A note on the maximal rank

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Abstract
We give an upper-bound for the X-rank of points with respect to a non-degenerate irreducible variety X in the case that sub-generic X-rank points generate a hypersurface.

Keywords Secant variety · Tensor rank · Maximum rank

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1 Introduction

All along the paper we will always work with an algebraically closed field $\mathbb{K}$ of characteristic 0 and a projective variety $X \subset \mathbb{P}^N$ which will be always assumed to be irreducible and non-degenerate. For a given point $P \in \mathbb{P}^N$ there is a well-defined notion of X-rank $r_X(P)$ of P which is the least number of points of X whose span contains P. Such a notion, before becoming part of the algebraic geometric language as X-rank (referring to the underlined variety X, cf. [4]), was previously used in the context of tensors (i.e. when X parametrizes particular type of tensors) and better known in the applied world as *structured rank* putting the accent on the particular structure of the tensors (cf. e.g. [17]). From the applied point of view the knowledge of the maximal possible X-rank that an element in $\langle X \rangle$ may have (e.g. [2, 20–22]) turns out to be extremely important. This raises a very interesting pure mathematical problem: being able to give a sharp upper bound on the maximal X-rank $r_{\max}$. One natural bound over an algebraically closed field of characteristic 0 is given by the

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codimension, i.e. $r_{\max} \leqslant \operatorname{codim} X + 1$ (cf. e.g. [25, Proposition 5.1, p. 348]). The next important result valid for any irreducible non-degenerate variety X is presented in [11]. Let g be the so-called *generic X-rank*, i.e., the first integer such that the Zariski closure of the set of points of rank smaller than or equal to g fills $\langle X \rangle$ (this is again a very much studied value, nowadays there are also numerical algorithms to compute it in certain cases, cf. e.g. [8, 23]). Then [11, Theorem 1, p. 1022] shows that $r_{\max} \leqslant 2g$. In the case in which X is not a hypersurface but the points of subgeneric rank generate a hypersurface, [11, Theorem 6, p. 1024] proved that $r_{\max} \leqslant 2g - 1$.

In [14, Theorems 3.7 and 3.9, p. 118] this last bound for the hypersurfaces cases is improved to $r_{\max} \leqslant 2g - 2$ in the special setting of X being either a curve or a homogeneous variety.

There are some other bounds worth noting in the case of maximum *symmetric* rank, i.e. the X-rank when X is a Veronese variety. The first two are due to Jelisiejew [24, Corollary 6, p. 331] and to Ballico–De Paris [5, Remark 4.18, p. 913], which both obtain a bound on the *open symmetric rank*, a higher notion of rank which is always greater than or equal to the symmetric one (we will recall them in (2) and (3) respectively). Eventually there is also a bound given by [19, Proposition 3.3, p. 28] (see (4)) for Veronese surfaces. An asymptotic bound is also presented.

In this note, we focus on the case of X being a variety such that the Zariski closure of the points of subgeneric rank $\sigma_{g-1}(X)$ is a hypersurface. In Theorem 2.4 we will show that the bound for $r_{\max}$ can be reduced to

$$r_{\max} \leqslant r_{\max, g-1} + 1,$$

where $r_{\max, g-1}$ is the maximum X-rank attained on the variety $\sigma_{g-1}(X)$.

We end the paper by comparing our bound with the existing ones highlighting the cases where our bound gives more accurate estimates than the known ones.

2 Notation and main result

Definition 2.1 Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate projective variety. The Zariski closure of the set $\sigma_s^0(X)$ of points of $\mathbb{P}^N$ of X-rank at most s is an irreducible projective variety called the *s-th secant variety of X* and denoted by $\sigma_s(X)$.

Secant varieties are nested and there exists an integer g such that $\sigma_g(X)$ fills the ambient space:

$$X \subset \sigma_2(X) \subset \cdots \subset \sigma_i(X) \subset \cdots \subset \sigma_g(X) = \mathbb{P}^N.$$

Definition 2.2 Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate projective variety. The least integer g such that $\sigma_g(X) = \mathbb{P}^N$ is the *generic X-rank*.

The generic X-rank may not coincide with the maximum X-rank appearing in $\mathbb{P}^N$. There are cases in which the generic is the same as the maximum X-rank and cases in which there exist points with X-rank greater than the generic one, for example points on tangent lines of a rational normal curve of degree $d > 2$ (cf. [31]). Hence one may

seek for a bound for the maximum rank and one may try to see if it is attained or not. When $\mathbb{K}$ is not algebraically closed there are different notions of X-rank, cf. [12].

Proposition 2.3 *Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate projective variety of dimension n and let W be a hypersurface strictly containing X. Let $P \in \mathbb{P}^N \setminus W$ be such that $r_X(P) \neq 2$, and let*

$$Y_P := \text{cone}(P, X) = \bigcup_{x \in X} \langle x, P \rangle.$$

Then $Y_P \cap W$ is reducible of dimension n. Moreover there exists a line in Y_P through P that meets X in one point only and $W \setminus X$ in at least one other point.

Proof The cone Y_P is irreducible of dimension $n + 1$, then the components of the intersection $Y_P \cap W$ have dimension at least $n + 1 + N - 1 - N = n = \dim(X)$ (cf. [30, Theorem 1.24, p. 75]). Actually, since Y_P is not contained in W, those components have dimension exactly n. Clearly $X \subset Y_P \cap W$. Assume for the sake of contradiction that $Y_P \cap W = X$. If this is the case, then every line contained in Y_P and passing through P meets W only on X and moreover such intersection is made by one point only, otherwise the point P would have X-rank 2 which is against our hypothesis. Now we show that the fact that every line $l \subset Y_P$ through P meets W only in one point of X, i.e.:

$$l \cap W = Q \in X$$

leads to a contradiction.

Since W is a hypersurface, it is cut out by a single homogeneous equation of degree $d > 1$, $W = V(f)$. As just shown every line $\overline{PQ} \subset Y_P$, with $Q \in X$, meets W only in Q. We can parametrize $\overline{PQ}$ as

$$\overline{PQ} = \{sP + tQ : [s : t] \in \mathbb{P}^1\} \cong \mathbb{P}^1$$

where in this notation the point Q is represented by the point $[0 : 1] \in \mathbb{P}^1$. Substituting the coordinates $sP + tQ$ of $\overline{PQ}$ in the equation f of the hypersurface we get a homogeneous equation of degree d in two variables which must vanish only at the point Q, i.e. we have an equation of the form

$$f(sP + tQ) = k_Q s^d \tag{1}$$

for some constant $k_Q \in \mathbb{K}$. We show that if this happens for every line $\overline{PQ}$, with $Q \in X$, then we will get a contradiction. Indeed, suppose that f can be written as the polynomial

$$f(x_0, \ldots, x_N) = \sum_{\substack{(b_0, \ldots, b_N) \in \mathbb{N}^{N+1} \\ b_0 + \cdots + b_N = d}} a_{(b_0, \ldots, b_N)} x_0^{b_0} \ldots x_N^{b_N}.$$

The coefficient of the monomial $s^{d-1}t$ after the substitution (1) has to be zero. Moreover it turns out to be the directional derivative of f at P in the direction of Q:

$$
\begin{aligned}
Df_P(Q) = a_{(b_0,\dots,b_N)} \big[& b_0 \cdot p_0^{b_0-1} p_1^{b_1} \dots p_N^{b_N} \cdot q_0 \\
& + b_1 \cdot p_0^{b_0} p_1^{b_1-1} p_2^{b_2} \dots p_N^{b_N} \cdot q_1 \\
& + \dots + b_N \cdot p_0^{b_0} \dots p_{N-1}^{b_{N-1}} p_N^{b_N-1} \cdot q_N \big].
\end{aligned}
$$

Since for a fixed P the form Df_P is linear in Q, X is contained in the hyperplane $Df_P = 0$, which is non-trivial by the Euler formula and the fact that the evaluation of f at P is different from zero. This is in contradiction with non-degeneracy hypothesis on X. Hence there exists a line l inside Y_P containing P that intersects W in at least another point in $W \setminus X$. □

Theorem 2.4 *Let $X \subset \mathbb{P}^N$ be a smooth non-degenerate projective variety of dimension n and let g be the generic X-rank. If $\sigma_{g-1}(X)$ is a hypersurface, $g > 2$, then*

$$
r_{\max} \leqslant r_{\max,g-1} + 1
$$

where $r_{\max,g-1}$ is the maximum X-rank achieved on $\sigma_{g-1}(X)$.

Proof Let $P \in \mathbb{P}^N \setminus \sigma_{g-1}(X)$ and $Y_P := \mathrm{cone}(P, X) = \bigcup_{x \in X} \langle x, P \rangle$ as in Proposition 2.3 with $W = \sigma_{g-1}(X)$, where we have shown that there exists a line of Y_P through P which intersects $\sigma_{g-1}(X)$ in at least two distinct points, say Q_1, Q_2 such that $Q_1 \in X$ and $Q_2 \in \sigma_{g-1}(X) \setminus X$. Therefore $P \in \langle Q_1, Q_2 \rangle$. If $Q_2 \in \sigma_{g-1}^0(X)$ then $r_X(P) \leqslant g$, while if the components of $Y_P \cap \sigma_{g-1}(X)$ different from X are all contained in $\sigma_{g-1}(X) \setminus \sigma_{g-1}^0(X)$ we can only say that $r_X(P) \leqslant r_{\max,g-1} + 1$. □

We would like to point out some interesting consequences of this result.

Remark 2.5 Suppose that $X \subset \mathbb{P}^N$ is a non-degenerate irreducible variety such that the last non-filling secant variety $\sigma_{g-1}(X)$ is hypersurface, with $g \neq 2$, and let $P \in \mathbb{P}^N \setminus \sigma_{g-1}(X)$. The intersection $Y_P \cap \sigma_{g-1}(X)$ cannot be contained in $\sigma_{g-2}^0(X)$ otherwise the point $P \in \mathbb{P}^N \setminus \sigma_{g-1}(X)$ must lie in $\sigma_{g-1}(X)$ which is impossible.

Remark 2.6 Suppose that $X \subset \mathbb{P}^N$ is a non-degenerate irreducible variety such that the last non-filling secant variety $\sigma_{g-1}(X)$ is a hypersurface, with $g \neq 2$, and let $P \in \mathbb{P}^N \setminus \sigma_{g-1}(X)$. The rank of the point $P \in \mathbb{P}^N \setminus \sigma_{g-1}(X)$ is g if and only if the intersection $Y_P \cap \sigma_{g-1}(X)$ contains at least one point of $\sigma_{g-1}^0(X)$ but we were not able to distinguish whether there exist points P for which $(Y_P \cap \sigma_{g-1}(X)) \subset (\sigma_{g-1}(X) \setminus \sigma_{g-1}^0(X))$. Of course if P is generic it is obvious that $Y_P \cap \sigma_{g-1}^0(X) \neq \varnothing$, and indeed the generic rank is g.

This leads us to the following conjecture.

Conjecture 2.7 *Let $X \subset \mathbb{P}^N$ be a smooth non-degenerate projective variety and let $g > 2$ be the generic X-rank. If $\sigma_{g-1}(X)$ is a hypersurface, then*

$$
r_{\max} = \max \{r_{\max,g-1}, g\}
$$

where $r_{\max,g-1}$ is the maximum X-rank achieved on $\sigma_{g-1}(X)$.

Table 1 Comparison of bounds on the maximum Waring rank when a secant variety of a Veronese variety is a hypersurface and where the maximum rank is known. See [9, Theorem 40, p. 48, and Theorem 44, p. 50] and [18]

	$2g-1$ in [11]	$2g-2$ [14]	[19]	[5]	[24]	Our bound	$r_{\max}$
$X_{2,2}$	5	4	4			3	3
$X_{2,3}$	7	6	7		9	6	5
$X_{2,4}$	11	10	10	17	18	8	7

3 Comparison

In this section we compare our bound with the existing ones on some known examples.

The most studied case is the one of symmetric tensors where the rank is with respect to a Veronese variety. Let $X_{n,d}$ be the Veronese embedding of $\mathbb{P}^n$ via $\mathcal{O}(d)$. To the best of our knowledge, some of the best upper bounds for the Waring rank are due to Jelisiejew [24, Corollary 6, p. 331] and to Ballico–De Paris [5, Remark 4.18, p. 913]. In both these works, the bounds are given on a different notion of rank known as *open Waring rank*, we refer to [24, Definition 2, p. 330] for a definition. Since the open Waring rank is always greater than or equal to the usual Waring rank, from [24] and [5] one gets these bounds on the maximum symmetric rank

$$r_{\max} \leqslant \binom{n+d-1}{n} - \binom{n+d-5}{n-2} \tag{2}$$

for $n \geqslant 2, d \geqslant 3$, and

$$r_{\max} \leqslant \binom{n+d-1}{n} - \binom{n+d-5}{n-2} - \binom{n+d-6}{n-2} \tag{3}$$

for $n \geqslant 2, d \geqslant 4$, respectively. Another known bound is given by [19, Proposition 3.3, p. 28] for all homogeneous polynomials of degree d in three variables. In this case [19] shows that

$$r_{\max} \leqslant \left\lfloor \frac{d^2 + 6d + 1}{4} \right\rfloor. \tag{4}$$

Eventually, consider any Veronese variety $X_{n,d}$. As pointed out in [11], it is worth noting that the bound which they give $r_{\max} \leqslant 2g$ in the general case is asymptotically better than (2) and (3), even though these last ones are better for small cases. See Table 1 for a comparison between all the bounds on the maximum symmetric rank.

For the skew-symmetric tensors case one has to study the rank with respect to Grassmann varieties $\mathrm{Gr}(\mathbb{P}^k, \mathbb{P}^n) \subset \mathbb{P}(\wedge^{k+1} V)$, $2k \leqslant n - 1$. We checked among $1 \leqslant n, r \leqslant 500$ and, if the conjecture on defectiveness of secant varieties of Grassmannians holds (cf. [6, 16], see also [1, 10, 15, 26]), we found only three cases in which there exists an r such that $\sigma_r(X)$ is a hypersurface: $\mathrm{Gr}(\mathbb{P}^1, \mathbb{P}^3)$ for $r = 1$ (but in such a case our theorem does not apply), $\mathrm{Gr}(\mathbb{P}^2, \mathbb{P}^6)$ for $r = 3$, $\mathrm{Gr}(\mathbb{P}^7, \mathbb{P}^{16})$ for $r = 333$.

The case of $\mathrm{Gr}(\mathbb{P}^1, \mathbb{P}^3)$ is trivial since elements of $\wedge^2 \mathbb{C}^4$ are skew-symmetric matrices. The second example is a defective case, the well-known $\sigma_3(\mathrm{Gr}(\mathbb{P}^2, \mathbb{P}^6))$ (cf. [29]

and [1, Section 5]). The maximum skew-symmetric rank of a point belonging to $\sigma_3(\mathrm{Gr}(\mathbb{P}^2, \mathbb{P}^6))$ is 3, so our main theorem shows that the maximum skew-symmetric rank of a point in $\mathbb{P}(\wedge^3 \mathbb{C}^7)$ is the generic one, i.e. 4. Indeed in [3] it is shown that the maximum rank is actually 4. This is an example that shows the sharpness of our result (remark that [11, Theorem 6, p. 3] in this case provides a bound of 7, while [14, Theorem 3.9, p. 118] gives 6).

For the case of Segre variety we highlight only the example of X being the Segre of three copies of $\mathbb{P}^2$ where the maximum rank is known to be 5 (cf. [13, Theorem 5.1, p. 412], [27, Theorem 4, p. 815]) and our Theorem 2.4 gives a bound of 6; this is not sharp but it is better than [11, Theorem 6, p. 3] which gives $r_{\max} \leqslant 9$. The reason why we highlight this example is that in this case our Conjecture 2.7 holds and by Remark 2.6 the intersection $Y_P \cap \sigma_4(X)$ contains at least one point of $\sigma_4^0(X)$.

For the case of flag varieties we underline only the case of the adjoint varieties of the Lie algebra $\mathfrak{sl}_{n+1}$. In this case the flag variety contained in $\mathbb{P}(\mathfrak{sl}_{n+1}) \simeq \mathbb{P}^{(n+1)^2 - 2}$ is the variety $F(\mathbb{P}^0, \mathbb{P}^{n-1}; \mathbb{P}^n)$ whose points are flags $\mathbb{P}^0 \subset \mathbb{P}^{n-1}$ in $\mathbb{P}^n$. It is a known fact that for any $n \geqslant 1$ this variety parametrizes the $(n + 1) \times (n + 1)$-traceless matrices of rank 1. As showed by [7, Theorem 1.1], the k-th secant variety of this flag variety is given by traceless matrices of rank at most k, for any $k \leqslant n + 1$. See also [28] for a description of this in terms of lower semi-continuous rank function. Moreover by [7, Corollary 1.2], it is easy to see that $\sigma_{n+1}(F(\mathbb{P}^0, \mathbb{P}^{n-1}; \mathbb{P}^n))$ fills the ambient space, and that the dimension of the k-th secant variety of X is $2k(n + 1) - k^2 - 2$. In particular for $k = n$ it is readily seen that $\sigma_n(F(\mathbb{P}^0, \mathbb{P}^{n-1}; \mathbb{P}^n))$ is a hypersurface in $\mathbb{P}(\mathfrak{sl}_{n+1})$. By what we have said we get that $r_{\max,n} = n$. By Theorem 2.4 we get that $r_{\max} \leqslant n + 1$. Applying the bound in [11, Theorem 6, p. 3] one gets $r_{\max} \leqslant 2(n + 1) - 1 = 2n + 1$, while by [14, Theorem 3.9, p. 118] one gets $r_{\max} \leqslant 2n$. Note that also in this case Conjecture 2.7 holds.

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