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Chow Quotients of $\mathbb{C}^*$-actions

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Abstract

Given an action of the one-dimensional torus on a projective variety, the associated Chow quotient arises as a natural parameter space of invariant 1-cycles, and dominates the GIT quotients of the variety. In this paper we investigate the relationship between the Chow and the GIT quotients, showing how to construct explicitly the former upon the latter via successive blowups under suitable assumptions. We also discuss conditions for the smoothness of the Chow quotient, and present some examples in which it is singular.

Key words: $\mathbb{C}^*$-actions, Chow Quotient, Birational Geometry, GIT.

1. Introduction

The relation of birational geometry with the theory of quotients of $\mathbb{C}^*$-actions has been acknowledged since the early days of Mori Theory and the Minimal Model Program; for early expositions on this matter see the contributions by Thaddeus and Reid (42), (40). Włodarczyk in (44) used the notion of birational cobordism, constructed by Morelli in the toric case (32), to prove the Weak Factorization Conjecture, which asserts that given a birational map of smooth projective varieties $Y \dashleftarrow\dashrightarrow Y'$, there exists a sequence of blowups and blowdowns along smooth centers (or, possibly, isomorphisms) resolving this map (see also (1) and (45) for a broader view on this topic). In Włodarczyk's proof a variety with a $\mathbb{C}^*$-action is constructed, the cobordism X, whose two distinguished quotients are the varieties Y and Y'; the crucial step in the argument is about elementary cobordisms associated to flips.

Recently, the relation between birational geometry and $\mathbb{C}^*$-actions has been reconsidered under the viewpoint of Mumford's Geometric Invariant Theory (GIT, see (33)), Variation of GIT (VGIT, see (16)), and the Białynicki-Birula decomposition (see (6; 9)). This viewpoints allow to go deeper into that relation: projective varieties endowed with $\mathbb{C}^*$-actions can be seen as geometric realizations of birational transformations, and linearizations of these actions provide natural factorizations of the given transformations (cf. (34; 35; 36; 37; 4)).

Given a $\mathbb{C}^*$-action on a projective variety X and a linearization of an ample line bundle L on X, the geometric quotients of the action can be described in terms of the weights $a_0, \dots, a_r$, which we associate to fixed point components of the action; we will accordingly denote them by $\mathcal{G}X_{i,i+1}$, for $i = 0, \dots, r-1$ (see Section 2.4). The positive integer r is called the criticality of the action. In particular, the two extremal quotients, $\mathcal{G}X_{0,1}$ and $\mathcal{G}X_{r-1,r}$ parametrize 1-dimensional orbits converging to points in the sink and the source of the action, respectively.

The action defines birational maps among the geometric quotients, in particular, between the two extremal ones; the birational map between them has a natural factorization via the intermediate quotients:

$$\begin{tikzcd} \mathcal{G}X_{0,1} \arrow[r, dashed, "\text{bir.}"] & \mathcal{G}X_{i,i+1} \arrow[r, dashed, "\text{bir}"] & \mathcal{G}X_{r-1,r} \end{tikzcd}$$

We prove that, under suitable assumptions, this factorization is related to the birational geometry of the variety X (see Section 3): all the geometric quotients $\mathcal{G}X_{i,i+1}$ can be seen as the extremal fixed point components of $\mathbb{C}^*$-equivariant birational modifications $X_{i,j}$, $0 \leq i < j \leq r$, of X, called *prunings*. We then have, for every $i < j$, a commutative diagram of rational maps:

$$\begin{tikzcd} \mathcal{G}X_{0,1} \arrow[r, dashed, "\text{bir.}"] & \mathcal{G}X_{i,i+1} \arrow[rr, dashed, "\text{bir}"] & & \mathcal{G}X_{j-1,j} \arrow[r, dashed, "\text{bir}"] & \mathcal{G}X_{r-1,r} \\ & & X_{i,j} \arrow[ul, dashed, "\text{GIT quot.}"'] \arrow[ur, dashed, "\text{GIT quot.}"] & & \\ & & X_{0,r} \arrow[uull, dashed, "\text{GIT quot.}"] \arrow[uurr, dashed, "\text{GIT quot.}"'] \arrow[u, dashed, "\text{bir.}"'] & & \end{tikzcd} \tag{1}$$

The technical conditions under which we prove this result are two. The first is the non existence of finite isotropy subgroups for the action –that we call *equalization*– which allows us to prove that the birational modifications $X_{i,j}$ are smooth. For instance, under this hypothesis, the variety $X_{0,r}$ is the blowup of X along its sink and source. Furthermore we assume the non existence of inner $\mathbb{C}^*$-invariant divisors in X, which implies that the birational modifications among the $\mathcal{G}X_{i,j}$'s and among the $X_{i,j}$'s are compositions of flips.

Given a projective variety X with a nontrivial $\mathbb{C}^*$-action, the *normalized Chow quotient* $\mathcal{C}X$ of the action is the normalization of the closure of the subscheme of $\mathrm{Chow}(X)$ parametrizing the closures of the general orbits of the action (see Section 4 for details). The normalized Chow quotient can be defined in the broader setting of actions by reductive groups; in the case of torus actions it is known to have a universal property, coming from the fact that it can be seen as the normalized limit quotient of the direct system of GIT quotients of the variety in question by the action of the torus (cf. (3, Corollary 2.6)).

We explore here this idea, from the viewpoint of birational geometry. All the geometric quotients of the action are birationally dominated by $\mathcal{C}X$; in particular, we get a resolution of the birational map $\psi : \mathcal{G}X_{0,1} \dashrightarrow \mathcal{G}X_{r-1,r}$:

$$\begin{tikzcd} & \mathcal{C}X \arrow[dl] \arrow[dr] & \\ \mathcal{G}X_{0,1} \arrow[rr, dashed, "\psi"] & & \mathcal{G}X_{r-1,r} \end{tikzcd}$$

It then makes sense to study the properties of this resolution and ask ourselves, for instance, whether it gives a Strong Factorization of ψ, that is whether the morphisms $\mathcal{C}X \to \mathcal{G}X_{0,1}$, $\mathcal{C}X \to \mathcal{G}X_{r-1,r}$ are compositions of blowups along smooth centers. What we prove is a structure theorem that shows how to obtain $\mathcal{C}X$ recursively out of the Chow quotients of the birational modifications $X_{i,j}$ described above, via subsequent blowups. More precisely, our main statement is the following (see §2.4 for the definition of $B^{\pm}(Y)$):

Theorem 1.1 *Let (X, L) be a polarized pair, with X smooth, endowed with an equalized nontrivial $\mathbb{C}^*$-action such that* $\mathrm{codim}(B^{\pm}(Y), X) > 1$ *for every inner fixed point component Y. Denote by r be the criticality of the action, by $X_{i,j}$, $0 \leq i < j \leq r$, the corresponding prunings of X, and by $\mathcal{C}X_{i,j}$ their normalized Chow* *quotients. Then $\mathcal{C}X = \mathcal{C}X_{0,r}$, and we have a commutative diagram:*

$$
\begin{tikzcd}[column sep=small]
 & & & & & \mathcal{C}X_{0,r} \arrow[dl, "s"'] \arrow[dr, "d"] & & & & & \\
 & & & & \mathcal{C}X_{0,r-1} \arrow[dl, "s"'] \arrow[dr, "d"] & & \mathcal{C}X_{1,r} \arrow[dl, "s"'] \arrow[dr, "d"] & & & & \\
 & & & \mathcal{C}X_{0,r-2} \arrow[dl, "s"'] \arrow[dr, "d"] & & \mathcal{C}X_{1,r-1} \arrow[dl, "s"'] \arrow[dr, "d"] & & \mathcal{C}X_{2,r} \arrow[dl, "s"'] \arrow[dr, "d"] & & & \\
 & & {} \arrow[dl, dotted, no head] & & {} \arrow[dl, dotted, no head] & & {} \arrow[dr, dotted, no head] & & {} \arrow[dr, dotted, no head] & & \\
 & \mathcal{C}X_{0,2} \arrow[dl, "s"'] \arrow[dr, "d"] & & \mathcal{C}X_{1,3} \arrow[dl, "s"'] & & \cdots & & \mathcal{C}X_{r-3,r-1} \arrow[dr, "d"] & & \mathcal{C}X_{r-2,r} \arrow[dl, "s"'] \arrow[dr, "d"] & \\
\mathcal{G}X_{0,1} \arrow[dr] & & \mathcal{G}X_{1,2} \arrow[dl] \arrow[dr] & & \cdots & & \cdots & & \mathcal{G}X_{r-2,r-1} \arrow[dl] \arrow[dr] & & \mathcal{G}X_{r-1,r} \arrow[dl] \\
 & \mathcal{G}X_{1,1} & & \mathcal{G}X_{2,2} & & \cdots & & \mathcal{G}X_{r-2,r-2} & & \mathcal{G}X_{r-1,r-1} &
\end{tikzcd}
\tag{2}
$$

where all the arrows denoted by s, d are blowups and every rhombus in the diagram is a normalized Cartesian square.

In particular, the whole diagram is constructed upon its first two layers (that contain the GIT quotients of X) by successive normalized Cartesian squares. In particular, this result implies that every chain of closures of orbits linking the sink and the source of X is parametrized by an element of $\mathcal{C}X$ (Corollary 5.9). Moreover, our proof of the statement will provide an explicit description of the centers of all the blowup maps in the diagram (Remark 5.10).

We also present an example that shows that $\mathcal{C}X$ can be singular (Example 6.1), and show that it is smooth if X is convex, i.e. if $\mathrm{H}^1(\mathbb{P}^1, \mu^*TX) = 0$ for every morphism $\mu : \mathbb{P}^1 \to X$. More precisely, we prove the following:

Theorem 1.2 *Under the assumptions of Theorem 1.1, assume moreover that X is convex. Then the varieties $\mathcal{C}X_{i,j}$ with $i = 0$ or $j = r$ are smooth, and the maps*

$$
\mathcal{G}X_{0,1} \xleftarrow{\ s\ } \mathcal{C}X_{0,2} \xleftarrow{\ s\ } \cdots \xleftarrow{\ s\ } \mathcal{C}X \xrightarrow{\ d\ } \cdots \xrightarrow{\ d\ } \mathcal{C}X_{r-2,r} \xrightarrow{\ d\ } \mathcal{G}X_{r-1,r}
$$

are smooth blowups, that is, we have a strong factorization of ψ.

The smoothness of the intermediate Chow quotients in the above factorization follows from using Theorem 1.1 together with some ideas from (19; 31), which allows us to prove the unobstructedness of deformations of the cycles parametrized by the elements of $\mathcal{C}X$ under the convexity assumption (see also Remark 6.3). The smoothness of the centers of the blowups follows then by a statement due to Andreatta and Wiśniewski (2, Corollary 4.11).

Outline

The structure of the paper is as follows. The first two sections provide preliminary material on $\mathbb{C}^*$-actions and prunings, primarily recalling useful results and notations presented in (36; 4). Note that the main result of Section 3 (Proposition 3.9) is presented under slightly different assumptions than in the original formulation (36). Section 4 is devoted to the Chow quotient of a $\mathbb{C}^*$-action; a key result that we prove is that under the equalization hypothesis, the Chow quotient is the normalization of a subscheme of the Hilbert scheme of X

(Remark 4.6). Furthermore, we show how to effectively write $\mathcal{C}X$ as the normalization of a certain subvariety of a Grassmannian (Corollary 4.16). The proofs of Theorems 1.1 and 1.2 are presented in Sections 5 and 6, respectively; the latter section also includes a toric example whose Chow quotient is not smooth. We conclude the paper with Section 7, in which we discuss the assumption on the codimension of $B^{\pm}(Y)$ for every inner fixed point component $Y \in \mathcal{Y}^{\circ}$, and show that Theorem 1.1 remains valid under milder hypotheses. As an illustrative example, we consider the Chow quotient of the diagonal $\mathbb{C}^*$-action of $(\mathbb{P}^1)^n$, which is known to be isomorphic to the Losev–Manin space (see (14) and references therein); our techniques allow us to describe the contractions of this variety appearing in Figure 2.

2. Preliminaries on $\mathbb{C}^*$-actions and GIT quotients

In this section X will denote a normal complex projective variety, with a nontrivial $\mathbb{C}^*$-action.

2.1. Fixed point components

We denote by $X^{\mathbb{C}^*}$ the fixed locus of the action, and by $\mathcal{Y}$ the set of irreducible fixed point components:

$$X^{\mathbb{C}^*} = \bigsqcup_{Y \in \mathcal{Y}} Y.$$

The *sink* and the *source* of the action are the fixed point components containing, respectively, the limiting points of a general $x \in X$:

$$\lim_{t \to 0} t^{-1}x, \quad \lim_{t \to 0} tx.$$

If X is smooth and projective, every $Y \in \mathcal{Y}$ is smooth (cf. (22)).

2.2. Linearizations and weight maps

Given a line bundle $L \in \mathrm{Pic}(X)$, one may find a linearization of the $\mathbb{C}^*$-action on it (cf. (25)), so that for every $Y \in \mathcal{Y}$, $\mathbb{C}^*$ acts on $L_{|Y}$ by multiplication with a character $m \in \mathrm{M}(\mathbb{C}^*) = \mathrm{Hom}(\mathbb{C}^*, \mathbb{C}^*)$, called *weight of the linearization on* Y. Fixing an isomorphism $\mathrm{M}(\mathbb{C}^*) \simeq \mathbb{Z}$, a linearization defines a map $\mu_L : \mathcal{Y} \to \mathbb{Z}$, such that

$$\mu_{kL}(Y) = k\mu_L(Y), \quad \mu_{L+L'}(Y) = \mu_L(Y) + \mu_{L'}(Y),$$

for every $L, L' \in \mathrm{Pic}(X)$, $k \in \mathbb{Z}$, $Y \in \mathcal{Y}$.

2.3. Actions on polarized pairs

A $\mathbb{C}^*$*-action on the polarized pair* (X, L) is a $\mathbb{C}^*$-action on a projective variety X, together with the weight map μ_L determined by an ample line bundle L. Since two linearizations differ by a character of $\mathbb{C}^*$, for $L \in \mathrm{Pic}(X)$ there exists a unique linearization (called *normalized*) whose smallest weight is equal to zero. In this case we denote by

$$0 = a_0 < \cdots < a_r,$$

the weights $\mu_L(Y)$, $Y \in \mathcal{Y}$, and set:

$$Y_i := \bigcup_{\mu_L(Y) = a_i} Y.$$

The values a_i will be called the *critical values* and the number r will be called the *criticality* of the action. The minimum and maximum of the critical values are achieved only at the sink and the source of the action, which are then equal to Y_0 and Y_r, respectively (see (37, Remark 2.12)). The value $\delta = a_r$ is called the *bandwidth* of the $\mathbb{C}^*$-action on (X, L). The fixed point components different from Y_0, Y_r are called *inner*, and their set is denoted by

$$\mathcal{Y}^{\circ} := \mathcal{Y} \setminus \{Y_0, Y_r\}.$$

2.4. Geometric quotients

It is a well known fact in Mumford's Geometric Invariant Theory that a $\mathbb{C}^*$-action on a polarized variety (X, L) allows to define projective quotients of X, as follows. Let

$$A := \bigoplus_{m \geq 0} \mathrm{H}^0(X, mL)$$

be the *graded* $\mathbb{C}$*-algebra of sections of* L, and set, for every $\tau \in [0, \delta] \cap \mathbb{Q}$:

$$A(\tau) := \bigoplus_{\substack{m \geq 0 \\ m\tau \in \mathbb{Z}}} \mathrm{H}^0(X, mL)_{m\tau}, \qquad \mathcal{G}X(\tau) := \mathrm{Proj}(A(\tau)).$$

We will call the $\mathcal{G}X(\tau)$'s the *GIT-quotients of the* $\mathbb{C}^*$*-action on* (X, L).

The above construction has been extended by Białynicki-Birula and Swięcicka to the construction of other (in general non projective) quotients of X, described in terms of semi-sections and sections of the $\mathbb{C}^*$-action (cf. (9); see also (36)). Their construction allows us to describe easily the open subsets $X^{\mathrm{ss}}(\tau) \subset X$ of semistable points on which the rational maps

$$X \dashrightarrow \mathcal{G}X(\tau)$$

are defined. In order to do so we need the language of *BB-cells*, provided by the *Białynicki-Birula decomposition* of X (see (6; 10)): given a fixed point component $Y \in \mathcal{Y}$, we denote by

$$X^{\pm}(Y) := \{x \in X | \lim_{t^{\pm 1} \to 0} tx \in Y\}, \qquad B^{\pm}(Y) := \overline{X^{\pm}(Y)}, \tag{3}$$

the *BB-cells* of the action and their closures. The equalities:

$$X = \bigsqcup_{Y \in \mathcal{Y}} X^+(Y) = \bigsqcup_{Y \in \mathcal{Y}} X^-(Y)$$

are called the $+$ and $-$ *BB-decomposition* of X.

To describe $X^{\mathrm{ss}}(\tau)$ for a $\mathbb{C}^*$-action on a polarized pair (X, L), let us define

$$B_i^{\pm} := \bigcup_{\mu_L(Y) = a_i} B^{\pm}(Y)$$

and, for notational reasons, $B_i^{\pm} = \emptyset$ for $i \in \mathbb{Z} \setminus \{0, \ldots, r\}$. Then, following (9), we may write:

$$X^{\mathrm{ss}}(\tau) = X \setminus \Big(\bigcup_{a_i < \tau} B_i^+ \cup \bigcup_{a_i > \tau} B_i^- \Big).$$

In particular, $X^{\mathrm{ss}}(\tau)$ is the same for any $\tau \in (a_i, a_{i+1}) \cap \mathbb{Q}$, and so we set:

- $\mathcal{G}X_{i,i} := \mathcal{G}X(\tau)$ for $\tau = a_i$;
- $\mathcal{G}X_{i,i+1} := \mathcal{G}X(\tau)$ for $\tau \in (a_i, a_{i+1}) \cap \mathbb{Q}$.

Note that $\mathcal{G}X_{i,i+1}$ is a geometric quotient of X for every i, whereas $\mathcal{G}X_{i,i}$ is only a good categorical quotient (see (15, p. 94)); following (7, p. 10) we say that $\mathcal{G}X_{i,i}$ is semigeometric. Moreover, by construction,

$$\mathcal{G}X_{0,0} = Y_0, \quad \mathcal{G}X_{r,r} = Y_r.$$

Furthermore, since the intersection of all the open sets $X^{\mathrm{ss}}(i, i+1)$ is nonempty, the varieties $\mathcal{G}X(i, i+1)$ are birationally equivalent. The natural birational maps among them fit in the following commutative diagram, whose diagonal arrows are contractions:

$$\begin{tikzcd}
& \mathcal{G}X_{0,1} \arrow[dl] \arrow[dr] \arrow[rr, dashed] & & \mathcal{G}X_{1,2} \arrow[dl] & \cdots & \mathcal{G}X_{r-2,r-1} \arrow[dr] \arrow[rr, dashed] & & \mathcal{G}X_{r-1,r} \arrow[dl] \arrow[dr] & \\
\mathcal{G}X_{0,0} & & \mathcal{G}X_{1,1} & \cdots & \cdots & & \mathcal{G}X_{r-1,r-1} & & \mathcal{G}X_{r,r}
\end{tikzcd} \tag{4}$$

The horizontal maps are Atiyah flips; this is a known fact which goes back to the work of Thaddeus, Reid and Włodarczyk ((45; 42; 40)).

Example 2.1 *An obvious source of examples of $\mathbb{C}^*$-actions is the category of toric varieties. An algebraic variety X is called* toric *if it admits an action of an algebraic torus $\mathbb{T} = (\mathbb{C}^*)^n$ with an open orbit. As usual, we denote by $M = \mathrm{M}(\mathbb{T}) := \mathrm{Hom}_{\mathrm{alg}}(\mathbb{T}, \mathbb{C}^*)$ and $N := \mathrm{Hom}_{\mathrm{alg}}(\mathbb{C}^*, \mathbb{T})$ the lattices of characters and 1-parameter subgroups of $\mathbb{T}$. If X is normal, it is completely described by a regular fan Σ in $N \otimes \mathbb{R}$. If X is projective and we choose an ample line bundle L on X, together with a linearization of the action of $\mathbb{T}$, then the weights of the torus action on the space of sections of L are lattice points in a lattice polytope $\Delta \subset M \otimes \mathbb{R}$; the lattice polytope determines the fan Σ, hence the variety X, so it makes sense to write $X = X(\Delta)$. The vertices of Δ represent the weights of the action of $\mathbb{T}$ on fibers of L over fixed points of the action on X.*

Actions of $\mathbb{C}^*$ on a projective toric variety $X = X(\Delta)$ as above can be obtained by considering 1-parameter subgroups $\nu \in N$, which are determined by projections $\nu : M \otimes \mathbb{R} \to \mathbb{Z} \otimes \mathbb{R}$. The critical values of the corresponding $\mathbb{C}^*$-action (together with the induced linearization on L) are the images of vertices of Δ under the projection ν. The associated GIT quotients of the action on (X, L) are then toric varieties, with respect to the quotient torus $T/\nu \otimes \mathbb{C}^*$. They can be described as the toric varieties determined by the polytopes $\nu^{-1}(a) \cap \Delta$, $a \in \mathbb{Q} \cap \nu(\Delta)$, obtained by slicing Δ with hyperplanes of the form $\nu^{-1}(a)$. We refer to (23) for details.

2.5. The smooth case

If X is smooth, the normal bundle in X of a fixed point component Y splits into two subbundles, on which $\mathbb{C}^*$ acts with positive and negative weights, respectively:

$$\mathcal{N}_{Y|X} \simeq \mathcal{N}^+(Y) \oplus \mathcal{N}^-(Y). \tag{5}$$

We set

$$\nu^{\pm}(Y) := \mathrm{rk}(\mathcal{N}^{\pm}(Y)), \qquad V^{\pm}(Y) := \mathbb{P}(\mathcal{N}^{\pm}(Y)^\vee). \tag{6}$$

2.6. Equalized actions

We say that a $\mathbb{C}^*$-action is *equalized* at $Y \in \mathcal{Y}$ if for every $x \in \big(X^-(Y) \cup X^+(Y)\big) \setminus Y$ the isotropy group of the action at x is trivial. In the smooth case, as shown in (36, Lemma 2.1), one may ask, equivalently, that for every fixed point component Y, $\mathbb{C}^*$ acts on $\mathcal{N}^+(Y)$ with all the weights equal to $+1$ and on $\mathcal{N}^-(Y)$ with all the weights equal to -1 ((41, Definition 1)).

Remark 2.2 *If X is smooth and the $\mathbb{C}^*$-action is equalized, the geometric quotients $\mathcal{G}X_{i,i+1}$ are smooth, as shown in (36, Lemma 2.15). Moreover, under these hypotheses, if Y is a fixed point component of weight $\mu_L(Y) = a_i$, then $V^+(Y)$ (respectively $V^-(Y)$) can be embedded into $\mathcal{G}X_{i-1,i}$ (resp. $\mathcal{G}X_{i,i+1}$) as the parameter space of orbits with source (resp. sink) at Y.*

The equalization hypothesis implies that the closure of any 1-dimensional orbit is a smooth rational curve, whose L-degree may be computed in terms of the weights at its extremal points. The following statement ((36, Lemma 2.2)) follows from the AM vs. FM formula presented in (41, Corollary 3.2):

Lemma 2.3 (AM vs. FM) *Let (X, L) be a polarized pair, with X smooth, supporting an equalized $\mathbb{C}^*$-action, and let C be the closure of a 1-dimensional orbit, whose sink and source are denoted by x_- and x_+. Then C is a smooth rational curve of L-degree equal to $\mu_L(x_+) - \mu_L(x_-)$.*

2.7. B-type actions and bordisms

Following (37, Section 3), we refer to a $\mathbb{C}^*$-action as a B-type action if its extremal fixed point components Y_0 and Y_r have codimension one.

If X is smooth and projective, by the Białynicki-Birula decomposition, the restriction maps $\mathrm{Pic}(X) \to \mathrm{Pic}(Y_j)$, $j = 0, r$ are surjective (cf. (11, Theorem 3)). We will say that the $\mathbb{C}^*$-action is a *bordism* (cf. (37, Definition 3.8)) if the restriction maps $\imath_j^* : \mathrm{Pic}(X) \to \mathrm{Pic}(Y_j)$, $j = 0, r$, fit into two short exact sequences:

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Z}[Y_r] & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & \mathrm{Pic}(Y_0) & \to & 0 \\ 0 & \to & \mathbb{Z}[Y_0] & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & \mathrm{Pic}(Y_r) & \to & 0 \end{array}$$

The following equivalence, which is a consequence of (11, Theorem 3) has been proved in (37, Corollary 3.7).

Lemma 2.4 *Let X be a smooth projective variety admitting a $\mathbb{C}^*$-action of B-type. The action is a bordism if and only if $\nu^{\pm}(Y) = \mathrm{codim}(B^{\pm}(Y)) \geq 2$ for every inner fixed point component $Y \in \mathcal{Y}^\circ$.*

2.8. A digression on Grassmannians and $\mathbb{C}^*$-actions

In this section we discuss an example that we will use later on.

Let V be a finite dimensional complex vector space with a nontrivial linear action of $\mathbb{C}^*$ with weights $a_0 < \cdots < a_s \in \mathbb{Z}$, inducing a decomposition:

$$V = \bigoplus_{k=0}^{s} V_k, \qquad V_k = \{v \in V \mid t(v) = t^{a_k} v \text{ for all } t \in \mathbb{C}^*\}.$$

The induced action on the (Grothendieck) projectivization $\mathbb{P}(V)$ of V has exactly $s+1$ fixed-point components $\mathbb{P}(V_k) \subset \mathbb{P}(V)$. The sink and the source of the action are $\mathbb{P}(V_0), \mathbb{P}(V_s)$, and there exists a linearization of the action on the tautological line bundle $\mathcal{O}(1)$ such that:

$$\mu_{\mathcal{O}(1)}(\mathbb{P}(V_k)) = a_k.$$

Let us denote by $\mathbb{G}(s, \mathbb{P}(V))$ the Grassmannian of s-dimensional projective subspaces of $\mathbb{P}(V)$ and consider the induced $\mathbb{C}^*$-action. One of its fixed point components is the variety parametrizing subspaces meeting each $\mathbb{P}(V_k)$ in one point, which is isomorphic to the Segre variety $\prod_{k=0}^{s} \mathbb{P}(V_k)$; we will denote it by

$$\Sigma(V) := \{\mathbb{P}(W) \subset \mathbb{P}(V) \mid \dim(\mathbb{P}(W) \cap \mathbb{P}(V_k)) = 0,\ \forall k\} \subset \mathbb{G}(s, \mathbb{P}(V)) \tag{7}$$

Now let us consider two given indices k_-, k_+ with $0 \leq k_- < k_+ \leq s$, and set:

$$V_{k_-,k_+} := \bigoplus_{k=k_-}^{k_+} V_k.$$

We have induced $\mathbb{C}^*$-actions on V_{k_-,k_+} (with weights $a_{k_-} < \cdots < a_{k_+}$), $\mathbb{P}(V_{k_-,k_+})$ and the Grassmannian $\mathbb{G}(k_+ - k_-, \mathbb{P}(V_{k_-,k_+}))$. One of the fixed-point components of the action on $\mathbb{G}(k_+ - k_-, \mathbb{P}(V_{k_-,k_+}))$ is $\Sigma(V_{k_-,k_+})$, which is isomorphic to $\prod_{k=k_-}^{k_+} \mathbb{P}(V_k)$; note also that the restriction of the Plücker embedding of the Grassmannian $\mathbb{G}(k_+ - k_-, \mathbb{P}(V_{k_-,k_+}))$ to $\Sigma(V_{k_-,k_+})$ is the Segre embedding.

The linear projection $\varphi_{k_-,k_+} : \mathbb{P}(V) \dashrightarrow \mathbb{P}(V_{k_-,k_+})$ is $\mathbb{C}^*$-equivariant and induces a surjective morphism

$$\pi_{k_-,k_+} : \Sigma(V) \longrightarrow \Sigma(V_{k_-,k_+}),$$

which corresponds to the natural projection

$$\prod_{k=0}^{s} \mathbb{P}(V_k) \longrightarrow \prod_{k=k_-}^{k_+} \mathbb{P}(V_k), \qquad (P_0, \ldots, P_s) \mapsto (P_{k_-}, \ldots, P_{k_+}).$$

3. Prunings

Following (36), the GIT quotients of X can be seen as the extremal fixed point components of $\mathbb{C}^*$-equivariant birational modifications of X, that are referred to as *prunings* of X in (4).

Definition 3.1 Let (X, L) be a polarized pair, and let $\tau_- < \tau_+$ be two rational numbers in $[0, \delta]$. We define the *pruning* of (X, L) with respect to τ_-, τ_+ as:

$$X(\tau_-, \tau_+) := \operatorname{Proj} A(\tau_-, \tau_+), \text{ where } A(\tau_-, \tau_+) = \bigoplus_{m \in d\mathbb{Z}_{\geq 0}} \bigoplus_{k \in [m\tau_-, m\tau_+]} \mathrm{H}^0(X, mL)_k,$$

and d is the minimum positive integer such that $d\tau_\pm \in \mathbb{Z}$.

Remark 3.2 *Note that $A(\tau_-, \tau_+)$ is finitely generated by (4, Lemma 3.2), guaranteeing the good definition of $X(\tau_-, \tau_+)$.*

Example 3.3 *Assume that $X = X(\Delta)$ is a projective toric variety, polarized with an ample line bundle L, associated with a lattice polytope $\Delta \subset M \otimes \mathbb{R}$, with a $\mathbb{C}^*$-action determined by a 1-parameter subgroup ν. Then, with the notation of Example 2.1, the prunings of X are the toric varieties associated with polytopes of the form $\nu^{-1}[a, b] \cap \Delta$ where $a < b$ are rationals contained in $\nu(\Delta)$.*

In the case in which X is smooth and the action is equalized, the varieties $X(\tau_-, \tau_+)$ can be easily written as birational modifications of X, as follows. Let us denote by $\beta : X^\flat \longrightarrow X$ the blowup of X along the sink and the source Y_0, Y_r. The $\mathbb{C}^*$-action on X extends via β to an action on $X^\flat$, whose sink and source are the exceptional divisors over Y_0, Y_r, that we denote by $Y_0^\flat, Y_r^\flat$. Let us define

$$L(\tau_-, \tau_+) := \beta^* L - \tau_- Y_0^\flat - (\delta - \tau_+) Y_r^\flat.$$

Then, following (36, Lemma 2.5), we can write:

$$A(\tau_-, \tau_+) = \bigoplus_{m \in d\mathbb{Z}_{\geq 0}} \mathrm{H}^0(X^\flat, mL(\tau_-, \tau_+)).$$

Remark 3.4 *In particular, for τ_-, τ_+ sufficiently close to 0, δ, the line bundle $L(\tau_-, \tau_+)$ is ample on $X^\flat$, so in this case $X(\tau_-, \tau_+) = X^\flat$.*

Furthermore, one can describe the base loci of the line bundles $mL(\tau_-, \tau_+)$ for m large enough and divisible, in terms of the subsets $B_i^\pm$. We refer the interested reader to (36, Corollary 2.7) for details on the proof. We note that in the cited reference there is a misprint in the formula preceding the statement of Corollary 2.7: the two inequalities in that formula should be $a_i < \tau_-$, $a_j > \tau_+$. The correct statement reads as follows:

Lemma 3.5 *Let (X, L) be a polarized pair, with X smooth, with an equalized $\mathbb{C}^*$-action. With the above notation, the base locus of $mL(\tau_-, \tau_+)$ is equal to the strict transform in $X^\flat$ of*

$$\bigcup_{a_i < \tau_-} B_i^+ \cup \bigcup_{a_j > \tau_+} B_j^-.$$

In particular, $\mathrm{Bs}(mL(\tau_-, \tau_+))$ depends only on:

$$i(\tau_-) := \min\{i |\ a_i \geq \tau_-\} - 1, \quad j(\tau_+) := \max\{j |\ a_j \leq \tau_+\} + 1,$$

so we may define:

$$\mathcal{B}_{i(\tau_-) j(\tau_+)} := \mathrm{Bs}(mL(\tau_-, \tau_+)) = \bigcup_{a_i < \tau_-} B_i^+ \cup \bigcup_{a_j > \tau_+} B_j^-,$$

and assert also that $\mathcal{B}_{i(\tau_-) j(\tau_+)}$ is equal to the *stable base locus of* $mL(\tau_-, \tau_+)$, that we denote by $\mathbb{B}(mL(\tau_-, \tau_+))$.

Notation 3.6 Given line bundles $L_1, L_2, \dots, L_i$ on a projective variety M, we will denote by

$$\langle L_1, L_2, \dots, L_i \rangle$$

the convex cone of nonnegative real linear combinations of the numerical classes of these line bundles in the space $N^1(M)$ (the real vector space generated by Cartier divisors in M modulo numerical equivalence).

We will now consider a $\mathbb{C}^*$-action on a polarized pair (X, L) with X smooth, for which we will use the notation of Section 2. Our goal will be to study the intersection of the movable cone of $X^\flat$ with $\langle \beta^* L, Y_0^\flat, Y_r^\flat \rangle$. For simplicity we will assume that the $\mathbb{C}^*$-action is a bordism, which, by our discussion above, is equivalent to say that $\mathbb{B}(mL(\tau_-, \tau_+))$ contains no codimension one subvarieties, for every τ_-, τ_+.

Remark 3.7 *Note that we are not assuming any particular value of the Picard number of X. Our arguments we will be similar to the ones considered in (36), where we assumed that* $\mathrm{Pic}(X) \simeq \mathbb{Z}$, *but considered also the non-bordism case.*

Lemma 3.8 *Let (X, L) be a polarized pair, with X smooth, endowed with an equalized $\mathbb{C}^*$-action such that the induced action on $X^\flat$ is a bordism. Then*

$$\overline{\mathrm{Mov}(X^\flat)} \cap \langle \beta^* L, Y_0^\flat, Y_r^\flat \rangle = \langle \beta^* L, L(0,0), L(\delta, \delta) \rangle.$$

Proof The proof follows verbatim (36, Proposition 4.7). Note that the assumption on the Picard number of X in the quoted reference was used only to guarantee that $\mathrm{codim}(B^\pm(Y), X) \geq 2$ for every $Y \in \mathcal{Y}^\circ$. $\square$

Combining Lemmas 3.8 and 3.5, we get the *stable base locus decomposition* of the cone $\overline{\mathrm{Mov}(X^\flat)} \cap \langle \beta^* L, Y_0^\flat, Y_r^\flat \rangle$, that we have represented in Figure 1; the stable base loci (open) chambers are denoted by $N_{i,j}$, and defined as:

$$N_{i,j} = \begin{cases} \mathrm{Int}\langle L(a_i, a_{j-1}), L(a_{i+1}, a_{j-1}), L(a_i, a_j), L(a_{i+1}, a_j) \rangle & \text{if } j - i \geq 2, \\ \mathrm{Int}\langle L(a_i, a_i), L(a_i, a_{i+1}), L(a_{i+1}, a_{i+1}) \rangle & \text{if } j = i + 1. \end{cases}$$

for $0 \leq i < j \leq r$.

Finally, following the lines of argumentation of (36, Proposition 4.11), we can prove that the stable base locus decomposition of the cone $\overline{\mathrm{Mov}(X^\flat)} \cap \langle \beta^* L, Y_0^\flat, Y_r^\flat \rangle$ coincides with its Mori chamber decomposition.

Proposition 3.9 *Let (X, L) be a polarized pair, with X smooth, with an equalized $\mathbb{C}^*$-action such that* $\mathrm{codim}(B^\pm(Y), X) > 1$ *for every inner fixed point component $Y \in \mathcal{Y}^\circ$. Then for every $0 \leq i \leq j \leq r$ there exist a smooth projective variety $X_{i,j}$ and a small $\mathbb{Q}$-factorial modification $\varphi_{i,j} : X^\flat \dashrightarrow X_{i,j}$ such that*

- $X_{i,j} = X(\tau_-, \tau_+)$ *whenever $\tau_- \in (a_i, a_{i+1})$, $\tau_+ \in (a_{j-1}, a_j)$;*
- $\mathrm{Nef}(X_{i,j}) \cap \langle \beta^* L, Y_0^\flat, Y_r^\flat \rangle = \overline{N_{i,j}}$.

Proof We start from $X_{0,r} := X^\flat$, that satisfies $\mathrm{Nef}(X_{0,r}) = \overline{N_{0,r}}$. Then for every pair of indices i, j such that $0 \leq i \leq j \leq r$, we construct a small $\mathbb{Q}$-factorial modification $\varphi_{i,j} : X_{0,r} \dashrightarrow X_{i,j}$ (called *pruning map*), such that

- the variety $X_{i,j}$ is smooth, projective, with a $\mathbb{C}^*$-action such that $\varphi_{i,j}$ is $\mathbb{C}^*$-equivariant;
- the set of inner fixed-point components of the action on $X_{i,j}$ is:

$$\{\varphi_{i,j}(Y) | \ Y \in \mathcal{Y}^\circ, \ a_i < \mu_L(Y) < a_j\},$$

and $\varphi_{i,j}$ is an isomorphism on an open set containing these components.

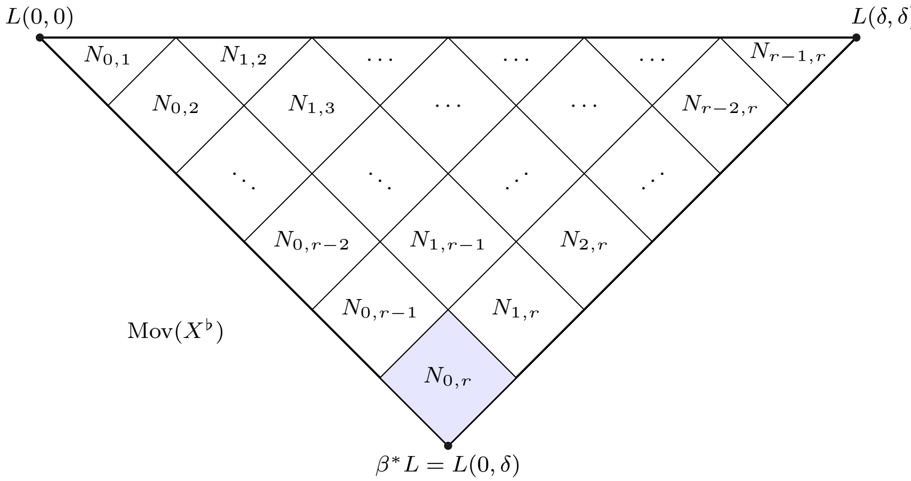

Fig. 1: SBL decomposition of $\overline{\mathrm{Mov}(X^\flat)} \cap \langle \beta^*L, Y_0^\flat, Y_r^\flat \rangle$.

These modifications are constructed recursively as follows: assuming we have already constructed the variety $X_{i,j}$, and the map $\varphi_{i,j}$ (with $i < j-1$), then the varieties $X_{i,j-1}, X_{i+1,j}$ are defined by means of small $\mathbb{Q}$-factorial modifications $s : X_{i,j} \dashrightarrow X_{i,j-1}$, $d : X_{i,j} \dashrightarrow X_{i+1,j}$ constructed as in (36, Theorem 3.1).

Namely $X_{i,j-1}$ (resp. $X_{i+1,j}$) is constructed as the smooth blowup $b_s : X^s_{i,j} \to X_{i,j}$ (resp. $b_d : X^d_{i,j} \to X_{i,j}$) along the strict transform into $X_{i,j}$ of B^-_{j-1} (resp. B^+_{i+1}), followed by a smooth blowdown with the same exceptional locus; this blowdown contracts the class of the closure of a 1-dimensional $\mathbb{C}^*$-orbit in $\mathrm{Exc}(b_s)$ (resp. $\mathrm{Exc}(b_d)$).

$$\begin{tikzcd}
& X^s_{i,j} \arrow[dl] \arrow[dr, "b_s"] & & X^d_{i,j} \arrow[dl, "b_d"'] \arrow[dr] & \\
X_{i,j-1} & & X_{i,j} \arrow[ll, dashed, "s"] \arrow[rr, dashed, "d"'] & & X_{i+1,j}
\end{tikzcd} \tag{8}$$

In particular, the indeterminacy loci of s, d are the strict transforms of B^-_{j-1}, B^+_{i+1} into $X_{i,j}$, respectively, and the criticality of the $\mathbb{C}^*$-action gets reduced by one via these maps. We then set

$$\varphi_{i+1,j} := d \circ \varphi_{i,j}, \qquad \varphi_{i,j-1} := s \circ \varphi_{i,j}.$$

Finally, arguing as in the proof of (36, Proposition 4.11) we can show that, identifying the space $\mathrm{N}^1(X_{i,j})$ with $\mathrm{N}^1(X^\flat)$ we have $\mathrm{Nef}(X_{i,j}) \cap \langle \beta^*L, Y_0^\flat, Y_r^\flat \rangle = \overline{N_{i,j}}$. In other words,

$$X_{i,j} = X(\tau_-, \tau_+), \text{ whenever } i(\tau_-) = i,\ j(\tau_+) = j.$$

This completes the proof. $\square$

Remark 3.10 *Note that by construction, the maps s, d commute, so that every map $\varphi_{i,j}$ can be written as a composition*

$$\varphi_{i,j} = s^i \circ d^{r-j}.$$

We will call s, d elementary pruning maps.

Let us finish this section relating extremal fixed point components of the varieties $X^s_{i,j}$ and $X^d_{i,j}$ appearing in the above proof with geometric quotients of X.

Remark 3.11 *In the recursive construction described in Diagram (8), the $\mathbb{C}^*$-action on the variety $X^d_{i,j}$ is of B-type.*

The morphism b_d sends the sink Y^d of $X^d_{i,j}$ onto the sink of $X_{i,j}$, which is, by construction, the geometric quotient $\mathcal{G}X_{i,i+1}$ of X; similarly the contraction $X^d_{i,j} \to X_{i+1,j}$ sends Y^d onto the geometric quotient $\mathcal{G}X_{i+1,i+2}$.

Since $X^d_{i,j}$ is the smooth blowup of $X_{i,j}$ along the strict transform of B^+_{i+1} into $X_{i,j}$, it follows that Y^d can be described as the smooth blowup of $\mathcal{G}X_{i,i+1}$ along the subvariety $V^+(Y_{i+1})$ parametrizing $\mathbb{C}^*$-orbits in X with source at Y_{i+1}. In a similar way, the source Y^s of $X^s_{i,j}$ is the smooth blowup of $\mathcal{G}X_{j-1,j}$ along $V^-(Y_{j-1})$.

4. Chow quotients

In this section we consider a nontrivial $\mathbb{C}^*$-action on a polarized pair (X, L) and we study the Chow quotient of X by $\mathbb{C}^*$, a concept due to Kapranov who, in his seminal work (24), defined it and identified it with a moduli of pointed curves when X is a Grassmannian endowed with an action of a complex torus. Slightly different parameter spaces for $\mathbb{C}^*$-invariant cycles were previously proposed by Białynicki-Birula and Sommese, (8), inspired by a construction by Fujiki (cf. (18)).

Following Kapranov's definition, we may find a nonempty open set $V \subset X$ of points x such that the cycles $\overline{\mathbb{C}^* \cdot x}$ belong to a single homology class in X. By shrinking V, if necessary, the geometric quotient of V by the action of $\mathbb{C}^*$ exists. Moreover, taking closures in X of the orbits of points of V, we obtain an algebraic irreducible family of rational curves in X, therefore a morphism $\psi : V/\mathbb{C}^* \to \mathrm{Chow}(X)$.

Definition 4.1 With the above notation, the *Chow quotient* of X by the action of $\mathbb{C}^*$ is defined as the closure:

$$\overline{\mathcal{C}X} := \overline{\psi(V/\mathbb{C}^*)} \subset \mathrm{Chow}(X).$$

We will denote by $\mathcal{C}X$ the normalization of $\overline{\mathcal{C}X}$, and call it the *normalized Chow quotient* of X by the $\mathbb{C}^*$-action. The pullback to $\mathcal{C}X$ of the universal family of $\mathrm{Chow}(X)$ will be denoted by $p : \mathcal{U} \to \mathcal{C}X$, with evaluation morphism $q : \mathcal{U} \to X$. We refer the interested reader to (26, Section I.3) (particularly Theorem 3.21) for generalities on the construction of the Chow scheme and its universal family, and to (24) for the construction of the Chow quotient. Note that, in our particular case, since there is a unique 1-dimensional orbit passing by the general point of X, the evaluation map $q : \mathcal{U} \to X$ is birational.

The $\mathbb{C}^*$-action on X extends naturally to an action on $\mathrm{Chow}(X)$ and the corresponding universal family of cycles. This action restricts to the trivial action on $V/\mathbb{C}^*$, hence to the trivial action on $\mathcal{C}X$. Consequently we have an action on $\mathcal{U}$, so that the maps $p : \mathcal{U} \to \mathcal{C}X$, $q : \mathcal{U} \to X$ are $\mathbb{C}^*$-equivariant, and every cycle parametrized by $\mathcal{C}X$ is $\mathbb{C}^*$-invariant.

Example 4.2 *In the case of $\mathbb{C}^*$-actions on toric varieties (see Example 2.1), Chow quotients are known to be toric varieties, that have been described combinatorially by Kapranov, Sturmfels and Zelevinsky in (23).*

Throughout the rest of the paper we will assume that the $\mathbb{C}^*$-action is equalized and that X is smooth. Let us start by observing that in this case every $\mathbb{C}^*$-invariant cycle parametrized by $\mathcal{C}X$ is a chain of rational curves with simple normal crossings (see (31, Lemma 3.12) for a similar statement for the parameter space of stable maps):

Lemma 4.3 *Let (X, L) be a polarized pair, with X smooth, admitting an equalized $\mathbb{C}^*$-action. Let $z \in \mathcal{C}X$ be any point, and let $Z = p^{-1}(z)$ be the corresponding cycle. Then Z is reduced, and its dual graph is of type* A.

Proof Note that Z is necessarily a connected cycle meeting the sink and the source, of L-degree equal to the bandwidth $\delta = a_r$ of the action, by Lemma 2.3. If Z_j is an irreducible component of Z, denoting by z_j^-, z_j^+

its sink and source, the equalization assumption implies that the L-degree of the reduced scheme of Z_j is equal to $\mu_L(z_j^+)-\mu_L(z_j^-)$ (again by Lemma 2.3), and this number is of type a_i-a_j for some $i,j\in\{0,\dots,r\}$. We conclude by summing on the components of Z. $\square$

Remark 4.4 *Note that the proof of Lemma 4.3 also implies that an irreducible component Z_k of $Z=p^{-1}(z)$ is a rational curve of degree $\mu_L(z_k^+)-\mu_L(z_k^-)$, where z_k^-, z_k^+ denote the sink and the source of Z_k, respectively. In particular, there are no components of Z contained in a fixed-point component.*

A consequence of the reducedness given by Lemma 4.3 is the following statement, which can be obtained by applying standard arguments on parameter spaces:

Lemma 4.5 *Let (X,L) be a polarized pair, with X smooth, with an equalized $\mathbb{C}^*$-action. Then the universal family $p:\mathcal{U}\to\mathcal{C}X$ is flat.*

Proof Let $Z=p^{-1}(z)$ be any fiber of p. By Lemma 4.3, the components of Z are all reduced, so that Z is normal at the general point of each component and, since $\mathcal{C}X$ is normal, we may apply (26, Theorem I.6.6) to claim that p is flat at the general point of any component of Z. We then conclude by applying (26, Theorem I.7.3.1). $\square$

Remark 4.6 *As a consequence of the flatness of $\mathcal{U}\to\mathcal{C}X$, we obtain a morphism $h:\mathcal{C}X\to\mathrm{Hilb}(X)$ such that $\mathcal{U}$ is the pullback to $\mathcal{C}X$ of the universal family over $\mathrm{Hilb}(X)$. We then recall that, following Białynicki-Birula and Sommese (cf. (5), see also (24, Section 0.5)) one defines the* Hilbert quotient *of X as the Zariski closure of the embedding into $\mathrm{Hilb}(X)$ of the quotient of a sufficiently small $\mathbb{C}^*$-invariant open subset in X (24, Definition 0.5.7); note that in our situation this coincides set-theoretically with $h(\mathcal{C}X)$. Its normalization will be denoted by $\mathcal{H}X$, so that h induces a morphism $h':\mathcal{C}X\to\mathcal{H}X$. Note that, by construction, $\mathcal{C}X$ and $\mathcal{H}X$ are reduced schemes, and that the natural cycle morphism $\mathrm{Hilb}(X)\to\mathrm{Chow}(X)$ (associating to every subscheme of X the corresponding cycle, cf. (26, Theorem I.6.3)), restricts to a morphism $g':\mathcal{H}X\to\mathcal{C}X$, which is birational, (24, 0.5.8). We then conclude that $h':\mathcal{C}X\to\mathcal{H}X$ is an isomorphism.*

Remark 4.7 *Given an equalized $\mathbb{C}^*$-action on X, every cycle parametrized by $\mathcal{C}X$ has a unique lift-up to the blowup $X^\flat$ of X along the sink and the source. In particular $\mathcal{C}X\simeq\mathcal{C}X^\flat$.*

Proposition 4.8 *If the action is of B-type and faithful, then $\mathcal{C}X$ is the normalization of the component of the Chow/Hilbert scheme containing the class of the closure of a general orbit of the action.*

Proof The faithfulness implies that the action is equalized at the sink and the source (see (37, Remark 3.2)). If $C\cong\mathbb{P}^1$ is the closure of a general orbit, then we can use (37, Lemma 2.16) to compute the splitting type of the normal bundle of C in X. Since the weights of the $\mathbb{C}^*$-action on $\mathcal{N}_{C/X}$ are $0^{(\dim X-1)}$ both at the sink and at the source of C, it follows that $\mathcal{N}_{C/X}\cong\mathcal{O}_{\mathbb{P}^1}^{\oplus n-1}$ and therefore the component of $\mathrm{Hilb}(X)$ containing the point $[C]$ is smooth at $[C]$ and of dimension $n-1=\dim\mathcal{C}X$. $\square$

4.1. The universal bundle of a $\mathbb{C}^*$-action

Let (X,L) be a polarized pair, with X smooth, supporting an equalized $\mathbb{C}^*$-action. We will now construct a vector bundle on $\mathcal{C}X$, whose fibers represent, roughly speaking, the linear spans of the cycles parametrized by $\mathcal{C}X$ in a certain embedding of X. This will allow us to identify $\mathcal{C}X$ with a subvariety of a Grassmannian.

Lemma 4.9 *Let (X,L) be a polarized pair, with X smooth, supporting an equalized $\mathbb{C}^*$-action. Then $p_*(q^*(L))$ is locally free of rank $\delta+1$.*

Proof We use the equalization of the action to ensure, by Lemma 4.5, that p is flat. In particular, $\chi(C, q^*L_{|C}) = \chi(C, L_{|C})$ is the same for every cycle C of the family parametrized by $\mathcal{C}X$, considered as a scheme with its reduced structure. We will show that $\mathrm{H}^1(C, L_{|C}) = 0$ for every C, so that $\dim \mathrm{H}^0(C, L_{|C})$ will be equal to $\delta + 1$, and we may then apply (20, Corollary III.12.9) to obtain the claimed result.

In order to do so, since the dual graph of C is of type A by Lemma 4.3, we may number the irreducible components of a given element C of $\mathcal{C}X$ as $C_1, \ldots, C_k$ so that C_i intersects (transversally) C_{i+1} in a point p_i, $i = 1, \ldots, k-1$, and $\{\mu_L(p_i)\}$ is strictly increasing. Note that we have an exact sequence:

$$0 \to \mathrm{H}^0(C, L_{|C}) \longrightarrow \bigoplus_{i=1}^{k} \mathrm{H}^0(C_i, L_{|C_i}) \longrightarrow \bigoplus_{i=1}^{k-1} \mathbb{C}_{p_i} \longrightarrow \mathrm{H}^1(C, L_{|C}) \to 0. \tag{9}$$

Then, since L is ample, one may easily check that the restriction map

$$\bigoplus_{i=1}^{k} \mathrm{H}^0(C_i, L_{|C_i}) \longrightarrow \bigoplus_{i=1}^{k-1} \mathbb{C}_{p_i}$$

is surjective, hence $\mathrm{H}^1(C, L_{|C})$ is equal to zero. This completes the proof. $\square$

Definition 4.10 The vector bundle

$$\mathcal{E} := p_*(q^*(L))$$

will be called the *universal bundle* associated to the $\mathbb{C}^*$-action on (X, L).

Lemma 4.11 *In the setting of Lemma 4.9, we have a $\mathbb{C}^*$-invariant decomposition*

$$\mathcal{E} = \bigoplus_{u=0}^{\delta} \mathcal{L}_u,$$

where $\mathcal{L}_u$ is a line bundle on which $\mathbb{C}^*$ acts with weight u.

Proof The action of $\mathbb{C}^*$ on X extends naturally to a linearization on the vector bundle $\mathcal{E}$, that we denote by μ. On the fiber C over a point $c \in \mathcal{C}X$ this is the induced action of $\mathbb{C}^*$ on the $(\delta+1)$-dimensional vector space $\mathrm{H}^0(C, L_{|C})$. For $c \in \mathcal{C}X$ general, by Lemma 2.3, $\mathrm{H}^0(C, L_{|C})$ is $\mathbb{C}^*$-equivariantly isomorphic to $\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\delta))$ equipped with the natural action of $\mathbb{C}^*$, which splits as a direct sum of 1-dimensional eigenspaces, associated to the weights $0, 1, \ldots, \delta$.

For every $u \in \{0, 1, \ldots, \delta\}$ we consider the set

$$\mathcal{L}_u := \bigcap_{t \in \mathbb{C}^*} \ker\big(\mu(t, \cdot) - u\,\mathrm{id} : \mathcal{E} \to \mathcal{E}\big),$$

so that over every point c we have an inclusion $\bigoplus_u \mathcal{L}_{u,c} \subset \mathcal{E}_c$. If c is general we have seen that the above inclusion is an equality and $\dim \mathcal{L}_{u,c} = 1$ for every u. Since by semicontinuity the dimension of $\mathcal{L}_{u,c}$ at the general point is the minimal one, it follows that $\dim \mathcal{L}_{u,c}$ is constant in c, that is, $\mathcal{L}_u$ is a vector subbundle of $\mathcal{E}$ of rank one, and we have a decomposition $\mathcal{E} = \bigoplus_u \mathcal{L}_u$. $\square$

Definition 4.12 We will say that (X, L) is *Chow projectively normal* with respect to the $\mathbb{C}^*$-action (*CPN*, for short) if L is very ample, q^*L is p-very ample, and

$$\mathrm{H}^1(X, L \otimes I_C) = 0, \text{ for every cycle } C \text{ of the family parametrized by } \mathcal{C}X.$$

Let us recall that, following (27, Definition 1.7.1), q^*L is p-very ample if

- the natural map $p^*\mathcal{E} = p^*p_*q^*L \to q^*L$ is surjective, and

- the induced morphism of $\mathcal{C}X$-schemes:
$$\mathcal{U} \to \mathbb{P}(p^*\mathcal{E}) \to \mathbb{P}(\mathcal{E})$$
is an embedding.

Moreover, the second condition implies that the restriction of global sections $\mathrm{H}^0(X, L) \to \mathrm{H}^0(C, L_{|C})$ is surjective for every C, so that the evaluation map
$$\mathrm{H}^0(X, L) \otimes \mathcal{O}_{\mathcal{C}X} = \mathrm{H}^0(\mathcal{C}X, \mathcal{E}) \otimes \mathcal{O}_{\mathcal{C}X} \longrightarrow \mathcal{E}$$
is surjective. Furthermore, since this map is $\mathbb{C}^*$-equivariant, it provides surjections:
$$\mathrm{H}^0(X, L)_u \otimes \mathcal{O}_{\mathcal{C}X} \longrightarrow \mathcal{L}_u, \tag{10}$$
for every weight $u \in \{0, 1, \dots, \delta\}$.

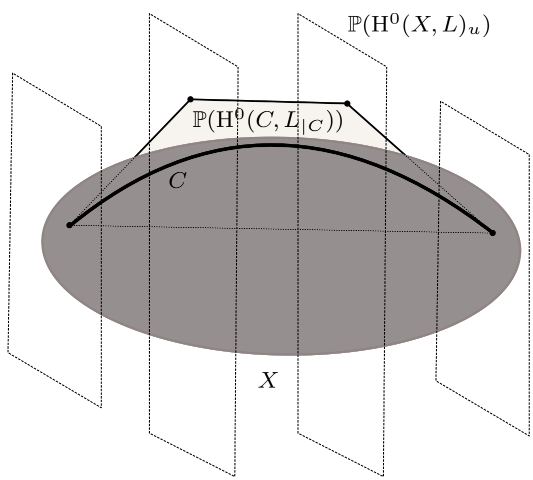

Fig. 2: A $\mathbb{C}^*$-invariant curve C is a rational normal curve in its linear span $\mathbb{P}(\mathrm{H}^0(C, L_{|C}))$, and this space meets each fixed point subspace $\mathbb{P}(\mathrm{H}^0(X, L)_u)$ at a point.

Remark 4.13 *The CPN hypothesis has the following geometric interpretation: via the embedding* $X \hookrightarrow \mathbb{P}(\mathrm{H}^0(X, L))$*, every* 1*-cycle* C *parametrized by* $\mathcal{C}X$ *is mapped to a reduced* $\mathbb{C}^*$*-invariant rational cycle of* L*-degree* δ *in* $\mathbb{P}(\mathrm{H}^0(C, L_{|C}))$*, on which the weights of the* $\mathbb{C}^*$*-action are* $0, 1, \dots, \delta$*. In particular, if* $C_1, \dots, C_k$ *are the irreducible components of* C*, with the notation of the proof of Lemma 4.9, from the exact sequence (9) it follows that the restriction map* $\mathrm{H}^0(X, L) \to \mathrm{H}^0(C_i, L_{|C_i})$ *is surjective, and so every component* C_i *embeds into* $\mathbb{P}(\mathrm{H}^0(C_i, L_{|C_i})) \subset \mathbb{P}(\mathrm{H}^0(X, L))$ *as a rational normal curve. Moreover, the* p*-very ampleness of* q^*L *yields that the embeddings* $C \hookrightarrow \mathbb{P}(\mathrm{H}^0(C, L_{|C}))$ *merge together into the embedding* $\mathcal{U} \hookrightarrow \mathbb{P}(\mathcal{E})$*. See Figure 2.*

Remark 4.14 *In the setting of Lemma 4.9, there exists an integer* m_0 *such that for* $m \geq m_0$ *the pair* (X, mL) *is CPN. In fact, note that, by (27, Theorem 1.7.8), the line bundle* q^*L *is* p*-ample. Then, by definition of ampleness and by (27, Theorem 1.7.6(iv)), there exists* m_1 *such that* mL *is very ample, and* mq^*L *is* p*-very ample for* $m \geq m_1$*. On the other hand, since* p *is flat, semicontinuity implies that there exists* m_2 *such that* $\mathrm{H}^1(X, L^{\otimes m} \otimes I_C) = 0$ *for every cycle* C *of the family parametrized by* $\mathcal{C}X$ *and every* $m \geq m_2$*.*

Lemma 4.15 *Let (X, L) be a polarized pair, with X smooth, supporting an equalized $\mathbb{C}^*$-action. Assume that (X, L) is CPN with respect to the $\mathbb{C}^*$-action. Then we have an isomorphism $\mathrm{H}^0(\mathcal{C}X, \mathcal{E}) \simeq \mathrm{H}^0(X, L)$, and the map $q : \mathcal{U} \to X$ composed with the embedding $X \hookrightarrow \mathbb{P}(\mathrm{H}^0(X, L))$ factors via the morphism $\mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathrm{H}^0(X, L))$ induced by the evaluation of global sections. In other words, we have a commutative diagram:*

$$\begin{tikzcd}
\mathcal{C}X & \mathcal{U} \arrow[l, "p"'] \arrow[r, hook, "\imath"'] \arrow[d, "q"] & \mathbb{P}(\mathcal{E}) \arrow[ll, bend right] \arrow[d] \\
& X \arrow[r, hook] & \mathbb{P}(\mathrm{H}^0(X, L))
\end{tikzcd}$$

Proof Note first that $\mathrm{H}^0(\mathcal{C}X, \mathcal{E}) \simeq \mathrm{H}^0(\mathcal{C}X, p_* q^* L) \simeq \mathrm{H}^0(\mathcal{U}, q^* L)$. Since $q \colon \mathcal{U} \to X$ is birational (see Definition 4.1) and X is normal, by Zariski's Main Theorem q has connected fibers. Then we have an isomorphism $\mathrm{H}^0(\mathcal{U}, q^* L) \simeq \mathrm{H}^0(X, L)$, and so the evaluation of global sections induces a map $\mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathrm{H}^0(X, L))$, that obviously commutes with the arrows in the diagram of the statement. □

Corollary 4.16 *Let (X, L) be a polarized pair, with X smooth, supporting an equalized $\mathbb{C}^*$-action. Assume that (X, L) is CPN with respect to the $\mathbb{C}^*$-action. Let δ be the bandwidth of the induced action on $\mathbb{P}(\mathrm{H}^0(X, L))$. Then we have a morphism*

$$\gamma : \mathcal{C}X \longrightarrow \mathbb{G}(\delta, \mathbb{P}(\mathrm{H}^0(X, L))),$$

such that the pullback via γ of the Plücker line bundle is $\det(\mathcal{E}) = \bigotimes_{u=0}^{\delta} \mathcal{L}_u$.

Proof This follows from the fact that, by Lemma 4.9, the vector bundle $\mathcal{E}$ has rank $\delta + 1$ and it is globally generated by $\mathrm{H}^0(X, L)$. In particular, the composition of γ with the Plücker embedding of $\mathbb{G}(\delta, \mathbb{P}(\mathrm{H}^0(X, L))$ into $\mathbb{P}\big(\bigwedge^{\delta+1} \mathrm{H}^0(X, L)\big)$ is the morphism associated with the line bundle $\det(\mathcal{E})$. □

Remark 4.17 *Note that the image of γ is contained in the Segre variety $\Sigma(\mathrm{H}^0(X, L))$. In fact, under our assumptions, for every $u \in \{0, 1, \dots, \delta\}$ we have surjective maps $\mathrm{H}^0(X, L)_u \otimes \mathcal{O}_{\mathcal{C}X} \to \mathcal{L}_u$. Geometrically this means that for every point $c \in \mathcal{C}X$, the projective space $\mathbb{P}(\mathcal{E}_c)$ is the linear span of the points $\mathbb{P}(\mathcal{L}_{u,c}) \in \mathbb{P}(\mathrm{H}^0(X, L)_u)$. Then the claim follows by the definition of $\Sigma(\mathrm{H}^0(X, L))$ – see Equation (7).*

Proposition 4.18 *In the situation of Corollary 4.16, the map γ is the normalization of its image.*

Proof Since $\mathcal{C}X$ is the normalization of the Chow quotient $\overline{\mathcal{C}X} \subset \mathrm{Chow}(X)$, it is enough to show that the morphism γ factors via an injective map from $\overline{\mathcal{C}X}$ to $\Sigma(\mathrm{H}^0(X, L))$. In other words, we need to prove that different $\mathbb{C}^*$-invariant cycles (corresponding to different points in $\overline{\mathcal{C}X}$) have different linear spans.

Let $C, C' \in \overline{\mathcal{C}X}$ be two $\mathbb{C}^*$-invariant cycles, and denote by $C_1, \dots, C_t$, $C'_1, \dots, C'_{t'}$ their irreducible components, ordered by the weights of their extremal fixed points as in the proof of Lemma 4.9. Let $P_u, P'_u \in \mathbb{P}(\mathrm{H}^0(X, L)_u)$ be the intersections of the linear spans of C, C' with $\mathbb{P}(\mathrm{H}^0(X, L)_u)$, for every weight $u = 0, 1, \dots, \delta$.

Assume now that the linear spans of C, C' coincide; equivalently, $P_u = P'_u$ for every u. Note that the first irreducible component C_1 of C can be parametrized as $\sum_{u=0}^{k} t^u P_u$, $t \in \mathbb{C}$, in a neighborhood of P_0; in particular, its tangent line at P_0 is $\langle P_0, P_1 \rangle$. Since the same happens for C'_1, and every 1-dimensional orbit is completely determined by the tangent direction at the sink, by the Białynicki-Birula theorem, the fact that $\langle P_0, P_1 \rangle = \langle P'_0, P'_1 \rangle$ tells us that $C_1 = C'_1$. Recursively, we prove that $C_i = C'_i$ for every i, that is $C = C'$. □

5. Proof of the main statement

This section is devoted to the proof of Theorem 1.1. First of all we will show the existence of morphisms among the normalized Chow quotients $\mathcal{C}X_{i,j}$ fitting in the commutative diagram (2), then we will prove (Proposition 5.6) that the morphisms in the diagram are blowups, whose centers are described in Remark 5.10. We finish the section discussing the smoothness of $\mathcal{C}X$.

Throughout the section (X, L) will be a polarized pair, with X smooth, endowed with an equalized $\mathbb{C}^*$-action of criticality r such that $\operatorname{codim}(B^{\pm}(Y), X) > 1$ for every inner fixed point component $Y \in \mathcal{Y}^\circ$. For every small $\mathbb{Q}$-factorial modification $X_{i,j}$ as in Proposition 3.9 we choose a $\mathbb{Q}$-divisor $L_{i,j} = L(\tau_{i,j}^-, \tau_{i,j}^+) \in \operatorname{Int}(N_{i,j})$, which is ample on $X_{i,j}$.

Lemma 5.1 *There exists an integer m such that (X, mL) and every $(X_{i,j}, mL_{i,j})$ are CPN with respect to the $\mathbb{C}^*$-action. Moreover, for every $0 \leq i \leq j \leq r$ we have $\mathbb{C}^*$-equivariant isomorphisms:*

$$\mathrm{H}^0(X_{i,j}, mL_{i,j}) \simeq \mathrm{H}^0(X, mL)_{m\tau_{i,j}^-, m\tau_{i,j}^+}.$$

Proof The first part of the statement follows from Remark 4.14. Moreover, since $X_{i,j}$ and $X^\flat$ are isomorphic in codimension one, for m large enough and divisible, we have $\mathbb{C}^*$-equivariant isomorphisms $\mathrm{H}^0(X_{i,j}, mL_{i,j}) \to \mathrm{H}^0(X^\flat, mL_{i,j})$. Then we conclude the proof observing that, by (36, Lemma 2.5), we have $\mathbb{C}^*$-equivariant isomorphisms $\mathrm{H}^0(X^\flat, mL_{i,j}) \simeq \mathrm{H}^0(X, mL)_{m\tau_{i,j}^-, m\tau_{i,j}^+}$. $\square$

As a consequence of Lemma 5.1, replacing L with a sufficiently large and divisible multiple, we may work, without loss of generality, in the following:

Setup 5.2 (X, L) is a polarized pair, with X smooth, endowed with an equalized $\mathbb{C}^*$-action of criticality r such that $\operatorname{codim}(B^{\pm}(Y), X) > 1$ for every inner fixed point component $Y \in \mathcal{Y}^\circ$, $L_{i,j} \in \operatorname{Pic}(X_{i,j})$ ample, for $0 \leq i \leq j \leq r$ and the polarized pairs (X, L) and $(X_{i,j}, L_{i,j})$ are CPN with respect to the $\mathbb{C}^*$-action.

In particular, for $0 \leq i \leq j \leq r$, we have embeddings $X_{i,j} \hookrightarrow \mathbb{P}(\mathrm{H}^0(X_{i,j}, L_{i,j}))$ which fit in the commutative diagram:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, dashed] & \mathbb{P}(\mathrm{H}^0(X, L)) \arrow[d, dashed, "\phi_{i,j}"] \\ X_{i,j} \arrow[r, hook] & \mathbb{P}(\mathrm{H}^0(X_{i,j}, L_{i,j})) \end{tikzcd}$$

In this diagram $\phi_{i,j}$ denotes the linear projection induced by the monomorphism:

$$\mathrm{H}^0(X_{i,j}, L_{i,j}) \simeq \mathrm{H}^0(X, L)_{\tau_{i,j}^-, \tau_{i,j}^+} \hookrightarrow \mathrm{H}^0(X, L),$$

and the left-hand-side vertical map denotes the composition of the blowup map and the pruning map (see Section 3):

$$X \overset{\beta^{-1}}{\dashrightarrow} X^\flat \overset{\varphi_{i,j}}{\dashrightarrow} X_{i,j}$$

Abusing notation, we will denote this composition by $\phi_{i,j}$, as well. Let us set:

$$V := \mathrm{H}^0(X, L), \qquad V_{i,j} := \mathrm{H}^0(X_{i,j}, L_{i,j});$$

we will use the notation introduced in Section 2.8, regarding Grassmannians of linear spaces in $\mathbb{P}(V)$ and $\mathbb{P}(V_{i,j})$, and we will consider the normalized Chow quotients $\mathcal{C}X_{i,j}$ of the varieties $X_{i,j}$. Note first that:

$$\mathcal{C}X_{i,i+1} \simeq \mathcal{G}X_{i,i+1}, \quad \text{for every } i = 0, \dots, r-1.$$

In fact, by (4, Theorem 3.6, Remark 3.5), $X_{i,i+1}$ is a $\mathbb{P}^1$-bundle over the smooth geometric quotient $\mathcal{G}X_{i,i+1}$. Then $\mathcal{G}X_{i,i+1}$ is also the Chow quotient of $X_{i,i+1}$. Moreover, by Corollary 4.16, for every $i < j$ we have a morphism

$$\gamma_{i,j} : \mathcal{C}X_{i,j} \to \mathbb{G}(\delta_{i,j}, \mathbb{P}(V_{i,j})),$$

where $\delta_{i,j}$ denotes the bandwidth of the induced $\mathbb{C}^*$-action on $(X_{i,j}, L_{i,j})$. The image of the above map is contained in the Segre variety $\Sigma(V_{i,j})$ by Remark 4.17. The next Lemma shows that the maps γ, $\gamma_{i,j}$ are compatible with the projection $\pi_{i,j} \colon \Sigma(V) \to \Sigma(V_{i,j})$ induced by the projection $\phi_{i,j}$ (see Section 2.8) and the natural map sending invariant cycles in X to invariant cycles in $\mathcal{C}X_{i,j}$; moreover, we show that this map is a morphism.

Lemma 5.3 *Let (X, L) be as in Setup 5.2. For $0 \le i \le j \le r$, the birational map $\phi_{i,j} : X \dashrightarrow X_{i,j}$ induces a morphism $\rho_{i,j} : \mathcal{C}X \to \mathcal{C}X_{i,j}$ fitting in the commutative diagram:*

$$\begin{tikzcd}
\mathcal{C}X \arrow[r] \arrow[d, "\rho_{i,j}"'] & \Sigma(V) \arrow[d, "\pi_{i,j}"] \\
\mathcal{C}X_{i,j} \arrow[r] & \Sigma(V_{i,j})
\end{tikzcd}$$

where $\pi_{i,j}$ denotes the projection between Segre varieties induced by $\varphi_{i,j}$.

Proof Let us consider the composition $\pi_{i,j} \circ \gamma : \mathcal{C}X \to \Sigma(V_{i,j})$. Since the projection via $\phi_{i,j}$ of a general –irreducible– $\mathbb{C}^*$-invariant 1-cycle in X is an irreducible $\mathbb{C}^*$-invariant 1-cycle in $X_{i,j}$, it follows that $\pi_{i,j}(\gamma(\mathcal{C}X))$ is contained in $\gamma_{i,j}(\mathcal{C}X_{i,j})$. By Proposition 4.18 $\mathcal{C}X_{i,j}$ is the normalization of its image via $\gamma_{i,j}$, thus the map $\pi_{i,j} \circ \gamma$ factors via $\mathcal{C}X_{i,j}$, as stated. □

Remark 5.4 *Denoting by $\delta_{i,j}$ the bandwidth of the induced $\mathbb{C}^*$-action on $(X_{i,j}, L_{i,j})$, the restriction to $\Sigma(V_{i,j})$ of the Plücker embedding of the Grassmannian $\mathbb{G}(\delta_{i,j}, \mathrm{H}^0(X_{i,j}, L_{i,j}))$ is the Segre embedding of that variety, identified with $\prod_u \mathbb{P}(\mathrm{H}^0(X, L)_u)$ where $u = \tau^-_{i,j}, \dots, \tau^+_{i,j}$. Moreover, $\pi_{i,j} : \Sigma(V) \to \Sigma(V_{i,j})$ can be identified with the canonical projection*

$$\textstyle\prod_{u=0}^{\delta} \mathbb{P}(\mathrm{H}^0(X, L)_u) \to \prod_{u=\tau^-_{i,j}}^{\tau^+_{i,j}} \mathbb{P}(\mathrm{H}^0(X, L)_u).$$

Together with Lemma 5.3 and Remark 4.17, this implies that the morphism $\pi_{i,j} \circ \gamma : \mathcal{C}X \to \Sigma(V_{i,j})$, and subsequently its normalization $\rho_{i,j} : \mathcal{C}X \to \mathcal{C}X_{i,j}$, is given by the line bundle $\bigotimes_u \mathcal{L}_u$, where $u = \tau^-_{i,j}, \dots, \tau^+_{i,j}$.

Remark 5.5 *Note that, by Remark 4.7, $\mathcal{C}X = \mathcal{C}X^\flat$, and that $X_{0,r} = X^\flat$ by definition (see proof of Proposition 3.9), so $\mathcal{C}X = \mathcal{C}X_{0,r}$. Moreover the above argument can be applied to any modification $X_{i,j}$ (recall that the $X_{i,j}$ are smooth, by Proposition 3.9), obtaining morphisms*

$$\rho^{i,j}_{k,\ell} : \mathcal{C}X_{i,j} \longrightarrow \mathcal{C}X_{k,\ell},$$

whenever $i \le k \le \ell \le j$. Abusing notation, we will call these morphisms pruning maps*, and denote*

$$s := \rho^{i,j}_{i,j-1}, \qquad d := \rho^{i,j}_{i+1,j}$$

(called elementary pruning maps*). As in Remark 3.11, s and d commute, so every pruning map $\rho^{i,j}_{k,\ell}$ can be written as $\rho^{i,j}_{k,\ell} = s^{i-k} \circ d^{j-\ell}$.*

$$
\begin{tikzcd}[column sep=small]
& & & & & \mathcal{C}X = \mathcal{C}X_{0,r} \arrow[dl, "s"'] \arrow[dr, "d"] & & & & & \\
& & & & \mathcal{C}X_{0,r-1} \arrow[dl, "s"'] \arrow[dr, "d"] & & \mathcal{C}X_{1,r} \arrow[dl, "s"'] \arrow[dr, "d"] & & & & \\
& & & \mathcal{C}X_{0,r-2} & & \mathcal{C}X_{1,r-1} & & \mathcal{C}X_{2,r} & & & \\
& & \iddots & \ddots & \iddots & \cdots & \ddots & \iddots & \ddots & & \\
& \mathcal{G}X_{0,1} \arrow[dl] \arrow[dr] & & \mathcal{G}X_{1,2} \arrow[dl] & & \cdots & & \mathcal{G}X_{r-2,r-1} \arrow[dr] & & \mathcal{G}X_{r-1,r} \arrow[dl] \arrow[dr] & \\
\mathcal{G}X_{0,0} & & \mathcal{G}X_{1,1} & & \cdots & & \cdots & & \mathcal{G}X_{r-1,r-1} & & \mathcal{G}X_{r,r}
\end{tikzcd}
\tag{11}
$$

Then these morphisms fit into the commutative diagram (11), which extends Diagram (4).

Proposition 5.6 *Let (X, L) be as in Setup 5.2. Then for every $i < j$ the maps:*

$$
\begin{tikzcd}
& \mathcal{C}X_{i,j} \arrow[dl, "s"'] \arrow[dr, "d"] & \\
\mathcal{C}X_{i,j-1} & & \mathcal{C}X_{i+1,j}
\end{tikzcd}
$$

are normalizations of blowups.

The proof will be done in three steps. First of all, we will consider the case of an action of criticality $r = 2$:

Lemma 5.7 *In the situation of Proposition 5.6, assume furthermore that $r = 2$. Then $s : \mathcal{C}X_{0,2} \to \mathcal{C}X_{0,1} = \mathcal{G}X_{0,1}$, $d : \mathcal{C}X_{0,2} \to \mathcal{C}X_{1,2} = \mathcal{G}X_{1,2}$ are blowups along smooth centers which are the exceptional loci of the birational maps $\mathcal{G}X_{0,1} \to \mathcal{G}X_{1,1}$, $\mathcal{G}X_{1,2} \to \mathcal{G}X_{1,1}$.*

Proof We will prove the statement for s; the case of d is analogous. We consider the blowup $X^d_{0,2}$ of $X_{0,2}$ along (the strict transform of) B^+_1, and denote the sink of the action on $X^d_{0,2}$ by Y^d. By Remark 3.11 the variety Y^d is the smooth blowup of $\mathcal{G}X_{0,1} = \mathcal{C}X_{0,1}$ along the subset parametrizing $\mathbb{C}^*$-orbits in $X_{0,2}$ with source at an inner fixed point component of weight a_1. It is then enough to show that $Y^d \simeq \mathcal{C}X_{0,2}$.

We note first that, if $Y^d_1 \subset X^d_{0,2}$ is an inner fixed point component of $X^d_{0,2}$, mapping via b_d onto a fixed point component Y_1 of weight a_1, then $\nu^-(Y^d_1) = 1$. This implies that, given the closure of any orbit with source P at Y^d_1, there exists a unique $\mathbb{C}^*$-invariant curve with sink at P (Cf. (35, Lemma 2.2)). In particular the morphism $\mathcal{C}X^d_{0,2} \to Y^d = \mathcal{G}X^d_{0,1}$ is bijective. By the normality of $\mathcal{C}X^d_{0,2}$ and Y^d, it follows that this map is an isomorphism.

On the other hand, the blowup morphism $b_d : X^d_{0,2} \to X_{0,2}$ is $\mathbb{C}^*$-equivariant, hence it induces a surjective morphism $\mathcal{C}X^d_{0,2} \to \mathcal{C}X_{0,2}$. We claim that this map is also injective and, consequently, an isomorphism. The morphism b_d is an isomorphism outside of the union of $\mathbb{C}^*$-invariant curves linking Y^d to inner fixed points component of weight a_1. It is then enough to show that for every connected $\mathbb{C}^*$-invariant 1-cycle $C = C_1 + C_2 \subset X_{0,2}$, with C_1 linking the sink to a fixed point component Y_1 of weight a_1, and C_2 linking Y_1 to the source, there exists a unique $\mathbb{C}^*$-invariant connected 1-cycle $C' = C'_1 + C'_2$ in $X^d_{0,2}$ such that $b_d(C'_i) = C_i$. Note that C'_2 is necessarily the strict transform of C_2 into $X^d_{0,2}$, which is unique. Let Y^d_1 be the inner fixed point component of $X^d_{0,2}$ mapping via b_d onto Y_1. Then C'_1 and C_1 are uniquely determined by their tangent directions at their sources $P' \in Y_1^d \subset X_{0,2}^d$, $P = b_d(P') \in Y_1 \subset X_{0,2}$. Since the differential of b_d maps $\mathcal{N}^+(Y_1^d)_{P'}$ isomorphically to $\mathcal{N}^+(Y_1)_P$, the uniqueness of C_1' follows. $\square$

As a consequence, we get that, in the Diagram (11), all the pruning maps:

$$\begin{tikzcd}[column sep=small] & \mathcal{C}X_{0,2} \arrow[dl, "s"'] \arrow[dr, "d"] & & \mathcal{C}X_{1,3} \arrow[dl, "s"'] & & \dots & & \mathcal{C}X_{r-3,r-1} \arrow[dr, "d"] & & \mathcal{C}X_{r-2,r} \arrow[dl, "s"'] \arrow[dr, "d"] & \\ \mathcal{G}X_{0,1} & & \mathcal{G}X_{1,2} & & \dots & & \dots & & \mathcal{G}X_{r-2,r-1} & & \mathcal{G}X_{r-1,r} \end{tikzcd} \tag{12}$$

are smooth blowups. The next statement tells us that the rest of the maps in Diagram (11) are, up to normalization, pullbacks of these blowups.

Lemma 5.8 *In the situation of Proposition 5.6, assume that $r > 2$ and consider the following diagram:*

$$\begin{tikzcd} & \mathcal{C}X_{0,r} \arrow[dl, "s^{r-2}"'] \arrow[dr, "d"] & \\ \mathcal{C}X_{0,2} \arrow[dr, "d"'] & & \mathcal{C}X_{1,r} \arrow[dl, "s^{r-2}"] \\ & \mathcal{G}X_{1,2} & \end{tikzcd}$$

Then $\mathcal{C}X_{0,r}$ is the normalization of the fiber product of $\mathcal{C}X_{0,2}$ and $\mathcal{C}X_{1,r}$ over $\mathcal{G}X_{1,2}$.

Proof For simplicity, we will set $\zeta := s^{r-2}$. We will show that the natural map from $\mathcal{C}X_{0,r}$ to the fiber product of $\mathcal{C}X_{0,2}$ and $\mathcal{C}X_{1,r}$ over $\mathcal{G}X_{1,2}$ is a bijection by showing that, for every $c \in \mathcal{C}X_{1,r}$, ζ induces a bijection between $d^{-1}(c)$ and $d^{-1}(\zeta(c))$.

We will describe the fiber $d^{-1}(c)$, for an element $c \in \mathcal{C}X_{1,r}$. We choose a preimage $\gamma \in d^{-1}(c) \subset \mathcal{C}X_{0,r}$ and we denote the irreducible components of the corresponding 1-cycles C and Γ by $C_1, \dots, C_k \subset X_{1,r}$ and $\Gamma_1, \dots, \Gamma_l \subset X_{0,r}$, respectively, ordered by the weights of their extremal fixed points. In particular, Γ_1 meets the sink of $X_{0,r}$, and we denote by P its source.

Let us consider another element $\delta \in d^{-1}(c)$, which corresponds to a 1-cycle $\Delta = \Delta_1 + \dots + \Delta_m$, whose components are numbered as above. We now distinguish two cases, according to the weight $\mu_L(P)$.

If $\mu_L(P) > a_1$, then, by Lemma 5.3, we have $m = l = k$ and $\varphi_{1,r}(\Gamma_i) = \varphi_{1,r}(\Delta_i) = C_i$ for every $i = 1, \dots, k$. Since $\mu_L(P) > a_1$, then $\Gamma_2, \dots, \Gamma_k$ and $\Delta_2, \dots, \Delta_k$ are contained in the open set where $\varphi_{1,r}$ is an isomorphism, and so we conclude that $\Gamma_j = \Delta_j$ for every $j \geq 2$. In particular the source of Γ_1 is P and, since $\varphi_{0,r}$ is a $\mathbb{C}^*$-equivariant isomorphism in a neighborhood of this point, it follows that $\Gamma_1 = \Delta_1$. Therefore $\Gamma = D'$ and $\gamma = \delta$. In particular $d^{-1}(\zeta(c))$ is a point.

If $\mu_L(P) = a_1$, a similar argument shows that $l = m = k + 1$, and that $\Gamma_j = \Delta_j$ for $j \geq 2$. On the other hand, in this case Δ_1 can be any irreducible $\mathbb{C}^*$-invariant (not fixed) curve with source at P. Since the action is equalized, it follows that $d^{-1}(c)$ is bijective to the projective space $\mathbb{P}(\mathcal{N}^+(Y_1)_P^\vee)$ (see Remark 2.2). Since $\varphi_{0,2}$ is an equivariant isomorphism in a neighborhood of P, it follows that ζ maps $\mathbb{P}(\mathcal{N}^+(Y_1)_P^\vee)$ isomorphically onto $d^{-1}(\zeta(c))$. This completes the proof. $\square$

Proof of Proposition 5.6 The blowup of a variety along a closed subset is uniquely determined by its universal property, namely that the inverse image of the closed subset is a Cartier divisor, and that it is a final object with this property ((20, II, Proposition 7.14)). Moreover, it is known (see (17, Proposition IV-21, Lemma IV-41)), that the fiber product of two blowup maps $b_1 : M_1 \to M$, $b_2 : M_2 \to M$ of a variety M along two subvarieties Z_1, Z_2 equals the blowup of M_1 along $b_1^{-1}(Z_2)$ and the blowup of M_2 along $b_2^{-1}(Z_1)$.

In our case, the exceptional locus of $d : \mathcal{C}X_{0,r} \to \mathcal{C}X_{1,r}$ is the inverse image of the exceptional locus of $d : \mathcal{C}X_{0,2} \to \mathcal{C}X_{1,2}$, which is a Cartier divisor by Lemma 5.7. The universality then follows easily from Lemma 5.8, by the universal property of the Cartesian square and the normalization.

A similar argument tells us that also the map $s : \mathcal{C}X_{0,r} \to \mathcal{C}X_{0,r-1}$ is the normalization of a blowup of $\mathcal{C}X_{0,r-1}$, and that this map is the pullback to $\mathcal{C}X_{0,r-1}$ of $s : \mathcal{C}X_{r-2,r} \to \mathcal{C}X_{r-2,r-1}$. Applying this to the Chow quotients of $X_{i,j}$ we obtain that also the morphisms $s : \mathcal{C}X_{i,j} \to \mathcal{C}X_{i,j-1}$, $d : \mathcal{C}X_{i,j} \to \mathcal{C}X_{i+1,j}$ are normalizations of blowups, and that the two parallelograms in the diagram:

$$\begin{tikzcd}
 & & \mathcal{C}X_{i,j} \arrow[dl] \arrow[dr] & & \\
 & \mathcal{C}X_{i,j-1} \arrow[dl] \arrow[ddrr] & & \mathcal{C}X_{i+1,j} \arrow[ddll] \arrow[dr] & \\
\mathcal{C}X_{i,i+2} \arrow[dr] & & & & \mathcal{C}X_{j-2,j} \arrow[dl] \\
 & \mathcal{C}X_{i+1,i+2} & & \mathcal{C}X_{j-2,j-1} &
\end{tikzcd}$$

are normalizations of Cartesian squares. It easily follows that all the parallelograms (in particular all the rhombuses) in Diagram (11) are normalizations of Cartesian squares, as well. Now, since all the maps in Diagram (11) are constructed by successive normalizations of fiber products upon Diagram (12), and all the maps in Diagram (12) are normalizations of blowups, then all the maps in Diagram (11) are normalizations of blowups. □

The proof of Theorem 1.1 is thus completed. One of the consequences of the Theorem is that $\mathcal{C}X$ parametrizes all the $\mathbb{C}^*$-invariant connected cycles with dual diagram of type A with extremal fixed point components in the sink and the source of X:

Corollary 5.9 *Let (X, L) be a smooth polarized pair endowed with an equalized $\mathbb{C}^*$-action of criticality r, such that $\nu^{\pm}(Y) \geq 2$ for every fixed point component of weight a_i with $2 \leq i \leq r-2$. Let C be a $\mathbb{C}^*$-invariant 1-dimensional reduced subscheme with irreducible components $C_1, \dots, C_k$ satisfying that*

- *(C1) every C_j is the (smooth) closure of a 1-dimensional orbit;*
- *(C2) the singular points of C are the (transversal) intersections $P_j := C_j \cap C_{j+1}$, are fixed points of weights a_{i_j}, and $0 < i_1 < \dots < i_{k-1} < r$;*
- *(C3) every P_j is the source of C_j and the sink of C_{j+1}, for $j = 1, \dots, k-1$;*
- *(C4) the sink of C_1 and the source of C_k belong to Y_0, Y_r, respectively.*

Then C is an element of the family parametrized by $\mathcal{C}X$.

Proof Consider a subscheme $C = C_1 \cup \dots \cup C_k$ satisfying the properties (C1–C4), and denote by $a_{i_0} = a_0, a_{i_1}, \dots, a_{i_k} = a_r$ the weights of its fixed points $P_0, \dots, P_k$. Every irreducible component C_t is the closure of an orbit parametrized by the geometric quotients $\mathcal{G}X_{i_t,i_t+1}, \dots, \mathcal{G}X_{i_{t+1}-1,i_{t+1}}$; let us denote the corresponding points by $g_{i_t}, \dots, g_{i_{t+1}-1}$, respectively. In this way to the subscheme C we can associate elements $g_i \in \mathcal{G}X_{i-1,i}$ for every $i = 1, \dots, r$. Property (C2) implies that for every $i = 1, \dots, r-1$ the source of the orbit parametrized by g_i coincides with the sink of the orbit parametrized by g_{i+1}, hence the images of this two elements into $\mathcal{G}X_{i,i}$ are equal, and we may conclude that they have a common preimage in $\mathcal{G}X_{i-1,i+1}$. Recursively, using Theorem 1.1, we get that the g_i's have a common preimage $g \in \mathcal{C}X_{0,r} = \mathcal{C}X$; in other words the image of $g \in \mathcal{C}X$ into $\mathrm{Hilb}(X)$ parametrizes a subscheme supported on C; since this scheme must be reduced by Lemma 2.3, it is precisely C. This concludes the proof. □

Remark 5.10 *Let us describe here the blowups given by the pruning maps. For every weight a_k we will denote by $\overline{Y}_k$ the image in $\mathcal{G}X_{k,k}$ of the union of the fixed point components of weight a_k. For every $i < k < j$, we denote by $S_{i,k} \subset \mathcal{C}X_{i,k}$ the (scheme-theoretical) inverse image of $\overline{Y}_k$ in $\mathcal{C}X_{i,k}$ and by $D_{k,j} \subset \mathcal{C}X_{k,j}$ the inverse image of $\overline{Y}_k$ in $\mathcal{C}X_{k,j}$. In other words $S_{i,k}$ (resp. $D_{k,j}$) is the subvariety of $\mathcal{C}X_{i,k}$ (resp. $\mathcal{C}X_{k,j}$) parametrizing invariant 1-cycles with source (resp. sink) in the image in $X_{i,k}$ (resp. in $X_{k,j}$) of the union of the fixed point components of weight a_k.*

The exceptional divisor of the map $s : \mathcal{C}X_{i,j} \to \mathcal{C}X_{i,j-1}$ consists of the points parametrizing cycles meeting an inner fixed component of $X_{i,j}$ of weight a_{j-1}, hence the center of the blowup $s : \mathcal{C}X_{i,j} \to \mathcal{C}X_{i,j-1}$ is $S_{i,j-1}$; similarly the center of the blowup $d : \mathcal{C}X_{i,j} \to \mathcal{C}X_{i+1,j}$ is $D_{i+1,j}$.

$$
\begin{tikzcd}
 & & \mathcal{C}X_{i,j} \arrow[dl, "s"'] \arrow[dr, "d"] & & \\
S_{i,j-1} \arrow[r, hook] \arrow[ddr] & \mathcal{C}X_{i,j-1} \arrow[dr, "d"'] & & \mathcal{C}X_{i+1,j} \arrow[dl, "s"] & D_{i+1,j} \arrow[l, hook'] \arrow[ddl] \\
 & & \mathcal{C}X_{i+1,j-1} & & \\
 & S_{i+1,j-1} \arrow[ur, hook] & & D_{i+1,j-1} \arrow[ul, hook'] &
\end{tikzcd}
\tag{13}
$$

By definition $S_{i,j-1} = d^{-1}(S_{i+1,j-1})$ and $D_{i+1,j} = s^{-1}(D_{i+1,j-1})$; they will be smooth if $S_{i+1,j-1}$ and $D_{i+1,j-1}$ are smooth and have transversal intersection. This is not true in general, and so the Chow quotient can be singular. In the following section we will present in Example 6.1 an equalized action on a smooth variety whose Chow quotient is singular.

6. Smoothness of the Chow quotient

In this section we will discuss the smoothness of the normalized Chow quotient $\mathcal{C}X$. We will first show (Example 6.1) that $\mathcal{C}X$ can be singular, even in the case in which the variety X is smooth and the action is equalized, that we are considering in this paper. In order to guarantee the smoothness of $\mathcal{C}X$, one needs to impose some positivity assumptions on X; we will show the smoothness of X –together with the smoothness of some of its partial quotients $\mathcal{C}X_{i,j}$– in the case of convex varieties (Section 6.1).

Example 6.1 *On the projective line $\mathbb{P}^1$ with homogeneous coordinates $(x : y)$ set $0 := (1 : 0)$, $\infty := (0 : 1)$ and consider the smooth projective variety G defined as the blowup of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ along the points $(0, \infty, 0)$, $(0, 0, \infty)$. We denote by ℓ_-, ℓ_+ the strict transform in G of the lines $\{0\} \times \mathbb{P}^1 \times \{0\}$, $\{0\} \times \{0\} \times \mathbb{P}^1$, respectively, which meet at the strict transform of the point $(0, 0, 0)$. The variety G admits two Atiyah flips ψ_-, ψ_+, with centers at ℓ_-, ℓ_+, respectively:*

$$
\begin{tikzcd}
Y_- & G \arrow[l, dashed, "\psi_-"'] \arrow[r, dashed, "\psi_+"] & Y_+
\end{tikzcd}
$$

We then consider two ample divisors $L_\pm$ in $Y_\pm$, identify them with two movable divisors in G, and consider the bi-graded finitely generated $\mathbb{C}$-algebra:

$$R := \bigoplus_{a_\pm \geq 0} \mathrm{H}^0(G, a_- L_- + a_+ L_+).$$

Following (4), we may choose $L_\pm$ and an integer grading in R so that $X := \mathrm{Proj}(R)$ (together with its natural polarization) is a projective variety with a $\mathbb{C}^*$-action of criticality 3, and three geometric quotients:

$$\mathcal{G}X_{0,1} = Y_-, \quad \mathcal{G}X_{1,2} = G, \quad \mathcal{G}X_{2,3} = Y_+.$$

The varieties Y_-, G, Y_+ are smooth toric threefolds, and the birational transformations among them are compatible with the torus action; we have represented very ample polytopes of the three varieties in question in Figure 3. Then the construction of a bordism X can be done in a toric way. In fact, one can check that the

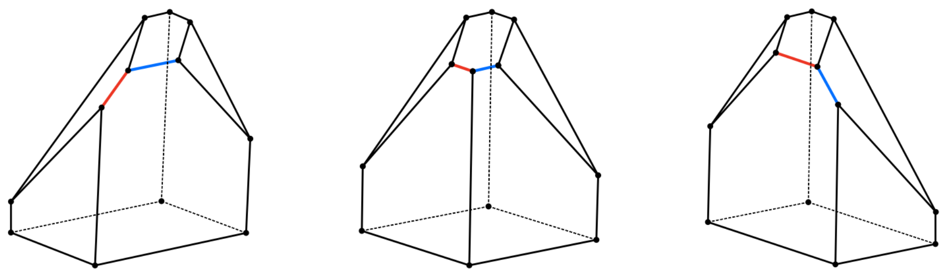

Fig. 3: Very ample polytopes of the varieties Y_-, G, Y_+.

polytope in $\mathbb{R}^4 = \mathbb{R} \otimes \mathbb{Z}^4$ whose vertices are the columns in the matrix below determines a smooth projective 4-dimensional variety X.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & -2 & -2 & -2 & -2 & -2 & -2 & -3 & -5 & 0 & 0 & 0 & 0 & 0 & -1 & -3 & -5 & -6 & -6 & -6 & -6 \\ 0 & 0 & -1 & -4 & -4 & 0 & 0 & 0 & -1 & -3 & -4 & -4 & -4 & -4 & 0 & 0 & -1 & -4 & -4 & 0 & -4 & -4 & 0 & 0 & -4 & -4 \\ 0 & -6 & 0 & -3 & -6 & 0 & -1 & -6 & 0 & 0 & -1 & -6 & 0 & 0 & 0 & -6 & 0 & -3 & -6 & 0 & 0 & 0 & -5 & -6 & -1 & -6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \end{pmatrix}$$

In the variety X we consider the $\mathbb{C}^*$-action corresponding to the fourth natural projection of the character lattice $\mathbb{Z}^4 \to \mathbb{Z}$. With respect to the embedding determined by the polytope, this action has bandwidth 4, the sink and the source are the varieties Y_-, Y_+, respectively, and we have two extra (inner) fixed points P_1, P_3 (of weights 1,3, respectively, corresponding to the vertices of the polytope highlighted in the matrix above). These two points correspond to the Atiyah flips $\psi_\pm$ in the following; the flip $\psi_-^{-1} : Y_- \to G$, for instance, substitutes the projective line parametrizing $\mathbb{C}^*$-orbits with source at P_1 (corresponding to tangent directions in $\mathcal{N}^+(P_1)$) by the projective line parametrizing $\mathbb{C}^*$-orbits with source at P_3 (corresponding to tangent directions in $\mathcal{N}^+(P_3)$). We have represented the $\mathbb{C}^*$-action on the variety X in Figure 4.

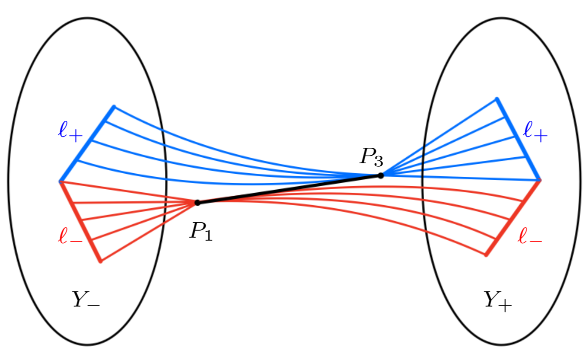

Fig. 4: The $\mathbb{C}^*$-action on the variety X.

Now we note that in the sink Y_- we have two distinguished rational curves, which are the intersections $B^+(P_1) \cap Y_-$ and $B^+(P_3) \cap Y_-$. Abusing notation, we denote them by ℓ_-, ℓ_+ (since ℓ_+ is the strict transform *of $\ell_+ \subset G$ and $\ell_- \in Y_-$ is the indeterminacy locus of $\psi_-^{-1} : Y_- \to G$). The intersection of $\ell_-, \ell_+ \in Y_-$ is a point.*

We claim that the Chow quotient of this action is not smooth. In fact, following the description presented in the proofs of Proposition 5.6 and Lemma 5.8, the Chow quotient $\mathcal{C}X$ may be constructed in two steps as follows:

- *The Chow quotient $\mathcal{C}X_{0,2}$ is the smooth blowup of Y_- along ℓ_-.*
- *The Chow quotient $\mathcal{C}X = \mathcal{C}X_{0,3}$ is the normalization of the blowup of $\mathcal{C}X_{0,2}$ along the inverse image of ℓ_+.*

Since the inverse image of ℓ_+ in $\mathcal{C}X_{0,2}$ consists of two $\mathbb{P}^1$'s meeting at a point, $\mathcal{C}X$ is not smooth.

6.1. The case of convex varieties

We finish this section by presenting a sufficient condition for the smoothness of the normalized Chow quotient $\mathcal{C}X$, which is the *convexity* of the variety X.

Definition 6.2 (19, 0.4) A smooth projective variety X is said to be *convex* if for every morphism $\mu : \mathbb{P}^1 \to X$, $H^1(\mathbb{P}^1, \mu^* T_X) = 0$, where T_X is the tangent bundle of X.

Remark 6.3 *Among smooth rationally connected varieties, the only known convex examples are rational homogeneous spaces G/P, where G is semisimple algebraic group and P is a parabolic subgroup (see (38)). Note, however, that we will use here only "$\mathbb{C}^*$-invariant convexity", i.e. that $H^1(C, T_{X|C}) = 0$, for closures C of 1-dimensional $\mathbb{C}^*$-orbits.*

We will now prove Theorem 1.2, using some ideas in (19; 31). The proof is divided in two steps; we will show first that the partial Chow quotients $\mathcal{C}X_{i,j}$, $i = 0$, or $j = r$, are smooth (Proposition 6.4), and then we will conclude by showing that the maps s, d among them are smooth blowups (Proposition 6.5).

Proposition 6.4 *Under the hypotheses of Theorem 1.1, assume moreover that X is convex. Then $\mathcal{C}X_{i,j}$ is smooth if $i = 0$ or $j = r$. In particular $\mathcal{C}X$ is smooth.*

Proof Let us prove the statement for the quotients $\mathcal{C}X_{0,j}$; the case of $\mathcal{C}X_{i,r}$ is analogous. Note that the variety $X_{0,j}$ is by definition the pruning $X(\tau_-, \tau_+)$ for $0 < \tau_- < a_1$, $a_{j-1} < \tau_+ < j$.

Let us consider the blowup of X along the sink Y_0 and the source Y_r of the action, $\beta : X^\flat \to X$, with exceptional divisors $Y_0^\flat$ and $Y_r^\flat$. The indeterminacy locus of the birational modification $X_{0,j} \dashrightarrow X^\flat$ is disjoint from $Y_0^\flat$, so there exists a smooth equivariant blowdown $X_{0,j} \to X'_{0,j}$ which contracts $Y_0^\flat$ to Y_0. Alternatively, one may describe $X'_{0,j}$ as the pruning $X(0, \tau_+)$. The Chow quotients $\mathcal{C}X_{0,j}$ and $\mathcal{C}X'_{0,j}$ are isomorphic (Cf. Remark 4.7), so we will show that $\mathcal{C}X'_{0,j}$ is smooth. To simplify notation, for the rest of the proof we will set $X' := X'_{0,j}$. The sink of the $\mathbb{C}^*$-action on X' is (isomorphic to) Y_0, and the source Y'_r is a divisor which is a birational modification of $Y_r^\flat$.

Let C be the closure of a 1-dimensional orbit of the action with sink and source $x_\pm$, and assume that $x_+ \in Y'_r$. Then the weights of the $\mathbb{C}^*$-action on T_{X',x_+} (respectively T_{X',x_-}) are all 0's and 1's (respectively 1's, 0's and (-1)'s). Then by (37, Lemma 2.16) we get that the degrees of the direct summands of $T_{X'|C}$ cannot be smaller than -1; thus the same holds for $\mathcal{N}_{C/X'}$, and we may conclude that

$$\mathrm{H}^1(C, T_{X'|C}) = \mathrm{H}^1(C, \mathcal{N}_{C/X'}) = 0. \tag{14}$$

Let us now denote by C_g the closure of the general orbit of the action in X', and by $[C_g]$ the corresponding point in $\mathrm{Hilb}(X')$; applying the above argument to C_g we get that $\mathrm{Hilb}(X')$ is smooth at $[C_g]$. In particular there exists a unique irreducible component of $\mathrm{Hilb}(X')$ containing it, that we denote here by $\mathcal{M}$. Note that $\mathcal{M}$ is a projective variety. Let us now consider the induced $\mathbb{C}^*$-action on $\mathcal{M}$; since $\mathcal{M}$ is smooth at $[C_g]$, by

(6, Theorem 2.5) there exists a unique fixed point component $H \subset \mathcal{M}$ containing $[C_g]$. We will show that $\mathcal{M}$ is smooth at every point of H, from which the smoothness of H will follow (again by (6, Theorem 2.5)).

By Remark 4.6, the normalization of H is $\mathcal{C}X'$; in particular, for every $[C] \in H$, the corresponding subscheme C is a reduced tree of smooth rational curves (Lemma 4.3). Furthermore, the intersection of two meeting components of C is transversal, since at every fixed point x' the positive and the negative part of $T_{X',x'}$ are transversal. In particular, the subscheme C is a local complete intersection, hence the conormal sheaf $\mathcal{I}_{C,X'}/\mathcal{I}^2_{C,X'}$ is locally free, and we have a short exact sequence:

$$0 \to \mathcal{I}_{C,X'}/\mathcal{I}^2_{C,X'} \longrightarrow \Omega_{X'|C} \longrightarrow \Omega_C \to 0$$

By applying the functor $\operatorname{Hom}(\_,\mathcal{O}_C)$, we get that $\operatorname{Ext}^1(\mathcal{I}_{C,X'}/\mathcal{I}^2_{C,X'},\mathcal{O}_C)$ is a quotient of $\operatorname{Ext}^1(\Omega_{X'|C},\mathcal{O}_C) = \mathrm{H}^1(C,T_{X'|C})$. To conclude that H is smooth at $[C]$ it is enough to show (see, for instance, (26, Theorem I.2.8)) that

$$\mathrm{H}^1(C,T_{X'|C}) = 0. \tag{15}$$

To prove (15) we write C as $C_1 \cup C_2$, with C_2 irreducible and meeting the source, and set $\{p\} = C_1 \cap C_2$. By (14), we know that $\mathrm{H}^1(C,T_{X'}|_{C_2}) = 0$, and so if $C = C_2$ the proof is finished. If else $C_1 \neq \emptyset$ we consider the exact sequence

$$0 \to T_{X'}|_{C_1}(-p) \longrightarrow T_{X'}|_C \longrightarrow T_{X'}|_{C_2} \to 0$$

Since X' is isomorphic to X in a neighborhood of C_1, and X is convex by assumption, we have $\mathrm{H}^1(C_1,T_{X'}|_{C_1}(-p)) = 0$ by (32) in the Proof of (19, Lemma 10), so (15) holds. □

Proposition 6.5 *Under the hypotheses of Proposition 6.4, we have a strong factorization:*

$$\begin{tikzcd}
\mathcal{G}X_{0,1} \arrow[rrrrrrr, dashed, bend left=15, "\psi"] & \mathcal{C}X_{0,2} \arrow[l, "s"'] & \cdots & \mathcal{C}X \arrow[l, "s"'] \arrow[r, "d"'] & \cdots & \mathcal{C}X_{r-2,r} \arrow[r, "d"'] & & \mathcal{G}X_{r-1,r}
\end{tikzcd}$$

Proof We know that the maps s, d above are normalizations of blowups (Proposition 5.6). Moreover the exceptional loci of these maps are divisors, whose connected components are irreducible (by Lemma 5.8 applied to the prunings $X_{i,j}$, $i = 0$ or $j = r$). Since for every component of the exceptional locus of each map s or d, the dimension of the fibers is constant, it follows by (2, Corollary 4.11) that the maps s, d are blowups with smooth centers. □

A natural problem that presents at this point is the full description of the Nef, Movable and Effective cones of $\mathcal{C}X$. Our results suggest that an answer can be given for the Chow quotients of a convex variety of Picard number one endowed with an equalized $\mathbb{C}^*$-action (see (36) for a description of these actions in the case of rational homogeneous spaces).

7. Final remarks

In Theorem 1.1 we assumed that $\operatorname{codim}(B^{\pm}(Y),X) > 1$ for every $Y \in \mathcal{Y}^{\circ}$, or equivalently, that the induced action on $X_{0,r}$ is a bordism. However, the pruning construction allows us to extend the statement to a broader setting. For instance, we may prove the following:

Corollary 7.1 *Let (X,L) be a polarized pair, with X smooth, endowed with an equalized nontrivial $\mathbb{C}^*$-action. Then the conclusions of Theorems 1.1 and 1.2 hold if the action satisfies the following condition:*

$$\operatorname{codim}(B^{\pm}(Y),X) > 1 \textit{ for every } Y \in \mathcal{Y} \textit{ of weight } a_i \textit{ with } 2 \leq i \leq r-2. \tag{$\star$}$$

Proof As already observed in Remark 5.5, $\mathcal{C}X = \mathcal{C}X_{0,r}$, so we can always assume that the action is of B-type. If Condition ($\star$) is fulfilled, by Theorem 1.1, we can assume that either $\operatorname{codim}(B^-(Y_1), X) = 1$, or $\operatorname{codim}(B^+(Y_{r-1}), X) = 1$. In the first case (the second is analogous) the arguments in the proof of Lemma 5.8 show that $\mathcal{C}X_{0,j} \simeq \mathcal{C}X_{1,j}$ for every $j \geq 1$. In particular $\mathcal{C}X_{0,r} \simeq \mathcal{C}X_{1,r}$ and we conclude observing that $X_{1,r}$ satisfies the hypotheses of Theorem 1.1. $\square$

Remark 7.2 *Note that Condition ($\star$) holds, for instance, in the case in which* $\operatorname{Pic}(X) \simeq \mathbb{Z}$ *and the extremal fixed point components of the action are isolated (see (37, Lemma 2.8 (2))). The $\mathbb{C}^*$-actions of this kind on rational homogeneous spaces have been described in (35, Section 3.4)). In particular, our result can be applied to the $\mathbb{C}^*$-actions whose Chow quotients are the classical spaces of complete collineations, complete quadrics and complete symplectic quadrics (see (43)). For these spaces a description of the Nef, Movable and Effective cones has been given in (29; 30; 12).*

Another relevant example in which Condition ($\star$) holds is the diagonal action on $X = (\mathbb{P}^1)^n$, whose Chow quotient $\mathcal{C}X$ has appeared extensively in the literature in different settings. For instance, it is isomorphic to a Losev–Manin moduli space of pointed stable curves of genus zero (see (28)), and its associated polytope (as a toric variety) is the permutahedron of order n, classically known and studied in combinatorics (cf. (39)). We will see that, for this example, all the intermediate Chow quotients $\mathcal{C}X_{i,j}$ are smooth and all the maps s and d appearing in Diagram 11 are blowups along smooth centers (or isomorphisms).

Example 7.3 *Let X be the projective variety $(\mathbb{P}^1)^n$. It is a toric variety with an action of the torus $\mathbb{T} := (\mathbb{C}^*)^n$. The associated fan in the lattice N of 1-parameter subgroups –with $\mathbb{Z}$-basis $\{e_1, \dots, e_n\}$– has $2n$ rays (generated by $\pm e_i$). It has 2^n maximal cones $\langle \pm e_1, \dots, \pm e_n \rangle$ (one cone for each sequence of signs $\pm$). We take the action associated to the 1-parameter subgroup $\nu = e_1 + \cdots + e_n$, denote it by $\mathbb{C}^*_\nu \subset \mathbb{T}$, and set $\mathbb{T}' := \mathbb{T}/\mathbb{C}^*_\nu$. Note that we have an induced action of $\mathbb{T}'$ on the GIT and Chow quotients of X by $\mathbb{C}^*_\nu$.*

We take the ample line bundle $\mathcal{O}_X(1, \dots, 1)$ and the linearization of the action of $\mathbb{T}$ such that the associated polytope Δ is the unit cube: If $\{e_1^*, \dots, e_n^*\}$ is the dual basis of $\{e_1, \dots, e_n\}$, then the vertices of the cube –which are associated to $\mathbb{T}$-fixed points– are the points of the form

$$e_I^* = \sum_{i \in I} e_i^*, \quad I \subseteq \{1, \dots n\}.$$

The $\mathbb{C}^*_\nu$-action is associated with a projection $M \otimes \mathbb{R} \to \mathbb{R}$, that we denote by ν as well. The criticality of the action is n, with critical values $0, \dots, n$. As described in Example 2.1, the semi-geometric quotient $\mathcal{G}X_{j,j}$ associated to the critical value $j \in \{0, \dots, n\}$ is the $(n-1)$-dimensional toric variety defined by the lattice polytope $\nu^{-1}(j) \cap \Delta$, associated with an ample line bundle $\mathcal{L}_{j,j} \in \operatorname{Pic}(\mathcal{G}X_{j,j})$ and a linearization of the $\mathbb{C}^*_\nu$-action. Note that the vertices of the polytope $\nu^{-1}(j) \cap \Delta$ are the elements of the form e_I^* with $|I| = j$.

Now we consider the $\mathbb{T}'$-action on $\mathcal{C}X = \mathcal{C}X_{0,n}$. Notice that the orbits of the $\mathbb{C}^*_\nu$-action on $X = X(\Delta)$ which are invariant with respect to the quotient torus $\mathbb{T}'$ correspond to the edges of the cube Δ. Therefore the $\mathbb{T}'$-fixed points of $\mathcal{C}X_{0,n}$ correspond to trees of $\mathbb{P}^1$'s associated to a sequence of consecutive edges of the cube linking $e_\emptyset^* = 0$ with $e_1^* + \cdots + e_n^*$; equivalently such a tree is determined by a sequence of cube vertices $v_0 = 0, v_1, \dots, v_n = e_1^* + \cdots + e_n^*$ such that there exists a (unique) permutation $\sigma \in S_n$ with

$$v_{n-i-1} = v_{n-i} - e^*_{\sigma(i)} \text{ for } i = 1, \dots, n.$$

Such a curve and its associated point in $\mathcal{C}X$ will be denoted by C_σ.

Let us now compute the weights of the $\mathbb{T}'$-action on $\mathcal{C}X$, with respect to a particular ample line bundle $\mathcal{L}$, defined as follows; abusing notation, we denote by $\mathcal{L}_{j,j} \in \operatorname{Pic}(\mathcal{C}X)$ the pullback of $\mathcal{L}_{j,j} \in \operatorname{Pic}(\mathcal{G}X_{j,j})$, and set:

$$\mathcal{L} := \mathcal{L}_{0,0} \otimes \cdots \otimes \mathcal{L}_{n,n} \in \operatorname{Pic}(\mathcal{C}X).$$

As Lemma 7.4 below shows, $\mathcal{L}$ is ample on $\mathcal{C}X$. Combining the $\mathbb{T}'$-linearizations on the bundles $\mathcal{L}_{j,j}$ (and their presentations as sections of the unit cube) we get a $\mathbb{T}'$-linearization on $\mathcal{L}$. In particular, we get a *linearization of the action of* $\mathbb{T}'$ *on* $\mathcal{L}$*, and the weight of the* $\mathbb{T}'$*-action with respect to* $\mathcal{L}$ *on the fixed point* C_σ *is*

$$v_n + (v_n - e^*_{\sigma(1)}) + \cdots + (v_n - (e^*_{\sigma(1)} + \cdots + e^*_{\sigma(n-1)})) = \sum_{i=1}^{n} i \cdot e^*_{\sigma(i)} \in \mathrm{M}(\mathbb{T}') \subset M.$$

This shows that the polytope associated to the pair $(\mathcal{C}X, \mathcal{L})$ *is the permutahedron of order* n *(see (46, 9.2)).*

Lemma 7.4 *Let* (X, L) *be a polarized pair, with* X *smooth, endowed with an equalized nontrivial* $\mathbb{C}^*$*-action, satisfying Condition* $(\star)$*. Let* $\mathcal{L}_{i,i+1} \in \mathrm{Pic}(\mathcal{C}X)$ *(resp.* $\mathcal{L}_{i,i} \in \mathrm{Pic}(\mathcal{C}X)$*) be pullbacks of ample line bundles on* $\mathcal{G}X_{i,i+1}$*, (resp. on* $\mathcal{G}X_{i,i}$*). Then the line bundles*

$$\mathcal{L}_{0,1} \otimes \cdots \otimes \mathcal{L}_{r-1,r}, \qquad \mathcal{L}_{0,0} \otimes \cdots \otimes \mathcal{L}_{r,r}$$

are ample on $\mathcal{C}X = \mathcal{C}X_{0,r}$*.*

Proof To prove that $\mathcal{L}_{0,1} \otimes \cdots \otimes \mathcal{L}_{r-1,r}$ is ample it is enough to show that the product morphism $\mathcal{C}X_{0,r} \to \Pi_i \mathcal{G}X_{i,i+1}$ is finite. This morphism maps a point $c \in \mathcal{C}X_{0,r}$ parametrizing a cycle C whose irreducible components $C_1, \ldots, C_s$ are closures of orbits, to a point of the form $([C_{j_0}], \ldots, [C_{j_{r-1}}])$, for certain $j_0, \ldots, j_s \in \{0, \ldots, r\}$. By definition C_{j_i} is the (unique) irreducible component of C such that the weights of L at its sink and source are smaller than or equal to a_i, and bigger than or equal to a_{i+1}, respectively.

Therefore two cycles in the inverse image of a point c' in the image of $\mathcal{C}X_{0,r} \to \Pi_i \mathcal{G}X_{i,i+1}$ have the same irreducible components; by Lemma 4.3 they have the same image in the Chow quotient $\overline{\mathcal{C}X} \subset \mathrm{Chow}(X)$. Since $\mathcal{C}X$ is the normalization of $\overline{\mathcal{C}X}$, the result follows.

The ampleness of $\mathcal{L}_{0,0} \otimes \cdots \otimes \mathcal{L}_{r,r}$ will follow from the first part if we show that the product morphisms $\mathcal{G}X_{i,i+1} \to \mathcal{G}X_{i,i} \times \mathcal{G}X_{i+1,i+1}$ are finite, for every i. Let us notice that two elements $[C]$ and $[C']$ of $\mathcal{G}X_{i,i+1}$ are mapped to the same element of $\mathcal{G}X_{i,i}$ (resp. $\mathcal{G}X_{i+1,i+1}$) if and only if they have the same sink, with L-weight a_i (resp. the same source, with L-weight a_{i+1}). Assume by contradiction that for some i there is a curve $B \subset \mathcal{G}X_{i+i+1}$ which is contracted by $\mathcal{G}X_{i,i+1} \to \mathcal{G}X_{i,i} \times \mathcal{G}X_{i+1,i+1}$. Then there is a one dimensional family $\{C_b, b \in B\}$ of closures of orbits with the same sink of L-weight a_i and the same source of L-weight a_{i+1}, hence, by Mori's Bend-and-Break (see (13, Proposition 3.2)) a reducible or non reduced $\mathbb{C}^*$-invariant cycle with sink of L-weight a_i and source of L-weight a_{i+1}, a contradiction. □

It is known (see (14, 1.1)) that the Chow quotient $\mathcal{C}X$ of $X = (\mathbb{P}^1)^n$ by the diagonal $\mathbb{C}^*$-action is smooth, and that it is isomorphic to the Losev-Manin space $\overline{L}_n$ (see (28)), hence it can be constructed starting from $\mathbb{P}^{n-1}$ (which is a GIT quotient of X) via a sequence of blowups along smooth centers (see (21, 6.1)). We will now show that also all the intermediate Chow quotients $\mathcal{C}X_{i,j}$ are smooth, and that the maps s, d among them are smooth blowups. In particular, $\mathcal{C}X$ can be constructed upon any geometric quotient $\mathcal{G}X_{i,i+1}$ of X by a certain sequence of smooth blowups.

Proposition 7.5 *For the action described in Example 7.3, all the intermediate Chow quotients* $\mathcal{C}X_{i,j}$ *are smooth and all the maps* s *and* d *appearing in Diagram (11) are blowups along smooth centers or isomorphisms.*

Proof In order to use inductive arguments, let us set $X^n := (\mathbb{P}^1)^n$. After blowing up the sink and the source of the action on X^n we have a B-type action on $X^n_{0,n}$ which satisfies Condition $(\star)$; as observed at the beginning of the section we have $\mathcal{C}X^n_{0,j} \simeq \mathcal{C}X^n_{1,j}$ for $j \geq 1$ and $\mathcal{C}X^n_{i,n-1} \simeq \mathcal{C}X^n_{i,n}$ for $i \leq n-1$ and $X^n_{1,n-1}$ satisfies the assumptions of Theorem 1.1.

Moreove we know, by Lemma 5.7, that $s : \mathcal{C}X^n_{i,i+2} \to \mathcal{G}X^n_{i,i+1}$ and $d : \mathcal{C}X^n_{i,i+2} \to \mathcal{G}X^n_{i+1,i+2}$ are blowups of smooth varieties along smooth centers. We will then consider only the "inner" triangle:

$$\begin{tikzcd}[column sep=small]
 & & & & \mathcal{C}X^n_{1,n-1} \arrow[dl, "s"'] \arrow[dr, "d"] & & & & \\
 & & & \mathcal{C}X^n_{1,n-2} \arrow[dl, "s"'] \arrow[dr, "d"] & & \mathcal{C}X^n_{2,n-1} \arrow[dl, "s"'] \arrow[dr, "d"] & & & \\
 & & \mathcal{C}X^n_{1,n-3} \arrow[dl, "s"'] & & \mathcal{C}X^n_{2,n-2} & & \mathcal{C}X^n_{3,n-1} \arrow[dr, "d"] & & \\
 & \arrow[dl, dotted, no head] & & & & & & \arrow[dr, dotted, no head] & \\
\mathcal{C}X^n_{1,3} & & \cdots & & \cdots & & \cdots & & \mathcal{C}X^n_{n-3,n-1}
\end{tikzcd}$$

Since we know that the varieties in the bottom row are smooth, and that the maps s, d are blowups, it will be enough to prove that the centers are smooth varieties. Note that, for $n = 4$ the "inner" triangle consists only of the smooth element $\mathcal{C}X^4_{1,3}$, so the statement is true. We will now assume that the statement holds for X^l, with $l \le n-1$, and prove it for X^n.

To describe the centers we start by describing the closure of the BB-cells of the action. Let x_k be a fixed point of weight k; then $B^+(x_k) \simeq X^k, B^-(x_k) \simeq X^{n-k}$. For every integer k such that $i \le k \le j$, the image of $B^\pm(x_k)$ in the pruning $X^n_{i,j}$ $(i \le k \le j)$ is the pruning of $B^\pm(x_k)$, that is:

$$\varphi_{i,j}(B^+(x_k)) \simeq X^k_{i,k}, \qquad \varphi_{i,j}(B^-(x_k)) \simeq X^{n-k}_{k,j}.$$

Now we observe that, with the notation of Remark 5.10, $S_{i,k}$ is the Chow quotient of the (disjoint) union over the fixed points of weight k of $\varphi_{i,k}(B^+(x_k)) \simeq X^k_{i,k}$ and, similarly, $D_{k,j}$ is the Chow quotient of the (disjoint) union over the fixed points of weight k of $\varphi_{k,j}(B^-(x_k)) \simeq X^k_{k,j}$, so the center of the blowup $s : \mathcal{C}X^n_{i,j} \to \mathcal{C}X^n_{i,j-1}$ is a disjoint union of varieties isomorphic to $\mathcal{C}X^{j-1}_{i,j-1}$, and the center of the blowup $d : \mathcal{C}X^n_{i,j} \to \mathcal{C}X^n_{i+1,j}$ is a disjoint union of varieties isomorphic to $\mathcal{C}X^{i+1}_{i+1,j}$. By the inductive assumption, since $1 \le i < j \le n-1$, the centers of the blowups are smooth and the proof is concluded. $\square$

Remark 7.6 *The geometric quotient $\mathcal{G}X^n_{1,2}$ is the sink of $X^n_{1,2}$, which is the projective space $\mathbb{P}^{n-1}$ blown up at n points. These points are the intersection of the exceptional divisor of the blow up of X^n along the sink with the varieties $B^+(x_1)$, where x_1 is a fixed point of weight one. It can be shown, recursively, that the blowups*

$$\mathcal{C}X = \mathcal{C}X^n_{1,n-1} \to \mathcal{C}X^n_{1,n-2} \to \cdots \to \mathcal{C}X^n_{1,3} \to \mathcal{G}X^n_{1,2}$$

are the ones giving the Losev–Manin space $\overline{L}_n$ in (21, Section 6.1).

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