can be viewed as a sequence of minimization problems, *i.e.* as a sequence functionals index by the microscale parameter ε , can be approached with Γ -convergence methods. Most notably, linearity or nonlinearity or even the existence of associated Euler-Lagrange equations is of no importance to Γ -convergence. The other main innovation in Marcellini’s contribution was the representation of the homogenized problem. Previous approaches, like *e.g.* the asymptotic expansion method I exposed earlier, always described both the homogenization limit of the classical homogenization problem (CHP) and the homogenized constitutive matrix A_{Hom} by means of – as I have to admit – complicated and rather unintuitive partial differential equations (see Definition 2.2 and Proposition 2.3). Instead, the Γ -convergence approach allows for a representation of the homogenized problem completely in terms of minimization problems. In fact, not only the homogenized equation (2.1) can be replaced by finding a minimizer to the functional \mathcal{E}_{Hom} as defined above, but also the equations leading to the homogenized constitutive matrix A_{Hom} . A close look to the definition of the auxiliary functions w_1, \dots, w_N from which A_{Hom} is computed (see Definition 2.2) and the very same arguments that revealed (CHP) and (2.1) to be Euler-Lagrange equations shows that

$$0 = \int_Y A(y)(e_i + \nabla_y w_i(y)) \cdot \nabla_y \psi(y) \, dy \quad \forall \psi \in C_{\text{per}}^\infty(Y) \quad \Leftrightarrow$$

$$w_i = \arg \min_{v \in W_{\text{per}}^{1,2}(Y)} \left(\int_Y \frac{1}{2} A(y)(e_i + \nabla_y v(y)) \cdot (e_i + \nabla_y v(y)) \, dy \right)$$

for all $i = 1, \dots, N$. More generally, for arbitrary $F \in \mathbb{R}^N$ with $w_F := \sum_{i=1}^N F_i w_i \in W_{\text{per}}^{1,2}(Y)$ one identifies the following minimization problem

$$0 = \int_Y A(y)(F + \nabla_y w_F(y)) \cdot \nabla_y \psi(y) \, dy \quad \forall \psi \in C_{\text{per}}^\infty(Y) \quad \Leftrightarrow$$

$$w_F = \arg \min_{v \in W_{\text{per}}^{1,2}(Y)} \left(\int_Y \frac{1}{2} A(y)(F + \nabla_y v(y)) \cdot (F + \nabla_y v(y)) \, dy \right).$$

Simply evaluating the corresponding minimal value leads to

$$\begin{aligned}
& \inf_{v \in \mathbb{W}_{\text{per}}^{1,2}(Y)} \left\{ \int_Y \frac{1}{2} A(y) (F + \nabla_y v(y)) \cdot (F + \nabla_y v(y)) \, dy \right\} \quad (2.24) \\
&= \int_Y \frac{1}{2} A(y) (F + \nabla_y w_F(y)) \cdot (F + \nabla_y w_F(y)) \, dy \\
&= \sum_{i,j=1}^N \int_Y \frac{1}{2} A(y) (F_i e_i + F_i \nabla_y w_i(y)) \cdot (F_j e_j + F_j \nabla_y w_j(y)) \, dy \\
&\quad \text{by the second statement of Proposition 2.3} \\
&= \frac{1}{2} \sum_{i,j=1}^N F_i A_{\text{Hom},ij} F_j = \frac{1}{2} A_{\text{Hom}} F \cdot F.
\end{aligned}$$

Not only is (2.24) a new characterization of the homogenized constitutive matrix A_{Hom} , it is also an equivalent statement of the integral kernel (also called ‘energy density’) of the homogenized functional \mathcal{E}_{Hom} introduced earlier in this subsection. Another thorough investigation of (2.24) reveals the integral kernel therein to be indeed the energy density $\frac{1}{2} A(\frac{\cdot}{\varepsilon}) F \cdot F$ of the functional \mathcal{E}_ε describing the problem with microstructure. By using a simpler notation

$$W : Y \times \mathbb{R}^N \rightarrow \mathbb{R}, \quad W(y, F) := \frac{1}{2} A(y) F \cdot F \quad \text{and} \quad (2.25)$$

$$W_{\text{Hom}} : \mathbb{R}^N \rightarrow \mathbb{R}, \quad W_{\text{Hom}}(F) := \frac{1}{2} A_{\text{Hom}} F \cdot F \quad (2.26)$$

for the respective energy densities of the functionals \mathcal{E}_ε and \mathcal{E}_{Hom} , one can now rewrite the sequence of minimization problems (2.23) like

$$\begin{aligned}
& \arg \min_{v \in \mathbb{W}_0^{1,2}(\Omega)} \left(\int_\Omega W\left(\frac{x}{\varepsilon}, \nabla v(x)\right) \, dx - \int_\Omega f(x)v(x) \, dx \right) = u_\varepsilon \\
& \xrightarrow{\varepsilon \rightarrow 0} u_{\text{Hom}} = \arg \min_{v \in \mathbb{W}_0^{1,2}(\Omega)} \left(\int_\Omega W_{\text{Hom}}(\nabla v(x)) \, dx - \int_\Omega f(x)v(x) \, dx \right). \quad (2.27)
\end{aligned}$$

Furthermore, from (2.24) one obtains the identity

$$W_{\text{Hom}}(F) = \inf \left\{ \int_Y W(y, F + \nabla_y v(y)) \, dy : v \in \mathbb{W}_{\text{per}}^{1,2}(Y) \right\}. \quad (2.28)$$

which nowadays is generally known as Marcellini’s *cell formula*. As the notation in (2.27) suggests, Marcellini’s cell formula can be applied to much more complicated situations than just the classical *linear* homogenization problem corresponding to a quadratic energy density $W(\frac{\cdot}{\varepsilon}, F) = \frac{1}{2} A(\frac{\cdot}{\varepsilon}) F \cdot F$ in the associated functional \mathcal{E}_ε . As a matter of fact, all homogenization problems that can be described by means of minimizing an energy functional like \mathcal{E}_ε with a *convex, coercive* energy density $W(\frac{\cdot}{\varepsilon}, F)$ of *quadratic growth* converge in the sense of (2.23) (respectively (2.27)). Furthermore, the

resulting energy density of the homogenized functional \mathcal{E}_{Hom} obeys (2.28). More precisely, it was shown by Paolo Marcellini in [Marcellini \[1978\]](#) that (2.23) follows from the Γ -convergence of $(\mathcal{E}_\varepsilon)_\varepsilon$ to \mathcal{E}_{Hom} and the fundamental properties of Γ -convergence.

At this point I would again like to include some personalized views and comments on homogenization by Γ -convergence before I actually provide the results from the literature as motivated just before.

The major strength of Γ -convergence approaches to problems of periodic homogenization is that the general theory of Γ -convergence requires basically no a priori assumptions on the sequence of functionals under investigation. In particular, it is by no means restricted to linear problems – *i.e.* quadratic energy functionals. Also, Γ -convergence methods apply to a wide range of problems in applied mathematics which go well beyond periodic homogenization. For instance, dimension reduction problems for thin domains like beams, plates or shells or the passage from discrete to continuous material theories (see the references given in Section 2.1). This however makes Γ -convergence particularly well-suited for situations in which multiple limit processes are considered simultaneously, such as homogenization and dimension reduction for periodically microstructured thin domains like interfaces (see *e.g.* [Del Vecchio \[1987\]](#); [Ansini \[2004\]](#); [Ansini et al. \[2007\]](#) and Chapter 3), thin films or plates (see *e.g.* [Braides et al. \[2000\]](#); [Neukamm \[2010\]](#)). Moreover, Γ -convergence methods are fairly ‘easy’ to deal with for sequences of convex functionals (like in the quadratic case) while sequences of nonconvex functionals often pose severe difficulties. See the following Section 2.3 for the fundamental results on ‘nonconvex homogenization’ of [Braides \[1985\]](#) and [Müller \[1987\]](#). Nonetheless, Γ -convergence approaches for homogenization problems are sometimes blamed for being of limited interest to real-life problems of continuum physics. Probably the most notable opponent is Luc Tartar who in [Tartar \[2009\]](#) refers to (energy) minimization approaches to homogenization problems (or continuum physics in general) as ‘fake mechanics’ [[Tartar, 2009](#), p. 31], and even goes far enough to call scientists employing such methods ‘experts in fake mechanics’ (see [[Tartar, 2009](#), p. 39]) pointing out that by the first principle of thermodynamics nature conserves energy rather than minimizing it. (Although one can name numerous applications in the engineering disciplines such as static considerations in elasticity or heat transfer where one is only interested in an equilibrium, steady state but not where the missing energy between initial and steady state actually went. Indeed, this must sometimes be considered even more honest than enforcing conservation of energy, especially when one cannot tell where and in which form energy is conserved. Also, one should recall that it is very well possible to formulate also time-dependent problems of continuum physics as minimization problems by the theory of optimal transportation, see [Benamou and Brenier \[2000\]](#) and [Li et al. \[2010\]](#) for several applications to continuum physics.)

Since Γ -convergence will play a major role in the remainder of the thesis, I will introduce it here in detail to avoid redundancy. The definitions and results that follow below are taken from the standard reference on Γ -convergence [Dal Maso \[1993\]](#) which is also a rich source for applications of Γ -convergence to homogenization problems (cf. [[Dal Maso, 1993](#), Chapters 24 and 25] and [[Dal Maso, 1993](#), pp. 277 – 283]). Further important and maybe more accessible references for Γ -convergence and its ap-

plication to periodic homogenization are [Braides and Defranceschi \[1998\]](#) and [Braides \[2000\]](#). In its most abstract (topological) form, the definition of Γ -convergence reads (cf. [\[Dal Maso, 1993, Definition 4.1\]](#)) as follows.

Definition 2.4. Let X be a topological space and let $\mathcal{U}(x)$ denote the set of all open neighborhoods of $x \in X$. For any sequence of functions $(\mathcal{F}_k)_{k \in \mathbb{N}}$ where $\mathcal{F}_k : X \rightarrow [-\infty, \infty]$, the *lower Γ -limit* and the *upper Γ -limit* are the functions from X into $[-\infty, \infty]$ respectively defined by

$$\begin{aligned} \left(\Gamma\text{-lim inf}_{k \rightarrow \infty} \mathcal{F}_k \right) (x) &:= \sup_{U \in \mathcal{U}(x)} \liminf_{k \rightarrow \infty} \inf_{y \in U} \mathcal{F}_k(y), \\ \left(\Gamma\text{-lim sup}_{k \rightarrow \infty} \mathcal{F}_k \right) (x) &:= \sup_{U \in \mathcal{U}(x)} \limsup_{k \rightarrow \infty} \inf_{y \in U} \mathcal{F}_k(y). \end{aligned}$$

If there exists a function $\mathcal{F}_\infty : X \rightarrow [-\infty, \infty]$ such that $\Gamma\text{-lim inf}_k \mathcal{F}_k = \mathcal{F}_\infty = \Gamma\text{-lim sup}_k \mathcal{F}_k$, one says that $(\mathcal{F}_k)_{k \in \mathbb{N}}$ Γ -converges to \mathcal{F}_∞ . In this case one writes $\Gamma\text{-lim}_k \mathcal{F}_k = \mathcal{F}_\infty$.

Since this definition is usually considered to be too unhandy in applications, the so-called ‘sequential characterization’ of Γ -convergence is often used (which though requires additional assumptions). For the following theorem I refer to [\[Dal Maso, 1993, Proposition 8.16\]](#) and [\[Dal Maso, 1993, Proposition 8.1\]](#).

Theorem 2.5. *For the space X and the sequence of functions $(\mathcal{F}_k)_{k \in \mathbb{N}}$ where $\mathcal{F}_k : X \rightarrow [-\infty, \infty]$ assume that one of the following assumptions is valid.*

- (i) (X, d) is a metric space,
- (ii) X is a reflexive Banach space endowed with its weak topology and the sequence $(\mathcal{F}_k)_{k \in \mathbb{N}}$ is equicoercive, i.e. there exists a lower semicontinuous function $\Psi : X \rightarrow [-\infty, \infty]$ with $\Psi(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$, such that $\mathcal{F}_k \geq \Psi$ on X for all $k \in \mathbb{N}$.

Then, for every $x \in X$ the lower Γ -limit can be equivalently expressed like

$$\left(\Gamma\text{-lim inf}_{k \rightarrow \infty} \mathcal{F}_k \right) (x) := \inf \left\{ \liminf_{k \rightarrow \infty} \mathcal{F}_k(x_k) : (x_k)_{k \in \mathbb{N}} \text{ in } X \text{ s.t. } x_k \rightarrow x \right\}$$

and there is a sequence $(x_k)_{k \in \mathbb{N}}$ converging to x that attains the infimum

and the upper Γ -limit like

$$\left(\Gamma\text{-lim sup}_{k \rightarrow \infty} \mathcal{F}_k \right) (x) := \inf \left\{ \limsup_{k \rightarrow \infty} \mathcal{F}_k(x_k) : (x_k)_{k \in \mathbb{N}} \text{ in } X \text{ s.t. } x_k \rightarrow x \right\}$$

and there is a sequence $(x_k)_{k \in \mathbb{N}}$ converging to x that attains the infimum.

Herein, the convergence of sequences in X is understood in the metric d in the case of (i) and as weak convergence in the case of (ii).

Corollary 2.6. *In the situation of Theorem 2.5, the sequence $(\mathcal{F}_k)_{k \in \mathbb{N}}$ Γ -converges to a function $\mathcal{F}_\infty : X \rightarrow [-\infty, \infty]$ if and only if for all $x \in X$ one has the*

Γ -lim inf-inequality: *for every sequence $(x_k)_{k \in \mathbb{N}}$ converging to x it is*

$$\mathcal{F}_\infty(x) \leq \liminf_{k \rightarrow \infty} \mathcal{F}_k(x_k),$$

and the

Existence of a recovery sequence: *there exists a sequence $(x_k)_{k \in \mathbb{N}}$ converging to x such that*

$$\mathcal{F}_\infty(x) = \lim_{k \rightarrow \infty} \mathcal{F}_k(x_k).$$

Any sequence $(x_k)_{k \in \mathbb{N}}$ with this property is called a ‘recovery sequence’ for x .

Like in Theorem 2.5, the convergences are understood in the metric or in the weak topology of X , respectively.

Remark 2.6. The existence of a recovery sequence is often equivalently formulated as the Γ -lim sup-inequality. In the situation of Corollary 2.6, the Γ -lim sup-inequality in some $x \in X$ states the existence of a recovery sequence $(x_k)_k$ in X converging to x such that $\mathcal{F}_\infty(x) \geq \limsup_k \mathcal{F}_k(x_k)$.

As announced previously, Γ -convergence of a sequence of functions is closely related to convergence of the minimizers. The result below is taken from [Dal Maso, 1993, Corollary 7.17] and [Dal Maso, 1993, Corollary 7.20].

Theorem 2.7. *Let X be a topological space and $(\mathcal{F}_k)_{k \in \mathbb{N}}$ be a sequence of functions where $\mathcal{F}_k : X \rightarrow [-\infty, \infty]$. Suppose that for every $k \in \mathbb{N}$ there is a minimizer $x_k \in X$ of \mathcal{F}_k . Then*

- (i) *If the sequence of minimizers $(x_k)_{k \in \mathbb{N}}$ converges to some $x \in X$, then x is a minimizer to both Γ -lim inf \mathcal{F}_k and Γ -lim sup \mathcal{F}_k and there holds*

$$\begin{aligned} \left(\Gamma\text{-lim inf}_{k \rightarrow \infty} \mathcal{F}_k \right) (x) &= \liminf_{k \rightarrow \infty} \mathcal{F}_k(x_k) \text{ and} \\ \left(\Gamma\text{-lim sup}_{k \rightarrow \infty} \mathcal{F}_k \right) (x) &= \limsup_{k \rightarrow \infty} \mathcal{F}_k(x_k). \end{aligned}$$

- (ii) *Assume $(\mathcal{F}_k)_{k \in \mathbb{N}}$ Γ -converges to some $\mathcal{F}_\infty : X \rightarrow [-\infty, \infty]$. If $x \in X$ is a cluster point of the sequence of minimizers $(x_k)_{k \in \mathbb{N}}$, then x is a minimizer of \mathcal{F}_∞ and there holds*

$$\mathcal{F}_\infty(x) = \limsup_{k \rightarrow \infty} \mathcal{F}_k(x_k).$$

If $(x_k)_{k \in \mathbb{N}}$ converges to $x \in X$, then x is a minimizer of \mathcal{F}_∞ and

$$\mathcal{F}_\infty(x) = \lim_{k \rightarrow \infty} \mathcal{F}_k(x_k).$$

Finally, the general theory of Γ -convergence also provides a compactness result for sequences of equi-coercive functions defined on a common separable metric space. In fact, in this situation Γ -convergence can be equivalently expressed as convergence in an appropriate metric space which is compact w.r.t. that metric. The result can be found in [Dal Maso, 1993, Theorem 10.22].

Theorem 2.8. *Let (X, d) be a separable metric space and $\Psi : X \rightarrow [-\infty, \infty]$ be a lower semicontinuous function satisfying $\Psi(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$. Furthermore, \mathcal{S}_Ψ shall denote the set of all lower semicontinuous functions $F : X \rightarrow [-\infty, \infty]$ such that $F \geq \Psi$ on X . Then there is a distance $\delta : \mathcal{S}_\Psi \times \mathcal{S}_\Psi \rightarrow [0, \infty)$ on \mathcal{S}_Ψ such that*

- (i) *the metric space $(\mathcal{S}_\Psi, \delta)$ is compact and*
- (ii) *a sequence of functions $(\mathcal{F}_k)_{k \in \mathbb{N}}$ where $\mathcal{F}_k : X \rightarrow [-\infty, \infty]$ Γ -converges to $\mathcal{F}_\infty : X \rightarrow [-\infty, \infty]$ if and only if $(\mathcal{F}_k)_{k \in \mathbb{N}}$ converges to \mathcal{F}_∞ in the metric space $(\mathcal{S}_\Psi, \delta)$.*

Remark 2.7. The distance δ in Theorem 2.8 can be constructed explicitly, cf. [Dal Maso, 1993, p. 123].

Now, in the language of Γ -convergence the result of Marcellini for problems of periodic homogenization which can be viewed as minimization problems is the following (compare [Marcellini, 1978, Theorems 4.1 and 4.4], but the version here is [Dal Maso, 1993, Corollary 24.5]).

Theorem 2.9. *Let Ω be an open and bounded subset of \mathbb{R}^N , $Y = [0, 1)^N$ be the unit cube and $W : Y \times \mathbb{R}^N \rightarrow \mathbb{R}$ be Y -periodic in its first argument and assume that the following three hypotheses hold:*

$$\text{for all } F \in \mathbb{R}^N \text{ the map } y \mapsto W(y, F) \text{ is measurable and } Y\text{-periodic,} \quad (W1)$$

$$\text{for a.e. } y \in Y \text{ the map } F \mapsto W(y, F) \text{ is convex,} \quad (W2)$$

$$\left\{ \begin{array}{l} \text{there exist positive constants } c, C \text{ such that} \\ c(|F|^2 - 1) \leq W(y, F) \leq C(1 + |F|^2) \\ \text{for a.e. } y \in Y \text{ and all } F \in \mathbb{R}^N. \end{array} \right. \quad (W3)$$

Furthermore, for some positive ε and $f \in L^2(\Omega)$ let the functionals $\mathcal{E}_\varepsilon, \mathcal{E}_{\text{Hom}} : W_0^{1,2}(\Omega) \rightarrow \mathbb{R}$ be given through

$$\mathcal{E}_\varepsilon(v) := \int_\Omega W\left(\frac{x}{\varepsilon}, \nabla v(x)\right) dx - \int_\Omega f(x) v(x) dx, \quad (2.29)$$

$$\mathcal{E}_{\text{Hom}}(v) := \int_\Omega W_{\text{Hom}}(\nabla v(x)) dx - \int_\Omega f(x) v(x) dx \quad (2.30)$$

where

$$W_{\text{Hom}}(F) := \inf \left\{ \int_Y W(y, F + \nabla_y v(y)) dy : v \in W_{\text{per}}^{1,2}(Y) \right\}.$$

Finally, let $\mathcal{F}_\varepsilon, \mathcal{F}_{\text{Hom}} : L^2(\Omega) \rightarrow (-\infty, \infty]$ be the extensions of \mathcal{E}_ε and \mathcal{E}_{Hom} to $L^2(\Omega)$ by ∞ . Then for every sequence $(\varepsilon_k)_{k \in \mathbb{N}}$ of positive real numbers converging to 0 one has

- (i) $(\mathcal{F}_{\varepsilon_k})_{k \in \mathbb{N}}$ Γ -converges to \mathcal{F}_{Hom} w.r.t. the $L^2(\Omega)$ -norm,
- (ii) $(\mathcal{E}_{\varepsilon_k})_{k \in \mathbb{N}}$ Γ -converges to \mathcal{E}_{Hom} w.r.t. the weak topology in $W_0^{1,2}(\Omega)$.

The convergence analysis of the classical homogenization problem (CHP) and its variational formulation (2.21) is now just a particular instance of this theorem.

Corollary 2.10. *Let $A : Y \rightarrow \mathbb{R}^{N \times N}$ satisfy the assumptions (A1) and (A2), and suppose A_{Hom} to be given like in Definition 2.2. Then for every vanishing sequence $(\varepsilon_k)_{k \in \mathbb{N}}$ of positive real numbers the sequence of functionals $(\mathcal{E}_{\varepsilon_k})_{k \in \mathbb{N}}$ defined through (2.21) Γ -converges to w.r.t. the weak topology in $W_0^{1,2}(\Omega)$ to \mathcal{E}_{Hom} as given in (2.22). Furthermore, the minimizers u_{ε_k} of $\mathcal{E}_{\varepsilon_k}$ converge to the minimizer u_{Hom} of \mathcal{E}_{Hom} weakly in $W_0^{1,2}(\Omega)$ and the minimal values converge also.*

Proof. From the assumptions on the constitutive Matrix A it is easy to verify that $W : Y \times \mathbb{R}^N \rightarrow \mathbb{R}$, $W(y, F) := \frac{1}{2}A(y)F \cdot F$ satisfies the requirements of Theorem 2.9. While the Γ -convergence of the sequence $(\mathcal{E}_{\varepsilon_k})_{k \in \mathbb{N}}$ is inferred from the second statement of Theorem 2.9, it follows from (2.24) and the definition of W that the Γ -limit is indeed (2.22).

From the ellipticity of the constitutive function $A : Y \rightarrow \mathbb{R}^{N \times N}$ by (A2) and standard a priori estimates for minimizers of coercive functionals it is easily deduced that the sequence of minimizers $(u_{\varepsilon_k})_{k \in \mathbb{N}}$ is uniformly bounded in $W_0^{1,2}(\Omega)$. Hence, any subsequence of $(u_{\varepsilon_k})_{k \in \mathbb{N}}$ contains another weakly convergent subsequence, which by the second statement of Theorem 2.7 and the Γ -convergence of $(\mathcal{E}_{\varepsilon_k})_{k \in \mathbb{N}}$ to \mathcal{E}_{Hom} converges to a minimizer of \mathcal{E}_{Hom} . Since \mathcal{E}_{Hom} has a unique minimizer u_{Hom} (cf. the ellipticity of A_{Hom} by Proposition 2.3), it follows that all subsequences of $(u_{\varepsilon_k})_{k \in \mathbb{N}}$ converge weakly in $W_0^{1,2}(\Omega)$ to the minimizer u_{Hom} of \mathcal{E}_{Hom} . Thus, also the whole sequence itself. The convergence of the minima is once more obtained from the second statement of Theorem 2.7. \square

Although this first chapter is supposed to be a short introduction to those who are new to methods of periodic homogenization, up to now I have only stated how to determine the homogenized constitutive relations for (2.1) from those in the problem with microstructure (CHP) (in Definition 2.2). Yet, no *interpretation* of the homogenization results has been given so far. The reason is that the classical result on the homogenized constitutive matrix in (2.1) as stated in Definition 2.2 comes as a set of partial differential equations, which are rather difficult to explain in the language of physics. Instead, with Marcellini's cell formula at hand the interpretation of the homogenized problem becomes fairly simple. In the case of the homogenized heat equation (2.1), it suffices to consider an infinitesimal, cubic volume element dx centered at x in the homogenized body Ω . For a smooth temperature distribution $u : \Omega \rightarrow \mathbb{R}$, the *overall* temperature can be regarded as linear over dx , *i.e.* locally it behaves like $y \mapsto u(x) + \nabla u(x) \cdot y$.

Now the value of the homogenized dissipation potential (2.22) in the volume element dx around x is by (2.24)

$$\begin{aligned} & W_{\text{Hom}}(\nabla u(x)) \\ &= \inf \left\{ \int_Y \frac{1}{2} A(y) (\nabla u(x) + \nabla_y v(y)) \cdot (\nabla u(x) + \nabla_y v(y)) \, dy : v \in W_{\text{per}}^{1,2}(Y) \right\} \\ &= \int_Y \frac{1}{2} A(y) (\nabla u(x) + \nabla_y u_1(x, y)) \cdot (\nabla u(x) + \nabla_y u_1(x, y)) \, dy \end{aligned}$$

where $u_1(x, \cdot) \in W_{\text{per}}^{1,2}(Y)$ is the (up to an additive constant) unique minimizer that attains the above infimum. (And for u_ε the solution to the homogenized problem (CHP) u_1 is nothing but the first order term in the formal asymptotic expansion (2.2). This is easily seen by noticing that the Euler-Lagrange equation to the above minimization problem is in fact (2.16).) Setting $U_1(x, y) := (u(x) + \nabla u(x) \cdot y) + u_1(x, y)$ one can further simplify

$$\begin{aligned} W_{\text{Hom}}(\nabla u(x)) &= \int_Y \frac{1}{2} A(y) \nabla_y U_1(x, y) \cdot \nabla_y U_1(x, y) \, dy \\ &= \inf \left\{ \int_Y \frac{1}{2} A(y) \nabla_y V(x, y) \cdot \nabla_y V(x, y) \, dy : \right. \\ &\quad \left. V(x, \cdot) \in (u(x) + \nabla u(x) \cdot y) + W_{\text{per}}^{1,2}(Y) \right\}. \end{aligned} \quad (2.31)$$

Thus, if one regards the periodicity cell Y as a ‘zoom’ on the cubic volume element dx around x (see Figure 2.4), then $W_{\text{Hom}}(\nabla u(x))$ is the ‘minimum response’ of the material in the microstructure’s periodicity cell to the (locally linear) temperature distribution of u in dx . To this end, note that the infimum in (2.31) is taken over $(u(x) + \nabla u(x) \cdot y) + W_{\text{per}}^{1,2}(Y)$, i.e. all periodic temperature distributions in Y that vary about the local temperature profile $y \mapsto (u(x) + \nabla u(x) \cdot y)$ of u . Hence, $U_1(x, \cdot)$ is the resulting *local equilibrium temperature distribution in the microstructure* of the volume element dx around x .

2.3 Passage to nonconvex homogenization: 80s

Up to now, in this introduction to the theory of periodic homogenization I have presented two classical methods to compute the homogenized constitutive relations (2.2) for the problem with microstructure (CHP). However, I have left unanswered a crucial question on the fundamentals of periodic homogenization: how does the *choice of the unit cell* affect the homogenization result? In fact, like illustrated in Figure 2.5, one can find various unit cells paving the same periodic microstructure. For instance, in the case of the classical homogenization problem (CHP) one could simply have chosen a bigger tile $kY = [0, k]^N$, $k \in \mathbb{N}$, instead of the unit cube $Y = [0, 1]^N$, but with the same constitutive function $A : Y \rightarrow \mathbb{R}^{N \times N}$. (Recall that I always asked A to be extended to the entire \mathbb{R}^N by Y -periodicity.) Then, when viewing the classical homogenization

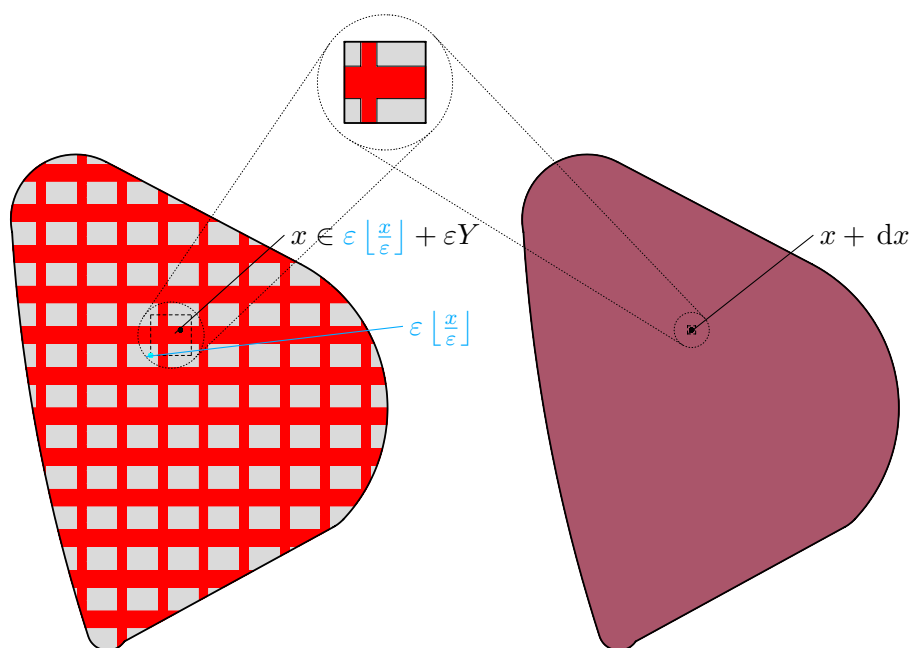


Figure 2.4: Interpretation of Marcellini's cell formula – zoom on the local microstructure

problem (CHP) by means of the functional \mathcal{E}_ε in (2.29) with a quadratic energy density $(y, F) \mapsto W(y, F)$ like in (2.25), Marcellini's cell formula applied to the periodicity cells Y and kY would yield two competing homogenized energy densities $W_{\text{Hom},k}$ and W_{Hom} for one and the same problem with microstructure:

$$W_{\text{Hom},k}(F) = \inf \left\{ \int_{kY} W(y, F + \nabla_y v(y)) \, dy : v \in W_{\text{per}}^{1,2}(kY) \right\} \quad \text{and}$$

$$W_{\text{Hom}}(F) = \inf \left\{ \int_Y W(y, F + \nabla_y v(y)) \, dy : v \in W_{\text{per}}^{1,2}(Y) \right\}.$$

However, it is again due to Paolo Marcellini and his Theorem 2.1 in Marcellini [1978] that these two quantities have been identified as equal all for convex energy densities W . In particular, for the quadratic case of the classical homogenization problem. For this result see also [Müller, 1987, Lemma 4.1].

Theorem 2.11. *Let $W : Y \times \mathbb{R}^N \rightarrow \mathbb{R}$, $Y := [0, 1]^N$, be convex and continuous in its second argument and satisfy the assumptions stated in Theorem 2.9. Then, for all $k \in \mathbb{N}$ the quantities $W_{\text{Hom},k}(F)$ and $W_{\text{Hom}}(F)$ defined above are equal for all $F \in \mathbb{R}^N$.*

On the other hand, what happens for nonconvex energy densities W ? This is a crucial question, since e.g. in the theory of geometrically and constitutively nonlinear elastostatics nonconvexity is not only common but even inevitable (cf. [Ciarlet, 1988, Theorem 4.8-1]). In that case, functionals like (2.29) and (2.30) describe the free energy

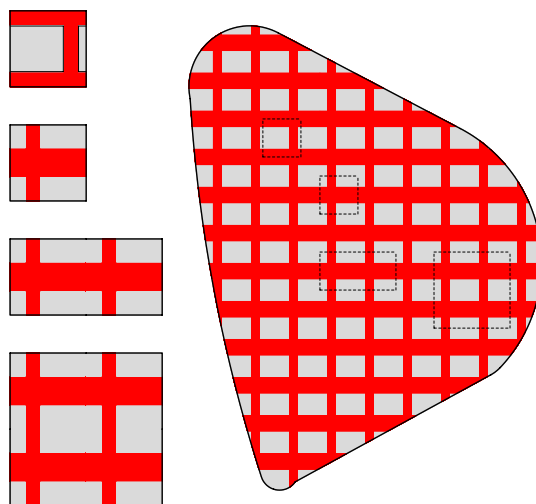


Figure 2.5: Various choices of the periodicity cell for one and the same microstructure

of hyperelastic materials with stored energy density W when exposed to volume loads f and deformed by a (vector-valued) deformation $v : \Omega \rightarrow \mathbb{R}^N$ (see e.g. [Ciarlet, 1988, Chapter 4] for the theory of hyperelastic materials). Unfortunately, a counterexample due to Stefan Müller (in the very context of nonlinear elastostatics) shows that Marcellini's Theorem 2.11 in general no longer holds true for nonconvex energy densities, cf. [Müller, 1987, Theorem 4.3].

Theorem 2.12. *There is a nonconvex energy density $W : Y \times \mathbb{R}^{N \times N} \rightarrow \mathbb{R}$, $Y := [0, 1)^N$, which is Y -periodic in its first argument and continuously differentiable and of polynomial growth of order $p > 2$ in its second argument such that for some $k \in \mathbb{N}$, $k > 1$*

$$\inf \left\{ \int_{kY} W(y, F + D_y v(y)) \, dy : v \in W_{\text{per}}^{1,p}(kY; \mathbb{R}^N) \right\} < \inf \left\{ \int_Y W(y, F + D_y v(y)) \, dy : v \in W_{\text{per}}^{1,p}(Y; \mathbb{R}^N) \right\}.$$

Remark 2.8. The energy density in W in Müller's counterexample is constructed explicitly in [Müller, 1987, Section 4].

Knowing Stefan Müller's counterexample, one would easily guess that the homogenization formula from the convex case in Theorem 2.9 no longer holds true for the nonconvex case. For the simple reason that this would make the homogenization result depend on the size of the actual periodicity cell chosen. Nonetheless, in the nonconvex case there is a different homogenization formula, as it has been proved independently by Andrea Braides in Braides [1985] and by Stefan Müller in Müller [1987]; the version stated here can be found in [Müller, 1987, Theorem 1.3]. In fact, this result may be regarded as the most important contribution to the theory of periodic homogenization in the entire decade from 1980 to 1989.

Theorem 2.13. *Assume that Ω is a bounded Lipschitz domain, $Y := [0, 1)^N$ and that $W : Y \times \mathbb{R}^{N \times N} \rightarrow \mathbb{R}$ satisfies*

$$(i) \quad \frac{1}{C}|F|^2 \leq W(y, F) \leq C(1 + |F|^2),$$

$$(ii) \quad |W(y, F) - W(y, M)| \leq C(1 + |F| + |M|)|F - M|$$

for all $F, M \in \mathbb{R}^{N \times N}$, $y \in Y$ and some constant $C > 0$, and moreover assume W to be extended in its first argument by Y -periodicity. Then, for every vanishing sequence of positive real numbers $(\varepsilon_k)_{k \in \mathbb{N}}$ the functionals $\mathcal{E}_{\varepsilon_k} : W^{1,2}(\Omega; \mathbb{R}^N) \rightarrow \mathbb{R}$ defined through

$$\mathcal{E}_{\varepsilon_k}(v) := \int_{\Omega} W\left(\frac{x}{\varepsilon_k}, Dv(x)\right) dx$$

Γ -converge w.r.t. the weak topology in $W^{1,2}(\Omega; \mathbb{R}^N)$ to the functional $\mathcal{E}_{\text{Hom}} : W^{1,2}(\Omega; \mathbb{R}^N) \rightarrow \mathbb{R}$,

$$\mathcal{E}_{\text{Hom}}(v) := \int_{\Omega} \bar{W}_{\text{Hom}}(Dv(x)) dx$$

where $\bar{W}_{\text{Hom}} : \mathbb{R}^{N \times N} \rightarrow \mathbb{R}$ is given through

$$\bar{W}_{\text{Hom}}(F) = \inf_{k \in \mathbb{N}} \inf \left\{ \int_{kY} W(y, F + D_y v(y)) dy : v \in W_{\text{per}}^{1,2}(kY; \mathbb{R}^N) \right\}.$$

Hence, for nonconvex problems one does not only have to consider the ‘minimum reaction’ of the microstructure to local gradients in one periodicity cell, but in an ever growing ensemble of periodicity cells. (Speaking in the language of structural mechanics, a nonconvex energy density allows structures to ‘buckle’ under compression loads whereas in the convex case they would only be compressed. Moreover, the larger a structure is the more ‘buckling modes’ it can exhibit and thus the better it can minimize its response to external loads. This is exactly the intuition that led Stefan Müller to his by now famous counterexample, showing that a microstructure can minimize its stored energy under an externally applied local gradient more efficiently over an ensemble of periodicity cells than just in a single periodicity cell.) Interestingly though, this is not the case for scalar problems, *i.e.* where the functionals in Theorem 2.13 take values in spaces of scalar functions. In fact, for scalar problems Marcellini’s theorem as formulated in Theorem 2.9 remains valid *even without any convexity conditions imposed on the energy density W* ; see [Müller, 1987, Section 4].

2.4 Formalization of homogenization theory: 90s

The late 1980s and the 1990s have witnessed most inspiring contributions and insights into the theory of periodic homogenization that most notably led to a new, revolutionary notion of convergence in homogenization theory, namely the so-called *two-scale*

convergence. In fact, the definitions and theorems on two-scale convergence provided by Gabriel Nguetseng and Grégoire Allaire – both representatives of the modern french school of homogenization – replaced by mathematical formalism much of the *genius* and intuition that once restricted the field of homogenization to a very limited number of specialists. As a consequence, homogenization theory became accessible for a far broader audience in the applied analysis community, which in turn resulted in rapidly growing number of contributions to the field. The start to this development was marked by Gabriel Nguetseng’s seminar work [Nguetseng \[1989\]](#), which I guess might with good faith be called as groundshaking and inspiring to periodic homogenization as was the fall of the Berlin wall in 1989 to world history.

2.4.1 Two-scale convergence

Previously, mathematically rigorous approaches to periodic homogenization – in particular G -convergence, H -convergence and Γ -convergence – provided definitions and compactness results always for the convergence of *problems* with periodic microstructure, that is of linear operators or functionals describing the periodic homogenization problem. *Convergence of the solutions* to the solution of the homogenized problem (as in the case of the classical homogenization problem (CHP)) was merely reduced to a corollary of more abstract results on the *convergence of the problems* with microstructure. This is the case for both G -convergence and H -convergence (see the references given earlier) and also for Γ -convergence (cf. Corollary 2.10). In contrast to that, the work of Gabriel Nguetseng focused exclusively on the convergence and compactness properties of the solutions to a problem with microstructure. The homogenized problem on the other hand would only result as a mean to ‘describe’ or ‘characterize’ the limit of the solutions. The key challenge in finding a satisfactory notion of convergence for a sequence of solutions $(u_\varepsilon)_\varepsilon$ in say $W_0^{1,2}(\Omega)$ to a problem with microstructure is that it should allow to derive the limit value of integrals like

$$\int_{\Omega} A\left(\frac{x}{\varepsilon}\right) \nabla u_\varepsilon(x) \cdot \nabla \psi(x) \, dx \quad (2.32)$$

for vanishing microscale parameter ε and a testfunction $\psi \in C_c^\infty(\Omega)$. Indeed, this type of integral is essential to the theory of periodic homogenization; notice that it also appears in the weak form (2.4) of the classical homogenization problem (CHP). From the standard assumptions (A1) and (A2) on the constitutive function $A : Y \rightarrow \mathbb{R}^{N \times N}$ and Poincaré’s inequality, one can a priori only state that the solutions $(u_\varepsilon)_\varepsilon$ to (CHP) are bounded in $W_0^{1,2}(\Omega)$. Hence, in general only weak convergence in $W_0^{1,2}(\Omega)$ along a suitable subsequence can be expected; in particular, the sequence of gradients $(\nabla u_\varepsilon)_\varepsilon$ would converge weakly in $L^2(\Omega; \mathbb{R}^N)$. However, also the function $A\left(\frac{\cdot}{\varepsilon}\right) \nabla \psi(\cdot)$ converges merely weakly in $L^2(\Omega; \mathbb{R}^N)$ (according to Lemma 2.1). Since the product of two weakly convergent sequences does in general not converge to the product of the weak limits, it is clear that one cannot naively pass to the limit in (2.32). Nevertheless, by the heuristic reasoning of the asymptotic expansion method (cf. the arguments

leading from the *ansatz* (2.2) and (2.4) to (2.15)), the limit of (2.32) is likely to be

$$\begin{aligned} \int_{\Omega} A\left(\frac{x}{\varepsilon}\right) \nabla u_{\varepsilon}(x) \cdot \nabla \psi(x) \, dx \\ \xrightarrow{\varepsilon \rightarrow 0} \int_{\Omega} \int_Y A(y) (\nabla u_0(x) + \nabla_y u_1(x, y)) \, dy \cdot \nabla \psi(x) \, dx. \end{aligned} \quad (2.33)$$

According to the *ansatz* (2.2) u_0 should be the weak limit of $(u_{\varepsilon})_{\varepsilon}$ in $W_0^{1,2}(\Omega)$ (possibly along a suitable subsequence of the vanishing microscale parameters ε) and $u_1 : \Omega \times Y \rightarrow \mathbb{R}$ some a priori unknown function that is differentiable in its second argument. To conclude, at least in the case of the classical homogenization problem (CHP) weak convergence in $W_0^{1,2}(\Omega)$ (respectively in $L^2(\Omega; \mathbb{R}^N)$) of the solutions $(u_{\varepsilon})_{\varepsilon}$ (respectively their gradients $(\nabla u_{\varepsilon})_{\varepsilon}$) is not sufficient to recover a limit behavior like (2.33). Yet, in his investigations on the convergence properties of integrals like (2.32), Gabriel Nguetseng showed in [Nguetseng, 1989, Theorem 3] that for any vanishing sequence of microscale parameters $(\varepsilon_k)_k$ and any bounded sequence $(u_{\varepsilon})_{\varepsilon}$ indexed by $\varepsilon = (\varepsilon_k)_k$ in $W^{1,2}(\Omega)$ there is a subsequence $(\varepsilon_{k_{\ell}})_{\ell}$ along which (2.33) holds true for all $\psi \in C_c^{\infty}(\Omega)$. Therein, u_0 is the weak limit of $(u_{\varepsilon_{k_{\ell}}})_{\ell}$ in $W^{1,2}(\Omega)$ and u_1 some function in $L^2(\Omega; W_{\text{per}}^{1,2}(Y))$. More generally, Nguetseng established that, given a vanishing sequence of microscale parameters $\varepsilon = (\varepsilon_k)_k$, every bounded sequence $(v_{\varepsilon})_{\varepsilon}$ in $L^2(\Omega)$ admits a subsequence $(\varepsilon_{k_{\ell}})_{\ell}$ and a function $v_0 \in L^2(\Omega \times Y)$ such that

$$\int_{\Omega} v_{\varepsilon}(x) \psi\left(x, \frac{x}{\varepsilon}\right) \, dx \xrightarrow{\varepsilon \rightarrow 0} \int_{\Omega} \int_Y v_0(x, y) \psi(x, y) \, dy \, dx, \quad (2.34)$$

along the subsequence $(\varepsilon_{k_{\ell}})_{\ell}$, where ψ is an arbitrary function in $C_c^{\infty}(\Omega; C_{\text{per}}^{\infty}(Y))$ (again assumed to be extended by Y -periodicity in its second argument). Sequences of functions $(v_{\varepsilon})_{\varepsilon}$ in $L^2(\Omega)$ having property (2.34) were later on called ‘two-scale convergent’ by Grégoire Allaire, and v_0 the ‘two-scale limit’ of the sequence $(v_{\varepsilon})_{\varepsilon}$. Moreover, Allaire showed in his celebrated paper [Allaire, 1992, Theorem 1.8] that under the assumption $\|v_{\varepsilon}\|_{L^2(\Omega)} \rightarrow \|v_0\|_{L^2(\Omega \times Y)}$ and some additional regularity $v_0 \in L^2(\Omega; C_{\text{per}}(Y))$ one has moreover the ‘corrector-result’

$$\|v_{\varepsilon}(x) - v_0\left(x, \frac{x}{\varepsilon}\right)\|_{L^2(\Omega)} \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (2.35)$$

In fact, as explained in [Allaire, 1992, Theorem 2.6] this yields another fundamental insight. If for the sequence of solutions to the classical homogenization problem (CHP), $(u_{\varepsilon})_{\varepsilon}$, and the two-scale limit of the sequence of gradients $(\nabla u_{\varepsilon})_{\varepsilon}$ which takes the form $\nabla u_0 + \nabla_y u_1$ (as it may be anticipated from (2.33)) one could show $\|\nabla_y u_1(\cdot, \frac{\cdot}{\varepsilon})\|_{L^2(\Omega; \mathbb{R}^N)} \rightarrow \|\nabla_y u_1\|_{L^2(\Omega \times Y; \mathbb{R}^N)}$ (and the same for u_1 and $\nabla_x u_1$ instead of $\nabla_y u_1$), then one would have

$$\|u_{\varepsilon}(x) - (u_0(x) + \varepsilon u_1(x, \frac{x}{\varepsilon}))\|_{W_0^{1,2}(\Omega)} \xrightarrow{\varepsilon \rightarrow 0} 0.$$

In other words, a mathematically rigorous justification for the (first term in the) heuristic asymptotic expansion (2.2) of the solution u_{ε} to the classical homogenization problem

(CHP). Eventually, with Nguetseng's and Allaire's notion of two-scale convergence at hand, just like with the method of asymptotic expansions it is just a matter of choosing the right testfunctions to derive the homogenization limit of classical homogenization problem (CHP). The same goes for the equations defining the homogenized constitutive relations given in Definition 2.2. Prior to demonstrating this again for the example of the classical homogenization problem, I once more add some comments on two-scale convergence.

As I already said before, the notion of two-scale convergence due to Gabriel Nguetseng and Grégoire Allaire marked a milestone in the theory of periodic homogenization. While previous approaches were originally defined only for certain problem classes (elliptic problems in the case of G -convergence and H -convergence, minimization problems for Γ -convergence), two-scale convergence now allowed to pass to the limit in all sorts of integrals featuring periodically oscillating testfunctions like $\psi(\cdot, \frac{\cdot}{\varepsilon})$ where $\psi \in C_c^\infty(\Omega; C_{\text{per}}^\infty(Y))$ (or more generally in $L^2(\Omega; C_{\text{per}}(Y))$), no matter which problem the integrals actually came from. As a matter of fact, integrals with periodically oscillating coefficients can be found basically in *all* weak forms of partial differential equations modeling periodic microstructures. In addition, two-scale convergence provides a mathematically rigorous and *universal* justification of the heuristic method of asymptotic expansions. Other than previous justifications which could only show the validity of asymptotic expansions for solutions to particular partial differential equations like (CHP); cf. *e.g.* [Cioranescu and Donato, 1999, Section 7.2]. Most important however, the definition of two-scale convergence together with the compactness and corrector results of Gabriel Nguetseng and Grégoire Allaire introduced a *problem-independent formalism* of how to pass to the limit of vanishing microscale parameter in problems featuring periodic microstructure. Thus, two-scale convergence rendered cumbersome and complicated calculations to actually identify the homogenization limit obsolete, in contrast to asymptotic expansions and G -convergence and H -convergence alike (not however the Γ -convergence approach). In the applied analysis community the development of two-scale convergence led to a 'wave' of new results for all kinds of homogenization problems. However, in recent years Doina Cioranescu, Alain Damlamian, George Griso and many other co-workers developed a new method for describing the limit behavior of integrals like (2.32), namely the so-called method of *periodic unfolding* (see the upcoming Section 2.5). Yet, since periodic unfolding may be regarded as an 'evolution' or 'generalization' of two-scale convergence, I would strongly suggest anyone who is about to study periodic homogenization more in detail to turn to periodic unfolding first. Mostly because two-scale convergence – which in its definition appears to be different from known concepts of functional analysis like weak convergence in $L^2(\Omega)$ – can by the method of periodic unfolding and quite simple calculations actually be identified as an 'ordinary' weak L^2 -convergence (however in a larger L^2 -space than $L^2(\Omega)$). Indeed, the periodic unfolding vista on two-scale convergence revealed many delicate results from two-scale convergence to be rather obvious corollaries from linear functional analysis. Still, two-scale convergence is a topic of its own interest in the field of periodic homogenization. Therefore and also because the fundamental contributions of Gabriel Nguetseng and Grégoire Allaire belong (at least in my opinion) to the

best ever written on periodic homogenization, I strongly recommend starters in periodic homogenization also to study [Lukkassen et al. \[2002\]](#), [Allaire \[1992\]](#) and [Nguetseng \[1989\]](#) (in this order); although there is no monograph dedicated to two-scale convergence, [[Cioranescu and Donato, 1999](#), Chapter 9] and [[Hornung, 1997](#), Appendix A] are a good sources of information on the topic (with the latter also including a very brief exposition of G -convergence, H -convergence and Γ -convergence).

Just like Γ -convergence, also two-scale convergence will be of great importance to the remainder of the thesis (most notably in Chapter 4), which is why I will expose two-scale convergence in detail. Since I already roughly stated the main motivation for two-scale convergence, I may immediately start with the definition of two-scale convergence (cf. [[Allaire, 1992](#), Definition 1.1] and [[Lukkassen et al., 2002](#), Definition 6]).

Definition 2.14. Let Ω be an open subset of \mathbb{R}^N , $Y = [0, 1)^N$ be the unit cube and $(\varepsilon_k)_{k \in \mathbb{N}}$ a sequence of positive real numbers converging to 0. A sequence $(v_{\varepsilon_k})_k$ in $L^2(\Omega)$ is said to (*weakly*) *two-scale converge* in $L^2(\Omega \times Y)$ to a limit $v_0 \in L^2(\Omega \times Y)$, if

$$\lim_{k \rightarrow \infty} \int_{\Omega} v_{\varepsilon_k}(x) \psi \left(x, \frac{x}{\varepsilon_k} \right) dx = \int_{\Omega} \int_Y v_0(x, y) \psi(x, y) dy dx \quad (2.36)$$

holds for all $\psi \in L^2(\Omega; C_{\text{per}}(Y))$. In this case one writes

$$v_{\varepsilon_k} \xrightarrow{2} v_0 \quad \text{in } L^2(\Omega \times Y).$$

If in addition $(v_{\varepsilon_k})_k$ satisfies $\lim_k \|v_{\varepsilon_k}\|_{L^2(\Omega)} = \|v_0\|_{L^2(\Omega \times Y)}$, then the sequence is said to be *strongly two-scale convergent* in $L^2(\Omega \times Y)$ to v_0 and one writes

$$v_{\varepsilon_k} \xrightarrow{2} v_0 \quad \text{in } L^2(\Omega \times Y).$$

Plainly speaking, in the definition of two-scale convergence sequences $(v_{\varepsilon})_{\varepsilon}$ are ‘sampled’ for oscillations of length ε through the testfunctions $x \mapsto \psi(x, x/\varepsilon)$, $\psi \in L^2(\Omega; C_{\text{per}}(Y))$. Hence, the two-scale limit $v_0(x, y)$ keeps track of these fine oscillations in the limit of $\varepsilon \rightarrow 0$ by means of its second variable as it returns asymptotically the same value when being ‘sampled’ with $(x, y) \mapsto \psi(x, y)$.

Example 2.1. Among the most important examples for two-scale convergence is the insight that for smooth functions $\psi \in C_c^\infty(\Omega; C_{\text{per}}^\infty(Y))$ extending Y -periodically in the second argument one observes *strong two-scale convergence* $\psi(x, x/\varepsilon_k) \xrightarrow{2} \psi(x, y)$ in $L^2(\Omega \times Y)$ for every sequence $(\varepsilon_k)_{k \in \mathbb{N}}$ of positive real numbers that converges to zero. In fact, this is an easy corollary of Lemma 2.1 which implies $\psi(x, x/\varepsilon_k) \xrightarrow{2} \psi(x, y)$ in $L^2(\Omega \times Y)$ and $\int_{\Omega} \psi(x, x/\varepsilon)^2 dx \rightarrow \int_{\Omega} \int_Y \psi(x, y)^2 dy dx$. For example, in one space dimension ($Y = [0, 1)$) one has $u_{\text{macro}}(x) \sin(2\pi x/\varepsilon) \xrightarrow{2} u_{\text{macro}}(x) \sin(2\pi y)$ in $L^2((0, 1) \times Y)$ for every $u_{\text{macro}} \in C((0, 1))$.

Obviously, by choosing testfunctions $\psi \in L^2(\Omega)$ independent of the second argument the above definition of two-scale convergence in $L^2(\Omega \times Y)$ reduces to the

definition of weak $L^2(\Omega)$ -convergence. Thus, one might wonder whether one can relate weak two-scale limits in $L^2(\Omega \times Y)$ and ‘ordinary’ weak or strong limits in $L^2(\Omega)$. The answer to this question is stated in the result below, for which I refer to [Lukkassen et al., 2002, Theorems 9, 10 and 17] and [Allaire, 1992, Proposition 1.6].

Proposition 2.15. *Again let Ω be an open subset of \mathbb{R}^N , $Y = [0, 1]^N$ and $(\varepsilon_k)_{k \in \mathbb{N}}$ be a sequence of positive real numbers converging to 0. Moreover, let $(v_{\varepsilon_k})_k$ be a sequence in $L^2(\Omega)$ and $v_0 \in L^2(\Omega \times Y)$.*

(i) *One has the implications*

$$\begin{aligned} v_{\varepsilon_k} \xrightarrow{2} v_0 \quad \text{in } L^2(\Omega \times Y) &\Rightarrow v_{\varepsilon_k} \xrightarrow{2} v_0 \quad \text{in } L^2(\Omega \times Y), \\ v_{\varepsilon_k} \xrightarrow{2} v_0 \quad \text{in } L^2(\Omega \times Y) &\Rightarrow v_{\varepsilon_k} \rightharpoonup \int_Y v_0(\cdot, y) \, dy \quad \text{in } L^2(\Omega). \end{aligned}$$

In particular, sequences that weakly two-scale converge in $L^2(\Omega \times Y)$ are bounded in $L^2(\Omega)$.

(ii) *If $(x, y) \mapsto v_0(x, y)$ is independent of its second argument y , i.e. $v_0 \in L^2(\Omega)$, then*

$$v_{\varepsilon_k} \rightharpoonup v_0 \quad \text{in } L^2(\Omega) \quad \Leftrightarrow \quad v_{\varepsilon_k} \xrightarrow{2} v_0 \quad \text{in } L^2(\Omega \times Y).$$

(iii) *In the case of weak two-scale convergence $v_{\varepsilon_k} \xrightarrow{2} v_0$ in $L^2(\Omega \times Y)$ there holds the estimate*

$$\|\bar{v}_0\|_{L^2(\Omega)} \leq \|v_0\|_{L^2(\Omega \times Y)} \leq \liminf_{k \rightarrow \infty} \|v_{\varepsilon_k}\|_{L^2(\Omega)}$$

where $\bar{v}_0 = \int_Y v_0(\cdot, y) \, dy$ is the $L^2(\Omega)$ -weak limit of the sequence $(v_{\varepsilon_k})_k$.

Hence, the weak limit $x \mapsto \bar{v}_0(x)$ of a sequence $(v_{\varepsilon})_{\varepsilon}$ in $L^2(\Omega)$ is just the average $\bar{v}_0(x) = \int_Y v_0(x, y) \, dy$ over the small-scale oscillations of the sequence as they are captured in the weak two-scale limit $(x, y) \mapsto v_0(x, y)$.

Remark 2.9. Interestingly, the original definition of two-scale convergence given in Allaire [1992] is somewhat ‘defective’ in that it takes testfunctions only from $C_c^\infty(\Omega; C_{\text{per}}^\infty(Y))$. In this case however, weak two-scale convergence of a sequence $(v_{\varepsilon})_{\varepsilon}$ in $L^2(\Omega)$ would not imply neither weak convergence nor boundedness in $L^2(\Omega)$ as it is shown by elementary counterexamples in [Lukkassen et al., 2002, Examples 11 and 12]. These examples exploit the fact that testfunctions in $C_c^\infty(\Omega; C_{\text{per}}^\infty(Y))$, i.e. having compact support in Ω , ignore any concentration of mass of the sequence $(v_{\varepsilon})_{\varepsilon}$ near the boundary. If however one a priori asks for boundedness of the sequence $(v_{\varepsilon})_{\varepsilon}$ in $L^2(\Omega)$, then it would again be sufficient to test for weak two-scale convergence with functions from $C_c^\infty(\Omega; C_{\text{per}}^\infty(Y))$ only (see [Lukkassen et al., 2002, Proposition 13]). It is actually funny to see that also the first definitions of periodic unfolding were ‘defective’ as they also ignored effects near the boundary of the domain Ω (see the next subsection or [Mielke and Timofte, 2008, Chapter 2]).

As one might expect from any meaningful definition of convergence, two-scale limits are unique (if necessary, see [Lukkassen et al., 2002, Chapter 3]). However, one should be aware of the fact that two-scale convergence and therefore also the two-scale limits *strongly depend on the particular sequence of microscale parameters* $(\varepsilon_k)_k$. Intuitively, this can already be seen from the definition of weak two-scale convergence: a testfunction $x \mapsto \psi(x, x/\varepsilon)$ (for say $\psi \in C_c^\infty(\Omega; C_{\text{per}}^\infty(Y))$) itself exhibits oscillations of length ε . Thus by evaluating the integral $\int_\Omega v_\varepsilon(x) \psi(x, x/\varepsilon) dx$ it ‘samples’ $v_\varepsilon \in L^2(\Omega)$ for oscillations of length ε . This should be compared with the definition of Fourier coefficients, wherein one samples a function for a certain frequency expressed by means of a sinusoidal testfunction of that very frequency. Consequently, it is obvious that *e.g.* oscillations of a function v_ε of length far smaller than the ‘sampling wave length’ ε cannot be captured by a testfunction $x \mapsto \psi(x, x/\varepsilon)$. In essence, this insight is explained best by the following example which may be found in [Visintin, 2006, Section 1].

Example 2.2. Let both $\Omega := (0, 1)$ and the periodicity cell $Y := [0, 1)$ be copies of the one-dimensional unit cube, and $(\varepsilon_k)_k$ be the special sequence $\varepsilon_k := \frac{1}{k}$. Then, by Lemma 2.1 it is easy to see that

$$\int_\Omega \sin\left(\frac{2\pi x}{\varepsilon_k}\right) \psi\left(x, \frac{x}{\varepsilon_k}\right) dx \xrightarrow{k \rightarrow \infty} \int_\Omega \int_Y \sin(2\pi y) \psi(x, y) dx$$

for every $\psi \in C_c^\infty(\Omega; C_{\text{per}}^\infty(Y))$. Hence, since $x \mapsto \sin(2\pi x/\varepsilon_k)$ is bounded in $L^\infty(\Omega)$ one has

$$\sin\left(\frac{2\pi x}{\varepsilon_k}\right) \xrightarrow{2} \sin(2\pi y) \quad \text{in } L^2(\Omega \times Y) \text{ w.r.t. } (\varepsilon_k)_k.$$

But for the sequence of functions $x \mapsto \sin(2\pi x/\varepsilon_k^2)$ things are quite different. From the much faster oscillations of these functions compared to the two-scale testfunctions $x \mapsto \psi(x, x/\varepsilon)$ one might already anticipate compensation effects in $\int_\Omega \sin(2\pi x/\varepsilon_k^2) \psi(x, x/\varepsilon_k) dx$, just like in the Riemann-Lebesgue Lemma on Fourier coefficients. Indeed, by using the very same arguments of the Riemann-Lebesgue Lemma one can prove that

$$\int_\Omega \sin\left(\frac{2\pi x}{\varepsilon_k^2}\right) \psi\left(x, \frac{x}{\varepsilon_k}\right) dx \xrightarrow{k \rightarrow \infty} 0$$

for every $\psi \in C_c^\infty(\Omega; C_{\text{per}}^\infty(Y))$ and therefore

$$\sin\left(\frac{2\pi x}{\varepsilon_k^2}\right) \xrightarrow{2} 0 \quad \text{in } L^2(\Omega \times Y) \text{ w.r.t. } (\varepsilon_k)_k.$$

To this end, let $\psi \in L^\infty(\Omega \times Y)$ be such that is extended by Y -periodicity in its second argument and moreover piecewise constant on the tiles $\varepsilon_k((p, q) + (\Omega \times Y))$, $p, q \in \mathbb{N}$. Then, by the particular choice $\varepsilon_k := \frac{1}{k}$ it is an elementary calculation to show that $\int_\Omega \sin(2\pi x/\varepsilon_k^2) \psi(x, x/\varepsilon_k) dx = 0$ for every $k \in \mathbb{N}$. For arbitrary $\psi \in C_c^\infty(\Omega; C_{\text{per}}^\infty(Y))$ the result follows by an approximation argument, *i.e.* choosing a piecewise constant function ψ_k like above such that $\|\psi - \psi_k\|_{L^\infty(\Omega \times Y)} \leq C\varepsilon_k$ for some positive constant C .

In the definition of two-scale convergence Grégoire Allaire distinguished between weak and strong two-scale convergence for a good reason. In fact, according to these definitions – just like in ‘ordinary’ linear functional analysis – products of weakly and strongly two-scale convergent sequences converge to the product of the two-scale limits. Also, (strong) two-scale convergence does not only relate a sequence of functions $(v_\varepsilon)_\varepsilon$ and its two-scale limit v_0 by the convergence of oscillating integrals (2.36) and the norm convergence $\|v_\varepsilon\|_{L^2(\Omega)} \rightarrow \|v_0\|_{L^2(\Omega \times Y)}$, but even by a particular $L^2(\Omega)$ -strong convergence (for both results see [Allaire, 1992, Theorem 1.8] or [Lukkassen et al., 2002, Theorem 18]).

Theorem 2.16. *Let Ω be an open subset of \mathbb{R}^N , $Y = [0, 1]^N$ be the unit cube and $(\varepsilon_k)_{k \in \mathbb{N}}$ a sequence of positive real numbers converging to 0. Moreover, let $(v_{\varepsilon_k})_k$ and $(w_{\varepsilon_k})_k$ be in sequences in $L^2(\Omega)$ such that $v_{\varepsilon_k} \xrightarrow{2} v_0$ in $L^2(\Omega \times Y)$ and $w_{\varepsilon_k} \xrightarrow{2} w_0$ in $L^2(\Omega \times Y)$.*

(i) *For every $\psi \in C_c^\infty(\Omega)$ it is*

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_{\Omega} v_{\varepsilon_k}(x) w_{\varepsilon_k}(x) \psi(x) \, dx \\ = \int_{\Omega} \left(\int_Y v_0(x, y) w_0(x, y) \, dy \right) \psi(x) \, dx. \end{aligned}$$

(ii) *If w_0 is an element of $L^2(\Omega; C_{\text{per}}(Y))$, then*

$$\lim_{k \rightarrow \infty} \|w_{\varepsilon_k}(x) - w_0(x, \frac{x}{\varepsilon})\|_{L^2(\Omega)} = 0.$$

The second statement of the above theorem is referred to as ‘Allaire’s corrector result’. Whereas Gabriel Nguetseng’s two-scale compactness result for bounded sequences in $L^2(\Omega)$ is the following (see [Nguetseng, 1989, Theorem 1] or [Allaire, 1992, Theorem 1.2]).

Theorem 2.17. *Let Ω be an open subset of \mathbb{R}^N , $Y = [0, 1]^N$ be the unit cube and $(\varepsilon_k)_{k \in \mathbb{N}}$ a sequence of positive real numbers converging to 0. Then for every bounded sequence $(v_{\varepsilon_k})_k$ in $L^2(\Omega)$ there exists a subsequence $(v_{\varepsilon_{k_\ell}})_\ell$ and a function $v_0 \in L^2(\Omega \times Y)$ such that*

$$v_{\varepsilon_{k_\ell}} \xrightarrow{2} v_0 \quad \text{in } L^2(\Omega \times Y).$$

Of particular interest is the case of two-scale convergence for $L^2(\Omega)$ -bounded sequences of gradients $(v_\varepsilon)_\varepsilon = (\nabla u_\varepsilon)_\varepsilon$. Given a sequence $(u_\varepsilon)_\varepsilon$ that is bounded in $W^{1,2}(\Omega)$, the weak two-scale limit of the sequence of gradients $(\nabla u_\varepsilon)_\varepsilon$ takes a particular form (see [Nguetseng, 1989, Theorem 3]) and [Allaire, 1992, Proposition 1.14].

Theorem 2.18. *Let Ω be an open subset of \mathbb{R}^N , $Y = [0, 1]^N$ be the unit cube and $(\varepsilon_k)_{k \in \mathbb{N}}$ a sequence of positive real numbers converging to 0. Then for every bounded sequence $(u_{\varepsilon_k})_k$ in $W^{1,2}(\Omega)$ there is a subsequence $(u_{\varepsilon_{k_\ell}})_\ell$ and functions $u_0 \in W^{1,2}(\Omega)$ and $u_1 \in L^2(\Omega; W_{\text{per}}^{1,2}(Y))$ such that*

$$\begin{aligned} u_{\varepsilon_{k_\ell}} &\rightharpoonup u_0 && \text{in } W^{1,2}(\Omega) \quad \text{and} \\ \nabla u_{\varepsilon_{k_\ell}} &\xrightarrow{2} \nabla u_0 + \nabla_y u_1 && \text{in } L^2(\Omega \times Y; \mathbb{R}^N). \end{aligned}$$

Herein, $\nabla_y u_1$ denotes the weak derivative of u_1 w.r.t. its second argument.

Having this theorem at hand, it would no longer be difficult to pass to the limit in the classical homogenization problem (CHP), more precisely in its weak form (2.4). Consider an arbitrary sequence of microscale parameters $(\varepsilon_k)_k$. Then, boundedness of the sequence of solutions $(u_{\varepsilon_k})_k$ to the classical homogenization problem (CHP) in $W_0^{1,2}(\Omega)$ is again easily inferred from the assumptions (A1) and (A2) on the constitutive function $A : Y \rightarrow \mathbb{R}^{N \times N}$ and Poincaré's inequality. Thus, in the spirit of the preceding Theorem 2.18 a weakly two-scale convergent subsequence of the gradients, say $\nabla u_{\varepsilon_{k_\ell}} \xrightarrow{2} \nabla u_0 + \nabla_y u_1$ in $L^2(\Omega \times Y; \mathbb{R}^N)$, may be chosen for u_0 the weak $W_0^{1,2}(\Omega)$ -limit of $(u_{\varepsilon_{k_\ell}})_\ell$ and $u_1 \in L^2(\Omega; W_{\text{per}}^{1,2}(Y))$. However, even for a smooth testfunction $\psi \in C_c^\infty(\Omega)$ in the weak form (2.4) the mapping $x \mapsto A(x/\varepsilon_k) \nabla \psi(x)$ is in general not an element of $L^2(\Omega; C_{\text{per}}(Y))$, hence no valid testfunction for two-scale convergence in the sense of Definition 2.14. This problem would most notably occur for a microstructure made from two homogeneous materials, resulting in a piecewise constant constitutive function A . Fortunately, two-scale convergence allows to pass to the limit with even more general oscillating testfunctions, as it is revealed by the next theorem (see [Lukkassen et al., 2002, Theorems 15 and 16]).

Theorem 2.19. *Let Ω be an open and bounded subset of \mathbb{R}^N , $Y = [0, 1]^N$ be the unit cube and $(\varepsilon_k)_{k \in \mathbb{N}}$ a sequence of positive real numbers converging to 0. Moreover, $(v_{\varepsilon_k})_k$ shall be a sequence in $L^2(\Omega)$ that weakly two-scale converges to $v_0 \in L^2(\Omega \times Y)$. Let $\psi : \Omega \times Y \rightarrow \mathbb{R}$ be a function which is extended by Y -periodicity in its second argument and assume that one of the following assumptions holds:*

- (i) ψ can be identified with an element from $L_{\text{per}}^2(Y; C(\overline{\Omega}))$,
- (ii) there are functions $\psi_1 \in L^{2s}(\Omega)$ and $\psi_2 \in L_{\text{per}}^{2t}(Y)$, $1 \leq s, t \leq \infty$ and $\frac{1}{s} + \frac{1}{t} = 1$, such that $\psi(x, y) = \psi_1(x) \psi_2(y)$.

Then there holds again

$$\lim_{k \rightarrow \infty} \int_{\Omega} v_{\varepsilon_k}(x) \psi\left(x, \frac{x}{\varepsilon_k}\right) dx = \int_{\Omega} \int_Y v_0(x, y) \psi(x, y) dy dx.$$

Eventually, as already explained in Subsection 2.2.2, in order to derive the homogenization limit it is essential to choose a testfunction ψ in the weak form (2.4) of the classical homogenization problem (CHP) that ‘samples’ the fine properties of the solutions $(u_{\varepsilon_k})_k$ for all ε_k -problems. If the weak two-scale limit u_1 of the subsequence of gradients $(\nabla u_{\varepsilon_{k_\ell}})_\ell$ were smooth, then by Allaire’s corrector result Theorem 2.16 and the weak $W_0^{1,2}(\Omega)$ -convergence of $(u_{\varepsilon_{k_\ell}})_\ell$ to u_0 one would obtain

$$\begin{aligned}
& \left\| u_{\varepsilon_{k_\ell}}(x) - \left(u_0(x) + \varepsilon_{k_\ell} u_1 \left(x, \frac{x}{\varepsilon_{k_\ell}} \right) \right) \right\|_{W_0^{1,2}(\Omega)}^2 \\
& \leq 2 \left(\left\| u_{\varepsilon_{k_\ell}}(x) - u_0(x) \right\|_{L^2(\Omega)}^2 \right. \\
& \quad \left. + \left\| \nabla u_{\varepsilon_{k_\ell}}(x) - \left(\nabla u_0(x) + \nabla_y u_1 \left(x, \frac{x}{\varepsilon_{k_\ell}} \right) \right) \right\|_{L^2(\Omega; \mathbb{R}^N)}^2 \right) \\
& \quad + 2\varepsilon_{k_\ell} \left(\left\| u_1 \left(x, \frac{x}{\varepsilon_{k_\ell}} \right) \right\|_{L^2(\Omega)}^2 + \left\| \nabla_x u_1 \left(x, \frac{x}{\varepsilon_{k_\ell}} \right) \right\|_{L^2(\Omega; \mathbb{R}^N)}^2 \right) \\
& \xrightarrow{\ell \rightarrow \infty} 0.
\end{aligned} \tag{2.37}$$

Therefore, testfunctions for solutions of (CHP) must at least capture the limit behavior (2.37), which is why one should choose testfunctions of the form $\psi(x) := \varphi(x) + \varepsilon_k \varphi_1(x, x/\varepsilon_k)$ for smooth functions $\varphi \in C_c^\infty(\Omega)$ and $\varphi_1 \in C_c^\infty(\Omega; C_{\text{per}}^\infty(Y))$. With the strong two-scale convergence

$$\begin{aligned}
& \nabla \varphi(x) + \varepsilon_k \nabla_x \varphi_1 \left(x, \frac{x}{\varepsilon_k} \right) + \nabla_y \varphi_1 \left(x, \frac{x}{\varepsilon_k} \right) \\
& \xrightarrow{2} \nabla \varphi(x) + \nabla_y \varphi_1(x, y) \quad \text{in } L^2(\Omega \times Y; \mathbb{R}^N)
\end{aligned}$$

by Example 2.1 and the weak two-scale convergence $\nabla u_{\varepsilon_{k_\ell}} \xrightarrow{2} \nabla u_0 + \nabla_y u_1$ in $L^2(\Omega \times Y; \mathbb{R}^N)$ it is easy to infer that the limit of the corresponding weak form of (CHP)

$$\begin{aligned}
& \int_{\Omega} A \left(\frac{x}{\varepsilon_{k_\ell}} \right) \nabla u_{\varepsilon_{k_\ell}}(x) \cdot \nabla \left(\varphi(x) + \varepsilon_{k_\ell} \varphi_1 \left(x, \frac{x}{\varepsilon_{k_\ell}} \right) \right) dx \\
& = \int_{\Omega} f(x) \left(\varphi(x) + \varepsilon_{k_\ell} \varphi_1 \left(x, \frac{x}{\varepsilon_{k_\ell}} \right) \right) dx
\end{aligned}$$

is according to Theorems 2.16 and 2.19

$$\begin{aligned}
& \int_{\Omega} \int_Y A(y) \left(\nabla u_0(x) + \nabla_y u_1(x, y) \right) \cdot \left(\nabla \varphi(x) + \nabla_y \varphi_1(x, y) \right) dy dx \\
& = \int_{\Omega} f(x) \varphi(x) dx. \tag{2.38}
\end{aligned}$$

Now, just like in Subsection 2.2.2 choosing $\varphi_1 \equiv 0$ yields

$$\int_{\Omega} \int_Y A(y) \left(\nabla u_0(x) + \nabla_y u_1(x, y) \right) dy \cdot \nabla \varphi(x) dx = \int_{\Omega} f(x) \varphi(x) dx$$

$$\forall \varphi \in C_c^\infty(\Omega)$$

which is nothing but (2.15). While the choice $\varphi \equiv 0$ and $\varphi_1(x, y) = \rho(x) \psi(y)$ for $\rho \in C_c^\infty(\Omega)$ and $\psi \in C_{\text{per}}^\infty(Y)$ results by the arbitrariness of ρ in

$$\int_Y A(y) \left(\nabla u_0(x) + \nabla_y u_1(x, y) \right) \cdot \nabla_y \psi(x) dy = 0 \quad \forall \psi \in C_{\text{per}}^\infty(Y)$$

for a.e. $x \in \Omega$, which is (2.16). In other words, by the very same arguments as employed in Subsection 2.2.2 for the asymptotic expansion method one can identify $u_1 \in L^2(\Omega; W_{\text{per}}^{1,2}(Y))$ to be given as in (2.18), *i.e.*

$$u_1(x, y) = \nabla u_0(x) \cdot \sum_{i=1}^N w_i(y) e_i = \begin{bmatrix} w_1(y) \\ \vdots \\ w_N(y) \end{bmatrix} \cdot \nabla u_0(x),$$

where the (up to an additive constant) unique $w_1, \dots, w_N \in W_{\text{per}}^{1,2}(Y)$ are stated in Definition 2.2. Similarly, the $W_0^{1,2}(\Omega)$ -weak limit u_0 of the subsequence $(u_{\varepsilon_{k_\ell}})_\ell$ of solutions to (CHP) is again nothing but the unique solution to the homogenization limit (2.1), *i.e.*

$$\int_{\Omega} f(x) \varphi(x) dx = \int_{\Omega} A_{\text{Hom}} \nabla u_0(x) \cdot \nabla \varphi(x) dx \quad \forall \varphi \in C_c^\infty(\Omega)$$

where the homogenized constitutive matrix A_{Hom} is as in Definition 2.2, that is

$$A_{\text{Hom}} = \int_Y A(y) \left(I + \left[\nabla_y w_1(y) \mid \cdots \mid \nabla_y w_N(y) \right] \right) dy.$$

In particular, neither the $W_0^{1,2}(\Omega)$ -weak limit of $(u_{\varepsilon_{k_\ell}})_\ell$ nor the weak two-scale limit $\nabla u_0 + \nabla_y u_1$ of $(\nabla u_{\varepsilon_{k_\ell}})_\ell$ depend on the particular (sub-)sequence of the microscale parameters $(\varepsilon_k)_k$. Thus, for *any* vanishing sequence of microscale parameters $(\varepsilon_k)_k$ there hold both the $W_0^{1,2}(\Omega)$ -weak convergence of the solutions $(u_{\varepsilon_k})_k$ to (CHP), as well as the weak two-scale convergence of the sequence of gradients $(\nabla u_{\varepsilon_k})_k$ along the *entire* sequence of microscale parameters.

Eventually, compressed in the form of a theorem Nguetseng's and Allaire's notion of two-scale convergence provides the following result on the classical homogenization problem (cf. [Nguetseng, 1989, Section 6] and [Allaire, 1992, Theorem 2.3]; see also [Cioranescu and Donato, 1999, Section 9.3]).

Theorem 2.20. *Let Ω be an open and bounded subset of \mathbb{R}^N , $Y = [0, 1]^N$ be the unit cube and let $A : Y \rightarrow \mathbb{R}^{N \times N}$ satisfy the assumptions (A1) and (A2). Suppose*

w_1, \dots, w_N and A_{Hom} to be given like in Definition 2.2. Then, for every vanishing sequence of positive real numbers $(\varepsilon_k)_k$ and $(u_{\varepsilon_k})_k$ the sequence of weak solutions in $W_0^{1,2}(\Omega)$ to (CHP) one has

$$\begin{aligned} u_{\varepsilon_k} &\rightharpoonup u_0 && \text{in } W_0^{1,2}(\Omega), \\ \nabla u_{\varepsilon_k} &\xrightarrow{2} \nabla u_0 + \nabla_y u_1 && \text{in } L^2(\Omega \times Y; \mathbb{R}^N) \end{aligned}$$

and u_0 and $u_1 \in L^2(\Omega; W_{\text{per}}^{1,2}(Y))$ and are uniquely defined through

$$\begin{aligned} \int_{\Omega} f(x) \varphi(x) \, dx &= \int_{\Omega} A_{\text{Hom}} \nabla u_0(x) \cdot \nabla \varphi(x) \, dx \quad \forall \varphi \in C_c^\infty(\Omega), \\ u_1(x, y) &= \nabla u_0(x) \cdot \sum_{i=1}^N w_i(y) e_i = \begin{bmatrix} w_1(y) \\ \vdots \\ w_N(y) \end{bmatrix} \cdot \nabla u_0(x). \end{aligned}$$

From this theorem one can easily see that while two-scale convergence itself may very well depend on the specific sequence of microscale parameters $(\varepsilon_k)_k$ chosen (see Example 2.2), the homogenization result usually does or better *should* not. There are indeed specific situations in which also the homogenization result varies with the specific sequence of microscale parameters. An example are the results obtained by Stefan Neukamm and myself in [Neukamm and Stelzig, 2011, Section 4]; see also the last Chapter 4 of this thesis, more precisely Section 4.3.

2.5 State of the Art: 21st century

Just like all preceding decades since the 1960s also the first decade of the 21st century has witnessed crucial contributions to the theory of periodic homogenization. Most notably it has seen a further formalization of what has been achieved by Gabriel Nguetseng and Grégoire Allaire in the 1990s through the notion of two-scale convergence. That is, starting in 2002 with the fundamental publication Cioranescu et al. [2002] another (mainly) french homogenization school – around such names as Doina Cioranescu, Alain Damlamian, Georges Griso, Riccardo DeArcangelis and Patrizia Donato, as well as my supervisor Augusto Visintin – has formed and created what is now called the *periodic unfolding* method. Plainly speaking, the periodic unfolding theory can be interpreted as the identification of two-scale convergence in the framework of classical functional analysis; more precisely, its identification as classical weak (or strong) convergence in an appropriate (larger) space. As a consequence, nowadays the theory of (convex) periodic homogenization can be regarded as a mature branch of applied analysis that – if taught appropriately – is accessible for *anyone* with a good knowledge of the classical weak convergence methods in functional analysis and the theory of partial differential equations. In other words, having a powerful but universal methodology for periodic homogenization problems at hand, more emphasis can finally be spent on the *application* of analysis to interesting problems coming from mathematical physics, rather than on *analysis* itself.

2.5.1 The periodic unfolding method

The periodic unfolding idea as well as its nomenclature goes back to [Cioranescu et al. \[2002\]](#) wherein the authors exploit a natural, although purely *mathematical* intuition. In fact, the following heuristic arguments are in substance unrelated to the classical homogenization problem (CHP) for the heat conduction in the microstructured body Ω that I used as a ‘showcase’ for the previously described methods of periodic homogenization. Let there be given a sequence of functions $(v_\varepsilon)_\varepsilon$ in $L^2(\Omega)$ that weakly two-scale converges to some $v_0 \in L^2(\Omega \times Y)$ as ε vanishes (where the periodicity cell Y shall for simplicity again be the unit cube $[0, 1]^N$). Thus, by Definition 2.14

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} v_\varepsilon(x) \psi\left(x, \frac{x}{\varepsilon}\right) dx = \int_{\Omega} \int_Y v_0(x, y) \psi(x, y) dy dx$$

for all $\psi \in L^2(\Omega; C_{\text{per}}(Y))$. As explained in the previous Subsection 2.4.1, the integral on the left hand side ‘samples’ the function v_ε for oscillations of length ε over the periodicity cells εY tiling the domain Ω (recall that $x \mapsto \psi(x, x/\varepsilon)$ is εY -periodic by the Y -periodicity of ψ in its second argument). Whereas in the two-scale limit $v_0(x, y)$ it is the second variable that keeps track of these oscillations in the limit $\varepsilon \rightarrow 0$, telling how the function $y \mapsto v_0(x, y)$ oscillates over the ‘infinitesimal periodicity cells $x + dY$ tiling Ω ($x \in \Omega$)’. In view of the above convergence one might wonder whether one could already describe the oscillations of v_ε over the ‘finite periodicity cells $\varepsilon a + \varepsilon Y$ tiling Ω ($a \in \mathbb{Z}^N$)’ in an analogous fashion by a two-variable function $(x, y) \mapsto \tilde{v}_\varepsilon(x, y)$. In the two-scale limit $v_0(x, y)$ the first variable $x \in \Omega$ indicates the location of the infinitesimal tile $x + dY$ it belongs to. Yet, in the case of finite tiles $\varepsilon a + \varepsilon Y$ only finitely many contribute to the tiling of the bounded domain Ω and many $x \in \Omega$ share one tile. Therefore, the first variable of such a function $\tilde{v}_\varepsilon(x, y)$ would for all x belonging to the tile $\varepsilon a + \varepsilon Y$ have to indicate the same location of the tile, namely $\varepsilon a = \varepsilon \lfloor \frac{x}{\varepsilon} \rfloor$. Here, $\lfloor \cdot \rfloor : \mathbb{R}^N \rightarrow \mathbb{Z}^N$ returns once more the integer part of its argument. Under heavy abuse of notation, for $x \in \Omega$ and $y \in Y$ this would then read as

$$\tilde{v}_\varepsilon(x, y) = \tilde{v}_\varepsilon\left(\varepsilon \left\lfloor \frac{x}{\varepsilon} \right\rfloor, y\right).$$

Now, in order to capture the oscillations of v_ε over the tile $\varepsilon a + \varepsilon Y \ni x$ one could simply restrict v_ε to this tile and describe its behavior on this tile by a variable coming from the periodicity cell Y . In symbols, for $x \in \Omega$ and $y \in Y$

$$\tilde{v}_\varepsilon(x, y) := (v_\varepsilon|_{\varepsilon a + \varepsilon Y})(\varepsilon a + \varepsilon y) = v_\varepsilon\left(\varepsilon \left\lfloor \frac{x}{\varepsilon} \right\rfloor + \varepsilon y\right).$$

Indeed, this is exactly the idea of the *periodic unfolding operator* from [Cioranescu et al. \[2002\]](#) which is defined as

$$\mathcal{T}_\varepsilon v_\varepsilon : \mathbb{R}^N \times Y \rightarrow \mathbb{R}, \quad (\mathcal{T}_\varepsilon v_\varepsilon)(x, y) := \begin{cases} v_\varepsilon\left(\varepsilon \left\lfloor \frac{x}{\varepsilon} \right\rfloor + \varepsilon y\right) & \text{if } \varepsilon \left\lfloor \frac{x}{\varepsilon} \right\rfloor + \varepsilon y \in \Omega, \\ 0 & \text{else.} \end{cases}$$

Just like a two-scale limit, by construction the first argument x of the *unfolded* function $(\mathcal{T}_\varepsilon v_\varepsilon)(x, y)$ indicates the finite periodicity cell $\varepsilon \lfloor \frac{x}{\varepsilon} \rfloor + \varepsilon Y$ it belongs to, while the

second argument y captures the function's behavior over this tile. Besides this qualitative behavior, the most important property of the unfolding operator \mathcal{T}_ε is its *isometry property*, in that it satisfies

$$\int_{\mathbb{R}^N} \int_Y (\mathcal{T}_\varepsilon v)(x, y) \, dy \, dx = \int_{\Omega} v(x) \, dx$$

for every $v \in L^1(\Omega)$ (compare Lemma 2.1). In particular, $\|\mathcal{T}_\varepsilon v\|_{L^2(\mathbb{R}^N \times Y)} = \|v\|_{L^2(\Omega)}$ for all $v \in L^2(\Omega)$. On the other hand, in view of the isometry property of the unfolding operator \mathcal{T}_ε and the boundedness of the sequence $(v_\varepsilon)_\varepsilon$ in $L^2(\Omega)$ by Proposition 2.15, the unfolded sequence $(\mathcal{T}_\varepsilon v_\varepsilon)_\varepsilon$ remains bounded in $L^2(\mathbb{R}^N \times Y)$. Hence, from the unfolded sequence one might extract a weakly convergent subsequence – for simplicity not relabeled – and some $\tilde{v}_0 \in L^2(\mathbb{R}^N \times Y)$ such that $\mathcal{T}_\varepsilon v_\varepsilon \rightharpoonup \tilde{v}_0$ in $L^2(\mathbb{R}^N \times Y)$. In other words, the bounded sequence $(v_\varepsilon)_\varepsilon$ in $L^2(\Omega)$ exhibits two competing limits: its weak two-scale limit $v_0 \in L^2(\Omega \times Y)$ and the weak $L^2(\mathbb{R}^N \times Y)$ -limit \tilde{v}_0 of the unfoldings $(\mathcal{T}_\varepsilon v_\varepsilon)_\varepsilon$. Now, the crucial insight of the periodic unfolding method is that these two limits are *equal*, in that (for domains Ω whose boundary has zero Lebesgue-measure)

$$\text{supp } \tilde{v}_0 \subseteq \overline{\Omega} \times Y \quad \text{and} \quad \tilde{v}_0(x, y) = \mathbb{1}_\Omega(x) v_0(x, y)$$

for a.e. $x \in \mathbb{R}^N$ and $y \in Y$ (see e.g. [Cioranescu et al., 2002, Proposition 1]). Surprisingly, this is not difficult to verify. For every ε the support of $\mathcal{T}_\varepsilon v_\varepsilon$ is obviously contained in $\{x : \text{dist}(x, \Omega) \leq \varepsilon 2\text{diam}(Y)\} \times Y$, cf. (2.11). Thus, as ε becomes smaller and smaller the supports of $(\mathcal{T}_\varepsilon v_\varepsilon)_\varepsilon$ shrink down to $\text{supp } \tilde{v}_0 \subseteq \overline{\Omega} \times Y$. Moreover, for all $\psi \in C_c^\infty(\Omega; C_{\text{per}}^\infty(Y))$ it is by the isometry property of the unfolding operator \mathcal{T}_ε

$$\begin{aligned} & \int_{\Omega} v_\varepsilon(x) \psi\left(x, \frac{x}{\varepsilon}\right) \, dx \\ &= \int_{\mathbb{R}^N} \int_Y \mathcal{T}_\varepsilon(v_\varepsilon \psi(\cdot, \frac{\cdot}{\varepsilon}))(x, y) \, dy \, dx \\ &= \int_{\mathbb{R}^N} \int_Y (\mathcal{T}_\varepsilon v_\varepsilon)(x, y) \psi(x, y) \, dy \, dx \\ & \quad + \int_{\mathbb{R}^N} \int_Y (\mathcal{T}_\varepsilon v_\varepsilon)(x, y) \left(\psi(x, y) - \mathcal{T}_\varepsilon(\psi(\cdot, \frac{\cdot}{\varepsilon}))(x, y)\right) \, dy \, dx. \end{aligned} \quad (2.39)$$

While the first integral obviously converges to $\int_{\mathbb{R}^N} \int_Y \tilde{v}_0(x, y) \psi(x, y) \, dy \, dx$, a simple calculation shows that by definition and the assumed Y -periodicity of ψ in its second argument

$$\psi(x, y) - \mathcal{T}_\varepsilon(\psi(\cdot, \frac{\cdot}{\varepsilon}))(x, y) = \psi(x, y) - \psi\left(\varepsilon \left\lfloor \frac{x}{\varepsilon} \right\rfloor + \varepsilon y, y\right).$$

The test function ψ being smooth, this term vanishes uniformly. Passing to the limit $\varepsilon \rightarrow 0$ in (2.39) and recalling the weak two-scale convergence $v_\varepsilon \xrightarrow{2} v_0$ in $L^2(\Omega \times Y)$ then implies

$$\int_{\Omega} \int_Y v_0(x, y) \psi(x, y) \, dy \, dx = \int_{\mathbb{R}^N} \int_Y \tilde{v}_0(x, y) \psi(x, y) \, dy \, dx$$

and from $\text{supp } \tilde{v}_0 \subseteq \bar{\Omega} \times Y$ and $\text{vol } \partial\Omega = 0$ it follows by the density of $C_c^\infty(\Omega; C_{\text{per}}^\infty(Y)) \ni \psi$ in $L^2(\Omega \times Y)$ that $\tilde{v}_0(x, y) = \mathbf{1}_\Omega(x)v_0(x, y)$ a.e. in $\mathbb{R}^N \times Y$.

This short proof of the equivalence between weak two-scale convergence and ‘ordinary’ weak convergence of the unfolded sequence prototypically demonstrates the major strength of the periodic unfolding method. In just a few lines of elementary integral calculus and basic concepts from linear functional analysis (even if one includes the isometry property of the unfolding operator, cf. the proof of Lemma 2.1) one can completely reproduce Nguetseng’s celebrated compactness result Theorem 2.17. Moreover, the equivalence of weak two-scale convergence $v_{\varepsilon_k} \xrightarrow{2} v_0$ in $L^2(\Omega \times Y)$ and the $L^2(\mathbb{R}^N \times Y)$ -weak convergence of the unfolded sequence $(\mathcal{T}_\varepsilon v_\varepsilon)_\varepsilon$ allows to define two-scale convergence by means of classical functional analysis methods. In fact, periodic unfolding can be understood as *the* explanation for two-scale convergence, which formerly was already known to have many similarities to classical weak convergence (definition via L^2 -scalar products with test functions, weak two-scale compactness, convergence of products of weakly and strongly two-scale convergent sequences), but could somehow not be aligned with known functional analysis. Whereas now, the embedding of two-scale convergence into a functional analysis context by means of the periodic unfolding method allows to derive two-scale analoga for many well-known results from functional analysis – by simply applying them to the unfolded sequence. This strength is exploited *e.g.* in Visintin [2006] and Visintin [2007] (for two-scale versions of compactness theorems, embeddings, inequalities, . . .). Another advantage of periodic unfolding is that basically the entire intuition and the convergence proofs based on Nguetseng’s and Allaire’s two-scale convergence can be reused in a periodic unfolding setting; often, it is only the two-scale argument that has to be replaced by a corresponding periodic unfolding statement. See *e.g.* the application of periodic unfolding to the classical homogenization problem (CHP) at the end of the present subsection. Eventually, the periodic unfolding idea can in principle be adapted to all sorts of periodic microstructures. For instance, periodically perforated domains (see Cioranescu et al. [2006b] and Cioranescu et al. [2011]), thin domains (see Neukamm [2010]), perforated thin domains or interfaces (see *e.g.* Cioranescu et al. [2008b] or the following Chapter 3) and many other situations. Yet, there are various different definitions of periodic unfolding operators available in the literature (cf. Cioranescu et al. [2008a]; Visintin [2006]; Mielke and Timofte [2008]); all are similar in spirit, but sometimes very different in substance due to seemingly negligible differences. This is nicely demonstrated by what one might call the ‘curse of the boundary’ in periodic homogenization (cf. Remark 2.9 and in particular [Mielke and Timofte, 2008, Chapter 2]).

Again, before stating the most important results on the periodic unfolding method and applying it to the ‘showcase’ of the classical homogenization problem (CHP), I would like to recall some personal experiences with the topic. In fact, the periodic unfolding method reduces many convex homogenization problems to a mere construction of an appropriate periodic unfolded operator. Whereas the actual convergence analysis becomes a corollary of classical weak convergence methods known from linear functional analysis. Hence, as already pointed out at the beginning of this section, more

emphasis can be spent on the homogenization problem itself rather than on the convergence analysis of the problem as the microscale parameter vanishes. The periodic unfolding method has also been applied to nonconvex homogenization problems as they were outlined in Section 2.3 (see Cioranescu et al. [2006a]), although in that context it did not prove to be handier than *e.g.* the methods used in Müller [1987]. Nevertheless, the periodic unfolding method has evolved as the main workhorse for convex homogenization problems and has in this function mostly displaced its ‘predecessor’ two-scale convergence (in its original definition). Also, due to its elementary and intuitive nature it is most suitable for teaching, since it does not require other than basic knowledge in Lebesgue- and Sobolev-spaces as well as linear functional analysis. In other words, requirements that every good undergraduate student should fulfill. Unfortunately, to my current knowledge there is no textbook available on periodic homogenization yet that introduces the periodic unfolding method. However, with the standard reference on periodic homogenization Cioranescu and Donato [1999] being written by two of the main drivers of the periodic unfolding method one might hope for a second extended edition of that volume. For the time being, the most suitable references for anyone who is new to periodic unfolding are to my humble opinion Cioranescu et al. [2002], Cioranescu et al. [2008a] and Visintin [2006], the most *precise* exposition of the topic being Mielke and Timofte [2008]. Yet, the periodic unfolding method is not free of ‘flaws’ – most notably, just like the asymptotic expansion method it shows a tendency of becoming ‘unreadable’ once a certain degree of complexity has been reached. For instance, Cioranescu et al. [2008b] studies a situation of periodically recurring voids in a domain, where the scale of the void diameter is different from the mutual distance of the voids (smaller), and moreover the voids concentrate near a hyperplane. Unfortunately then, although being very well-written and of interesting content, the paper becomes very hard to read due to the sheer notational effort that is needed to cope with the various different microscales involved.

Due to the importance of the periodic unfolding method for the remainder of the thesis I will introduce the most relevant definitions and theorems in a similar level of detail as I did for Γ -convergence and two-scale convergence. The start is marked by the definition of the periodic unfolding operator like found in Mielke and Timofte [2008] (also the notation used here leans to that work).

Definition 2.21. Let Ω be an open subset of \mathbb{R}^N , $Y = [0, 1)^N$ be the unit cube and $\varepsilon > 0$ be some positive real number. Then the periodic unfolding operator $\mathcal{T}_\varepsilon : L^2(\Omega) \rightarrow L^2(\mathbb{R}^N \times Y)$ is defined as

$$\begin{aligned} (\mathcal{T}_\varepsilon v)(x, y) &:= \begin{cases} v\left(\varepsilon \lfloor \frac{x}{\varepsilon} \rfloor + \varepsilon y\right) & \text{if } \varepsilon \lfloor \frac{x}{\varepsilon} \rfloor + \varepsilon y \in \Omega, \\ 0 & \text{else.} \end{cases} \\ &= v^{\text{Ext}}\left(\varepsilon \lfloor \frac{x}{\varepsilon} \rfloor + \varepsilon y\right), \end{aligned}$$

where $(\cdot)^{\text{Ext}} : L^2(\Omega) \rightarrow L^2(\mathbb{R}^N)$, $v^{\text{Ext}} := \mathbb{1}_\Omega v$ denotes the extension of v to \mathbb{R}^N by 0.

As indicated previously, the most essential feature of the periodic unfolding operator

is its isometry property (cf. [Cioranescu et al., 2002, Proposition 1], [Visintin, 2006, Lemma 1.1] and [Mielke and Timofte, 2008, Proposition 2.1]).

Proposition 2.22. *Again let Ω be an open subset of \mathbb{R}^N , $Y = [0, 1]^N$ be the unit cube and $\varepsilon > 0$ be some positive real number. Then, for the periodic unfolding operator $\mathcal{T}_\varepsilon : L^2(\Omega) \rightarrow L^2(\mathbb{R}^N \times Y)$ of Definition 2.21 there holds*

$$\int_{\mathbb{R}^N} \int_Y (\mathcal{T}_\varepsilon v)(x, y) \, dy \, dx = \int_\Omega v(x) \, dx,$$

and \mathcal{T}_ε is an isometry.

In view of the above isometry property and the usual weak compactness of bounded sequences in Hilbert-spaces, it is basically trivial to infer the following statement. (Recall the origin of the regularity assumption on the domain boundary having zero Lebesgue measure as explained in the motivation of the periodic unfolding method.)

Proposition 2.23. *Let there be given Ω an open and bounded subset of \mathbb{R}^N such that $\partial\Omega$ has zero Lebesgue measure, $Y = [0, 1]^N$ the unit cube and $(\varepsilon_k)_{k \in \mathbb{N}}$ a vanishing sequence of positive real numbers, and for every ε_k the periodic unfolding operator $\mathcal{T}_{\varepsilon_k} : L^2(\Omega) \rightarrow L^2(\mathbb{R}^N \times Y)$ from Definition 2.21. Then, for every bounded sequence $(v_{\varepsilon_k})_{k \in \mathbb{N}}$ in $L^2(\Omega)$ there is a subsequence $(\varepsilon_{k_\ell})_{\ell \in \mathbb{N}}$ and a function $v_0 \in L^2(\Omega \times Y)$ such that*

$$\mathcal{T}_{\varepsilon_{k_\ell}} v_{\varepsilon_{k_\ell}} \rightharpoonup E v_0 \quad \text{in } L^2(\mathbb{R}^N \times Y).$$

where $E : L^2(\Omega \times Y) \rightarrow L^2(\mathbb{R}^N \times Y)$ is the 0-extension defined as $(Ev)(x, y) := \mathbb{1}_\Omega(x)v(x, y) = (v^{\text{Ext}}(\cdot, y))(x)$.

Now, in view of the weak two-scale compactness in $L^2(\Omega \times Y)$ of $L^2(\Omega)$ -bounded sequences and the weak compactness of their unfoldings in $L^2(\mathbb{R}^N \times Y)$, the natural question on the relation between these two concepts of convergence is answered by the next proposition.

Proposition 2.24. *Once more assume to be given Ω an open and bounded subset of \mathbb{R}^N such that $\partial\Omega$ has zero Lebesgue measure, $Y = [0, 1]^N$ the unit cube and $(\varepsilon_k)_{k \in \mathbb{N}}$ a vanishing sequence of positive real numbers, and for every ε_k the periodic unfolding operator $\mathcal{T}_{\varepsilon_k} : L^2(\Omega) \rightarrow L^2(\mathbb{R}^N \times Y)$ from Definition 2.21. Moreover, let $(v_{\varepsilon_k})_{k \in \mathbb{N}}$ be some sequence in $L^2(\Omega)$ and $v_0 \in L^2(\Omega \times Y)$. Then one has the relations*

$$v_{\varepsilon_k} \xrightarrow{2} v_0 \quad \text{in } L^2(\Omega \times Y) \quad \Leftrightarrow \quad \mathcal{T}_{\varepsilon_k} v_{\varepsilon_k} \rightharpoonup E v_0 \quad \text{in } L^2(\mathbb{R}^N \times Y)$$

and

$$v_{\varepsilon_k} \xrightarrow{2} v_0 \quad \text{in } L^2(\Omega \times Y) \quad \Leftrightarrow \quad \mathcal{T}_{\varepsilon_k} v_{\varepsilon_k} \rightarrow E v_0 \quad \text{in } L^2(\mathbb{R}^N \times Y).$$

Here, $E v_0$ is the 0-extension of v_0 defined in Proposition 2.23.

Proof. The first statement may be proved as outlined at the beginning of this subsection, using the basic properties of two-scale convergence stated in Subsection 2.4.1. Alternatively, one can turn to [Visintin, 2006, Proposition 2.5]. The second statement is an easy corollary of the first assertion and the isometry property of the unfolding operator, upon taking into account the fact that strong convergence in Hilbert spaces is equivalent to weak convergence and convergence of the norm. \square

Given the equivalence in the proposition above, the relations of traditional two-scale convergence with ordinary weak and strong convergence in $L^2(\Omega)$ – like stated in Proposition 2.15 – should also hold in a periodic unfolding setting.

Proposition 2.25. *Let Ω be an open subset of \mathbb{R}^N such that $\partial\Omega$ has zero Lebesgue measure, $Y = [0, 1]^N$ be the unit cube and $(\varepsilon_k)_{k \in \mathbb{N}}$ a vanishing sequence of positive real numbers. Moreover, let $(v_{\varepsilon_k})_k$ be a sequence in $L^2(\Omega)$ and $v_0 \in L^2(\Omega \times Y)$.*

(i) *One has the implications*

$$\begin{aligned} \mathcal{T}_{\varepsilon_k} v_{\varepsilon_k} \rightharpoonup E v_0 \quad \text{in } L^2(\mathbb{R}^N \times Y) &\Rightarrow \mathcal{T}_{\varepsilon_k} v_{\varepsilon_k} \rightharpoonup E v_0 \quad \text{in } L^2(\mathbb{R}^N \times Y), \\ \mathcal{T}_{\varepsilon_k} v_{\varepsilon_k} \rightharpoonup E v_0 \quad \text{in } L^2(\mathbb{R}^N \times Y) &\Rightarrow v_{\varepsilon_k} \rightharpoonup \int_Y v_0(\cdot, y) \, dy \quad \text{in } L^2(\Omega). \end{aligned}$$

(ii) *If $(x, y) \mapsto v_0(x, y)$ is independent of its second argument y , i.e. $v_0 \in L^2(\Omega)$, then*

$$v_{\varepsilon_k} \rightharpoonup v_0 \quad \text{in } L^2(\Omega) \quad \Leftrightarrow \quad \mathcal{T}_{\varepsilon_k} v_{\varepsilon_k} \rightharpoonup E v_0 \quad \text{in } L^2(\mathbb{R}^N \times Y).$$

(iii) *In the case of weak convergence $\mathcal{T}_{\varepsilon_k} v_{\varepsilon_k} \rightharpoonup E v_0$ in $L^2(\mathbb{R}^N \times Y)$ there holds the estimate*

$$\|\bar{v}_0\|_{L^2(\Omega)} \leq \|v_0\|_{L^2(\Omega \times Y)} \leq \liminf_{k \rightarrow \infty} \|v_{\varepsilon_k}\|_{L^2(\Omega)}$$

where $\bar{v}_0 = \int_Y v_0(\cdot, y) \, dy$ is the $L^2(\Omega)$ -weak limit of the sequence $(v_{\varepsilon_k})_k$.

Although one might turn to [Visintin, 2006, Theorem 1.3], it should be noticed that these results are basically trivial to prove in view of the definition of the unfolding operator \mathcal{T}_ε , its isometry property and standard results from linear functional analysis. Actually, in this fashion one can prove numerous results for two-scale convergence, the core idea being always the same: unfold sequences from $L^2(\Omega)$ into the larger space $L^2(\mathbb{R}^N \times Y)$ and then apply well-known results from functional analysis to the unfolded sequence. Although being in practice often far more delicate than such trivial corollaries like Proposition 2.25, this procedure is *e.g.* applied in Visintin [2006] where numerous powerful two-scale versions of compactness theorems, convergence theorems from the Lebesgue-calculus and embedding theorems are derived.

Another major advantage of periodic unfolding is the treatment of *periodically perforated domains*. That is, for Ω a smoothly bounded subset of \mathbb{R}^N , V some void contained in $Y = [0, 1]^N$ and ε a microscale parameter, the periodically perforated domain

Ω_ε is defined as $\Omega \setminus \bigcup_{a \in \mathbb{Z}^N} \varepsilon(a + V)$. In other words, the domain Ω from which small voids εV arranged along the periodic \mathbb{Z}^N -grid have been removed. Equivalently, upon defining the *perforated periodicity cell* $Y^* := Y \setminus V$, the indicator function of the periodically perforated domain Ω_ε can be expressed like $\mathbb{1}_{\Omega_\varepsilon}(x) = \mathbb{1}_\Omega(x) \mathbb{1}_{Y^*}(\frac{x}{\varepsilon})$ where $\mathbb{1}_{Y^*}$ is supposed repeated over the entire \mathbb{R}^N by Y -periodicity. By applying the periodic unfolding operator \mathcal{T}_ε to the 0-extension of a function $v \in L^2(\Omega_\varepsilon)$ to \mathbb{R}^N (again denoted $v^{\text{Ext}} = \mathbb{1}_{\Omega_\varepsilon} v$), it is easily seen that

$$\begin{aligned} (\mathcal{T}_\varepsilon v^{\text{Ext}})(x, y) &= \mathcal{T}_\varepsilon \left(\mathbb{1}_{Y^*} \left(\frac{\cdot}{\varepsilon} \right) v^{\text{Ext}} \right) (x, y) \\ &= \mathbb{1}_{Y^*}(y) v^{\text{Ext}} \left(\varepsilon \left\lfloor \frac{x}{\varepsilon} \right\rfloor + \varepsilon y \right). \end{aligned}$$

Hence, the right hand side can be identified with a function in $L^2(\mathbb{R}^N \times Y^*)$ (*i.e.* the second argument is taken from the perforated periodicity cell Y^* only). By the isometry property of the unfolding operator \mathcal{T}_ε one can further write

$$\begin{aligned} \int_{\Omega_\varepsilon} v(x) \, dx &= \int_{\mathbb{R}^N} v^{\text{Ext}}(x) \, dx \\ &= \int_{\mathbb{R}^N} \int_{Y^*} (\mathcal{T}_\varepsilon v^{\text{Ext}})(x, y) \, dy = \int_{\mathbb{R}^N} \int_{Y^*} v^{\text{Ext}} \left(\varepsilon \left\lfloor \frac{x}{\varepsilon} \right\rfloor + \varepsilon y \right) \, dy. \end{aligned}$$

In other words, by restricting the periodic unfolding operator \mathcal{T}_ε to $L^2(\mathbb{R}^N \times Y^*)$ one obtains an operator $\mathcal{T}_\varepsilon^* : L^2(\Omega_\varepsilon) \rightarrow L^2(\mathbb{R}^N \times Y^*)$ which is again isometric and unfolds any function over the ε -dependent, periodically perforated domain Ω_ε to the ε -independent domain $\mathbb{R}^N \times Y^*$. In particular, a sequence $(v_\varepsilon)_\varepsilon$ whose elements are respectively taken from the sequence of spaces $(L^2(\Omega_\varepsilon))_\varepsilon$ is unfolded into one common space $L^2(\mathbb{R}^N \times Y^*)$. Consequently, classical properties of this common space $L^2(\mathbb{R}^N \times Y^*)$ (like *e.g.* weak compactness) immediately apply to the sequence of unfoldings. This essential feature of the periodic unfolding method is outlined in detail in Cioranescu et al. [2006b] and Cioranescu et al. [2011], but also used in the following Chapter 3.

The periodic unfolding idea also simplifies the two-scale analysis of sequences of $L^2(\Omega)$ -bounded gradients (cf. Theorem 2.18). That is, if $v_\varepsilon = \nabla u_\varepsilon$ for some $u_\varepsilon \in W^{1,2}(\Omega)$, the definition of the periodic unfolding operator

$$(\mathcal{T}_\varepsilon u_\varepsilon)(x, y) = u_\varepsilon^{\text{Ext}} \left(\varepsilon \left\lfloor \frac{x}{\varepsilon} \right\rfloor + \varepsilon y \right) \in L^2(\mathbb{R}^N \times Y)$$

yields for a.e. $x \in \mathbb{R}^N$ such that $\varepsilon \left\lfloor \frac{x}{\varepsilon} \right\rfloor + \varepsilon Y \subseteq \Omega$ a function

$$\begin{aligned} (\mathcal{T}_\varepsilon u_\varepsilon)(x, \cdot) &\in W^{1,2}(Y) \quad \text{with} \\ \nabla_y (\mathcal{T}_\varepsilon u_\varepsilon)(x, y) &= \varepsilon \nabla u_\varepsilon \left(\varepsilon \left\lfloor \frac{x}{\varepsilon} \right\rfloor + \varepsilon y \right) = \varepsilon \mathcal{T}_\varepsilon (\nabla u_\varepsilon)(x, y). \end{aligned}$$

Now, as expected from the analogy of weak convergence of the unfoldings and weak two-scale converge by Proposition 2.24, also the unfoldings of gradients of $W^{1,2}(\Omega)$ -bounded sequences obey a weak compactness property (see *e.g.* [Cioranescu et al., 2002, Theorem 1], [Cioranescu et al., 2008a, Theorem 3.5] or [Mielke and Timofte, 2008, Theorem 2.8]).

Theorem 2.26. *Let Ω be an open subset of \mathbb{R}^N such that $\partial\Omega$ has zero Lebesgue measure, $Y = [0, 1]^N$ be the unit cube and $(\varepsilon_k)_{k \in \mathbb{N}}$ a sequence of positive real numbers converging to 0. Then for every bounded sequence $(u_{\varepsilon_k})_k$ in $W^{1,2}(\Omega)$ there is a subsequence $(u_{\varepsilon_{k_\ell}})_\ell$ and functions $u_0 \in W^{1,2}(\Omega)$ and $u_1 \in L^2(\Omega; W_{\text{per}}^{1,2}(Y))$ such that*

$$\begin{aligned} u_{\varepsilon_{k_\ell}} &\rightharpoonup u_0 && \text{in } W^{1,2}(\Omega) \quad \text{and} \\ \mathcal{T}_{\varepsilon_{k_\ell}}(\nabla u_{\varepsilon_{k_\ell}}) &\rightharpoonup E(\nabla u_0) + E(\nabla_y u_1) && \text{in } L^2(\mathbb{R}^N \times Y; \mathbb{R}^N). \end{aligned}$$

Herein, $E(\nabla u_0)$ and $E(\nabla_y u_1)$ shall be the (componentwise) 0-extensions of ∇u_0 and $\nabla_y u_1$ to $\mathbb{R}^N \times Y$ like in Proposition 2.23, and $\nabla_y u_1$ denotes the weak derivative of u_1 w.r.t. its second argument.

Remark 2.10. Due to the notational confusion caused by the $(\cdot)^{\text{Ext}}$ superscript (cf. Definition 2.21) and the 0-extension E (cf. Proposition 2.23), I will omit these symbols whenever possible and appropriate. Mostly, this will lead to a clearer exposition rather than causing additional confusion.

Having seen how the periodic unfolding method simplifies the theory of two-scale convergence, an application to the ‘showcase’ of this introduction, *i.e.* the classical homogenization problem (CHP) will also show its impact on applications to actual homogenization problems. That is, suppose once more to be given an arbitrary sequence of microscale parameters $(\varepsilon_k)_k$ and let $(u_{\varepsilon_k})_k$ be the sequence of solutions in $W_0^{1,2}(\Omega)$ to the weak form (2.4) of (CHP). Again, the boundedness of this sequence in $W_0^{1,2}(\Omega)$ is again easily inferred from the assumptions (A1) and (A2) on the constitutive function $A : Y \rightarrow \mathbb{R}^{N \times N}$ and Poincaré’s inequality. Then, by Theorem 2.26 there is a subsequence $(\varepsilon_{k_\ell})_\ell$, a function $u_0 \in W^{1,2}(\Omega)$ and some $u_1 \in L^2(\Omega; W_{\text{per}}^{1,2}(Y))$ such that $\mathcal{T}_{\varepsilon_{k_\ell}}(\nabla u_{\varepsilon_{k_\ell}}) \rightharpoonup E(\nabla u_0) + E(\nabla_y u_1)$ in $L^2(\mathbb{R}^N \times Y; \mathbb{R}^N)$. Turning to the weak form (2.4) of the classical homogenization problem, an application of the isometry property of the unfolding operator yields

$$\begin{aligned} \int_{\mathbb{R}^N} \int_Y (\mathcal{T}_{\varepsilon_{k_\ell}} \nabla u_{\varepsilon_{k_\ell}})(x, y) \cdot \mathcal{T}_{\varepsilon_{k_\ell}} \left(A \left(\frac{\cdot}{\varepsilon_{k_\ell}} \right) \nabla \psi \right) (x, y) \, dy \, dx \\ = \int_{\mathbb{R}^N} \int_Y (\mathcal{T}_{\varepsilon_{k_\ell}} f)(x, y) (\mathcal{T}_{\varepsilon_{k_\ell}} \psi)(x, y) \, dy \, dx \quad (2.40) \end{aligned}$$

for all $\psi \in C_c^\infty(\Omega)$. Given the equivalence of two-scale convergence and convergence of the unfoldings, it apparently makes sense to choose the same testfunctions $\psi(x) := \varphi(x) + \varepsilon_k \varphi_1(x, x/\varepsilon_k)$ for smooth functions $\varphi \in C_c^\infty(\Omega)$ and $\varphi_1 \in C_c^\infty(\Omega; C_{\text{per}}^\infty(Y))$. Indeed, by exploiting the definition of the periodic unfolding operator $\mathcal{T}_{\varepsilon_{k_\ell}}$ and the smoothness of φ and φ_1 , some elementary calculations reveal that for this choice of the testfunction ψ

$$\begin{aligned} (\mathcal{T}_{\varepsilon_{k_\ell}} \psi)(x, y) &= (\mathcal{T}_{\varepsilon_{k_\ell}} \varphi)(x, y) + \varepsilon_{k_\ell} \varphi_1 \left(\varepsilon \left\lfloor \frac{x}{\varepsilon} \right\rfloor + \varepsilon y, y \right) \\ &\rightarrow \varphi(x) \quad \text{in } L^2(\mathbb{R}^N \times Y) \end{aligned}$$

and

$$\begin{aligned} (\mathcal{T}_{\varepsilon_{k_\ell}} \nabla \psi)(x, y) &= (\mathcal{T}_{\varepsilon_{k_\ell}} \nabla \varphi)(x, y) \\ &\quad + \varepsilon_{k_\ell} \nabla_x \varphi_1 \left(\varepsilon \left\lfloor \frac{x}{\varepsilon} \right\rfloor + \varepsilon y, y \right) + \nabla_y \varphi_1 \left(\varepsilon \left\lfloor \frac{x}{\varepsilon} \right\rfloor + \varepsilon y, y \right) \\ &\rightarrow \nabla \varphi(x) + \nabla_y \varphi_1(x, y) \quad \text{in } L^2(\mathbb{R}^N \times Y). \end{aligned}$$

Another simple calculation for the left hand side of (2.40) shows

$$\begin{aligned} \mathcal{T}_{\varepsilon_{k_\ell}} \left(A \left(\frac{\cdot}{\varepsilon_{k_\ell}} \right) \nabla \psi \right) (x, y) &= A(y) (\mathcal{T}_{\varepsilon_{k_\ell}} \nabla \psi)(x, y) \\ &\rightarrow A(y) (\nabla \varphi(x) + \nabla_y \varphi_1(x, y)) \quad \text{in } L^2(\mathbb{R}^N \times Y). \end{aligned}$$

These two strong convergences in $L^2(\mathbb{R}^N \times Y)$ together with the $L^2(\mathbb{R}^N \times Y)$ -weak convergence of $\mathcal{T}_{\varepsilon_{k_\ell}}(\nabla u_{\varepsilon_{k_\ell}}) \rightharpoonup E(\nabla u_0) + E(\nabla_y u_1)$ allows to pass to the limit of vanishing microscale parameter ε_{k_ℓ} in the unfolded classical homogenization problem (2.40). The resulting limiting equation reads

$$\begin{aligned} \int_{\mathbb{R}^N} \int_Y (E(\nabla u_0(x)) + E(\nabla_y u_1(x, y))) \cdot A(y) (\nabla \varphi(x) + \nabla_y \varphi_1(x, y)) \, dy \, dx \\ = \int_{\mathbb{R}^N} \int_Y (Ef)(x) \varphi(x) \, dy \, dx \end{aligned}$$

which can further be simplified to

$$\begin{aligned} \int_{\Omega} \int_Y A(y) (\nabla u_0(x) + \nabla_y u_1(x, y)) \cdot (\nabla \varphi(x) + \nabla_y \varphi_1(x, y)) \, dy \, dx \\ = \int_{\Omega} f(x) \varphi(x) \, dx \end{aligned}$$

for all $\varphi \in C_c^\infty(\Omega)$ and all $\varphi_1 \in C_c^\infty(\Omega; C_{\text{per}}^\infty(Y))$. Recalling (2.38) from Subsection 2.4.1 one can by the same arguments employed there identify $u_1 \in L^2(\Omega; W_{\text{per}}^{1,2}(Y))$ as

$$u_1(x, y) = \nabla u_0(x) \cdot \sum_{i=1}^N w_i(y) e_i = \begin{bmatrix} w_1(y) \\ \vdots \\ w_N(y) \end{bmatrix} \cdot \nabla u_0(x),$$

where the (up to an additive constant) unique $w_1, \dots, w_N \in W_{\text{per}}^{1,2}(Y)$ are stated in Definition 2.2. Furthermore, the $W_0^{1,2}(\Omega)$ -weak limit u_0 of the subsequence $(u_{\varepsilon_{k_\ell}})_\ell$ of solutions to (CHP) is again revealed to be the unique solution of the homogenization limit (2.1) with A_{Hom} being given as in Definition 2.2. Since these arguments – repeated for any arbitrary subsequence of the vanishing sequence $(\varepsilon_k)_k$ – yield always the same limit behavior, one can furthermore infer that the above convergences hold along the entire sequence $(\varepsilon_k)_k$. To conclude, the periodic unfolding method applied to the classical homogenization problem (CHP) reveals the following convergence result (with rather elementary methods compared to the asymptotic expansion method, the Γ -convergence approach or a two-scale analysis).

Theorem 2.27. *Let Ω be an open and bounded subset of \mathbb{R}^N such that $\partial\Omega$ has zero Lebesgue measure, $Y = [0, 1]^N$ be the unit cube and let $A : Y \rightarrow \mathbb{R}^{N \times N}$ satisfy the assumptions (A1) and (A2). Suppose w_1, \dots, w_N and A_{Hom} to be given like in Definition 2.2. Then, for every vanishing sequence of positive real numbers $(\varepsilon_k)_k$ and $(u_{\varepsilon_k})_k$ the sequence of weak solutions in $W_0^{1,2}(\Omega)$ to (CHP) one has*

$$\begin{aligned} u_{\varepsilon_k} &\rightharpoonup u_0 && \text{in } W_0^{1,2}(\Omega), \\ \mathcal{T}_{\varepsilon_k}(\nabla u_{\varepsilon_k}) &\rightharpoonup E(\nabla u_0) + E(\nabla_y u_1) && \text{in } L^2(\mathbb{R}^N \times Y; \mathbb{R}^N) \end{aligned}$$

and u_0 and $u_1 \in L^2(\Omega; W_{\text{per}}^{1,2}(Y))$ and are respectively uniquely defined through

$$\begin{aligned} \int_{\Omega} f(x) \varphi(x) \, dx &= \int_{\Omega} A_{\text{Hom}} \nabla u_0(x) \cdot \nabla \varphi(x) \, dx \quad \forall \varphi \in C_c^\infty(\Omega), \\ u_1(x, y) &= \nabla u_0(x) \cdot \sum_{i=1}^N w_i(y) e_i = \begin{bmatrix} w_1(y) \\ \vdots \\ w_N(y) \end{bmatrix} \cdot \nabla u_0(x). \end{aligned}$$

Again, $E(\nabla u_0)$ and $E(\nabla_y u_1)$ denote the (componentwise) 0-extensions of ∇u_0 and $\nabla_y u_1$ to $\mathbb{R}^N \times Y$ like in Proposition 2.23.

2.5.2 Periodic unfolding for non-translatory microstructures

As explained in Remark 2.3 and illustrated in Figure 2.3 there are relevant examples for periodic microstructures whose fine scale does not stem from pure translations of a common periodicity cell, but also involves rotations – or maybe even more general transformations. The periodic nature of such microstructures can in general not be obtained by rescaling a suitable tiling of the entire space like it is done in the case of translatory microstructures (see the basic modeling principles for periodic microstructures outlined in Subsection 2.2.1). For instance, in euclidean coordinates the constitutive properties of some point x in the functionally graded annular disk shown in Figure 2.3 cannot be expressed through a term $A\left(\frac{x}{\varepsilon}\right)$ like in the classical homogenization problem (CHP), where $A : \mathbb{R}^N \rightarrow \mathbb{R}^N$ and a ε is an *arbitrary* positive microscale parameter. (Yet, one could switch to polar coordinates, where rotational periodicity in euclidean coordinates would result as translational periodicity in the angle variable.) In particular, for such situations neither the method of asymptotic expansions nor two-scale convergence could be applied without having performed a suitable change of variables beforehand.

Despite this ‘shortcoming’ of the traditional methods, the idea of the periodic unfolding can be adapted to this new situation. In fact, the intuition behind the periodic unfolding method does *not* require any periodicity assumption, in particular no translatory periodicity. In fact, the previously explained periodic unfolding idea can in a natural fashion be generalized to bodies $\Omega \subseteq \mathbb{R}^N$ whose microstructure is merely given as the pairwise disjoint union of ‘copies’ $\{T_\ell(Y) : \ell \in \mathbb{N}\} =: \mathcal{P}$ of a *reference tile* $Y \subseteq \mathbb{R}^N$ such that $\text{vol } Y < \infty$. Here, the $\mathcal{T}_\ell : Y \rightarrow \mathbb{R}^N$ shall be sufficiently regular and bijective transformations of the reference tile Y . In analogy to the periodic unfolding

idea of Cioranescu et al. [2002] outlined in the previous subsection, to describe a function $v \in L^2(\Omega)$ over a particular tile of the microstructure \mathcal{P} I would use two operators. A first operator, which I call the *macrolocalization operator* $\mathcal{N}_{\mathcal{P}} : \Omega \rightarrow \mathbb{N}$, indicates the tile $T_{\ell}(Y)$ a point $x \in \Omega$ is located in. While a second operator $\mathcal{R}_{\mathcal{P}} : \Omega \rightarrow Y$ defined as $\mathcal{R}_{\mathcal{P}}(x) := T_{\ell}^{-1}(x)$, with $\ell = \mathcal{N}_{\mathcal{P}}(x)$, indicates the relative position of a point $x \in \Omega$ in its respective tile. This operator I will henceforth call the *microlocalization operator*. Then, just like in the traditional periodic unfolding method from Cioranescu et al. [2002], the behavior of the function v in a tile of the microstructure \mathcal{P} containing $x \in \Omega$ is captured by

$$Y \ni y \mapsto \left(v|_{T_{\mathcal{N}_{\mathcal{P}}(x)}(Y)} \right) (T_{\mathcal{N}_{\mathcal{P}}(x)}(y)) = v(\mathcal{S}_{\mathcal{P}}(x, y))$$

where

$$\mathcal{S}_{\mathcal{P}} : \Omega \times Y \rightarrow \Omega, \quad \mathcal{S}_{\mathcal{P}}(x, y) := T_{\mathcal{N}_{\mathcal{P}}(x)}(y).$$

The map $\mathcal{S}_{\mathcal{P}}$ shall be called the *composition map* associated with the microstructure \mathcal{P} .

Remark 2.11. At this point I would like to emphasize that the notation of the present subsection heavily leans to Visintin [2006], while the term composition map is inspired by Mielke and Timofte [2008].

Assuming the maps $T_{\ell} : Y \rightarrow \mathbb{R}^N$, $\ell \in \mathbb{N}$, to be sufficiently regular and the microstructure $\mathcal{P} = \{T_{\ell}(Y) : \ell \in \mathbb{N}\}$ of Ω to be such that $\text{vol}(\Omega \setminus \bigcup\{\mathcal{P}\}) = 0$ one can state

$$\begin{aligned} \int_{\Omega} v(x) \, dx &= \sum_{\ell \in \mathbb{N}} \int_{T_{\ell}(Y)} v(x) \, dx = \\ &= \sum_{\ell \in \mathbb{N}} \int_Y v(T_{\ell}(y)) \, |\det T_{\ell}(y)| \, dy \\ &= \sum_{\ell \in \mathbb{N}} \int_{T_{\ell}(Y)} \int_Y v(T_{\ell}(y)) \, |\det T_{\ell}(y)| \frac{\text{vol} Y}{\text{vol} T_{\ell}(Y)} \, dy \, dx \\ &= \sum_{\ell \in \mathbb{N}} \int_{T_{\ell}(Y)} \int_Y v(\mathcal{S}_{\mathcal{P}}(x, y)) \, |\det T_{\ell}(y)| \frac{\text{vol} Y}{\text{vol} T_{\ell}(Y)} \, dy \, dx. \end{aligned}$$

Upon defining

$$\begin{aligned} \mathcal{D}_{\mathcal{P}} : \Omega \times Y \rightarrow \mathbb{R}, \quad \mathcal{D}_{\mathcal{P}}(x, y) &:= \left| \det T_{\ell}(y) \right| \frac{\text{vol} Y}{\text{vol} T_{\ell}(Y)} \\ &\quad \text{for } \ell \in \mathbb{N} \text{ such that } x \in T_{\ell}(Y) \\ &= \left| \det T_{\mathcal{N}_{\mathcal{P}}(x)}(y) \right| \frac{\text{vol} Y}{\text{vol} T_{\mathcal{N}_{\mathcal{P}}(x)}(Y)}, \end{aligned}$$

it follows that

$$\int_{\Omega} v(x) \, dx = \int_{\Omega} \int_Y v(\mathcal{S}_{\mathcal{P}}(x, y)) \mathcal{D}_{\mathcal{P}}(x, y) \, dy \, dx,$$

which looks very much like the well-known isometry property of the periodic unfolding operator, cf. Proposition 2.22. Recalling from the previous Subsection 2.5.1 that it is the isometry property and the resulting isometric embedding into a larger L^2 -space that allows to derive many powerful statements on unfolded sequences, one may hope that the same strategies apply to the present adaption of the unfolding method.

Expressed in the form of a precise definition, the just described concepts and operators read as follows.

Definition 2.28. Let Ω and Y be open subsets of \mathbb{R}^N and assume $\text{vol } Y < \infty$. An essential partition \mathcal{P} of Ω , i.e. a set of measurable subsets of Ω such that $\text{vol}(\Omega \setminus \bigcup\{\mathcal{P}\}) = 0$, is called a *generalized tiling* of Ω with *reference tile* Y and *pattern* $(T_\ell)_{\ell \in \mathbb{N}}$, if the following assumptions hold.

- (i) For every $\ell \in \mathbb{N}$ the map $T_\ell : Y \rightarrow \mathbb{R}^N$ is a Lipschitz diffeomorphism.
- (ii) $\mathcal{P} = \{T_\ell(Y) : \ell \in \mathbb{N}\}$.

This definition is most intuitively illustrated by the following Figure 2.6.

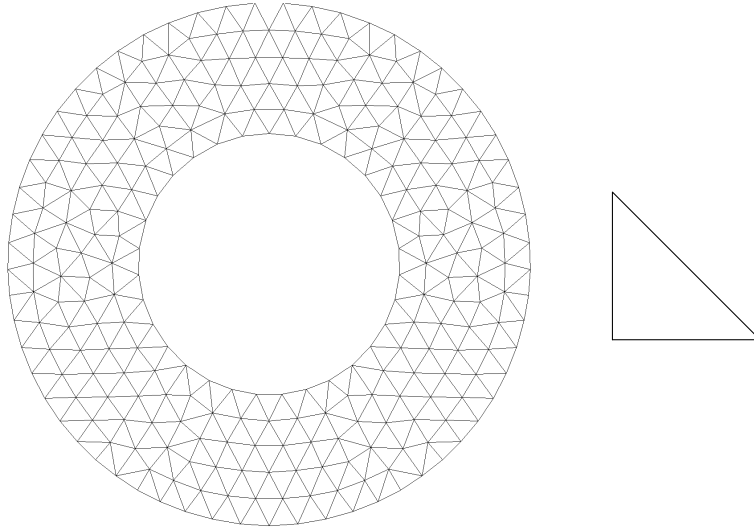


Figure 2.6: A triangulation of an annular disk and a reference triangle (tile). The mesh has been generated with code from Persson and Strang [2004]

Definition 2.29. Let Ω and Y be open subsets of \mathbb{R}^N , $\text{vol } Y < \infty$, and let \mathcal{P} be a generalized tiling of Ω with reference tile Y and pattern $(T_\ell)_{\ell \in \mathbb{N}}$. Then

- (i) the *macrolocalization operator* associated with the generalized tiling \mathcal{P} is defined as

$$\mathcal{N}_{\mathcal{P}} : \Omega \rightarrow \mathbb{N}, \quad \mathcal{N}_{\mathcal{P}}(x) := \ell \text{ such that } x \in T_\ell(Y),$$

(ii) the *microlocalization operator* associated with the generalized tiling \mathcal{P} is the map

$$\mathcal{R}_{\mathcal{P}} : \Omega \rightarrow Y, \quad \mathcal{R}_{\mathcal{P}}(x) := T_{\mathcal{N}_{\mathcal{P}}(x)}^{-1}(x),$$

(iii) the *composition map* associated with the generalized tiling \mathcal{P} is

$$\mathcal{S}_{\mathcal{P}} : \Omega \times Y \rightarrow \Omega, \quad \mathcal{S}_{\mathcal{P}}(x, y) := T_{\mathcal{N}_{\mathcal{P}}(x)}(y)$$

(iv) the *micro-mass density operator* associated with the generalized tiling \mathcal{P} is given through

$$\mathcal{D}_{\mathcal{P}} : \Omega \times Y \rightarrow \mathbb{R}, \quad \mathcal{D}_{\mathcal{P}}(x, y) := \left| \det T_{\mathcal{N}_{\mathcal{P}}(x)}(y) \right| \frac{\text{vol } Y}{\text{vol } T_{\mathcal{N}_{\mathcal{P}}(x)}(Y)}$$

(v) the *unfolding operator for generalized tilings* associated with the generalized tiling \mathcal{P} shall for measurable $v : \Omega \rightarrow \mathbb{R}$ be defined as

$$\mathcal{T}_{\mathcal{P}} : \Omega \times Y \rightarrow \mathbb{R}, \quad (\mathcal{T}_{\mathcal{P}}v)(x, y) := v(\mathcal{S}_{\mathcal{P}}(x, y)).$$

Remark 2.12. The seemingly awkward name ‘micro-mass density operator’ $\mathcal{D}_{\mathcal{P}}$ associated with a generalized tiling \mathcal{P} is justified by the following heuristic. For some $x \in \Omega$ such that x results as an element of the tile $T_{\ell}(Y)$, one can write

$$\mathcal{D}_{\mathcal{P}}(x, y) = \left| \det T_{\ell}(y) \right| \frac{\text{vol } Y}{\text{vol } T_{\ell}(Y)} = \frac{\frac{\text{vol } T_{\ell}(dy)}{\text{vol } dy}}{\frac{\text{vol } T_{\ell}(Y)}{\text{vol } Y}}.$$

In other words, in a transformed tile $T_{\ell}(Y)$ the micro-mass density operator measures the *local* change of the mass density induced by the transformation T_{ℓ} *inside* the tile, relative to the change in the mass density over the *entire* transformed tile $T_{\ell}(Y)$.

As the following statements shall play no further role in the remainder of this thesis, I will only state some properties of the just introduced unfolding operator for generalized tilings. Proofs will not be included, but may be subject of a follow-up work. An exception is the following statement, whose proof idea has been given previously in the motivation of the unfolding operator for generalized tilings.

Proposition 2.30. *Let Ω and Y be open subsets of \mathbb{R}^N , $\text{vol } Y < \infty$, and let \mathcal{P} be a generalized tiling of Ω with reference tile Y and pattern $(T_{\ell})_{\ell \in \mathbb{N}}$. Then*

(i) *for every $f \in L^1(\Omega; L^{\infty}(Y))$ it is*

$$\int_{\Omega} f(x, \mathcal{R}_{\mathcal{P}}(x)) \, dx = \int_{\Omega} \int_Y f(\mathcal{S}_{\mathcal{P}}(x, y), y) \mathcal{D}_{\mathcal{P}}(x, y) \, dy \, dx.$$

In particular, for $v \in L^2(\Omega)$ one has

$$\int_{\Omega} v(x) \, dx = \int_{\Omega} \int_Y (\mathcal{T}_{\mathcal{P}}v)(x, y) \mathcal{D}_{\mathcal{P}}(x, y) \, dy \, dx.$$

- (ii) assume that for the micro-mass density operator it holds $\|\frac{1}{\mathcal{D}_{\mathcal{P}}}\|_{L^\infty(\Omega \times Y)} < \infty$. Then, the unfolding operator $\mathcal{T}_{\mathcal{P}}$ associated with the generalized tiling \mathcal{P} is linear,

$$\mathcal{T}_{\mathcal{P}} : L^2(\Omega) \rightarrow L^2(\Omega \times Y)$$

and moreover

$$\|\mathcal{T}_{\mathcal{P}}v\|_{L^2(\Omega \times Y)} \leq \left(\text{vol } Y \left\| \frac{1}{\mathcal{D}_{\mathcal{P}}} \right\|_{L^\infty(\Omega \times Y)} \right)^{\frac{1}{2}} \|v\|_{L^2(\Omega)}.$$

Now, one may consider *sequences* of generalized tilings $(\mathcal{P}_k)_k$ sharing one *common* reference tile Y . Just like in the case of homogenization problems this is to describe the microstructure of a body Ω whose characteristic size (in the classical periodic case quantified by the microscale parameter) becomes finer and finer. Given the unfolding operator for generalized tilings and knowing about the equivalence between periodic unfolding and two-scale convergence, one can naturally define two-scale convergence also in the setting of generalized tilings.

Definition 2.31. Assume to be given open subsets Ω and Y of \mathbb{R}^N , $\text{vol } Y < \infty$, and let $(\mathcal{P}_k)_{k \in \mathbb{N}}$ be a sequence of generalized tilings of Ω with common reference tile Y , whereas for every $k \in \mathbb{N}$ the pattern of an individual tiling \mathcal{P}_k shall be denoted $(T_{k,\ell})_{\ell \in \mathbb{N}}$. Let $(v_k)_{k \in \mathbb{N}}$ be a sequence in $L^2(\Omega)$ and $v_0 \in L^2(\Omega \times Y)$. Then $(v_k)_{k \in \mathbb{N}}$ shall be called *weakly two-scale convergent to v_0 in $L^2(\Omega \times Y)$ w.r.t. $(\mathcal{P}_k)_{k \in \mathbb{N}}$* , in symbols $v_k \xrightarrow{(\mathcal{P}_k)_k} v_0$ in $L^2(\Omega \times Y)$, if

$$\mathcal{T}_{\mathcal{P}_k}v_k \rightharpoonup v_0 \quad \text{in } L^2(\Omega \times Y).$$

Similarly, $(v_k)_{k \in \mathbb{N}}$ is said to *strongly two-scale converge to v_0 in $L^2(\Omega \times Y)$ w.r.t. $(\mathcal{P}_k)_{k \in \mathbb{N}}$* , in symbols $v_k \xrightarrow{(\mathcal{P}_k)_k} v_0$ in $L^2(\Omega \times Y)$, if

$$\mathcal{T}_{\mathcal{P}_k}v_k \rightarrow v_0 \quad \text{in } L^2(\Omega \times Y).$$

In fact, also for this adaption of the unfolding idea to sequences of generalized tilings sharing a common reference tile one can prove analogous results to those cited in Propositions 2.23 and 2.25 for periodic unfolding in the sense of Cioranescu et al. [2002].

Proposition 2.32. Let Ω and Y be open subsets of \mathbb{R}^N , $\text{vol } Y < \infty$, and let $(\mathcal{P}_k)_{k \in \mathbb{N}}$ be a sequence of generalized tilings of Ω with common tile Y , and let the pattern corresponding to an individual tiling \mathcal{P}_k , $k \in \mathbb{N}$, be denoted $(T_{k,\ell})_{\ell \in \mathbb{N}}$. Moreover, $(v_k)_{k \in \mathbb{N}}$ shall be a sequence in $L^2(\Omega)$ and $v_0 \in L^2(\Omega \times Y)$. For the generalized tilings

$(\mathcal{P}_k)_{k \in \mathbb{N}}$ the assumptions $(\mathcal{P}1), \dots, (\mathcal{P}4)$ below shall be defined. For some positive constant $C > 0$

$$\left\{ \begin{array}{l} \sup_{(x,y) \in \Omega \times Y} |\mathcal{S}_{\mathcal{P}_k}(x,y) - x| \leq C \quad \text{for all } k \in \mathbb{N}, \\ \text{and } \mathcal{S}_{\mathcal{P}_k}(x,y) \rightarrow x \quad \text{pointwise a.e. in } \Omega \times Y \end{array} \right\}, \quad (\mathcal{P}1)$$

$$\mathcal{D}_{\mathcal{P}_k}(x,y) \geq \frac{1}{C} \quad \text{for a.e. } (x,y) \in \Omega \times Y, \quad (\mathcal{P}2)$$

$$\mathcal{D}_{\mathcal{P}_k}(x,y) \leq C \quad \text{for a.e. } (x,y) \in \Omega \times Y, \quad (\mathcal{P}3)$$

$$\mathcal{D}_{\mathcal{P}_k}(x,y) \rightarrow \mathcal{D} \quad \text{pointwise a.e. in } \Omega \times Y \text{ for some } \mathcal{D} : \Omega \times Y \rightarrow \mathbb{R}. \quad (\mathcal{P}4)$$

Then one can prove the following statements.

(i) If $(\mathcal{P}2)$ holds and $(v_k)_{k \in \mathbb{N}}$ is bounded in $L^2(\Omega)$, then the unfolded sequence $(\mathcal{T}_{\mathcal{P}_k} v_k)_{k \in \mathbb{N}}$ is bounded in $L^2(\Omega \times Y)$, and in particular sequentially weakly compact. Hence, one can extract a subsequence of $(v_k)_{k \in \mathbb{N}}$ that is weakly two-scale convergent in $L^2(\Omega \times Y)$ w.r.t. $(\mathcal{P}_k)_{k \in \mathbb{N}}$.

(ii) If $(\mathcal{P}1)$, $(\mathcal{P}2)$ and $(\mathcal{P}3)$ hold, then

$$v_k \rightarrow v_0 \quad \text{in } L^2(\Omega) \quad \Rightarrow \quad v_k \xrightarrow{(\mathcal{P}_k)_k} v_0 \quad \text{in } L^2(\Omega \times Y).$$

(iii) If $(\mathcal{P}1)$ and $(\mathcal{P}3)$ hold, then

$$\left. \begin{array}{l} v_k \xrightarrow{(\mathcal{P}_k)_k} v_0 \quad \text{in } L^2(\Omega \times Y) \\ \text{and } v_0 \text{ is independent of } y \end{array} \right\} \Rightarrow v_k \rightarrow v_0 \quad \text{in } L^2(\Omega).$$

(iv) If $(\mathcal{P}1)$, $(\mathcal{P}3)$ and $(\mathcal{P}4)$ hold, then

$$v_k \xrightarrow{(\mathcal{P}_k)_k} v_0 \quad \text{in } L^2(\Omega \times Y) \quad \Rightarrow \quad v_k \rightharpoonup \int_Y v_0(\cdot, y) \mathcal{D}(\cdot, y) dy \quad \text{in } L^2(\Omega).$$

Remark 2.13. A question I could not fully answer yet is the case of sequences of gradients $(v_k)_k = (\nabla u_k)_k$ in $L^2(\Omega; \mathbb{R}^N)$ for $(u_k)_k$ a bounded sequence in $W^{1,2}(\Omega)$. However, even if one could prove a compactness result like Theorem 2.26 for sequences of generalized tilings $(\mathcal{P}_k)_k$, i.e. the existence of a subsequence and some $U_0 \in L^2(\Omega; W^{1,2}(Y))$ such that

$$\nabla u_{k_m} \xrightarrow{(\mathcal{P}_k)_k} \nabla u_0 + \nabla_y U_0 \quad \text{in } L^2(\Omega \times Y; \mathbb{R}^N),$$

then one could still *not* expect something like periodicity of U_0 in its second argument. This is due to the fact that for some general reference tile (already for a triangle, see Figure 2.6) the traditional periodicity concept of cubic tiles does not apply. Unfortunately

then, the mapping $x \mapsto U_0(x, \mathcal{R}_{\mathcal{P}_k}(x))$ results no longer as an element of $W^{1,2}(\Omega)$ (like in the case of cubic periodicity cells, see Example 2.1), but merely as a piecewise smooth function over the tiles $(\mathcal{T}_{k,\ell})_\ell$. However, a possible option to obtain from U_0 a $W^{1,2}$ -function over the entire domain Ω would be *interpolation* which is often applied in the theory of periodic unfolding (see *e.g.* Cioranescu et al. [2006b] or Section 3.4 of this thesis). This idea came to my mind only recently, but looks rather promising in particular for triangular tilings (also because one could exploit the finite element theory on interpolations over triangular meshes).

Chapter 3

Homogenization in perforated thin domains and applications to interface problems

In this chapter I give a contribution to a class of problems from continuum physics that has in recent years caught once more the attention of the applied analysis community. That is, the simplified description of interfaces of finite thickness connecting two adjacent media by means of asymptotically equivalent boundary conditions, but *in the presence of periodically recurring voids enclosed in the interface*. (Here, ‘finite’ is to be understood in the language of physicists, *i.e.* the antonym of ‘infinitesimal’.) This special type of interface problems is particularly interesting for the engineering disciplines. Therein, one often seeks to manipulate the constitutive properties of thin interface layers connecting the adjacent media, thus the way the media interact, by introducing microstructures into the interface layer. Important examples for microstructured interfaces are such featuring periodically recurring voids like pores or channels, or interfaces that merely consist of a set of periodically arranged ‘spots’ of interface material. The mathematical challenge here is finding an asymptotically equivalent description that tells the engineer how the interface makes the adjacent media interact over their common boundary. In particular, how the microstructure in the interface affects this interaction. From the viewpoint of analysis this means combining reducing a thin domain (the interface) to a condition imposed on the media’s common boundary *and*, simultaneously, homogenizing the microstructure in the interface in order to obtain a homogeneous constitutive relation for the interaction over the common boundary. Over the past five years, the asymptotic analysis of microstructured interfaces of finite thickness has become more accessible with the development of new mathematical methods both in dimension reduction (starting with the seminal work [Friesecke et al. \[2002\]](#)), the adaption of two-scale convergence and periodic unfolding for in-plane oscillations in thin domains due to Stefan Neukamm (see [\[Neukamm, 2010, Chapter 6\]](#)) and the combination of both in the context of finite elasticity, again a work of Stefan Neukamm (cf. [\[Neukamm, 2010, Chapter 7\]](#)). One should notice in the context of thin microstruc-

tured interfaces the presence of *three* different scales, which would be the two small scales of the interface (thickness and microstructure size) and the macroscopic scale of the adjacent media connected by the interface.

To be more specific, in this chapter I present the asymptotic analysis for two particular types of interfaces which transmit heat between two adjacent bodies while being made from material with relatively high thermal conductivity. Here, the one interface type encloses tubular voids reaching over the entire interface, which is to mimick channels for heating wires embedded in the interface. This way, the heat flow between the two connected bodies may be manipulated by providing external heat to specific parts of the bodies' common boundary via the interface layer; due to the high conductivity of the interface material, this would be done in a smooth fashion. Actually, this configuration is similar to a real-life optical device; details are given in the following subsection. The second interface type features the same constitutive properties like the first, whereas the interface now includes pore-like voids arranged in a periodic fashion. The main difference between the two types of interfaces, besides the fact that the pore-type follows no specific technical application, lies in the mathematical concepts involved in the asymptotic analysis. In the case of the interface with periodically recurring pore-like voids I employ a version of two-scale convergence due to Stefan Neukamm and extension theorems like found in [Oleřnik et al. \[1992\]](#) to extend the temperature fields in the interface over the pores, while classical Γ -convergence arguments then allow me to pass to the limit of vanishing interface thickness. In the case of tubular voids however, *i.e.* voids crossing the cubic periodicity cell and piercing two opposite faces, the use of extension theorems turns out to be of little help. Instead, I adapt the notion of periodic unfolding as it was outlined in Subsection 2.5.1 to thin and perforated domains, where the perforation may touch the periodicity cell's boundaries. At this point I would like to advice the reader that the main ideas behind the periodic unfolding operator for perforated domains are due to Patrizia Donato and her collaborators (see [Cioranescu et al. \[2006b\]](#) and the recent preprint [Cioranescu et al. \[2011\]](#)).

3.1 Introduction and outline

In the engineering disciplines, the general interest in interfaces and their mathematical description may be explained as follows. Interfaces, in particular engineered interfaces change the way in which two media interact. Joining of two materials is often achieved by inserting a thin layer of glue in between the materials which increases the adhesion. While in other situations a thin layer is inserted between two materials to decrease adhesion, as it is the case in the seemingly trivial example of bakery paper. Interfaces are often also used for damping purposes or vibration reduction. Therein, vibrations are generated in one medium and reach the other only after having been damped while traveling through a dissipative interface connecting both media (often made from rubber). Another important application where interfaces between adjacent media play a crucial role is filtering. Therein, a thin filter slows down or even prevents the transportation of mass between the media; examples include sieves or water filters made from fab-

ric. Interfaces are also a widespread mean to manipulate diffusion processes between adjacent media. For example, in cooling devices thin interfaces of highly conductive material can be employed to transmit heat smoothly between two media in order to avoid local temperature peaks (which is why thermal grease is usually inserted between CPUs and the heat sink on top). Whereas interfaces can also be used as barriers to diffusion, say ceramic coatings as they cover the surfaces of certain components of combustion engines.

Here, the natural question an engineer would ask is what are the principal means in the design of interfaces in order to achieve a particular interaction of two media connected by a layer of interface material? In fact, there are various parameters that influence the nature of interfaces. Of course, there is the geometry of the interface (most importantly its thickness) but also the interface's constitutive properties. The latter can obviously be modified by choosing different interface materials, but also by using a mixture of different materials or introducing a geometric microstructure in the interface. Finally, another way would be the active control of the interface's constitutive properties, *e.g.* through external control of a microstructure in the interface.

In this contribution I focus on interfaces whose constitutive properties are modified by the introduction of *periodic geometric microstructures*. In fact, periodic geometric microstructures are a common mean to manipulate an interface's constitutive properties. Coming back to the previously given examples for the use of interfaces in engineering, joining interfaces sometimes exhibit a periodic geometry whenever the adhesive is too expensive to cover the entire joining surfaces of the adherends. Instead, the adhesive may be applied in small quantities over periodically recurring patterns. Another reason is that by appropriately choosing the periodic pattern the interface can easily be made anisotropic, although the adhesive itself may very well be isotropic. Another nice example for interfaces with periodic geometric microstructure is again modern bakery paper, which features not only non-stick surface coatings, but also a periodically oscillating cross-section to reduce the actual contact area between to media it connects. One might name the use of interfaces made from perforated rubber to damp the relative movement of two bodies that are in contact along a common surface. In this case, periodically recurring voids in the rubber interface render the interface compressible, while rubber itself is technically speaking incompressible. Thus, an interface made from perforated rubber can be deformed more easily and consequently can be expected to be more effective in damping vibrations traveling through the interface. Sieves and other metallic filters are mostly nothing but periodically perforated interfaces that manipulate the flow of mass between the media they connect. Also in diffusion processes interfaces with periodic microstructures can be found. For instance, interfaces that are to deliver or subtract heat from the surrounding media sometimes shows periodically recurring channels for heating or cooling fluid, or for heating wires. Moreover, similarly to interfaces of adhesive where one can employ periodic microstructures to achieve a specially anisotropic transmission of stresses between the adherends across the interface, one could distribute a thin periodic layer of thermal grease on the joint surfaces of two contacting bodies in order to obtain a specific anisotropic flow of heat from one body into the other.

From the mathematical point of view, the main challenge in the theory of interfaces with periodic microstructures is to eliminate the small scales involved. More precisely, the small thickness of the interface and the small size of the periodically recurring pattern quantified by the microscale parameter. The reason is again that otherwise in numerical computations one would have to use a very fine discretization *in* the interface, with the mesh size being of size smaller than both the interface thickness and the microscale parameter. A particular problem in the discretization of microstructured interfaces, compared to microstructured thin films or plates, is that the discretization has to be similarly fine also *near* the interface, *i.e.* in the adjacent media. This is because in the case of interfaces of finite thickness only the constitutive properties change between the adjacent media and the interface, whereas the physical and mathematical model (*e.g.* the heat equation) is the same. Thus, for finite thickness the solution extends smoothly into the interface which is why any meaningful discretization must (at least asymptotically) allow for discrete solutions that also extend *continuously* from the adjacent media into the interface layer. Plainly speaking in terms of triangular finite element discretizations, the triangulation of the *entire* domain, *i.e.* adjacent media and thin interface layer, must be *regular* (see *e.g.* Brenner and Carstensen [2004]). A way to get rid of both the small scales in interface problems with periodic microstructure is to find conditions on the *boundaries of the adjacent media* to the interface that describe an *asymptotically equivalent* but *homogeneous* constitutive behavior. In other words, these conditions should be such that in the limit of small interface thickness and small microscale parameter they ensure the same interaction between the adjacent media like the microstructured interface with finite thickness. In this case, any discretization of the adjacent media would neither have to respect the interface thickness nor the size of the microstructure, thus can be far more coarse and therefore less degrees of freedom would be required.

Reducing a microstructured interface of finite thickness to homogeneous boundary conditions is known to the applied analysis community since the early 1980s, the first contribution being the work of Evariste Sanchez-Palencia on the stationary flow through a periodic sieve, cf. Sanchez-Palencia [1982]. Similar situations have been studied in Murat [1985], Damlamian [1985] and Picard [1987]. However, all these contributions considered periodic sieves of *zero* thickness, thus had to eliminate only the small scale of the periodic microstructure. Mathematical models of this type are nowadays designated as ‘thin Neumann sieves’. A first study on the stationary flow through a sieve of finite thickness was carried out by Teresa del Vecchio in Del Vecchio [1987], who studied also different ratios of sieve thickness and microscale parameter. At this point, the reader should notice well that in the case of a flow through a sieve, the flow extends only *into* the holes of the sieve. Other than say in the case of heat, which would flow through the sieve’s material but *around* the (empty) holes of the sieve.

Generally speaking, eliminating both the small scales of periodic interfaces with finite thickness requires on the one hand the asymptotic analysis of mathematical models over thin domains, and on the other hand methods from periodic homogenization. Over the last ten to fifteen years there has been significant progress in both fields, most notably through Γ -convergence methods applied to thin films and plates (see the references

in Section 2.1) and the concept of periodic unfolding in periodic homogenization (see Subsection 2.5.1). These two developments might explain why the asymptotic analysis of periodic interfaces has recently once more caught the attention of the applied mathematics community (see the upcoming literature review).

3.1.1 A short literature review on interface problems

In this subsection I state a brief review of the applied analysis community's contributions to the description of interfaces of finite thickness by means of boundary conditions which ensure – in the limit of zero interface thickness – the same interaction between the media adjacent to the interface. This problem class was in the case of *homogeneous* interfaces extensively studied by Giuseppe Geymonat and his many collaborators. Whereas the analysis of periodic interfaces is a recent development where most of the contributions – at least to my knowledge – have been published over the last five years.

Homogeneous interfaces can be classified into three different categories. In the first category, the interface material has constitutive properties of the same order as the media the interface connects. This situation was studied in the context of linear elasticity in [Lebon and Rizzoni \[2010\]](#). Therein, the authors showed that up to zero order an interface of similar elastic properties like the two adjacent bodies makes them behave as if they simply were *one* body made from the two materials in the corresponding subdomains defined by the now flattened, two-dimensional interface. Yet, studying first order effects near the interface, Lebon and Rizzoni could again describe the influence of the interface's constitutive relations on the interaction of the two bodies over the flattened interface. In short words, for interfaces of similar elastic properties both the displacements and the stresses are continuous over the flattened interface, while the boundary conditions on the flattened interface only affect the derivatives of the stresses (more precisely their jumps) over the interface. The second category contains those situations in which the interface material is much 'stronger' than the surrounding material. This means that variations of the physical quantities under consideration are energetically much more unfavorable in the interface material than in the surrounding media. Again in the context of linear elasticity, corresponding to interface material of far higher elastic modulus, this interface type was extensively studied by Françoise Krasucki and Anne-Laure Bessoud in collaboration with others, see [Bessoud et al. \[2009\]](#) and [Bessoud \[2009\]](#), but also [Bessoud et al. \[2010, 2011\]](#). They found that in the limit of zero thickness the interface may be replaced by boundary conditions on the flattened interface, across which the displacements are continuous while the stresses are subject to newly found boundary conditions of the Ventcel type. In the last category the interface material is much 'weaker' than the material in the surrounding media. In analogy to the case of a strong interface, this is equivalent to saying that variations of the physical quantities under consideration can be maintained in the interface with much less external work than in the surrounding media. For this situation I refer again in the context of linear elasticity to [Geymonat et al. \[1999\]](#) (and to [Krasucki et al. \[2004\]](#) for nonlinear elasticity), where the authors proved that in the zero thickness limit the interface can be

equivalently expressed by a boundary condition on the flattened interface which allows for jumps in the displacement, but enforces continuity of the stresses. Phenomenologically, this model was proposed already in 1944 in [Goland and Reissner \[1944\]](#) who modelled a weak elastic interface as an array of infinitesimal springs connecting the surfaces of the adjacent bodies. I would also like to make the reader aware of an interesting approach performed in [Åslund \[2005\]](#) where the author studies the elastic behavior of two thin plates connected by an even thinner film of adhesive and derives a limit plate theory for the entire bonded structure.

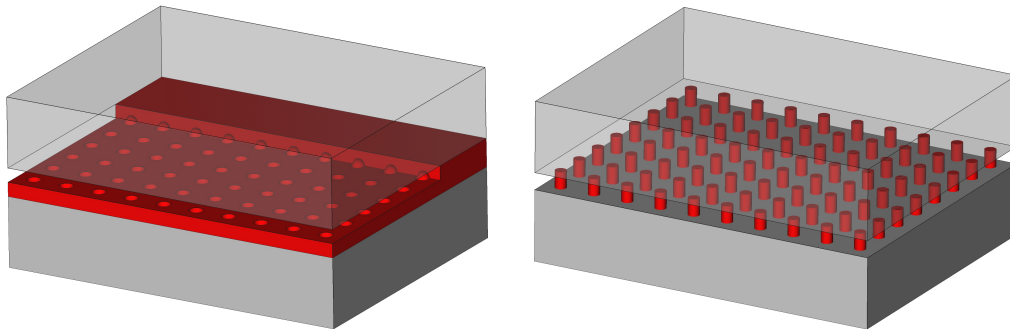


Figure 3.1: Pore-type (left; cutaway drawing) and spot-type periodic interfaces (right)

Interfaces of finite thickness that have a periodic microstructure have been studied mostly in recent years. The available literature, as it is known to me today, treats the following categories of periodically microstructured interfaces. Interfaces of finite thickness with periodically varying constitutive properties that completely fill space between the adjacent media (*i.e.* there are no voids in the interface layer) are studied in [Donato and Piatnitski \[2010\]](#). Therein, two bodies are joined along a rough, sawtooth-like periodic surface and interact through a Fourier-like law over the zig-zagged contact surface (*i.e.* the (heat) flux is proportional to the jump (in the temperature) across the surface). Depending on the surface roughness, the authors find different asymptotically equivalent boundary conditions on the *flattened smooth* interface to describe the interaction between the bodies. In the most interesting case of moderate surface roughness they observe that the Fourier-like behavior between the bodies is conserved. Only in February 2012 it has come to my attention that in [Moussa and Zlaïji \[2012\]](#) the authors analyzed in the context of nonlinear elasticity the problem of an interface with finite thickness that completely fills space between the adjacent media and shows a periodic microstructure. Depending on the ratio between interface thickness and interface rigidity, the authors identify different limit problems. The most interesting corresponds to rigidity and thickness being of the same order, which can be viewed as the homogenized and nonlinear analogon of the results in [Bessoud et al. \[2009\]](#).

The case of interfaces of finite thickness with periodically recurring voids has attracted more attention in the applied analysis community. Here, one has to distinguish between voids which leave a connected interface layer, what I am going to call ‘pore-type’ periodic interfaces (see [Figure 3.1](#) (left)), and such in which the interface is consti-

tuted through periodically recurring, isolated spots of interface material, in the remainder of this section called ‘spot-type’ periodic interfaces (see Figure 3.1 (right)). A pore-type periodic interface of finite thickness is studied in Rohan and Lukeš [2010], more precisely the traveling of acoustic waves in two adjacent media through a connecting periodically perforated interface layer. For this problem the authors derive asymptotically equivalent boundary conditions on the flattened interface, which relate the normal derivatives of the pressure on opposite sides of the flattened interface to a fictitious acoustic transverse velocity in the interface. Pore-type situations, now in the context of linear elasticity, are studied in Marigo and Pideri [2011] and Geymonat et al. [2010]. In both articles it is assumed that in an elastic body a periodic array of holes concentrates near a hyperplane, thus creating a perforated interface layer between the solid parts of the elastic body. Since the authors of both articles assumed neither a weak nor a strong interface, like in the homogeneous case Lebon and Rizzoni [2010], higher order terms have to be incorporated in order to derive the correct asymptotically equivalent boundary conditions on the flattened interface. In both cases it is revealed that the first order terms of displacement and stress jump across the flattened interface.

However, many contributions focus on the analysis of spot-type periodic interfaces, the most prominent example of which is the flow through a sieve (Neumann’s sieve). In the case of finite interface thickness, the start was marked by Del Vecchio [1987] who studied a model related to a stationary, potential flow between two media through a sieve of finite thickness where the constitutive properties are identical in the adjacent media and the sieve’s holes. Depending on the ratio of the sieve’s thickness and the diameter of the sieve’s cylindrical holes different boundary conditions on the flattened sieve are recovered. That is, either no flow restriction at all, a perfect barrier to the flow or once more a Fourier-type boundary condition, which states a linear relation between the flow through the sieve and the jump of the flow potential over the flattened sieve. Similar results have been obtained in a series of articles by Daniel Onofrei, Doina Cioranescu and their collaborators, see Onofrei [2006, 2007] and Cioranescu et al. [2008b]. While not going into the details of the countless situations considered in these papers, the reader might notice that the authors employed the method of periodic unfolding as the main workhorse and adapted it to the study of spot-type (sieve-like) periodic interfaces of finite thickness, again for different ratios of the diameter of the spots (the sieve’s holes) and their mutual distance. The authors also analyze the case of zero thickness, again with the help of the periodic unfolding method, where analogous results to the case of finite thickness are proved. Another example for the reduction of spot-like interfaces to asymptotically equivalent boundary conditions is Ansini [2004]. Also in this paper the author studies a model inspired by stationary, potential flows through a sieve, but for the case of zero interface thickness and general non-quadratic flow potentials. Finally, the case of spot-type interfaces of finite thickness made from material of extremal strength (weak or strong) compared to the media adjacent to the interface will in the context of elasticity be analyzed in a forthcoming work by François Krasucki, Giuseppe Geymonat and myself Geymonat et al. [2012].

3.1.2 On two problems arising in continuum mechanics

Although the two specific problems of pore-type periodic interfaces I study in this chapter are essentially the product of my imagination, they are in principle very similar and are in fact loosely inspired by an actual application in optomechanics. More precisely, by a product innovation of Leica Geosystems that allows to compensate for thermomechanical deformations of lenses through controlled external heat supply via an interface layer between the lens's mounting and the lens itself. For details on the technical implementation of this concept I refer to the presentation of [Aebischer and Braunecker \[2004\]](#). The heat supply itself can be realized by embedding heating wires into the interface material, leading to a thin interface layer of the pore-type with periodically recurring tubular voids; cf. Figure 3.2.

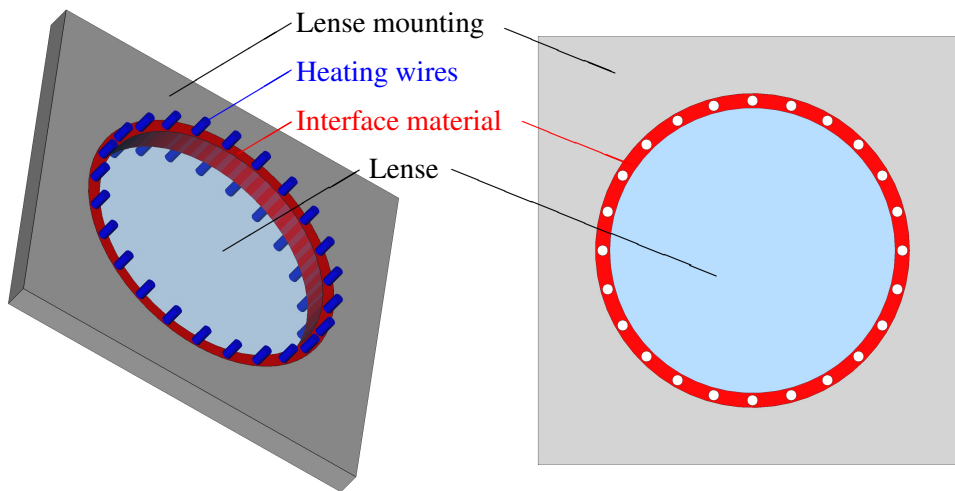


Figure 3.2: Schematic configuration of a lens with external heat supply

The motivation behind this kind of application is that many technical devices are operated under thermally varying conditions, leading to thermally induced residual stresses and deformations and, as a consequence, often to a decrease in precision and performance. The geometry of a deformed lens yields different refraction of light than in the undeformed state, and may therefore lead to unsatisfactory image quality. Similarly, devices conducting hot fluid or gas exhibit strong temperature gradients and thermally induced residual deformations, which in turn may lead to moving parts getting jammed, leakages occurring at joints, and other problems. Here, thermal management, *i.e.* the controlled supply of external heat opens up a possibility to control the temperature distribution in a device and consequently the thermally induced stresses and deformations. A natural way to deliver heat rapidly and in a controlled fashion is to dissipate electric energy in electrical resistors, *e.g.* heating wires attached to the device's surface or *embedded* into an interface layer connecting two bodies.

To this end, the remainder of this chapter focuses on heat conduction problems where external heat is supplied to two adjacent homogeneous bodies over a pore-type

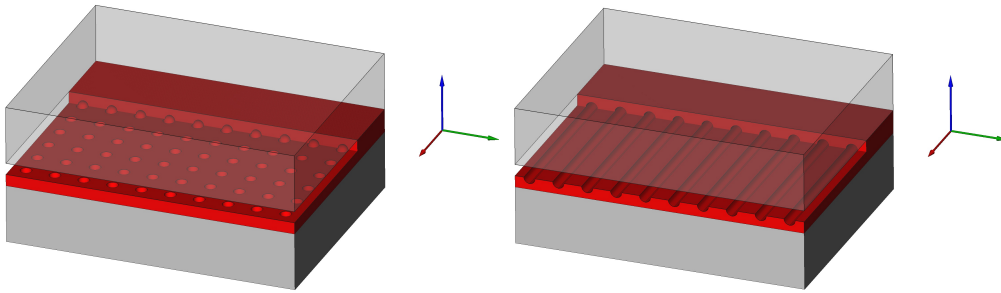


Figure 3.3: Highly conductive interfaces of the pore-type with spherical voids (left) and tubular voids (right)

periodic interface of finite thickness. In what follows, the interface material is assumed to have far higher thermal conductivity than the material occupying the bodies adjacent to the interface. This assumption is due to the fact that thermal management measures should *reduce* thermally induced stresses in the adjacent bodies through external heat supplied via the interface. In particular, the temperature distribution in the interface itself should vary only smoothly, which is most likely to be guaranteed if the interface conducts heat well between its embedded heat sources. More precisely, the thermal conductivity will be assumed to scale like the inverse of the interface thickness. A simplification to the application in optomechanics described above is that I will assume the bodies connected by the interface to have *flat* contacting surfaces and therefore assume a *translatory* periodic microstructure of the pore-type in the interface. In a first toy problem in Section 3.3 (see also Figure 3.3 (left)) I will analyze the problem of a highly conductive interface with periodically recurring voids that are compactly contained in a cubic periodicity cell, *i.e.* the voids do not touch the periodicity cell's boundary. Whereas in a second, more realistic setting in Section 3.4 I will consider a situation in which the interface is perforated by long tubular voids reaching over the whole interface (see Figure 3.3 (right)). These are to mimick the presence of heating wires, with Fourier type boundary conditions for the heat flow into the interface imposed on the voids' boundaries. Mathematically speaking, the first case can be solved by using a combination of classical methods like extension operators for periodically perforated domains and Γ -convergence approaches for dimension reduction, together with a recent version of two-scale convergence due to Stefan Neukamm (the so-called 'two-scale convergence for in-plane oscillations' see [Neukamm, 2010, Chapter 6]). Whereas the second case requires a different approach, since extension theorems for voids that extend over the whole interface turn out to be of little help. Yet, this problem can be overcome by suitably adapting the periodic unfolding method to perforated *thin* domains. This together with a Γ -convergence argument is the key to reducing the highly conductive interface with periodically recurring tubular voids to a boundary condition imposed on the boundaries of the bodies separated by the interface. Here, I would like to advise the reader that the Γ -convergence approach to strong *homogeneous* interfaces of finite thickness is due to Bessoud et al. [2009].

The basic notation and preliminary assumptions used throughout this chapter will be outlined in the upcoming Section 3.2. To conclude the introduction of this chapter I would like to state some remarks.

- Remark 3.1.*
1. The interface problem for the heating of lenses is a prototypical example for a non-translatory microstructure (cf. the illustration in Figure 3.2). Again the periodic unfolding method appears most suitable to cope with both the non-translatory microstructure (see Subsection 2.5.2) and the voids that reach over the entire interface. When preparing a research article on this topic I intend to extend the methodology presented here (in particular in Section 3.4) from translatory microstructures to the special non-translatory microstructure depicted in Figure 3.2. Due to limited time it was not possible to analyze this problem in the context of the present thesis.
 2. Although the heating of lenses has been analyzed mathematically by experts of Leica Geosystems, cf. Aebischer [2007], the main focus of their analysis was to find an explicit solution for the heat equation in the lense-interface-mounting assembly. This was mainly possible by making simplifying but reasonable assumptions on the geometry (radial symmetry), thus effectively reducing the problem to one space dimension. Yet, the small scales involved were not eliminated, thus convergence of the problem for vanishing small scales was not studied in Aebischer [2007].
 3. The mathematical setting of the interface problem studied in this chapter may be compared to the one in Moussa and Zlaïji [2012]. Therein, the authors consider a microstructured interface of finite thickness in the context of nonlinear elasticity (more precisely, in the quasiconvex setting of Braides [1985] and Müller [1987], which includes many problems of nonlinear elasticity). However, the microstructure they study is of purely constitutive nature and fills the entire space between the bodies adjacent to the interface. Other than in my situation, where the microstructure stems from periodically recurring voids in the interface layer while the interface material is the same everywhere in the interface. Another crucial difference is that the homogenization results in Moussa and Zlaïji [2012] rely on classical methods of the Italian school of calculus of variations (the so-called ‘Blow-up’ technique), whereas in this thesis methods from french homogenization schools like two-scale convergence and an adaption of the periodic unfolding method to perforated thin domains are used.

3.2 Notation and preliminaries

This section contains all the main definitions and notation used for the geometry of the two interface problems studied in this chapter. Furthermore, also the constitutive assumptions on the materials involved are specified here, as well as some function spaces that will be used later on in this chapter.

3.2.1 Geometry and periodicity cell

The notation I use here in this chapter is similar to that found in many contributions to interface problems; more precisely, I lean to the nomenclature of Bessoud et al. [2009]. In the remainder of this chapter Ω shall denote a three-dimensional Lipschitz domain such that the intersection $\Omega \cap \{x : x_3 = 0\}$ has nonempty relative interior and the domains $\Omega^\pm := \Omega \cap \{x : \pm x_3 > 0\}$ are again Lipschitz. Moreover, $\omega \subseteq \mathbb{R}^2$ shall be such that $\omega \times \{0\} = \Omega \cap \{x : x_3 = 0\}$. Generally, in what follows the coordinates of an arbitrary point $x \in \mathbb{R}^3$ are split like $x = (\hat{x}, x_3)$. To further simplify the notation let $\Omega_{\text{tr}}^\pm := \pm \frac{1}{2}e_3 + \Omega^\pm$ and $\Gamma_{\text{tr}}^\pm := \pm \frac{1}{2}e_3 + \Gamma^\pm$.

Furthermore, geometric interface quantities are referred to by the letter B (which as in Bessoud et al. [2009] stands for ‘bond’). It turns out useful to define $B := \omega \times [-\frac{1}{2}, \frac{1}{2}]$ and the interface of thickness ε without voids $B_\varepsilon = \omega \times [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$. The interior of the latter shall be referred to like B° and B_ε° .

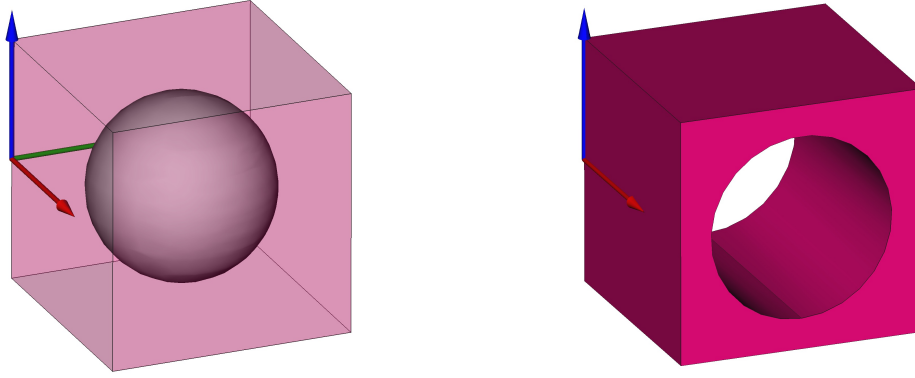


Figure 3.4: Perforated periodicity cells for voids compactly contained in the periodicity cell (left) and tubular voids touching the boundary (right)

To simplify the notation, the unit cube Y shall in the remainder of this chapter be shifted along the axis of the third component to $Y := [0, 1)^2 \times [-\frac{1}{2}, \frac{1}{2}] = \hat{Y} \times [-\frac{1}{2}, \frac{1}{2}]$, where $\hat{Y} = [0, 1)^2$ is the standard *two-dimensional* unit cube. The (open) void in the unit cube shall be denoted $V \subseteq Y$, and finally the perforated periodicity cell by $Y^* := Y \setminus \bar{V}$. For both the situations of a void that is compactly contained in the periodicity cell and where V is a tubular void touching the periodicity cell’s lateral boundaries I refer to Figure 3.4.

Assuming the interface thickness and the size of voids enclosed in the interface to be of the same size ε , the pore-type periodic interface $B_\varepsilon^{\varepsilon\text{-perf}}$ is defined as

$$\begin{aligned} B_\varepsilon^{\varepsilon\text{-perf}} &:= B_\varepsilon \cap \bigcup_{\hat{a} \in \mathbb{Z}^2} \varepsilon ([\hat{a}, 0]^T + Y^*) \\ &= B_\varepsilon \setminus \bigcup_{\hat{a} \in \mathbb{Z}^2} \varepsilon ([\hat{a}, 0]^T + \bar{V}). \end{aligned}$$

Here, the subscript index of $B_\varepsilon^{\varepsilon\text{-perf}}$ refers to the height of the interface layer ε , while

the superscript ε -perf refers to the voids recurring in periods of ε in interface direction. A simplification to avoid both unnecessary and annoying technicalities caused by imperfections of the microstructure at the lateral boundaries of the interface layer $B_\varepsilon^{\varepsilon\text{-perf}}$ is that I will henceforth assume that the microstructure always fits the interface layer B_ε . That is, the intersection ω of Ω with the hyperplane $\{x : x_3 = 0\}$ and the microscale parameter ε shall always be such that

$$\omega = \bigcup_{\hat{a} \in Z_\varepsilon} \varepsilon(\hat{a} + \hat{Y}) \quad \text{for some finite } Z_\varepsilon \subseteq \mathbb{Z}^2. \quad (\varepsilon\text{-Tiling})$$

This yields in particular $B_\varepsilon^{\varepsilon\text{-perf}} = \bigcup_{\hat{a} \in Z_\varepsilon} \varepsilon([\hat{a}, 0]^T + Y^*)$. Assumption (ε -Tiling) is undoubtedly a strong restriction and could be relaxed (likely without changing the results for sufficiently smooth lateral boundary $\partial\omega \times [-\frac{1}{2}, \frac{1}{2}]$). However, in this fashion the exposition of the analysis becomes clearer. Also, for the application of heating wires being embedded into the tubular voids in the interface this assumption makes sense as one would not want to cut the wires over an oscillating boundary.

Then, the whole assembly composed of the two bodies $\Omega_\varepsilon^\pm := \pm \frac{\varepsilon}{2} e_3 + \Omega^\pm$ and the perforated interface $B_\varepsilon^{\varepsilon\text{-perf}}$ becomes

$$\Omega_\varepsilon := \Omega_\varepsilon^+ \cup B_\varepsilon^{\varepsilon\text{-perf}} \cup \Omega_\varepsilon^-.$$

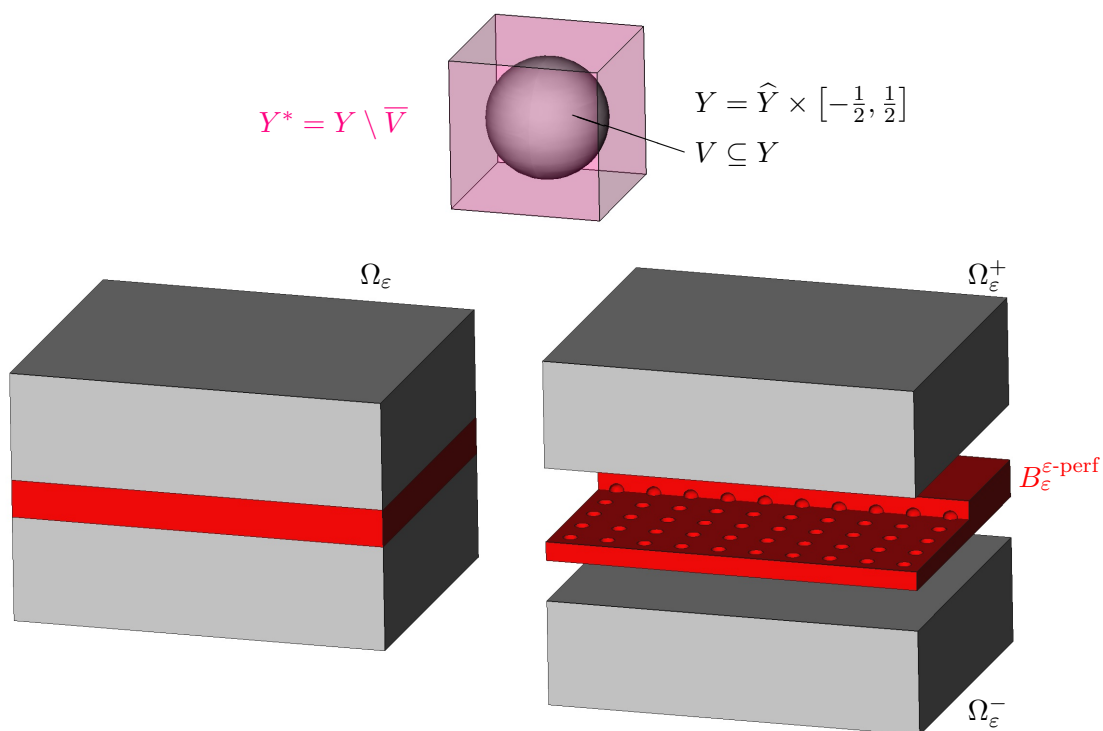
Finally, Γ^\pm shall be relatively open subsets of the boundaries $\partial\Omega \cap \{x : \pm x_3 > 0\}$, such that on $\Gamma_\varepsilon^\pm := \pm \frac{\varepsilon}{2} e_3 + \Gamma^\pm$ homogeneous Dirichlet boundary conditions for the heat flow in the assembly Ω_ε will be described.

All the above notation is once more illustrated in 3.5 below.

3.2.2 Constitutive assumptions

The bodies Ω_ε^\pm are assumed to be occupied by materials of similar constitutive behavior, each of which shall be described by means of a conductivity matrix $A^\pm \in \mathbb{R}^{3 \times 3}$ that satisfies the requirements of symmetry and positive definiteness. Qualitatively, the same assumptions shall hold for the conductivity of the material occupying the perforated interface layer $B_\varepsilon^{\varepsilon\text{-perf}}$ in between Ω_ε^+ and Ω_ε^- . However, as explained in the previous section the interface material shall conduct heat considerably better than the adjacent bodies in order to obtain a smooth temperature profile in the thin interface layer. That is, in the sequel it will be assumed that the thermal conductivity of the interface material scales with the inverse thickness ε of the interface layer, leading to a thermal conductivity matrix $\frac{1}{\varepsilon} A^{\text{interf}}$, where $A^{\text{interf}} \in \mathbb{R}^{3 \times 3}$ is again symmetric and positive definite.

Since the asymptotic analysis of the two interface problems sketched in the previous section will be carried out with the help of Γ -convergence arguments, one has to work with dissipation potentials instead of the static heat equation in the interface assembly Ω_ε (cf. Subsection 2.2.3). That is, instead of describing the constitutive properties of the bodies Ω_ε^\pm or the interface material $B_\varepsilon^{\varepsilon\text{-perf}}$ by means of their conductivity matrices A^\pm and $\frac{1}{\varepsilon} A^{\text{interf}}$, one has to work with the densities $F \mapsto \frac{1}{2} A^\pm F \cdot F$ and $F \mapsto \frac{1}{2\varepsilon} A^{\text{interf}} F \cdot F$, $F \in \mathbb{R}^3$, of the corresponding dissipation potentials. But for

Figure 3.5: The notation used for the interface structure Ω_ε

the sake of a simple notation it turns out useful to replace these quadratic energy densities by generic densities $F \mapsto W^\pm(F)$ and $F \mapsto \frac{1}{\varepsilon}W^{\text{interf}}(F)$, being convex and of quadratic growth from below and above. In other words, $W^\pm, W^{\text{interf}} : \mathbb{R}^3 \rightarrow \mathbb{R}$ must satisfy (W2) and (W3), *i.e.*

$$\left\{ \begin{array}{l} \text{the maps } F \mapsto W^\pm(F), W^{\text{interf}}(F) \text{ are convex,} \\ \text{there exist positive constants } c, C \text{ such that} \\ c(|F|^2 - 1) \leq W^\pm(F), W^{\text{interf}}(F) \leq C(1 + |F|^2) \\ \text{for all } F \in \mathbb{R}^3. \end{array} \right.$$

3.2.3 Some function spaces

For some open and bounded subset U of \mathbb{R}^N with Lipschitz boundary ∂U and $\Gamma \subset \partial U$, the set $W_\Gamma^{1,2}(U)$ contains all elements of $W^{1,2}(U)$ whose trace on Γ vanishes. In this fashion one has to understand the notation $W_{\Gamma_\varepsilon^+ \cup \Gamma_\varepsilon^-}^{1,2}(\Omega_\varepsilon)$ or $W_{\Gamma^+ \cup \Gamma^-}^{1,2}(\Omega^+ \cup B \cup \Omega^-)$.

While $W_{\text{per}}^{1,2}(Y)$ is as before defined to be the space of all $W^{1,2}(Y)$ -functions having identical trace on opposite faces of the unit cube Y , the space $W_{\widehat{Y}\text{-per}}^{1,2}(Y)$ contains all those functions in $W^{1,2}(Y)$ that are only asked to have identical trace on opposite faces of the *lateral* boundary of the unit cube Y . Mathematically, $u \in W_{\widehat{Y}\text{-per}}^{1,2}(Y)$ if and only

if $u \in W^{1,2}(Y)$ and $u(y + e_\alpha) = u(y)$ for all $y \in \partial Y$ with $y_\alpha = 0$, $\alpha \in 1, 2$. Similarly, the space $W_{\hat{Y}\text{-per}}^{1,2}(Y^*)$ shall contain all those elements in $W^{1,2}(Y^*)$ that have identical trace on the *intersection* of opposite faces of the lateral boundary of Y^* . This is because in a perforated periodicity cell $Y^* = Y \setminus \bar{V}$ a void V may leave ‘holes’ in the lateral boundary of Y that do not coincide on opposite faces. In mathematical notation then, $W_{\hat{Y}\text{-per}}^{1,2}(Y^*)$ contains all those $u \in W^{1,2}(Y^*)$ such that for all $y \in \partial Y^* \cap \{y : y_\alpha = 0\}$ satisfying $y + e_\alpha \in \partial Y^*$, $\alpha = 1, 2$, one has $u(y + e_\alpha) = u(y)$.

Because of its importance for what follows, I shall define the vector space

$$\mathcal{V}(\Omega^+, \Omega^-, B) := \left\{ u : u \in W^{1,2}(\Omega_{\text{tr}}^+ \cup B \cup \Omega_{\text{tr}}^-) \quad \text{and} \quad \partial_3 u = 0 \text{ in } B \right\}.$$

and $\mathcal{V}_{\Gamma^+, \Gamma^-}(\Omega^+, \Omega^-, B)$ as those functions in $\mathcal{V}(\Omega^+, \Omega^-, B)$ that vanish on Γ_{tr}^\pm . In fact, these spaces are frequently encountered in the analysis of interface problems if the interface is made from much ‘stronger’ material than the surrounding media (see *e.g.* [Bessoud et al. \[2009\]](#); [Bessoud \[2009\]](#) or [Moussa and Zlaïji \[2012\]](#)). One should notice that $\mathcal{V}(\Omega^+, \Omega^-, B)$ can be identified with

$$\mathcal{V}(\Omega^+, \Omega^-, \omega) := \left\{ u : u \in W^{1,2}(\Omega), u|_\omega \in W^{1,2}(\omega) \right\},$$

i.e. those functions in $W^{1,2}(\Omega)$ whose inner trace on the hyperplane $\omega \times \{0\} = \Omega \cap \{x : x_3 = 0\}$ is weakly differentiable and has square integrable derivatives on ω . Similarly, $\mathcal{V}_{\Gamma^+, \Gamma^-}(\Omega^+, \Omega^-, B)$ can be identified with the space $\mathcal{V}_{\Gamma^+, \Gamma^-}(\Omega^+, \Omega^-, \omega)$ containing all functions in $\mathcal{V}(\Omega^+, \Omega^-, \omega)$ that vanish on Γ^+ and on Γ^- .

3.3 Voids compactly contained in the periodicity cell

The current section considers the case of the void V being compactly contained in the unit cell Y , in symbols $V \Subset Y$. Furthermore, we assume that the void V has a sufficiently smooth boundary, say Lipschitz-regular. That is, the interface $B_\varepsilon^{\varepsilon\text{-perf}}$ in the assembly Ω_ε is perforated by periodically recurring, isolated voids with diameter of order ε . For a graphical representation of this configuration see [Figure 3.3](#) (left).

On the boundaries Γ_ε^+ and Γ_ε^- of Ω_ε there shall for simplicity hold homogeneous Dirichlet boundary conditions for the heat flow in the assembly Ω_ε . On the remaining boundaries $\partial\Omega_\varepsilon \setminus (\Gamma_\varepsilon^+ \cup \Gamma_\varepsilon^-)$ I will assume homogeneous Neumann boundary conditions, in particular on the boundaries of the voids enclosed in the interface $B_\varepsilon^{\varepsilon\text{-perf}}$. The reason for this is more of practical than of mathematical nature: since the voids in the interface are isolated from each other, one cannot reach into the voids (and not even see them from the outside when looking at the assembly Ω_ε). Thus, it makes little sense to impose boundary conditions other than homogeneous Neumann boundary conditions. Here I assumed implicitly that the heat flow from the interface into the void is negligible compared to the heat flow inside the interface (which is again a reasonable assumption due to the high conductivity of the interface material).

Given the constitutive assumptions from the previous Subsection 3.2.2, the dissipation potential \mathcal{E}_ε for the assembly Ω_ε reads as

$$\mathcal{E}_\varepsilon : W_{\Gamma_\varepsilon^+ \cup \Gamma_\varepsilon^-}^{1,2}(\Omega_\varepsilon) \rightarrow \mathbb{R}$$

with

$$\begin{aligned} \mathcal{E}_\varepsilon(u) := & \int_{\Omega_\varepsilon^+} W^+(\nabla u(x)) \, dx + \int_{\Omega_\varepsilon^-} W^-(\nabla u(x)) \, dx \\ & + \int_{B_\varepsilon^{\varepsilon\text{-perf}}} \frac{1}{\varepsilon} W^{\text{interf}}(\nabla u(x)) \, dx. \end{aligned} \quad (3.1)$$

Remark 3.2. In the formulation of the heat flow in the assembly Ω_ε by means of the above dissipation potential \mathcal{E}_ε , bulk and surface heat source terms in the bodies Ω_ε^+ and Ω_ε^- (respectively on their outer surfaces $\Omega_\varepsilon^\pm \cap \{x : \pm x_3 > \varepsilon/2\}$) are neglected, assuming that they would be independent of the interface thickness and void diameter ε . The reason is that otherwise the dissipation potential \mathcal{E}_ε would simply be added terms like $\int_{\Omega_\varepsilon^\pm} f^\pm(\hat{x}, x_3 \mp \frac{\varepsilon}{2} e_3) u(x) \, dx$ or $\int_{\Gamma_\varepsilon^{\text{Neu},\pm}} g^\pm(\hat{x}, x_3 \mp \frac{\varepsilon}{2} e_3) u(x) \, da(x)$, for some $f \in L^2(\Omega^\pm)$ and $g \in L^2(\Gamma^{\text{Neu},\pm})$. Herein, $\Gamma_\varepsilon^{\text{Neu},\pm} \subseteq (\partial\Omega \setminus \Gamma^\pm) \cap \{x : \pm x_3 > 0\}$ and $\Gamma_\varepsilon^{\text{Neu},\pm} := \pm \frac{\varepsilon}{2} e_3 + \Gamma^{\text{Neu},\pm}$. However, this would certainly not affect the limit behavior of the dissipation potential \mathcal{E}_ε .

Clearly, just like already said in Subsection 2.2.3, the equilibrium temperature distribution in the assembly Ω_ε is nothing but the minimizer of the dissipation potential \mathcal{E}_ε .

Heuristically, one might anticipate the limit behavior of the functional \mathcal{E}_ε as the interface thickness and the microstructure, both quantified by ε , become smaller and smaller. More precisely, by some intuition one can qualitatively describe a dissipation potential for the situation where the perforated interface $B_\varepsilon^{\varepsilon\text{-perf}}$ connecting the two bodies Ω_ε^+ and Ω_ε^- is replaced by an asymptotically equivalent boundary condition for the heat flow between Ω^+ and Ω^- over the flattened interface ω . In fact, since the conductive behavior in the bodies adjacent to the interface is not affected by the interface $B_\varepsilon^{\varepsilon\text{-perf}}$ – the material properties Ω^+ and Ω^- remain the same for all ε – the bulk terms $\int_{\Omega^\pm} W^\pm(\nabla u(x)) \, dx$ are likely to enter the limit dissipation potential unchanged. Similar arguments tell that also the homogeneous Dirichlet boundary conditions for the heat flow on Γ_ε^\pm are likely to transfer into the limit as homogeneous Dirichlet boundary conditions on Γ^\pm . Things are quite different when it comes to the term $\int_{B_\varepsilon^{\varepsilon\text{-perf}}} \frac{1}{\varepsilon} W^{\text{interf}}(\nabla u(x)) \, dx$ describing the highly conductive interface layer. First, due to the assumption of high conductivity it is unlikely that heat accumulates at the interface. Thus, the temperature will not jump over the flattened interface ω . Furthermore, the high conductivity will cause the temperature in the interface to vary smoothly in interface direction ω as sharp temperature gradients would dissipate rapidly in the interface. Moreover, the presence of periodically recurring voids in the interface will most probably not affect the just described behavior as long as they are not too large compared to the area of the interface layer ω . At least not qualitatively. Whereas quantitatively the voids will for sure reduce the conductivity in interface direction as the heat

has to pass by the voids when flowing through the interface: the bigger the voids the less the amount of heat conducting material in the interface. In short words, it seems reasonable that the integral over the interface layer $B_\varepsilon^{\varepsilon\text{-perf}}$ behaves for vanishing interface thickness and void diameter ε asymptotically like an integral over the flattened interface ω that penalizes sharp temperature gradients in interface direction. A first guess for the limit integral describing the interface would be $\int_\omega \widehat{W}^{\text{interf}}(\widehat{\nabla}u(\widehat{x}, 0)) \, da(\widehat{x})$ for a smooth temperature distribution $u : \Omega \rightarrow \mathbb{R}$ and some *homogenized* interfacial energy density $\widehat{W}^{\text{interf}} : \mathbb{R}^2 \rightarrow \mathbb{R}$, presumably also of quadratic growth. Note, that $\widehat{\nabla}(\cdot) = [\partial_1(\cdot), \partial_2(\cdot)]^T$ is only the gradient in interface direction ω . Naturally, the size and geometry of the voids will enter the homogenized interfacial energy density $\widehat{W}^{\text{interf}}$. To conclude, one might expect that in the limit of small interface thickness and void diameter ε the dissipation potential \mathcal{E}_ε behaves like

$$\mathcal{E}_{\text{Hom}} : \mathcal{V}_{\Gamma^+, \Gamma^-}(\Omega^+, \Omega^-, \omega) \rightarrow \mathbb{R}$$

where

$$\begin{aligned} \mathcal{E}_{\text{Hom}}(u) := & \int_{\Omega^+} W^+(\nabla u(x)) \, dx + \int_{\Omega^-} W^-(\nabla u(x)) \, dx \\ & + \int_\omega \widehat{W}^{\text{interf}}(\widehat{\nabla}u(\widehat{x}, 0)) \, da(\widehat{x}). \end{aligned} \quad (3.2)$$

In the next subsection the above functional \mathcal{E}_{Hom} will indeed be revealed as a suitable Γ -limit of the dissipation potentials \mathcal{E}_ε describing to the heat flow in the assembly Ω_ε with the highly conductive interface layer $B_\varepsilon^{\varepsilon\text{-perf}}$.

3.3.1 Statement of the limit problem and outline of the proof

Regarding the formulation of the static heat flow in the assembly Ω_ε by means of the dissipation potential \mathcal{E}_ε , *i.e.* by means of a *minimization problem*, the natural way to study the asymptotics of \mathcal{E}_ε for vanishing interface thickness and void diameter ε is Γ -convergence (see Subsection 2.2.3). However, studying Γ -convergence of a sequence of functionals requires a common topological space over which all elements of the sequence are defined. Here however, the domain of the functional \mathcal{E}_ε , *i.e.* $W_{\Gamma_\varepsilon^+ \cup \Gamma_\varepsilon^-}^{1,2}(\Omega_\varepsilon)$, varies with the geometry of the assembly Ω_ε , *i.e.* with both the thickness and the voids of the interface layer $B_\varepsilon^{\varepsilon\text{-perf}}$. In order to eliminate the dependency on the varying interface thickness, the commonly used technique – originally due to [Ciarlet and Destuynder \[1979\]](#), see also [Ciarlet \[1997\]](#) – is to perform a rescaling of the interface layer to fixed thickness. That is, one performs the change of variables

$$\begin{aligned} R_\varepsilon : \Omega_{\text{tr}}^+ \cup B \cup \Omega_{\text{tr}}^- & \rightarrow \Omega_\varepsilon^+ \cup B_\varepsilon \cup \Omega_\varepsilon^-, \\ R_\varepsilon(\widehat{x}, x_3) := & \begin{cases} (\widehat{x}, x_3 + (-\frac{1}{2} + \frac{\varepsilon}{2}) e_3) & \text{if } x \in \Omega_{\text{tr}}^+, \\ (\widehat{x}, \varepsilon x_3) & \text{if } x \in B, \\ (\widehat{x}, x_3 + (\frac{1}{2} - \frac{\varepsilon}{2}) e_3) & \text{if } x \in \Omega_{\text{tr}}^-. \end{cases} \end{aligned} \quad (3.3)$$

Given a function $u \in W_{\Gamma_\varepsilon^+ \cup \Gamma_\varepsilon^-}^{1,2}(\Omega_\varepsilon)$, consider $v(x) := u(R_\varepsilon(x))$ for $x \in R_\varepsilon^{-1}(\Omega_\varepsilon) = \Omega_{\text{tr}}^+ \cup B^{\varepsilon\text{-perf}} \cup \Omega_{\text{tr}}^-$, with $B^{\varepsilon\text{-perf}} = R_\varepsilon^{-1}(B_\varepsilon^{\varepsilon\text{-perf}})$. Thus, $v \in W_{\Gamma_{\text{tr}}^+ \cup \Gamma_{\text{tr}}^-}^{1,2}(\Omega_{\text{tr}}^+ \cup B^{\varepsilon\text{-perf}} \cup \Omega_{\text{tr}}^-)$. Note that due to the rescaling of $B_\varepsilon^{\varepsilon\text{-perf}}$ the voids in $B^{\varepsilon\text{-perf}}$ are found to be extremely stretched in the x_3 -direction, cf. Figure 3.6 below. With a slight abuse of

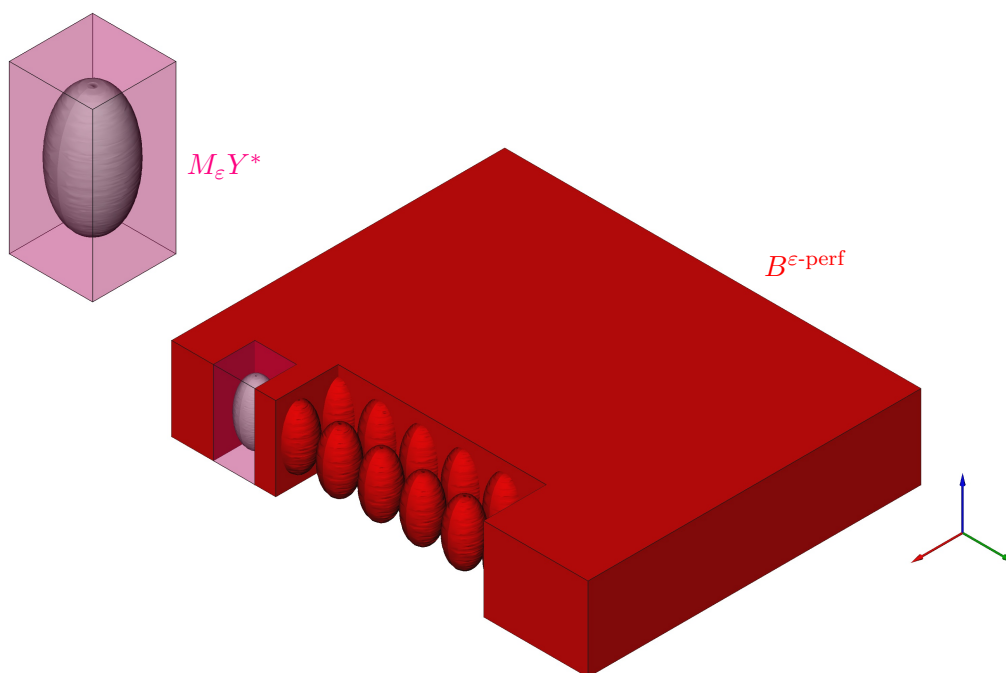


Figure 3.6: Extremely stretched voids in the rescaled interface layer $B^{\varepsilon\text{-perf}}$ (partial cutaway drawing) and a rescaled perforated periodicity cell $M_\varepsilon Y^*$ (enlarged view)

notation, I denote by ∇v the gradient of v w.r.t. the rescaled coordinates in $\Omega_\varepsilon^+ \cup B^{\varepsilon\text{-perf}} \cup \Omega_\varepsilon^-$ and by $\nabla u = [\widehat{\nabla} u, \partial_3 u]^T$ the gradient of u w.r.t. the original coordinates in Ω_ε . Then, there holds

$$\nabla v(x) = \begin{cases} \nabla u(R_\varepsilon(x)) & \text{if } x \in \Omega_{\text{tr}}^+, \\ [\widehat{\nabla} u, \varepsilon \partial_3 u](R_\varepsilon(x)) = \text{diag}(1, 1, \varepsilon) \nabla u(R_\varepsilon(x)) & \text{if } x \in B^{\varepsilon\text{-perf}}, \\ \nabla u(R_\varepsilon(x)) & \text{if } x \in \Omega_{\text{tr}}^-. \end{cases} \quad (3.4)$$

Applying this rescaling to the argument of the dissipation potential \mathcal{E}_ε leads to

$$\begin{aligned}
\mathcal{E}_\varepsilon(u) &= \int_{\Omega_{\text{tr}}^+} W^+(\nabla u(R_\varepsilon(x))) \, dx + \int_{\Omega_{\text{tr}}^-} W^-(\nabla u(R_\varepsilon(x))) \, dx \\
&\quad + \int_{B^{\varepsilon\text{-perf}}} \frac{1}{\varepsilon} W^{\text{interf}}(\nabla u(R_\varepsilon(x))) \, \varepsilon \, dx \\
&= \int_{\Omega_{\text{tr}}^+} W^+(\nabla v(x)) \, dx + \int_{\Omega_{\text{tr}}^-} W^-(\nabla v(x)) \, dx \\
&\quad + \int_{B^{\varepsilon\text{-perf}}} W^{\text{interf}}\left(\text{diag}(1, 1, \frac{1}{\varepsilon})\nabla v(x)\right) \, dx \\
&= \int_{\Omega_{\text{tr}}^+} W^+(\nabla v(x)) \, dx + \int_{\Omega_{\text{tr}}^-} W^-(\nabla v(x)) \, dx \\
&\quad + \int_{B^{\varepsilon\text{-perf}}} W^{\text{interf}}\left(\widehat{\nabla}v(x), \frac{1}{\varepsilon}\partial_3 v(x)\right) \, dx \\
&=: \mathcal{F}_\varepsilon(v).
\end{aligned}$$

Yet, the arguments v of \mathcal{F}_ε are still taken from a space, namely $W_{\Gamma_{\text{tr}}^+ \cup \Gamma_{\text{tr}}^-}^{1,2}(\Omega_{\text{tr}}^+ \cup B^{\varepsilon\text{-perf}} \cup \Omega_{\text{tr}}^-)$, that varies with the mutual distance ε of the voids in the rescaled interface layer $B^{\varepsilon\text{-perf}}$. At this point, it turns out useful to simply extend the rescaled dissipation potential \mathcal{F}_ε by ∞ to the larger space $L^2(\Omega_{\text{tr}}^+ \cup B \cup \Omega_{\text{tr}}^-)$ equipped with the usual L^2 -distance. Mathematically, the extended and rescaled dissipation potential

$$\mathcal{F}_\varepsilon : L^2(\Omega_{\text{tr}}^+ \cup B \cup \Omega_{\text{tr}}^-) \rightarrow (-\infty, \infty]$$

is now defined as

$$\mathcal{F}_\varepsilon(v) := \begin{cases} \int_{\Omega_{\text{tr}}^+} W^+(\nabla v(x)) \, dx + \int_{\Omega_{\text{tr}}^-} W^-(\nabla v(x)) \, dx \\ \quad + \int_{B^{\varepsilon\text{-perf}}} W^{\text{interf}}\left(\widehat{\nabla}v(x), \frac{1}{\varepsilon}\partial_3 v(x)\right) \, dx \\ \quad \text{if } v \in W_{\Gamma_{\text{tr}}^+ \cup \Gamma_{\text{tr}}^-}^{1,2}(\Omega_{\text{tr}}^+ \cup B^{\varepsilon\text{-perf}} \cup \Omega_{\text{tr}}^-), \\ \infty \quad \text{else.} \end{cases} \quad (3.5)$$

While being – up to the rescaling argument – equivalent to the original dissipation potential \mathcal{E}_ε for the assembly Ω_ε , the rescaled and extended dissipation potential \mathcal{F}_ε is immediately accessible for a Γ -convergence analysis as ε , *i.e.* the thickness and void diameter of the interface layer $B_\varepsilon^{\varepsilon\text{-perf}}$ in Ω_ε , vanishes. In fact, the previous heuristic reasoning, stating that the dissipation potential \mathcal{E}_ε behaves for small interface thickness and void diameter ε like \mathcal{E}_{Hom} (see (3.1)), is confirmed by the following convergence result.

Theorem 3.1. *Let there be given the notation and the assumptions from Section 3.2, and suppose $(\varepsilon_k)_k$ to be a vanishing sequence of positive real numbers whose elements*

such that ε_k and ω are compatible with (ε -Tiling). Moreover, let the void $V \Subset Y$ be compactly contained in the cubic periodicity cell Y and have a smooth boundary. Then, the rescaled dissipation potentials $(\mathcal{F}_{\varepsilon_k})_k$ defined in (3.5) Γ -converge with respect to the $L^2(\Omega_{\text{tr}}^+ \cup B \cup \Omega_{\text{tr}}^-)$ -norm, in symbols

$$\Gamma\text{-}\lim_{k \rightarrow \infty} \mathcal{F}_{\varepsilon_k} = \mathcal{F}_{\text{Hom}}$$

where $\mathcal{F}_{\text{Hom}} : L^2(\Omega_{\text{tr}}^+ \cup B \cup \Omega_{\text{tr}}^-) \rightarrow (-\infty, \infty]$ obeys

$$\mathcal{F}_{\text{Hom}}(v) = \begin{cases} \int_{\Omega_{\text{tr}}^+} W^+(\nabla v(x)) \, dx + \int_{\Omega_{\text{tr}}^-} W^-(\nabla v(x)) \, dx \\ \quad + \int_{\omega} \widehat{W}^{\text{interf}}(\widehat{\nabla} v(\widehat{x}, 0)) \, da(\widehat{x}) \\ \quad \text{if } v \in \mathcal{V}_{\Gamma^+, \Gamma^-}(\Omega^+, \Omega^-, B), \\ \infty \quad \text{else.} \end{cases} \quad (3.6)$$

Therein, $\widehat{W}^{\text{interf}} : \mathbb{R}^2 \rightarrow \mathbb{R}$ is defined through

$$\widehat{W}^{\text{interf}}(\widehat{F}) = \inf \left\{ \int_{Y^*} W^{\text{interf}}\left([\widehat{F}, 0]^T + \nabla_y w(y)\right) \, dy : w \in W_{\widehat{Y}\text{-per}}^{1,2}(Y^*) \right\}. \quad (3.7)$$

Remembering that the space $\mathcal{V}_{\Gamma^+, \Gamma^-}(\Omega^+, \Omega^-, B)$ can be identified with

$$\mathcal{V}_{\Gamma^+, \Gamma^-}(\Omega^+, \Omega^-, \omega) = \left\{ u : u \in W^{1,2}(\Omega), \right. \\ \left. u|_{\omega} \in W^{1,2}(\omega) \text{ and } u = 0 \text{ on } \Gamma^+ \cup \Gamma^- \right\},$$

the Γ -limit \mathcal{F}_{Hom} in the above theorem is indeed nothing but the heuristically motivated dissipation potential \mathcal{E}_{Hom} from (3.2). As expected, the highly conductive perforated interface layer $B_{\varepsilon}^{\varepsilon\text{-perf}}$ in the assembly Ω_{ε} can – in the limit of vanishing interface thickness and void diameter ε – be equivalently expressed through a boundary condition on the flattened interface ω . That is, through the interfacial energy density $\widehat{W}^{\text{interf}}$ that penalizes temperature gradients within the flattened interface. Moreover, the interfacial energy density can be computed from a cell problem similar to the one of Marcellini (cf. (2.28)). However, the above cell formula (3.7) is different in that the infimum is taken over a bigger set of microscopic temperature distributions, namely those having periodic boundary conditions only on the *lateral faces* of the *perforated* periodicity cell Y^* .

As another simple but important remark I would like to emphasize that the interfacial energy density $\widehat{W}^{\text{interf}}$ can also be written as

$$\widehat{W}^{\text{interf}}(\widehat{F}) = \inf \left\{ \int_Y \mathbb{1}_{Y^*}(y) W^{\text{interf}}\left([\widehat{F}, 0]^T + \nabla_y w(y)\right) \, dy : w \in W_{\widehat{Y}\text{-per}}^{1,2}(Y) \right\}. \quad (3.8)$$

This can be easily verified since by the assumed smoothness of the void's boundary ∂V every element of $W_{\hat{Y}\text{-per}}^{1,2}(Y^*)$ can be extended to an element of $W_{\hat{Y}\text{-per}}^{1,2}(Y)$ (see also Theorem 3.2 in the upcoming subsection).

The proof of Theorem 3.1 mainly relies on two ingredients. One is the use of classical extension operators to extend functions defined over the rescaled perforated interface $B^{\varepsilon\text{-perf}}$ to the whole of B . Yet, since the voids in $B^{\varepsilon\text{-perf}}$ are strongly distorted due to the rescaling in interface thickness direction, extension operators have to be used with care. Also, extension operators for functions over the rescaled perforated interface $B^{\varepsilon\text{-perf}}$ call for suitable a priori-estimates that have to be established. This will be done in the following Subsection 3.3.2. The other main ingredient is a recent concept for two-scale convergence developed by Stefan Neukamm and introduced in his PhD thesis Neukamm [2010] that is specifically designed for capturing oscillations of functions defined over thin domains or, more precisely, over *rescaled* thin domains. By combining the two concepts one obtains a method to capture the fine oscillations of the *rescaled* temperature gradients within the rescaled interface $B^{\varepsilon\text{-perf}}$ (see (3.5)) caused by the periodically recurring voids. When proving Γ -convergence for the sequence of rescaled dissipation potentials $(\mathcal{F}_{\varepsilon_k})_k$ like stated in Theorem 3.1, the Γ -lim inf-inequality along a sequence of temperature distributions $(v_{\varepsilon_k})_k$ respectively taken from the spaces $W_{\Gamma_{\text{tr}}^+ \cup \Gamma_{\text{tr}}^-}^{1,2}(\Omega_{\text{tr}}^+ \cup B^{\varepsilon\text{-perf}} \cup \Omega_{\text{tr}}^-)$ can be inferred from the two-scale convergence properties of $([\nabla v_{\varepsilon_k}, \frac{1}{\varepsilon_k} \partial_3 v_{\varepsilon_k}])_k$ and standard lower semicontinuity arguments. As concerns the Γ -lim sup-inequality, some minor technicalities have to be overcome. That is, one has to adapt standard techniques for the construction of recovery sequences in the context of periodic homogenization (see *e.g.* Allaire [1992]; Visintin [2007] or Section 4.3) to the present situation, where the recovery sequence extends over a microstructured (the rescaled interface layer $B^{\varepsilon\text{-perf}}$) and a homogeneous domain (the bodies Ω_{tr}^{\pm} adjacent to the rescaled interface layer).

While the following Subsection 3.3.2 is dedicated to a collection of auxiliary results for the proof (extension operators, a priori-estimates and Stefan Neukamm's adaption of two-scale convergence for in-plane oscillations in rescaled thin domains), the proof of Theorem 3.1 is stated in Subsection 3.3.3.

Remark 3.3. Here and also in the upcoming Section 3.4 I will slightly abuse the notation in that I will omit the index of ε_k in the symbols $B_{\varepsilon}^{\varepsilon\text{-perf}}$ and $B^{\varepsilon\text{-perf}}$. For instance, the expression

$$u_{\varepsilon_k} \in W^{1,2}(B_{\varepsilon}^{\varepsilon\text{-perf}}) \quad \text{actually means} \quad u_{\varepsilon_k} \in W^{1,2}(B_{\varepsilon_k}^{\varepsilon_k\text{-perf}}),$$

and similarly

$$u_{\varepsilon_k} \in W^{1,2}(B^{\varepsilon\text{-perf}}) \quad \text{actually means} \quad v_{\varepsilon_k} \in W^{1,2}(B^{\varepsilon_k\text{-perf}}).$$

This is to make the notation somewhat easier, as expressions like $v_{\varepsilon_k} \in W^{1,2}(B^{\varepsilon_k\text{-perf}})$ or even $\|v_{\varepsilon_k}\|_{W^{1,2}(B^{\varepsilon_k\text{-perf}})}$ are very difficult to read. In other words, in what follows the indexing of a function shall already indicate its domain.

3.3.2 Extension operators, a priori-estimates and two-scale convergence in thin domains

The construction of an extension operator extending functions from the rescaled interface with periodically recurring voids, $B^{\varepsilon\text{-perf}}$, to the entire rescaled interface B , i.e. from $W^{1,2}(B^{\varepsilon\text{-perf}})$ to $W^{1,2}(B)$, relies on the following classical result from [Oleřnik et al., 1992, Lemma 4.1].

Proposition 3.2. *In the situation of Theorem 3.1, there exists an extension operator $P : W^{1,2}(Y^*) \rightarrow W^{1,2}(Y)$ such that for all $v \in W^{1,2}(Y^*)$*

$$\begin{aligned} \|Pv\|_{W^{1,2}(Y)} &\leq C(V) \|v\|_{W^{1,2}(Y^*)}, \\ \|\nabla(Pv)\|_{L^2(Y;\mathbb{R}^3)} &\leq C(V) \|\nabla v\|_{L^2(Y^*;\mathbb{R}^3)}. \end{aligned}$$

The constant $C(V)$ depends on the void V in the perforated periodicity cell $Y^* = Y \setminus \bar{V}$ only.

With this extension operator at hand, it is no longer difficult to define the desired extension operator $P_\varepsilon : W^{1,2}(B^{\varepsilon\text{-perf}}) \rightarrow W^{1,2}(B)$. According to assumption (ε -Tiling), the perforated interface layer becomes $B_\varepsilon^{\varepsilon\text{-perf}} = \bigcup_{\hat{a} \in Z_\varepsilon} \varepsilon ([\hat{a}, 0]^T + Y^*)$ for some finite subset $Z_\varepsilon \subseteq \mathbb{Z}^2$. After performing the rescaling in interface thickness direction, one can still write

$$B^{\varepsilon\text{-perf}} = \bigcup_{\hat{a} \in Z_\varepsilon} M_\varepsilon ([\hat{a}, 0]^T + Y^*)$$

where

$$M_\varepsilon = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (3.9)$$

Now, for $\hat{a} \in Z_\varepsilon$ it turns out useful to introduce the mapping

$$T_{\hat{a},\varepsilon} : M_\varepsilon ([\hat{a}, 0]^T + Y) \rightarrow Y, \quad T_{\hat{a},\varepsilon}(y) := M_\varepsilon^{-1}y - \begin{bmatrix} \hat{a} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\varepsilon}\hat{y} - \hat{a} \\ y_3 \end{bmatrix}$$

which obviously ensures $T_{\hat{a},\varepsilon}(M_\varepsilon ([\hat{a}, 0]^T + Y^*)) = Y^*$. The extension operator $P_\varepsilon : W^{1,2}(B^{\varepsilon\text{-perf}}) \rightarrow W^{1,2}(B)$ can now be defined as follows. Let $v \in W^{1,2}(B^{\varepsilon\text{-perf}})$ and $x \in B$ be arbitrary, and let $\hat{a} \in Z_\varepsilon$ be such that $x \in M_\varepsilon ([\hat{a}, 0]^T + Y)$. Then

$$(P_\varepsilon v)(x) := P(v \circ T_{\hat{a},\varepsilon}^{-1})(T_{\hat{a},\varepsilon}(x)) \quad (3.10)$$

and, obviously, since $P(v \circ T_{\hat{a},\varepsilon}^{-1}) \in W^{1,2}(Y)$ and P is an extension operator one obtains both

$$(P_\varepsilon v)|_{B^{\varepsilon\text{-perf}}} = v \quad \text{and} \quad P_\varepsilon v \in W^{1,2}(B).$$

While the definition of P_ε is indeed rather natural, it requires some effort to show that P_ε has suitable continuity properties. To this end, one should notice that given a temperature distribution in the rescaled interface layer $B^{\varepsilon\text{-perf}}$, say $v \in W^{1,2}(B^{\varepsilon\text{-perf}})$, the interface term in the dissipation potential \mathcal{F}_ε (see (3.5)) allows by the coercivity (W3) of W^{interf} to estimate

$$\begin{aligned} \int_{B^{\varepsilon\text{-perf}}} W^{\text{interf}} \left(\widehat{\nabla} v(x), \frac{1}{\varepsilon} \partial_3 v(x) \right) dx \\ \geq c \left(\|\widehat{\nabla} v\|_{L^2(B^{\varepsilon\text{-perf}}; \mathbb{R}^2)}^2 + \left\| \frac{1}{\varepsilon} \partial_3 v \right\|_{L^2(B^{\varepsilon\text{-perf}})}^2 - \text{vol } B \right). \end{aligned}$$

It would therefore be desirable if the extension $P_\varepsilon v$ of v allowed for a similar estimate by means of the interface term in the dissipation potential. This is in fact the case as revealed by the following proposition.

Proposition 3.3. *Let there be given the notation and assumptions stated in Theorem 3.1, a small positive real number ε such that ε and ω are compatible with assumption (ε -Tiling). Moreover, let there be given the extension operators P from Proposition 3.2 and P_ε as defined in (3.10). Then for every $v \in W^{1,2}(B^{\varepsilon\text{-perf}})$ and sufficiently small ε there hold the estimates*

$$\|P_\varepsilon v\|_{L^2(B)}^2 \leq C(V) \left(\|v\|_{L^2(B^{\varepsilon\text{-perf}})}^2 + \|\varepsilon \widehat{\nabla} v\|_{L^2(B^{\varepsilon\text{-perf}}; \mathbb{R}^2)}^2 + \|\partial_3 v\|_{L^2(B^{\varepsilon\text{-perf}})}^2 \right)$$

and

$$\begin{aligned} \|\widehat{\nabla}(P_\varepsilon v)\|_{L^2(B; \mathbb{R}^2)}^2 + \left\| \frac{1}{\varepsilon} \partial_3(P_\varepsilon v) \right\|_{L^2(B)}^2 \\ \leq C(V) \left(\|\widehat{\nabla} v\|_{L^2(B^{\varepsilon\text{-perf}}; \mathbb{R}^2)}^2 + \left\| \frac{1}{\varepsilon} \partial_3 v \right\|_{L^2(B^{\varepsilon\text{-perf}})}^2 \right). \end{aligned}$$

Here, $C(V)$ is a positive constant depending on the void V in the perforated periodicity cell $Y^* = Y \setminus \overline{V}$ only.

Proof. The proof of Proposition 3.3 splits into three parts. In a first step I will show that one has

$$\begin{aligned} \|\nabla(P_\varepsilon v)\|_{L^2(B; \mathbb{R}^2)}^2 + \|\partial_3(P_\varepsilon v)\|_{L^2(B)}^2 \\ \leq C(V) \left(\|\widehat{\nabla} v\|_{L^2(B^{\varepsilon\text{-perf}}; \mathbb{R}^2)}^2 + \left\| \frac{1}{\varepsilon} \partial_3 v \right\|_{L^2(B^{\varepsilon\text{-perf}})}^2 \right), \quad (3.11) \end{aligned}$$

while the second step reveals that

$$\|\partial_3(P_\varepsilon v)\|_{L^2(B)}^2 \leq C(V) \left(\|\varepsilon \widehat{\nabla} v\|_{L^2(B^{\varepsilon\text{-perf}}; \mathbb{R}^2)}^2 + \|\partial_3 v\|_{L^2(B^{\varepsilon\text{-perf}})}^2 \right). \quad (3.12)$$

As it is easily seen, adding (3.11) and (3.12) (multiplied by $\frac{1}{\varepsilon^2}$) yields the second estimate of the proposition. The third and last step is to prove the first inequality stated in the proposition.

Step 1. To prove (3.11) one starts with

$$\begin{aligned}
& \int_B |\nabla(P_\varepsilon v)(x)|^2 dx = \sum_{\widehat{a} \in Z_\varepsilon} \int_{T_{\widehat{a},\varepsilon}^{-1}(Y)} \left| M_\varepsilon^{-1} \nabla \left(P(v \circ T_{\widehat{a},\varepsilon}^{-1}) \right) (T_{\widehat{a},\varepsilon}(x)) \right|^2 dx \\
&= \sum_{\widehat{a} \in Z_\varepsilon} \int_Y \left| M_\varepsilon^{-1} \nabla \left(P(v \circ T_{\widehat{a},\varepsilon}^{-1}) \right) (y) \right|^2 \varepsilon^2 dy \\
&= \sum_{\widehat{a} \in Z_\varepsilon} \varepsilon^2 \int_Y \left| \frac{1}{\varepsilon} \widehat{\nabla} \left(P(v \circ T_{\widehat{a},\varepsilon}^{-1}) \right) (y) \right|^2 + \left| \partial_3 \left(P(v \circ T_{\widehat{a},\varepsilon}^{-1}) \right) (y) \right|^2 dy,
\end{aligned}$$

where the first equality is due to the assumption (ε -Tiling). Now, since $\varepsilon \ll 1$ one can ‘brutally’ estimate

$$\begin{aligned}
& \int_B |\nabla(P_\varepsilon v)(x)|^2 dx \\
&\leq \sum_{\widehat{a} \in Z_\varepsilon} \varepsilon^2 \int_Y \left| \frac{1}{\varepsilon} \widehat{\nabla} \left(P(v \circ T_{\widehat{a},\varepsilon}^{-1}) \right) (y) \right|^2 + \left| \frac{1}{\varepsilon} \partial_3 \left(P(v \circ T_{\widehat{a},\varepsilon}^{-1}) \right) (y) \right|^2 dy \\
&= \sum_{\widehat{a} \in Z_\varepsilon} \int_Y \left| \nabla \left(P(v \circ T_{\widehat{a},\varepsilon}^{-1}) \right) (y) \right|^2 dy.
\end{aligned}$$

By the continuity properties of the extension operator P stated in Proposition 3.2 one further infers

$$\begin{aligned}
& \int_B |\nabla(P_\varepsilon v)(x)|^2 dx \\
&\leq \sum_{\widehat{a} \in Z_\varepsilon} C(V) \int_{Y^*} \left| \nabla \left(v \circ T_{\widehat{a},\varepsilon}^{-1} \right) (y) \right|^2 dy \\
&= C(V) \sum_{\widehat{a} \in Z_\varepsilon} \int_{T_{\widehat{a},\varepsilon}^{-1}(Y^*)} \left| \nabla \left(v \circ T_{\widehat{a},\varepsilon}^{-1} \right) (T_{\widehat{a},\varepsilon}(x)) \right|^2 \frac{1}{\varepsilon^2} dx \\
&= C(V) \sum_{\widehat{a} \in Z_\varepsilon} \int_{T_{\widehat{a},\varepsilon}^{-1}(Y^*)} \left| \frac{1}{\varepsilon} M_\varepsilon \nabla v(x) \right|^2 dx \\
&= C(V) \sum_{\widehat{a} \in Z_\varepsilon} \int_{T_{\widehat{a},\varepsilon}^{-1}(Y^*)} \left| \begin{bmatrix} 1 & & \\ & 1 & \\ & & \frac{1}{\varepsilon} \end{bmatrix} \nabla v(x) \right|^2 dx \\
&= C(V) \int_{B^{\varepsilon\text{-perf}}} |\widehat{\nabla} v(x)|^2 + \left| \frac{1}{\varepsilon} \partial_3 v(x) \right|^2 dx
\end{aligned}$$

and (3.11) is proved.

Step 2. By using similar arguments like in the previous step, one first transforms the

left hand side of (3.12) as follows:

$$\begin{aligned}
& \int_B |\partial_3(P_\varepsilon v)(x)|^2 dx \\
&= \sum_{\widehat{a} \in Z_\varepsilon} \int_{T_{\widehat{a}, \varepsilon}^{-1}(Y)} \left| \nabla \left(P(v \circ T_{\widehat{a}, \varepsilon}^{-1}) \right) (T_{\widehat{a}, \varepsilon}(x)) \cdot \partial_3 T_{\widehat{a}, \varepsilon}(x) \right|^2 dx \\
&= \sum_{\widehat{a} \in Z_\varepsilon} \int_{T_{\widehat{a}, \varepsilon}^{-1}(Y)} \left| \nabla \left(P(v \circ T_{\widehat{a}, \varepsilon}^{-1}) \right) (T_{\widehat{a}, \varepsilon}(x)) \cdot e_3 \right|^2 dx \\
&= \sum_{\widehat{a} \in Z_\varepsilon} \int_Y \left| \partial_3 \left(P(v \circ T_{\widehat{a}, \varepsilon}^{-1}) \right) (y) \right|^2 \varepsilon^2 dy.
\end{aligned}$$

Another ‘brutal’ estimation gives

$$\int_B |\partial_3(P_\varepsilon v)(x)|^2 dx \leq \sum_{\widehat{a} \in Z_\varepsilon} \varepsilon^2 \int_Y \left| \nabla \left(P(v \circ T_{\widehat{a}, \varepsilon}^{-1}) \right) (y) \right|^2 dy$$

and applying Proposition 3.2 once more results in

$$\begin{aligned}
& \int_B |\partial_3(P_\varepsilon v)(x)|^2 dx \leq \sum_{\widehat{a} \in Z_\varepsilon} \varepsilon^2 C(V) \int_{Y^*} \left| \nabla(v \circ T_{\widehat{a}, \varepsilon}^{-1})(y) \right|^2 dy \\
&= C(V) \sum_{\widehat{a} \in Z_\varepsilon} \int_{Y^*} \left| M_\varepsilon \nabla v(T_{\widehat{a}, \varepsilon}^{-1}(y)) \right|^2 \varepsilon^2 dy \\
&= C(V) \sum_{\widehat{a} \in Z_\varepsilon} \int_{T_{\widehat{a}, \varepsilon}^{-1}(Y^*)} \left| M_\varepsilon \nabla v(x) \right|^2 dx \\
&= C(V) \sum_{\widehat{a} \in Z_\varepsilon} \int_{T_{\widehat{a}, \varepsilon}^{-1}(Y^*)} \varepsilon^2 \left| \widehat{\nabla} v(x) \right|^2 + \left| \partial_3 v(x) \right|^2 dx \\
&= C(V) \int_{B^{\varepsilon\text{-perf}}} \left| \varepsilon \widehat{\nabla} v(x) \right|^2 + \left| \partial_3 v(x) \right|^2 dx.
\end{aligned}$$

Hence, also (3.12) is correct, which completes the proof of the second inequality stated in the proposition.

Step 3. Just like in the previous steps one can easily show that

$$\int_B |(P_\varepsilon v)(x)|^2 dx = \sum_{\widehat{a} \in Z_\varepsilon} \varepsilon^2 \int_Y \left| P(v \circ T_{\widehat{a}, \varepsilon}^{-1})(y) \right|^2 dy.$$

Then, Proposition 3.2 allows to further estimate

$$\begin{aligned}
& \int_B |(P_\varepsilon v)(x)|^2 dx \\
& \leq \sum_{\hat{a} \in Z_\varepsilon} \varepsilon^2 C(V) \left(\int_{Y^*} |(v \circ T_{\hat{a}, \varepsilon}^{-1})(y)|^2 dy + \int_{Y^*} |\nabla(v \circ T_{\hat{a}, \varepsilon}^{-1})(y)|^2 dy \right) \\
& = C(V) \sum_{\hat{a} \in Z_\varepsilon} \left(\int_{T_{\hat{a}, \varepsilon}^{-1}(Y^*)} |v(x)|^2 dx + \int_{T_{\hat{a}, \varepsilon}^{-1}(Y^*)} |M_\varepsilon \nabla v(x)|^2 dx \right) \\
& = C(V) \left(\int_{B^{\varepsilon\text{-perf}}} |v(x)|^2 dx + \int_{B^{\varepsilon\text{-perf}}} |\varepsilon \widehat{\nabla} v(x)|^2 + |\partial_3 v(x)|^2 dx \right)
\end{aligned}$$

where the last equality can be verified by performing the same integral transformations as before. This concludes the proof of the proposition. \square

As announced before, the second tool that is essential to the proof of Theorem 3.1 is an adaption of two-scale convergence in rescaled thin domains like B , called ‘two-scale convergence for in-plane oscillations’. The definitions and results stated here are due to Stefan Neukamm and taken from his thesis Neukamm [2010]. However, to ease the presentation I chose not to cite them in all detail, but rather in form that is tailored to the present context and notation. The core definition for Stefan Neukamm’s adaption of two-scale convergence for in-plane oscillations is the following unfolding operator (see [Neukamm, 2010, Definition 6.2.1]).

Definition 3.4. Let U be some open and bounded subset of \mathbb{R}^3 and $\widehat{Y} = [0, 1)^2$ be defined like in Subsection 3.2.1, and let ε be a small positive real number. For any $v \in L^2(U)$ the periodic unfolding operator for in-plane oscillations $\widehat{\mathcal{T}}_\varepsilon v \in L^2(\mathbb{R}^3 \times \widehat{Y})$ is defined as

$$(\widehat{\mathcal{T}}_\varepsilon v)(x, \widehat{y}) := \begin{cases} v\left(\varepsilon \lfloor \frac{\widehat{x}}{\varepsilon} \rfloor + \varepsilon \widehat{y}, x_3\right) & \text{if } \left(\varepsilon \lfloor \frac{\widehat{x}}{\varepsilon} \rfloor + \varepsilon \widehat{y}, x_3\right) \in U, \\ 0 & \text{else.} \end{cases}$$

Here, $\lfloor \cdot \rfloor : \mathbb{R}^2 \rightarrow \mathbb{Z}^2$ returns the integer part of its argument.

Remark 3.4. The designation ‘in-plane oscillations’ stems from the insight that in the case of U being a rescaled thin domain like $B = \omega \times [-\frac{1}{2}, \frac{1}{2}]$ with a microstructure extending *in in-plane direction*, oscillations of functions caused by this microstructure must foremost be expected in the in-plane direction of the rescaled thin domain. For instance, oscillations in interface direction of the rescaled temperature distribution in $B^{\varepsilon\text{-perf}}$ that are caused by the presence of periodically recurring voids (see in Figure 3.6). Two-scale convergence for in-plane oscillations has specifically been designed to capture such in-plane oscillations.

This periodic unfolding operator yields the following notion of two-scale convergence (cf. [Neukamm, 2010, Definition 6.2.3]).

Definition 3.5. Assume once more U to be some open and bounded subset of \mathbb{R}^3 and $\widehat{Y} = [0, 1]^2$ to be defined like in Subsection 3.2.1. Let $(\varepsilon_k)_k$ be an arbitrary sequence of positive real numbers converging to zero. A sequence $(v_{\varepsilon_k})_k$ in $L^2(U)$ is said to weakly (respectively strongly) two-scale converge w.r.t. in-plane oscillations to some $v_0 \in L^2(U \times \widehat{Y})$, if $\widehat{\mathcal{T}}_{\varepsilon_k} v_{\varepsilon_k}$ weakly (respectively strongly) converges to v_0 in $L^2(\mathbb{R}^3 \times \widehat{Y})$ (where v_0 is supposed to be extended to $\mathbb{R}^3 \times Y$ by 0 in its first argument). In symbols weak two-scale convergence w.r.t. in-plane oscillations is denoted

$$v_{\varepsilon_k} \xrightarrow[\widehat{Y}]{2} v_0 \quad \text{in } L^2(U \times \widehat{Y}),$$

and strong two-scale convergence w.r.t. in-plane oscillations

$$v_{\varepsilon_k} \xrightarrow[\widehat{Y}]{2} v_0 \quad \text{in } L^2(U \times \widehat{Y}).$$

Just like usual two-scale convergence and periodic unfolding, also the above adaption to in-plane oscillations in rescaled thin domains possesses the usual isometry (see Proposition 2.22) and compactness properties (see Theorem 2.17 and Proposition 2.23). This is revealed by the next proposition (again see [Neukamm, 2010, Lemma 6.2.2]).

Proposition 3.6. *Suppose U to be some open and bounded subset of \mathbb{R}^3 and $\widehat{Y} = [0, 1]^2$ to be defined like in Subsection 3.2.1. Then*

- (i) *for every positive real number ε the periodic unfolding operator for in-plane oscillations $\widehat{\mathcal{T}}_\varepsilon : L^2(U) \rightarrow L^2(\mathbb{R}^3 \times \widehat{Y})$ satisfies*

$$\int_U v(x) \, dx = \int_{\mathbb{R}^3} \int_{\widehat{Y}} (\widehat{\mathcal{T}}_\varepsilon v)(x, \widehat{y}) \, d\widehat{y} \, dx \quad \text{for all } v \in L^2(U).$$

In particular, $\widehat{\mathcal{T}}_\varepsilon$ is an isometry.

- (ii) *given an arbitrary vanishing sequence of positive real numbers $(\varepsilon_k)_k$, every bounded sequence in $L^2(U)$ contains a weakly two-scale convergent subsequence w.r.t. in-plane oscillations.*

One can establish relations between the notion of two-scale convergence for in-plane oscillations in rescaled thin domains and traditional weak and strong convergence, but also with the traditional two-scale convergence of Nguetseng and Allaire.

Proposition 3.7. *Let U be some open and bounded subset of \mathbb{R}^3 , $\widehat{Y} = [0, 1]^2$ and $Y = \widehat{Y} \times [-\frac{1}{2}, \frac{1}{2}]$ be defined like in Subsection 3.2.1 and let $(\varepsilon_k)_k$ be an arbitrary sequence of positive real numbers converging to zero. Moreover, let $(v_{\varepsilon_k})_k$ be a sequence in $L^2(U)$. Then*

(i) for $v_0 \in L^2(U \times Y)$ one has the implications

$$\begin{aligned} v_{\varepsilon_k} &\xrightarrow{2} v_0 \quad \text{in } L^2(U \times Y) \\ &\Rightarrow v_{\varepsilon_k} \xrightarrow{\widehat{Y}} \int_{(-\frac{1}{2}, \frac{1}{2})} v_0(\cdot, \cdot, y_3) dy_3 \quad \text{in } L^2(U \times \widehat{Y}), \\ v_{\varepsilon_k} &\xrightarrow{2} v_0 \quad \text{in } L^2(U \times Y) \\ &\Rightarrow v_{\varepsilon_k} \xrightarrow{\widehat{Y}} \int_{(-\frac{1}{2}, \frac{1}{2})} v_0(\cdot, \cdot, y_3) dy_3 \quad \text{in } L^2(U \times \widehat{Y}), \end{aligned}$$

(ii) for $v_0 \in L^2(U \times \widehat{Y})$ there holds

$$v_{\varepsilon_k} \xrightarrow{\widehat{Y}} v_0 \quad \text{in } L^2(U \times \widehat{Y}) \quad \Rightarrow \quad v_{\varepsilon_k} \rightharpoonup \int_{\widehat{Y}} v_0(\cdot, \widehat{y}) d\widehat{y} \quad \text{in } L^2(U),$$

(iii) if $v_0 \in L^2(U \times \widehat{Y})$ and v_0 is independent of its second argument, i.e. $v_0 \in L^2(U)$, then

$$v_{\varepsilon_k} \rightarrow v_0 \quad \text{in } L^2(U) \quad \Leftrightarrow \quad v_{\varepsilon_k} \xrightarrow{\widehat{Y}} v_0 \quad \text{in } L^2(U \times \widehat{Y}),$$

(iv) if $v_0 \in L^2(U \times \widehat{Y})$ such that $v_{\varepsilon_k} \xrightarrow{\widehat{Y}} v_0$ in $L^2(U \times \widehat{Y})$ and $(w_{\varepsilon_k})_k$ is another sequence in $L^2(U)$ satisfying $w_{\varepsilon_k} \xrightarrow{\widehat{Y}} w_0$ in $L^2(U \times \widehat{Y})$ for some $w_0 \in L^2(U \times \widehat{Y})$, it is

$$\int_U v_{\varepsilon_k}(x) w_{\varepsilon_k}(x) dx \rightarrow \int_U \int_{\widehat{Y}} v_0(x, \widehat{y}) w_0(x, \widehat{y}) d\widehat{y} dx.$$

For the first statement in this proposition I refer to [Neukamm, 2010, Proposition 6.2.5] (whereas the other statements are simple adaptations of the arguments found *e.g.* in [Visintin, 2006, Theorem 1.3]).

The key results in Neukamm [2010] on two-scale convergence for in-plane oscillations concern the situation of a rescaled thin domain like $B = \omega \times [-\frac{1}{2}, \frac{1}{2}]$ and a sequence in $L^2(B; \mathbb{R}^3)$ that can be identified as a sequence of gradients $v_\varepsilon = \nabla u_\varepsilon$. Here, the identification of the corresponding two-scale limit for in-plane oscillations leads to a quite different result than in the case of common two-scale convergence (compare in Theorem 2.18). The next theorem is only a special case of a more general result found in [Neukamm, 2010, Theorem 6.3.3].

Theorem 3.8. Assume $B := \omega \times [-\frac{1}{2}, \frac{1}{2}]$, $\widehat{Y} = [0, 1)^2$ and $Y := Y \times [-\frac{1}{2}, \frac{1}{2}]$ to be defined like in Subsection 3.2.1 and let $(\varepsilon_k)_k$ be an arbitrary sequence of positive real

numbers converging to zero. Moreover, let $(u_{\varepsilon_k})_k$ be a weakly convergent sequence in $W^{1,2}(B)$ with limit u_0 satisfying the estimate

$$\|\widehat{\nabla} u_{\varepsilon_k}\|_{L^2(B;\mathbb{R}^2)}^2 + \left\| \frac{1}{\varepsilon_k} \partial_3 u_{\varepsilon_k} \right\|_{L^2(B)}^2 \leq C \quad (3.13)$$

for all ε_k and some positive constant C . Then u_0 is independent of x_3 and there exist a subsequence $(\varepsilon_{k_\ell})_\ell$ and a $U_0 \in L^2(\omega; W_{\widehat{Y}\text{-per}}^{1,2}(Y; \mathbb{R}^3))$ such that

$$\left[\widehat{\nabla} u_{\varepsilon_{k_\ell}}, \frac{1}{\varepsilon_{k_\ell}} \partial_3 u_{\varepsilon_{k_\ell}} \right] \xrightarrow{\widehat{Y}} \left[\widehat{\nabla} u_0(x) + \widehat{\nabla}_y U_0(\widehat{x}, (\widehat{y}, x_3)), \partial_{y_3} U_0(\widehat{x}, (\widehat{y}, x_3)) \right] \\ \text{in } L^2(B \times \widehat{Y}; \mathbb{R}^3),$$

where $\partial_{y_1} U_0, \dots, \partial_{y_3} U_0$ denote the partial weak derivatives of U_0 w.r.t. the components of its second argument and $\widehat{\nabla}_y U_0 = [\partial_{y_1} U_0, \partial_{y_2} U_0]^T$.

Given the results on extension operators and on two-scale convergence for in-plane oscillations in rescaled thin domains, one can prove the following lemma which will play a central role in the proof of the main Theorem 3.1.

Lemma 3.9. *Let there be given the notation and assumptions stated in Theorem 3.1, a sequence of positive real numbers $(\varepsilon_k)_k$ whose elements and ω satisfy $(\varepsilon\text{-Tiling})$, and moreover the extension operators P from Theorem 3.2 and P_ε as defined in (3.10). Furthermore, let $(v_{\varepsilon_k})_k$ be a sequence in $L^2(B)$ such that $v_{\varepsilon_k} \in W^{1,2}(B^{\varepsilon\text{-perf}})$ and suppose*

$$\|\widehat{\nabla} v_{\varepsilon_k}\|_{L^2(B^{\varepsilon\text{-perf}};\mathbb{R}^2)}^2 + \left\| \frac{1}{\varepsilon_k} \partial_3 v_{\varepsilon_k} \right\|_{L^2(B^{\varepsilon\text{-perf}})}^2 \leq C$$

for all ε_k and some positive constant C . Set $u_{\varepsilon_k} := P_{\varepsilon_k} v_{\varepsilon_k} \in W^{1,2}(B)$. Then

- (i) $(\widehat{\nabla} u_{\varepsilon_k})_k$ and $(\partial_3 u_{\varepsilon_k})_k$ satisfy estimate (3.13) for some (possibly different) ε_k -independent constant C .
- (ii) Suppose furthermore $v_{\varepsilon_k} \rightarrow v_0$ in $L^2(B)$. Then $v_0 \in W^{1,2}(B)$, v_0 is independent of x_3 and

$$u_{\varepsilon_k} \rightharpoonup v_0 \quad \text{in } W^{1,2}(B).$$

Proof. Obviously, the first assertion of the lemma immediately follows from the second inequality stated in Proposition 3.3 and the assumptions of the lemma.

To prove the second assertion, one again employs Proposition 3.3 to infer together with the assumptions of the lemma, *i.e.* the uniform boundedness of both $\|v_{\varepsilon_k}\|_{L^2(B^{\varepsilon\text{-perf}})}$ and $\|\widehat{\nabla} v_{\varepsilon_k}\|_{L^2(B^{\varepsilon\text{-perf}};\mathbb{R}^2)}^2 + \|\frac{1}{\varepsilon_k} \partial_3 v_{\varepsilon_k}\|_{L^2(B^{\varepsilon\text{-perf}})}^2$, that indeed also $\|u_{\varepsilon_k}\|_{L^2(B)}$ is uniformly bounded. This in combination with the validity of (3.13) yields the uniform boundedness of $(u_{\varepsilon_k})_k$ in $W^{1,2}(B)$. Hence, there is a $u_0 \in W^{1,2}(B)$ such that

$$u_{\varepsilon_{k_\ell}} \rightharpoonup u_0 \quad \text{in } W^{1,2}(B) \quad (3.14)$$

along a suitable subsequence $(k_\ell)_\ell$. Obviously, as $(\widehat{\nabla} u_{\varepsilon_k})_k$ and $(\partial_3 u_{\varepsilon_k})_k$ satisfy estimate (3.13), it is $\partial_3 u_{\varepsilon_k} \rightarrow 0$ in $L^2(B)$. Thus, $\partial_3 u_0 = 0$ and u_0 is independent of x_3 .

All that remains to do now is to show that u_0 can be identified with v_0 a.e. in B . To this end, let U be an arbitrary open subset of B . Since $u_{\varepsilon_k} = P_{\varepsilon_k} v_{\varepsilon_k}$ is the extension of v_{ε_k} from $B^{\varepsilon\text{-perf}}$ to B it is obvious that

$$\int_{U \cap B^{\varepsilon\text{-perf}}} u_{\varepsilon_k}(x) \, dx = \int_{U \cap B^{\varepsilon\text{-perf}}} v_{\varepsilon_k}(x) \, dx. \quad (3.15)$$

Turning to the integral on the left hand side, one first notices that the indicator function of $B^{\varepsilon\text{-perf}}$ can be written like

$$\mathbb{1}_{B^{\varepsilon\text{-perf}}}(x) = \mathbb{1}_B(x) \mathbb{1}_{Y^*} \left(\frac{\widehat{x}}{\varepsilon_k}, x_3 \right), \quad (3.16)$$

where it is assumed that $Y \ni y \mapsto \mathbb{1}_{Y^*}(\widehat{y}, y_3)$ is extended to $\mathbb{R}^2 \times [-\frac{1}{2}, \frac{1}{2}]$ by \widehat{Y} -periodicity in its \widehat{y} -argument. Hence

$$\begin{aligned} \int_{U \cap B^{\varepsilon\text{-perf}}} u_{\varepsilon_k}(x) \, dx &= \int_B \mathbb{1}_U(x) \mathbb{1}_{B^{\varepsilon\text{-perf}}}(x) u_{\varepsilon_k}(x) \, dx \\ &= \int_B \mathbb{1}_U(x) \mathbb{1}_{Y^*} \left(\frac{\widehat{x}}{\varepsilon_k}, x_3 \right) u_{\varepsilon_k}(x) \, dx. \end{aligned} \quad (3.17)$$

The function $(x, \widehat{y}) \mapsto \mathbb{1}_U(x) \mathbb{1}_{Y^*}(\widehat{y}, x_3)$ is an element of $L^\infty(B; L^\infty_{\text{per}}(\widehat{Y}))$ and can – given some arbitrary $\eta > 0$ – be approximated with $\psi \in C_c^\infty(B; C^\infty_{\text{per}}(\widehat{Y}))$ such that

$$\left\| \psi(x, \widehat{y}) - \mathbb{1}_U(x) \mathbb{1}_{Y^*}(\widehat{y}, x_3) \right\|_{L^2(B \times \widehat{Y})} \leq \eta. \quad (3.18)$$

Just like in the case of ‘normal’ two-scale convergence, it is easily checked that $\psi \left(x, \frac{\widehat{x}}{\varepsilon_k} \right) \xrightarrow[2]{\widehat{Y}} \psi(x, \widehat{y})$ in $L^2(B \times \widehat{Y})$ (or inferred from the strong two-scale convergence $\psi \left(x, \frac{\widehat{x}}{\varepsilon_k} \right) \xrightarrow[2]{Y} \psi(x, \widehat{y})$ in $L^2(B \times Y)$ by Example 2.1 and the first statement of Proposition 3.7). Then, by applying the third statement of the same proposition to $u_{\varepsilon_{k_\ell}} \rightarrow u_0$ in $L^2(B)$ (itself inferred from (3.14)) it is the fourth statement of Proposition 3.7 that leads to

$$\begin{aligned} \int_B \psi \left(x, \frac{\widehat{x}}{\varepsilon_{k_\ell}} \right) u_{\varepsilon_{k_\ell}}(x) \, dx &\rightarrow \int_B \int_{\widehat{Y}} \psi(x, \widehat{y}) u_0(x) \, d\widehat{y} \, dx \\ &= \int_B u_0(x) \left(\int_{\widehat{Y}} \psi(x, \widehat{y}) \, d\widehat{y} \right) \, dx. \end{aligned} \quad (3.19)$$

On the other hand, according to the isometry property of the unfolding operator for

in-plane oscillations $\widehat{\mathcal{T}}_{\varepsilon_{k_\ell}} : L^2(B) \rightarrow L^2(B \times \widehat{Y})$ stated in Proposition 3.6 it is

$$\begin{aligned} & \left| \int_B \psi \left(x, \frac{\widehat{x}}{\varepsilon_{k_\ell}} \right) u_{\varepsilon_{k_\ell}}(x) \, dx - \int_B \mathbf{1}_U(x) \mathbf{1}_{Y^*} \left(\frac{\widehat{x}}{\varepsilon_{k_\ell}}, x_3 \right) u_{\varepsilon_{k_\ell}}(x) \, dx \right| \\ &= \left| \int_{\mathbb{R}^3} \int_{\widehat{Y}} \left(\widehat{\mathcal{T}}_{\varepsilon_{k_\ell}} \psi \left(\cdot, \frac{\widehat{\cdot}}{\varepsilon_{k_\ell}} \right) - \widehat{\mathcal{T}}_{\varepsilon_{k_\ell}} \mathbf{1}_U \widehat{\mathcal{T}}_{\varepsilon_{k_\ell}} \mathbf{1}_{Y^*} \left(\frac{\widehat{\cdot}}{\varepsilon_{k_\ell}}, (\cdot)_3 \right) \right) (x, \widehat{y}) \right. \\ & \quad \left. (\widehat{\mathcal{T}}_{\varepsilon_{k_\ell}} u_{\varepsilon_{k_\ell}})(x, \widehat{y}) \, d\widehat{y} \, dx \right| \\ &\leq \left\| \widehat{\mathcal{T}}_{\varepsilon_{k_\ell}} \psi \left(\cdot, \frac{\widehat{\cdot}}{\varepsilon_{k_\ell}} \right) - \widehat{\mathcal{T}}_{\varepsilon_{k_\ell}} \mathbf{1}_U \widehat{\mathcal{T}}_{\varepsilon_{k_\ell}} \mathbf{1}_{Y^*} \left(\frac{\widehat{\cdot}}{\varepsilon_{k_\ell}}, (\cdot)_3 \right) \right\|_{L^2(\mathbb{R}^3 \times \widehat{Y})} \left\| \widehat{\mathcal{T}}_{\varepsilon_{k_\ell}} u_{\varepsilon_{k_\ell}} \right\|_{L^2(\mathbb{R}^3 \times \widehat{Y})} \end{aligned}$$

Furthermore, by applying the definition of the periodic unfolding operator $\widehat{\mathcal{T}}_{\varepsilon_{k_\ell}}$ one finds that

$$\begin{aligned} \left(\widehat{\mathcal{T}}_{\varepsilon_{k_\ell}} \mathbf{1}_{Y^*} \left(\frac{\widehat{\cdot}}{\varepsilon_{k_\ell}}, (\cdot)_3 \right) \right) (x, \widehat{y}) &= \mathbf{1}_B \left(\varepsilon_{k_\ell} \lfloor \frac{\widehat{x}}{\varepsilon_{k_\ell}} \rfloor + \varepsilon_{k_\ell} \widehat{y}, x_3 \right) \mathbf{1}_{Y^*}(\widehat{y}, x_3) \\ &= \mathbf{1}_B(x) \mathbf{1}_{Y^*}(\widehat{y}, x_3) \end{aligned} \quad (3.20)$$

where the last equality follows from the simplifying assumption (ε -Tiling) on the geometry of ω and the values of ε_k . Thus, one can further estimate

$$\begin{aligned} & \left| \int_B \psi \left(x, \frac{\widehat{x}}{\varepsilon_{k_\ell}} \right) u_{\varepsilon_{k_\ell}}(x) \, dx - \int_B \mathbf{1}_U(x) \mathbf{1}_{Y^*} \left(\frac{\widehat{x}}{\varepsilon_{k_\ell}}, x_3 \right) u_{\varepsilon_{k_\ell}}(x) \, dx \right| \\ &\leq \left\| \psi \left(\left(\varepsilon_{k_\ell} \lfloor \frac{\widehat{x}}{\varepsilon_{k_\ell}} \rfloor + \varepsilon_{k_\ell} \widehat{y}, x_3 \right), \widehat{y} \right) \right. \\ & \quad \left. - \mathbf{1}_U \left(\varepsilon_{k_\ell} \lfloor \frac{\widehat{x}}{\varepsilon_{k_\ell}} \rfloor + \varepsilon_{k_\ell} \widehat{y}, x_3 \right) \mathbf{1}_B(x) \mathbf{1}_{Y^*}(\widehat{y}, x_3) \right\|_{L^2(\mathbb{R}^3 \times \widehat{Y})} \|u_{\varepsilon_{k_\ell}}\|_{L^2(B)} \\ &\leq \left(\left\| \psi \left(\left(\varepsilon_{k_\ell} \lfloor \frac{\widehat{x}}{\varepsilon_{k_\ell}} \rfloor + \varepsilon_{k_\ell} \widehat{y}, x_3 \right), \widehat{y} \right) - \psi(x, \widehat{y}) \right\|_{L^2(\mathbb{R}^3 \times \widehat{Y})} \right. \\ & \quad + \left\| \psi(x, \widehat{y}) - \mathbf{1}_U(x) \mathbf{1}_{Y^*}(\widehat{y}, x_3) \right\|_{L^2(\mathbb{R}^3 \times \widehat{Y})} \\ & \quad \left. + \left\| \mathbf{1}_U(x) \mathbf{1}_{Y^*}(\widehat{y}, x_3) - \mathbf{1}_U \left(\varepsilon_{k_\ell} \lfloor \frac{\widehat{x}}{\varepsilon_{k_\ell}} \rfloor + \varepsilon_{k_\ell} \widehat{y}, x_3 \right) \mathbf{1}_B(x) \mathbf{1}_{Y^*}(\widehat{y}, x_3) \right\|_{L^2(\mathbb{R}^3 \times \widehat{Y})} \right) \\ & \quad \cdot \|u_{\varepsilon_{k_\ell}}\|_{L^2(B)}. \end{aligned}$$

In view of (3.18), passing to the limit $\varepsilon_{k_\ell} \rightarrow 0$ yields

$$\begin{aligned} \limsup_{\ell \rightarrow \infty} \left| \int_B \psi \left(x, \frac{\widehat{x}}{\varepsilon_{k_\ell}} \right) u_{\varepsilon_{k_\ell}}(x) \, dx \right. \\ \left. - \int_B \mathbf{1}_U(x) \mathbf{1}_{Y^*} \left(\frac{\widehat{x}}{\varepsilon_{k_\ell}}, x_3 \right) u_{\varepsilon_{k_\ell}}(x) \, dx \right| \leq \eta \|u_0\|_{L^2(B)}. \end{aligned} \quad (3.21)$$

Having this at hand and recalling (3.17), it is now easy to infer

$$\begin{aligned}
& \limsup_{\ell \rightarrow \infty} \left| \int_{U \cap B^{\varepsilon\text{-perf}}} u_{\varepsilon_{k_\ell}}(x) \, dx - \int_B u_0(x) \left(\int_{\widehat{Y}} \mathbf{1}_U(x) \mathbf{1}_{Y^*}(\widehat{y}, x_3) \, d\widehat{y} \right) dx \right| \\
& \leq \limsup_{\ell \rightarrow \infty} \left| \int_B \mathbf{1}_U(x) \mathbf{1}_{Y^*} \left(\frac{\widehat{x}}{\varepsilon_{k_\ell}}, x_3 \right) u_{\varepsilon_{k_\ell}}(x) \, dx - \int_B \psi \left(x, \frac{\widehat{x}}{\varepsilon_{k_\ell}} \right) u_{\varepsilon_{k_\ell}}(x) \, dx \right| \\
& \quad + \lim_{\ell \rightarrow \infty} \left| \int_B \psi \left(x, \frac{\widehat{x}}{\varepsilon_{k_\ell}} \right) u_{\varepsilon_{k_\ell}}(x) \, dx - \int_B u_0(x) \left(\int_{\widehat{Y}} \psi(x, \widehat{y}) \, d\widehat{y} \right) dx \right| \\
& \quad + \left| \int_B u_0(x) \left(\int_{\widehat{Y}} \psi(x, \widehat{y}) \, d\widehat{y} \right) dx - \int_B u_0(x) \left(\int_{\widehat{Y}} \mathbf{1}_U(x) \mathbf{1}_{Y^*}(\widehat{y}, x_3) \, d\widehat{y} \right) dx \right| \\
& \leq \eta \|u_0\|_{L^2(B)} + 0 + \eta \|u_0\|_{L^2(B \times \widehat{Y})}
\end{aligned}$$

upon taking into account (3.21), (3.19) and (3.18). By choosing η arbitrarily small one now has

$$\int_{U \cap B^{\varepsilon\text{-perf}}} u_{\varepsilon_{k_\ell}}(x) \, dx \rightarrow \int_B u_0(x) \left(\int_{\widehat{Y}} \mathbf{1}_U(x) \mathbf{1}_{Y^*}(\widehat{y}, x_3) \, d\widehat{y} \right) dx$$

as ε_{k_ℓ} vanishes. In fact, by employing the very same arguments as before one can also infer

$$\int_{U \cap B^{\varepsilon\text{-perf}}} v_{\varepsilon_{k_\ell}}(x) \, dx \rightarrow \int_B v_0(x) \left(\int_{\widehat{Y}} \mathbf{1}_U(x) \mathbf{1}_{Y^*}(\widehat{y}, x_3) \, d\widehat{y} \right) dx.$$

Thus, in view of (3.15) it follows that

$$\int_U u_0(x) \left(\int_{\widehat{Y}} \mathbf{1}_{Y^*}(\widehat{y}, x_3) \, d\widehat{y} \right) dx = \int_U v_0(x) \left(\int_{\widehat{Y}} \mathbf{1}_{Y^*}(\widehat{y}, x_3) \, d\widehat{y} \right) dx,$$

which by the arbitrariness of the open subset U of B results in

$$u_0(x) \left(\int_{\widehat{Y}} \mathbf{1}_{Y^*}(\widehat{y}, x_3) \, d\widehat{y} \right) = v_0(x) \left(\int_{\widehat{Y}} \mathbf{1}_{Y^*}(\widehat{y}, x_3) \, d\widehat{y} \right) \quad \text{for a.e. } x \in B.$$

Now, since $Y^* = Y \setminus \overline{V}$ and by assumption V is compactly contained in Y one can deduce $\int_{\widehat{Y}} \mathbf{1}_{Y^*}(\widehat{y}, x_3) \, d\widehat{y} > 0$ for all $x \in B$ and therefore $u_0(x) = v_0(x)$ for a.e. $x \in B$. In particular, $v_0 \in W^{1,2}(B)$ as claimed.

Since the above arguments reveal in fact that *all* weakly convergent subsequences of $(u_{\varepsilon_k})_k$ in $W^{1,2}(B)$ have the *same* weak limit v_0 , it follows that the entire sequence $(u_{\varepsilon_k})_k$ weakly converges to v_0 in $W^{1,2}(B)$. This finishes the proof of the lemma. \square

With all these preparatory results at hand one can finally prove the main Theorem 3.1.

3.3.3 Proof of the main result

Since the rescaled dissipation potentials $(\mathcal{F}_{\varepsilon_k})_k$ as defined in (3.5) are posed over a common *metric space*, i.e. $L^2(\Omega_{\text{tr}}^+ \cup B \cup \Omega_{\text{tr}}^-)$ equipped with the strong convergence, in the situation of Theorem 3.1 it suffices to verify the sequential characterization of Γ -convergence (according to Corollary 2.6). Therefore, the proof as presented below will be split in two parts, one dealing with the Γ -lim inf-inequality and the other providing a proof for the Γ -lim sup-inequality.

Step 1. Γ -lim inf-inequality. Let there be given a sequence $(v_{\varepsilon_k})_k$ in $L^2(\Omega_{\text{tr}}^+ \cup B \cup \Omega_{\text{tr}}^-)$ such that $v_{\varepsilon_k} \rightarrow v_0$ in $L^2(\Omega_{\text{tr}}^+ \cup B \cup \Omega_{\text{tr}}^-)$ and $\liminf_k \mathcal{F}_{\varepsilon_k}(v_{\varepsilon_k}) < \infty$, and let $(\varepsilon_{k_\ell})_\ell$ be a subsequence satisfying $\liminf_k \mathcal{F}_{\varepsilon_k}(v_{\varepsilon_k}) = \lim_\ell \mathcal{F}_{\varepsilon_{k_\ell}}(v_{\varepsilon_{k_\ell}})$. Hence, it is $v_{\varepsilon_{k_\ell}} \in W_{\Gamma_{\text{tr}}^+ \cup \Gamma_{\text{tr}}^-}^{1,2}(\Omega_{\text{tr}}^+ \cup B^{\varepsilon\text{-perf}} \cup \Omega_{\text{tr}}^-)$ for all $\ell \in \mathbb{N}$.

From the constitutive assumption (W3) on the energy densities W^\pm and W^{interf} as stated in Section 3.2 and the boundedness of $(\mathcal{F}_{\varepsilon_{k_\ell}}(v_{\varepsilon_{k_\ell}}))_\ell$, it is now easily inferred that the sequence $(v_{\varepsilon_{k_\ell}})_\ell$ satisfies the a priori estimates

$$\|\nabla v_{\varepsilon_{k_\ell}}\|_{L^2(\Omega_{\text{tr}}^\pm)}^2 \leq C \quad (3.22)$$

and

$$\|\widehat{\nabla} v_{\varepsilon_{k_\ell}}\|_{L^2(B^{\varepsilon\text{-perf}}; \mathbb{R}^2)}^2 + \left\| \frac{1}{\varepsilon_{k_\ell}} \partial_3 v_{\varepsilon_{k_\ell}} \right\|_{L^2(B^{\varepsilon\text{-perf}})}^2 \leq C \quad (3.23)$$

for all $\ell \in \mathbb{N}$ and some positive constant $C > 0$. At this point, it turns out useful to extend the sequence $(v_{\varepsilon_{k_\ell}})_\ell, v_{\varepsilon_{k_\ell}} \in W_{\Gamma_{\text{tr}}^+ \cup \Gamma_{\text{tr}}^-}^{1,2}(\Omega_{\text{tr}}^+ \cup B^{\varepsilon\text{-perf}} \cup \Omega_{\text{tr}}^-)$, to a sequence in the ε_k -independent space $W_{\Gamma_{\text{tr}}^+ \cup \Gamma_{\text{tr}}^-}^{1,2}(\Omega^+ \cup B \cup \Omega^-)$. This can be done by setting

$$u_{\varepsilon_{k_\ell}}(x) := \begin{cases} v_{\varepsilon_{k_\ell}}(x) & \text{if } x \in \Omega_{\text{tr}}^+ \cup \Omega_{\text{tr}}^-, \\ (P_{\varepsilon_{k_\ell}} v_{\varepsilon_{k_\ell}})(x) & \text{if } x \in B^{\varepsilon\text{-perf}} \end{cases}$$

where $P_{\varepsilon_{k_\ell}} : W^{1,2}(B^{\varepsilon\text{-perf}}) \rightarrow W^{1,2}(B)$ is the extension operator defined in (3.10). Moreover, by the above a priori-estimates (3.22) and (3.23) on $(v_{\varepsilon_{k_\ell}})_\ell$ and the strong convergence $v_{\varepsilon_{k_\ell}} \rightarrow v_0$ in $L^2(\Omega_{\text{tr}}^+ \cup B \cup \Omega_{\text{tr}}^-)$ it is easily verified that indeed

$$u_{\varepsilon_{k_\ell}}|_{\Omega_{\text{tr}}^\pm} \equiv v_{\varepsilon_{k_\ell}}|_{\Omega_{\text{tr}}^\pm} \rightarrow v_0|_{\Omega_{\text{tr}}^\pm} \quad \text{in } W_{\Gamma_{\text{tr}}^\pm}^{1,2}(\Omega_{\text{tr}}^\pm) \quad (3.24)$$

and from Lemma 3.9 one obtains that

$$u_{\varepsilon_{k_\ell}}|_B \rightarrow v_0|_B \quad \text{in } W^{1,2}(B) \quad \text{and} \quad v_0 \text{ is independent of } x_3 \text{ in } B.$$

In particular, $v_0|_{\Omega_{\text{tr}}^\pm} \in W_{\Gamma_{\text{tr}}^\pm}^{1,2}(\Omega_{\text{tr}}^\pm)$ and $v_0|_B \in W^{1,2}(B)$. Since by construction $u_{\varepsilon_{k_\ell}}|_{B^{\varepsilon\text{-perf}}} \equiv v_{\varepsilon_{k_\ell}}|_{B^{\varepsilon\text{-perf}}}$ for all $\ell \in \mathbb{N}$, the traces of $u_{\varepsilon_{k_\ell}}|_{\Omega_{\text{tr}}^\pm}$ and $u_{\varepsilon_{k_\ell}}|_B$ on the upper and lower faces of B (i.e. on $\omega \times \{\pm \frac{1}{2}\}$, which is completely contained in $B^{\varepsilon\text{-perf}}$), coincide. By the continuity of the trace operator, this yields equality of the

traces of $v_0|_{\Omega_{\text{tr}}^\pm}$ and $v_0|_B$ on the upper and lower boundaries of B and therefore $v_0 \in W_{\Gamma_{\text{tr}}^+ \cup \Gamma_{\text{tr}}^-}^{1,2}(\Omega^+ \cup B \cup \Omega^-)$. Recalling the independence of v_0 of x_3 in B as established above, one can conclude $v_0 \in \mathcal{V}_{\Gamma^+, \Gamma^-}(\Omega^+, \Omega^-, B)$.

As concerns the limiting behavior of the rescaled dissipation potential in the bodies adjacent to the interface $B^{\varepsilon\text{-perf}}$, standard lower semicontinuity arguments for convex integral functionals, together with the construction $u_{\varepsilon_{k_\ell}}|_{\Omega_{\text{tr}}^\pm} = v_{\varepsilon_{k_\ell}}|_{\Omega_{\text{tr}}^\pm}$, the weak convergences (3.24) and the assumed convexity of W^\pm (see (W2)), lead to

$$\begin{aligned} \int_{\Omega_{\text{tr}}^\pm} W^\pm(\nabla v_0(x)) \, dx &\leq \liminf_{\ell \rightarrow \infty} \int_{\Omega_{\text{tr}}^\pm} W^\pm(\nabla u_{\varepsilon_{k_\ell}}(x)) \, dx \\ &= \liminf_{\ell \rightarrow \infty} \int_{\Omega_{\text{tr}}^\pm} W^\pm(\nabla v_{\varepsilon_{k_\ell}}(x)) \, dx \end{aligned} \quad (3.25)$$

Whereas the asymptotics of the interface term in $\mathcal{F}_{\varepsilon_{k_\ell}}$ requires some more attention. Thanks to (3.23) and the construction of $u_{\varepsilon_{k_\ell}} = P_{\varepsilon_{k_\ell}} v_{\varepsilon_{k_\ell}}$ in B , by the first assertion of Lemma 3.9 one knows that the sequence $(u_{\varepsilon_{k_\ell}}|_B)_k$ satisfies assumption (3.13) in $B = \omega \times [-\frac{1}{2}, \frac{1}{2}]$. Hence, by the compactness result stated in Theorem 3.8 one can find a subsequence (here without loss of generality again $(\varepsilon_{k_\ell})_\ell$) and some $U_0 \in L^2(\omega; W_{\widehat{Y}\text{-per}}^{1,2}(Y; \mathbb{R}^3))$ such that

$$\begin{aligned} \left[\widehat{\nabla} u_{\varepsilon_{k_\ell}}, \frac{1}{\varepsilon_{k_\ell}} \partial_3 u_{\varepsilon_{k_\ell}} \right] &\xrightarrow{\widehat{Y}} \left[\widehat{\nabla} v_0(x) + \widehat{\nabla}_y U_0(\widehat{x}, (\widehat{y}, x_3)), \partial_{y_3} U_0(\widehat{x}, (\widehat{y}, x_3)) \right] \\ &\text{in } L^2(B \times \widehat{Y}; \mathbb{R}^3), \end{aligned}$$

According to the definition of two-scale convergence for in-plane oscillations, this means that

$$\begin{aligned} &\left[\widehat{\mathcal{T}}_{\varepsilon_{k_\ell}}(\widehat{\nabla} u_{\varepsilon_{k_\ell}}), \widehat{\mathcal{T}}_{\varepsilon_{k_\ell}}\left(\frac{1}{\varepsilon_{k_\ell}} \partial_3 u_{\varepsilon_{k_\ell}}\right) \right] \\ &\rightarrow \left[\widehat{\nabla} v_0(x) + \widehat{\nabla}_y U_0(\widehat{x}, (\widehat{y}, x_3)), \partial_{y_3} U_0(\widehat{x}, (\widehat{y}, x_3)) \right] \quad \text{in } L^2(\mathbb{R}^3 \times \widehat{Y}; \mathbb{R}^3), \end{aligned} \quad (3.26)$$

where $\widehat{\mathcal{T}}_{\varepsilon_{k_\ell}}$ is the corresponding unfolding operator for in-plane oscillations from Definition 3.4. While from (3.16) and (3.20) it follows that

$$\left(\widehat{\mathcal{T}}_{\varepsilon_{k_\ell}} \mathbb{1}_{B^{\varepsilon\text{-perf}}} \right) (x, \widehat{y}) = \mathbb{1}_B(x) \mathbb{1}_{Y^*}(\widehat{y}, x_3) \quad (3.27)$$

the isometry property of the unfolding operator for in-plane oscillations $\widehat{\mathcal{T}}_{\varepsilon_{k_\ell}}$ stated in

Proposition 3.6 gives

$$\begin{aligned}
& \int_{B^{\varepsilon\text{-perf}}} W^{\text{interf}} \left(\widehat{\nabla} v_{\varepsilon_{k_\ell}}(x), \frac{1}{\varepsilon_{k_\ell}} \partial_3 v_{\varepsilon_{k_\ell}}(x) \right) dx \\
&= \int_B \mathbb{1}_{B^{\varepsilon\text{-perf}}}(x) W^{\text{interf}} \left(\widehat{\nabla} u_{\varepsilon_{k_\ell}}(x), \frac{1}{\varepsilon_{k_\ell}} \partial_3 u_{\varepsilon_{k_\ell}}(x) \right) dx \\
&= \int_{\mathbb{R}^3} \int_{\widehat{Y}} (\widehat{\mathcal{T}}_{\varepsilon_{k_\ell}} \mathbb{1}_{B^{\varepsilon\text{-perf}}})(x, \widehat{y}) \\
&\quad W^{\text{interf}} \left(\widehat{\mathcal{T}}_{\varepsilon_{k_\ell}} (\widehat{\nabla} u_{\varepsilon_{k_\ell}})(x, \widehat{y}), \widehat{\mathcal{T}}_{\varepsilon_{k_\ell}} \left(\frac{1}{\varepsilon_{k_\ell}} \partial_3 u_{\varepsilon_{k_\ell}} \right)(x, \widehat{y}) \right) d\widehat{y} dx \\
&= \int_B \int_{\widehat{Y}} \mathbb{1}_{Y^*}(\widehat{y}, x_3) \\
&\quad W^{\text{interf}} \left(\widehat{\mathcal{T}}_{\varepsilon_{k_\ell}} (\widehat{\nabla} u_{\varepsilon_{k_\ell}})(x, \widehat{y}), \widehat{\mathcal{T}}_{\varepsilon_{k_\ell}} \left(\frac{1}{\varepsilon_{k_\ell}} \partial_3 u_{\varepsilon_{k_\ell}} \right)(x, \widehat{y}) \right) d\widehat{y} dx.
\end{aligned}$$

Now, the weak convergence (3.26), the assumed convexity of W^{interf} and standard lower semicontinuity arguments for convex integral functionals allow to pass to the limit $\varepsilon_{k_\ell} \rightarrow 0$, resulting in

$$\begin{aligned}
& \liminf_{\ell \rightarrow \infty} \int_{B^{\varepsilon\text{-perf}}} W^{\text{interf}} \left(\widehat{\nabla} v_{\varepsilon_{k_\ell}}(x), \frac{1}{\varepsilon_{k_\ell}} \partial_3 v_{\varepsilon_{k_\ell}}(x) \right) dx \\
&\geq \int_B \int_{\widehat{Y}} \mathbb{1}_{Y^*}(\widehat{y}, x_3) \\
&\quad W^{\text{interf}} \left(\widehat{\nabla} v_0(x) + \widehat{\nabla}_y U_0(\widehat{x}, (\widehat{y}, x_3)), \partial_{y_3} U_0(\widehat{x}, (\widehat{y}, x_3)) \right) d\widehat{y} dx \\
&= \int_\omega \int_{(-\frac{1}{2}, \frac{1}{2})} \int_{\widehat{Y}} \mathbb{1}_{Y^*}(\widehat{y}, x_3) \\
&\quad W^{\text{interf}} \left(\widehat{\nabla} v_0(\widehat{x}, x_3) + \widehat{\nabla}_y U_0(\widehat{x}, (\widehat{y}, x_3)), \partial_{y_3} U_0(\widehat{x}, (\widehat{y}, x_3)) \right) d\widehat{y} dx_3 d\widehat{x}.
\end{aligned}$$

Since v_0 is independent of x_3 in B as stated previously, the expression on the right hand

side can be further simplified like (recall that $Y = \widehat{Y} \times [-\frac{1}{2}, \frac{1}{2}]$)

$$\begin{aligned}
& \liminf_{\ell \rightarrow \infty} \int_{B^{\varepsilon\text{-perf}}} W^{\text{interf}} \left(\widehat{\nabla} v_{\varepsilon_{k_\ell}}(x), \frac{1}{\varepsilon_{k_\ell}} \partial_3 v_{\varepsilon_{k_\ell}}(x) \right) dx \\
& \geq \int_{\omega} \left(\int_{(-\frac{1}{2}, \frac{1}{2})} \int_{\widehat{Y}} \mathbb{1}_{Y^*}(\widehat{y}, x_3) \right. \\
& \quad \left. W^{\text{interf}} \left(\widehat{\nabla} v_0(\widehat{x}, 0) + \widehat{\nabla}_y U_0(\widehat{x}, (\widehat{y}, x_3)), \partial_{y_3} U_0(\widehat{x}, (\widehat{y}, x_3)) \right) d\widehat{y} dx_3 \right) d\widehat{x} \\
& = \int_{\omega} \left(\int_Y \mathbb{1}_{Y^*}(y) \right. \\
& \quad \left. W^{\text{interf}} \left(\widehat{\nabla} v_0(\widehat{x}, 0) + \widehat{\nabla}_y U_0(\widehat{x}, y), \partial_{y_3} U_0(\widehat{x}, y) \right) dy \right) d\widehat{x} \\
& \geq \int_{\omega} \inf \left\{ \int_Y \mathbb{1}_{Y^*}(y) \right. \\
& \quad \left. W^{\text{interf}} \left([\widehat{\nabla} v_0(\widehat{x}, 0), 0]^T + \nabla_y w(y) \right) dy : w \in W_{\widehat{Y}\text{-per}}^{1,2}(Y) \right\} d\widehat{x}.
\end{aligned}$$

Finally, by means of (3.8) the last inequality reveals that

$$\begin{aligned}
\liminf_{\ell \rightarrow \infty} \int_{B^{\varepsilon\text{-perf}}} W^{\text{interf}} \left(\widehat{\nabla} v_{\varepsilon_{k_\ell}}(x), \frac{1}{\varepsilon_{k_\ell}} \partial_3 v_{\varepsilon_{k_\ell}}(x) \right) dx \\
\geq \int_{\omega} \widehat{W}^{\text{interf}} \left(\widehat{\nabla} v_0(\widehat{x}, 0) \right) d\widehat{x}. \quad (3.28)
\end{aligned}$$

To conclude, from (3.25), (3.28) and the fact that $v_0 \in \mathcal{V}_{\Gamma^+, \Gamma^-}(\Omega^+, \Omega^-, B)$ it follows that

$$\begin{aligned}
& \liminf_{k \rightarrow \infty} \mathcal{F}_{\varepsilon_k}(v_{\varepsilon_k}) = \liminf_{\ell \rightarrow \infty} \mathcal{F}_{\varepsilon_{k_\ell}}(v_{\varepsilon_{k_\ell}}) \\
& = \liminf_{\ell \rightarrow \infty} \left(\int_{\Omega_{\text{tr}}^+} W^+(\nabla v_{\varepsilon_{k_\ell}}(x)) dx + \int_{\Omega_{\text{tr}}^-} W^-(\nabla v_{\varepsilon_{k_\ell}}(x)) dx \right. \\
& \quad \left. + \int_{B^{\varepsilon\text{-perf}}} W^{\text{interf}} \left(\widehat{\nabla} v_{\varepsilon_{k_\ell}}(x), \frac{1}{\varepsilon_{k_\ell}} \partial_3 v_{\varepsilon_{k_\ell}}(x) \right) dx \right) \\
& \geq \int_{\Omega_{\text{tr}}^+} W^+(\nabla v_0(x)) dx + \int_{\Omega_{\text{tr}}^-} W^-(\nabla v_0(x)) dx \\
& \quad + \int_{\omega} \widehat{W}^{\text{interf}} \left(\widehat{\nabla} v_0(\widehat{x}, 0) \right) d\widehat{x} \\
& = \mathcal{F}_{\text{Hom}}(v_0).
\end{aligned}$$

In other words, the functional \mathcal{F}_{Hom} stated in Theorem 3.1 satisfies the Γ -lim inf-inequality for the sequence of rescaled dissipation potentials $(\mathcal{F}_{\varepsilon_k})_k$ as claimed.

Step 2. Γ -lim sup-inequality. Let there be given some $v_0 \in L^2(\Omega_{\text{tr}}^+ \cup B \cup \Omega_{\text{tr}}^-)$ and assume without loss of generality $\mathcal{F}_{\text{Hom}}(v_0) < \infty$. Thus $v_0 \in \mathcal{V}_{\Gamma^+, \Gamma^-}(\Omega^+, \Omega^-, B)$ meaning $v_0 \in W_{\Gamma^+ \cup \Gamma^-}^{1,2}(\Omega_{\text{tr}}^+ \cup B \cup \Omega_{\text{tr}}^-)$ and $\partial_3 v_0|_B = 0$. Regarding the definition of $\widehat{W}^{\text{interf}}$ stated in Theorem 3.1 and the coercivity and convexity assumptions (W2) and (W3) made on W^{interf} in Section 3.2, it is easily seen that the set

$$\Lambda(\widehat{x}) := \left\{ w : w \in W_{\widehat{Y}\text{-per}}^{1,2}(Y^*), \int_{Y^*} w(y) dy = 0, \right. \\ \left. \int_{Y^*} W^{\text{interf}}([\widehat{\nabla} v_0(\widehat{x}, 0), 0]^T + \nabla_y w(y)) dy = \widehat{W}^{\text{interf}}(\widehat{\nabla} v_0(\widehat{x}, 0)) \right\}$$

is nonempty for all $\widehat{x} \in \omega$. From Proposition 6.3 and Theorem 6.5 of [Fonseca and Leoni \[2007\]](#) one obtains the existence of a measurable selection $V_0 : \omega \rightarrow W_{\widehat{Y}\text{-per}}^{1,2}(Y^*)$, and from the coercivity (W3) of W^{interf} and Poincaré's inequality one can deduce that in fact $V_0 \in L^2(\omega; W_{\widehat{Y}\text{-per}}^{1,2}(Y^*))$. Using the extension operator $P : W^{1,2}(Y^*) \rightarrow W^{1,2}(Y)$ from Proposition 3.2, the selection V_0 can be extended to some $U_0 : \omega \rightarrow W_{\widehat{Y}\text{-per}}^{1,2}(Y)$ by setting

$$U_0(\widehat{x}, y) := (PV_0(\widehat{x}, \cdot))(y),$$

and Proposition 3.2 also reveals that $U_0 \in L^2(\omega; W_{\widehat{Y}\text{-per}}^{1,2}(Y))$. Recalling that P is an extension operator implies $U_0(\widehat{x}, y) = V_0(\widehat{x}, y)$ for all $y \in Y^*$, $\widehat{x} \in \omega$, and a simple calculation shows

$$\begin{aligned} & \int_Y \mathbb{1}_{Y^*}(y) W^{\text{interf}}([\widehat{\nabla} v_0(\widehat{x}, 0), 0]^T + \nabla_y U_0(\widehat{x}, y)) dy \\ &= \int_Y \mathbb{1}_{Y^*}(y) W^{\text{interf}}([\widehat{\nabla} v_0(\widehat{x}, 0), 0]^T + \nabla_y (PV_0(\widehat{x}, \cdot))(y)) dy \\ &= \int_Y \mathbb{1}_{Y^*}(y) W^{\text{interf}}([\widehat{\nabla} v_0(\widehat{x}, 0), 0]^T + \nabla_y V_0(\widehat{x}, y)) dy \\ &= \widehat{W}^{\text{interf}}(\widehat{\nabla} v_0(\widehat{x}, 0)). \end{aligned} \quad (3.29)$$

For δ some small positive real number let $\Psi_\delta \in C_c^\infty(\omega; C_{\widehat{Y}\text{-per}}^\infty(Y))$ be such that

$$\|\Psi_\delta - U_0\|_{L^2(\omega \times Y)} + \|\nabla_y \Psi_\delta - \nabla_y U_0\|_{L^2(\omega \times Y; \mathbb{R}^3)} \leq \delta. \quad (3.30)$$

Furthermore, assume every $\Psi_\delta(\widehat{x}, \cdot)$ to be extended from $Y = \widehat{Y} \times [-\frac{1}{2}, \frac{1}{2}]$ to the larger cuboid $\widehat{Y} \times [-\frac{3}{2}, \frac{3}{2}]$ by reflection w.r.t. the upper and lower faces of Y , i.e. $\widehat{Y} \times \{-\frac{1}{2}\}$ and $\widehat{Y} \times \{\frac{1}{2}\}$. Let $\rho_\delta \in C_c^\infty(-\frac{3}{2}, \frac{3}{2})$ be a cut-off function with values in $[0, 1]$ such that it equals 1 on $[-\frac{1}{2}, \frac{1}{2}]$ and has its support contained in $(-\frac{1}{2} - \delta, \frac{1}{2} + \delta)$. Finally, set

$$\Phi_\delta : \omega \times \left(\widehat{Y} \times \left[-\frac{3}{2}, \frac{3}{2}\right]\right) \rightarrow \mathbb{R}, \quad \Phi_\delta(\widehat{x}, y) := \rho_\delta(y_3) \Psi_\delta(\widehat{x}, y)$$

and define a sequence $v_{\varepsilon_k, \delta} \in W_{\Gamma_{\text{tr}}^+ \cup \Gamma_{\text{tr}}^-}^{1,2}(\Omega_{\text{tr}}^+ \cup B \cup \Omega_{\text{tr}}^-)$ through

$$v_{\varepsilon_k, \delta}(x) := v_0(x) + \varepsilon_k \Phi_\delta \left(\widehat{x}, \left(\frac{\widehat{x}}{\varepsilon_k}, x_3 \right) \right), \quad x \in \Omega_{\text{tr}}^+ \cup B \cup \Omega_{\text{tr}}^-.$$

Obviously then

$$v_{\varepsilon_k, \delta} \rightarrow v_0 \quad \text{in } L^2(\Omega_{\text{tr}}^+ \cup B \cup \Omega_{\text{tr}}^-) \text{ as } \varepsilon_k \rightarrow 0. \quad (3.31)$$

Also, for $x \in \Omega_{\text{tr}}^+ \cup B \cup \Omega_{\text{tr}}^-$ a simple calculation shows that

$$\begin{aligned} \widehat{\nabla} v_{\varepsilon_k, \delta}(x) &= \widehat{\nabla} v_0(x) \\ &\quad + \varepsilon_k \widehat{\nabla}_x \Phi_\delta \left(\widehat{x}, \left(\frac{\widehat{x}}{\varepsilon_k}, x_3 \right) \right) + \widehat{\nabla}_y \Phi_\delta \left(\widehat{x}, \left(\frac{\widehat{x}}{\varepsilon_k}, x_3 \right) \right) \\ &= \widehat{\nabla} v_0(x) \\ &\quad + \varepsilon_k \rho_\delta(x_3) \widehat{\nabla}_x \Psi_\delta \left(\widehat{x}, \left(\frac{\widehat{x}}{\varepsilon_k}, x_3 \right) \right) + \rho_\delta(x_3) \widehat{\nabla}_y \Psi_\delta \left(\widehat{x}, \left(\frac{\widehat{x}}{\varepsilon_k}, x_3 \right) \right), \end{aligned} \quad (3.32)$$

and

$$\begin{aligned} \partial_3 v_{\varepsilon_k, \delta}(x) &= \partial_3 v_0(x) \\ &\quad + \varepsilon_k \partial_{y_3} \Phi_\delta \left(\widehat{x}, \left(\frac{\widehat{x}}{\varepsilon_k}, x_3 \right) \right) \\ &= \partial_3 v_0(x) \\ &\quad + \varepsilon_k \rho'_\delta(x_3) \Psi_\delta \left(\widehat{x}, \left(\frac{\widehat{x}}{\varepsilon_k}, x_3 \right) \right) + \varepsilon_k \rho_\delta(x_3) \partial_{y_3} \Psi_\delta \left(\widehat{x}, \left(\frac{\widehat{x}}{\varepsilon_k}, x_3 \right) \right). \end{aligned} \quad (3.33)$$

In view of $\partial_3 v_0 = 0$ in B and $\rho_\delta = 1$ in $[-\frac{1}{2}, \frac{1}{2}]$ the above derivatives simplify for $x = (\widehat{x}, x_3) \in B = \omega \times [-\frac{1}{2}, \frac{1}{2}]$. That is,

$$\begin{aligned} \widehat{\nabla} v_{\varepsilon_k, \delta}(x) &= \widehat{\nabla} v_0(x) \\ &\quad + \varepsilon_k \widehat{\nabla}_x \Psi_\delta \left(\widehat{x}, \left(\frac{\widehat{x}}{\varepsilon_k}, x_3 \right) \right) + \widehat{\nabla}_y \Psi_\delta \left(\widehat{x}, \left(\frac{\widehat{x}}{\varepsilon_k}, x_3 \right) \right) \quad \text{for } x \in B, \end{aligned} \quad (3.34)$$

$$\partial_3 v_{\varepsilon_k, \delta}(x) = \varepsilon_k \partial_{y_3} \Psi_\delta \left(\widehat{x}, \left(\frac{\widehat{x}}{\varepsilon_k}, x_3 \right) \right) \quad \text{for } x \in B. \quad (3.35)$$

Inserting $v_{\varepsilon_k, \delta}$ into the corresponding rescaled dissipation potential $\mathcal{F}_{\varepsilon_k}$ and passing to the limit $\varepsilon_k \rightarrow 0$ leads to the following results. Regarding the limiting behavior in the bodies Ω_{tr}^+ and Ω_{tr}^- adjacent to the rescaled interface $B^{\varepsilon\text{-perf}}$, one first notices that by (3.32), (3.33) and the smoothness of Ψ_δ and ρ_δ

$$\begin{aligned} \widehat{\nabla} v_{\varepsilon_k, \delta} &\xrightarrow[\widehat{Y}]{2} \widehat{\nabla} v_0 + \rho_\delta(x_3) \widehat{\nabla}_y \Psi_\delta(\widehat{x}, (\widehat{y}, x_3)) && \text{in } L^2(\Omega_{\text{tr}}^\pm \times \widehat{Y}; \mathbb{R}^2), \\ \partial_3 v_{\varepsilon_k, \delta} &\rightarrow \partial_3 v_0 && \text{in } L^2(\Omega_{\text{tr}}^\pm). \end{aligned}$$

as $\varepsilon_k \rightarrow 0$. From the above strong two-scale convergence in $L^2(\Omega_{\text{tr}}^\pm \times \widehat{Y}; \mathbb{R}^2)$, the isometry property of the unfolding operator for in-plane oscillations $\mathcal{T}_{\varepsilon_k} : L^2(\Omega_{\text{tr}}^\pm) \rightarrow$

$L^2(\Omega_{\text{tr}}^\pm \times \widehat{Y})$ (see Proposition 3.6) and the continuity and growth of W^\pm (given the assumptions (W2) and (W3); continuity from the convexity of W^\pm on the entire \mathbb{R}^3) it follows that

$$\begin{aligned} & \int_{\Omega_{\text{tr}}^\pm} W^\pm(\nabla v_{\varepsilon_k, \delta}(x)) \, dx \\ &= \int_{\mathbb{R}^3} \int_{\widehat{Y}} \mathbf{1}_{\Omega_{\text{tr}}^\pm}(\varepsilon_k \lfloor \frac{\widehat{x}}{\varepsilon_k} \rfloor + \varepsilon_k \widehat{y}, x_3) \\ & \quad W^\pm\left(\widehat{\mathcal{T}}_{\varepsilon_k}(\widehat{\nabla} v_{\varepsilon_k, \delta})(x, \widehat{y}), \widehat{\mathcal{T}}_{\varepsilon_k}(\partial_3 v_{\varepsilon_k, \delta})(x, \widehat{y})\right) \, d\widehat{y} \, dx \\ & \rightarrow \int_{\Omega_{\text{tr}}^\pm} \int_{\widehat{Y}} W^\pm\left(\widehat{\nabla} v_0(x) + \rho_\delta(x_3) \widehat{\nabla}_y \Psi_\delta(\widehat{x}, (\widehat{y}, x_3)), \partial_3 v_0(x)\right) \, d\widehat{y} \, dx \end{aligned} \quad (3.36)$$

as ε_k vanishes. Since Ψ_δ has been extended to $\omega \times (-\frac{3}{2}, \frac{3}{2})$ by reflection arguments, it follows from (3.30) that

$$\widehat{\nabla}_y \Psi_\delta(\widehat{x}, (\widehat{y}, x_3)) \rightarrow \nabla_y U_0 \quad \text{in } L^2\left((\Omega_{\text{tr}}^\pm \cap \{\frac{1}{2} \leq \pm x_3 \leq \frac{3}{2}\}) \times \widehat{Y}; \mathbb{R}^3\right)$$

as the small parameter δ vanishes. Also, by assumption it is $\text{supp } \rho_\delta \subseteq (-\frac{1}{2} - \delta, \frac{1}{2} + \delta)$. Thus, $\rho_\delta(x_3) \widehat{\nabla}_y \Psi_\delta(\widehat{x}, (\widehat{y}, x_3))$ as a function of $L^2((\Omega_{\text{tr}}^\pm \cap \{\frac{1}{2} \leq \pm x_3 \leq \frac{3}{2}\}) \times \widehat{Y})$ has its support contained in $(\Omega_{\text{tr}}^\pm \cap \{\frac{1}{2} \leq \pm x_3 \leq \frac{1}{2} + \delta\}) \times \widehat{Y}$ and therefore

$$\rho_\delta(x_3) \widehat{\nabla}_y \Psi_\delta(\widehat{x}, (\widehat{y}, x_3)) \rightarrow 0 \quad \text{in measure over } (\Omega_{\text{tr}}^\pm \cap \{\frac{1}{2} \leq \pm x_3 \leq \frac{3}{2}\}) \times \widehat{Y}$$

as δ tends to 0. From these two observations one infers that

$$\rho_\delta(x_3) \widehat{\nabla}_y \Psi_\delta(\widehat{x}, (\widehat{y}, x_3)) \rightarrow 0 \quad \text{in } L^2\left((\Omega_{\text{tr}}^\pm \cap \{\frac{1}{2} \leq \pm x_3 \leq \frac{3}{2}\}) \times \widehat{Y}; \mathbb{R}^3\right)$$

as δ becomes smaller and smaller, thus also L^2 -strongly as a function of $L^2(\Omega_{\text{tr}}^\pm \times \widehat{Y})$. In other words, for $(\delta_\ell)_\ell$ some vanishing sequence of positive real numbers one infers from (3.36)

$$\begin{aligned} & \lim_{\ell \rightarrow \infty} \left(\lim_{k \rightarrow \infty} \int_{\Omega_{\text{tr}}^\pm} W^\pm(\nabla v_{\varepsilon_k, \delta_\ell}(x)) \, dx \right) \\ & \quad \int_{\Omega_{\text{tr}}^\pm} \int_{\widehat{Y}} W^\pm\left(\widehat{\nabla} v_0(x), \partial_3 v_0(x)\right) \, d\widehat{y} \, dx = \int_{\Omega_{\text{tr}}^\pm} W^\pm(\nabla v_0(x)) \, dx. \end{aligned} \quad (3.37)$$

When it comes to the limiting behavior of the interface term in $\mathcal{F}_{\varepsilon_k}$, one observes that from (3.34) and (3.35) it follows

$$\begin{aligned} \widehat{\nabla} v_{\varepsilon_k, \delta} & \xrightarrow{\frac{2}{\widehat{Y}}} \widehat{\nabla} v_0(x) + \widehat{\nabla}_y \Psi_\delta(\widehat{x}, (\widehat{y}, x_3)) & \text{in } L^2(B \times \widehat{Y}; \mathbb{R}^2), \\ \frac{1}{\varepsilon_k} \partial_3 v_{\varepsilon_k, \delta} & \xrightarrow{\frac{2}{\widehat{Y}}} \partial_{y_3} \Psi_\delta(\widehat{x}, (\widehat{y}, x_3)) & \text{in } L^2(B \times \widehat{Y}) \end{aligned}$$

as ε_k vanishes. Employing once more the isometry property of the unfolding operator for in-plane oscillations $\widehat{\mathcal{T}}_{\varepsilon_k} : L^2(B) \rightarrow L^2(B \times \widehat{Y})$ and recalling (3.27) these convergences yield

$$\begin{aligned}
& \int_{B^{\varepsilon\text{-perf}}} W^{\text{interf}} \left(\widehat{\nabla} v_{\varepsilon_k, \delta}(x), \frac{1}{\varepsilon_k} \partial_3 v_{\varepsilon_k, \delta}(x) \right) dx \\
&= \int_B \mathbb{1}_{B^{\varepsilon\text{-perf}}}(x) W^{\text{interf}} \left(\widehat{\nabla} v_{\varepsilon_k, \delta}(x), \frac{1}{\varepsilon_k} \partial_3 v_{\varepsilon_k, \delta}(x) \right) dx \\
&= \int_{\mathbb{R}^3} \int_{\widehat{Y}} (\widehat{\mathcal{T}}_{\varepsilon_k} \mathbb{1}_{B^{\varepsilon\text{-perf}}})(x, \widehat{y}) \\
&\quad W^{\text{interf}} \left(\widehat{\mathcal{T}}_{\varepsilon_k} (\widehat{\nabla} v_{\varepsilon_k, \delta})(x, \widehat{y}), \widehat{\mathcal{T}}_{\varepsilon_k} \left(\frac{1}{\varepsilon_k} \partial_3 v_{\varepsilon_k, \delta} \right)(x, \widehat{y}) \right) d\widehat{y} dx \\
&= \int_B \int_{\widehat{Y}} \mathbb{1}_{Y^*}(\widehat{y}, x_3) \\
&\quad W^{\text{interf}} \left(\widehat{\mathcal{T}}_{\varepsilon_k} (\widehat{\nabla} v_{\varepsilon_k, \delta})(x, \widehat{y}), \widehat{\mathcal{T}}_{\varepsilon_k} \left(\frac{1}{\varepsilon_k} \partial_3 v_{\varepsilon_k, \delta} \right)(x, \widehat{y}) \right) d\widehat{y} dx \\
&\rightarrow \int_B \int_{\widehat{Y}} \mathbb{1}_{Y^*}(\widehat{y}, x_3) \\
&\quad W^{\text{interf}} \left(\widehat{\nabla} v_0(x) + \widehat{\nabla}_y \Psi_\delta(\widehat{x}, (\widehat{y}, x_3)), \partial_{y_3} \Psi_\delta(\widehat{x}, (\widehat{y}, x_3)) \right) d\widehat{y} dx
\end{aligned}$$

as ε_k tends to zero. Recalling that v_0 is independent of x_3 in B , this can be further simplified like

$$\begin{aligned}
& \lim_{k \rightarrow \infty} \int_{B^{\varepsilon\text{-perf}}} W^{\text{interf}} \left(\widehat{\nabla} v_{\varepsilon_k, \delta}(x), \frac{1}{\varepsilon_k} \partial_3 v_{\varepsilon_k, \delta}(x) \right) dx \\
&= \int_\omega \left(\int_{(-\frac{1}{2}, \frac{1}{2})} \int_{\widehat{Y}} \mathbb{1}_{Y^*}(\widehat{y}, x_3) \right. \\
&\quad \left. W^{\text{interf}} \left(\widehat{\nabla} v_0(\widehat{x}, 0) + \widehat{\nabla}_y \Psi_\delta(\widehat{x}, (\widehat{y}, x_3)), \partial_{y_3} \Psi_\delta(\widehat{x}, (\widehat{y}, x_3)) \right) d\widehat{y} dx_3 \right) d\widehat{x} \\
&= \int_\omega \left(\int_Y \mathbb{1}_{Y^*}(y) W^{\text{interf}} \left(\widehat{\nabla} v_0(\widehat{x}, 0) + \widehat{\nabla}_y \Psi_\delta(\widehat{x}, y), \partial_{y_3} \Psi_\delta(\widehat{x}, y) \right) dy \right) d\widehat{x}
\end{aligned}$$

Remembering (3.30) one can now pass to the limit $\delta_\ell \rightarrow 0$ and obtains

$$\begin{aligned}
& \lim_{\ell \rightarrow \infty} \left(\lim_{k \rightarrow \infty} \int_{B^{\varepsilon\text{-perf}}} W^{\text{interf}} \left(\widehat{\nabla} v_{\varepsilon_k, \delta_\ell}(x), \frac{1}{\varepsilon_k} \partial_3 v_{\varepsilon_k, \delta_\ell}(x) \right) dx \right) \\
&= \lim_{\ell \rightarrow \infty} \left(\int_{\omega} \left(\int_Y \mathbb{1}_{Y^*}(y) \right. \right. \\
&\quad \left. \left. W^{\text{interf}} \left(\widehat{\nabla} v_0(\widehat{x}, 0) + \widehat{\nabla}_y \Psi_{\delta_\ell}(\widehat{x}, y), \partial_{y_3} \Psi_{\delta_\ell}(\widehat{x}, y) \right) dy \right) d\widehat{x} \right) \\
&= \int_{\omega} \left(\int_Y \mathbb{1}_{Y^*}(y) \right. \\
&\quad \left. W^{\text{interf}} \left(\widehat{\nabla} v_0(\widehat{x}, 0) + \widehat{\nabla}_y U_0(\widehat{x}, y), \partial_{y_3} U_0(\widehat{x}, y) \right) dy \right) d\widehat{x},
\end{aligned}$$

which in view of (3.29) is nothing but

$$\begin{aligned}
& \lim_{\ell \rightarrow \infty} \left(\lim_{k \rightarrow \infty} \int_{B^{\varepsilon\text{-perf}}} W^{\text{interf}} \left(\widehat{\nabla} v_{\varepsilon_k, \delta_\ell}(x), \frac{1}{\varepsilon_k} \partial_3 v_{\varepsilon_k, \delta_\ell}(x) \right) dx \right) \\
&= \int_{\omega} \widehat{W}^{\text{interf}} \left(\widehat{\nabla} v_0(\widehat{x}, 0) \right) d\widehat{x}, \quad (3.38)
\end{aligned}$$

Hence, from (3.37) and (3.38) it follows that

$$\begin{aligned}
& \lim_{\ell \rightarrow \infty} \left(\lim_{k \rightarrow \infty} \mathcal{F}_{\varepsilon_k}(v_{\varepsilon_k, \delta_\ell}) \right) \\
&= \lim_{\ell \rightarrow \infty} \left(\lim_{k \rightarrow \infty} \int_{\Omega_{\text{tr}}^+} W^+(\nabla v_{\varepsilon_k, \delta_\ell}(x)) dx + \lim_{k \rightarrow \infty} \int_{\Omega_{\text{tr}}^-} W^-(\nabla v_{\varepsilon_k, \delta_\ell}(x)) dx \right. \\
&\quad \left. + \lim_{k \rightarrow \infty} \int_{B^{\varepsilon\text{-perf}}} W^{\text{interf}} \left(\widehat{\nabla} v_{\varepsilon_k, \delta_\ell}(x), \frac{1}{\varepsilon_k} \partial_3 v_{\varepsilon_k, \delta_\ell}(x) \right) dx \right) \\
&= \int_{\Omega_{\text{tr}}^+} W^+(\nabla v_0(x)) dx + \int_{\Omega_{\text{tr}}^-} W^-(\nabla v_0(x)) dx \\
&\quad + \int_{\omega} \widehat{W}^{\text{interf}} \left(\widehat{\nabla} v_0(\widehat{x}, 0) \right) d\widehat{x} \\
&= \mathcal{F}_{\text{Hom}}(v_0).
\end{aligned}$$

In other words, by successively passing to the limit $\varepsilon_k \rightarrow 0$ and $\delta_\ell \rightarrow 0$ the doubly indexed sequence $(\mathcal{F}_{\varepsilon_k}(v_{\varepsilon_k, \delta_\ell}))_{k, \ell}$ would recover the limit energy $\mathcal{F}_{\text{Hom}}(v_0)$ in $v_0 \in \mathcal{V}_{\Gamma^+, \Gamma^-}(\Omega^+, \Omega^-, B)$. Now, in order to obtain a recovery sequence from $(v_{\varepsilon_k, \delta_\ell})_{k, \ell}$ one has to choose a suitable diagonal sequence. This can be done with the help of [Attouch,

1984, Corollary 1.18]. To this end, define

$$\begin{aligned}
c_{\ell,k} := & \left\| v_{\varepsilon_k, \delta_\ell} - v_0 \right\|_{L^2(\Omega^+ \cup B \cup \Omega^-)} \\
& + \left| \int_{\Omega_{\text{tr}}^+} W^+(\nabla v_{\varepsilon_k, \delta_\ell}(x)) \, dx - \int_{\Omega_{\text{tr}}^+} W^+(\nabla v_0(x)) \, dx \right| \\
& + \left| \int_{\Omega_{\text{tr}}^-} W^-(\nabla v_{\varepsilon_k, \delta_\ell}(x)) \, dx - \int_{\Omega_{\text{tr}}^-} W^-(\nabla v_0(x)) \, dx \right| \\
& + \left| \int_{B^{\varepsilon\text{-perf}}} W^{\text{interf}} \left(\widehat{\nabla} v_{\varepsilon_k, \delta_\ell}(x), \frac{1}{\varepsilon_k} \partial_3 v_{\varepsilon_k, \delta_\ell}(x) \right) \, dx \right. \\
& \quad \left. - \int_{\omega} \widehat{W}^{\text{interf}} \left(\widehat{\nabla} v_0(\widehat{x}, 0) \right) \, d\widehat{x} \right|,
\end{aligned}$$

and obtain from (3.31), (3.37) and (3.38) that

$$\lim_{\ell \rightarrow \infty} \left(\lim_{k \rightarrow \infty} c_{\ell,k} \right) = 0.$$

Then, [Attouch, 1984, Corollary 1.18] states the existence of a subsequence $(\ell(k))_k$ such that

$$\lim_{k \rightarrow \infty} c_{\ell(k),k} = \lim_{\ell \rightarrow \infty} \left(\lim_{k \rightarrow \infty} c_{\ell,k} \right) = 0.$$

Finally, setting

$$v_{\varepsilon_k} := v_{\varepsilon_k, \delta_{\ell(k)}} \in W_{\Gamma_{\text{tr}}^+ \cup \Gamma_{\text{tr}}^-}^{1,2}(\Omega_{\text{tr}}^+ \cup B \cup \Omega_{\text{tr}}^-)$$

yields the desired recovery sequence for v_0 , as it is revealed by

$$\left\| v_{\varepsilon_k} - v_0 \right\|_{L^2(\Omega^+ \cup B \cup \Omega^-)} \leq c_{\ell(k),k} \quad \text{and} \quad \left| \mathcal{F}_{\varepsilon_k}(v_{\varepsilon_k}) - \mathcal{F}_{\text{Hom}}(v_0) \right| \leq c_{\ell(k),k}$$

and $\lim_k c_{\ell(k),k} = 0$. This finishes the proof of Theorem 3.1. \square

3.4 Voids touching the periodicity cell's faces

Like already outlined in Subsection 3.1.2 and depicted in Figure 3.3 (right) it remains to study the case of tubular voids that stretch (*e.g.* in x_1 -direction) over the entire interface $B_\varepsilon^{\varepsilon\text{-perf}}$. In the notation outlined in Subsection 3.2.1 this corresponds to a cylindrical void V of the form $V = [0, 1) \times H$ where $H \Subset [0, 1) \times [-\frac{1}{2}, \frac{1}{2}]$ shall be smooth, and a perforated periodicity cell $Y^* = Y \setminus \overline{V}$ like shown in Figure 3.4. In other words, the interface $B_\varepsilon^{\varepsilon\text{-perf}}$ in the assembly Ω_ε is perforated by periodically recurring, parallel tubular voids where both the mutual distance and the cross-sectional diameter of the voids are of order ε .

Just like in the previous case of voids compactly contained in the periodicity cell Y I will impose homogeneous Dirichlet boundary conditions on Γ_ε^+ and Γ_ε^- . Whereas on the boundaries of the voids enclosed in the perforated interface $B_\varepsilon^{\varepsilon\text{-perf}}$ there shall hold suitable boundary conditions to mimick the presence of heating wires filling the voids (cf. the motivation in Subsection 3.1.2). More precisely, assuming that these heating wires are held on constant temperature $T > 0$, the heat flux into the interface material occupying $B_\varepsilon^{\varepsilon\text{-perf}}$ is supposed to be proportional to the difference between wire temperature and interface temperature. Thus, given the thermal conductivity matrix $\frac{1}{\varepsilon}A^{\text{interf}}$ of the interface material as described in Subsection 3.2.2 – recall that the conductivity of the interface material was assumed to scale with the inverse of the interface thickness ε – any temperature distribution $u : \Omega_\varepsilon \rightarrow \mathbb{R}$ in the assembly Ω_ε has to obey

$$\begin{aligned} \left(\frac{1}{\varepsilon}A^{\text{interf}}\nabla u(x) \right) \cdot n(x; \partial B_\varepsilon^{\varepsilon\text{-perf}}) \\ = -\alpha(u(x) - T) \quad \text{for all } x \in \partial B_\varepsilon^{\varepsilon\text{-perf}} \cap B_\varepsilon^\circ. \end{aligned} \quad (3.39)$$

Therein, $n(\cdot; \partial B_\varepsilon^{\varepsilon\text{-perf}})$ denotes the outward unit normal to $B_\varepsilon^{\varepsilon\text{-perf}}$ and $\alpha > 0$ some positive parameter. With the generic notation $W^{\text{interf}}(F) = \frac{1}{2}A^{\text{interf}}F \cdot F$, $F \in \mathbb{R}^3$, and

$$G : \mathbb{R} \rightarrow \mathbb{R}, \quad G(v) := \frac{1}{2}\alpha(v - T)^2 \quad (3.40)$$

the boundary condition (3.39) can equivalently be written like

$$\begin{aligned} \left(\frac{1}{\varepsilon} \frac{\partial W^{\text{interf}}}{\partial F}(\nabla u(x)) \right) \cdot n(x; \partial B_\varepsilon^{\varepsilon\text{-perf}}) \\ = -\frac{\partial G}{\partial v}(u(x)) \quad \text{for all } x \in \partial B_\varepsilon^{\varepsilon\text{-perf}} \cap B_\varepsilon^\circ. \end{aligned}$$

Finally, on the other parts of the boundary of the assembly Ω_ε , *i.e.* on $(\partial\Omega_\varepsilon^\pm \cap \{x : \pm x_3 > \frac{\varepsilon}{2}\}) \setminus \Gamma_\varepsilon^\pm$ there shall simply hold homogeneous Neumann boundary conditions.

Expressing the resulting temperature distribution in the assembly Ω_ε under the above described boundary conditions by means of a dissipation potential

$$\mathcal{E}_\varepsilon : W_{\Gamma_\varepsilon^+ \cup \Gamma_\varepsilon^-}^{1,2}(\Omega_\varepsilon) \rightarrow \mathbb{R}$$

yields

$$\begin{aligned} \mathcal{E}_\varepsilon(u) := \int_{\Omega_\varepsilon^+} W^+(\nabla u(x)) \, dx + \int_{\Omega_\varepsilon^-} W^-(\nabla u(x)) \, dx \\ + \int_{B_\varepsilon^{\varepsilon\text{-perf}}} \frac{1}{\varepsilon} W^{\text{interf}}(\nabla u(x)) \, dx + \int_{\partial B_\varepsilon^{\varepsilon\text{-perf}} \cap B_\varepsilon^\circ} G(u(x)) \, da(x) \end{aligned} \quad (3.41)$$

Herein, like outlined in Subsection 3.2.2, the constitutive properties of the materials occupying the bodies Ω_ε^\pm and the interface $B_\varepsilon^{\varepsilon\text{-perf}}$ are described through generic energy

densities $W^\pm : \mathbb{R}^3 \rightarrow \mathbb{R}$ and $W^{\text{interf}} : \mathbb{R}^3 \rightarrow \mathbb{R}$. In fact, by identifying $W^\pm(F) = \frac{1}{2}A^\pm F \cdot F$ and $W^{\text{interf}}(F) = \frac{1}{2}A^{\text{interf}} F \cdot F$, $F \in \mathbb{R}^3$, with A^\pm and A^{interf} the respective positive definite and symmetric conductivity matrices, it is an easy task to show that the unique minimizer of \mathcal{E}_ε is the equilibrium temperature distribution in the assembly Ω_ε under the previously described boundary conditions (and satisfies in particular (3.39)).

In analogy to the previous situation of voids that are compactly contained in the periodicity cell, also in the present case of tubular voids enclosed in the highly conductive interface layer $B_\varepsilon^{\varepsilon\text{-perf}}$ one can anticipate the limit behavior of the dissipation potential (3.41) as the interface thickness and void diameter ε vanishes. By the very same arguments as used in Section 3.3, it is clear that the bulk terms $\int_{\Omega^\pm} W^\pm(\nabla u(x)) dx$ will enter the limit dissipation potential unchanged. Also, the homogeneous Dirichlet boundary conditions on Γ_ε^+ and Γ_ε^- are likely to translate into homogeneous Dirichlet boundary conditions on Γ^+ and Γ^- . Moreover, even in the presence of tubular voids in $B_\varepsilon^{\varepsilon\text{-perf}}$ stretching over the entire interface the interface layer still forms a connected set. Hence, the high conductivity of the interface material will again prevent heat from accumulating in the interface but rather lead to a temperature distribution that is continuous over the flattened interface ω . Also, any temperature distribution is likely to vary smoothly in interface direction ω as sharp temperature gradients would dissipate rapidly in the interface. In other words, most probably the term $\int_{B_\varepsilon^{\varepsilon\text{-perf}}} \frac{1}{\varepsilon} W^{\text{interf}}(\nabla u(x)) dx$ describing the interface layer behaves in the limit of ε becoming smaller and smaller again like some $\int_\omega \widehat{W}^{\text{interf}}(\widehat{\nabla} u(\widehat{x}, 0)) da(\widehat{x})$ for $u : \Omega \rightarrow \mathbb{R}$ a smooth temperature distribution. Clearly, the geometry of the tubular voids in $B_\varepsilon^{\varepsilon\text{-perf}}$ will enter the *homogenized* interfacial energy density $\widehat{W}^{\text{interf}} : \mathbb{R}^2 \rightarrow \mathbb{R}$, as it is clear that $\widehat{W}^{\text{interf}}$ will in general be strongly anisotropic. As concerns the source term $\int_{\partial B_\varepsilon^{\varepsilon\text{-perf}} \cap B_\varepsilon^\circ} G(u(x)) da(x)$ in the above dissipation potential \mathcal{E}_ε , the constant heat provided by heating wires leads to a preferred temperature T on the inner boundaries of the voids $\partial B_\varepsilon^{\varepsilon\text{-perf}} \cap B_\varepsilon^\circ$, with deviations from T being penalized (see the definition of G in (3.39)). Now, since the interface material in between two neighboring voids is by assumption highly conductive and the mutual distance of the voids (being of order ε) also becomes smaller and smaller, it appears reasonable to conclude that T becomes – for small interface thickness and void diameter ε – the preferred temperature in the *entire* interface. Consequently, one might expect that the source term $\int_{\partial B_\varepsilon^{\varepsilon\text{-perf}} \cap B_\varepsilon^\circ} G(u(x)) da(x)$ can asymptotically be replaced by another source term $\int_\omega \widehat{G}(u(\widehat{x}, 0)) da(\widehat{x})$ over the flattened interface ω , where $\widehat{G} : \mathbb{R} \rightarrow \mathbb{R}$ is some function penalizing deviations from the wire temperature T . In view of these arguments, one is led to conclude that in the limit of small interface thickness and void diameter ε the dissipation potential \mathcal{E}_ε behaves like

$$\mathcal{E}_{\text{Hom}} : \mathcal{V}_{\Gamma^+, \Gamma^-}(\Omega^+, \Omega^-, \omega) \rightarrow \mathbb{R}$$

where

$$\begin{aligned} \mathcal{E}_{\text{Hom}}(u) := & \int_{\Omega^+} W^+(\nabla u(x)) \, dx + \int_{\Omega^-} W^-(\nabla u(x)) \, dx \\ & + \int_{\omega} \widehat{W}^{\text{interf}}(\widehat{\nabla} u(\widehat{x}, 0)) \, da(\widehat{x}) + \int_{\omega} \widehat{G}(u(\widehat{x}, 0)) \, da(\widehat{x}). \end{aligned} \quad (3.42)$$

Indeed, suitable Γ -convergence arguments reveal this heuristic derivation to be correct; see Theorem 3.10 in the upcoming subsection.

3.4.1 Statement of the limit problem and outline of the proof

In order to make the dissipation potentials \mathcal{E}_ε for the case of tubular voids in the perforated interface layer $B_\varepsilon^{\varepsilon\text{-perf}}$ accessible to a Γ -convergence analysis, one needs to transform them such that for all possible values of the interface thickness and the void diameter ε they take arguments from one and the same topological vector space. This is once more done like in Subsection 3.3.1 by applying Ciarlet's rescaling argument to bring the interface layer $B_\varepsilon^{\varepsilon\text{-perf}}$ from thickness ε to fixed thickness 1. Let there be given some $u \in W_{\Gamma_\varepsilon^+ \cup \Gamma_\varepsilon^-}^{1,2}(\Omega_\varepsilon)$ and the change of variables $R_\varepsilon : \Omega_{\text{tr}}^+ \cup B \cup \Omega_{\text{tr}}^- \rightarrow \Omega_\varepsilon^+ \cup B_\varepsilon \cup \Omega_\varepsilon^-$ defined in (3.3). Then, set $v(x) := u(R_\varepsilon(x))$ for $x \in R_\varepsilon^{-1}(\Omega_\varepsilon) = \Omega_{\text{tr}}^+ \cup B^{\varepsilon\text{-perf}} \cup \Omega_{\text{tr}}^-$, with $B^{\varepsilon\text{-perf}} = R_\varepsilon^{-1}(B_\varepsilon^{\varepsilon\text{-perf}})$. In particular, v results again as an element of $W_{\Gamma_{\text{tr}}^+ \cup \Gamma_{\text{tr}}^-}^{1,2}(\Omega_{\text{tr}}^+ \cup B^{\varepsilon\text{-perf}} \cup \Omega_{\text{tr}}^-)$. Applying this change of variables to the single integral terms of \mathcal{E}_ε as defined in (3.41) and recalling (3.4) yields

$$\begin{aligned} \mathcal{E}_\varepsilon(u) &= \int_{\Omega_{\text{tr}}^+} W^+(\nabla u(R_\varepsilon(x))) \, dx + \int_{\Omega_{\text{tr}}^-} W^-(\nabla u(R_\varepsilon(x))) \, dx \\ &\quad + \int_{B^{\varepsilon\text{-perf}}} \frac{1}{\varepsilon} W^{\text{interf}}(\nabla u(R_\varepsilon(x))) \, \varepsilon \, dx \\ &\quad + \int_{\partial B^{\varepsilon\text{-perf}} \cap B^\circ} G(u(R_\varepsilon(x))) \left| \text{Cof}(DR_\varepsilon(x)) n(x; \partial B^{\varepsilon\text{-perf}}) \right| da(x) \\ &= \int_{\Omega_{\text{tr}}^+} W^+(\nabla v(x)) \, dx + \int_{\Omega_{\text{tr}}^-} W^-(\nabla v(x)) \, dx \\ &\quad + \int_{B^{\varepsilon\text{-perf}}} W^{\text{interf}}\left(\widehat{\nabla} v(x), \frac{1}{\varepsilon} \partial_3 v(x)\right) \, dx \\ &\quad + \int_{\partial B^{\varepsilon\text{-perf}} \cap B^\circ} G(v(x)) \left| M_\varepsilon n(x; \partial B^{\varepsilon\text{-perf}}) \right| da(x) \\ &=: \mathcal{F}_\varepsilon(v). \end{aligned}$$

Here, the change of variables in the source term $\int_{\partial B_\varepsilon^{\varepsilon\text{-perf}} \cap B_\varepsilon^\circ} G(u(x)) \, da(x)$ follows Nanson's formula (see e.g. [Ciarlet, 1988, Theorem 1.7-1]). In particular, for invertible matrices $M \in \mathbb{R}^{3 \times 3}$ the Cofactor-matrix is defined as $\text{Cof } M := (\det M) M^{-T}$ and a simple calculation shows that

$$\text{Cof}(DR_\varepsilon(x)) = \text{Cof}(\text{diag}(1, 1, \varepsilon)) = \text{diag}(\varepsilon, \varepsilon, 1) = M_\varepsilon$$

with M_ε as in (3.9). Furthermore, $n(\cdot; \partial B^{\varepsilon\text{-perf}})$ denotes the outward unit normal to $B^{\varepsilon\text{-perf}}$. Having removed the dependency on the interface thickness by the above rescaling argument, the domain of the resulting functional $\mathcal{F}_\varepsilon : W_{\Gamma_{\text{tr}}^+ \cup \Gamma_{\text{tr}}^-}^{1,2}(\Omega_{\text{tr}}^+ \cup B^{\varepsilon\text{-perf}} \cup \Omega_{\text{tr}}^-) \rightarrow \mathbb{R}$ still varies with the voids in the rescaled perforated interface $B^{\varepsilon\text{-perf}}$. This obstacle is again overcome by extending \mathcal{F}_ε to the larger space $L^2(\Omega_{\text{tr}}^+ \cup B \cup \Omega_{\text{tr}}^-)$ by ∞ . Thus, the extended and rescaled dissipation potential

$$\mathcal{F}_\varepsilon : L^2(\Omega_{\text{tr}}^+ \cup B \cup \Omega_{\text{tr}}^-) \rightarrow (-\infty, \infty]$$

becomes

$$\mathcal{F}_\varepsilon(v) := \begin{cases} \int_{\Omega_{\text{tr}}^+} W^+(\nabla v(x)) \, dx + \int_{\Omega_{\text{tr}}^-} W^-(\nabla v(x)) \, dx \\ \quad + \int_{B^{\varepsilon\text{-perf}}} W^{\text{interf}}\left(\widehat{\nabla} v(x), \frac{1}{\varepsilon} \partial_3 v(x)\right) \, dx \\ \quad + \int_{\partial B^{\varepsilon\text{-perf}} \cap B^0} G(v(x)) \Big|_{M_\varepsilon n(x; \partial B^{\varepsilon\text{-perf}})} \Big| \, da(x) & \text{if } v \in W_{\Gamma_{\text{tr}}^+ \cup \Gamma_{\text{tr}}^-}^{1,2}(\Omega_{\text{tr}}^+ \cup B^{\varepsilon\text{-perf}} \cup \Omega_{\text{tr}}^-), \\ \infty & \text{else.} \end{cases} \quad (3.43)$$

Again it should be noticed that the rescaled and extended dissipation potential \mathcal{F}_ε is still equivalent to the original dissipation potential \mathcal{E}_ε from (3.41). Yet, it is directly accessible to a Γ -convergence analysis which shows that also in the present case of tubular voids stretching over the entire interface the heuristically motivated limit (3.42) is obtained as a suitable Γ -limit for vanishing interface thickness and void diameter ε .

Theorem 3.10. *Let there be given the notation and the assumptions from Section 3.2, and suppose $(\varepsilon_k)_k$ to be a vanishing sequence of positive real numbers whose elements ε_k and ω are compatible with (ε -Tiling). Moreover, assume ω to be convex and let $V = [0, 1) \times H$ be a tubular void extending in x_1 -direction over the periodicity cell Y , leaving a perforated periodicity cell $Y^* = Y \setminus \overline{V}$. Here, $H \Subset [0, 1) \times [-\frac{1}{2}, \frac{1}{2}]$ shall have a smooth boundary. Then, the rescaled dissipation potentials $(\mathcal{F}_{\varepsilon_k})_k$ defined in (3.43) Γ -converge with respect to the $L^2(\Omega_{\text{tr}}^+ \cup B \cup \Omega_{\text{tr}}^-)$ -norm, in symbols*

$$\Gamma\text{-}\lim_{k \rightarrow \infty} \mathcal{F}_{\varepsilon_k} = \mathcal{F}_{\text{Hom}}$$

where $\mathcal{F}_{\text{Hom}} : L^2(\Omega_{\text{tr}}^+ \cup B \cup \Omega_{\text{tr}}^-) \rightarrow (-\infty, \infty]$ obeys

$$\mathcal{F}_{\text{Hom}}(v) = \begin{cases} \int_{\Omega_{\text{tr}}^+} W^+(\nabla v(x)) \, dx + \int_{\Omega_{\text{tr}}^-} W^-(\nabla v(x)) \, dx \\ \quad + \int_{\omega} \widehat{W}^{\text{interf}}\left(\widehat{\nabla} v(\widehat{x}, 0)\right) \, da(\widehat{x}) \\ \quad + \text{vol}_2(\partial V \cap \text{int } Y) \int_{\omega} G(v(\widehat{x}, 0)) \, da(\widehat{x}) & \text{if } v \in \mathcal{V}_{\Gamma^+, \Gamma^-}(\Omega^+, \Omega^-, B), \\ \infty & \text{else.} \end{cases} \quad (3.44)$$

Therein, $\widehat{W}^{\text{interf}} : \mathbb{R}^2 \rightarrow \mathbb{R}$ is defined through

$$\widehat{W}^{\text{interf}}(\widehat{F}) = \inf \left\{ \int_{Y^*} W^{\text{interf}} \left([\widehat{F}, 0]^T + \nabla_y w(y) \right) dy : w \in W_{\widehat{Y}\text{-per}}^{1,2}(Y^*) \right\}. \quad (3.45)$$

Remark 3.5. The restriction that ω – in addition to the assumption (ε -Tiling) on ω and the vanishing sequence $(\varepsilon_k)_k$ – be convex makes it actually an axis-parallel rectangle in \mathbb{R}^2 . This is easily seen, since by assumption (ε -Tiling) ω results as a polyhedron with axis parallel faces and the convexity hypothesis forbids inward pointing corners.

As it is immediately noticed, the formula (3.45) for the limiting interfacial energy $\widehat{W}^{\text{interf}}$ on the flattened interface ω is the same for tubular voids that touch the periodicity cell's lateral boundaries and for such that are compactly contained in the periodicity cell (compare the situation in Theorem 3.1 and in (3.7)). Similarly, just like in Theorem 3.1 the identification of $\mathcal{V}_{\Gamma^+, \Gamma^-}(\Omega^+, \Omega^-, B)$ with

$$\mathcal{V}_{\Gamma^+, \Gamma^-}(\Omega^+, \Omega^-, \omega) = \left\{ u : u \in W^{1,2}(\Omega), \right. \\ \left. u|_{\omega} \in W^{1,2}(\omega) \text{ and } u = 0 \text{ on } \Gamma^+ \cup \Gamma^- \right\},$$

reveals the Γ -limit \mathcal{F}_{Hom} in the above theorem to be nothing but the heuristically motivated dissipation potential \mathcal{E}_{Hom} from (3.42). Hence, for both types of perforations in the highly conductive interface $B_{\varepsilon}^{\varepsilon\text{-perf}}$ one obtains the same qualitative limiting behavior, telling that for small interface thickness and void diameter ε temperature distributions extend continuously over the flattened interface ω from one body into the other, *i.e.* from Ω^+ into Ω^- . Whereas inside the flattened interface the temperature varies smoothly and minimizes temperature gradients in interface direction due to the interface material's high conductivity.

Yet, the ingredients involved in the proof of Theorem 3.10 are quite different from those used in Subsection 3.3.3 for the proof of Theorem 3.1. In fact, in the present context of tubular voids, *i.e.* such touching the periodicity cell's faces, traditional extension operators turn out to be unsuitable to extend functions from the rescaled perforated interface layer $B^{\varepsilon\text{-perf}}$ to the entire rescaled interface B . This is because applying the extension operator from Proposition 3.2 to two neighboring cells sharing a face that is 'pierced' by a tubular void $V = [0, 1) \times H$, *e.g.* the face $([0, 1) \times [-\frac{1}{2}, \frac{1}{2}]) \setminus \overline{H}$, yields extensions of class $W^{1,2}$ in the two individual cells. But there is a priori no guarantee that the extensions into the 'hole' H of the common face match, too. In fact, the traces of the individual extensions on that face are in general going to differ in the hole H . Here, a major advantage of the periodic unfolding method comes into play: periodically perforated domains can by suitably defined unfolding operators directly be unfolded into a perforated periodicity cell, like proposed by Doina Cioranescu, Patrizia Donato and Rachad Zaki in Cioranescu et al. [2006b]. Therefore, extension operators are no longer required. Instead, the authors of Cioranescu et al. [2006b] make use of *interpolation*

operators to prove compactness for $W^{1,2}$ -bounded sequences of functions over periodically perforated domains, showing that there is a suitable weak $W^{1,2}$ -limit defined over the entire domain without perforations (see [Cioranescu et al., 2006b, Section 3]). Unfortunately, these results cannot simply be applied in the present context. In the case of the thin perforated interface layer $B_\varepsilon^{\varepsilon\text{-perf}}$, the domain thickness varies with the mutual distance of the perforations ε , which is not covered in Cioranescu et al. [2006b] (therein, the focus rests on perforated domains where the voids are periodically distributed over a fixed domain). Furthermore, after having eliminated the dependency on the domain thickness by the above rescaling argument one has to work with L^2 -bounded sequences of *rescaled* gradients over $B^{\varepsilon\text{-perf}}$ (cf. the interface terms in \mathcal{F}_ε as defined in (3.43)), a situation that the authors in Cioranescu et al. [2006b] did not consider. On the other hand, the periodic unfolding method for L^2 -bounded sequences of rescaled gradients was extensively studied in Neukamm [2010] (see also the previous Section 3.3), however in non-perforated rescaled domains. Therefore, in the proof of Theorem 3.10 I will mainly rely on bringing together the interpolation techniques from Cioranescu et al. [2006b] and ideas for periodic unfolding in rescaled thin domains from Neukamm [2010]. The resulting periodic unfolding operator for rescaled perforated thin domains, as presented in the upcoming Subsection 3.4.2, together with a compactness result for unfolded L^2 -bounded sequences of rescaled gradients over rescaled perforated thin domains then allows to infer the Γ -lim inf-inequality for $(\mathcal{F}_{\varepsilon_k})_k$ and \mathcal{F}_{Hom} . The weak convergence arguments employed in the proof of the Γ -lim inf-inequality are actually similar to those in the proof of Theorem 3.1. Finally, also the Γ -lim sup-inequality for $(\mathcal{F}_{\varepsilon_k})_k$ and \mathcal{F}_{Hom} can be inferred by similar steps as performed in Subsection 3.3.3 in the context of Theorem 3.1.

The next Subsection 3.4.2 provides the definition of a periodic unfolding operator for rescaled perforated thin domains and some of its basic properties. As announced before, it also features interpolation error estimates and a compactness result for sequences of functions over rescaled perforated thin domains with L^2 -bounded rescaled gradients. Nevertheless, in order to ease the presentation I will prove these results only under the simplifying assumption (ε -Tiling) made in Subsection 3.2.1. This restriction could of course be relaxed, but to my opinion only little new insight would be gained while the technical efforts would increase significantly and the notation become even more confusing. The proof of Theorem 3.10 is then outlined in Subsection 3.4.3.

3.4.2 Periodic unfolding in rescaled perforated thin domains

The definition of the unfolding operator for rescaled perforated thin domains like $B^{\varepsilon\text{-perf}}$ follows a rather natural intuition. Recalling the simplifying assumption (ε -Tiling) on the flattened interface ω and the possible values of the interface thickness and void diameter ε , the rescaled perforated thin domain $B^{\varepsilon\text{-perf}}$ can be expressed like

$$B^{\varepsilon\text{-perf}} = \bigcup_{\hat{a} \in Z_\varepsilon} M_\varepsilon([\hat{a}, 0]^T + Y^*),$$

where Z_ε is some finite subset of \mathbb{Z}^2 , $Y^* = Y \setminus \bar{V}$ and $V = [0, 1) \times H$ is the tubular void from the previous subsection ($H \Subset [0, 1) \times [-\frac{1}{2}, \frac{1}{2}]$; for the definition of M_ε see (3.9)). In other words, $B^{\varepsilon\text{-perf}}$ is the union of finitely many translates of the rescaled perforated periodicity cell $M_\varepsilon Y^*$, which appears extremely stretched into the x_3 -direction (or squeezed in interface-direction; compare Figure 3.7 below). Like depicted in the

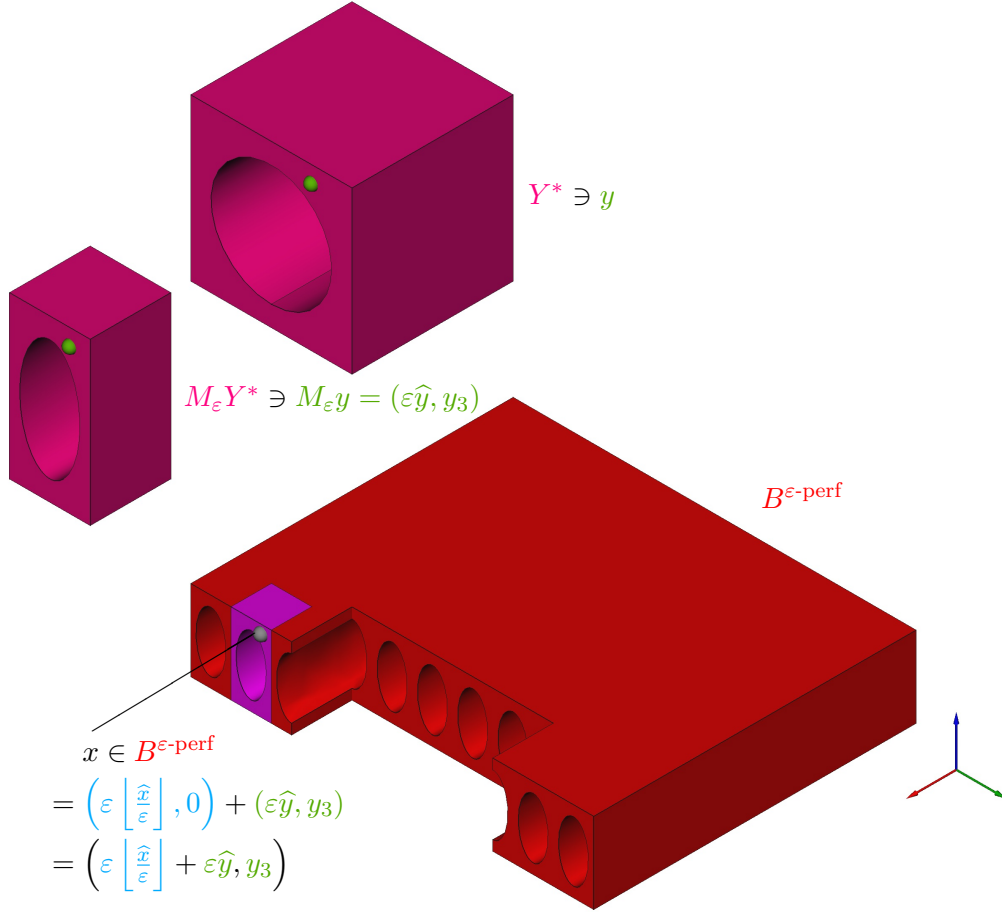


Figure 3.7: Extremely stretched voids in the rescaled perforated thin domain $B^{\varepsilon\text{-perf}}$ and a periodic unfolding idea for $B^{\varepsilon\text{-perf}}$

figure, the periodic unfolding idea for rescaled perforated thin domains is now to split any point $x \in B^{\varepsilon\text{-perf}}$ into a part that tells the observer in which translated and rescaled perforated periodicity cell $M_\varepsilon ([\hat{a}, 0]^T + Y^*)$ to look for x , and a part that gives the exact relative position of x in the rescaled perforated periodicity cell $M_\varepsilon Y^*$. That is,

$$x \in M_\varepsilon ([\hat{a}, 0]^T + Y^*) \subseteq B^{\varepsilon\text{-perf}} \text{ is unfolded into } M_\varepsilon [\hat{a}, 0]^T + M_\varepsilon y,$$

or equivalently, recalling $M_\varepsilon = \text{diag}(\varepsilon, \varepsilon, 1)$ and that $\lfloor \cdot \rfloor : \mathbb{R}^2 \rightarrow \mathbb{Z}^2$ returns the integer part of its argument (componentwise),

$$x \in B^{\varepsilon\text{-perf}} \text{ is unfolded into } \left(\varepsilon \left\lfloor \frac{\hat{x}}{\varepsilon} \right\rfloor + \varepsilon \hat{y}, y_3 \right).$$

This intuition then yields the desired unfolding operator for rescaled perforated thin domains.

Definition 3.11. Let ω be some open and bounded subset of \mathbb{R}^2 , $B := \omega \times [-\frac{1}{2}, \frac{1}{2}]$ and $Y = \hat{Y} \times [-\frac{1}{2}, \frac{1}{2}]$ with $\hat{Y} = [0, 1]^2$ be defined like in Section 3.2. Moreover, let ε be some positive small parameter and assume that ω and ε satisfy assumption (ε -Tiling). Finally, let $Y^* := Y \setminus \bar{V}$ for some $V \subseteq Y$ and set

$$B^{\varepsilon\text{-perf}} := B \cap \bigcup_{\hat{a} \in \mathbb{Z}^2} M_\varepsilon \left([\hat{a}, 0]^T + Y^* \right)$$

where $M_\varepsilon = \text{diag}(\varepsilon, \varepsilon, 1)$. Then, for any $v \in L^2(B^{\varepsilon\text{-perf}})$ the periodic unfolding operator $\widehat{\mathcal{T}}_\varepsilon^* v \in L^2(\omega \times Y^*)$ for the rescaled perforated thin domain $B^{\varepsilon\text{-perf}}$ is defined as

$$(\widehat{\mathcal{T}}_\varepsilon^* v)(\hat{x}, y) := v \left(\varepsilon \left\lfloor \frac{\hat{x}}{\varepsilon} \right\rfloor + \varepsilon \hat{y}, y_3 \right).$$

Here, $\lfloor \cdot \rfloor : \mathbb{R}^2 \rightarrow \mathbb{Z}^2$ returns the integer part of its argument.

Remark 3.6. I would like to emphasize that a similar unfolding idea was already introduced in [Cioranescu et al., 2008b, Section 2.3] and applied to both Neumann sieve problems (see Subsection 3.1.1 for Neumann sieve problems) and pore-type interface problems where the pores do not touch the boundary of the periodicity cell. However, the authors of Cioranescu et al. [2008b] studied quite different scalings. More precisely, various *geometric* scalings, *i.e.* different ratios of interface thickness and diameter of the voids included in the interface layer. In contrast, the interface problems analyzed in this thesis address *constitutive* scalings of the material occupying the interface layer (cf. the constitutive assumptions made in Subsection 3.2.2). Consequently, both the homogenization results Theorems 3.1 and 3.10 and the results on the periodic unfolding operators stated below (most notably Propositions 3.19, 3.20 and Lemma 3.21) are quite different from those in [Cioranescu et al., 2008b, Sections 4 and 5] (cf. also the literature review on interface problems in Subsection 3.1.1).

Just like other periodic unfolding operators, also the present adaption enjoys the usual isometry property and consequently also the same compactness.

Proposition 3.12. *Let there be given the periodic unfolding operator $\widehat{\mathcal{T}}_\varepsilon^*$ for the rescaled perforated thin domain $B^{\varepsilon\text{-perf}}$ as in Definition 3.11.*

(i) *The periodic unfolding operator $\widehat{\mathcal{T}}_\varepsilon^* : L^2(B^{\varepsilon\text{-perf}}) \rightarrow L^2(\omega \times Y^*)$ satisfies*

$$\int_{B^{\varepsilon\text{-perf}}} v(x) \, dx = \int_\omega \int_{Y^*} (\widehat{\mathcal{T}}_\varepsilon^* v)(\hat{x}, y) \, dy \, d\hat{x} \quad \text{for all } v \in L^2(B^{\varepsilon\text{-perf}}).$$

In particular, $\widehat{\mathcal{T}}_\varepsilon^$ is an isometry.*

- (ii) Suppose furthermore to be given a vanishing sequence of positive real numbers $(\varepsilon_k)_k$ whose elements ε_k together with ω satisfy $(\varepsilon\text{-Tiling})$, and a sequence $(v_{\varepsilon_k})_k$ such that $v_{\varepsilon_k} \in L^2(B^{\varepsilon\text{-perf}})$ and $(\|v_{\varepsilon_k}\|_{L^2(B^{\varepsilon\text{-perf}})})_k$ is bounded. Then there is a $v_0 \in L^2(\omega \times Y^*)$ and a subsequence $(\varepsilon_{k_\ell})_\ell$ such that

$$\widehat{\mathcal{T}}_{\varepsilon_{k_\ell}}^* v_{\varepsilon_{k_\ell}} \rightharpoonup v_0 \quad \text{in } L^2(\omega \times Y^*).$$

Proof. By the assumption $(\varepsilon\text{-Tiling})$ made on B and ε , there is a finite subset Z_ε of \mathbb{Z}^2 such that

$$B^{\varepsilon\text{-perf}} = \bigcup_{\widehat{a} \in Z_\varepsilon} M_\varepsilon \left([\widehat{a}, 0]^T + Y^* \right).$$

Now, the proof of the first assertion pretty much follows the lines of Lemma 2.1, itself an adaption of [Visintin, 2006, Lemma 1.1]. For arbitrary $v \in L^2(B^{\varepsilon\text{-perf}})$ it is

$$\begin{aligned} \int_{B^{\varepsilon\text{-perf}}} v(x) \, dx &= \sum_{\widehat{a} \in Z_\varepsilon} \int_{M_\varepsilon([\widehat{a}, 0]^T + Y^*)} v(x) \, dx \\ &= \sum_{\widehat{a} \in Z_\varepsilon} \varepsilon^2 \int_{Y^*} v(\varepsilon(\widehat{a} + \widehat{y}), y_3) \, dy \\ &= \sum_{\widehat{a} \in Z_\varepsilon} \int_{\varepsilon(\widehat{a} + \widehat{Y})} \int_{Y^*} v(\varepsilon(\widehat{a} + \widehat{y}), y_3) \, dy \, d\widehat{x} \\ &= \sum_{\widehat{a} \in Z_\varepsilon} \int_{\varepsilon(\widehat{a} + \widehat{Y})} \int_{Y^*} v\left(\varepsilon \left\lfloor \frac{\widehat{x}}{\varepsilon} \right\rfloor + \varepsilon \widehat{y}, y_3\right) \, dy \, d\widehat{x} \\ &= \int_{\omega} \int_{Y^*} v\left(\varepsilon \left\lfloor \frac{\widehat{x}}{\varepsilon} \right\rfloor + \varepsilon \widehat{y}, y_3\right) \, dy \, d\widehat{x} = \int_{\omega} \int_{Y^*} (\widehat{\mathcal{T}}_\varepsilon^* v)(\widehat{x}, y) \, dy \, d\widehat{x}. \end{aligned}$$

This proves the first assertion, from which the second is obtained as a rather obvious consequence. \square

Likewise, the periodic unfolding operator $\widehat{\mathcal{T}}_\varepsilon^*$ for the rescaled perforated thin domain $B^{\varepsilon\text{-perf}}$ shows general relations with common weak or strong convergence in $L^2(B)$.

Proposition 3.13. *Suppose once more the assumptions stated in Definition 3.11 to be valid, and let $\widehat{\mathcal{T}}_\varepsilon^*$ be the periodic unfolding operator for the rescaled perforated thin domain $B^{\varepsilon\text{-perf}}$ defined therein. Let $(\varepsilon_k)_k$ be a vanishing sequence of positive real numbers whose elements ε_k together with ω satisfy $(\varepsilon\text{-Tiling})$, and let $(v_{\varepsilon_k})_k$ be a sequence such that $v_{\varepsilon_k} \in L^2(B^{\varepsilon\text{-perf}})$. Then*

- (i) for $v_0 \in L^2(\omega \times Y^*)$ one has the implication

$$\begin{aligned} \widehat{\mathcal{T}}_{\varepsilon_k}^* v_{\varepsilon_k} \rightharpoonup v_0 \quad \text{in } L^2(\omega \times Y^*) \\ \Rightarrow \quad \mathbb{1}_{B^{\varepsilon\text{-perf}}} v_{\varepsilon_k} \rightharpoonup \int_{\widehat{Y}} \mathbb{1}_{Y^*}(\widehat{y}, x_3) v_0(\widehat{x}, (\widehat{y}, x_3)) \, d\widehat{y} \quad \text{in } L^2(B), \end{aligned}$$

(ii) if $(v_{\varepsilon_k})_k$ can actually be identified as a sequence in $L^2(B)$, then for $v_0 \in L^2(B)$ one has

$$v_{\varepsilon_k} \rightarrow v_0 \quad \text{in } L^2(B) \quad \Rightarrow \quad \widehat{\mathcal{T}}_{\varepsilon_k}^* v_{\varepsilon_k} \rightarrow v_0(\widehat{x}, y_3) \quad \text{in } L^2(\omega \times Y^*),$$

(iii) if $v_0 \in L^2(\omega \times Y^*)$ such that $\widehat{\mathcal{T}}_{\varepsilon_k}^* v_{\varepsilon_k} \rightarrow v_0$ in $L^2(\omega \times Y^*)$ and $(w_{\varepsilon_k})_k$ is another sequence satisfying $w_{\varepsilon_k} \in L^2(B^{\varepsilon\text{-perf}})$, $\widehat{\mathcal{T}}_{\varepsilon_k}^* w_{\varepsilon_k} \rightarrow w_0$ in $L^2(\omega \times Y^*)$ for some $w_0 \in L^2(\omega \times Y^*)$, it follows that

$$\int_{B^{\varepsilon\text{-perf}}} v_{\varepsilon_k}(x) w_{\varepsilon_k}(x) dx \rightarrow \int_{\omega} \int_{Y^*} v_0(\widehat{x}, y) w_0(\widehat{x}, y) dy d\widehat{x}.$$

Proof. To prove the first statement of the proposition, by the isometry property of the unfolding operator $\widehat{\mathcal{T}}_{\varepsilon_k}^*$ stated in Proposition 3.12 and the boundedness of weakly convergent sequence $(\widehat{\mathcal{T}}_{\varepsilon_k}^* v_{\varepsilon_k})_k$ in $L^2(\omega \times Y^*)$ one realizes that also $\|v_{\varepsilon_k}\|_{L^2(B^{\varepsilon\text{-perf}})}$ is bounded. Hence, the sequence $(\mathbb{1}_{B^{\varepsilon\text{-perf}}} v_{\varepsilon_k})_k$ is bounded in $L^2(B)$. As a consequence, it suffices to test $(\mathbb{1}_{B^{\varepsilon\text{-perf}}} v_{\varepsilon_k})_k$ for weak convergence in $L^2(B)$ with functions from $C_c^\infty(B)$. Hence, given $\psi \in C_c^\infty(B)$ it is

$$\begin{aligned} & \int_B \mathbb{1}_{B^{\varepsilon\text{-perf}}}(x) v_{\varepsilon_k}(x) \psi(x) dx = \int_{B^{\varepsilon\text{-perf}}} v_{\varepsilon_k}(x) \psi(x) dx \\ &= \int_{\omega} \int_{Y^*} (\widehat{\mathcal{T}}_{\varepsilon_k}^* v_{\varepsilon_k})(\widehat{x}, y) (\widehat{\mathcal{T}}_{\varepsilon_k}^* \psi)(\widehat{x}, y) dy d\widehat{x} \\ &= \int_{\omega} \int_{Y^*} (\widehat{\mathcal{T}}_{\varepsilon_k}^* v_{\varepsilon_k})(\widehat{x}, y) \psi\left(\varepsilon \left\lfloor \frac{\widehat{x}}{\varepsilon} \right\rfloor + \varepsilon \widehat{y}, y_3\right) dy d\widehat{x}. \end{aligned}$$

By the smoothness of ψ , it follows that

$$\psi\left(\varepsilon \left\lfloor \frac{\widehat{x}}{\varepsilon} \right\rfloor + \varepsilon \widehat{y}, y_3\right) \rightarrow \psi(\widehat{x}, y_3) \quad \text{uniformly on } \omega \times Y.$$

Hence, by the assumed weak convergence of $(\widehat{\mathcal{T}}_{\varepsilon_k}^* v_{\varepsilon_k})_k$ and $Y = \widehat{Y} \times [-\frac{1}{2}, \frac{1}{2}]$

$$\begin{aligned} & \lim_{k \rightarrow \infty} \int_B \mathbb{1}_{B^{\varepsilon\text{-perf}}}(x) v_{\varepsilon_k}(x) \psi(x) dx \\ &= \int_{\omega} \int_{Y^*} v_0(\widehat{x}, y) \psi(\widehat{x}, y_3) dy d\widehat{x} = \int_{\omega} \int_Y \mathbb{1}_{Y^*}(y) v_0(\widehat{x}, y) \psi(\widehat{x}, y_3) dy d\widehat{x} \\ &= \int_{\omega} \left(\int_{(-\frac{1}{2}, \frac{1}{2})} \int_{\widehat{Y}} \mathbb{1}_{Y^*}(\widehat{y}, y_3) v_0(\widehat{x}, (\widehat{y}, y_3)) \psi(\widehat{x}, y_3) d\widehat{y} dy_3 \right) d\widehat{x} \\ &= \int_B \left(\int_{\widehat{Y}} \mathbb{1}_{Y^*}(\widehat{y}, x_3) v_0(\widehat{x}, (\widehat{y}, x_3)) d\widehat{y} \right) \psi(x) dx. \end{aligned}$$

The second assertion is rather straightforward to show. To this end, one writes

$$\begin{aligned} & \|\widehat{\mathcal{T}}_{\varepsilon_k}^* v_{\varepsilon_k} - v_0(\widehat{x}, y_3)\|_{L^2(\omega \times Y^*)} \leq \\ & \quad \|\widehat{\mathcal{T}}_{\varepsilon_k}^* v_{\varepsilon_k} - \widehat{\mathcal{T}}_{\varepsilon_k}^* v_0\|_{L^2(\omega \times Y^*)} + \|\widehat{\mathcal{T}}_{\varepsilon_k}^* v_0 - v_0(\widehat{x}, y_3)\|_{L^2(\omega \times Y^*)}. \end{aligned}$$

By the isometry property of $\widehat{\mathcal{T}}_{\varepsilon_k}^*$ stated in Proposition 3.12, the first term on the right hand side is nothing but $\|v_{\varepsilon_k} - v_0\|_{L^2(B^{\varepsilon\text{-perf}})}$, thus vanishes by assumption as ε_k tends to zero. As concerns the second term,

$$\|\widehat{\mathcal{T}}_{\varepsilon_k}^* v_0 - v_0(\widehat{x}, y_3)\|_{L^2(\omega \times Y^*)}^2 = \int_{\omega} \int_{Y^*} \left| v_0 \left(\varepsilon \left\lfloor \frac{\cdot}{\varepsilon} \right\rfloor + \varepsilon \widehat{y}, y_3 \right) - v_0(\widehat{x}, y_3) \right|^2 dy d\widehat{x}$$

which would certainly vanish if v_0 were smooth. Yet, the same holds for arbitrary $v_0 \in L^2(B)$ as it is easily inferred by the density of $C_c^\infty(B)$ in $L^2(B)$ and an approximation argument.

Finally, the third assertion is an easy consequence of the isometry property stated in Proposition 3.12 and the convergence of products of weakly and strongly convergent sequences in Hilbert spaces. \square

Remark 3.7. Other than for periodic unfolding operators over non-perforated domains, in general the inverse implication in the second statement of the Proposition is wrong (compare the Propositions 2.25 and 3.7). This is due to the fact that the periodic unfolding operator $\widehat{\mathcal{T}}_{\varepsilon}^*$ cannot control what happens inside the perforations. As a counterexample one might consider

$$v_{\varepsilon} := \mathbb{1}_{B^{\varepsilon\text{-perf}}} \psi + \frac{1}{\varepsilon} (1 - \mathbb{1}_{B^{\varepsilon\text{-perf}}})$$

for a $\psi \in C_c^\infty(B)$ which is independent of x_3 , *i.e.* with a slight abuse of notation $\psi(x) = \psi(\widehat{x})$. Then, a simple calculation shows that $\widehat{\mathcal{T}}_{\varepsilon}^* v_{\varepsilon} = \widehat{\mathcal{T}}_{\varepsilon}^* \psi \rightarrow \psi(\widehat{x}, y_3) = \psi(\widehat{x})$ in $L^2(\omega \times Y^*)$. Hence, $(\widehat{\mathcal{T}}_{\varepsilon}^* v_{\varepsilon})_{\varepsilon}$ converges strongly in $L^2(\omega \times Y^*)$, its limit is independent of the Y^* -argument, but $\|v_{\varepsilon}\|_{L^2(B)} = \|\psi\|_{L^2(B^{\varepsilon\text{-perf}})} + \frac{1}{\varepsilon} \text{vol}(B \setminus B^{\varepsilon\text{-perf}})^{1/2}$ is not even bounded.

However, the most interesting situations occur when the periodic unfolding operator $\widehat{\mathcal{T}}_{\varepsilon}^*$ for the rescaled perforated thin domain $B^{\varepsilon\text{-perf}}$ is applied to $L^2(B^{\varepsilon\text{-perf}})$ -bounded sequences of rescaled gradients like $[\widehat{\nabla} v_{\varepsilon}, \frac{1}{\varepsilon} \partial_3 v_{\varepsilon}]$, where $v_{\varepsilon} \in W^{1,2}(B^{\varepsilon\text{-perf}})$. As it was revealed in Subsection 3.3.3, such sequences arise naturally in the analysis of the interface problems studied in this thesis (more precisely, through a priori estimates, see (3.23)). If the voids in the rescaled perforated thin domain $B^{\varepsilon\text{-perf}}$ are rescaled copies of a void V that is compactly contained in the periodicity cell Y – *i.e.* the situation analyzed in the previous Section 3.3, see Figure 3.6 – then it was revealed by Lemma 3.9 how extension operators $P : W^{1,2}(Y^*) \rightarrow W^{1,2}(Y)$ could help to establish essential compactness results. Both in $W^{1,2}(B)$ for the extensions of v_{ε} to $W^{1,2}(B)$, and for the (unfolded) sequence of rescaled gradients $[\widehat{\nabla} v_{\varepsilon}, \frac{1}{\varepsilon} \partial_3 v_{\varepsilon}]$. Yet, in the present situation the tubular voids in the rescaled perforated thin domain $B^{\varepsilon\text{-perf}}$ are assembled from copies of a void V that touches the periodicity cell's boundary as depicted in Figure 3.7. Therefore, extension operators turn out to be unsuitable to extend v_{ε} from $B^{\varepsilon\text{-perf}}$ to the entire rescaled thin domain B . This is easily seen when applying an extension operator (*e.g.* a rescaled version of P from Proposition 3.2; compare (3.10)) to two neighboring copies of the rescaled perforated periodicity cell $M_{\varepsilon} Y^*$ whose common face is pierced by the void V . Then, there is a priori no guarantee that the traces of the extensions in

the respective cells match on the part of the common face that is occupied by the void. To circumvent this issue, the authors of Cioranescu et al. [2006b] have resorted to the use of *interpolation operators* instead of *extension operators*. More precisely, they employed polynomial interpolations in the cubic periodicity cell which in Finite Element Theory are known as $Q1$ -interpolations (see e.g. [Knabner and Angermann, 2003, Section 3.3]). The $Q1$ -interpolation operator takes a cube's eight vertices as interpolation nodes and fits a trilinear polynomial to the values given in these nodes. In this fashion the $Q1$ -interpolation results as continuous over faces of neighboring cubic periodicity cells whenever the values corresponding to the nodes located on that face are identical in both cells, and therefore as of class $W^{1,\infty}$ over neighboring cells. By suitably condensing a function $v_\varepsilon \in W^{1,2}(B^{\varepsilon\text{-perf}})$ to scalar values in the vertices of the rescaled perforated periodicity cells $M_\varepsilon Y^*$ from which $B^{\varepsilon\text{-perf}}$ is assembled, $Q1$ -interpolation over the single cells yields an interpolation operator $Q_\varepsilon : W^{1,2}(B^{\varepsilon\text{-perf}}) \rightarrow W^{1,2}(B)$. In other words, this allows to embed the elements of a sequence $v_\varepsilon \in W^{1,2}(B^{\varepsilon\text{-perf}})$ as $Q_\varepsilon v_\varepsilon$ into $W^{1,2}(B)$, and thus to exploit common compactness properties of this space. Indeed, $Q1$ -interpolations are found quite frequently in the theory of periodic unfolding, see e.g. Cioranescu et al. [2002, 2006b] but also Visintin [2006] where a $Q1$ -interpolation operator is used to define two-scale convergence in spaces of continuous functions. In the present context of the rescaled perforated thin domain $B^{\varepsilon\text{-perf}}$ however, the $Q1$ -interpolation has to be used with care on the extremely distorted rescaled perforated periodicity cells $M_\varepsilon Y^*$ forming $B^{\varepsilon\text{-perf}}$. This calls for a detailed characterization of the interpolation error $\|Q_\varepsilon v_\varepsilon - v_\varepsilon\|_{L^2(B^{\varepsilon\text{-perf}})}$, $v_\varepsilon \in W^{1,2}(B^{\varepsilon\text{-perf}})$. Finally, with the help of the interpolation operator Q_ε a compactness result for the (unfolded) sequence of rescaled gradients $[\widehat{\nabla} v_\varepsilon, \frac{1}{\varepsilon} \partial_3 v_\varepsilon]$ can be established, in spirit being similar to Lemma 3.9.

The remainder of this subsection is now concerned with the statement and proof of this compactness result. To this end, basic properties of the $Q1$ -interpolation operator are stated and a definition of the resulting interpolation operator $Q_\varepsilon : W^{1,2}(B^{\varepsilon\text{-perf}}) \rightarrow W^{1,2}(B)$ is given. Then, following a careful analysis of the interpolation error $\|Q_\varepsilon v_\varepsilon - v_\varepsilon\|_{L^2(B^{\varepsilon\text{-perf}})}$ for $v_\varepsilon \in W^{1,2}(B^{\varepsilon\text{-perf}})$, the essential compactness result for sequences of rescaled gradients $[\widehat{\nabla} v_\varepsilon, \frac{1}{\varepsilon} \partial_3 v_\varepsilon]$ unfolded by $\widehat{\mathcal{T}}_\varepsilon^*$ is stated and proved.

Definition 3.14. Let the cubic periodicity cell $Y = \widehat{Y} \times [-\frac{1}{2}, \frac{1}{2}]$, $\widehat{Y} = [0, 1)^2$, be defined like in Subsection 3.2.1. Moreover, let the vertices of the periodicity cell Y be numbered like shown in Figure 3.8. Then, the shapefunction $q_i : Y \rightarrow \mathbb{R}$ is the unique trilinear polynomial in $\text{lin}\{1, y_1, y_2, y_3, y_1 y_2, y_1 y_3, y_2 y_3, y_1 y_2 y_3\}$ satisfying

$$q_i(\text{' Node } j') = \delta_{ij}.$$

Moreover, the $Q1$ -interpolation operator Q is defined through

$$Q : \mathbb{R}^8 \rightarrow W^{1,2}(Y), \quad Qb := \sum_{i=1}^8 b_i q_i.$$

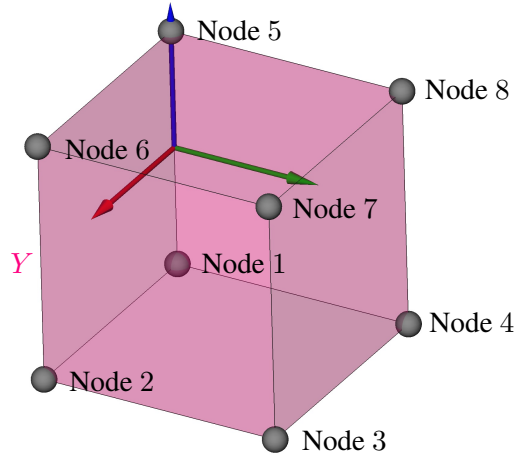


Figure 3.8: The nodes 1, . . . , 8 in the cubic periodicity cell Y

Proposition 3.15. *In the situation of Definition 3.14, the $Q1$ -interpolation operator Q enjoys the following properties. There is a positive constant C_Q such that for every $b \in \mathbb{R}^8$*

$$\begin{aligned} \|Qb\|_{L^2(Y)} &\leq C_Q |b|, \\ \|\partial_{y_1}(Qb)\|_{L^2(Y)} &\leq C_Q \left| [b_2 - b_1, b_4 - b_3, b_6 - b_5, b_8 - b_7] \right|, \\ \|\partial_{y_2}(Qb)\|_{L^2(Y)} &\leq C_Q \left| [b_4 - b_1, b_3 - b_2, b_7 - b_6, b_8 - b_5] \right|, \\ \|\partial_{y_3}(Qb)\|_{L^2(Y)} &\leq C_Q \left| [b_5 - b_1, b_6 - b_2, b_7 - b_3, b_8 - b_4] \right|. \end{aligned}$$

Proof. The first statement is trivial and the other estimates are basically corollaries from the explicit forms of $\partial_{y_1}(Qa)$, $\partial_{y_2}(Qa)$ and $\partial_{y_3}(Qa)$. In fact, this can easily be checked by computing the derivatives of the shapefunctions q_1, \dots, q_8 (by hand or using a computer algebra system). \square

Although the following adaption is fairly simple, for the sake of a brief notation it turns out useful to introduce the $Q1$ -interpolation operator on the rescaled periodicity cell $M_\varepsilon Y$.

Definition 3.16. The $Q1$ -interpolation operator \tilde{Q}_ε on the rescaled periodicity cell $M_\varepsilon Y = [0, \varepsilon)^2 \times [-\frac{1}{2}, \frac{1}{2}]$ is defined as

$$\tilde{Q}_\varepsilon : \mathbb{R}^8 \rightarrow W^{1,2}(M_\varepsilon Y), \quad \tilde{Q}_\varepsilon b := (Qb) \left(\frac{\cdot}{\varepsilon}, (\cdot)_3 \right).$$

In view of Proposition 3.15 it is straightforward to prove the properties of \tilde{Q}_ε listed below.

Proposition 3.17. *In the situation of Definition 3.16, the $Q1$ -interpolation operator \tilde{Q}_ε on the rescaled periodicity cell $M_\varepsilon Y$ enjoys the following properties. There is a positive constant C_Q such that for every $b \in \mathbb{R}^8$*

$$\begin{aligned} \|\tilde{Q}_\varepsilon b\|_{L^2(M_\varepsilon Y)} &\leq \varepsilon C_Q |b|, \\ \|\partial_{y_1}(\tilde{Q}_\varepsilon b)\|_{L^2(M_\varepsilon Y)} &\leq \varepsilon C_Q \left| \left[\frac{b_2-b_1}{\varepsilon}, \frac{b_4-b_3}{\varepsilon}, \frac{b_6-b_5}{\varepsilon}, \frac{b_8-b_7}{\varepsilon} \right] \right|, \\ \|\partial_{y_2}(\tilde{Q}_\varepsilon b)\|_{L^2(M_\varepsilon Y)} &\leq \varepsilon C_Q \left| \left[\frac{b_4-b_1}{\varepsilon}, \frac{b_3-b_2}{\varepsilon}, \frac{b_7-b_6}{\varepsilon}, \frac{b_8-b_5}{\varepsilon} \right] \right|, \\ \|\partial_{y_3}(\tilde{Q}_\varepsilon b)\|_{L^2(M_\varepsilon Y)} &\leq \varepsilon C_Q |b_5 - b_1, b_6 - b_2, b_7 - b_3, b_8 - b_4|. \end{aligned}$$

Having these preparatory definitions and properties at hand, one can now define a suitable interpolation operator $Q_\varepsilon : W^{1,2}(B^{\varepsilon\text{-perf}}) \rightarrow W^{1,2}(B)$ as follows.

Definition 3.18. Let $Y = \hat{Y} \times [-\frac{1}{2}, \frac{1}{2}]$, $\hat{Y} = [0, 1]^2$ be defined like in Section 3.2. Moreover, let $B = \omega \times [-\frac{1}{2}, \frac{1}{2}]$, where ω is a convex open and bounded subset of \mathbb{R}^2 and ε a small positive parameter such that (ε -Tiling) is satisfied. Eventually, let V be a subset of Y that does not touch the edges of Y and set $Y^* := Y \setminus \bar{V}$ and let $r > 0$ be such that $r < \frac{1}{2} \text{dist}(V, E)$ for all edges E of Y . Moreover, given the numbering ①, ..., ⑧ for the nodes of the rescaled periodicity cell $M_\varepsilon Y$ as shown in Figure 3.9, set

$$\begin{aligned} K_{\varepsilon r, r}(\textcircled{i}) &:= \textcircled{i} + [-\varepsilon r, \varepsilon r]^2 \times [0, r] && \text{for } i \in \{1, \dots, 4\} \\ K_{\varepsilon r, r}(\textcircled{i}) &:= \textcircled{i} + [-\varepsilon r, \varepsilon r]^2 \times [-r, 0] && \text{for } i \in \{5, \dots, 8\} \end{aligned}$$

and observe that by assumption (ε -Tiling)

$$\begin{aligned} B^{\varepsilon\text{-perf}} &:= B \cap \bigcup_{\hat{a} \in \mathbb{Z}^2} M_\varepsilon \left([\hat{a}, 0]^T + Y^* \right) \\ &= \bigcup_{\hat{a} \in Z_\varepsilon} M_\varepsilon \left([\hat{a}, 0]^T + Y^* \right). \end{aligned}$$

Then, for $v \in W^{1,2}(B^{\varepsilon\text{-perf}})$ and $x \in M_\varepsilon([\hat{a}, 0]^T + Y)$ the interpolation operator $Q_\varepsilon v \in W^{1,2}(B)$ is defined as

$$(Q_\varepsilon v)(x) = (\tilde{Q}_\varepsilon b)(x - M_\varepsilon[\hat{a}, 0]^T)$$

where

$$b := \left[\int_{B \cap ([\varepsilon \hat{a}, 0]^T + K_{\varepsilon r, r}(\textcircled{1}))} v(\xi) \, d\xi, \dots, \int_{B \cap ([\varepsilon \hat{a}, 0]^T + K_{\varepsilon r, r}(\textcircled{8}))} v(\xi) \, d\xi \right] \in \mathbb{R}^8.$$

Here, \tilde{Q}_ε is the $Q1$ -interpolation operator from Definition 3.16.

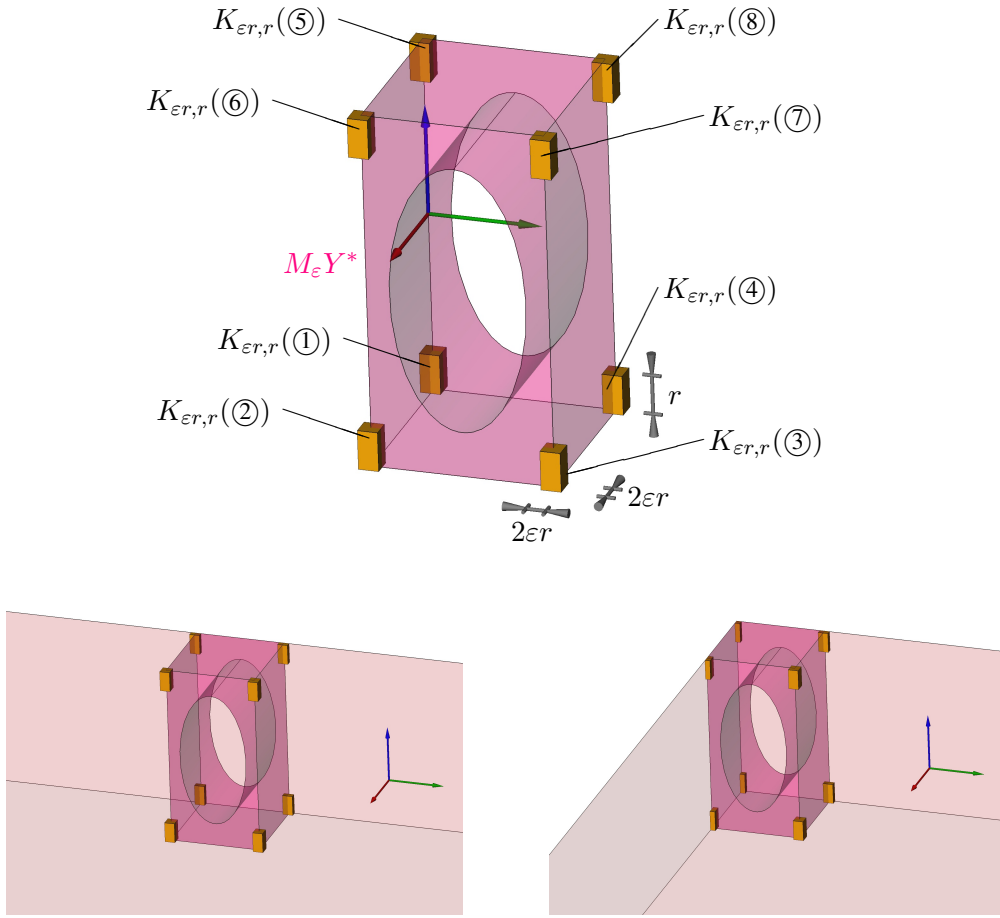


Figure 3.9: Sets for taking mean values of $v_\varepsilon \in W^{1,2}(B^{\varepsilon\text{-perf}})$ around the vertices of the rescaled periodicity cells $M_\varepsilon Y^*$ that are contained *inside* $B^{\varepsilon\text{-perf}}$ (top) or touch the lateral boundary of B (bottom)

Remark 3.8. The sets over which mean values of $v_\varepsilon \in W^{1,2}(B^{\varepsilon\text{-perf}})$ are taken to obtain the required node values for the Q_1 -interpolation $Q_\varepsilon v_\varepsilon$ in the respective rescaled perforated periodicity cells $M_\varepsilon([\hat{a}, 0]^T + Y^*)$ may not be entirely contained in $B^{\varepsilon\text{-perf}}$. This observation is illustrated in Figure 3.9 (bottom). In fact, this effect occurs for cells $M_\varepsilon([\hat{a}, 0]^T + Y^*)$ that touch the lateral boundary of B . Yet, due to the assumption (ε -Tiling) and the convexity of ω it follows that the base ω is an axis-parallel rectangle (see Remark 3.5). Moreover, from (ε -Tiling) it also follows that the cells $M_\varepsilon([\hat{a}, 0]^T + Y^*)$ match $B^{\varepsilon\text{-perf}}$, which is why only the two situations shown in Figure 3.9 (bottom) can be observed at the lateral boundary of B . However, one should notice well that for all rescaled perforated periodicity cells $M_\varepsilon([\hat{a}, 0]^T + Y^*)$ in $B^{\varepsilon\text{-perf}}$ that do *not* touch the lateral boundary of B one has $[\varepsilon\hat{a}, 0]^T + K_{\varepsilon,r}(i) \subseteq B^{\varepsilon\text{-perf}}$ for all $i = 1, \dots, 8$.

Similar to what Proposition 3.3 stated for extension operators $P_\varepsilon : W^{1,2}(B^{\varepsilon\text{-perf}}) \rightarrow$

$W^{1,2}(B)$ over rescaled perforated thin domains $B^{\varepsilon\text{-perf}}$ where the voids are rescaled copies of some $V \in Y$, also the interpolation operator $Q_\varepsilon : W^{1,2}(B^{\varepsilon\text{-perf}}) \rightarrow W^{1,2}(B)$ allows for estimates on the L^2 -norms of the interpolated function $Q_\varepsilon v_\varepsilon$ and its gradient $\nabla(Q_\varepsilon v_\varepsilon)$, $v_\varepsilon \in W^{1,2}(B^{\varepsilon\text{-perf}})$.

Proposition 3.19. *Let the notation and assumptions of Definition 3.18 hold and assume $v \in W^{1,2}(B^{\varepsilon\text{-perf}})$. Then, the interpolation operator $Q_\varepsilon : W^{1,2}(B^{\varepsilon\text{-perf}}) \rightarrow W^{1,2}(B)$ satisfies the estimates*

$$\|Q_\varepsilon v\|_{L^2(B)}^2 \leq C(V) \|v\|_{L^2(B^{\varepsilon\text{-perf}})}^2$$

and

$$\begin{aligned} \|\widehat{\nabla}(Q_\varepsilon v)\|_{L^2(B; \mathbb{R}^2)}^2 + \left\| \frac{1}{\varepsilon} \partial_3(Q_\varepsilon v) \right\|_{L^2(B)}^2 \\ \leq C(V) \left(\|\widehat{\nabla}v\|_{L^2(B^{\varepsilon\text{-perf}}; \mathbb{R}^2)}^2 + \left\| \frac{1}{\varepsilon} \partial_3 v \right\|_{L^2(B^{\varepsilon\text{-perf}})}^2 \right). \end{aligned}$$

Here, $C(V)$ is a positive constant depending on the void V in the perforated periodicity cell $Y^* = Y \setminus \overline{V}$ only.

Proof. In order to prove the first estimate stated in the proposition, consider $M_\varepsilon([\widehat{a}, 0]^T + Y) \subseteq B$ for some $\widehat{a} \in Z_\varepsilon$ (compare the assumption (ε -Tiling)). According to the Definition 3.18 of the interpolation operator Q_ε it is

$$(Q_\varepsilon v)(x) = (\widetilde{Q}_\varepsilon b)(x - M_\varepsilon[\widehat{a}, 0]^T) \quad \text{for } x \in M_\varepsilon([\widehat{a}, 0]^T + Y)$$

where

$$b := \left[\int_{B \cap ([\widehat{a}, 0]^T + K_{\varepsilon r, r}(\textcircled{1}))} v(\xi) \, d\xi, \dots, \int_{B \cap ([\widehat{a}, 0]^T + K_{\varepsilon r, r}(\textcircled{8}))} v(\xi) \, d\xi \right] \in \mathbb{R}^8.$$

Hence, it follows from Proposition 3.17 that

$$\begin{aligned} \|Q_\varepsilon v\|_{L^2(M_\varepsilon([\widehat{a}, 0]^T + Y))}^2 &\leq C_Q \varepsilon^2 |b|^2 \\ &= C_Q \varepsilon^2 \sum_{i=1}^8 \left| \int_{B \cap ([\varepsilon \widehat{a}, 0]^T + K_{\varepsilon r, r}(\textcircled{i}))} v(x) \, dx \right|. \end{aligned}$$

Applying Jensen's inequality then leads to

$$\begin{aligned} \|Q_\varepsilon v\|_{L^2(M_\varepsilon([\widehat{a}, 0]^T + Y))}^2 \\ \leq C_Q \varepsilon^2 \sum_{i=1}^8 \frac{1}{\text{vol}(B \cap ([\varepsilon \widehat{a}, 0]^T + K_{\varepsilon r, r}(\textcircled{i})))} \int_{B \cap ([\varepsilon \widehat{a}, 0]^T + K_{\varepsilon r, r}(\textcircled{i}))} |v(x)|^2 \, dx. \end{aligned}$$

Moreover, according to Remark 3.8 it is due to assumption (ε -Tiling) that for all $i = 1, \dots, 8$ the intersection $B \cap ([\varepsilon \widehat{a}, 0]^T + K_{\varepsilon r, r}(\textcircled{i}))$ leaves at least 'one quarter' of $[\varepsilon \widehat{a}, 0]^T + K_{\varepsilon r, r}(\textcircled{i})$ (see also Figure 3.9 (bottom)). Thus,

$$\text{vol}(B \cap ([\varepsilon \widehat{a}, 0]^T + K_{\varepsilon r, r}(\textcircled{i}))) \geq \text{vol}([0, \varepsilon r]^2 \times [0, r]) = \varepsilon^2 r^3$$

and furthermore

$$\begin{aligned} \|Q_\varepsilon v\|_{L^2(M_\varepsilon([\widehat{a}, 0]^T + Y))}^2 &\leq C_Q \sum_{i=1}^8 \frac{1}{r^3} \int_{B \cap ([\varepsilon \widehat{a}, 0]^T + K_{\varepsilon r, r}(\widehat{\cdot}))} |v(x)|^2 dx \\ &= C(V) \sum_{i=1}^8 \int_{B \cap ([\varepsilon \widehat{a}, 0]^T + K_{\varepsilon r, r}(\widehat{\cdot}))} |v(x)|^2 dx. \end{aligned}$$

Finally, recalling that $B = \bigcup_{\widehat{a} \in Z_\varepsilon} M_\varepsilon([\widehat{a}, 0]^T + Y)$ and what has been said in Remark 3.8, summing the above inequality over all $\widehat{a} \in Z_\varepsilon$ yields

$$\begin{aligned} \|Q_\varepsilon v\|_{L^2(B)}^2 &= \sum_{\widehat{a} \in Z_\varepsilon} \|Q_\varepsilon v\|_{L^2(M_\varepsilon([\widehat{a}, 0]^T + Y))}^2 \\ &\leq C(V) \sum_{\widehat{a} \in Z_\varepsilon} \sum_{i=1}^8 \int_{B \cap ([\varepsilon \widehat{a}, 0]^T + K_{\varepsilon r, r}(\widehat{\cdot}))} |v(x)|^2 dx \leq C(V) \|v\|_{L^2(B^{\varepsilon\text{-perf}})}^2 \end{aligned}$$

for some constant $C(V)$ depending on the void V only. This finishes the proof of the first assertion.

As concerns the second statement of the proposition, the proof will be split into several steps. In the first step it will be proved that for all $\widehat{a} \in Z_\varepsilon$ such that $M_\varepsilon([\widehat{a}, 0]^T + Y)$ does not touch the lateral boundary of B one has

$$\|\partial_1(Q_\varepsilon v)\|_{L^2(M_\varepsilon([\widehat{a}, 0]^T + Y))}^2 \leq C(V) \|\partial_1 v\|_{L^2([\varepsilon \widehat{a}, 0]^T + E_{\varepsilon r}^1)}^2 \quad (3.46)$$

$$\|\partial_2(Q_\varepsilon v)\|_{L^2(M_\varepsilon([\widehat{a}, 0]^T + Y))}^2 \leq C(V) \|\partial_2 v\|_{L^2([\varepsilon \widehat{a}, 0]^T + E_{\varepsilon r}^2)}^2 \quad (3.47)$$

and

$$\|\partial_3(Q_\varepsilon v)\|_{L^2(M_\varepsilon([\widehat{a}, 0]^T + Y))}^2 \leq C(V) \|\partial_3 v\|_{L^2([\varepsilon \widehat{a}, 0]^T + E_{\varepsilon r}^3)}. \quad (3.48)$$

The sets $E_{\varepsilon r}^1$, $E_{\varepsilon r}^2$ and $E_{\varepsilon r}^3$ are depicted in Figure 3.10 below. A second step will then reveal that for those $\widehat{a} \in Z_\varepsilon$ such that $M_\varepsilon([\widehat{a}, 0]^T + Y)$ touches the lateral boundary of B one has similar estimates, namely

$$\begin{aligned} \|\partial_1(Q_\varepsilon v)\|_{L^2(M_\varepsilon([\widehat{a}, 0]^T + Y))}^2 &\leq C(V) \left(\|\partial_1 v\|_{L^2(B \cap ([\varepsilon \widehat{a}, 0]^T + E_{\varepsilon r}^1))}^2 \right. \\ &\quad \left. + \sum_{i=1}^8 \int_{B \cap ([\varepsilon \widehat{a}, 0]^T + K_{\varepsilon r, r}(\widehat{\cdot}))} \left| [\widehat{\nabla} v(x), \frac{1}{\varepsilon} \partial_3 v(x)] \right|^2 d\xi \right), \quad (3.49) \end{aligned}$$

$$\begin{aligned} \|\partial_2(Q_\varepsilon v)\|_{L^2(M_\varepsilon([\widehat{a}, 0]^T + Y))}^2 &\leq C(V) \left(\|\partial_2 v\|_{L^2(B \cap ([\varepsilon \widehat{a}, 0]^T + E_{\varepsilon r}^2))}^2 \right. \\ &\quad \left. + \sum_{i=1}^8 \int_{B \cap ([\varepsilon \widehat{a}, 0]^T + K_{\varepsilon r, r}(\widehat{\cdot}))} \left| [\widehat{\nabla} v(x), \frac{1}{\varepsilon} \partial_3 v(x)] \right|^2 d\xi \right) \quad (3.50) \end{aligned}$$

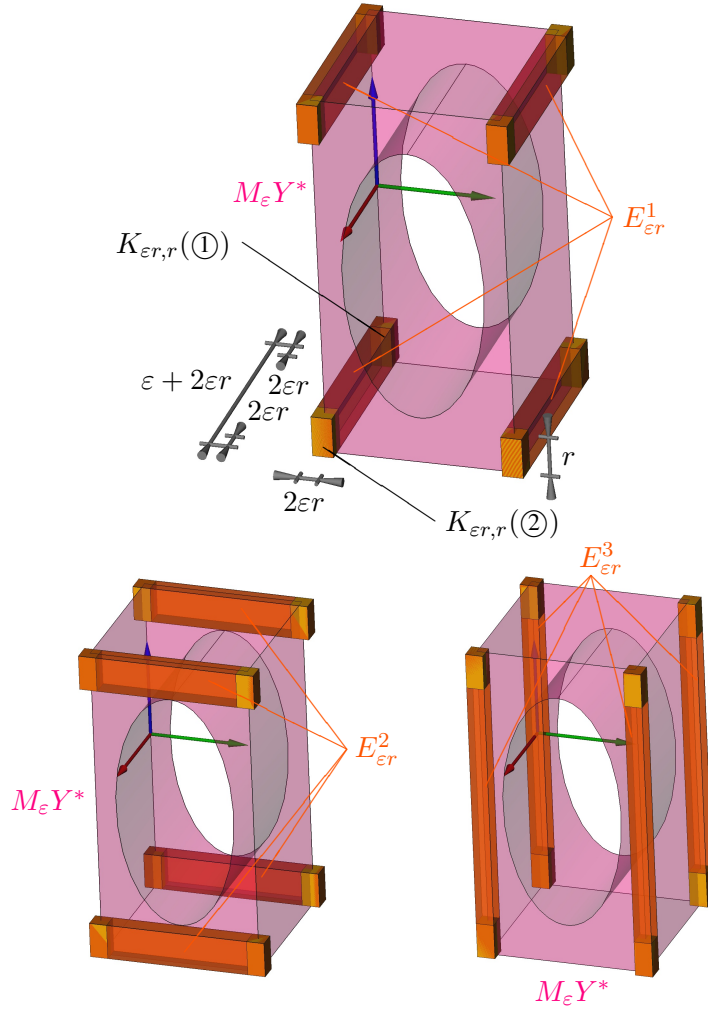


Figure 3.10: The sets $E_{\epsilon r}^1$, $E_{\epsilon r}^2$ and $E_{\epsilon r}^3$ in the rescaled perforated periodicity cell $M_{\epsilon} Y^*$

and moreover

$$\|\partial_3(Q_{\epsilon} v)\|_{L^2(M_{\epsilon}([\hat{a}, 0]^T + Y))}^2 \leq C(V) \|\partial_3 v\|_{L^2(B \cap ([\epsilon \hat{a}, 0]^T + E_{\epsilon r}^3))}^2. \quad (3.51)$$

Then, summing the estimates (3.46), (3.47) and (3.48) (multiplied by $\frac{1}{\epsilon^2}$) over all $\hat{a} \in Z_{\epsilon}$ such that the corresponding cell $M_{\epsilon}([\hat{a}, 0]^T + Y)$ does not touch the lateral boundary of B , and furthermore adding the sum of (3.49), (3.50) and (3.51) (multiplied by $\frac{1}{\epsilon^2}$) for those $\hat{a} \in Z_{\epsilon}$ such that $M_{\epsilon}([\hat{a}, 0]^T + Y)$ touches the lateral boundary of B yields the second assertion. Again it should be noticed that by the assumption (ϵ -Tiling) on ω and ϵ it is $B = \bigcup_{\hat{a} \in Z_{\epsilon}} M_{\epsilon}([\hat{a}, 0]^T + Y)$ and $B^{\epsilon\text{-perf}} = \bigcup_{\hat{a} \in Z_{\epsilon}} M_{\epsilon}([\hat{a}, 0]^T + Y^*)$.

Step 1. For simplicity I will only prove (3.46), since both (3.47) and (3.48) can be verified by the same strategy. Let $\hat{a} \in Z_{\epsilon}$ be such that $M_{\epsilon}([\hat{a}, 0]^T + Y)$ does not touch the lateral boundary of B . Then, according to the Definition 3.18 of the interpolation

operator Q_ε it is

$$Q_\varepsilon(x) = (\tilde{Q}_\varepsilon b)(x - M_\varepsilon[\hat{a}, 0]^T)$$

and by Proposition 3.17 one has

$$\|\partial_1(Q_\varepsilon)\|_{L^2(M_\varepsilon([\hat{a}, 0]^T + Y))}^2 \leq \varepsilon^2 C_Q \left(\left(\frac{b_2-b_1}{\varepsilon}\right)^2 + \left(\frac{b_4-b_3}{\varepsilon}\right)^2 + \left(\frac{b_6-b_5}{\varepsilon}\right)^2 + \left(\frac{b_8-b_7}{\varepsilon}\right)^2 \right) \quad (3.52)$$

where

$$b := \left[\int_{[\varepsilon\hat{a}, 0]^T + K_{\varepsilon r, r}(\textcircled{1})} v(\xi) \, d\xi, \dots, \int_{[\varepsilon\hat{a}, 0]^T + K_{\varepsilon r, r}(\textcircled{8})} v(\xi) \, d\xi \right] \in \mathbb{R}^8.$$

Exemplarily I will consider the term $((b_2 - b_1)/\varepsilon)^2$ while noticing that the others can be treated in an analogous manner. Also for the sake of a brief notation I will assume $\hat{a} = 0$. Then, by the definition of b it follows

$$\begin{aligned} \left(\frac{b_2-b_1}{\varepsilon}\right)^2 &= \frac{1}{\varepsilon^2} \left| \frac{1}{\text{vol } K_{\varepsilon r, r}(\textcircled{2})} \int_{K_{\varepsilon r, r}(\textcircled{2})} v(x) \, dx - \frac{1}{\text{vol } K_{\varepsilon r, r}(\textcircled{1})} \int_{K_{\varepsilon r, r}(\textcircled{1})} v(x) \, dx \right|^2 \\ &= \frac{1}{\varepsilon^2} \left| \frac{1}{\text{vol } K_{\varepsilon r, r}(\textcircled{1})} \int_{K_{\varepsilon r, r}(\textcircled{1})} v(x + \varepsilon e_1) - v(x) \, dx \right|^2, \end{aligned} \quad (3.53)$$

where I used that by definition $K_{\varepsilon r, r}(\textcircled{2}) = \varepsilon e_1 + K_{\varepsilon r, r}(\textcircled{1})$ (see Figure 3.9 (left)). As for almost all $x \in K_{\varepsilon r, r}(\textcircled{1})$ one can write

$$v(x + \varepsilon e_1) - v(x) \, dx = \int_0^\varepsilon \partial_1 v(x + se_1) \, ds,$$

twice applying Jensen's inequality leads to

$$\begin{aligned} \left(\frac{b_2-b_1}{\varepsilon}\right)^2 &= \frac{1}{\varepsilon^2} \left| \frac{1}{\text{vol } K_{\varepsilon r, r}(\textcircled{1})} \int_{K_{\varepsilon r, r}(\textcircled{1})} \int_0^\varepsilon \partial_1 v(x + se_1) \, ds \, dx \right|^2 \\ &\leq \frac{1}{\varepsilon^2} \frac{1}{\text{vol } K_{\varepsilon r, r}(\textcircled{1})} \int_{K_{\varepsilon r, r}(\textcircled{1})} \left| \int_0^\varepsilon \partial_1 v(x + se_1) \, ds \right|^2 \, dx \\ &\leq \frac{1}{\varepsilon^2} \frac{1}{\text{vol } K_{\varepsilon r, r}(\textcircled{1})} \int_{K_{\varepsilon r, r}(\textcircled{1})} \varepsilon \int_0^\varepsilon |\partial_1 v(x + se_1)|^2 \, ds \, dx. \end{aligned}$$

Here, a quite simple but crucial trick comes into play. In fact, enlarging the line integral for any $x \in K_{\varepsilon r, r}(\textcircled{1}) = \textcircled{1} + [-\varepsilon r, \varepsilon r]^2 \times [0, r]$ like

$$\begin{aligned} \int_0^\varepsilon |\partial_1 v(x + se_1)|^2 \, ds \\ = \int_{x_1}^{x_1+\varepsilon} |\partial_1 v(s, x_2, x_3)|^2 \, ds \leq \int_{-\varepsilon r}^{\varepsilon r+\varepsilon} |\partial_1 v(s, x_2, x_3)|^2 \, ds \end{aligned}$$

makes it *independent* of x_1 . Thus, one obtains

$$\begin{aligned}
\left(\frac{b_2-b_1}{\varepsilon}\right)^2 &\leq \frac{1}{\varepsilon} \frac{1}{\text{vol } K_{\varepsilon r, r}(\textcircled{1})} \int_{K_{\varepsilon r, r}(\textcircled{1})} \int_{-\varepsilon r}^{\varepsilon r + \varepsilon} |\partial_1 v(s, x_2, x_3)|^2 \, ds \, dx \\
&= \frac{1}{\varepsilon} \frac{2\varepsilon r}{\text{vol } K_{\varepsilon r, r}(\textcircled{1})} \int_{[-\varepsilon r, \varepsilon r] \times [-\frac{1}{2}, -\frac{1}{2} + r]} \int_{-\varepsilon r}^{\varepsilon r + \varepsilon} |\partial_1 v(s, x_2, x_3)|^2 \, ds \, d(x_2, x_3) \\
&= \frac{2r}{\text{vol } K_{\varepsilon r, r}(\textcircled{1})} \int_{[-\varepsilon r, \varepsilon + \varepsilon r] \times [-\varepsilon r, \varepsilon r] \times [-\frac{1}{2}, -\frac{1}{2} + r]} |\partial_1 v(x_1, x_2, x_3)|^2 \, dx \\
&= \frac{2r}{\text{vol } K_{\varepsilon r, r}(\textcircled{1})} \int_{\textcircled{1} + [-\varepsilon r, \varepsilon + \varepsilon r] \times [-\varepsilon r, \varepsilon r] \times [0, r]} |\partial_1 v(x)|^2 \, dx,
\end{aligned}$$

where the set $\textcircled{1} + [-\varepsilon r, \varepsilon + \varepsilon r] \times [-\varepsilon r, \varepsilon r] \times [0, r]$ is easily identified as a the subset of $E_{\varepsilon r}^1$ along the $\textcircled{1} - \textcircled{2}$ edge in $M_\varepsilon Y^*$ as depicted in Figure 3.9. Repeating this argument for $((b_4 - b_3)/\varepsilon)^2$, $((b_6 - b_5)/\varepsilon)^2$ and $((b_8 - b_7)/\varepsilon)^2$ leads by means of (3.52) to

$$\begin{aligned}
\|\partial_1(Q_\varepsilon v)\|_{L^2(M_\varepsilon Y)}^2 &\leq \varepsilon^2 C_Q \frac{2r}{\text{vol } K_{\varepsilon r, r}(\textcircled{1})} \int_{E_{\varepsilon r}^1} |\partial_1 v(x)|^2 \, dx \\
&\leq C_Q \frac{2r}{4r^3} \int_{E_{\varepsilon r}^1} |\partial_1 v(x)|^2 \, dx
\end{aligned}$$

which is nothing but (3.46) (for $\widehat{a} = 0$).

Step 2. Again, for simplicity I will only prove (3.49) – both (3.50) and (3.51) can be proved by similar arguments. Let $\widehat{a} \in Z_\varepsilon$ be such that $M_\varepsilon([\widehat{a}, 0]^T + Y)$ touches the lateral boundary of B . Then, just like in the previous step the Definition 3.18 of the interpolation operator Q_ε over $M_\varepsilon([\widehat{a}, 0]^T + Y)$ gives

$$Q_\varepsilon(x) = (\tilde{Q}_\varepsilon b)(x - M_\varepsilon[\widehat{a}, 0]^T)$$

and Proposition 3.17 once more reveals

$$\|\partial_1(Q_\varepsilon)\|_{L^2(M_\varepsilon([\widehat{a}, 0]^T + Y))}^2 \leq \varepsilon^2 C_Q \left(\left(\frac{b_2-b_1}{\varepsilon}\right)^2 + \left(\frac{b_4-b_3}{\varepsilon}\right)^2 + \left(\frac{b_6-b_5}{\varepsilon}\right)^2 + \left(\frac{b_8-b_7}{\varepsilon}\right)^2 \right) \quad (3.54)$$

where

$$b := \left[\int_{B \cap ([\varepsilon \widehat{a}, 0]^T + K_{\varepsilon r, r}(\textcircled{1}))} v(\xi) \, d\xi, \dots, \int_{B \cap ([\varepsilon \widehat{a}, 0]^T + K_{\varepsilon r, r}(\textcircled{8}))} v(\xi) \, d\xi \right] \in \mathbb{R}^8.$$

The difference to the situation analyzed in the previous step is that the mean values defining the various entries of b are now taken over the *intersection* of $([\varepsilon \widehat{a}, 0]^T + K_{\varepsilon r, r}(\textcircled{i}))$, $i = 1, \dots, 8$, with B . In particular, these sets are no longer simple translates of one and the same set, which is why the argument used in (3.53) cannot be simply

transferred to the present situation. Yet, like shown in Figure 3.9 (bottom), due to the base ω of B being a rectangle that is tiled by copies of $\varepsilon\widehat{Y}$, only two different situations for relative position of the rescaled perforated periodicity cell $M_\varepsilon([\widehat{a}, 0]^T + Y^*)$ and the lateral boundary of B can occur. Either $M_\varepsilon([\widehat{a}, 0]^T + Y^*)$ comes to lie in a corner of B or shares one of its faces with the lateral boundary of B . As it can be seen from Figure

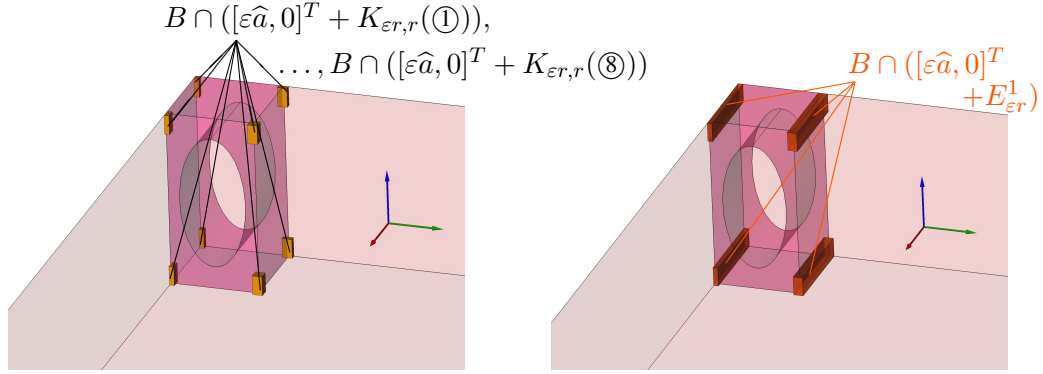


Figure 3.11: The situation of $M_\varepsilon([\widehat{a}, 0]^T + Y^*)$ lying in a corner of B 's lateral boundary

3.9, the lateral boundary of B then cuts the sets $B \cap ([\varepsilon\widehat{a}, 0]^T + K_{\varepsilon r, r}(\textcircled{1})), \dots, B \cap ([\varepsilon\widehat{a}, 0]^T + K_{\varepsilon r, r}(\textcircled{8}))$ always such that neighboring sets are either congruent or one is congruent to a subset of the other. For instance, consider the situation shown in Figure 3.11 – all other cases can actually be treated in an analogous manner. Therein, $B \cap ([\varepsilon\widehat{a}, 0]^T + K_{\varepsilon r, r}(\textcircled{1}))$ is congruent to a subset of $B \cap ([\varepsilon\widehat{a}, 0]^T + K_{\varepsilon r, r}(\textcircled{2}))$. Evaluating the term $((b_2 - b_1)/\varepsilon)^2$ like in Step 1 now leads to (again, for a clearer exposition let $\widehat{a} = 0$)

$$\left(\frac{b_2 - b_1}{\varepsilon}\right)^2 \quad (3.55)$$

$$\begin{aligned} &= \frac{1}{\varepsilon^2} \left| \frac{1}{\text{vol}(B \cap K_{\varepsilon r, r}(\textcircled{2}))} \int_{B \cap K_{\varepsilon r, r}(\textcircled{2})} v(x) \, dx - \frac{1}{\text{vol}(B \cap K_{\varepsilon r, r}(\textcircled{1}))} \int_{B \cap K_{\varepsilon r, r}(\textcircled{1})} v(x) \, dx \right|^2 \\ &\leq \frac{2}{\varepsilon^2} \left| \frac{1}{\text{vol}(B \cap K_{\varepsilon r, r}(\textcircled{1}))} \int_{\textcircled{2} - \textcircled{1} + B \cap K_{\varepsilon r, r}(\textcircled{1})} v(x) \, dx \right. \\ &\quad \left. - \frac{1}{\text{vol}(B \cap K_{\varepsilon r, r}(\textcircled{1}))} \int_{B \cap K_{\varepsilon r, r}(\textcircled{1})} v(x) \, dx \right|^2 \quad (3.56) \end{aligned}$$

$$\begin{aligned} &+ \frac{2}{\varepsilon^2} \left| \frac{1}{\text{vol}(B \cap K_{\varepsilon r, r}(\textcircled{2}))} \int_{B \cap K_{\varepsilon r, r}(\textcircled{2})} v(x) \, dx \right. \\ &\quad \left. - \frac{1}{\text{vol}(B \cap K_{\varepsilon r, r}(\textcircled{1}))} \int_{\textcircled{2} - \textcircled{1} + B \cap K_{\varepsilon r, r}(\textcircled{1})} v(x) \, dx \right|^2 \quad (3.57) \end{aligned}$$

Now, having noticed that in the present situation depicted in Figure 3.11 (left) $B \cap$

$K_{\varepsilon r, r}(\textcircled{1}) = \textcircled{1} + [0, \varepsilon r]^2 \times [0, r]$ and $\textcircled{2} - \textcircled{1} = \varepsilon e_1$ in the rescaled periodicity cell $M_\varepsilon Y$ (see Figure 3.9 (top)), the same methods as employed in the first step show

$$\begin{aligned}
& \frac{1}{\varepsilon^2} \left| \frac{1}{\text{vol}(B \cap K_{\varepsilon r, r}(\textcircled{1}))} \int_{\textcircled{2} - \textcircled{1} + B \cap K_{\varepsilon r, r}(\textcircled{1})} v(x) \, dx \right. \\
& \quad \left. - \frac{1}{\text{vol}(B \cap K_{\varepsilon r, r}(\textcircled{1}))} \int_{B \cap K_{\varepsilon r, r}(\textcircled{1})} v(x) \, dx \right|^2 \\
&= \frac{1}{\varepsilon^2} \left| \frac{1}{\text{vol}(B \cap K_{\varepsilon r, r}(\textcircled{1}))} \int_{\textcircled{1} + [0, \varepsilon r]^2 \times [0, r]} v(x + \varepsilon e_1) - v(x) \, dx \right|^2 \\
&\leq \frac{r}{\text{vol}(B \cap K_{\varepsilon r, r}(\textcircled{1}))} \int_{\textcircled{1} + [0, \varepsilon + \varepsilon r] \times [0, \varepsilon r] \times [0, r]} |\partial_1 v(x)|^2 \, dx \\
&= \frac{r}{\text{vol}(B \cap K_{\varepsilon r, r}(\textcircled{1}))} \int_{B \cap (\textcircled{1} + [-\varepsilon r, \varepsilon + \varepsilon r] \times [-\varepsilon r, \varepsilon r] \times [0, r])} |\partial_1 v(x)|^2 \, dx.
\end{aligned}$$

As concerns the term (3.57), turning to Figure 3.11 (left) allows to identify

$$\begin{aligned}
B \cap K_{\varepsilon r, r}(\textcircled{2}) &= \textcircled{2} + [-\varepsilon r, \varepsilon r] \times [0, \varepsilon r] \times [0, r], \\
\textcircled{2} - \textcircled{1} + B \cap K_{\varepsilon r, r}(\textcircled{1}) &= \textcircled{2} + [0, \varepsilon r] \times [0, \varepsilon r] \times [0, r],
\end{aligned}$$

and a simple rescaling argument $\tilde{v}_\varepsilon(x) := v(M_\varepsilon x + \textcircled{2})$ yields

$$\begin{aligned}
& \left| \frac{1}{\text{vol}(B \cap K_{\varepsilon r, r}(\textcircled{2}))} \int_{B \cap K_{\varepsilon r, r}(\textcircled{2})} v(x) \, dx \right. \\
& \quad \left. - \frac{1}{\text{vol}(B \cap K_{\varepsilon r, r}(\textcircled{1}))} \int_{\textcircled{2} - \textcircled{1} + B \cap K_{\varepsilon r, r}(\textcircled{1})} v(x) \, dx \right|^2 \\
&= \left| \frac{\varepsilon^2}{\text{vol}(B \cap K_{\varepsilon r, r}(\textcircled{2}))} \int_{[-r, r] \times [0, r] \times [0, r]} \tilde{v}_\varepsilon(x) \, dx \right. \\
& \quad \left. - \frac{\varepsilon^2}{\text{vol}(B \cap K_{\varepsilon r, r}(\textcircled{1}))} \int_{[0, r] \times [0, r] \times [0, r]} \tilde{v}_\varepsilon(x) \, dx \right|^2 \\
&= \left| \int_{[-r, r] \times [0, r] \times [0, r]} \tilde{v}_\varepsilon(x) \, dx - \int_{[0, r] \times [0, r] \times [0, r]} \tilde{v}_\varepsilon(x) \, dx \right|^2 \\
&\leq \frac{1}{\text{vol}([-r, r] \times [0, r] \times [0, r])} \int_{[-r, r] \times [0, r] \times [0, r]} \left| \tilde{v}_\varepsilon(x) - \int_{[0, r] \times [0, r] \times [0, r]} \tilde{v}_\varepsilon(\xi) \, d\xi \right|^2 \, dx.
\end{aligned}$$

Eventually, the term on the right hand side can be estimated by means of a suitable version of Poincaré's inequality, like *e.g.* found in [Jost, 2007, Corollary 9.1.4]. More

precisely

$$\begin{aligned}
& \left| \frac{1}{\text{vol}(B \cap K_{\varepsilon r, r}(\textcircled{2}))} \int_{B \cap K_{\varepsilon r, r}(\textcircled{2})} v(x) \, dx \right. \\
& \quad \left. - \frac{1}{\text{vol}(B \cap K_{\varepsilon r, r}(\textcircled{1}))} \int_{\textcircled{2} - \textcircled{1} + B \cap K_{\varepsilon r, r}(\textcircled{1})} v(x) \, dx \right|^2 \\
& \leq C(r) \int_{[-r, r] \times [0, r] \times [0, r]} |\nabla \tilde{v}_\varepsilon(x)|^2 \, dx \\
& = C(r) \int_{[-r, r] \times [0, r] \times [0, r]} |M_\varepsilon \nabla v(M_\varepsilon x + \textcircled{2})|^2 \, dx \\
& = C(r) \frac{1}{\varepsilon^2} \int_{\textcircled{2} + [-\varepsilon r, \varepsilon r] \times [0, \varepsilon r] \times [0, r]} |M_\varepsilon \nabla v(x)|^2 \, dx \\
& = C(r) \int_{B \cap K_{\varepsilon r, r}(\textcircled{2})} \left| [\widehat{\nabla} v(x), \frac{1}{\varepsilon} \partial_3 v(x)] \right|^2 \, dx,
\end{aligned}$$

where $C(r)$ is a positive constant depending on r only. Returning to (3.55) one can now further estimate (3.56) and (3.57) like

$$\begin{aligned}
\left(\frac{b_2 - b_1}{\varepsilon} \right)^2 & \leq \frac{2r}{\text{vol}(B \cap K_{\varepsilon r, r}(\textcircled{1}))} \int_{B \cap (\textcircled{1} + [-\varepsilon r, \varepsilon + \varepsilon r] \times [-\varepsilon r, \varepsilon r] \times [0, r])} |\partial_1 v(x)|^2 \, dx \\
& \quad + \frac{2C(r)}{\varepsilon^2} \int_{B \cap K_{\varepsilon r, r}(\textcircled{2})} \left| [\widehat{\nabla} v(x), \frac{1}{\varepsilon} \partial_3 v(x)] \right|^2 \, dx \\
& = \frac{C(r)}{\varepsilon^2} \left(\int_{B \cap (\textcircled{1} + [-\varepsilon r, \varepsilon + \varepsilon r] \times [-\varepsilon r, \varepsilon r] \times [0, r])} |\partial_1 v(x)|^2 \, dx \right. \\
& \quad \left. + \int_{B \cap K_{\varepsilon r, r}(\textcircled{2})} \left| [\widehat{\nabla} v(x), \frac{1}{\varepsilon} \partial_3 v(x)] \right|^2 \, dx \right).
\end{aligned}$$

for $C(r)$ again a positive constant depending on r only. By repeating the same argument for $((b_4 - b_3)/\varepsilon)^2$, $((b_6 - b_5)/\varepsilon)^2$ and $((b_8 - b_7)/\varepsilon)^2$ one can by means of (3.54) and the definition of $E_{\varepsilon r}^1$ as shown in Figure 3.10 infer

$$\begin{aligned}
\|\partial_1(Q_\varepsilon v)\|_{L^2(M_\varepsilon Y)}^2 & \leq \varepsilon^2 C_Q \frac{C(r)}{\varepsilon^2} \left(\int_{B \cap E_{\varepsilon r}^1} |\partial_1 v(x)|^2 \, dx \right. \\
& \quad \left. + \sum_{i=1}^8 \int_{B \cap K_{\varepsilon r, r}(\textcircled{i})} \left| [\widehat{\nabla} v(x), \frac{1}{\varepsilon} \partial_3 v(x)] \right|^2 \, dx \right) \\
& = C(V) \left(\int_{B \cap E_{\varepsilon r}^1} |\partial_1 v(x)|^2 \, dx \right. \\
& \quad \left. + \sum_{i=1}^8 \int_{B \cap K_{\varepsilon r, r}(\textcircled{i})} \left| [\widehat{\nabla} v(x), \frac{1}{\varepsilon} \partial_3 v(x)] \right|^2 \, dx \right)
\end{aligned}$$

which is nothing but (3.49) (with $\widehat{a} = 0$). \square

Another important result on the interpolation operator $Q_\varepsilon : W^{1,2}(B^{\varepsilon\text{-perf}}) \rightarrow W^{1,2}(B)$ is the following estimation of the interpolation error measured in the L^2 -norm over $B^{\varepsilon\text{-perf}}$.

Proposition 3.20. *Again assume the notation and assumptions of Definition 3.18 to hold and consider $v \in W^{1,2}(B^{\varepsilon\text{-perf}})$. The interpolation error $Q_\varepsilon v - v$ of the interpolation operator $Q_\varepsilon : W^{1,2}(B^{\varepsilon\text{-perf}}) \rightarrow W^{1,2}(B)$ can over $B^{\varepsilon\text{-perf}}$ be estimated like*

$$\|Q_\varepsilon v - v\|_{L^2(B^{\varepsilon\text{-perf}})}^2 \leq \varepsilon^2 C(V) \left(\|\widehat{\nabla} v\|_{L^2(B^{\varepsilon\text{-perf}})}^2 + \left\| \frac{1}{\varepsilon} \partial_3 v \right\|_{L^2(B^{\varepsilon\text{-perf}})}^2 \right)$$

where $C(V)$ is a positive constant depending on the void V in the periodicity cell Y only.

Proof. By the assumption (ε -Tiling) on the base ω of $B = \omega \times [-\frac{1}{2}, \frac{1}{2}]$ one has again

$$B = \bigcup_{\widehat{a} \in Z_\varepsilon} M_\varepsilon \left([\widehat{a}, 0]^T + Y \right) \quad \text{and} \quad B^{\varepsilon\text{-perf}} = \bigcup_{\widehat{a} \in Z_\varepsilon} M_\varepsilon \left([\widehat{a}, 0]^T + Y^* \right).$$

Now, for some \widehat{a} in Z_ε the interpolation error over the rescaled perforated periodicity cell $M_\varepsilon([\widehat{a}, 0]^T + Y^*)$ is

$$\begin{aligned} & \int_{M_\varepsilon([\widehat{a}, 0]^T + Y^*)} |(Q_\varepsilon v)(x) - v(x)|^2 dx \\ &= \int_{M_\varepsilon([\widehat{a}, 0]^T + Y^*)} |(\widetilde{Q}_\varepsilon b)(x - M_\varepsilon[\widehat{a}, 0]^T) - v(x)|^2 dx \\ &= \varepsilon^2 \int_{Y^*} |(Qb)(x) - v(M_\varepsilon[\widehat{a}, 0]^T + M_\varepsilon x)|^2 dx \end{aligned}$$

where $b \in \mathbb{R}^8$ is defined through

$$b := \left[\int_{B \cap ([\varepsilon \widehat{a}, 0]^T + K_{\varepsilon r, r}(\textcircled{1}))} v(\xi) d\xi, \dots, \int_{B \cap ([\varepsilon \widehat{a}, 0]^T + K_{\varepsilon r, r}(\textcircled{8}))} v(\xi) d\xi \right] \in \mathbb{R}^8.$$

Again see Figure 3.9 for an illustration of the sets $K_{\varepsilon r, r}(\textcircled{1}), \dots, K_{\varepsilon r, r}(\textcircled{8})$. Inserting the Definition 3.14 for the Q_1 -interpolation operator Q leads to

$$\begin{aligned} & \int_{M_\varepsilon([\widehat{a}, 0]^T + Y^*)} |(Q_\varepsilon v)(x) - v(x)|^2 dx \\ &= \varepsilon^2 \int_{Y^*} \left| \sum_{i=1}^8 b_i q_i(x) - v(M_\varepsilon[\widehat{a}, 0]^T + M_\varepsilon x) \right|^2 dx \\ &= \varepsilon^2 \int_{Y^*} \left| \sum_{i=1}^8 b_i q_i(x) - \widetilde{v}_\varepsilon(x) \right|^2 dx \end{aligned} \tag{3.58}$$

for $\tilde{v}_\varepsilon(x) := v(M_\varepsilon[\widehat{a}, 0]^T + M_\varepsilon x)$. Furthermore, a simple calculation shows that

$$b := \left[\int_{(M_\varepsilon^{-1}B - [\widehat{a}, 0]^T) \cap K_r(\textcircled{1})} \tilde{v}_\varepsilon(\xi) \, d\xi, \dots, \int_{(M_\varepsilon^{-1}B - [\widehat{a}, 0]^T) \cap K_r(\textcircled{8})} \tilde{v}_\varepsilon(\xi) \, d\xi \right] \quad (3.59)$$

where $K_r(\textcircled{i}) := M_\varepsilon^{-1}K_{\varepsilon r, r}(\textcircled{i})$. See also the following Figure 3.12. As the poly-

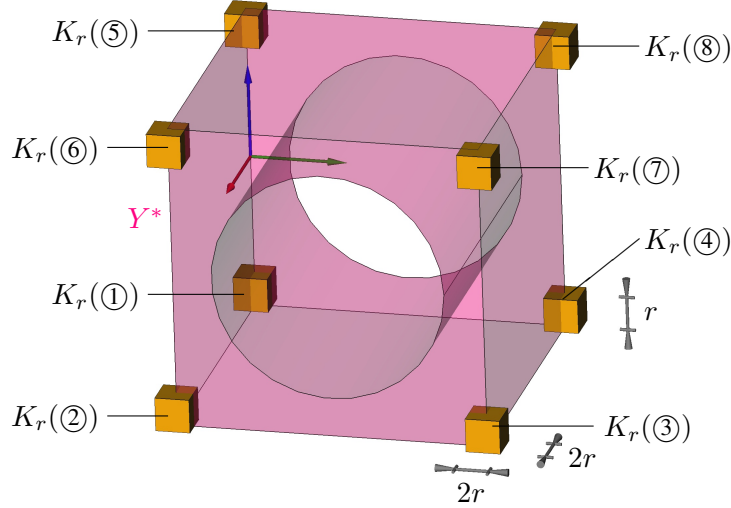


Figure 3.12: The congruent sets $K_r(\textcircled{1}), \dots, K_r(\textcircled{8})$ and their relative position to the perforated periodicity cell Y^*

mial shape functions q_i ($i = 1, \dots, 8$; see Definition 3.14) take values in $[0, 1]$ only and sum up to the constant function 1 over the periodicity cell Y , it follows from (3.58)

$$\begin{aligned} & \int_{M_\varepsilon([\widehat{a}, 0]^T + Y^*)} |(Q_\varepsilon v)(x) - v(x)|^2 \, dx \\ &= \varepsilon^2 \int_{Y^*} \left| \sum_{i=1}^8 (b_i - \tilde{v}_\varepsilon(x)) q_i(x) \right|^2 \, dx \\ &\leq \varepsilon^2 \int_{Y^*} \left(\sum_{i=1}^8 |b_i - \tilde{v}_\varepsilon(x)|^2 \right) \left(\sum_{i=1}^8 |q_i(x)|^2 \right) \, dx \\ &\leq 8\varepsilon^2 \sum_{i=1}^8 \int_{Y^*} |\tilde{v}_\varepsilon(x) - b_i|^2 \, dx. \end{aligned} \quad (3.60)$$

Just like the proof of Proposition 3.19, also the remainder of this proof will be split into two steps. The first step considers the situation of rescaled perforated periodicity cells $M_\varepsilon([\widehat{a}, 0]^T + Y^*)$ that do not touch the lateral boundary of B , whereas the second step considers such that do touch the lateral boundary.

Step 1. In the situation where the rescaled periodicity cell $M_\varepsilon([\widehat{a}, 0]^T + Y^*)$ does not touch the lateral boundary of B , it follows that $B \cap ([\varepsilon\widehat{a}, 0]^T + K_{\varepsilon r, r}(\textcircled{i})) =$

$[\varepsilon \widehat{a}, 0]^T + K_{\varepsilon r, r}(\odot)$ for all $i = 1, \dots, 8$, hence also after the above change of variables $(M_\varepsilon^{-1}B - [\widehat{a}, 0]^T) \cap K_r(\odot) = K_r(\odot)$ for all $i = 1, \dots, 8$. Now, let

$$\tilde{P}_i : W^{1,2}(Y^* \cup K_r(\odot)) \rightarrow W^{1,2}([-r, 1+r]^2 \times [-\frac{1}{2}, \frac{1}{2}])$$

be an extension operator according to [Oleřnik et al., 1992, Lemma 4.1]. Turning to the term $\int_{Y^*} |b_i - \tilde{v}_\varepsilon(x)|^2 dx$ in (3.60) and recalling (3.59) yields

$$\begin{aligned} & \int_{Y^*} |\tilde{v}_\varepsilon(x) - b_i|^2 dx \\ &= \int_{Y^*} \left| \tilde{v}_\varepsilon(x) - \int_{K_r(\odot)} \tilde{v}_\varepsilon(\xi) d\xi \right|^2 dx \\ &= \int_{Y^*} \left| (\tilde{P}_i \tilde{v}_\varepsilon)(x) - \int_{K_r(\odot)} (\tilde{P}_i \tilde{v}_\varepsilon)(\xi) d\xi \right|^2 dx \\ &\leq \int_{[-r, 1+r]^2 \times [-\frac{1}{2}, \frac{1}{2}]} \left| (\tilde{P}_i \tilde{v}_\varepsilon)(x) - \int_{K_r(\odot)} (\tilde{P}_i \tilde{v}_\varepsilon)(\xi) d\xi \right|^2 dx. \end{aligned}$$

Here, it is once more the Poincaré inequality as stated in [Jost, 2007, Corollary 9.1.4] that allows to further estimate

$$\int_{Y^*} |\tilde{v}_\varepsilon(x) - b_i|^2 dx \leq C(r, i) \int_{[-r, 1+r]^2 \times [-\frac{1}{2}, \frac{1}{2}]} |\nabla(\tilde{P}_i \tilde{v}_\varepsilon)(x)|^2 dx.$$

for another positive constant $C(r, i)$ depending r and i through the enlarged periodicity cell $[-r, 1+r]^2 \times [-\frac{1}{2}, \frac{1}{2}]$ and the set $K_r(\odot)$ only. Now, due to the continuity of the extension operator \tilde{P}_i as stated in [Oleřnik et al., 1992, Lemma 4.1] the right hand side can be bounded from above like

$$\int_{Y^*} |\tilde{v}_\varepsilon(x) - b_i|^2 dx \leq C(r, i) \int_{Y^* \cup K_r(\odot)} |\nabla(\tilde{v}_\varepsilon)(x)|^2 dx$$

for another constant $C(r, i)$. Since by definition $\tilde{v}_\varepsilon(x) := v(M_\varepsilon[\widehat{a}, 0]^T + M_\varepsilon x)$ it follows $\nabla(\tilde{v}_\varepsilon)(x) = M_\varepsilon \nabla v(M_\varepsilon[\widehat{a}, 0]^T + M_\varepsilon x)$, and reverting the change of variables in v implies

$$\begin{aligned} & \int_{Y^*} |\tilde{v}_\varepsilon(x) - b_i|^2 dx \\ &\leq \frac{1}{\varepsilon^2} C(r, i) \int_{M_\varepsilon([\widehat{a}, 0]^T + Y^* \cup K_{\varepsilon r, r}(\odot))} |M_\varepsilon \nabla v(x)|^2 dx \\ &= C(r, i) \int_{M_\varepsilon([\widehat{a}, 0]^T + Y^* \cup K_{\varepsilon r, r}(\odot))} \left| [\widehat{\nabla} v(x), \frac{1}{\varepsilon} \partial_3 v(x)] \right|^2 dx. \end{aligned}$$

This together with (3.60) finally leads to

$$\begin{aligned}
& \int_{M_\varepsilon([\widehat{a}, 0]^T + Y^*)} |(Q_\varepsilon v)(x) - v(x)|^2 dx \\
& \leq 8\varepsilon^2 \sum_{i=1}^8 C(r, i) \int_{M_\varepsilon([\widehat{a}, 0]^T + Y^* \cup K_{\varepsilon r, r}(\widehat{\circledast}_i))} \left| [\widehat{\nabla} v(x), \frac{1}{\varepsilon} \partial_3 v(x)] \right|^2 dx \\
& \leq \varepsilon^2 C(V) \int_{M_\varepsilon([\widehat{a}, 0]^T + Y^* \cup K_{\varepsilon r, r}(\widehat{\circledast}_1) \cup \dots \cup K_{\varepsilon r, r}(\widehat{\circledast}_8))} \left| [\widehat{\nabla} v(x), \frac{1}{\varepsilon} \partial_3 v(x)] \right|^2 dx \quad (3.61)
\end{aligned}$$

where the constant $C(V)$ depends on V only (through $r > 0$ and the extension operators $\tilde{P}_1, \dots, \tilde{P}_8$; yet, these quantities are completely determined by the geometry of V , cf. Definition 3.18).

Step 2. Consider the case of $M_\varepsilon([\widehat{a}, 0]^T + Y^*)$ touching the lateral boundary of B . Other than in the previous step, the sets $B \cap ([\varepsilon \widehat{a}, 0]^T + K_{\varepsilon r, r}(\widehat{\circledast}_i))$ in the statement (3.59) of b result now in general as genuine subsets of $[\varepsilon \widehat{a}, 0]^T + K_{\varepsilon r, r}(\widehat{\circledast}_i)$. Nevertheless, if one could again find an extension operator

$$\begin{aligned}
\tilde{P}_i : W^{1,2}(Y^* \cup ((M_\varepsilon^{-1}B - [\widehat{a}, 0]^T) \cap K_r(\widehat{\circledast}_i))) \\
\rightarrow W^{1,2}([-r, 1+r]^2 \times [-\frac{1}{2}, \frac{1}{2}])
\end{aligned}$$

in the sense of [Oleřnik et al., 1992, Lemma 4.1], then the very same arguments as used in the first step would lead by (3.59) and (3.60) to

$$\begin{aligned}
& \int_{Y^*} |\tilde{v}_\varepsilon(x) - b_i|^2 dx \\
& = \int_{Y^*} \left| \tilde{v}_\varepsilon(x) - \int_{(M_\varepsilon^{-1}B - [\widehat{a}, 0]^T) \cap K_r(\widehat{\circledast}_i)} \tilde{v}_\varepsilon(\xi) d\xi \right|^2 dx \\
& = \int_{Y^*} \left| (\tilde{P}_i \tilde{v}_\varepsilon)(x) - \int_{(M_\varepsilon^{-1}B - [\widehat{a}, 0]^T) \cap K_r(\widehat{\circledast}_i)} (\tilde{P}_i \tilde{v}_\varepsilon)(\xi) d\xi \right|^2 dx \\
& \leq \int_{[-r, 1+r]^2 \times [-\frac{1}{2}, \frac{1}{2}]} \left| (\tilde{P}_i \tilde{v}_\varepsilon)(x) - \int_{(M_\varepsilon^{-1}B - [\widehat{a}, 0]^T) \cap K_r(\widehat{\circledast}_i)} (\tilde{P}_i \tilde{v}_\varepsilon)(\xi) d\xi \right|^2 dx.
\end{aligned}$$

and another application of [Jost, 2007, Corollary 9.1.4] yields

$$\begin{aligned}
& \int_{Y^*} |\tilde{v}_\varepsilon(x) - b_i|^2 dx \\
& \leq C(r, (M_\varepsilon^{-1}B - [\widehat{a}, 0]^T) \cap K_r(\widehat{\circledast}_i)) \int_{[-r, 1+r]^2 \times [-\frac{1}{2}, \frac{1}{2}]} \left| \nabla (\tilde{P}_i \tilde{v}_\varepsilon)(x) \right|^2 dx. \quad (3.62)
\end{aligned}$$

for a positive constant $C(r, (M_\varepsilon^{-1}B - [\widehat{a}, 0]^T) \cap K_r(\widehat{\circledast}_i))$ depending on $r > 0$ and the volume of the set $(M_\varepsilon^{-1}B - [\widehat{a}, 0]^T) \cap K_r(\widehat{\circledast}_i)$ only. Yet, as already outlined in Remark

3.8 and illustrated in Figure 3.9, due to the assumption (ε -Tiling) on the base ω of B and the convexity of ω , only two different geometric situations for the intersection of $M_\varepsilon([\hat{a}, 0]^T + Y^*)$ with the lateral boundary of B can occur. For the sets

$$(M_\varepsilon^{-1}B - [\hat{a}, 0]^T) \cap K_r(\textcircled{i}) = (M_\varepsilon^{-1}(\cdot) - [\hat{a}, 0]^T)(B \cap ([\varepsilon\hat{a}, 0]^T + K_{\varepsilon r, r}(\textcircled{i}))) \quad (3.63)$$

this means that the lateral boundary of B can either cut them in half or leaves a quarter of the original set $K_r(\textcircled{i})$ (cf. Figure 3.11). In other words, for all rescaled perforated periodicity cells $M_\varepsilon([\hat{a}, 0]^T + Y^*)$ touching the lateral boundary of B and all ε the sets (3.63) describe only a finite number of different geometries. This not only means that there is a common constant $C(V)$ in (3.62) for all $i = 1, \dots, 8$, but that the number of extension operators \tilde{P}_i is also finite and independent of ε . Thus, there is a common constant in the sense of [Oleřnik et al., 1992, Lemma 4.1] such that from (3.62) one can further deduce

$$\int_{Y^*} |\tilde{v}_\varepsilon(x) - b_i|^2 dx \leq C(V) \int_{Y^* \cup ((M_\varepsilon^{-1}B - [\hat{a}, 0]^T) \cap K_r(\textcircled{i}))} |\nabla(\tilde{v}_\varepsilon)(x)|^2 dx$$

for some common constant $C(V)$ depending on V only. Recalling $\nabla(\tilde{v}_\varepsilon)(x) = M_\varepsilon \nabla v(M_\varepsilon[\hat{a}, 0]^T + M_\varepsilon x)$ and reverting the change of variables $\tilde{v}_\varepsilon(x) := v(M_\varepsilon[\hat{a}, 0]^T + M_\varepsilon x)$ leads to

$$\begin{aligned} & \int_{Y^*} |\tilde{v}_\varepsilon(x) - b_i|^2 dx \\ & \leq \frac{1}{\varepsilon^2} C(V) \int_{B \cap M_\varepsilon([\hat{a}, 0]^T + Y^* \cup K_{\varepsilon r, r}(\textcircled{i}))} |M_\varepsilon \nabla v(x)|^2 dx \\ & = C(V) \int_{B \cap M_\varepsilon([\hat{a}, 0]^T + Y^* \cup K_{\varepsilon r, r}(\textcircled{i}))} \left| [\widehat{\nabla} v(x), \frac{1}{\varepsilon} \partial_3 v(x)] \right|^2 dx. \end{aligned}$$

Finally, from (3.60) one obtains

$$\begin{aligned} & \int_{M_\varepsilon([\hat{a}, 0]^T + Y^*)} |(Q_\varepsilon v)(x) - v(x)|^2 dx \\ & \leq 8\varepsilon^2 \sum_{i=1}^8 C(V) \int_{B \cap M_\varepsilon([\hat{a}, 0]^T + Y^* \cup K_{\varepsilon r, r}(\textcircled{i}))} \left| [\widehat{\nabla} v(x), \frac{1}{\varepsilon} \partial_3 v(x)] \right|^2 dx \\ & \leq \varepsilon^2 C(V) \int_{B \cap M_\varepsilon([\hat{a}, 0]^T + Y^* \cup K_{\varepsilon r, r}(\textcircled{1}) \cup \dots \cup K_{\varepsilon r, r}(\textcircled{8}))} \left| [\widehat{\nabla} v(x), \frac{1}{\varepsilon} \partial_3 v(x)] \right|^2 dx \quad (3.64) \end{aligned}$$

where again the constant $C(V)$ depends on V only.

Step 3. The proof of the proposition is now complete when summing (3.61) for all $\hat{a} \in Z_\varepsilon$ such that $M_\varepsilon([\hat{a}, 0]^T + Y^*)$ does not touch the lateral boundary of B and adding the sum of (3.64) over all $\hat{a} \in Z_\varepsilon$ such that $M_\varepsilon([\hat{a}, 0]^T + Y^*)$ touches the lateral boundary of B . \square

In the proof of Theorem 3.10 I mainly rely on the following result.

Lemma 3.21. *Assume the assumptions of Definition 3.18 to be valid and let $(\varepsilon_k)_k$ be a vanishing sequence of positive real numbers such that each of its elements ε_k is compatible with these assumptions. Also, let there be given the periodic unfolding operator $\widehat{\mathcal{T}}_{\varepsilon_k}^* : L^2(B^{\varepsilon\text{-perf}}) \rightarrow L^2(\omega \times Y^*)$ for the rescaled perforated thin domain $B^{\varepsilon\text{-perf}}$. Moreover, let $(v_{\varepsilon_k})_k$ be a sequence having bounded $L^2(B^{\varepsilon\text{-perf}})$ -norm and suppose $v_{\varepsilon_k} \in W^{1,2}(B^{\varepsilon\text{-perf}})$. Also, assume that there is a positive constant $C > 0$ such that*

$$\|\widehat{\nabla} v_{\varepsilon_k}\|_{L^2(B^{\varepsilon\text{-perf}}; \mathbb{R}^2)}^2 + \left\| \frac{1}{\varepsilon_k} \partial_3 v_{\varepsilon_k} \right\|_{L^2(B^{\varepsilon\text{-perf}})}^2 \leq C$$

for all ε_k . Then there is a subsequence $(\varepsilon_{k_\ell})_\ell$ and a function $v_0 \in W^{1,2}(B)$ which is independent of x_3 such that

$$\begin{aligned} Q_{\varepsilon_{k_\ell}} v_{\varepsilon_{k_\ell}} &\rightharpoonup v_0 && \text{in } W^{1,2}(B), \\ \widehat{\mathcal{T}}_{\varepsilon_{k_\ell}}^* v_{\varepsilon_{k_\ell}} &\rightharpoonup v_0(\widehat{x}, 0) && \text{in } L^2(\omega; W^{1,2}(Y^*)) \end{aligned}$$

and some $V_0 \in L^2(\omega; W_{\widehat{Y}\text{-per}}^{1,2}(Y^*))$ satisfying

$$\begin{aligned} &\left[\widehat{\mathcal{T}}_{\varepsilon_{k_\ell}}^* (\widehat{\nabla} v_{\varepsilon_{k_\ell}}), \widehat{\mathcal{T}}_{\varepsilon_{k_\ell}}^* \left(\frac{1}{\varepsilon_{k_\ell}} \partial_3 v_{\varepsilon_{k_\ell}} \right) \right] \\ &\quad \rightarrow \left[\widehat{\nabla} v_0(\widehat{x}, 0), 0 \right]^T + \nabla_y V_0(\widehat{x}, y) \quad \text{in } L^2(\omega \times Y^*; \mathbb{R}^3). \end{aligned}$$

Here, the space $W_{\widehat{Y}\text{-per}}^{1,2}(Y^*)$ is like in Subsection 3.2.3 the set of all $W^{1,2}(Y^*)$ -functions w that satisfy $w(y+e_\alpha) = w(y)$ for all $y \in \partial Y^* \cap \{y : y_\alpha = 0\}$ such that $y+e_\alpha \in \partial Y^*$, $\alpha = 1, 2$.

Proof. As concerns the first assertion of the lemma, from Proposition 3.19 and the $L^2(B^{\varepsilon\text{-perf}})$ -bounds on v_{ε_k} and $[\widehat{\nabla} v_{\varepsilon_k}, \frac{1}{\varepsilon_k} \partial_3 v_{\varepsilon_k}]$ one obtains

$$\|Q_{\varepsilon_k} v_{\varepsilon_k}\|_{L^2(B)} \leq C$$

and

$$\|\widehat{\nabla} (Q_{\varepsilon_k} v_{\varepsilon_k})\|_{L^2(B; \mathbb{R}^2)}^2 + \left\| \frac{1}{\varepsilon_k} \partial_3 (Q_{\varepsilon_k} v_{\varepsilon_k}) \right\|_{L^2(B^{\varepsilon\text{-perf}})}^2 \leq C$$

for all ε_k and some positive constant C . Hence, as claimed there is subsequence $(\varepsilon_{k_\ell})_\ell$ and a v_0 in $W^{1,2}(B)$ such that $Q_{\varepsilon_{k_\ell}} v_{\varepsilon_{k_\ell}} \rightharpoonup v_0$ in $W^{1,2}(B)$. Moreover, by the boundedness of $\left\| \frac{1}{\varepsilon_k} \partial_3 (Q_{\varepsilon_k} v_{\varepsilon_k}) \right\|_{L^2(B^{\varepsilon\text{-perf}})}$ one infers $\partial_3 v_0 = 0$ in B , i.e. v_0 is independent of x_3 in B .

To prove the second assertion, first one employs the definition of the unfolding operator $\widehat{\mathcal{T}}_{\varepsilon_k}^*$ for the rescaled perforated thin domain $B^{\varepsilon\text{-perf}}$ to infer

$$(\widehat{\mathcal{T}}_{\varepsilon_k}^* v_{\varepsilon_k})(\widehat{x}, y) = v_{\varepsilon_k} \left(\varepsilon_k \left[\frac{\widehat{x}}{\varepsilon_k} \right] + \varepsilon_k \widehat{y}, y_3 \right)$$

and

$$\begin{aligned}\nabla_y(\widehat{\mathcal{T}}_{\varepsilon_k}^* v_{\varepsilon_k})(\widehat{x}, y) &= \left[\varepsilon_k \widehat{\nabla} v_{\varepsilon_k} \left(\varepsilon_k \left[\frac{\widehat{x}}{\varepsilon_k} \right] + \varepsilon_k \widehat{y}, y_3 \right), \partial_3 v_{\varepsilon_k} \left(\varepsilon_k \left[\frac{\widehat{x}}{\varepsilon_k} \right] + \varepsilon_k \widehat{y}, y_3 \right) \right] \\ &= \varepsilon_k \left[\widehat{\mathcal{T}}_{\varepsilon_k}^* (\widehat{\nabla} v_{\varepsilon_k})(\widehat{x}, y), \widehat{\mathcal{T}}_{\varepsilon_k}^* \left(\frac{1}{\varepsilon_k} \partial_3 v_{\varepsilon_k} \right) (\widehat{x}, y) \right].\end{aligned}$$

Then, the isometry property of $\widehat{\mathcal{T}}_{\varepsilon_k}^*$ stated in Proposition 3.12 yields

$$\begin{aligned}& \|\widehat{\mathcal{T}}_{\varepsilon_k}^* v_{\varepsilon_k}\|_{L^2(\omega; W^{1,2}(Y^*))}^2 \\ &= \int_{\omega} \int_{Y^*} \left| (\widehat{\mathcal{T}}_{\varepsilon_k}^* v_{\varepsilon_k})(\widehat{x}, y) \right|^2 + \left| \nabla_y (\widehat{\mathcal{T}}_{\varepsilon_k}^* v_{\varepsilon_k})(\widehat{x}, y) \right|^2 dy d\widehat{x} \\ &= \int_{\omega} \int_{Y^*} \left| (\widehat{\mathcal{T}}_{\varepsilon_k}^* v_{\varepsilon_k})(\widehat{x}, y) \right|^2 dy d\widehat{x} \\ &\quad + \varepsilon_k^2 \int_{\omega} \int_{Y^*} \left| \widehat{\mathcal{T}}_{\varepsilon_k}^* (\widehat{\nabla} v_{\varepsilon_k})(\widehat{x}, y) \right|^2 + \left| \widehat{\mathcal{T}}_{\varepsilon_k}^* \left(\frac{1}{\varepsilon_k} \partial_3 v_{\varepsilon_k} \right) (\widehat{x}, y) \right|^2 dy d\widehat{x} \\ &= \|v_{\varepsilon_k}\|_{L^2(B^{\varepsilon\text{-perf}})}^2 \\ &\quad + \varepsilon_k^2 \|\widehat{\nabla} v_{\varepsilon_k}\|_{L^2(B^{\varepsilon\text{-perf}}; \mathbb{R}^2)}^2 + \varepsilon_k^2 \left\| \frac{1}{\varepsilon_k} \partial_3 v_{\varepsilon_k} \right\|_{L^2(B^{\varepsilon\text{-perf}})}^2.\end{aligned}$$

Thus, according to the assumptions on $(v_{\varepsilon_k})_k$ stated in the lemma the sequence $(\widehat{\mathcal{T}}_{\varepsilon_k}^* v_{\varepsilon_k})_k$ is bounded in $L^2(\omega; W^{1,2}(Y^*))$. Without loss of generality one may therefore assume that there is some $U \in L^2(\omega; W^{1,2}(Y^*))$ such that

$$\widehat{\mathcal{T}}_{\varepsilon_{k_\ell}}^* v_{\varepsilon_{k_\ell}} \rightharpoonup U \quad \text{in } L^2(\omega; W^{1,2}(Y^*))$$

along the same subsequence $(v_{\varepsilon_{k_\ell}})_\ell$. Yet, one also has the identity

$$\widehat{\mathcal{T}}_{\varepsilon_{k_\ell}}^* v_{\varepsilon_{k_\ell}} = \widehat{\mathcal{T}}_{\varepsilon_{k_\ell}}^* (Q_{\varepsilon_{k_\ell}} v_{\varepsilon_{k_\ell}}) + \widehat{\mathcal{T}}_{\varepsilon_{k_\ell}}^* (v_{\varepsilon_{k_\ell}} - Q_{\varepsilon_{k_\ell}} v_{\varepsilon_{k_\ell}}).$$

According to the second assertion of Proposition 3.13, the strong convergence $Q_{\varepsilon_{k_\ell}} v_{\varepsilon_{k_\ell}} \rightarrow v_0$ in $L^2(B)$ and the independence of v_0 from its third argument, one obtains

$$\widehat{\mathcal{T}}_{\varepsilon_{k_\ell}}^* (Q_{\varepsilon_{k_\ell}} v_{\varepsilon_{k_\ell}}) \rightarrow v_0(\widehat{x}, 0) \quad \text{in } L^2(\omega \times Y^*). \quad (3.65)$$

Whereas the term $\widehat{\mathcal{T}}_{\varepsilon_{k_\ell}}^* (v_{\varepsilon_{k_\ell}} - Q_{\varepsilon_{k_\ell}} v_{\varepsilon_{k_\ell}})$ can by means of the isometry of $\widehat{\mathcal{T}}_{\varepsilon_{k_\ell}}^*$ and the interpolation error estimate from Proposition 3.20 be bounded from above like

$$\begin{aligned}\|\widehat{\mathcal{T}}_{\varepsilon_{k_\ell}}^* (v_{\varepsilon_{k_\ell}} - Q_{\varepsilon_{k_\ell}} v_{\varepsilon_{k_\ell}})\|_{L^2(\omega \times Y^*)}^2 &= \|v_{\varepsilon_{k_\ell}} - Q_{\varepsilon_{k_\ell}} v_{\varepsilon_{k_\ell}}\|_{L^2(B^{\varepsilon\text{-perf}})}^2 \\ &\leq \varepsilon_{k_\ell}^2 C(V) \left(\|\widehat{\nabla} v_{\varepsilon_{k_\ell}}\|_{L^2(B^{\varepsilon\text{-perf}}; \mathbb{R}^2)}^2 + \left\| \frac{1}{\varepsilon_{k_\ell}} \partial_3 v_{\varepsilon_{k_\ell}} \right\|_{L^2(B^{\varepsilon\text{-perf}})}^2 \right).\end{aligned}$$

Then, the assumed $L^2(B^{\varepsilon\text{-perf}})$ -boundedness of $[\widehat{\nabla} v_{\varepsilon_k}, \frac{1}{\varepsilon_k} \partial_3 v_{\varepsilon_k}]$ together with (3.65) yields

$$\begin{aligned}\widehat{\mathcal{T}}_{\varepsilon_{k_\ell}}^* v_{\varepsilon_{k_\ell}} &= \widehat{\mathcal{T}}_{\varepsilon_{k_\ell}}^* (Q_{\varepsilon_{k_\ell}} v_{\varepsilon_{k_\ell}}) + \widehat{\mathcal{T}}_{\varepsilon_{k_\ell}}^* (v_{\varepsilon_{k_\ell}} - Q_{\varepsilon_{k_\ell}} v_{\varepsilon_{k_\ell}}) \\ &\rightarrow v_0(\widehat{x}, 0) \quad \text{in } L^2(\omega \times Y^*)\end{aligned}$$

as ε_{k_ℓ} vanishes. Therefore, the weak $L^2(\omega; W^{1,2}(Y^*))$ -limit U of $(\widehat{\mathcal{T}}_{\varepsilon_{k_\ell}}^* v_{\varepsilon_{k_\ell}})_\ell$ can actually be identified as $v_0(\widehat{x}, 0)$ which proves the second assertion of the proposition.

Eventually, the proof of the third statement follows well-known arguments from the theory of periodic unfolding, as outlined *e.g.* in [Cioranescu et al., 2008a, Section 3]. To this end, define the local averaging operator

$$\widehat{\mathcal{M}}_{\varepsilon_k}^* : L^2(B^{\varepsilon\text{-perf}}) \rightarrow L^2(\omega \times Y^*), \quad (\widehat{\mathcal{M}}_{\varepsilon_k}^* v)(\widehat{x}, y) := \int_{Y^*} (\widehat{\mathcal{T}}_{\varepsilon_k}^* v)(\widehat{x}, y) dy.$$

In particular, with the help of the local averaging operator $\widehat{\mathcal{M}}_{\varepsilon_k}^*$ one can state

$$\int_{Y^*} (\widehat{\mathcal{T}}_{\varepsilon_k}^* v - \widehat{\mathcal{M}}_{\varepsilon_k}^* v)(\widehat{x}, y) dy = 0$$

for a.e. $\widehat{x} \in \omega$. Now, since $Y^* \ni y \mapsto (\widehat{\mathcal{M}}_{\varepsilon_k}^* v_{\varepsilon_k})(\widehat{x}, y)$ is constant, the function

$$w_{\varepsilon_k} := \frac{1}{\varepsilon_k} (\widehat{\mathcal{T}}_{\varepsilon_k}^* v_{\varepsilon_k} - \widehat{\mathcal{M}}_{\varepsilon_k}^* v_{\varepsilon_k})$$

results as an element of $L^2(\omega; W^{1,2}(Y^*))$ and has the property of vanishing integral mean over Y^* for a.e. $\widehat{x} \in \omega$. Hence, one can apply Poincaré's inequality and obtains

$$\begin{aligned} \|w_{\varepsilon_k}\|_{L^2(\omega; W^{1,2}(Y^*))}^2 &= \int_{\omega} \|w_{\varepsilon_k}(\widehat{x}, \cdot)\|_{W^{1,2}(Y^*)}^2 d\widehat{x} \\ &\leq C(Y^*) \int_{\omega} \|\nabla_y w_{\varepsilon_k}(\widehat{x}, \cdot)\|_{L^2(Y^*; \mathbb{R}^3)}^2 d\widehat{x} \\ &= C(Y^*) \|\nabla_y w_{\varepsilon_k}\|_{L^2(\omega \times Y^*; \mathbb{R}^3)}^2. \end{aligned} \quad (3.66)$$

Using the definition of the unfolding operator $\widehat{\mathcal{T}}_{\varepsilon_k}^*$ for the rescaled perforated thin domain $B^{\varepsilon\text{-perf}}$, a simple calculation shows that the derivatives of w_{ε_k} w.r.t. the second argument are given through

$$\widehat{\nabla}_y w_{\varepsilon_k} = \frac{1}{\varepsilon_k} \widehat{\nabla}_y (\widehat{\mathcal{T}}_{\varepsilon_k}^* v_{\varepsilon_k}) = \widehat{\mathcal{T}}_{\varepsilon_k}^* (\widehat{\nabla} v_{\varepsilon_k})$$

and

$$\partial_{y_3} w_{\varepsilon_k} = \frac{1}{\varepsilon_k} \partial_{y_3} (\widehat{\mathcal{T}}_{\varepsilon_k}^* v_{\varepsilon_k}) = \widehat{\mathcal{T}}_{\varepsilon_k}^* \left(\frac{1}{\varepsilon_k} \partial_3 v_{\varepsilon_k} \right).$$

Thus, once more by the isometry property of $\widehat{\mathcal{T}}_{\varepsilon_k}^*$ the assumed $L^2(B^{\varepsilon\text{-perf}})$ -boundedness of $\widehat{\nabla} v_{\varepsilon_k}$ and $\frac{1}{\varepsilon_k} \partial_3 v_{\varepsilon_k}$ indicates that $(\nabla_y w_{\varepsilon_k})_k$ is bounded in $L^2(\omega \times Y^*; \mathbb{R}^3)$. This together with (3.66) yields the boundedness of $(w_{\varepsilon_k})_k$ in $L^2(\omega; W^{1,2}(Y^*))$. Then, setting

$$y^c := \int_{Y^*} y dy$$

(just like in [Cioranescu et al., 2008a, Section 3]), there is some $V_0 \in L^2(\omega; W^{1,2}(Y^*))$ such that along $(\varepsilon_{k_\ell})_\ell$ (upon possibly choosing another subsequence) it is

$$w_{\varepsilon_{k_\ell}} - \nabla v_0(\hat{x}, 0) \cdot (y - y^c) \rightharpoonup V_0 \quad \text{in } L^2(\omega; W^{1,2}(Y^*)). \quad (3.67)$$

In particular, by the independence of v_0 from x_3 and the above explicit characterization of $\nabla_y w_{\varepsilon_{k_\ell}} = [\widehat{\mathcal{T}}_{\varepsilon_{k_\ell}}^*(\widehat{\nabla} v_{\varepsilon_{k_\ell}}), \widehat{\mathcal{T}}_{\varepsilon_{k_\ell}}^*(\frac{1}{\varepsilon_{k_\ell}} \partial_3 v_{\varepsilon_{k_\ell}})]$,

$$\begin{aligned} \left[\widehat{\mathcal{T}}_{\varepsilon_{k_\ell}}^*(\widehat{\nabla} v_{\varepsilon_{k_\ell}}), \widehat{\mathcal{T}}_{\varepsilon_{k_\ell}}^*\left(\frac{1}{\varepsilon_{k_\ell}} \partial_3 v_{\varepsilon_{k_\ell}}\right) \right] &\rightharpoonup \nabla v_0(\hat{x}, 0) + \nabla_y V_0(\hat{x}, y) \quad \text{in } L^2(\omega \times Y^*; \mathbb{R}^3) \\ &= \left[\widehat{\nabla} v_0(\hat{x}, 0), 0 \right]^T + \nabla_y V_0(\hat{x}, y). \end{aligned}$$

The proof of the lemma would be finished if one could show that in fact $V_0(\hat{x}, \cdot) \in W_{\widehat{Y}\text{-per}}^{1,2}(Y^*)$ for a.e. $\hat{x} \in \omega$. To this end, I again adapt the ideas of [Cioranescu et al., 2008a, Section 3]). Set

$$Y_{V,1} := \{y : y_1 = 0, y \in \partial Y^* \text{ and } y + e_1 \in \partial Y^*\},$$

i.e. $Y_{V,1}$ is the intersection of the vertical faces of Y^* that are parallel to the hyperplane $\{x : x_1 = 0\}$. Furthermore, let $\psi \in C_c^\infty(\omega \times Y_{V,1})$. Some calculations then reveal that

$$\begin{aligned} &\int_\omega \int_{Y_{V,1}} \left(w_{\varepsilon_{k_\ell}}(\hat{x}, y) - w_{\varepsilon_{k_\ell}}(\hat{x}, y + e_1) \right) \psi(\hat{x}, y) \, da(y) \, d\hat{x} \\ &= \int_\omega \int_{Y_{V,1}} \frac{1}{\varepsilon_{k_\ell}} \left((\widehat{\mathcal{T}}_{\varepsilon_{k_\ell}}^* v_{\varepsilon_{k_\ell}})(\hat{x}, y) - (\widehat{\mathcal{T}}_{\varepsilon_{k_\ell}}^* v_{\varepsilon_{k_\ell}})(\hat{x}, y + e_1) \right) \psi(\hat{x}, y) \, da(y) \, d\hat{x} \\ &= \int_\omega \int_{Y_{V,1}} (\widehat{\mathcal{T}}_{\varepsilon_{k_\ell}}^* v_{\varepsilon_{k_\ell}})(\hat{x}, y) \frac{1}{\varepsilon_{k_\ell}} \left(\psi(\hat{x}, y) - \psi((x_1 - \varepsilon_{k_\ell}, x_2), y) \right) \, da(y) \, d\hat{x} \\ &\rightarrow \int_\omega \int_{Y_{V,1}} v_0(\hat{x}, 0) \partial_{x_1} \psi(\hat{x}, y) \, da(y) \, d\hat{x} \end{aligned} \quad (3.68)$$

where the convergence follows from the second assertion of the lemma and the continuity of the trace operator. Similarly, it is

$$\begin{aligned} &\int_\omega \int_{Y_{V,1}} \left(\nabla v_0(\hat{x}, 0) \cdot (y - y^c) - \nabla v_0(\hat{x}, 0) \cdot (y + e_1 - y^c) \right) \psi(\hat{x}, y) \, da(y) \, d\hat{x} \\ &= \int_\omega \int_{Y_{V,1}} -\partial_{x_1} v_0(\hat{x}, 0) \psi(\hat{x}, y) \, da(y) \, d\hat{x} \\ &= \int_\omega \int_{Y_{V,1}} v_0(\hat{x}, 0) \partial_{x_1} \psi(\hat{x}, y) \, da(y) \, d\hat{x}. \end{aligned} \quad (3.69)$$

Furthermore, by the convergence (3.67) one has

$$\begin{aligned} & \int_{\omega} \int_{Y_{V,1}} \left(w_{\varepsilon_{k_\ell}}(\hat{x}, y) - \nabla v_0(\hat{x}, 0) \cdot (y - y^c) \right) \psi(\hat{x}, y) \, da(y) \, d\hat{x} \\ & \quad \rightarrow \int_{\omega} \int_{Y_{V,1}} V_0(\hat{x}, y) \psi(\hat{x}, y) \, da(y) \, d\hat{x} \quad \text{and} \\ & \int_{\omega} \int_{Y_{V,1}} \left(w_{\varepsilon_{k_\ell}}(\hat{x}, y + e_1) - \nabla v_0(\hat{x}, 0) \cdot (y + e_1 - y^c) \right) \psi(\hat{x}, y) \, da(y) \, d\hat{x} \\ & \quad \rightarrow \int_{\omega} \int_{Y_{V,1}} V_0(\hat{x}, y + e_1) \psi(\hat{x}, y) \, da(y) \, d\hat{x}. \end{aligned}$$

This, together with (3.68) and (3.69) leads to the conclusion

$$\int_{\omega} \int_{Y_{V,1}} \left(V_0(\hat{x}, y) - V_0(\hat{x}, y + e_1) \right) \psi(\hat{x}, y) \, da(y) \, d\hat{x} = 0$$

for all $\psi \in C_c^\infty(\omega \times Y_{V,1})$, hence

$$V_0(\hat{x}, y) = V_0(\hat{x}, y + e_1) \quad \text{for a.e. } y \in Y_{V,1}$$

and a.e. $\hat{x} \in \omega$. The same arguments repeated for the faces of Y^* that are parallel to the hyperplane $\{x : x_2 = 0\}$ finally reveal that $V_0(\hat{x}, \cdot) \in W_{Y\text{-per}}^{1,2}(Y^*)$ for a.e. $\hat{x} \in \omega$ and the proof of the lemma is finished. \square

With all these results at hand, one can now turn to the proof of Theorem 3.10.

3.4.3 Proof of the main results

The rescaled dissipation potentials $(\mathcal{F}_{\varepsilon_k})_k$ from (3.43) corresponding to the rescaled perforated thin domain $B^{\varepsilon\text{-perf}}$ with tubular voids stretching over the entire rescaled interface layer (see Figure 3.3, Figure 3.4 and Figure 3.7) are posed over the common metric space $L^2(\Omega_{\text{tr}}^+ \cup B \cup \Omega_{\text{tr}}^-)$ equipped with the strong convergence. Hence, to prove Theorem 3.10 it suffices to verify the sequential characterization of Γ -convergence for the sequence $(\mathcal{F}_{\varepsilon_k})_k$ (again by means of Corollary 2.6). The following proof will be split into two parts, one for the Γ -lim inf-inequality and the other for the Γ -lim sup-inequality.

Step 1. Γ -lim inf-inequality. Let there be given a sequence $(v_{\varepsilon_k})_k$ in $L^2(\Omega_{\text{tr}}^+ \cup B \cup \Omega_{\text{tr}}^-)$ such that $v_{\varepsilon_k} \rightarrow v_0$ in $L^2(\Omega_{\text{tr}}^+ \cup B \cup \Omega_{\text{tr}}^-)$ and $\liminf_k \mathcal{F}_{\varepsilon_k}(v_{\varepsilon_k}) < \infty$, and let $(\varepsilon_{k_\ell})_\ell$ be a subsequence satisfying $\liminf_k \mathcal{F}_{\varepsilon_k}(v_{\varepsilon_k}) = \lim_\ell \mathcal{F}_{\varepsilon_{k_\ell}}(v_{\varepsilon_{k_\ell}})$. Hence, $v_{\varepsilon_{k_\ell}} \in W_{\Gamma_{\text{tr}}^+ \cup \Gamma_{\text{tr}}^-}^{1,2}(\Omega_{\text{tr}}^+ \cup B^{\varepsilon\text{-perf}} \cup \Omega_{\text{tr}}^-)$ for all $\ell \in \mathbb{N}$.

As in the proof of Theorem 3.1, from the constitutive assumption (W3) on the energy densities W^\pm and W^{interf} as stated in Section 3.2 and the boundedness of $(\mathcal{F}_{\varepsilon_k}(v_{\varepsilon_{k_\ell}}))_\ell$, it is easily inferred that the sequence $(v_{\varepsilon_{k_\ell}})_\ell$ satisfies the a priori estimates

$$\|\nabla v_{\varepsilon_{k_\ell}}\|_{L^2(\Omega_{\text{tr}}^\pm)}^2 \leq C \tag{3.70}$$

and

$$\|\widehat{\nabla} v_{\varepsilon_{k_\ell}}\|_{L^2(B^{\varepsilon\text{-perf}}; \mathbb{R}^2)}^2 + \left\| \frac{1}{\varepsilon_{k_\ell}} \partial_3 v_{\varepsilon_{k_\ell}} \right\|_{L^2(B^{\varepsilon\text{-perf}})}^2 \leq C \quad (3.71)$$

for all $\ell \in \mathbb{N}$ and some positive constant C . From the strong convergence $v_{\varepsilon_k} \rightarrow v_0$ in $L^2(\Omega_{\text{tr}}^+ \cup B \cup \Omega_{\text{tr}}^-)$, the fact that $v_{\varepsilon_{k_\ell}}|_{\Omega_{\text{tr}}^\pm} \in W_{\Gamma_{\text{tr}}^\pm}^{1,2}(\Omega_{\text{tr}}^\pm)$ for all $\ell \in \mathbb{N}$ and the boundedness of (3.70) one concludes that

$$v_{\varepsilon_{k_\ell}}|_{\Omega_{\text{tr}}^\pm} \rightharpoonup v_0|_{\Omega_{\text{tr}}^\pm} \quad \text{in } W_{\Gamma_{\text{tr}}^\pm}^{1,2}(\Omega_{\text{tr}}^\pm). \quad (3.72)$$

In particular, $v_0|_{\Omega_{\text{tr}}^\pm} \in W_{\Gamma_{\text{tr}}^\pm}^{1,2}(\Omega_{\text{tr}}^\pm)$. Moreover, from estimate (3.71) and Lemma 3.21 one obtains the existence of a $\tilde{v}_0 \in W^{1,2}(B)$ independent of x_3 such that

$$Q_{\varepsilon_{k_\ell}} \left(v_{\varepsilon_{k_\ell}}|_{B^{\varepsilon\text{-perf}}} \right) \rightharpoonup \tilde{v}_0 \quad \text{in } W^{1,2}(B), \quad (3.73)$$

$$\widehat{\mathcal{T}}_{\varepsilon_{k_\ell}}^* \left(v_{\varepsilon_{k_\ell}}|_{B^{\varepsilon\text{-perf}}} \right) \rightharpoonup \tilde{v}_0(\widehat{x}, 0) \quad \text{in } L^2(\omega; W^{1,2}(Y^*)), \quad (3.74)$$

where $Q_{\varepsilon_{k_\ell}} : W^{1,2}(B^{\varepsilon\text{-perf}}) \rightarrow W^{1,2}(B)$ is the interpolation operator from Definition 3.18 and $\widehat{\mathcal{T}}_{\varepsilon_{k_\ell}}^* : L^2(B^{\varepsilon\text{-perf}}) \rightarrow L^2(\omega \times Y^*)$ the unfolding operator for the rescaled perforated thin domain $B^{\varepsilon\text{-perf}}$ from Definition 3.11. The same lemma also states the existence of a $V_0 \in L^2(\omega; W_{\widehat{Y}\text{-per}}^{1,2}(Y^*))$ such that

$$\begin{aligned} & \left[\widehat{\mathcal{T}}_{\varepsilon_{k_\ell}}^* (\widehat{\nabla} v_{\varepsilon_{k_\ell}}), \widehat{\mathcal{T}}_{\varepsilon_{k_\ell}}^* \left(\frac{1}{\varepsilon_{k_\ell}} \partial_3 v_{\varepsilon_{k_\ell}} \right) \right] \\ & \rightarrow \left[\widehat{\nabla} v_0(\widehat{x}, 0), 0 \right]^T + \nabla_y V_0(\widehat{x}, y) \quad \text{in } L^2(\omega \times Y^*; \mathbb{R}^3). \end{aligned} \quad (3.75)$$

Now, the assumed strong convergence $v_{\varepsilon_{k_\ell}}|_B \rightarrow v_0|_B$ in $L^2(B)$ and Proposition 3.13 reveal that

$$\widehat{\mathcal{T}}_{\varepsilon_{k_\ell}}^* \left(v_{\varepsilon_{k_\ell}}|_{B^{\varepsilon\text{-perf}}} \right) \rightarrow v_0|_B(\widehat{x}, y_3) \quad \text{in } L^2(\omega \times Y^*).$$

This together with (3.74) yields the identity

$$v_0|_B(\widehat{x}, y_3) = \tilde{v}_0(\widehat{x}, 0) \quad \text{for a.e. } (\widehat{x}, y) \in \omega \times Y^*.$$

Hence, $v_0(\widehat{x}, \cdot)$ is constant for a.e. $\widehat{x} \in \omega$ and therefore

$$v_0|_B(x) = \tilde{v}_0(\widehat{x}, 0) \quad \text{for a.e. } x \in B. \quad (3.76)$$

To conclude $v_0|_{\Omega_{\text{tr}}^\pm} \in W_{\Gamma_{\text{tr}}^\pm}^{1,2}(\Omega_{\text{tr}}^\pm)$, $v_0|_B \in W^{1,2}(B)$ and v_0 is independent of x_3 in B . What remains to show now is that the traces of $v_0|_{\Omega_{\text{tr}}^\pm}$ and $v_0|_B$ on the faces $\omega \times \{\pm \frac{1}{2}\}$ of B coincide. In this case one would conclude $v_0 \in W_{\Gamma_{\text{tr}}^+ \cup \Gamma_{\text{tr}}^-}^{1,2}(\Omega_{\text{tr}}^+ \cup B \cup \Omega_{\text{tr}}^-)$ and v_0 is independent of x_3 in B , thus $v_0 \in \mathcal{V}_{\Gamma^+, \Gamma^-}(\Omega^+, \Omega^-, B)$. By the assumption on the

tubular void $V = [0, 1) \times H$ for some $H \in [0, 1) \times [-\frac{1}{2}, \frac{1}{2}]$, it follows that there is an $r > 0$ such that the layer $L := \omega \times [\frac{1}{2} - r, \frac{1}{2}]$ does not touch the voids in $B^{\varepsilon\text{-perf}}$. Moreover, it is according to (3.71) $(v_{\varepsilon_{k_\ell}}|_L)_\ell$ a bounded sequence in $W^{1,2}(L)$, thus converges without loss of generality weakly in that space. However,

$$\begin{aligned} & \left\| v_{\varepsilon_{k_\ell}}|_L - v_0|_L \right\|_{L^2(L)} \\ & \leq \left\| v_{\varepsilon_{k_\ell}}|_{B^{\varepsilon\text{-perf}}} - Q_{\varepsilon_{k_\ell}} \left(v_{\varepsilon_{k_\ell}}|_{B^{\varepsilon\text{-perf}}} \right) \right\|_{L^2(L)} + \left\| Q_{\varepsilon_{k_\ell}} \left(v_{\varepsilon_{k_\ell}}|_{B^{\varepsilon\text{-perf}}} \right) - v_0|_B \right\|_{L^2(L)} \\ & \leq \left\| v_{\varepsilon_{k_\ell}}|_{B^{\varepsilon\text{-perf}}} - Q_{\varepsilon_{k_\ell}} \left(v_{\varepsilon_{k_\ell}}|_{B^{\varepsilon\text{-perf}}} \right) \right\|_{L^2(B^{\varepsilon\text{-perf}})} + \left\| Q_{\varepsilon_{k_\ell}} \left(v_{\varepsilon_{k_\ell}}|_{B^{\varepsilon\text{-perf}}} \right) - v_0|_B \right\|_{L^2(B)} \\ & \longrightarrow 0 + 0 \end{aligned}$$

as ε_{k_ℓ} vanishes. This can be easily deduced on the one hand from (3.73) and (3.76), and on the other hand from Proposition 3.20 and the a priori-estimate (3.71). Hence, $v_{\varepsilon_{k_\ell}}|_L \rightharpoonup v_0|_L$ in $W^{1,2}(L)$. Since $v_{\varepsilon_{k_\ell}} \in W_{\Gamma_{\text{tr}}^+ \cup \Gamma_{\text{tr}}^-}^{1,2}(\Omega_{\text{tr}}^+ \cup B^{\varepsilon\text{-perf}} \cup \Omega_{\text{tr}}^-)$ for all $\ell \in \mathbb{N}$, the traces of $v_{\varepsilon_{k_\ell}}|_{\Omega_{\text{tr}}^+}$ and $v_{\varepsilon_{k_\ell}}|_L$ on the face $\omega \times \{\frac{1}{2}\}$ of B coincide for all ℓ . Finally, due to the weak continuity of the trace operator, it also follows that the limits

$$\begin{aligned} \lim_{\ell \rightarrow \infty} \text{tr}_{\omega \times \{\frac{1}{2}\}} \left(v_{\varepsilon_{k_\ell}}|_{\Omega_{\text{tr}}^+} \right) &= \text{tr}_{\omega \times \{\frac{1}{2}\}} \left(v_0|_{\Omega_{\text{tr}}^+} \right) \quad \text{and} \\ \lim_{\ell \rightarrow \infty} \text{tr}_{\omega \times \{\frac{1}{2}\}} \left(v_{\varepsilon_{k_\ell}}|_L \right) &= \text{tr}_{\omega \times \{\frac{1}{2}\}} \left(v_0|_L \right) \end{aligned}$$

coincide. In other words, $v_0|_{\Omega_{\text{tr}}^+}$ and $v_0|_B$ have identical traces on $\omega \times \{\frac{1}{2}\}$, and by similar arguments it follows that also $v_0|_{\Omega_{\text{tr}}^-}$ and $v_0|_B$ have identical traces on $\omega \times \{-\frac{1}{2}\}$. This yields $v_0 \in W_{\Gamma_{\text{tr}}^+ \cup \Gamma_{\text{tr}}^-}^{1,2}(\Omega_{\text{tr}}^+ \cup B \cup \Omega_{\text{tr}}^-)$ and v_0 is independent of x_3 in B , hence $v_0 \in \mathcal{V}_{\Gamma^+, \Gamma^-}(\Omega^+, \Omega^-, B)$ as claimed.

As concerns the limiting behavior of the rescaled dissipation potential $\mathcal{F}_{\varepsilon_{k_\ell}}$ in the bodies adjacent to the interface $B^{\varepsilon\text{-perf}}$, again standard lower semicontinuity arguments for convex integral functionals, together with the weak convergences (3.72) and the assumed convexity of W^\pm (see (W2)) lead to

$$\int_{\Omega_{\text{tr}}^\pm} W^\pm(\nabla v_0(x)) \, dx \leq \liminf_{\ell \rightarrow \infty} \int_{\Omega_{\text{tr}}^\pm} W^\pm(\nabla v_{\varepsilon_{k_\ell}}(x)) \, dx. \quad (3.77)$$

With the help of the unfolding operator $\widehat{\mathcal{T}}_{\varepsilon_{k_\ell}}^*$ the asymptotics of the interface term in the rescaled dissipation potential $\mathcal{F}_{\varepsilon_{k_\ell}}$ can be easily described. In fact, by the isometry property of $\widehat{\mathcal{T}}_{\varepsilon_{k_\ell}}^*$ stated in Proposition 3.12 it is

$$\begin{aligned} & \int_{B^{\varepsilon\text{-perf}}} W^{\text{interf}} \left(\widehat{\nabla} v_{\varepsilon_{k_\ell}}(x), \frac{1}{\varepsilon_{k_\ell}} \partial_3 v_{\varepsilon_{k_\ell}}(x) \right) \, dx \\ &= \int_{\omega} \int_{Y^*} W^{\text{interf}} \left(\widehat{\mathcal{T}}_{\varepsilon_{k_\ell}}^* \left(\widehat{\nabla} v_{\varepsilon_{k_\ell}} \right) (\widehat{x}, y), \widehat{\mathcal{T}}_{\varepsilon_{k_\ell}}^* \left(\frac{1}{\varepsilon_{k_\ell}} \partial_3 v_{\varepsilon_{k_\ell}} \right) (\widehat{x}, y) \right) \, dy \, d\widehat{x} \end{aligned}$$

Just like in the proof of Theorem 3.1, the weak convergence (3.75), the assumed convexity of W^{interf} and standard lower semicontinuity arguments for convex integral functionals allow to pass to the limit $\varepsilon_{k_\ell} \rightarrow 0$, resulting in

$$\begin{aligned}
& \liminf_{\ell \rightarrow \infty} \int_{B^{\varepsilon\text{-perf}}} W^{\text{interf}} \left(\widehat{\nabla} v_{\varepsilon_{k_\ell}}(x), \frac{1}{\varepsilon_{k_\ell}} \partial_3 v_{\varepsilon_{k_\ell}}(x) \right) dx \\
&= \liminf_{\ell \rightarrow \infty} \int_{\omega} \int_{Y^*} W^{\text{interf}} \left(\widehat{\mathcal{T}}_{\varepsilon_{k_\ell}}^* (\widehat{\nabla} v_{\varepsilon_{k_\ell}})(\widehat{x}, y), \widehat{\mathcal{T}}_{\varepsilon_{k_\ell}}^* \left(\frac{1}{\varepsilon_{k_\ell}} \partial_3 v_{\varepsilon_{k_\ell}} \right)(\widehat{x}, y) \right) dy d\widehat{x} \\
&\geq \int_{\omega} \int_{Y^*} W^{\text{interf}} \left([\widehat{\nabla} v_0(\widehat{x}, 0), 0]^T + \nabla_y V_0(\widehat{x}, y) \right) dy d\widehat{x} \\
&\geq \int_{\omega} \inf \left\{ \int_{Y^*} W^{\text{interf}} \left([\widehat{\nabla} v_0(\widehat{x}, 0), 0]^T + \nabla_y w(y) \right) dy : w \in W_{\widehat{Y}\text{-per}}^{1,2}(Y^*) \right\} d\widehat{x} \\
&= \int_{\omega} \widehat{W}^{\text{interf}} \left(\widehat{\nabla} v_0(\widehat{x}, 0) \right) d\widehat{x}. \tag{3.78}
\end{aligned}$$

Before analyzing the source term in the rescaled dissipation potential $\mathcal{F}_{\varepsilon_{k_\ell}}$, one has to perform some minor computations. Again, by the assumption (ε -Tiling) on ω and the elements of the sequence $(\varepsilon_k)_k$

$$\begin{aligned}
& \int_{\partial B^{\varepsilon\text{-perf}} \cap B^\circ} G(v_{\varepsilon_{k_\ell}}(x)) \Big|_{M_{\varepsilon_{k_\ell}} n(x; \partial B^{\varepsilon\text{-perf}})} \Big| da(x) \\
&= \sum_{\widehat{a} \in Z_\varepsilon} \int_{M_{\varepsilon_{k_\ell}}([\widehat{a}, 0]^T + \partial V \cap \text{int } Y)} G(v_{\varepsilon_{k_\ell}}(x)) \Big|_{M_{\varepsilon_{k_\ell}} n(x; M_{\varepsilon_{k_\ell}}([\widehat{a}, 0]^T + \partial V))} \Big| da(x). \tag{3.79}
\end{aligned}$$

Applying the unfolding $\widehat{\mathcal{T}}_{\varepsilon_{k_\ell}}^*$ in the single integrals of the above sum is in fact equivalent to performing the change of variables

$$S_{\varepsilon_{k_\ell}} : Y \rightarrow M_{\varepsilon_{k_\ell}}([\widehat{a}, 0]^T + Y), \quad S_{\varepsilon_{k_\ell}}(y) = M_{\varepsilon_{k_\ell}}([\widehat{a}, 0]^T + y)$$

in the respective terms. Again with the help of *Nanson's formula* (see [Ciarlet, 1988, Theorem 1.7-1]) this amounts to

$$\begin{aligned}
& \int_{M_{\varepsilon_{k_\ell}}([\widehat{a}, 0]^T + \partial V \cap \text{int } Y)} G(v_{\varepsilon_{k_\ell}}(x)) \Big|_{M_{\varepsilon_{k_\ell}} n(x; M_{\varepsilon_{k_\ell}}([\widehat{a}, 0]^T + \partial V))} \Big| da(x) \\
&= \int_{\partial V \cap \text{int } Y} G(v_{\varepsilon_{k_\ell}}(S_{\varepsilon_{k_\ell}}(y))) \Big|_{M_{\varepsilon_{k_\ell}} n(S_{\varepsilon_{k_\ell}}(y); M_{\varepsilon_{k_\ell}}([\widehat{a}, 0]^T + \partial V))} \Big| \\
&\quad \Big| \text{Cof}(\nabla S_{\varepsilon_{k_\ell}}(y)) n(y; \partial V) \Big| da(y) \\
&= \int_{\partial V \cap \text{int } Y} G(v_{\varepsilon_{k_\ell}}(\varepsilon_{k_\ell} \widehat{a} + \varepsilon_{k_\ell} \widehat{y}, y_3)) \Big|_{M_{\varepsilon_{k_\ell}} n(M_{\varepsilon_{k_\ell}} y; M_{\varepsilon_{k_\ell}} \partial V)} \Big| \\
&\quad \Big| \varepsilon_{k_\ell}^2 M_{\varepsilon_{k_\ell}}^{-1} n(y; \partial V) \Big| da(y). \tag{3.80}
\end{aligned}$$

On the other hand, a corollary of Nanson's formula [Ciarlet, 1988, p. 41] allows to identify

$$\begin{aligned} n\left(M_{\varepsilon_{k_\ell}} y; M_{\varepsilon_{k_\ell}} \partial V\right) &= \frac{\text{Cof}\left(M_{\varepsilon_{k_\ell}}\right) n(y, \partial V)}{\left|\text{Cof}\left(M_{\varepsilon_{k_\ell}}\right) n(y, \partial V)\right|} \\ &= \frac{M_{\varepsilon_{k_\ell}}^{-1} n(y, \partial V)}{\left|M_{\varepsilon_{k_\ell}}^{-1} n(y, \partial V)\right|}. \end{aligned}$$

Using this one can further deduce

$$\begin{aligned} &\int_{M_{\varepsilon_{k_\ell}}([\hat{a}, 0]^T + \partial V \cap \text{int } Y)} G(v_{\varepsilon_{k_\ell}}(x)) \left| M_{\varepsilon_{k_\ell}} n\left(x; M_{\varepsilon_{k_\ell}}([\hat{a}, 0]^T + \partial V)\right) \right| \text{da}(x) \\ &= \int_{\partial V \cap \text{int } Y} G\left(v_{\varepsilon_{k_\ell}}(\varepsilon_{k_\ell} \hat{a} + \varepsilon_{k_\ell} \hat{y}, y_3)\right) \left| M_{\varepsilon_{k_\ell}} \frac{M_{\varepsilon_{k_\ell}}^{-1} n(y, \partial V)}{\left|M_{\varepsilon_{k_\ell}}^{-1} n(y, \partial V)\right|} \right| \\ &\quad \left| \varepsilon_{k_\ell}^2 M_{\varepsilon_{k_\ell}}^{-1} n(y; \partial V) \right| \text{da}(y) \\ &= \varepsilon_{k_\ell}^2 \int_{\partial V \cap \text{int } Y} G\left(v_{\varepsilon_{k_\ell}}(\varepsilon_{k_\ell} \hat{a} + \varepsilon_{k_\ell} \hat{y}, y_3)\right) \left| n(y; \partial V) \right| \text{da}(y) \\ &= \varepsilon_{k_\ell}^2 \int_{\partial V \cap \text{int } Y} G\left(v_{\varepsilon_{k_\ell}}(\varepsilon_{k_\ell} \hat{a} + \varepsilon_{k_\ell} \hat{y}, y_3)\right) \text{da}(y). \end{aligned}$$

Inserting this into (3.80) yields

$$\begin{aligned} &\int_{\partial B^{\varepsilon\text{-perf}} \cap B^\circ} G(v_{\varepsilon_{k_\ell}}(x)) \left| M_{\varepsilon_{k_\ell}} n(x; \partial B^{\varepsilon\text{-perf}}) \right| \text{da}(x) \\ &= \sum_{\hat{a} \in Z_\varepsilon} \varepsilon_{k_\ell}^2 \int_{\partial V \cap \text{int } Y} G\left(v_{\varepsilon_{k_\ell}}(\varepsilon_{k_\ell} \hat{a} + \varepsilon_{k_\ell} \hat{y}, y_3)\right) \text{da}(y) \\ &= \sum_{\hat{a} \in Z_\varepsilon} \int_{\varepsilon_{k_\ell}(\hat{a} + \hat{Y})} \int_{\partial V \cap \text{int } Y} G\left(v_{\varepsilon_{k_\ell}}(\varepsilon_{k_\ell} \hat{a} + \varepsilon_{k_\ell} \hat{y}, y_3)\right) \text{da}(y) \text{d}\hat{x} \\ &= \sum_{\hat{a} \in Z_\varepsilon} \int_{\varepsilon_{k_\ell}(\hat{a} + \hat{Y})} \int_{\partial V \cap \text{int } Y} G\left(v_{\varepsilon_{k_\ell}}\left(\varepsilon_{k_\ell} \left\lfloor \frac{\hat{x}}{\varepsilon_{k_\ell}} \right\rfloor + \varepsilon_{k_\ell} \hat{y}, y_3\right)\right) \text{da}(y) \text{d}\hat{x} \\ &= \int_{\omega} \int_{\partial V \cap \text{int } Y} G\left((\hat{\mathcal{T}}_{\varepsilon_{k_\ell}}^* v_{\varepsilon_{k_\ell}})(\hat{x}, y)\right) \text{da}(y) \text{d}\hat{x}. \end{aligned} \tag{3.81}$$

Since from (3.74) and (3.76) and the continuity of the trace operator it also follows

$$\hat{\mathcal{T}}_{\varepsilon_{k_\ell}}^* \left(v_{\varepsilon_{k_\ell}} \Big|_{B^{\varepsilon\text{-perf}}} \right) \rightharpoonup v_0(\hat{x}, 0) \quad \text{in } L^2(\omega; L^2(\partial V)),$$

one can in view of the convexity of $G : \mathbb{R} \rightarrow \mathbb{R}$ like defined in (3.40) conclude

$$\begin{aligned}
& \liminf_{\ell \rightarrow \infty} \int_{\partial B^{\varepsilon\text{-perf}} \cap B^\circ} G(v_{\varepsilon_{k_\ell}}(x)) \Big| M_{\varepsilon_{k_\ell}} n(x; \partial B^{\varepsilon\text{-perf}}) \Big| \, da(x) \\
&= \liminf_{\ell \rightarrow \infty} \int_{\omega} \int_{\partial V \cap \text{int } Y} G\left(\widehat{\mathcal{T}}_{\varepsilon_{k_\ell}}^* v_{\varepsilon_{k_\ell}}(\widehat{x}, y)\right) \, da(y) \, d\widehat{x} \\
&\geq \int_{\omega} \int_{\partial V \cap \text{int } Y} G(v_0(\widehat{x}, 0)) \, da(y) \, d\widehat{x} \\
&= \text{vol}_2(\partial V \cap \text{int } Y) \int_{\omega} G(v_0(\widehat{x}, 0)) \, d\widehat{x}. \tag{3.82}
\end{aligned}$$

Finally, from $v_0 \in \mathcal{V}_{\Gamma^+, \Gamma^-}(\Omega^+, \Omega^-, B)$, (3.77), (3.78) and (3.82) it is now easy to infer

$$\begin{aligned}
\liminf_{\ell \rightarrow \infty} \mathcal{F}_{\varepsilon_{k_\ell}}(v_{\varepsilon_{k_\ell}}) &\geq \int_{\Omega_{\text{tr}}^+} W^+(\nabla v_0(x)) \, dx + \int_{\Omega_{\text{tr}}^-} W^-(\nabla v(x)) \, dx \\
&\quad + \int_{\omega} \widehat{W}^{\text{interf}}\left(\widehat{\nabla} v_0(\widehat{x}, 0)\right) \, d\widehat{x} \\
&\quad + \text{vol}_2(\partial V \cap \text{int } Y) \int_{\omega} G(v_0(\widehat{x}, 0)) \, d\widehat{x} \\
&= \mathcal{F}_{\text{Hom}}(v_0),
\end{aligned}$$

which concludes the proof of the Γ -lim inf-inequality.

Step 2. Γ -lim sup-inequality. The proof of the Γ -lim sup-inequality is very similar to what has been done in the case of Theorem 3.1. In fact, since many of the arguments employed therein can be applied literally also in the present situation, I will only sketch the proof of the Γ -lim sup-inequality. Let $v_0 \in L^2(\Omega_{\text{tr}}^+ \cup B \cup \Omega_{\text{tr}}^-)$ and assume without loss of generality $\mathcal{F}_{\text{Hom}}(v_0) < \infty$. Thus $v_0 \in \mathcal{V}_{\Gamma^+, \Gamma^-}(\Omega^+, \Omega^-, B)$, in other words $v_0 \in W_{\Gamma_{\text{tr}}^+ \cup \Gamma_{\text{tr}}^-}^{1,2}(\Omega_{\text{tr}}^+ \cup B \cup \Omega_{\text{tr}}^-)$ and $\partial_3 v_0|_B = 0$. By the same arguments as used in the proof of Theorem 3.1 one obtains the existence of some $V_0 \in L^2(\omega; W_{\widehat{Y}\text{-per}}^{1,2}(Y^*))$ such that

$$\begin{aligned}
V_0(\widehat{x}, \cdot) &\in \left\{ w : w \in W_{\widehat{Y}\text{-per}}^{1,2}(Y^*), \int_{Y^*} w(y) \, dy = 0, \right. \\
&\quad \left. \int_{Y^*} W^{\text{interf}}\left([\widehat{\nabla} v_0(\widehat{x}, 0), 0]^T + \nabla_y w(y)\right) \, dy = \widehat{W}^{\text{interf}}\left(\widehat{\nabla} v_0(\widehat{x}, 0)\right) \right\}. \tag{3.83}
\end{aligned}$$

Moreover, for δ some small positive real number one can also find a smooth function $\Psi_\delta \in C_c^\infty(\omega; C_{\widehat{Y}\text{-per}}^\infty(Y^*))$ such that

$$\|\Psi_\delta - V_0\|_{L^2(\omega \times Y^*)} + \|\nabla_y \Psi_\delta - \nabla_y V_0\|_{L^2(\omega \times Y^*; \mathbb{R}^3)} \leq \delta. \tag{3.84}$$

Like in Step 1 of the present proof, by the assumption on the tubular void $V = [0, 1) \times H$ for some $H \in [0, 1) \times [-\frac{1}{2}, \frac{1}{2}]$, it follows that there is an $r > 0$ such that both

$\widehat{Y} \times [\frac{1}{2} - r, \frac{1}{2}]$ and $\widehat{Y} \times [-\frac{1}{2}, -\frac{1}{2} + r]$ do not touch the void in $Y^* = Y \setminus \overline{V}$. Therefore, one may assume Ψ_δ to be extended to the set $(\widehat{Y} \times [-\frac{1}{2} - r, \frac{1}{2} + r]) \setminus \overline{V}$ by reflection of $\Psi_\delta|_{\widehat{Y} \times [\frac{1}{2} - r, \frac{1}{2}]}$ and $\Psi_\delta|_{\widehat{Y} \times [-\frac{1}{2}, -\frac{1}{2} + r]}$ w.r.t. the top and bottom faces of Y^* , i.e. w.r.t. $\widehat{Y} \times \{\frac{1}{2}\}$ and $\widehat{Y} \times \{-\frac{1}{2}\}$. Furthermore, $\rho_\delta \in C_c^\infty([-\frac{1}{2} - r, \frac{1}{2} + r])$ shall be a cut-off function with values in $[0, 1]$ such that it equals 1 on $[-\frac{1}{2}, \frac{1}{2}]$ and has its support contained in $[-\frac{1}{2} - \delta, \frac{1}{2} + \delta]$. Again, one defines a sequence $v_{\varepsilon_k, \delta} \in W_{\Gamma_{\text{tr}}^+ \cup \Gamma_{\text{tr}}^-}^{1,2}(\Omega_{\text{tr}}^+ \cup B^{\varepsilon\text{-perf}} \cup \Omega_{\text{tr}}^-)$ through

$$v_{\varepsilon_k, \delta}(x) := v_0(x) + \varepsilon_k \rho_\delta(x_3) \Psi_\delta \left(\widehat{x}, \left(\frac{\widehat{x}}{\varepsilon_k}, x_3 \right) \right), \quad x \in \Omega_{\text{tr}}^+ \cup B \cup \Omega_{\text{tr}}^-.$$

Obviously,

$$v_{\varepsilon_k, \delta} \rightarrow v_0 \quad \text{in } L^2(\Omega_{\text{tr}}^+ \cup B \cup \Omega_{\text{tr}}^-) \text{ as } \varepsilon_k \rightarrow 0.$$

Once more for $x \in \Omega_{\text{tr}}^+ \cup B \cup \Omega_{\text{tr}}^-$, a simple calculation show that

$$\begin{aligned} \widehat{\nabla} v_{\varepsilon_k, \delta}(x) &= \widehat{\nabla} v_0(x) \\ &\quad + \varepsilon_k \rho_\delta(x_3) \widehat{\nabla}_x \Psi_\delta \left(\widehat{x}, \left(\frac{\widehat{x}}{\varepsilon_k}, x_3 \right) \right) + \rho_\delta(x_3) \widehat{\nabla}_y \Psi_\delta \left(\widehat{x}, \left(\frac{\widehat{x}}{\varepsilon_k}, x_3 \right) \right), \end{aligned}$$

and

$$\begin{aligned} \partial_3 v_{\varepsilon_k, \delta}(x) &= \partial_3 v_0(x) \\ &\quad + \varepsilon_k \rho'_\delta(x_3) \Psi_\delta \left(\widehat{x}, \left(\frac{\widehat{x}}{\varepsilon_k}, x_3 \right) \right) + \varepsilon_k \rho_\delta(x_3) \partial_{y_3} \Psi_\delta \left(\widehat{x}, \left(\frac{\widehat{x}}{\varepsilon_k}, x_3 \right) \right). \end{aligned}$$

In view of $\partial_3 v_0 = 0$ in B and $\rho_\delta = 1$ in $[-\frac{1}{2}, \frac{1}{2}]$ the above derivatives simplify for $x = (\widehat{x}, x_3) \in B = \omega \times [-\frac{1}{2}, \frac{1}{2}]$. That is,

$$\begin{aligned} \widehat{\nabla} v_{\varepsilon_k, \delta}(x) &= \widehat{\nabla} v_0(x) \\ &\quad + \varepsilon_k \widehat{\nabla}_x \Psi_\delta \left(\widehat{x}, \left(\frac{\widehat{x}}{\varepsilon_k}, x_3 \right) \right) + \widehat{\nabla}_y \Psi_\delta \left(\widehat{x}, \left(\frac{\widehat{x}}{\varepsilon_k}, x_3 \right) \right) \quad \text{for } x \in B, \end{aligned} \quad (3.85)$$

$$\partial_3 v_{\varepsilon_k, \delta}(x) = \varepsilon_k \partial_{y_3} \Psi_\delta \left(\widehat{x}, \left(\frac{\widehat{x}}{\varepsilon_k}, x_3 \right) \right) \quad \text{for } x \in B. \quad (3.86)$$

While not going into details, by similar arguments to those used in the proof of Theorem 3.1 one can in fact show that

$$\begin{aligned} &\lim_{\ell \rightarrow \infty} \left(\lim_{k \rightarrow \infty} \int_{\Omega_{\text{tr}}^\pm} W^\pm(\nabla v_{\varepsilon_k, \delta_\ell}(x)) \, dx \right) \\ &= \int_{\Omega_{\text{tr}}^\pm} \int_{\widehat{Y}} W^\pm(\widehat{\nabla} v_0(x), \partial_3 v_0(x)) \, d\widehat{y} \, dx = \int_{\Omega_{\text{tr}}^\pm} W^\pm(\nabla v_0(x)) \, dx, \end{aligned} \quad (3.87)$$

where $(\delta_\ell)_\ell$ shall be some arbitrary vanishing sequence of positive real numbers.

Regarding the limiting behavior of the interface term in the rescaled dissipation potential $\mathcal{F}_{\varepsilon_k}(v_{\varepsilon_k})$, one first notices that by (3.85)

$$\begin{aligned}\widehat{\mathcal{T}}_{\varepsilon_k}^*(\widehat{\nabla}v_{\varepsilon_k,\delta}) &= \widehat{\mathcal{T}}_{\varepsilon_k}^*(\widehat{\nabla}v_0)(\widehat{x}, y) + \varepsilon_k \widehat{\nabla}_x \Psi_\delta \left(\varepsilon_k \left\lfloor \frac{\widehat{x}}{\varepsilon_k} \right\rfloor + \varepsilon_k \widehat{y}, y \right) \\ &\quad + \widehat{\nabla}_y \Psi_\delta \left(\varepsilon_k \left\lfloor \frac{\widehat{x}}{\varepsilon_k} \right\rfloor + \varepsilon_k \widehat{y}, y \right) \\ &\rightarrow \widehat{\nabla}v_0(\widehat{x}, 0) + \widehat{\nabla}_y \Psi_\delta(\widehat{x}, y) \quad \text{in } L^2(\omega \times Y^*; \mathbb{R}^2)\end{aligned}$$

and by (3.86)

$$\begin{aligned}\widehat{\mathcal{T}}_{\varepsilon_k}^*\left(\frac{1}{\varepsilon_k} \partial_3 v_{\varepsilon_k,\delta}\right) &= \partial_{y_3} \Psi_\delta \left(\varepsilon_k \left\lfloor \frac{\widehat{x}}{\varepsilon_k} \right\rfloor + \varepsilon_k \widehat{y}, y \right) \\ &\rightarrow \partial_{y_3} \Psi_\delta(\widehat{x}, y) \quad \text{in } L^2(\omega \times Y^*).\end{aligned}$$

With these two strong convergences at hand, it is again easy to infer that

$$\begin{aligned}&\lim_{k \rightarrow \infty} \int_{B^{\varepsilon\text{-perf}}} W^{\text{interf}} \left(\widehat{\nabla}v_{\varepsilon_k,\delta}(x), \frac{1}{\varepsilon_k} \partial_3 v_{\varepsilon_k,\delta}(x) \right) dx \\ &= \lim_{k \rightarrow \infty} \int_\omega \int_{Y^*} W^{\text{interf}} \left(\widehat{\mathcal{T}}_{\varepsilon_k}^*(\widehat{\nabla}v_{\varepsilon_k,\delta})(\widehat{x}, y), \widehat{\mathcal{T}}_{\varepsilon_k}^*\left(\frac{1}{\varepsilon_k} \partial_3 v_{\varepsilon_k,\delta}\right)(\widehat{x}, y) \right) dy d\widehat{x} \\ &= \int_\omega \int_{Y^*} W^{\text{interf}} \left([\widehat{\nabla}v_0(\widehat{x}, 0), 0]^T + \nabla_y \Psi_\delta(\widehat{x}, y) \right) dy d\widehat{x}.\end{aligned}$$

Then, with the help of (3.84) one can pass to limit $\delta_\ell \rightarrow 0$ and obtains

$$\begin{aligned}&\lim_{\ell \rightarrow \infty} \left(\lim_{k \rightarrow \infty} \int_{B^{\varepsilon\text{-perf}}} W^{\text{interf}} \left(\widehat{\nabla}v_{\varepsilon_k,\delta_\ell}(x), \frac{1}{\varepsilon_k} \partial_3 v_{\varepsilon_k,\delta_\ell}(x) \right) dx \right) \\ &= \lim_{\ell \rightarrow \infty} \left(\int_\omega \int_{Y^*} W^{\text{interf}} \left([\widehat{\nabla}v_0(\widehat{x}, 0), 0]^T + \nabla_y \Psi_{\delta_\ell}(\widehat{x}, y) \right) dy d\widehat{x} \right) \\ &= \int_\omega \left(\int_{Y^*} W^{\text{interf}} \left([\widehat{\nabla}v_0(\widehat{x}, 0), 0]^T + \nabla_y V_0(\widehat{x}, y) \right) dy \right) d\widehat{x},\end{aligned}$$

which by (3.83) is nothing but

$$\begin{aligned}\lim_{\ell \rightarrow \infty} \left(\lim_{k \rightarrow \infty} \int_{B^{\varepsilon\text{-perf}}} W^{\text{interf}} \left(\widehat{\nabla}v_{\varepsilon_k,\delta_\ell}(x), \frac{1}{\varepsilon_k} \partial_3 v_{\varepsilon_k,\delta_\ell}(x) \right) dx \right) \\ = \int_\omega \widehat{W}^{\text{interf}}(\widehat{\nabla}v_0(\widehat{x}, 0)) d\widehat{x}. \quad (3.88)\end{aligned}$$

Furthermore, the source term can be treated as follows. By performing the same

steps leading to (3.81) one has

$$\begin{aligned}
& \int_{\partial B^{\varepsilon\text{-perf}} \cap B^\circ} G(v_{\varepsilon_k, \delta}(x)) \left| M_{\varepsilon_k} n(x; \partial B^{\varepsilon\text{-perf}}) \right| da(x) \\
&= \int_{\omega} \int_{\partial V \cap \text{int } Y} G\left(\widehat{\mathcal{T}}_{\varepsilon_k}^* v_{\varepsilon_k, \delta}(\widehat{x}, y)\right) da(y) d\widehat{x} \\
&= \int_{\omega} \int_{\partial V \cap \text{int } Y} G\left(v_0(\widehat{x}, 0) + \varepsilon_k \Psi_\delta\left(\varepsilon_k \left\lfloor \frac{\widehat{x}}{\varepsilon_k} \right\rfloor + \varepsilon_k \widehat{y}, y\right)\right) da(y) d\widehat{x} \\
&\rightarrow \int_{\omega} \int_{\partial V \cap \text{int } Y} G(v_0(\widehat{x}, 0)) da(y) d\widehat{x} \\
&= \text{vol}_2(\partial V \cap \text{int } Y) \int_{\omega} G(v_0(\widehat{x}, 0)) d\widehat{x}. \tag{3.89}
\end{aligned}$$

as ε_k vanishes.

Now, by collecting the convergences (3.87), (3.88) and (3.89) one can conclude

$$\begin{aligned}
& \lim_{\ell \rightarrow \infty} \left(\lim_{k \rightarrow \infty} \mathcal{F}_{\varepsilon_k}(v_{\varepsilon_k, \delta_\ell}) \right) \\
&= \lim_{\ell \rightarrow \infty} \left(\lim_{k \rightarrow \infty} \int_{\Omega_{\text{tr}}^+} W^+(\nabla v_{\varepsilon_k, \delta_\ell}(x)) dx + \lim_{k \rightarrow \infty} \int_{\Omega_{\text{tr}}^-} W^-(\nabla v_{\varepsilon_k, \delta_\ell}(x)) dx \right. \\
&\quad + \lim_{k \rightarrow \infty} \int_{B^{\varepsilon\text{-perf}}} W^{\text{interf}}\left(\widehat{\nabla} v_{\varepsilon_k, \delta_\ell}(x), \frac{1}{\varepsilon_k} \partial_3 v_{\varepsilon_k, \delta_\ell}(x)\right) dx \\
&\quad \left. + \lim_{k \rightarrow \infty} \int_{\partial B^{\varepsilon\text{-perf}} \cap B^\circ} G(v_{\varepsilon_k, \delta_\ell}(x)) \left| M_{\varepsilon_k} n(x; \partial B^{\varepsilon\text{-perf}}) \right| da(x) \right) \\
&= \int_{\Omega_{\text{tr}}^+} W^+(\nabla v_0(x)) dx + \int_{\Omega_{\text{tr}}^-} W^-(\nabla v_0(x)) dx \\
&\quad + \int_{\omega} \widehat{W}^{\text{interf}}\left(\widehat{\nabla} v_0(\widehat{x}, 0)\right) d\widehat{x} \\
&\quad + \text{vol}_2(\partial V \cap \text{int } Y) \int_{\omega} G(v_0(\widehat{x}, 0)) d\widehat{x} \\
&= \mathcal{F}_{\text{Hom}}(v_0).
\end{aligned}$$

The proof of the Γ -lim sup-inequality can finally be completed by applying the very same diagonalization argument already used in the proof of Theorem 3.1.

Chapter 4

Nonconvergence in convex homogenization by the use of multiple coordinate frames

This last chapter of my thesis is the result of a collaboration with my friend and colleague Stefan Neukamm that started already in summer 2008 when we discovered ‘by accident’ a somewhat particular property of two-scale convergence. That is, we proved that two-scale convergence of a sequence of functions (see Subsection 2.4.1) actually depends on the specific *coordinate frame* in which the functions’ domain is described. To the extent that two-scale convergent sequences in one coordinate frame may result as non two-scale convergent in another coordinate frame. Moreover, we could precisely describe how a change of variables to a translated coordinate frame affects two-scale convergence. In fact, as a result of an intense exchange of eMails between Trento and Munich in May 2008 (Stefan was PhD student at the Technische Universität München back then while I was already staying in Trento) that lasted only two days we could already establish the main result of our collaboration. However, following the advice of my supervisor, Prof. Augusto Visintin, we kept on elaborating the main result and finally managed to solve a new kind of homogenization problem involving multiple coordinate frames in which we observed general nonconvergence, depending on the specific sequence of microscale parameters chosen. These results were published in *Asymptotic Analysis* in 2011, vol. 71(3), pp. 163–183 with the title ‘On the interplay of two-scale convergence and translation’ (cf. [Neukamm and Stelzig \[2011\]](#)).

4.1 Introduction and outline

Two-scale convergence, as it was already introduced in detail in Subsection 2.4.1, is an already well-established, well-studied and important tool in the theory of periodic homogenization, and its relation to ‘traditional’ notions of convergence appears to be well-understood (see the Propositions 2.15 and 2.24). Yet, we were quite surprised by the effects on two-scale convergence in $L^2(\Omega)$ caused by the simplest change of

variables one can think of: a translation of the underlying coordinate frame describing the domain Ω . While the translation operator is continuous w.r.t. strong and weak convergence in $L^2(\Omega)$, its interplay with two-scale convergence turns out to be more subtle. As an illustration one might consider for one space dimension $\Omega = \mathbb{R}$ and the periodicity cell $Y = [0, 1)$ the sequence

$$u_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}, \quad u_\varepsilon(x) := u_{\text{macro}}(x) u_{\text{oscill}}\left(\frac{x}{\varepsilon}\right),$$

parametrized by vanishing microscale parameters ε , with u_{macro} being an arbitrary function in $L^2(\mathbb{R})$ and u_{oscill} a smooth, Y -periodic function s.t. $\int_Y u_{\text{oscill}}(y) dy = 1$. It is well-known that

$$\begin{aligned} u_{\varepsilon_k}(x) &\rightharpoonup u_{\text{macro}}(x) && \text{weakly in } L^2(\mathbb{R}) \\ u_{\varepsilon_k}(x) &\overset{2}{\rightharpoonup} u_{\text{macro}}(x) u_{\text{oscill}}(y) && \text{weakly two-scale in } L^2(\mathbb{R} \times Y) \end{aligned}$$

for any vanishing sequence of positive real numbers $(\varepsilon_k)_k$ (see the Example 2.1 and e.g. Proposition 2.15). Clearly, the translated sequence $(u_{\varepsilon_k}(x+t))_k$ converges to $u_{\text{macro}}(x+t)$ weakly in $L^2(\mathbb{R})$. In contrast to this, $(u_{\varepsilon_k}(x+t))_k$ may not even be weakly two-scale convergent. Still, as it remains bounded in $L^2(\Omega)$, the translated sequence exhibits weakly two-scale convergent subsequences (by the classical two-scale compactness result in Theorem 2.17). For instance, if $\varepsilon_k = \frac{2}{k}$ and $t = 1$, then the translated sequence solely exhibits the weak two-scale cluster points $u_{\text{macro}}(x+1) u_{\text{oscill}}(y)$ and $u_{\text{macro}}(x+1) u_{\text{oscill}}\left(y + \frac{1}{2}\right)$. From this one might with some optimism ‘deduce’ that for *any* sequence $(\varepsilon_k)_k$ and *any* translation t ,

$$\begin{aligned} \text{each weak two-scale cluster point of } (u_{\varepsilon_k}(x+t))_k \\ \text{is of the form } u_{\text{macro}}(x+t) u_{\text{oscill}}(y+r) \end{aligned}$$

for some $r \in \overline{Y}$, which we call a ‘microtranslation’.

Indeed, we are going to prove the following: Let $(u_{\varepsilon_k})_k$ be a sequence in $L^2(\Omega)$ (Ω an open subset of \mathbb{R}^N and $Y = [0, 1)^N$) that weakly two-scale converges to $u \in L^2(\Omega \times Y)$ w.r.t. the vanishing sequence $(\varepsilon_k)_k$ and consider an arbitrary translation $t \in \mathbb{R}^N$. Then each weak two-scale cluster point of the translated sequence $(u_{\varepsilon_k}(\cdot+t))_k$ has the form $u(\cdot+t, \cdot+r)$ for some microtranslation r . That is, the original weak two-scale limit not only shifted in its ‘macrovariable’ by the ‘macrotranslation’ t , but also in the ‘microvariable’ by the microtranslation r . More precisely, we show that the set \mathcal{C}_w of all weak two-scale cluster points of $(u(\cdot+t)_{\varepsilon_k})_k$ can be characterized according to

$$\mathcal{C}_w = \left\{ u(\cdot+t, \cdot+r) : r \in \mathcal{M} \right\}$$

where \mathcal{M} denotes the set of all attainable microtranslations. It turns out, that \mathcal{M} is completely determined by the translation t and the vanishing sequence $(\varepsilon_k)_k$. In the case where $(u_{\varepsilon_k})_k$ strongly two-scale converges to u , the very same representation holds true for the set \mathcal{C}_s of all strong two-scale cluster points of the translated sequence. Thus translation of the coordinate frame considerably affects two-scale convergence in $L^2(\Omega)$, in

a way that *only* depends on the translation t and the vanishing sequence $(\varepsilon_k)_k$ but *not* on the particular sequence of functions itself. Finally, we will show that $\mathcal{C}_w, \mathcal{C}_s$ and \mathcal{M} are compact and illustrate the dependence of \mathcal{M} on the choice of t and $(\varepsilon_k)_k$.

Later on in Section 4.3, we apply these results to a homogenization problem that involves two distinct coordinate frames being translated by a constant vector. Our goal is to characterize the asymptotic behavior of the oscillating functional

$$\mathcal{F}_\varepsilon : W_0^{1,2}(\Omega; \mathbb{R}^N) \rightarrow [0, +\infty),$$

$$\mathcal{F}_\varepsilon(u) := \int_{\Omega} W\left(\frac{x}{\varepsilon}, Du(x)\right) dx + \int_{\Omega-t} W\left(\frac{x}{\varepsilon}, Du(x+t)\right) dx,$$

$t \in \mathbb{R}^N$ being an arbitrary translation, as the size ε of the oscillations tends to zero. Here, the energy density $W(y, F)$ shall be convex, continuous and of quadratic growth in F , and Y -periodic in y . In the case $t = 0$, this kind of problem has been extensively studied, most notably by Marcellini in his fundamental contribution [Marcellini \[1978\]](#) and later on also in non-convex situations related to elasticity by Müller [Müller \[1987\]](#) (see the introduction of this thesis, Subsection 2.2.3). Since the early 1990s, many authors have revisited the above problem with trivial translation $t = 0$ employing methods related to two-scale convergence in order to elegantly prove Γ -convergence of \mathcal{F}_ε to a homogenized limiting problem (see in particular [Allaire \[1992\]](#); [Cioranescu et al. \[2006a\]](#); [Visintin \[2007\]](#)). In the new situation with nontrivial translation $t \neq 0$ on the other hand, Γ -convergence generally fails as we show explicitly in Section 4.4. However, thanks to our characterization of weak two-scale cluster points of translated sequences we can explicitly state the lower and upper Γ -limits of \mathcal{F}_ε (as ε vanishes) in terms of associated microtranslations. In passing we note that in Subsection 4.3.2 we provide an abstract result on Γ -convergence that may be of its own interest.

4.2 Two-scale convergence in translated coordinate frames

Here and hereafter we will denote by Ω an open subset of \mathbb{R}^N . Given an arbitrary vector $t \in \mathbb{R}^N$, the set $\Omega - t$ is to be understood as the set $\{x - t : x \in \Omega\}$. As the periodicity cell Y we take the N -dimensional unit cube $[0, 1)^N$ and any function defined on Y in one of its variables is assumed to be extended to \mathbb{R}^N by Y -periodicity. Moreover, we will frequently encounter the sequence $(\varepsilon_k)_k$, which shall be an arbitrary but *fixed* vanishing sequence of positive real numbers. Following ideas from [Cioranescu et al. \[2002\]](#) and adopting the notation used in [Visintin \[2006\]](#), we define $\mathcal{N} : \mathbb{R}^N \rightarrow \mathbb{Z}^N$ and $\mathcal{R} : \mathbb{R}^N \rightarrow Y$ by setting

$$\mathcal{N}(x) := \max\{z \in \mathbb{Z}^N : z_i \leq x_i, i = 1, \dots, N\}, \quad \mathcal{R}(x) := x - \mathcal{N}(x)$$

wherein max is taken componentwise.

4.2.1 The occurrence of microtranslations

As explained in the Section 4.1, our intention is to understand the behavior of weakly or strongly two-scale convergent sequences under translation of the coordinate frame.

To this end, it is convenient to define the following:

Definition 4.1. Let $(u_{\varepsilon_k})_k$ be a sequence in $L^2(\Omega)$ and $t \in \mathbb{R}^N$.

- (i) The set of all *weak two-scale cluster points* of the sequence $(u_{\varepsilon_k})_k$ translated by t is defined by

$$\mathcal{C}_w := \left\{ v \in L^2((\Omega-t) \times Y) : \text{there exists a subsequence } (k_\ell)_\ell \text{ s.t.} \right. \\ \left. u_{\varepsilon_{k_\ell}}(\cdot + t) \xrightarrow{2} v \text{ in } L^2((\Omega-t) \times Y) \right\}.$$

The set of all *strong two-scale cluster points* of the sequence $(u_{\varepsilon_k})_k$ translated by t is defined by

$$\mathcal{C}_s := \left\{ v \in L^2((\Omega-t) \times Y) : \text{there exists a subsequence } (k_\ell)_\ell \text{ s.t.} \right. \\ \left. u_{\varepsilon_{k_\ell}}(\cdot + t) \xrightarrow{2} v \text{ in } L^2((\Omega-t) \times Y) \right\}.$$

- (ii) The set of all *microtranslations* emerging from the translation t and $(\varepsilon_k)_k$ is defined by

$$\mathcal{M} := \left\{ r \in \bar{Y} : r \text{ is a cluster point of } \left(\mathcal{R}\left(\frac{t}{\varepsilon_k}\right) \right)_k \text{ in } \mathbb{R}^N \right\}.$$

Moreover, we call a subsequence $(k_\ell)_\ell$ a (t, r) -subsequence, if $\lim_{\ell \rightarrow \infty} \mathcal{R}\left(\frac{t}{\varepsilon_{k_\ell}}\right) = r$.

When considering a weakly or strongly two-scale convergent sequence $(u_{\varepsilon_k})_k$ in $L^2(\Omega)$, the reasoning in the previous section suggests that we cannot expect two-scale convergence of the translated sequence $(u_{\varepsilon_k}(\cdot + t))_k$. In particular, the sets \mathcal{C}_w and \mathcal{C}_s are going to be nontrivial. However, they can be characterized in a very precise manner:

Theorem 4.2. Let $(u_{\varepsilon_k})_k$ be a sequence in $L^2(\Omega)$ and $u \in L^2(\Omega \times Y)$.

- (i) If $u_{\varepsilon_k} \xrightarrow{2} u$ in $L^2(\Omega \times Y)$, then

$$\mathcal{C}_w = \left\{ u(\cdot + t, \cdot + r) : r \in \mathcal{M} \right\}.$$

- (ii) If $u_{\varepsilon_k} \xrightarrow{2} u$ in $L^2(\Omega \times Y)$, then

$$\mathcal{C}_s = \left\{ u(\cdot + t, \cdot + r) : r \in \mathcal{M} \right\}.$$

Regarding this result, we urge the reader to notice that the set of all weak (strong) two-scale cluster points \mathcal{C}_w (\mathcal{C}_s) of a two-scale convergent sequence $(u_{\varepsilon_k})_k$ translated by t does *not* depend on the sequence itself, but only on the vanishing sequence $(\varepsilon_k)_k$, the translation t and the original two-scale limit u . Indeed, since all two-scale cluster points can be obtained by doubly translating the original limit u by t in the macrovariable and by $r \in \mathcal{M}$ in the microvariable, we refer to \mathcal{M} as the set of all microtranslations emerging from the translation t and $(\varepsilon_k)_k$.

The main ingredient in the proof of Theorem 4.2 will be the next lemma, which can be regarded as a result of its own interest.

Lemma 4.3. *Let $(k_\ell)_\ell$ be a (t, r) -subsequence and $(u_{\varepsilon_{k_\ell}})_\ell$ a weakly (respectively strongly) two-scale convergent sequence in $L^2(\Omega \times Y)$ with limit u . Then the sequence $(u_{\varepsilon_{k_\ell}}(\cdot + t))_\ell$ is weakly (respectively strongly) two-scale convergent in $L^2((\Omega - t) \times Y)$ to the function*

$$u(\cdot + t, \cdot + r) \in L^2((\Omega - t) \times Y).$$

Proof. The proof of this statement will be two-stage.

Step 1. We first consider the case of weak two-scale convergence $u_{\varepsilon_{k_\ell}} \xrightarrow{2} u$. Since the translated sequence remains bounded, it suffices to prove

$$\lim_{\ell \rightarrow \infty} \int_{\Omega - t} u_{\varepsilon_{k_\ell}}(x + t) \varphi\left(x, \frac{x}{\varepsilon_{k_\ell}}\right) dx = \int_{\Omega - t} \int_Y u(x + t, y + r) \varphi(x, y) dy dx \quad (4.1)$$

for all $\varphi \in C_c^\infty(\Omega - t; C_{\text{per}}^\infty(Y))$. By the change of variables $x \mapsto x - t$ we can write the integral on the left hand side as

$$\begin{aligned} \int_{\Omega - t} u_{\varepsilon_{k_\ell}}(x + t) \varphi\left(x, \frac{x}{\varepsilon_{k_\ell}}\right) dx &= \int_{\Omega} u_{\varepsilon_{k_\ell}}(x) \varphi\left(x - t, \frac{x}{\varepsilon_{k_\ell}} - \frac{t}{\varepsilon_{k_\ell}}\right) dx \\ &= \int_{\Omega} u_{\varepsilon_{k_\ell}}(x) \varphi\left(x - t, \frac{x}{\varepsilon_{k_\ell}} - r\right) dx + \int_{\Omega} u_{\varepsilon_{k_\ell}}(x) \varphi_{\varepsilon_{k_\ell}}(x) dx, \end{aligned} \quad (4.2)$$

where

$$\varphi_{\varepsilon_{k_\ell}}(x) := \varphi\left(x - t, \frac{x}{\varepsilon_{k_\ell}} - \frac{t}{\varepsilon_{k_\ell}}\right) - \varphi\left(x - t, \frac{x}{\varepsilon_{k_\ell}} - r\right).$$

Since $\varphi(\cdot - t, \cdot - r) \in C_c^\infty(\Omega; C_{\text{per}}^\infty(Y))$ is an admissible two-scale testfunction and $u_{\varepsilon_{k_\ell}} \xrightarrow{2} u$, we can pass to the limit in the first integral of (4.2) and obtain (after retransformation)

$$\lim_{\ell \rightarrow \infty} \int_{\Omega} u_{\varepsilon_{k_\ell}}(x) \varphi\left(x - t, \frac{x}{\varepsilon_{k_\ell}} - r\right) dx = \int_{\Omega - t} \int_Y u(x + t, y + r) \varphi(x, y) dy dx. \quad (4.3)$$

Now, it remains to show that the second integral in (4.2) vanishes in the limit. Therefore, we prove that $\varphi_{\varepsilon_{k_\ell}} \rightarrow 0$ uniformly. Due to the decomposition

$$\frac{t}{\varepsilon_{k_\ell}} = \mathcal{N}\left(\frac{t}{\varepsilon_{k_\ell}}\right) + \mathcal{R}\left(\frac{t}{\varepsilon_{k_\ell}}\right)$$

and the Y -periodicity of φ in the second variable, we deduce

$$\varphi_{\varepsilon_{k_\ell}}(x) = \varphi\left(x - t, \frac{x}{\varepsilon_{k_\ell}} - r + r_\ell\right) - \varphi\left(x - t, \frac{x}{\varepsilon_{k_\ell}} - r\right),$$

where $r_\ell := r - \mathcal{R}\left(\frac{t}{\varepsilon_{k_\ell}}\right)$.

Now, the condition of $(k_\ell)_\ell$ being a (t, r) -subsequence and the smoothness of φ implies first that $r_\ell \rightarrow 0$ and secondly, that $\varphi_{\varepsilon_{k_\ell}}$ vanishes uniformly. Consequently, assertion (4.1) follows.

Step 2. Assume now strong two-scale convergence $u_{\varepsilon_{k_\ell}} \xrightarrow{2} u$. From Step 1 we already know that the translated sequence weakly two-scale converges to $u(\cdot + t, \cdot + r)$. However, this is already sufficient to infer strong two-scale convergence of the translated sequence, since $\|v\|_{L^2(\Omega)} = \|v(\cdot + t)\|_{L^2(\Omega-t)}$ for all $v \in L^2(\Omega)$ and thus $\|u_{\varepsilon_{k_\ell}}(\cdot + t)\|_{L^2(\Omega-t)} \rightarrow \|u(\cdot + t, \cdot + r)\|_{L^2((\Omega-t) \times Y)}$. \square

Having this preparatory result at hand, the proof of the main Theorem 4.2 is no longer difficult:

Proof of Theorem 4.2. We confine ourselves to proving the statement for \mathcal{C}_w , since the proof for \mathcal{C}_s is similar.

First, we show the inclusion $\{u(\cdot + t, \cdot + r) : r \in \mathcal{M}\} \subseteq \mathcal{C}_w$. Let $r \in \mathcal{M}$ and $(k_\ell)_\ell$ a corresponding (t, r) -subsequence. By Lemma 4.3 we obtain $u_{\varepsilon_{k_\ell}}(\cdot + t) \xrightarrow{2} u(\cdot + t, \cdot + r)$ and conclude $u(\cdot + t, \cdot + r) \in \mathcal{C}_w$.

In order to show the opposite inclusion, let $v \in \mathcal{C}_w$ and choose a subsequence $(k_\ell)_\ell$ such that $u_{\varepsilon_{k_\ell}}(\cdot + t) \xrightarrow{2} v$. By the compactness of \bar{Y} we can assume without loss of generality that

$$\mathcal{R}\left(\frac{t}{\varepsilon_{k_\ell}}\right) \rightarrow r.$$

Hence, $(k_\ell)_\ell$ is a (t, r) -subsequence with $r \in \mathcal{M}$ and Lemma 4.3 implies that $v = u(\cdot + t, \cdot + r)$. \square

4.2.2 Strong compactness of in the set of cluster points

The next result states that the set of all weak (respectively strong) two-scale cluster points \mathcal{C}_w (respectively \mathcal{C}_s) of a translated weakly (respectively strongly) two-scale convergent sequence and the set of all microtranslations \mathcal{M} are *compact*:

Proposition 4.4. *The set \mathcal{M} is a compact subset of \mathbb{R}^N . In the situation of the first (respectively second) statement of Theorem 4.2, the set \mathcal{C}_w (respectively \mathcal{C}_s) is compact w.r.t. strong convergence in $L^2((\Omega - t) \times Y)$.*

Proof. The proof is split into two parts, the first dealing with the compactness of \mathcal{M} , the second stating the compactness of \mathcal{C}_w . As the compactness of \mathcal{C}_s can be proved similarly to \mathcal{C}_w , we do not go into the details of its proof.

Step 1. First, let us remark that

$$\mathcal{M} = \left\{ r \in \mathbb{R}^N : \liminf_{k \rightarrow \infty} \left| \mathcal{R}\left(\frac{t}{\varepsilon_k}\right) - r \right| = 0 \right\}. \quad (4.4)$$

Obviously \mathcal{M} is bounded. We will see that it is also closed: Consider an arbitrary sequence $(r_\ell)_\ell$ in \mathcal{M} that converges to some $r \in \mathbb{R}^N$. We define the quantity

$$c_{\ell, k} := \left| \mathcal{R}\left(\frac{t}{\varepsilon_k}\right) - r_\ell \right| + |r_\ell - r|.$$

Since $r_\ell \in \mathcal{M}$, equation (4.4) implies that $\liminf_k c_{\ell,k} = |r_\ell - r|$ for all $\ell \in \mathbb{N}$, and therefore

$$\limsup_{\ell \rightarrow \infty} \left(\liminf_{k \rightarrow \infty} c_{\ell,k} \right) = \limsup_{\ell \rightarrow \infty} |r_\ell - r| = 0. \tag{4.5}$$

Utilizing [Attouch, 1984, Lemma 1.17], we can choose a subsequence $(\ell(k))_\ell$ such that

$$\liminf_{k \rightarrow \infty} c_{\ell(k),k} \leq \limsup_{\ell \rightarrow \infty} \left(\liminf_{k \rightarrow \infty} c_{\ell,k} \right)$$

and deduce by (4.5) that the left hand side is equal to 0. A simple triangle inequality then leads to

$$\begin{aligned} \liminf_{k \rightarrow \infty} \left| \mathcal{R} \left(\frac{t}{\varepsilon_k} \right) - r \right| &\leq \liminf_{k \rightarrow \infty} \left(\left| \mathcal{R} \left(\frac{t}{\varepsilon_k} \right) - r_{\ell(k)} \right| + |r_{\ell(k)} - r| \right) \\ &= \liminf_{k \rightarrow \infty} c_{\ell(k),k} = 0 \end{aligned}$$

and again with the help of (4.4) we deduce that $r \in \mathcal{M}$.

Step 2. Like in the situation of Theorem 4.2, let u denote the weak two-scale limit of the sequence $(u_{\varepsilon_k})_k$. We introduce the mapping

$$\Phi : \mathcal{M} \rightarrow L^2((\Omega - t) \times Y), \quad r \mapsto u(\cdot + t, \cdot + r).$$

By Theorem 4.2 one can easily see that $\Phi(\mathcal{M})$ is equal to \mathcal{C}_w . Since \mathcal{M} is compact by step 1, it is sufficient to prove that Φ is continuous. But this is true, since for every sequence $(r_\ell)_\ell$ in \overline{Y} converging to some r we observe $u(\cdot + t, \cdot + r_\ell) \rightarrow u(\cdot + t, \cdot + r)$ in $L^2((\Omega - t) \times Y)$. \square

Before proceeding with the application of the results of the present section to a novel type of variational convex homogenization, we shall state several explanatory comments.

Remark 4.1. Since $(\mathcal{R}(\frac{t}{\varepsilon_k}))_k$ is a sequence in the compact set \overline{Y} , we in particular conclude

$$\mathcal{M} \neq \emptyset \quad \text{and thus} \quad \mathcal{C}_w \neq \emptyset$$

for every weakly two-scale convergent sequence $(u_{\varepsilon_k})_k$ in $L^2(\Omega)$. The same observation is true for a strongly two-scale convergent sequence $(u_{\varepsilon_k})_k$ and \mathcal{C}_s .

Indeed, the sets \mathcal{C}_w and \mathcal{C}_s of all two-scale cluster points can be very rich, as it is revealed by the following example: We consider the case $N = 1$. Let $t \neq 0$ be an arbitrary translation vector in \mathbb{R} and define $\varepsilon_k := \frac{|t|}{k + \text{sgn}(t) \cdot q(k)}$, where $q : \mathbb{N} \rightarrow \mathbb{Q} \cap [0, 1]$ is a surjective map and sgn returns the sign of its argument. In this case we obtain $\mathcal{M} = [0, 1] = \overline{Y}$.

Remark 4.2. The result of this section can be easily extended to converging translations: Let $\Omega = \mathbb{R}^N$, $t_k \rightarrow t$ in \mathbb{R}^N and consider a subsequence $(k_\ell)_\ell$ satisfying

$$\lim_{\ell \rightarrow \infty} \left| \mathcal{R} \left(\frac{t_{k_\ell}}{\varepsilon_{k_\ell}} \right) - r \right| = 0$$

for some $r \in \overline{Y}$. Then for any weakly two-scale convergent sequence $(u_{\varepsilon_{k_\ell}})_\ell$ in $L^2(\mathbb{R}^N)$ with limit u we have

$$u_{\varepsilon_{k_\ell}}(\cdot + t_{k_\ell}) \xrightarrow{2} u(\cdot + t, \cdot + r) \text{ in } L^2(\mathbb{R}^N \times Y).$$

This can be seen by modifying the definition of $\varphi_{\varepsilon_{k_\ell}}$ in equation (4.2) in the proof of Lemma 4.3:

$$\begin{aligned} \int_{\mathbb{R}^N} u_{\varepsilon_{k_\ell}}(x + t_{k_\ell}) \varphi \left(x, \frac{x}{\varepsilon_{k_\ell}} \right) dx &= \int_{\mathbb{R}^N} u_{\varepsilon_{k_\ell}}(x) \varphi \left(x - t_{k_\ell}, \frac{x}{\varepsilon_{k_\ell}} - \frac{t_{k_\ell}}{\varepsilon_{k_\ell}} \right) dx \\ &= \int_{\mathbb{R}^N} u_{\varepsilon_{k_\ell}}(x) \varphi \left(x - t, \frac{x}{\varepsilon_{k_\ell}} - r \right) dx + \int_{\mathbb{R}^N} u_{\varepsilon_{k_\ell}}(x) \varphi_{\varepsilon_{k_\ell}}(x) dx \end{aligned}$$

with

$$\varphi_{\varepsilon_{k_\ell}}(x) := \varphi \left(x - t_{k_\ell}, \frac{x}{\varepsilon_{k_\ell}} - \frac{t_{k_\ell}}{\varepsilon_{k_\ell}} \right) - \varphi \left(x - t, \frac{x}{\varepsilon_{k_\ell}} - r \right).$$

Since φ is smooth, $\varphi_{\varepsilon_{k_\ell}}$ vanishes uniformly.

Remark 4.3. In Meunier and Van Schaftingen [2005] Meunier and van Schaftingen introduce a modification of the periodic unfolding operator from Cioranescu et al. [2002] that features additive perturbations on the microscale ε , which they call microscopic translations. As concerns the objective of their contribution, Meunier and van Schaftingen prove that vanishing microscopic translations do not alter the two-scale limit behavior of a sequence of functions. Upon assuming that these microscopic translations originate from ‘very small macroscopic translations’, their insight corresponds to the previous remark for the case of macroscopic translations $(t_\varepsilon)_\varepsilon$ satisfying $t_\varepsilon/\varepsilon \rightarrow 0$ and consequently $\mathcal{M} = \{0\}$.

However, we would like to remark that in contrast to our analysis the subtle connection between macroscopic translations and the corresponding microscopic translations is not discussed in Meunier and Van Schaftingen [2005].

Remark 4.4. Translations occurring in the microscopic variable of homogenized quantities sometimes play a crucial role in the understanding of the underlying homogenization processes, see *e.g.* our results in Section 4.3 below. Let us remark that microtranslations may also originate from phenomena other than macroscopic translations of the coordinate frame. For instance, this is the case for the Bloch wave homogenization method due to Allaire and Conca Allaire and Conca [1996]. The purpose of this method is to characterize the asymptotic behavior of the spectrum associated to linear second order elliptic PDEs with periodically oscillating coefficients as the size of the period tends to zero. Therein, microtranslations emerge from a Bloch wave decomposition (cf. Allaire and Conca [1995]) and play a key role in the resulting cell problems (see Allaire and Conca [1996]; Castro and Zuazua [1996]).

4.3 A novel problem in variational convex homogenization

In this section we present an application of our previous insights to a novel homogenization problem. To this end, we will provide a convergence study for an oscillating convex integral functional, the oscillating arguments of which are translated by a nonzero macroscopic quantity. Whereas in the absence of translation one can prove Γ -convergence of the integral functionals, in its presence Γ -convergence no longer holds true in general. Nevertheless, the results of the Section 4.2 allow us to explicitly identify the lower and the upper Γ -limit.

4.3.1 An oscillating integral functional involving two coordinate frames

For the notation used in the sequel we also refer to the preceding section. Let t be an arbitrary but fixed translation in \mathbb{R}^N and Ω be an open and bounded subset of \mathbb{R}^N . We consider the functional

$$\mathcal{F}_\varepsilon : W_0^{1,2}(\Omega; \mathbb{R}^N) \rightarrow \mathbb{R}, \quad u \mapsto \int_\Omega W\left(\frac{x}{\varepsilon}, Du(x)\right) dx + \int_{\Omega-t} W\left(\frac{x}{\varepsilon}, Du(x+t)\right) dx$$

for a positive microscale parameter ε and seek to describe its behavior as ε vanishes. We assume that the integrand

$$W : Y \times \mathbb{R}^{N \times N} \rightarrow \mathbb{R}, \quad (y, F) \mapsto W(y, F)$$

satisfies the properties (W1), (W2) and (W3) stated in Subsection 2.2.3 (in a vector-valued fashion though), *i.e.*

for all $F \in \mathbb{R}^{N \times N}$ the map $y \mapsto W(y, F)$ is measurable and Y -periodic,
for a.e. $y \in Y$ the map $F \mapsto W(y, F)$ is convex,

$$\left\{ \begin{array}{l} \text{there exist positive constants } c, C \text{ such that} \\ c(|F|^2 - 1) \leq W(y, F) \leq C(1 + |F|^2) \\ \text{for a.e. } y \in Y \text{ and all } F \in \mathbb{R}^{N \times N}. \end{array} \right.$$

Indeed, for vanishing translation $t = 0$ the homogenization of integral functionals like $(\mathcal{F}_\varepsilon)_\varepsilon$ with integrands satisfying the above assumptions using Γ -convergence methods is classical. Besides the introduction of this thesis (see Subsection 2.2.3 and in particular Theorem 2.9) we refer to Marcellini [1978]; Müller [1987]; Allaire [1992]; Cioranescu et al. [2006a]; Visintin [2007] for this homogenization result. However, for nonzero translations t the homogenization of $(\mathcal{F}_\varepsilon)_\varepsilon$ becomes far more delicate as we will show hereafter.

Prior to stating the homogenization result, we introduce the following objects, which depend on a microtranslation r and naturally arise in the subsequent analysis: For $r \in \bar{Y}$, $y \in Y$ and $F \in \mathbb{R}^{N \times N}$ we set

$$W_r(y, F) := W(y, F) + W(y - r, F)$$

$$W_{\text{Hom},r}(F) := \inf \left\{ \int_Y W_r(y, F + D_y \varphi(y)) dy : \varphi \in W_{\text{per}}^{1,2}(Y; \mathbb{R}^N), \int_Y \varphi(y) dy = 0 \right\}$$

and refer to W_r as the microtranslated energy density and to $W_{\text{Hom},r}$ as its homogenization. Moreover, we define the corresponding two-scale energy functional $\overline{\mathcal{F}}_r$, as well as the homogenized microtranslated energy $\mathcal{F}_{\text{Hom},r}$ according to

$$\begin{aligned}\overline{\mathcal{F}}_r(u, u_1) &:= \int_{\Omega} \int_Y W_r(y, Du(x) + D_y u_1(x, y)) \, dy \, dx, \\ \mathcal{F}_{\text{Hom},r}(u) &:= \int_{\Omega} W_{\text{Hom},r}(Du(x)) \, dx\end{aligned}$$

for every $u \in W_0^{1,2}(\Omega; \mathbb{R}^N)$ and every $u_1 \in L^2(\Omega; W_{\text{per}}^{1,2}(Y; \mathbb{R}^N))$.

Theorem 4.5. *Let $(\varepsilon_k)_k$ be an arbitrary vanishing sequence of positive real numbers. Then*

$$\begin{aligned}\Gamma\text{-}\liminf_{k \rightarrow \infty} \mathcal{F}_{\varepsilon_k} &= \inf_{r \in \mathcal{M}} \mathcal{F}_{\text{Hom},r} \\ \Gamma\text{-}\limsup_{k \rightarrow \infty} \mathcal{F}_{\varepsilon_k} &= \sup_{r \in \mathcal{M}} \mathcal{F}_{\text{Hom},r}\end{aligned}$$

w.r.t. the weak topology in $W_0^{1,2}(\Omega; \mathbb{R}^N)$. Like in Definition 4.1, \mathcal{M} denotes the set of all microtranslations emerging from the translation t and $(\varepsilon_k)_k$.

In fact, Theorem 4.5 reveals that for a nonzero translation t one can no longer expect Γ -convergence of the sequence $(\mathcal{F}_{\varepsilon_k})_k$. An explicit example therefor is found in the upcoming Section 4.4. Our result also contrasts the classical homogenization problem for convex integral functionals, where there indeed holds Γ -convergence (see Subsection 2.2.3, Theorem 2.9). It should again be noticed that our Theorem 4.5 recovers the classical result Theorem 2.9 by choosing the trivial translation $t = 0$, hence $\mathcal{M} = \{0\}$ according its Definition 4.1.

For the sake of a brief notation, in the following we set for every $u \in W_0^{1,2}(\Omega; \mathbb{R}^N)$

$$\mathcal{F}_{\text{Hom}}^-(u) := \inf_{r \in \mathcal{M}} \mathcal{F}_{\text{Hom},r}(u) \quad \text{and} \quad \mathcal{F}_{\text{Hom}}^+(u) := \sup_{r \in \mathcal{M}} \mathcal{F}_{\text{Hom},r}(u).$$

There are two main insights that we will rely on in the proof of Theorem 4.5.

First, we observe that $(\mathcal{F}_{\varepsilon_k})_k$ is Γ -convergent along (t, r) -subsequences. More precisely, for $r \in \mathcal{M}$ we will show that $\Gamma\text{-}\lim_{\ell} \mathcal{F}_{\varepsilon_{k_{\ell}}} = \mathcal{F}_{\text{Hom},r}$ along every (t, r) -subsequence $(k_{\ell})_{\ell}$. This will be done by combining Theorem 4.2 and general (lower semi-) continuity properties of convex integral functionals with oscillating integrands stated in Proposition 4.6 below. For a systematic investigation of the convex oscillating integral functionals with methods related to two-scale convergence we refer to Visintin [2007].

Secondly, we will provide an abstract result, which allows us to identify the lower and upper Γ -limit of a sequence by falling back to Γ -convergent subsequences. That is, if $(f_k)_k$ is a sequence of functions on a topological space X , such that the sequential characterization of Γ -convergence is valid for $(f_k)_k$ and the set $\mathcal{L}(x) := \{f(x) : f \text{ is } \Gamma\text{-limit of a subsequence of } (f_k)_k\}$ attains its extrema for all $x \in X$, then the lower

and upper Γ -limit of $(f_k)_k$ are given pointwise as $(\Gamma\text{-lim inf}_k f_k)(x) = \min \mathcal{L}(x)$ and $(\Gamma\text{-lim sup}_k f_k)(x) = \max \mathcal{L}(x)$, respectively (see Theorem 4.10). We will see that this observation together with the Γ -convergence of $(\mathcal{F}_{\varepsilon_k})_k$ along (t, r) -subsequences allows us to prove Theorem 4.5.

The following result and similar statements can be found in Visintin [2007] and Allaire [1992]; Cioranescu et al. [2006a, 2002].

Proposition 4.6. *Let W satisfy the conditions (W1), ..., (W3) and let U be an open and bounded subset of \mathbb{R}^N . For $\varepsilon > 0$ define the functionals*

$$\begin{aligned} \mathcal{J}_\varepsilon : L^2(U; \mathbb{R}^{N \times N}) &\rightarrow \mathbb{R}, & F &\mapsto \int_U W\left(\frac{x}{\varepsilon}, F(x)\right) dx, \\ \overline{\mathcal{J}} : L^2(U \times Y; \mathbb{R}^{N \times N}) &\rightarrow \mathbb{R}, & F &\mapsto \int_U \int_Y W(y, F(x, y)) dy dx. \end{aligned}$$

Then:

- (i) *The functionals \mathcal{J}_ε and $\overline{\mathcal{J}}$ are continuous w.r.t. strong convergence and lower semicontinuous w.r.t. weak convergence. Moreover, the functionals are convex and finite.*
- (ii) *Let $F \in L^2(U \times Y; \mathbb{R}^{N \times N})$ and $(F_{\varepsilon_k})_k$ be a sequence in $L^2(U; \mathbb{R}^{N \times N})$. If $F_{\varepsilon_k} \xrightarrow{2} F$ in $L^2(U \times Y; \mathbb{R}^{N \times N})$, then*

$$\overline{\mathcal{J}}(F) \leq \liminf_{k \rightarrow \infty} \mathcal{J}_{\varepsilon_k}(F_{\varepsilon_k}).$$

Moreover, if $F_{\varepsilon_k} \xrightarrow{2} F$ in $L^2(U \times Y; \mathbb{R}^{N \times N})$, it is

$$\overline{\mathcal{J}}(F) = \lim_{k \rightarrow \infty} \mathcal{J}_{\varepsilon_k}(F_{\varepsilon_k}).$$

An immediate consequence of the foregoing proposition and Theorem 4.2 is the observation below.

Corollary 4.7. *Let $F \in L^2(\Omega \times Y; \mathbb{R}^{N \times N})$, let $(F_{\varepsilon_k})_k$ be a sequence in $L^2(\Omega; \mathbb{R}^{N \times N})$, $r \in \mathcal{M}$ and $(k_\ell)_\ell$ a corresponding (t, r) -subsequence. Then*

- (i) *if $F_{\varepsilon_k} \xrightarrow{2} F$ in $L^2(\Omega \times Y; \mathbb{R}^{N \times N})$, then*

$$\begin{aligned} &\int_\Omega \int_Y W_r(y, F(x, y)) dy dx \leq \\ &\liminf_{\ell \rightarrow \infty} \left(\int_\Omega W\left(\frac{x}{\varepsilon_{k_\ell}}, F_{\varepsilon_{k_\ell}}(x)\right) dx + \int_{\Omega-t} W\left(\frac{x}{\varepsilon_{k_\ell}}, F_{\varepsilon_{k_\ell}}(x+t)\right) dx \right), \end{aligned}$$

- (ii) *if $F_{\varepsilon_k} \xrightarrow{2} F$ in $L^2(\Omega \times Y; \mathbb{R}^{N \times N})$, then*

$$\begin{aligned} &\int_\Omega \int_Y W_r(y, F(x, y)) dy dx = \\ &\lim_{\ell \rightarrow \infty} \left(\int_\Omega W\left(\frac{x}{\varepsilon_{k_\ell}}, F_{\varepsilon_{k_\ell}}(x)\right) dx + \int_{\Omega-t} W\left(\frac{x}{\varepsilon_{k_\ell}}, F_{\varepsilon_{k_\ell}}(x+t)\right) dx \right). \end{aligned}$$

With this result at hand, we can finally prove Γ -convergence of $(\mathcal{F}_{\varepsilon_k})_k$ along (t, r) -subsequences. Note that by the growth assumption (W3) and Poincaré's inequality the sequence $(\mathcal{F}_{\varepsilon_k})_k$ is equicoercive. Hence, by Theorem 2.5 Γ -convergence of $(\mathcal{F}_{\varepsilon_k})_k$ and its subsequences can equivalently be verified in terms of sequential Γ -convergence.

Proposition 4.8. *Let $r \in \mathcal{M}$ and $(k_\ell)_\ell$ be a corresponding (t, r) -subsequence. Then the following statements are valid.*

- (i) *For every $u \in W_0^{1,2}(\Omega; \mathbb{R}^N)$ and every sequence $(u_{\varepsilon_{k_\ell}})_\ell$ weakly converging to u in $W_0^{1,2}(\Omega; \mathbb{R}^N)$ we have*

$$\mathcal{F}_{\text{Hom},r}(u) \leq \liminf_{k \rightarrow \infty} \mathcal{F}_{\varepsilon_{k_\ell}}(u_{\varepsilon_{k_\ell}}).$$

- (ii) *For every $u \in W_0^{1,2}(\Omega; \mathbb{R}^N)$ there is a sequence $(u_{\varepsilon_{k_\ell}})_\ell$ weakly converging to u in $W_0^{1,2}(\Omega; \mathbb{R}^N)$ such that*

$$\mathcal{F}_{\text{Hom},r}(u) = \lim_{\ell \rightarrow \infty} \mathcal{F}_{\varepsilon_{k_\ell}}(u_{\varepsilon_{k_\ell}}).$$

Proof. The statements of the proposition will be proved separately in the following two steps.

Step 1. Let $(u_{\varepsilon_{k_\ell}})_\ell$ be a sequence that converges to u weakly in $W_0^{1,2}(\Omega; \mathbb{R}^N)$ and let $(\eta_n)_n$ be an arbitrary subsequence of $(\varepsilon_{k_\ell})_\ell$. In view of the compactness result Theorem 2.18 we find another subsequence $(\eta_{n_m})_m$ and a function $u_1 \in L^2(\Omega; W_{\text{per}}^{1,2}(Y; \mathbb{R}^N))$ with $\int_Y u_1(x, y) \, dy = 0$ for a.e. $x \in \Omega$, such that

$$Du_{\eta_{n_m}} \xrightarrow{2} Du + D_y u_1 \quad \text{in } L^2(\Omega \times Y; \mathbb{R}^{N \times N}).$$

Upon recalling the definitions of $\mathcal{F}_{\text{Hom},r}$ and $W_{\text{Hom},r}$, an application of Corollary 4.7 leads to

$$\mathcal{F}_{\text{Hom},r}(u) \leq \liminf_{m \rightarrow \infty} \mathcal{F}_{\eta_{n_m}}(u_{\eta_{n_m}}). \quad (4.6)$$

We see that every subsequence of $(\varepsilon_{k_\ell})_\ell$ has a further subsequence satisfying (4.6). Consequently, inequality (4.6) remains valid for the whole sequence $(\varepsilon_{k_\ell})_\ell$ and we infer the validity of the first assertion.

Step 2. Consider an arbitrary $u \in W_0^{1,2}(\Omega; \mathbb{R}^N)$. We start with the observation that the functional $\mathcal{F}_{\text{hom},r}$ can be characterized by means of $\overline{\mathcal{F}}_r$ in the following way (see Remark 4.5 for details).

$$\begin{aligned} \mathcal{F}_{\text{Hom},r}(u) &= \overline{\mathcal{F}}_r(u, u_1) \quad \text{for some } u_1 \in L^2(\Omega; W_{\text{per}}^{1,2}(Y; \mathbb{R}^N)) \\ &\quad \text{with } \int_Y u_1(x, y) \, dy = 0 \text{ for a.e. } x \in \Omega. \end{aligned} \quad (4.7)$$

By density we now find a sequence $(v_n)_n$ in $C_c^\infty(\Omega; C_{\text{per}}^\infty(Y; \mathbb{R}^N))$ such that

$$v_n \rightarrow u_1 \quad \text{in } L^2(\Omega \times Y; \mathbb{R}^N), \quad D_y v_n \rightarrow D_y u_1 \quad \text{in } L^2(\Omega \times Y; \mathbb{R}^{N \times N}) \quad (4.8)$$

subject to $\int_Y v_n(\cdot, y) dy = 0$. We define the doubly indexed sequence of functions $(u_{n,\ell})_{n,\ell}$ in $W_0^{1,2}(\Omega; \mathbb{R}^N)$ by

$$u_{n,\ell}(x) := u(x) + \varepsilon_{k_\ell} v_n(x, \frac{x}{\varepsilon_{k_\ell}}).$$

Invoking the technique of two-scale decomposition (cf. *e.g.* Visintin [2006]) we could easily find a diagonal sequence $u_{\varepsilon_{k_\ell}} := u_{n(\ell),\ell}$ such that $(u_{\varepsilon_{k_\ell}})_\ell \rightharpoonup u$ weakly in $W_0^{1,2}(\Omega; \mathbb{R}^N)$ and $(Du_{\varepsilon_{k_\ell}})_\ell \xrightarrow{2} Du + D_y u_1$ in $L^2(\Omega \times Y; \mathbb{R}^{N \times N})$. In view of Corollary 4.7 this sequence would clearly recover the limit energy and prove the assertion. However, for the reader's convenience we will present a self-contained construction of the recovery sequence.

To this end we start with the observation that for every $n \in \mathbb{N}$ we have

$$u_{n,\ell} \xrightarrow{\ell \rightarrow \infty} u \quad \text{in } W_0^{1,2}(\Omega; \mathbb{R}^N), \quad (4.9)$$

$$Du_{n,\ell} \xrightarrow{\ell \rightarrow \infty} Du + D_y v_n \quad \text{in } L^2(\Omega \times Y; \mathbb{R}^{N \times N}), \quad (4.10)$$

where the latter follows from the smoothness of v_n (see *e.g.* Example 2.1 or Visintin [2006]). Hence, we can apply Corollary 4.7 and infer

$$\lim_{\ell \rightarrow \infty} \mathcal{F}_{\varepsilon_{k_\ell}}(u_{n,\ell}) = \overline{\mathcal{F}}_r(u, v_n).$$

Now, the strong convergence (4.8) and the assumptions (W2), (W3) imply

$$\lim_{n \rightarrow \infty} \left(\lim_{\ell \rightarrow \infty} \mathcal{F}_{\varepsilon_{k_\ell}}(u_{n,\ell}) \right) = \overline{\mathcal{F}}_r(u, u_1) \stackrel{(4.7)}{=} \mathcal{F}_{\text{Hom},r}(u). \quad (4.11)$$

The previous reasoning suggests that we may obtain the recovery sequence $(u_{\varepsilon_{k_\ell}})_\ell$ by carefully choosing a diagonal sequence of $(u_{n,\ell})_{n,\ell}$. With this intention in mind, we define the quantity

$$c_{n,\ell} := \left| \mathcal{F}_{\varepsilon_{k_\ell}}(u_{n,\ell}) - \mathcal{F}_{\text{Hom},r}(u) \right| + \|u_{n,\ell} - u\|_{L^2(\Omega; \mathbb{R}^N)}.$$

By means of Rellich's compactness theorem, (4.9) and (4.11) imply

$$\lim_{n \rightarrow \infty} \left(\lim_{\ell \rightarrow \infty} c_{n,\ell} \right) = 0.$$

Referring to [Attouch, 1984, Corollary 1.18], we find a subsequence $(n(\ell))_\ell$ with $\lim_\ell c_{n(\ell),\ell} = 0$ and consequently the sequence $u_{\varepsilon_{k_\ell}} := u_{n(\ell),\ell}$ recovers the limit energy, *i.e.*

$$\lim_{\ell \rightarrow \infty} \mathcal{F}_{\varepsilon_{k_\ell}}(u_{\varepsilon_{k_\ell}}) = \mathcal{F}_{\text{Hom},r}(u),$$

and converges to u w.r.t. strong convergence in $L^2(\Omega; \mathbb{R}^N)$. It remains to prove

$$u_{\varepsilon_{k_\ell}} \xrightarrow{\ell \rightarrow \infty} u \quad \text{in } W_0^{1,2}(\Omega; \mathbb{R}^N). \quad (4.12)$$

First, we observe that the sequence $(u_{\varepsilon_{k_\ell}})_k$ must be norm bounded in $W_0^{1,2}(\Omega; \mathbb{R}^N)$ due to the growth condition (W3) and the boundedness of $(\mathcal{F}_{\varepsilon_{k_\ell}}(u_{\varepsilon_{k_\ell}}))_\ell$. Hence, we can extract from any subsequence of $(u_{\varepsilon_{k_\ell}})_\ell$ a further subsequence, which is weakly convergent in $W_0^{1,2}(\Omega; \mathbb{R}^N)$. But any weak limit of these subsequences must coincide with u , because of the strong convergence of $(u_{\varepsilon_{k_\ell}})_\ell$ to u in $L^2(\Omega; \mathbb{R}^N)$; thus, (4.12) is valid and $(u_{\varepsilon_{k_\ell}})_\ell$ is a recovery sequence. \square

Before proceeding further, we analyze the dependence of the homogenized microtranslated energy $\mathcal{F}_{\text{Hom},r}$ on the microtranslations r .

Lemma 4.9. *For all $F \in \mathbb{R}^{N \times N}$ and $u \in W_0^{1,2}(\Omega; \mathbb{R}^N)$ the maps*

$$r \mapsto W_{\text{Hom},r}(F) \quad \text{and} \quad r \mapsto \mathcal{F}_{\text{Hom},r}(u)$$

are continuous.

Proof. We remark that due to the growth condition (W3) it is sufficient to prove the continuity of $r \mapsto W_{\text{Hom},r}(F)$. Let $F \in \mathbb{R}^{N \times N}$ be arbitrary and define for all $r \in \bar{Y}$ the functional

$$\mathcal{E}_r : \left\{ \varphi \in W_{\text{per}}^{1,2}(Y; \mathbb{R}^N) : \int_Y \varphi(y) \, dy = 0 \right\} \rightarrow \mathbb{R}, \quad \varphi \mapsto \int_Y W_r(y, F + D_y \varphi(y)) \, dy.$$

By the assumptions on W and Poincaré's inequality, we see that $\varphi \mapsto \mathcal{E}_r(\varphi)$ is convex, coercive and continuous w.r.t. to strong convergence in $W_{\text{per}}^{1,2}(Y; \mathbb{R}^N)$. Therefore, for all $r \in \bar{Y}$ the functional \mathcal{E}_r admits a minimizer $\varphi_r \in W_{\text{per}}^{1,2}(Y; \mathbb{R}^N)$ with $\int_Y \varphi_r(y) \, dy = 0$, thus

$$\mathcal{E}_r(\varphi_r) = W_{\text{Hom},r}(F).$$

We recall the growth condition (W3) and infer that

$$2c \left(\|F + D_y \varphi_r\|_{L^2(Y; \mathbb{R}^{N \times N})}^2 - 1 \right) \leq \mathcal{E}_r(\varphi_r) \leq \int_Y 2W(y, F) \, dy \leq 2C(1 + |F|^2).$$

As a consequence, Poincaré's inequality implies the boundedness of the sequence $(\varphi_r)_r$ in $W_{\text{per}}^{1,2}(Y; \mathbb{R}^N)$. Now, consider an arbitrary sequence $(r_n)_n$ in \bar{Y} with $r_n \rightarrow r$. Due to the previous considerations, we can extract a subsequence $(r_{n_m})_m$ such that

$$\liminf_{n \rightarrow \infty} W_{\text{hom},r_n}(F) = \lim_{m \rightarrow \infty} W_{\text{hom},r_{n_m}}(F)$$

and in addition $\varphi_{r_{n_m}} \rightharpoonup \varphi_0$ weakly in $W_{\text{per}}^{1,2}(Y; \mathbb{R}^N)$ for a function $\varphi_0 \in W_{\text{per}}^{1,2}(Y; \mathbb{R}^N)$ with $\int_Y \varphi_0(y) \, dy = 0$. As it is easily seen, we also have $\varphi_{r_{n_m}}(\cdot + r_{n_m}) \rightharpoonup \varphi_0(\cdot + r)$ weakly in $W_{\text{per}}^{1,2}(Y; \mathbb{R}^N)$. Hence, we can exploit the weak lower semicontinuity of the functional $\varphi \mapsto \int_Y W(y, F + D_y \varphi(y)) \, dy$ and obtain

$$\mathcal{E}_r(\varphi_0) \leq \liminf_{m \rightarrow \infty} \mathcal{E}_{r_{n_m}}(\varphi_{r_{n_m}}) = \liminf_{n \rightarrow \infty} W_{\text{hom},r_n}(F). \quad (4.13)$$

On the other hand there holds

$$\limsup_{n \rightarrow \infty} W_{\text{hom}, r_n}(F) \leq \limsup_{n \rightarrow \infty} \mathcal{E}_{r_n}(\varphi_r) = \mathcal{E}_r(\varphi_r). \quad (4.14)$$

Herein, the last identity can be inferred from the observation, that the functional $\varphi \mapsto \int_Y W(y, F + D_y \varphi(y)) \, dy$ is continuous w.r.t. strong convergence and $\varphi_r(\cdot + r_n) \rightarrow \varphi_r(\cdot + r)$ strongly in $W_{\text{per}}^{1,2}(Y; \mathbb{R}^N)$ as $n \rightarrow \infty$. In view of (4.13) and (4.14), we have just shown that

$$\mathcal{E}_r(\varphi_0) \leq \liminf_{n \rightarrow \infty} W_{\text{hom}, r_n}(F) \leq \limsup_{n \rightarrow \infty} W_{\text{hom}, r_n}(F) \leq W_{\text{Hom}, r}(F). \quad (4.15)$$

Upon recalling $W_{\text{Hom}, r}(F) = \inf_{\varphi} \mathcal{E}_r(\varphi) \leq \mathcal{E}_r(\varphi_0)$, we realize that the inequalities in (4.15) are indeed equalities. This completes the proof. \square

So far we have shown Γ -convergence of $(\mathcal{F}_{\varepsilon_k})_k$ to $\mathcal{F}_{\text{Hom}, r}$ along (t, r) -subsequences and the continuous dependence of $\mathcal{F}_{\text{Hom}, r}$ on the microtranslation r (see Proposition 4.8). As we will prove now, this already allows us to reduce the proof of Theorem 4.5 to the situation considered in the upcoming Subsection 4.3.2. Therein, we show how lower and upper Γ -limits of a sequence can be characterized by means of its subsequences' Γ -limits (see Theorem 4.10).

Proof of Theorem 4.5. Define the set

$$\mathcal{L} := \left\{ \mathcal{F} : \mathcal{F} = \Gamma\text{-}\lim_{\ell \rightarrow \infty} \mathcal{F}_{\varepsilon_{k_\ell}} \text{ for a subsequence } (k_\ell)_\ell \right\}.$$

In view of Proposition 4.8 and the fact, that every subsequence $(k_\ell)_\ell$ contains a (t, r) -subsequence for a certain $r \in \mathcal{M}$, we immediately conclude that

$$\mathcal{L} = \{ \mathcal{F}_{\text{Hom}, r} : r \in \mathcal{M} \}.$$

Note that for any $u \in W_0^{1,2}(\Omega; \mathbb{R}^N)$ the map $r \mapsto \mathcal{F}_{\text{Hom}, r}(u)$ is continuous due to Lemma 4.9. Since \mathcal{M} is a compact subset of \mathbb{R}^N by Proposition 4.4, there exist r^+ and r^- in \mathcal{M} such that

$$\mathcal{F}_{\text{Hom}}^+(u) = \mathcal{F}_{\text{hom}, r^+}(u) \quad \text{and} \quad \mathcal{F}_{\text{Hom}}^-(u) = \mathcal{F}_{\text{hom}, r^-}(u).$$

This enables us to conclude the proof by applying the abstract Theorem 4.10 found in the next subsection, which might be regarded a result of its own interest. \square

Before turning to the missing abstract Γ -convergence result in Subsection 4.3.2 and stating an explicit example illustrating the general nonconvergence for convex homogenization problems involving multiple coordinate frames (in Section 4.4), we state two concluding remarks. While the first remark provides a technical detail which we used in the proof of Proposition 4.8, the second remark indicates a slight generalization of our homogenization result Theorem 4.5.

Remark 4.5. Let $r \in \overline{Y}$ and $F \in L^2(\Omega; \mathbb{R}^{N \times N})$. Then there exists a function $v \in L^2(\Omega; W_{\text{per}}^{1,2}(Y; \mathbb{R}^N))$ with $\int_Y v(x, y) dy = 0$ for a.e. $x \in \Omega$, such that

$$\int_{\Omega} W_{\text{Hom},r}(F(x)) dx = \int_{\Omega} \int_Y W_r(F(x) + D_y v(x, y)) dy dx. \quad (4.16)$$

Due to the convexity and continuity (W2) and the growth (W3) of W , for a.e. $x \in \Omega$ there exists $\varphi \in W_{\text{per}}^{1,2}(Y; \mathbb{R}^N)$ with $\int_Y \varphi(y) dy = 0$ and

$$W_{\text{Hom},r}(F(x)) = \int_Y W_r(y, F(x) + D_y \varphi(y)) dy,$$

thus the multifunction

$$\Lambda(x) := \left\{ \varphi \in W_{\text{per}}^{1,2}(Y; \mathbb{R}^N) : \int_Y W_r(y, F(x) + D_y \varphi(y)) dy = W_{\text{Hom},r}(F(x)) \quad \text{and} \quad \int_Y \varphi(y) dy = 0 \right\}$$

is well-defined. One can now show (for instance by applying Proposition 6.3 and Theorem 6.5 of Fonseca and Leoni [2007]) that Λ possesses a measurable selection $x \mapsto v(x) \in \Lambda(x)$. Moreover, the growth condition (W3) and Poincaré's inequality imply that $v \in L^2(\Omega; W_{\text{per}}^{1,2}(Y; \mathbb{R}^N))$. The definition of Λ then immediately implies the claimed identity (4.16). Note that (4.16) in particular establishes for every $u \in W_0^{1,2}(\Omega; \mathbb{R}^N)$ the existence of a $u_1 \in L^2(\Omega; W_{\text{per}}^{1,2}(Y; \mathbb{R}^N))$ with $\int_Y u_1(x, y) dy = 0$ for a.e. $x \in \Omega$, such that $\mathcal{F}_{\text{Hom},r}(u) = \overline{\mathcal{F}}_r(u, u_1)$.

Remark 4.6. The restriction in Theorem 4.5 to the space $W_0^{1,2}(\Omega; \mathbb{R}^N)$ implies that the sequence $(\mathcal{F}_{\varepsilon_k})_k$ is equicoercive w.r.t. to weak convergence in $W_0^{1,2}(\Omega; \mathbb{R}^N)$ and therefore the sequential characterization of Γ -convergence (which we exploited in the proof of Theorem 4.5) is valid for $(\mathcal{F}_{\varepsilon_k})_k$. Likely, results similar to Theorem 4.5 hold in more general situations. For instance let \mathcal{G} be a function from $W^{1,2}(\Omega; \mathbb{R}^N)$ into \mathbb{R} and consider the functionals

$$\mathcal{F}_{\varepsilon} : W^{1,2}(\Omega; \mathbb{R}^N) \rightarrow \mathbb{R}, \quad u \mapsto \int_{\Omega} W\left(\frac{x}{\varepsilon}, Du(x)\right) dx + \int_{\Omega-t} W\left(\frac{x}{\varepsilon}, Du(x+t)\right) dx,$$

where $t \in \mathbb{R}^N$ is an arbitrary translation and W fulfills (W1), ..., (W3). If \mathcal{G} satisfies one of the following conditions

- (G1) \mathcal{G} is finite, continuous w.r.t. to weak convergence in $W^{1,2}(\Omega; \mathbb{R}^N)$ and coercive w.r.t. to the strong convergence in $L^2(\Omega; \mathbb{R}^N)$,
- (G2) there exists $g \in W^{1,2}(\Omega; \mathbb{R}^N)$ such that $\mathcal{G}(u) = 0$ if $u - g \in W_0^{1,2}(\Omega; \mathbb{R}^N)$ and $+\infty$ otherwise,

then

$$\begin{aligned} \Gamma\text{-lim inf}_{k \rightarrow \infty} (\mathcal{F}_{\varepsilon_k} + \mathcal{G}) &= \mathcal{F}_{\text{Hom}}^- + \mathcal{G} \\ \Gamma\text{-lim sup}_{k \rightarrow \infty} (\mathcal{F}_{\varepsilon_k} + \mathcal{G}) &= \mathcal{F}_{\text{Hom}}^+ + \mathcal{G} \end{aligned}$$

with respect to weak convergence in $W^{1,2}(\Omega; \mathbb{R}^N)$. Herein, $\mathcal{F}_{\text{Hom}}^+$ and $\mathcal{F}_{\text{Hom}}^-$ are defined as before.

4.3.2 A general Γ -convergence result

In Proposition 4.8 we showed that the sequence $(\mathcal{F}_{\varepsilon_k})_k$ Γ -converges along (t, r) -subsequences, and the proof of Theorem 4.5 tells us that this is sufficient to explicitly characterize *all* possible clusterpoints of $(\mathcal{F}_{\varepsilon_k})_k$ in the topology of Γ -convergence. In particular, due to the strong dependence of the subsequences' Γ -limits on emerging microtranslations $r \in \mathcal{M}$ these clusterpoints do not coincide in general (see Section 4.4 for a counterexample), thus one cannot expect Γ -convergence along the entire sequence $(\mathcal{F}_{\varepsilon_k})_k$. At this point one might then wonder, whether one can at least infer the *lower and upper Γ -limits* of a sequence of functionals in case one explicitly knew all clusterpoints in the topology of Γ -convergence. In fact, as we now show that for certain situations this is the case.

Theorem 4.10. *Let (X, \mathcal{T}) be a topological vector space and $(f_k)_k$ a sequence of functions from X into $[-\infty, +\infty]$. We define the set*

$$\mathcal{L} := \left\{ f : f = \Gamma\text{-}\lim_{\ell \rightarrow \infty} f_{k_\ell} \text{ for a subsequence } (k_\ell)_\ell \right\}$$

and assume that

(i) *for all $x \in X$ there exists a sequence $(x_k)_k$ converging to x in X such that*

$$\left(\Gamma\text{-}\liminf_{k \rightarrow \infty} f_k \right) (x) = \liminf_{k \rightarrow \infty} f_k(x_k),$$

(ii) *any subsequence of $(f_k)_k$ has a Γ -convergent subsequence,*

(iii) *for all $x \in X$ there exist $f^-, f^+ \in \mathcal{L}$ such that*

$$f^-(x) \leq f(x) \leq f^+(x) \quad \text{for every } f \in \mathcal{L}.$$

Then there holds

$$\left(\Gamma\text{-}\liminf_{k \rightarrow \infty} f_k \right) (x) = \min_{f \in \mathcal{L}} f(x) \quad \text{and} \quad \left(\Gamma\text{-}\limsup_{k \rightarrow \infty} f_k \right) (x) = \max_{f \in \mathcal{L}} f(x).$$

for every $x \in X$.

Proof. Let $x \in X$ and $f^+, f^- \in \mathcal{L}$ according to assumption (iii). We remind the reader of the definition of the lower and upper Γ -limit given in Section 2.2.3, which reads as

$$\begin{aligned} \left(\Gamma\text{-}\liminf_{k \rightarrow \infty} f_k \right) (x) &:= \sup_{U \in \mathcal{U}(x)} \liminf_{k \rightarrow \infty} \inf_{y \in U} f_k(y), \\ \left(\Gamma\text{-}\limsup_{k \rightarrow \infty} f_k \right) (x) &:= \sup_{U \in \mathcal{U}(x)} \limsup_{k \rightarrow \infty} \inf_{y \in U} f_k(y), \end{aligned}$$

where $\mathcal{U}(x)$ denotes the set of all open neighborhoods of x . One immediately realizes that

$$\liminf_{k \rightarrow \infty} \inf_{y \in \tilde{U}} f_k(y) \leq \liminf_{\ell \rightarrow \infty} \inf_{y \in \tilde{U}} f_{k_\ell}(y) \quad \text{and} \quad \limsup_{k \rightarrow \infty} \inf_{y \in \tilde{U}} f_k(y) \geq \limsup_{\ell \rightarrow \infty} \inf_{y \in \tilde{U}} f_{k_\ell}(y)$$

for any subsequence $(k_\ell)_\ell$ and any $\tilde{U} \in \mathcal{U}(x)$. From this we easily infer the estimates

$$\left(\Gamma\text{-lim inf}_{k \rightarrow \infty} f_k \right) (x) \leq f^-(x) \quad \text{and} \quad \left(\Gamma\text{-lim sup}_{k \rightarrow \infty} f_k \right) (x) \geq f^+(x)$$

by first considering particular subsequences satisfying $\Gamma\text{-lim}_\ell f_{k_\ell} = f^-$ (respectively $\Gamma\text{-lim}_\ell f_{k_\ell} = f^+$) and, secondly, taking the supremum over all $\tilde{U} \in \mathcal{U}(x)$ on both sides.

In order to show the characterization of the lower Γ -limit of $(f_k)_k$ stated above, it remains to show $(\Gamma\text{-lim inf}_k f_k)(x) \geq f^-(x)$. By assumption (i) and (ii) we can find a sequence $(x_k)_k$ converging to x in X , a subsequence $(k_\ell)_\ell$ and $f \in \mathcal{L}$ such that

$$\lim_{\ell \rightarrow \infty} f_{k_\ell}(x_{k_\ell}) = \left(\Gamma\text{-lim inf}_{k \rightarrow \infty} f_k \right) (x) \quad \text{and} \quad \Gamma\text{-lim}_{\ell \rightarrow \infty} f_{k_\ell} = f.$$

Since Γ -limits naturally satisfy the 'lim inf-inequality' (also in the case where the sequential characterization is not valid), we infer $\liminf_{\ell \rightarrow \infty} f_{k_\ell}(x_{k_\ell}) \geq f(x)$ and by applying condition (iii) we finally obtain

$$\left(\Gamma\text{-lim inf}_{k \rightarrow \infty} f_k \right) (x) = \liminf_{\ell \rightarrow \infty} f_{k_\ell}(x_{k_\ell}) \geq f(x) \geq f^-(x).$$

For the characterization of the upper Γ -limit of $(f_k)_k$, we need to prove that

$$(\Gamma\text{-lim sup}_{k \rightarrow \infty} f_k)(x) \leq f^+(x).$$

Let $\tilde{U} \in \mathcal{U}(x)$. Again due to assumption (ii) we can switch to a subsequence $(k_\ell)_\ell$ with

$$\limsup_{k \rightarrow \infty} \inf_{y \in \tilde{U}} f_k(y) = \lim_{\ell \rightarrow \infty} \inf_{y \in \tilde{U}} f_{k_\ell}(y) \quad \text{and} \quad \Gamma\text{-lim}_{\ell \rightarrow \infty} f_{k_\ell} = f,$$

where $f \in \mathcal{L}$. In passing we realize

$$\limsup_{k \rightarrow \infty} \inf_{y \in \tilde{U}} f_k(y) \leq \sup_{U \in \mathcal{U}(x)} \limsup_{\ell \rightarrow \infty} \inf_{y \in U} f_{k_\ell}(y) = f(x) \leq f^+(x),$$

where the last inequality follows by assumption (iii). Since this estimate is true for all $\tilde{U} \in \mathcal{U}(x)$, it remains valid if we pass to the supremum over all $\tilde{U} \in \mathcal{U}(x)$. This completes the proof. \square

We remark that in the situation of Theorem 2.5 the sequential characterization of the lower and upper Γ -limit is valid and assumption (i) is fulfilled (see [Dal Maso, 1993, Proposition 8.16] for details). Thus, Proposition 4.10 can indeed be applied in the proof of Theorem 4.5.

4.4 An example of nonconvergence

In this final section we state an explicit example showing that in general the lower and upper Γ -limits of the energy functionals $(\mathcal{F}_{\varepsilon_k})_k$ are different in the presence of nontrivial translations t , as it may be anticipated from our homogenization result Theorem 4.5. Here, we highlight once more the importance of the particular sequence of microscale parameters $(\varepsilon_k)_k$ – different choices of $(\varepsilon_k)_k$ result in different limit behavior of the sequence $(\mathcal{F}_{\varepsilon_k})_k$.

In the example below we consider the special situation of quadratic energy functionals. To this end, let \mathbb{L} be a Y -periodic function from \mathbb{R}^N into the space of fourth order tensors over \mathbb{R} and suppose that

$$\begin{aligned} \langle \mathbb{L}(y)F, G \rangle &= \langle \mathbb{L}(y)G, F \rangle, \\ c|F|^2 &\leq \langle \mathbb{L}(y)F, F \rangle \leq C|F|^2 \end{aligned}$$

for all $F, G \in \mathbb{R}^{N \times N}$, a.e. $y \in Y$ and some positive constants c, C . The energy density shall now be given as

$$W(y, F) := \langle \mathbb{L}(y)F, F \rangle,$$

which obviously complies with the properties (W1), ..., (W3). Observe that

$$W_{\text{Hom},r}(F) = \int_Y W(y, F + D_y \varphi_{r,F}(y)) \, dy,$$

where $\varphi_{r,F} \in W_{\text{per}}^{1,2}(Y; \mathbb{R}^N)$ is the unique solution of the linear problem

$$\int_Y \langle \mathbb{L}(y)(F + D_y \varphi(y)), D_y \psi(y) \rangle \, dy = 0 \quad \text{for all } \psi \in W_{\text{per}}^{1,2}(Y; \mathbb{R}^N)$$

subject to $\int_Y \varphi(y) \, dy = 0$. In the particular situation where $N = 1$, $\mathcal{M} = \{0, \frac{1}{2}\}$ and

$$\mathbb{L}(y) = \alpha(y) = \begin{cases} \alpha_1 & \text{if } y \in [k, k + \frac{1}{2}) \text{ for } k \in \mathbb{Z} \\ \alpha_2 & \text{else} \end{cases}$$

we can explicitly compute $\mathcal{F}_{\text{Hom}}^+$ and $\mathcal{F}_{\text{Hom}}^-$. The above set of microtranslations $\mathcal{M} = \{0, \frac{1}{2}\}$ occurs e.g. in the case of $\varepsilon_k = \frac{2}{k}$ and $t = 1$. A calculations shows

$$W_{\text{Hom},r}(F) = \alpha_{\text{Hom},r} |F|^2, \quad \text{where} \quad \alpha_{\text{Hom},r} := \left(\int_Y \frac{1}{\alpha(y) + \alpha(y-r)} \, dy \right)^{-1}.$$

In the case of $r \in \{0, \frac{1}{2}\}$ we infer

$$\alpha_{\text{hom},0} = \frac{4\alpha_1\alpha_2}{\alpha_1 + \alpha_2}, \quad \alpha_{\text{hom},\frac{1}{2}} = \alpha_1 + \alpha_2 \quad \text{and} \quad \alpha_{\text{hom},\frac{1}{2}} - \alpha_{\text{hom},0} = \frac{(\alpha_1 - \alpha_2)^2}{\alpha_1 + \alpha_2} \geq 0.$$

With the help of the homogenization result Theorem 4.5 one obtains the explicit representations

$$\begin{aligned}\left(\Gamma\text{-}\liminf_{k \rightarrow \infty} \mathcal{F}_{\varepsilon_k}\right)(u) &= \frac{4\alpha_1\alpha_2}{\alpha_1 + \alpha_2} \int_{\Omega} |u'(x)|^2 dx, \\ \left(\Gamma\text{-}\limsup_{k \rightarrow \infty} \mathcal{F}_{\varepsilon_k}\right)(u) &= (\alpha_1 + \alpha_2) \int_{\Omega} |u'(x)|^2 dx,\end{aligned}$$

for all $u \in W_0^{1,2}(\Omega)$. Finally, we conclude that $(\mathcal{F}_{\varepsilon_k})_k$ is Γ -convergent, if and only if $\alpha_1 = \alpha_2$.

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Bibliography

- Acerbi, E., Buttazzo, G., and Percivale, D. (1991). A variational definition of the strain energy for an elastic string. *J. Elasticity*, 25(2):137–148.
- Aebischer, B. (2007). Heat conduction in lenses. *Mathematical Problems in Engineering*.
- Aebischer, B. and Braunecker, B. (2004). Thermal management of large scale optical systems. In *105. Tagung der DGaO 2004*. Available electronically at http://www.leica-geosystems.com/media/new/product_solution/DGaOvortrag_22June2004_en.pdf.
- Allaire, G. (1992). Homogenization and two-scale convergence. *SIAM J. Math. Anal.*, 23(6):1482–1518.
- Allaire, G. and Conca, C. (1995). Analyse asymptotique spectrale de l'équation des ondes. Homogénéisation par ondes de Bloch. *C. R. Acad. Sci. Paris Sér. I Math.*, 321(3):293–298.
- Allaire, G. and Conca, C. (1996). Bloch-wave homogenization for a spectral problem in fluid-solid structures. *Arch. Rational Mech. Anal.*, 135(3):197–257.
- Ansini, N. (2004). The nonlinear sieve problem and applications to thin films. *Asymptot. Anal.*, 39(2):113–145.
- Ansini, N., Babadjian, J.-F. c., and Zeppieri, C. I. (2007). The Neumann sieve problem and dimensional reduction: A multiscale approach. *Math. Models Methods Appl. Sci.*, 17(5):681–735.
- Antman, S. S. (2005). *Nonlinear problems of elasticity*, volume 107 of *Applied Mathematical Sciences*. Springer-Verlag, New York.
- Åslund, J. (2005). Asymptotic analysis of adhesively bonded nonlinearly elastic plates. *ZAMM, Z. Angew. Math. Mech.*, 85(1):23–35.
- Attouch, H. (1984). *Variational convergence for functions and operators*. Applicable Mathematics Series. Boston - London - Melbourne: Pitman Advanced Publishing Program.

- Bakhvalov, N. and Panasenko, G. (1989). *Homogenisation: averaging processes in periodic media*, volume 36 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht.
- Benamou, J.-D. and Brenier, Y. (2000). A computational fluid mechanics solution to the monge-kantorovich mass transfer problem. *Numerische Mathematik*, 84:375–393. 10.1007/s002110050002.
- Bensoussan, A., Lions, J.-L., and Papanicolaou, G. (1978). *Asymptotic analysis for periodic structures*, volume 5 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam.
- Bessoud, A. L. (2009). *Modélisations mathématiques d'un multi-matériau*. PhD thesis, Université Montpellier II.
- Bessoud, A. L., Krasucki, F., and Michaille, G. (2009). Multi-materials with strong interface: variational modelings. *Asymptot. Anal.*, 61(1):1–19.
- Bessoud, A. L., Krasucki, F., and Michaille, G. (2010). A relaxation process for bifunctionals of displacement-Young measure state variables: a model of multi-material with micro-structured strong interface. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire*, 27(2):447–469.
- Bessoud, A.-L., Krasucki, F. c., and Serpilli, M. (2011). Asymptotic analysis of shell-like inclusions with high rigidity. *J. Elasticity*, 103(2):153–172.
- Braides, A. (1985). Homogenization of some almost periodic coercive functional. *Rend. Accad. Naz. Sci. XL Mem. Mat. (5)*, 9(1):313–321.
- Braides, A. (2000). Non-local variational limits of discrete systems. *Commun. Contemp. Math.*, 2(2):285–297.
- Braides, A. and Defranceschi, A. (1998). *Homogenization of multiple integrals*, volume 12 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press Oxford University Press, New York.
- Braides, A., Fonseca, I., and Francfort, G. (2000). 3D-2D asymptotic analysis for inhomogeneous thin films. *Indiana Univ. Math. J.*, 49(4):1367–1404.
- Braides, A. and Gelli, M. (2005). The passage from discrete to continuous variational problems: a nonlinear homogenization process. In Castañeda, P., Telega, J., and Gambin, B., editors, *Nonlinear Homogenization and its Applications to Composites, Polycrystals and Smart Materials*, volume 170 of *NATO Science Series*, pages 45–63. Springer Netherlands.
- Braides, A. and Gelli, M. S. (2006). From discrete systems to continuous variational problems: an introduction. In *Topics on concentration phenomena and problems with multiple scales*, volume 2 of *Lect. Notes Unione Mat. Ital.*, pages 3–77. Springer, Berlin.

- Braides, A., Riey, G., and Solci, M. (2009). Homogenization of penrose tilings. *Comptes Rendus Mathématique*, 347(11-12):697 – 700.
- Brenner, S. C. and Carstensen, C. (2004). Finite element methods. In Stein, E., de Borst, R., and Hughes, T. J. R., editors, *Encyclopedia of computational mechanics. Vol. 1. Fundamentals*. John Wiley & Sons.
- Castro, C. and Zuazua, E. (1996). Une remarque sur l’analyse asymptotique spectrale en homogénéisation. *C. R. Acad. Sci. Paris Sér. I Math.*, 322(11):1043–1047.
- Chacha, D. and Sanchez-Palencia, E. (1992). Overall behavior of elastic plates with periodically distributed fissures. *Asymptotic Anal.*, 5(5):381–396.
- Ciarlet, P. G. (1988). *Mathematical elasticity. Vol. I. Three-dimensional elasticity*, volume 20 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam.
- Ciarlet, P. G. (1997). *Mathematical elasticity. Vol. II*, volume 27 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam. Theory of plates.
- Ciarlet, P. G. and Destuynder, P. (1979). A justification of the two-dimensional linear plate model. *J. Mécanique*, 18(2):315–344.
- Cioranescu, D., Damlamian, A., and De Arcangelis, R. (2006a). Homogenization of quasiconvex integrals via the periodic unfolding method. *SIAM J. Math. Anal.*, 37(5):1435–1453 (electronic).
- Cioranescu, D., Damlamian, A., Donato, P., Griso, G., and Zaki, R. (2011). The periodic unfolding method in domains with holes. Preprint available electronically at hal.archives-ouvertes.fr/docs/00/59/16/32/PDF/TrousDV2010-12-11.pdf.
- Cioranescu, D., Damlamian, A., and Griso, G. (2002). Periodic unfolding and homogenization. *C. R. Math. Acad. Sci. Paris*, 335(1):99–104.
- Cioranescu, D., Damlamian, A., and Griso, G. (2008a). The periodic unfolding method in homogenization. *SIAM J. Math. Anal.*, 40(4):1585–1620.
- Cioranescu, D., Damlamian, A., Griso, G., and Onofrei, D. (2008b). The periodic unfolding method for perforated domains and Neumann sieve models. *J. Math. Pures Appl. (9)*, 89(3):248–277.
- Cioranescu, D. and Donato, P. (1999). *An introduction to homogenization*, volume 17 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press Oxford University Press, New York.
- Cioranescu, D., Donato, P., and Zaki, R. (2006b). The periodic unfolding method in perforated domains. *Port. Math. (N.S.)*, 63(4):467–496.

- Dal Maso, G. (1993). *An introduction to Γ -convergence*. Progress in Nonlinear Differential Equations and their Applications, 8. Birkhäuser Boston Inc., Boston, MA.
- Damlamian, A. (1985). Le problème de la passoire de Neumann. (The Neumann sieve problem). *Rend. Semin. Mat., Torino*, 43:427–450.
- Davoli, E. (2011). Thin-walled beams with a cross-section of arbitrary geometry: derivation of linear theories starting from 3D nonlinear elasticity. Preprint available electronically at <http://cvgmt.sns.it/cgi/get.cgi/papers/dav/Davoli.pdf>.
- De Giorgi, E. and Franzoni, T. (1975). Su un tipo di convergenza variazionale. *Atti Accad. Naz. Lincei, VIII. Ser., Rend., Cl. Sci. Fis. Mat. Nat.*, 58:842–850.
- Del Vecchio, T. (1987). The thick Neumann’s sieve. *Ann. Mat. Pura Appl. (4)*, 147:363–402.
- Donato, P. and Piatnitski, A. (2010). On the effective interfacial resistance through rough surfaces. *Commun. Pure Appl. Anal.*, 9(5):1295–1310.
- Fonseca, I. and Leoni, G. (2007). *Modern methods in the calculus of variations: L^p spaces*. Springer Monographs in Mathematics. Springer, New York.
- Friesecke, G. and James, R. D. (2000). A scheme for the passage from atomic to continuum theory for thin films, nanotubes and nanorods. *J. Mech. Phys. Solids*, 48(6-7):1519–1540. The J. R. Willis 60th anniversary volume.
- Friesecke, G., James, R. D., Mora, M. G., and Müller, S. (2003). Derivation of nonlinear bending theory for shells from three-dimensional nonlinear elasticity by Gamma-convergence. *C. R. Math. Acad. Sci. Paris*, 336(8):697–702.
- Friesecke, G., James, R. D., and Müller, S. (2002). A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity. *Comm. Pure Appl. Math.*, 55(11):1461–1506.
- Friesecke, G., James, R. D., and Müller, S. (2006). A hierarchy of plate models derived from nonlinear elasticity by Gamma-convergence. *Arch. Ration. Mech. Anal.*, 180(2):183–236.
- Geymonat, G., Hendili, S., Krasucki, F., and Vidrascu, M. (2010). The matched asymptotic expansion for the computation of the effective behavior of an elastic structure with a thin layer of holes. *Int. J. Mult. Comp. Eng.* To appear.
- Geymonat, G., Krasucki, F., and Lenci, S. (1999). Mathematical analysis of a bonded joint with a soft thin adhesive. *Mathematics and Mechanics of Solids*, 4(2):201–225.
- Geymonat, G., Krasucki, F., and Stelzig, P. E. (2012). In preparation.

- Goland, M. and Reissner, E. (1944). The stresses in cemented joints. *Journal of Applied Mechanics*, 11.
- Hornung, U., editor (1997). *Homogenization and porous media*, volume 6 of *Interdisciplinary Applied Mathematics*. Springer-Verlag, New York.
- Jikov, V. V., Kozlov, S. M., and Oleĭnik, O. A. (1994). *Homogenization of differential operators and integral functionals*. Springer-Verlag, Berlin. Translated from the Russian by G. A. Yosifian [G. A. Iosif'yan].
- Jost, J. (2007). *Partial differential equations. 2nd ed.* New York, NY: Springer.
- Knabner, P. and Angermann, L. (2003). *Numerical methods for elliptic and parabolic partial differential equations*. New York, NY: Springer.
- Krasucki, F., Münch, A., and Ousset, Y. (2004). Mathematical analysis of nonlinear bonded joint models. *Math. Models Methods Appl. Sci.*, 14(4):535–556.
- Le Dret, H. and Raoult, A. (1995). The nonlinear membrane model as variational limit of nonlinear three-dimensional elasticity. *J. Math. Pures Appl. (9)*, 74(6):549–578.
- Lebon, F. and Rizzoni, R. (2010). Asymptotic analysis of a thin interface: the case involving similar rigidity. *Int. J. Eng. Sci.*, 48(5):473–486.
- Lemaitre, J. and Chaboche, J. L. (1990). *Mechanics of solid materials. Transl. from the French by B. Shrivastava. Foreword to the French edition by Paul Germain, foreword to the English edition by Fred Leckie*. Cambridge etc.: Cambridge University Press.
- Lewicka, M., Mora, M. G., and Pakzad, M. R. (2010). Shell theories arising as low energy Γ -limit of 3d nonlinear elasticity. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 9(2):253–295.
- Li, B., Habbal, F., and Ortiz, M. (2010). Optimal transportation meshfree approximation schemes for fluid and plastic flows. *International Journal for Numerical Methods in Engineering*, 83(12):1541–1579.
- Lukkassen, D., Nguetseng, G., and Wall, P. (2002). Two-scale convergence. *Int. J. Pure Appl. Math.*, 2(1):35–86.
- Marcellini, P. (1978). Periodic solutions and homogenization of non linear variational problems. *Ann. Mat. Pura Appl., IV. Ser.*, 117:139–152.
- Marigo, J. and Pideri, C. (2011). The effective behavior of elastic bodies containing microcracks or microholes localized on a surface. *International Journal of Damage Mechanics*.
- Meunier, N. and Van Schaftingen, J. (2005). Periodic reiterated homogenization for elliptic functions. *J. Math. Pures Appl. (9)*, 84(12):1716–1743.

- Mielke, A. and Timofte, A. (2008). Two-scale homogenization for evolutionary variational inequalities via the energetic formulation. *SIAM Journal on Mathematical Analysis*, 39(2):642–668.
- Mora, M. G. and Müller, S. (2003). Derivation of the nonlinear bending-torsion theory for inextensible rods by Γ -convergence. *Calc. Var. Partial Differential Equations*, 18(3):287–305.
- Mora, M. G. and Müller, S. (2004). A nonlinear model for inextensible rods as a low energy Γ -limit of three-dimensional nonlinear elasticity. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 21(3):271–293.
- Moussa, A. A. and Zlăjji, L. (2012). Homogenization of nonlinear variational problems with thin inclusions. *Math. J. Okayama Univ.*, 54:97–131.
- Müller, S. (1987). Homogenization of nonconvex integral functionals and cellular elastic materials. *Arch. Rational Mech. Anal.*, 99(3):189–212.
- Murat, F. (1985). The neumann sieve. In *Nonlinear variational problems. Elba/Italy 1983*, volume 127 of *Research Notes in Mathematics*, pages 24–32. Pitman, Boston.
- Neukamm, S. (2010). *Homogenization, linearization and dimension reduction in elasticity with variational methods*. PhD thesis, Technische Universität München. Thesis available electronically at <http://nbn-resolving.de/urn/resolver.pl?urn:nbn:de:bvb:91-diss-20100927-976438-1-8>.
- Neukamm, S. and Stelzig, P. E. (2011). On the interplay of two-scale convergence and translation. *Asymptot. Anal.*, 71(3):163–183.
- Nguetseng, G. (1989). A general convergence result for a functional related to the theory of homogenization. *SIAM J. Math. Anal.*, 20(3):608–623.
- Nguetseng, G. (2003). Homogenization structures and applications. I. *Z. Anal. Anwend.*, 22(1):73–107.
- Nguetseng, G. (2004a). Homogenization in perforated domains beyond the periodic setting. *J. Math. Anal. Appl.*, 289(2):608–628.
- Nguetseng, G. (2004b). Homogenization structures and applications. II. *Z. Anal. Anwend.*, 23(3):483–508.
- Oleĭnik, O. A., Shamaev, A. S., and Yosifian, G. A. (1992). *Mathematical problems in elasticity and homogenization*, volume 26 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam.
- Onofrei, D. (2006). The unfolding operator near a hyperplane and its application to the Neumann sieve model. *Adv. Math. Sci. Appl.*, 16(1):239–258.

- Onofrei, D. (2007). *New results in the multiscale analysis on perforated domains and applications*. PhD thesis, Worcester Polytechnic Institute.
- Parnell, W. J. (2007). Effective wave propagation in a prestressed nonlinear elastic composite bar. *IMA J. Appl. Math.*, 72(2):223–244.
- Persson, P.-O. and Strang, G. (2004). A simple mesh generator in MATLAB. *SIAM Rev.*, 46(2):329–345.
- Picard, C. (1987). Analyse limite d'équations variationnelles dans un domaine contenant une grille. (Limit analysis of variational equations in a domain containing a sieve). *RAIRO, Modélisation Math. Anal. Numér.*, 21:293–326.
- Reuss, A. (1929). Berechnung der Fließgrenze von Mischkristallen auf Grund der Plastizitätsbedingung für Einkristalle. *ZAMM – Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik*, 9(1):49–58.
- Rohan, E. and Lukeš, V. (2010). Homogenization of the acoustic transmission through a perforated layer. *Journal of Computational and Applied Mathematics*, 234(6):1876 – 1885.
- Sánchez-Palencia, E. (1980). *Nonhomogeneous media and vibration theory*, volume 127 of *Lecture Notes in Physics*. Springer-Verlag, Berlin.
- Sanchez-Palencia, E. (1982). Boundary value problems in domains containing perforated walls. In *Nonlinear partial differential equations and their applications. Collège de France Seminar, III.*, volume 70 of *Research Notes in Mathematics*, pages 309–325. Pitman, Boston.
- Schmidt, B. (2006). A derivation of continuum nonlinear plate theory from atomistic models. *Multiscale Model. Simul.*, 5(2):664–694 (electronic).
- Schmidt, B. (2008). On the passage from atomic to continuum theory for thin films. *Arch. Ration. Mech. Anal.*, 190(1):1–55.
- Schmidt, B. (2009). On the derivation of linear elasticity from atomistic models. *Netw. Heterog. Media*, 4(4):789–812.
- Spagnolo, S. (1969). Sulla convergenza di soluzioni di equazioni paraboliche ed ellittiche. *Ann. Sc. Norm. Super. Pisa, Sci. Fis. Mat., III. Ser. 22*, (1968):571–597.
- Tartar, L. (1984). Étude des oscillations dans les équations aux dérivées partielles non linéaires. In *Trends and applications of pure mathematics to mechanics. Paris/France 1983*, volume 195 of *Lecture Notes in Physics*, pages 384–412. Springer, Berlin.
- Tartar, L. (2009). *The general theory of homogenization. A personalized introduction*, volume 7 of *Lecture Notes of the Unione Matematica Italiana*. Springer-Verlag, Berlin.

- Visintin, A. (2006). Towards a two-scale calculus. *ESAIM, Control Optim. Calc. Var.*, 12:371–397.
- Visintin, A. (2007). Two-scale convergence of some integral functionals. *Calc. Var. Partial Differ. Equ.*, 29(2):239–265.
- Yeung, Y. A., Friesecke, G., and Schmidt, B. (2009). Minimizing atomic configurations of short range pair potentials in two dimensions: crystallization in the Wulff shape. *ArXiv e-prints*.