Rational homogeneous spaces as geometric realizations of birational transformations

Gianluca Occhetta[1] · Eleonora A. Romano[2] · Luis Eduardo Solá Conde[1] · Jarosław A. Wiśniewski[3]

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Abstract

In this paper we study geometric realizations of some classic birational maps, such as inversion maps, special Cremona transformations, special birational transformations of type (2, 1), by considering $\mathbb{C}^*$-actions on rational homogeneous spaces and their subvarieties.

1 Introduction

A remarkable idea present in modern birational geometry is the one that two birationally equivalent varieties should be obtained as GIT quotients of the same algebraic variety. This connection between Geometric Invariant Theory and the Minimal Model Program arose from the seminal work of Thaddeus and Reid in the 1990s (see [32, 36]). Since then, it has led to the concept of Mori Dream Spaces, whose small $\mathbb{Q}$-factorial modifications can be obtained

✉ Luis Eduardo Solá Conde
   eduardo.solaconde@unitn.it

   Gianluca Occhetta
   gianluca.occhetta@unitn.it

   Eleonora A. Romano
   eleonoraanna.romano@unige.it

   Jarosław A. Wiśniewski
   J.Wisniewski@uw.edu.pl

1 Dipartimento di Matematica, Università degli Studi di Trento, Via Sommarive 14, 38123 Povo di Trento, TN, Italy

2 Dipartimento di Matematica, Università degli Studi di Genova, Via Dodecaneso 35, 16146 Genova, GE, Italy

3 Instytut Matematyki UW, Banacha 2, 02-097 Warszawa, Poland

as quotients by the action of a torus on the spectrum of a finitely generated algebra, the so called Cox ring of the Mori Dream Space.

On the other hand, an action of a complex 1-dimensional torus $\mathbb{C}^*$ on a projective variety induces a birational transformation between the associated geometric quotients (cf. [27, Lemma 3.4], [28, Remark 2.13]). Then it makes sense to ask whether any birational transformation may be obtained in this way. On the one hand, this question may be dealt with $\mathbb{C}^*$-equivariant projective compactifications of the cobordisms studied by Morelli and Włodarczyk (cf. [22, 38]). On the other hand, one would like to construct explicitly a projective variety with an action of $\mathbb{C}^*$ inducing a given birational map ψ, i.e., a *geometric realization of ψ*, which is the main motivation of this paper. More precisely, a geometric realization of a birational map ψ among two complex projective varieties is a variety X equipped with a $\mathbb{C}^*$-action inducing ψ as the natural birational map among two extremal geometric quotients. This concept has been introduced in [26], where geometric realizations for a certain simple class of birational transformations, called bispecial, have been constructed; a similar construction had been proved to work in the case of Atiyah flips (cf. [29]). Morever, recent results in [2] provide a way to construct geometric realizations starting from Mori dream pairs.

Another possible idea to construct geometric realizations is to start with some reasonable class of $\mathbb{C}^*$-actions on projective varieties, and then to study their $\mathbb{C}^*$-invariant subvarieties and the associated birational maps. In this paper we consider $\mathbb{C}^*$-actions on rational homogeneous spaces, and describe the birational maps induced by them. Furthermore, by considering certain $\mathbb{C}^*$-invariant subsets of rational homogeneous spaces we are able to produce geometric realizations of other birational transformations, such as the special transformations of type $(2, 1)$ classified by Fu and Hwang, [13]. We stick to the case in which the action is equalized (see 2.1.9 below), a condition that guarantees the smoothness of the birationally equivalent varieties associated with the action. Remarkably, the list of transformations that we obtain in this way contains many classically interesting examples, such as the inversion of matrices, and the special quadro-quadric Cremona transformations (see [9]). As a consequence of our discussion, we finally pose two problems.

Problem 1 Classify G-equivariant Cremona transformations among two irreducible projective representations of a semisimple group G, constructing geometric realizations for them.

One may observe that the standard Cremona transformation from $\mathbb{P}^{n-1}$ to $\mathbb{P}^{n-1}$ is a restriction of the projectivization of the inversion map of $n \times n$ matrices (cf. [11, Example 1.1] and [21, Lemma 3.6]). Moreover, its geometric realization given by the diagonal natural $\mathbb{C}^*$-action on $(\mathbb{P}^1)^n$ can be $\mathbb{C}^*$-equivariantly embedded into $A_{2n-1}(n)$. We would like to present here the problem of classifying the Cremona transformations that satisfy this property. Philosophically speaking, one would like to understand how far is the inverse map from being a "universal" Cremona transformation. See [11] for some recent progress concerning the following problem.

Problem 2 Classify Cremona transformations that are restrictions of the inversion map (Example 4.2), studying when their geometric realizations may be embedded $\mathbb{C}^*$-equivariantly into $A_{2n-1}(n)$.

Outline. After recalling some basic facts on torus actions on rational homogeneous varieties (Sect. 2), we focus on the classification of equalized actions with isolated extremal fixed point components, and study their associated birational transformations (Sect. 3). Section 4 contains descriptions of the Cremona transformations obtained in this way. In particular, our discussion provides the following statement:

Theorem *Let $\psi : \mathbb{P}(M_{n \times n}(\mathbb{C})) \dashrightarrow \mathbb{P}(M_{n \times n}(\mathbb{C}))$ be the projectivization of the inversion map of $n \times n$ matrices, and let ψ_s, ψ_a be its restrictions to the projetivizations of the spaces of symmetric and skew-symmetric matrices, respectively. Then there exist geometric realizations of ψ, ψ_s, ψ_a (with n even in this case), given by an equalized $\mathbb{C}^*$-action on, respectively, the following rational homogeneous varieties (see Sect. 3.1 for notation):*

$$A_{2n-1}(n), \qquad C_n(n), \qquad D_n(n).$$

Note that the inversion maps ψ, ψ_s, ψ_a induced by the equalized $\mathbb{C}^*$-action on $A_5(3)$, $C_3(3)$, $D_6(6)$ are quadro-quadric special Cremona transformations as described in [9]; the list is completed with the map induced by the equalized $\mathbb{C}^*$-action on $E_7(7)$, see Remark 3.19.

In the final Sect. 5 we study induced $\mathbb{C}^*$-actions on certain fiber bundles contained in rational homogeneous varieties, and show that some of them are geometric realizations of special birational transformations. More concretely, our arguments in that section allow us to prove the following statement:

Theorem *Let F be a Fano manifold of Picard number one, and assume that there exists a special birational transformation $\psi : F \dashrightarrow \mathbb{P}^{\dim(F)}$ of type $(1, 2)$. Then there exists a geometric realization of ψ, given by an equalized $\mathbb{C}^*$-action on a locally trivial F-bundle $X \to \mathbb{P}^1$. Furthermore, F can be $\mathbb{C}^*$-equivariantly embedded in a rational homogeneous variety with an equalized $\mathbb{C}^*$-action.*

2 Preliminaries on $\mathbb{C}^*$-actions

Throughout the paper all varieties will be defined over the field of complex numbers. This section contains some preliminary results, notation and conventions regarding $\mathbb{C}^*$-actions. We refer the interested reader to [1, 4, 8, 27, 28, 33] for details, original references, and further results. In particular we will recall the concept of geometric realization of a birational map, introduced in [26]. In the remaining sections, X will be a smooth projective variety with a $\mathbb{C}^*$-action.

2.1 Generalities on $\mathbb{C}^*$-actions

Let X be a complex, smooth, projective variety admitting a non-trivial (morphical) action of the multiplicative group $\mathbb{C}^*$, which will be denoted as follows:

$$\mathbb{C}^* \times X \longrightarrow X \qquad (t, x) \mapsto tx.$$

2.1.1. $X^{\mathbb{C}^*} \subset X$ will denote the set of fixed points of the $\mathbb{C}^*$-action, which is a closed, smooth subset of X, and $\mathcal{Y}$ the set of irreducible components of $X^{\mathbb{C}^*}$.

2.1.2. For every $x \in X$ we set

$$x^{\pm} := \lim_{t^{\pm 1} \to 0} tx \in X^{\mathbb{C}^*}$$

and call them the *sink* and the *source of the orbit* $\mathbb{C}^* x$, respectively.

2.1.3. Y_-, $Y_+ \in \mathcal{Y}$ will denote the *sink* and the *source* of the action, defined as the fixed point components satisfying $x^{\pm} \in Y_{\pm}$ for the general point $x \in X$. We will call them the extremal fixed point components of the action. The rest of the fixed point components are called *inner*, and their set is denoted by $\mathcal{Y}^{\circ}$.

2.1.4. For every subset $S \subset X^{\mathbb{C}^*}$, we denote by $X^{\pm}(S) \subset X$ the $\mathbb{C}^*$-invariant subsets:

$$X^{\pm}(S) := \{x \in X | \ x^{\pm} \in S\}.$$

In the case of a fixed point component $Y \in \mathcal{Y}$, the sets $X^{\pm}(Y) \subset X$ are called the *positive/negative BB-cells* (short form of Białynicki-Birula cell) associated with Y. They are open subsets in their Zariski closures, and the maps from $X^{\pm}(Y)$ to Y sending each point x to $x^{\pm}$ are known to be morphisms (cf. [4, Theorem 4.3]).

2.1.5. For every $Y \in \mathcal{Y}$, we denote by $\mathcal{N}_{Y|X}$ the normal bundle of Y in X. The action of $\mathbb{C}^*$ on X induces a fiberwise linear $\mathbb{C}^*$-action on the fibers of $\mathcal{N}_{Y|X}$ over Y, and we may decompose it as a direct sum of vector bundles

$$\mathcal{N}_{Y|X} = \mathcal{N}^-(Y, X) \oplus \mathcal{N}^+(Y, X),$$

according to the sign of the weights of the $\mathbb{C}^*$-action. When the ambient variety X is clear in the context, we will simply write $\mathcal{N}^{\pm}(Y) := \mathcal{N}^{\pm}(Y, X)$. By the Białynicki-Birula theorem [4, Theorem 4.2], we know that for every point $x \in Y$ there exists an open neighborhood $U \subset Y$ of x such that $X^{\pm}(U)$ are $\mathbb{C}^*$-equivariantly isomorphic over U to $\mathcal{N}^{\pm}(Y)_{|U}$. This is not equivalent to saying that there exist equivariant isomorphisms from $\mathcal{N}^{\pm}(Y)$ to $X^{\pm}(Y)$, since the morphisms $X^{\pm}(Y) \to Y$ are only known to be affine bundles, in general (cf. [4, Remark, p.491]).

The rank of $\mathcal{N}^{\pm}(Y)$, which is equal to $\dim(X^{\pm}(Y)) - \dim(Y)$, will be denoted by $\nu^{\pm}(Y)$. Note that

$$\nu^{\mp}(Y_{\pm}) = 0,$$
$$\nu^+(Y) + \nu^-(Y) = \dim(X) - \dim(Y), \ \text{for every } Y \in \mathcal{Y}, \tag{1}$$
$$\nu^{\pm}(Y) = 0 \text{ if and only if } Y = Y_{\mp}.$$

for every $Y \in \mathcal{Y}$.

2.1.6. A *linearization* of the action of $\mathbb{C}^*$ on a line bundle L on X is a fiberwise linear $\mathbb{C}^*$-action on L such that the natural projection is $\mathbb{C}^*$-equivariant. Given $Y \in \mathcal{Y}$, there exists an integer $\mu_L(Y) \in \mathbb{Z}$, called the *weight of the linearization on* Y, such that the action of $\mathbb{C}^*$ on every fiber of $L_{|Y} \to Y$ is given by $(t, v) \mapsto t^{\mu_L(Y)} v$.

2.1.7. If L is ample, the $\mathbb{C}^*$-action together with a linearization on L will be referred to as a $\mathbb{C}^*$-*action on the polarized pair* (X, L). In this case the minimum and maximum value of the weights are achieved at the sink and the source of the action, respectively (see [27, Remark 2.12]). By multiplying the linearization with a character we may assume that $\mu_L(Y_-) = 0$, and then $\delta := \mu_L(Y_+)$ is called the *bandwidth* of the $\mathbb{C}^*$-action on (X, L). Furthermore, the set

$$\mu_L(\mathcal{Y}) = \{a_0 = 0, a_1, \ldots, a_r = \delta\}, \quad (a_0 < a_1 < \cdots < a_r)$$

is called the set of *critical values* of the $\mathbb{C}^*$-action on (X, L). The integer r is called the *criticality* of the action.

2.1.8. A linearization of the $\mathbb{C}^*$-action on an ample line bundle L over X gives a weight decomposition on $H^0(X, L)$, namely:

$$H^0(X, L) = \bigoplus_{u \in \mathbb{Z}} H^0(X, L)_u,$$

where $H^0(X, L)_u \subset H^0(X, L)$ stands for the vector subspace of $\mathbb{C}^*$-weight u. If L is globally generated, the extremal values for which $H^0(X, L)_u \neq \{0\}$ are $\mu_L(Y_-) = 0$, $\mu_L(Y_+) = \delta$.

2.1.9. The action of $\mathbb{C}^*$ on X is said to be *equalized* if and only if the induced action on $\mathcal{N}^{\pm}(Y)$ has all its weights equal to ± 1; that is to say that the action on $\mathcal{N}^{\pm}(Y)$ is faithful and fiberwise homothetical, for every $Y \in \mathcal{Y}$. Equivalently, the $\mathbb{C}^*$-action has no proper non-trivial isotropy groups.

2.1.10. If the $\mathbb{C}^*$-action is equalized, then the closure of any 1-dimensional orbit is a smooth rational curve, whose L-degree (with respect to $L \in \text{Pic}(X)$ ample) can be computed in terms of the weights at its extremal points:

$$L \cdot \overline{\mathbb{C}^* x} = \mu_L(x_+) - \mu_L(x_-). \qquad \text{(AM vs FM)}$$

2.1.11. The action of $\mathbb{C}^*$ on X is said to be of *B-type* if and only if $\nu^{\pm}(Y_{\pm}) = 1$, i.e., if $Y_{\pm}$ are divisors in X.

2.1.12. If the $\mathbb{C}^*$-action on X is equalized, then it extends to a B-type $\mathbb{C}^*$-action on the blowup

$$\beta : X^{\flat} \to X,$$

of X along $Y_{\pm}$, whose sink and source are the exceptional divisors $\mathbb{P}_{Y_{\pm}}(\mathcal{N}^{\vee}_{Y_{\pm}|X})$.

2.1.13. An important consequence of equalization (that follows from [4, Remark, p.491]) is that under this hypothesis the BB-cells $X^{\pm}(Y)$ are vector bundles $\mathbb{C}^*$-equivariantly isomorphic to $\mathcal{N}^{\pm}(Y)$, for every $Y \in \mathcal{Y}$. Applied to the extremal fixed point components $Y_{\pm}$, this tells us that $\mathbb{P}_{Y_{\pm}}(\mathcal{N}^{\vee}_{Y_{\pm}|X})$ is isomorphic to the geometric quotient of the open set $X^{\pm}(Y_{\pm}) \setminus Y_{\pm} \subset X$ by the action of $\mathbb{C}^*$.

Furthermore, the nonempty intersection of the open sets $X^{\pm}(Y_{\pm}) \setminus Y_{\pm}$ induces a birational map among the corresponding quotients:

$$\psi : (X^-(Y_-) \setminus Y_-)/\mathbb{C}^* \dashrightarrow (X^+(Y_+) \setminus Y_+)/\mathbb{C}^*,$$

that in the equalized case is identified with a birational map:

$$\psi : \mathbb{P}_{Y_-}(\mathcal{N}_{Y_-|X}) \dashrightarrow \mathbb{P}_{Y_+}(\mathcal{N}_{Y_+|X}). \qquad (2)$$

Note that in the case in which the $\mathbb{C}^*$-action is of B-type, the map ψ goes from Y_- to Y_+. Note also that the birational maps induced by an equalized action on X and on its blowup along its sink and source are the same.

2.1.14. Conversely, let $\psi : E_- \dashrightarrow E_+$ be a birational map. A *geometric realization* of ψ is a variety X endowed with a $\mathbb{C}^*$-action such that $E_{\pm} \simeq (X^{\pm}(Y_{\pm}) \setminus Y_{\pm})/\mathbb{C}^*$, and the induced birational map from E_- to E_+ coincides with ψ.

2.2 Exceptional locus of the induced birational transformation

In this section we will describe the exceptional locus of the birational map associated with a faithful $\mathbb{C}^*$-action on a smooth projective variety X. For simplicity we will assume that the action is of B-type (see 2.1.12 above), so that the $\mathbb{C}^*$-action induces a birational map:

$$\psi : Y_- \dashrightarrow Y_+,$$

which assigns to a general point $y \in Y_-$ the limit as $t \to 0$ of the unique orbit having limit as $t^{-1} \to 0$ equal to y.

Given an inner fixed point component $Y \in \mathcal{Y}^{\circ}$ we consider the invariant closed subvarieties of X defined recursively as follows:

$$C_1^{\pm}(Y) := \overline{X^{\pm}(Y)}, \qquad C_k^{\pm}(Y) := \overline{X^{\pm}(C_{k-1}^{\pm}(Y))}, \ k \geq 2$$

Note that $C_{k-1}^{\pm}(Y) \subseteq C_k^{\pm}(Y)$, and that the inclusion is strict if and only if there exists a 1-dimensional orbit $\mathbb{C}^* x$ not contained in $C_{k-1}^{\pm}(Y)$ such that $\lim_{t \to 0} t^{\pm 1} x \in C_{k-1}^{\pm}(Y)$. Then for every inner fixed point component Y there exists $m \geq 1$ such that $C_{m-1}^{\pm}(Y) = C_m^{\pm}(Y)$, and so we may define:

$$C^{\pm}(Y) := \lim_{k \to +\infty} C_k^{\pm}(Y), \ Y \in \mathcal{Y}^{\circ}.$$

In other words, $C^{\pm}(Y)$ can be defined as the union of all the connected 1-cycles consisting of closures of 1-dimensional orbits linking Y with $Y_{\mp}$, whose existence is guaranteed by [27, Corollary 2.15]. We finally set:

$$Z_{\mp}(Y) := C^{\pm}(Y) \cap Y_{\mp} \subset Y_{\mp}, \ Y \in \mathcal{Y}^{\circ}, \quad Z_{\mp} := \bigcup_{Y \in \mathcal{Y}^{\circ}} Z_{\mp}(Y) \subset Y_{\mp}.$$

Lemma 2.1 *In the above notation, the map ψ restricts to an isomorphism:*

$$\psi : Y_- \setminus Z_- \longrightarrow Y_+ \setminus Z_+,$$

Proof Denote by $\pi_{\pm} : X^{\pm}(Y_{\pm}) \simeq \mathcal{N}_{Y_{\pm}|X} \to Y_{\pm}$ the projection map given by the Białynicki-Birula theorem. The open subsets:

$$U_{\pm} = X^{\pm}(Y_{\pm}) \setminus \left(Y_{\pm} \cup \pi_{\pm}^{-1}(Z_{\pm}) \right)$$

are equal since, by definition, they can be identified with the set of orbits of the action having limiting points at Y_- and Y_+. It then follows that their quotients by the action of $\mathbb{C}^*$, $Y_- \setminus Z_-$ and $Y_+ \setminus Z_+$, are isomorphic. $\qquad \square$

This statement tells us that the exceptional locus of ψ is contained in

$$\bigcup_{Y \in \mathcal{Y}^{\circ}} Z_-(Y).$$

However, in general $\mathrm{Exc}(\psi)$ does not coincide with this set; in fact one may show the following:

Lemma 2.2 *Let X be a smooth, projective variety together with an equalized B-type $\mathbb{C}^*$-action, and let $\mathcal{C}$ be the set of inner fixed point components Y of the action with $\nu^-(Y) > 1$. Then:*

$$\mathrm{Exc}(\psi) \subseteq \bigcup_{Y \in \mathcal{C}} Z_-(Y).$$

Proof Let $x_- \in Y_- \setminus \bigcup_{Y \in \mathcal{C}} Z_-(Y)$. We start by claiming that there exists a unique sequence $\Gamma_1, \Gamma_2, \ldots, \Gamma_r$ of closures of 1-dimensional orbits such that $x_0 := x_- \in \Gamma_1$, every Γ_i intersects Γ_{i+1} at a fixed point x_i, Γ_r intersects Y_+ at a point $x_r = x_+$, and $\mu_L(x_i) < \mu_L(x_{i+1})$ for every i.

In order to prove the claim, we note first that, by the Białynicki-Birula decomposition, there exists a unique 1-dimensional $\mathbb{C}^*$-orbit having x_- as sink; let us denote by Γ_1 its closure, by x_1 its source, and by Y_1 the unique fixed point component containing x_1. If there were at least two 1-dimensional orbits having x_1 as sink, the Białynicki-Birula decomposition would tell us that $\nu^-(Y_1) > 1$, that is $Y_1 \in \mathcal{C}$; but then $x_- \in Z_-(Y_1)$ would belong to $\bigcup_{Y \in \mathcal{C}} Z_-(Y)$, a contradiction. We proceed now recursively, proving that there exists a unique 1-dimensional

orbit having x_1 as sink, whose closure is denoted by Γ_2, etc. The fact that the weight $\mu_L(x_i)$ grows at every step follows by 2.1.10.

We now prove that the map ψ extends to x_-. To this end, let us consider the unique sequence of $\mathbb{C}^*$-invariant curves $\Gamma_1, \ldots, \Gamma_r$ linking x_- with $x_+ \in Y_+$ as above. Let $\gamma(s)$ be a holomorphic curve converging to x_- when s goes to 0, with $\gamma(s) \in Y_- \setminus Z_-$ for $s \neq 0$. For every $s \neq 0$, the image $\psi(\gamma(s))$ is defined as the source of the unique orbit $O(s)$ having $\gamma(s)$ as sink when t goes to ∞. When s goes to 0, the closure $\overline{O(s)}$ must converge to an effective and connected $\mathbb{C}^*$-invariant 1-cycle Γ' passing by x_- and meeting Y_+; moreover, by the AM vs FM formula (2.1.10) we know that $L \cdot \Gamma' = L \cdot \overline{O(s)}$ equals the bandwidth δ. In particular it contains a 1-cycle $\sum_{i=1}^{m} r_i \Gamma'_i$ with $x_- \in \Gamma'_1$, $\Gamma'_i \cap \Gamma'_{i+1} = \{x'_i\}$, $\Gamma'_m \cap Y_+ \neq \emptyset$. By 2.1.10, the equalization of the action, and the equality $L \cdot \Gamma' = \delta$, we may conclude that $r_i = 1$ for every i, that $\Gamma' = \sum_{i=1}^{m} \Gamma'_i$ and that $\mu_L(x'_i) < \mu_L(x'_{i+1})$ for every i. In other words, Γ' equals $\Gamma_1 + \cdots + \Gamma_r$, and from this it follows that

$$\lim_{s \to 0} \psi(\gamma(s)) = (\Gamma_1 + \cdots + \Gamma_r) \cap Y_+ = x_+.$$

In particular, since this limit does not depend on the choice of the curve γ, it follows that the map ψ extends to x_-. This finishes the proof. □

2.3 Local description of $\mathbb{C}^*$-invariant divisors

We end this section by describing locally the invariant divisors determined by sections $\sigma \in$ $\mathrm{H}^0(X, L)_u$, paying attention to their smoothness. The following result is a restatement of [7, Lemma 2.17]; the notation we will use is consistent with that paper.

Lemma 2.3 *Let X be a smooth proper variety together with a $\mathbb{C}^*$-action, linearized on a line bundle L on X. Let $Y \subset X$ be an irreducible fixed point component, and $y \in Y$ be a point. Let $D_\sigma \subset X$ be a $\mathbb{C}^*$-invariant divisor associated with a $\mathbb{C}^*$-equivariant section $\sigma \in \mathrm{H}^0(X, L)_u$, with $u \in \mathbb{Z}$. Then there exist local coordinates $x_1, \ldots, x_{d_+}, y_1, \ldots, y_{d_0}, z_1, \ldots, z_{d_-}$ around $y = \{x_i = y_j = z_k = 0\}$ such that $Y = \{x_i = z_k = 0\}$, and the divisor D_σ is described as the zero set of a power series $f \in \mathbb{C}[[x_i, y_j, z_k]]$, homogeneous of degree $\mu_L(Y) - u$ with respect to the grading of $\mathbb{C}[[x_i, y_j, z_k]]$ induced by the $\mathbb{C}^*$-action.*

Proof Without loss of generality, we may assume that $\mu_L(Y) > 0$. We consider the $\mathbb{C}^*$-action on the line bundle L, identifying X and Y with their corresponding images into L via the zero section. Note that in this way $X \subset L$ is a $\mathbb{C}^*$-invariant subvariety and, by the hypothesis on μ_L, $Y \subset L$ is a $\mathbb{C}^*$-fixed point component.

Given a point $y \in Y$, we may apply [4, Theorem 2.5] to find a $\mathbb{C}^*$-invariant neighborhood U of $y \in L$ equivariantly isomorphic to $(Y \cap U) \times V$, where V is a $\mathbb{C}^*$-module. By differentiating at y, V may be identified with $\mathcal{N}_{Y|L,y} = \mathcal{N}_{Y|X,y} \oplus L_y$. The action of $\mathbb{C}^*$ on V diagonalizes with weights $p_1, \ldots, p_{d_+}, n_1, \ldots, n_{d_-}, \mu_L(Y)$ ($p_i > 0, n_k < 0$), on a set of coordinates $(x_1, \ldots, x_{d_+}, z_1, \ldots, z_{d_-}, \lambda)$. We may complete this set of coordinates with local coordinates $y_1, \ldots, y_{d_0}$ of Y around y, and consider the rings of local coordinates $\mathbb{C}[[x_i, y_j, z_k]]$ of X around y, and $\mathbb{C}[[x_i, y_j, z_k, \lambda]]$ of L around y.

In these local coordinates a section $\sigma \in \mathrm{H}^0(X, L)$ reads as $\sigma(x_i, y_j, z_k) = (x_i, y_j, z_k, f(x_i, y_j, z_k))$ where $f \in \mathbb{C}[[x_i, y_j, z_k]]$. The proof is then concluded by noting that saying that σ is invariant of weight u means that

$$f(t(x_i, y_j, z_k)) = t^{\mu_L(Y) - u} f(x_i, y_j, z_k).$$

□

A straightforward consequence of the above Lemma is the following:

Corollary 2.4 *In the situation of the above lemma suppose that the action is equalized. Then the function f vanishes along Y with multiplicity $\geq |u - \mu_L(Y)|$. If moreover Y is either the sink or the source, then the inequality becomes an equality.*

Proof By Lemma 2.3, the function f can be written as a series:

$$f = \sum_{I,J,K} a_{IJK} x^I y^J z^K$$

where

$$x^I = x_1^{i_1} \ldots x_{d_+}^{i_{d_+}}, \quad y^J = y_1^{j_1} \ldots y_{d_+}^{j_{d_0}}, \quad z^K = z_1^{k_1} \ldots z_{d_-}^{k_{d_-}},$$

homogeneous of degree $u - \mu_L(Y)$ with respect to the action of $\mathbb{C}^*$. Set $\overline{I}$, $\overline{K}$ to be the sum of the exponents in I, K, respectively. Since the action is equalized, we have that $\deg x_i = 1$, $\deg z_k = -1$ for all i and k, and so

$$u - \mu_L(Y) = \deg(f) = \deg(x^I y^J z^K) = \overline{I} - \overline{K}, \text{ for } a_{IJK} \neq 0.$$

Then the vanishing multiplicity of f along Y, which is $\min\{\overline{I} + \overline{K}, \; a_{IJK} \neq 0\}$ is bigger than or equal to $|\overline{I} - \overline{K}| = |u - \mu_L(Y)|$. For the second part we simply note that if Y is an extremal component, then either I or K are empty. $\qquad\square$

The following statement is an application of the above result to a particular situation, in the form of a smoothness criterion for invariant sections of L.

Proposition 2.5 *Let (X, L) be a smooth polarized pair with an equalized $\mathbb{C}^*$-action of bandwidth two. Let $\sigma \in \mathrm{H}^0(X, L)_1$ be a section, with zero locus D_σ, such that D_σ is smooth at the points of $D_\sigma \cap Y_1$. Then the differential $d\sigma$ defines sections of $N^\vee_{Y_\pm/X} \otimes L$; if these sections are nowhere vanishing then $D_\sigma \subset X$ is smooth and the induced action on $(D_\sigma, L_{|D_\sigma})$ has bandwidth two.*

Remark 2.6 The condition on the non-vanishing of the section of $N^\vee_{Y_\pm/X} \otimes L$ defined by σ is satisfied for a general σ if L is very ample and, either $\dim Y_\pm < \frac{1}{2} \dim X$, or $\dim Y_\pm = \frac{1}{2} \dim X$ and the top Chern class of $N^\vee_{Y_\pm} \otimes L$ is zero.

Proof of Proposition 2.5 Since the section σ is $\mathbb{C}^*$-invariant, it is enough to check its smoothness at the fixed point locus. In fact, the singular locus of the section is closed and $\mathbb{C}^*$-invariant; then if it is non-empty, it contains a singular fixed point. We will check smoothness locally so that we can write the expansion of σ in coordinates.

By assumption we just need to check the smoothness of D_σ at its intersection with the source (smoothness at the sink is analogous). We then take y to be a point of the source Y_+, and use Lemma 2.3 to obtain local coordinates y_i's of weight zero and z_i's of weight one. Therefore, if f is a local description of σ, it vanishes with multiplicity one at Y_+, by Corollary 2.4. In particular, $df_{|Y_+}$ is not identically zero, and the zero locus of f is singular at the points of Y_+ on which this differential is zero.

Moreover, since f vanishes at Y_+, we may write:

$$df_{|Y_+} = \sum_k f_k(y_1, \ldots, y_{d_0}) dz_k.$$

In other words $df_{|Y_+}$, as a section of $(\Omega_X \otimes L)_{|Y_+}$, lies in the kernel of the differential map $(\Omega_X \otimes L)_{|Y_+} \to \Omega_{Y_+} \otimes L_{|Y_+}$, which is $N^\vee_{Y_+/X} \otimes L_{|Y_+}$. This finishes the proof. $\qquad\square$

3 Torus actions on rational homogeneous spaces

Representation theory provides a framework in which one may construct many examples of projective varieties supporting $\mathbb{C}^*$-actions (see [5, II, Chapter 3]), that can be thought of as geometric realizations of some birational transformations. This section is devoted to the description of $\mathbb{C}^*$-actions on rational homogeneous varieties. We will pay special attention to those having isolated extremal fixed points that give rise to Cremona transformations whenever they are equalized (that we will study in Sect. 4 below). In particular, we will show how to re-construct in this setting the geometric realizations of the special Cremona transformations within the list of Ein and Shepherd-Barron.

3.1 Notation and basic facts on rational homogeneous varieties

In this section we will recall some basic facts on rational homogeneous varieties, and introduce the notation we will use to describe them. We will use some standard representation theory of semisimple algebraic groups, for which we refer the interested reader to [15, 16].

3.1.1 Rational homogeneous varieties

Let G be a semisimple algebraic group and $H \subset G$ a Cartan subgroup, with associated Lie algebras $\mathfrak{h} \subset \mathfrak{g}$; assume that G is the adjoint group of $\mathfrak{g}$, so that the group $\mathrm{M}(H) := \mathrm{Hom}(H, Z)$ of characters of H coincides with the root lattice of G with respect to H, generated by the root system $\Phi \subset \mathrm{M}(H)$ of $\mathfrak{g}$ (with respect to H). We will denote by $\Delta = \{\alpha_1, \ldots, \alpha_n\} \subset \Phi$ a base of positive simple roots of Φ, and by $\Phi^+ \subset \Phi$ the set of roots that are non-negative integral combinations of elements of Δ, so that $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$ is a Borel subalgebra of $\mathfrak{g}$, corresponding to a Borel subgroup $B \subset G$ containing H. It is then known that any projective variety admitting a transitive action of G (a so-called *rational homogeneous G-variety*) can be written as G/P, where P is a parabolic subgroup of G containing B.

The Weyl group of G is defined as the finite group $W := \mathrm{N}_G(H)/H$; its natural action on $\mathrm{M}(H) \otimes_{\mathbb{Z}} \mathbb{R}$ preserves the inner product induced by the Killing form $\kappa(\cdot, \cdot)$ of $\mathfrak{g}$ (which gives $\mathrm{M}(H) \otimes_{\mathbb{Z}} \mathbb{R}$ the structure of Euclidean space). In this way W can be described as the group generated by the reflections r_i with respect to the positive simple roots α_i. Given $w \in W$, the minimum number k of reflections r_i such that we can write $w = r_{i_1} \circ \cdots \circ r_{i_k}$ is called the *length* of w; it is known that there exists a unique element of W of maximal length, that is called the *longest element* of W, which will be denoted by w_0. We will denote by $\mathcal{D}$ the Dynkin diagram of G, and by D its set of nodes, which is in one to one correspondence with the base of positive simple roots Δ.

Remark 3.1 The bijection of the root system Φ given by w_0, in the cases in which $\mathfrak{g}$ is simple, is described in [6, Planche I–IX]. Essentially, w_0 equals $-id$ whenever $\mathcal{D}$ has no non-trivial automorphisms (B_n, C_n, E_7, E_8, F_4, G_2) and in the case D_n, n even; in the cases A_n, E_6 and D_n, n odd, it is the composition of $-id$ with the homomorphism of $\mathrm{M}(H)$ determined by the permutation of Δ induced by the non-trivial automorphism of $\mathcal{D}$.

Every rational homogeneous G-variety G/P is determined by the marking of $\mathcal{D}$ on a set of nodes $\{i_1, \ldots, i_k\} \subset D$ (corresponding to positive simple roots $\alpha_{i_1}, \ldots, \alpha_{i_k}$), where k equals the Picard number of X; we refer to [25, Section 2] for details. We will then set:

$$\mathcal{D}(I) := G/P, \text{ for } I = \{i_1, \ldots, i_k\}.$$

Fig. 1 The Dynkin diagram E$_6$
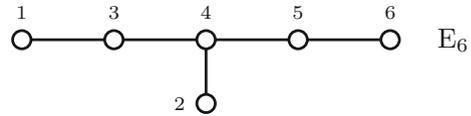
The parabolic subgroup P associated with I can be described as $P = BW(D \setminus I)B \supset B$, where $W(D \setminus I) \subset W$ is the subgroup of W generated by the reflections r_j, $j \in D \setminus I$.

With this notation, any inclusion $I \subset I' \subset D$ gives rise to a natural projection

$$\mathcal{D}(I') \to \mathcal{D}(I),$$

whose fibers are rational homogeneous varieties of the form $\mathcal{D}_I(I' \setminus I)$, where $\mathcal{D}_I$ denotes the Dynkin subdiagram of $\mathcal{D}$ obtained by deleting the nodes corresponding to the indices in I.

The marking of the Dynkin diagram $\mathcal{D}$, of a semisimple group G as above on the whole set of nodes D corresponds to the quotient of G by the Borel subgroup B containing H, which is usually called the complete flag variety associated with G. Complete flags can be characterized (cf. [30]) by the property of being smooth projective varieties having as many independent $\mathbb{P}^1$-bundle structures as their Picard number; in the notation we have just introduced, these structures are precisely the natural projections:

$$\mathcal{D}(D) \to \mathcal{D}(D \setminus \{i\}), \qquad i = 1, \ldots n.$$

Later on, we will denote the numerical class of the fibers of these maps by Γ_i.

3.1.2 Examples of rational homogeneous varieties

For the connected Dynkin diagrams we will use the numbering proposed by Bourbaki ([6, Planche I–IX]); for instance, the diagram E$_6$ is numbered as in Fig. 1.

Algebraic geometers usually denote rational homogeneous varieties in a way that reflects one of their geometric descriptions. The advantage of the above "group-minded" notation comes from the fact that it may be applied to all rational homogeneous varieties. For the reader's convenience, we include here a list of rational homogeneous varieties, their geometric descriptions, and the notation we will use for them.

- The projective space $\mathbb{P}^n$ is written as A$_n(1)$, and its dual as A$_n(n)$; more generally, the Grassmannian of $(k-1)$-linear subspaces of $\mathbb{P}^n$ is A$_n(k)$, and so A$_n(k_1, \ldots, k_r)$, $k_1 < \cdots < k_r$ is the variety of flags of linear spaces $\mathbb{P}^{k_1-1} \subset \cdots \subset \mathbb{P}^{k_r-1} \subset \mathbb{P}^n$.
- Smooth quadrics of dimension $2n - 1$ (resp. $2n - 2$) are denoted by B$_n(1)$ (resp. D$_n(1)$). Varieties of the form B$_n(k_1, \ldots, k_r)$ parametrize flags of linear subspaces of B$_n(1)$. The case of flags in D$_n(1)$ is essentially analogous, but there exist two disjoint irreducible families of $(n-1)$-dimensional linear subspaces, denoted by D$_n(n-1)$, D$_n(n)$ (the so-called spinor varieties); the family of $(n-2)$-dimensional linear subspaces in D$_n(1)$ is D$_n(n-1, n)$.
- Varieties of the form C$_n(k_1, \ldots, k_r)$ parametrize flags of linear subspaces in $\mathbb{P}(V) = \mathbb{P}^{2n-1}$ that are isotropic with respect to a maximal rank skew-symmetric form in V. The variety C$_n(1)$ is nothing but $\mathbb{P}(V)$, and the varieties C$_n(k)$ are usually called isotropic Grassmannians.
- The rational homogeneous varieties of types E$_k$, F$_4$, G$_2$ can be described projectively in terms of the algebra of complexified octonions; we refer the interested reader to

[19] and the references therein. Here it will be enough to remark that the variety $E_6(1)$ is the octonionic projective plane, sometimes called the Cartan variety. It can also be described as the Severi variety of dimension 16 in Zak's classification, [39, IV. Theorem 4.7]. The rest of varieties of type E_6 can be seen as families of subvarieties (or flags of subvarieties) in $E_6(1)$. For instance $E_6(6)$ (which is isomorphic to $E_6(1)$) parametrizes smooth 8-dimensional quadrics in $E_6(1)$; furthermore $E_6(3)$ parametrizes the lines in (the minimal projective embedding of) $E_6(1)$, so that we may assert, by our previous arguments, that the family of lines in $E_6(1)$ passing through a point is isomorphic to the spinor variety $D_5(5)$. In a similar way, the family of lines passing through a point in the variety $E_7(7)$ is isomorphic to $E_6(1)$.

3.1.3 Line bundles on rational homogeneous varieties

Let us finish this section by describing the Picard group of the complete flag $\mathcal{D}(D)$, that contains $\mathrm{Pic}(\mathcal{D}(I))$ via the corresponding pullback map, for every $I \subset D$.

Let us denote by G' the (unique) simply connected semisimple group with Lie algebra $\mathfrak{g}$. One has a surjective finite homomorphism of algebraic groups $\phi : G' \to G$ such that $B' := \phi^{-1}(B)$ is a Borel subgroup of G' (so that $G'/B' \simeq G/B = \mathcal{D}(D)$), and $H' := \phi^{-1}(H) \subset G'$ is a maximal torus; moreover the restriction of ϕ to H' induces an inclusion of $\mathrm{M}(H)$ into $\mathrm{M}(H')$ as a sublattice of finite index. Furthermore after embedding these two lattices in the Euclidean space $\mathrm{M}(H) \otimes_{\mathbb{Z}} \mathbb{R} = \mathrm{M}(H') \otimes_{\mathbb{Z}} \mathbb{R}$, one may identified $\mathrm{M}(H')$ with the lattice of abstract weights of $\mathfrak{g}$, defined as the set of vectors $v \in \mathrm{M}(H) \otimes_{\mathbb{Z}} \mathbb{R}$ satisfying:

$$2\frac{\kappa(v, \alpha)}{\kappa(\alpha, \alpha)} \in \mathbb{Z}, \quad \text{for all } \alpha \in \Phi.$$

For every element $\lambda \in \mathrm{M}(H')$ one may consider its composition with the projection $\xi : B' \to H'$ (defined by quotienting B' by its unipotent radical) to obtain a homomorphism from B' to $\mathbb{C}^*$ defining an action of B' on $\mathbb{C}$ and, subsequently, a line bundle on $\mathcal{D}(D)$:

$$L := G' \times^{B'} \mathbb{C} = (G' \times \mathbb{C})/\sim, \quad (g', v) \sim (g'b', \xi(b')^{\lambda}v) \text{ for every } b' \in B'.$$

Furthermore, one may prove (see for instance [34, Theorem 6.4]) that:

Proposition 3.2 *With the notation as above, the correspondence $\lambda \mapsto L$ induces an isomorphism of lattices:*

$$\mathrm{M}(H') \simeq \mathrm{Pic}(\mathcal{D}(D)).$$

Finally we will denote by $\{\lambda_1, \ldots, \lambda_n\} \in \mathrm{M}(H')$ the set of fundamental weights of $\mathfrak{g}$, defined by:

$$2\frac{\kappa(\alpha_i, \lambda_j)}{\kappa(\alpha_i, \alpha_i)} = \delta_{ij}, \text{ for every } i, j,$$

and by $L_1, \ldots, L_n \in \mathrm{Pic}(\mathcal{D}(D))$ the line bundles associated with $\lambda_1, \ldots, \lambda_n$ (which constitute a $\mathbb{Z}$-basis of $\mathrm{Pic}(\mathcal{D}(D))$). Note that we have:

$$L_i \cdot \Gamma_j = \delta_{ij}, \quad \text{for every } i, j.$$

The numerical classes of the $L_i's$ (also denoted by L_i) generate the nef cone of $\mathcal{D}(D)$ (cf. [24, Proposition 1]), $\mathrm{Nef}(\mathcal{D}(D)) \subset \mathrm{N}^1(\mathcal{D}(D))$, and more generally, denoting by $\pi_I : \mathcal{D}(D) \to \mathcal{D}(I)$ the natural projection, we may write:

$$\pi_I^*(\mathrm{Nef}(\mathcal{D}(I))) = \mathbb{R}_{\geq 0}\{L_i, \ i \in I\}.$$

By abuse of notation, for every $i \in D$ we will denote also by L_i the line bundle on $\mathcal{D}(i)$ whose pullback to $\mathcal{D}(D)$ is $L_i \in \operatorname{Pic}(\mathcal{D}(D))$. They are known to be very ample and to generate $\operatorname{Pic}(\mathcal{D}(i))$. The action of $\mathbb{C}^*$ on $\mathcal{D}(i)$ extends to an action on the projective space $\mathbb{P}(\mathrm{H}^0(\mathcal{D}(i), L_i)^\vee)$, in which $\mathcal{D}(i)$ is embedded. Although it is not true in general that the family of lines in $\mathcal{D}(i)$ is G-homogeneous (this happens whenever i is not an exposed short node of $\mathcal{D}$, see [19, Theorem 4.3]), there is a covering family of lines parametrized by a G-homogeneous projective variety; more concretely, one considers the set $\operatorname{ngb}(i)$ of nodes linked to i in the Dynkin diagram $\mathcal{D}$, so that the natural maps:

$$\mathcal{D}(\operatorname{ngb}(i) \cup \{i\})$$

$$\mathcal{D}(\operatorname{ngb}(i)) \qquad\qquad \mathcal{D}(i)$$
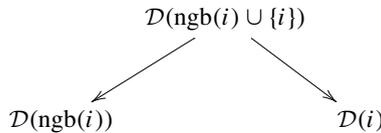
can be regarded as a family of rational curves in $\mathcal{D}(i)$ and the corresponding evaluation. This is called the family of G-*isotropic lines* in $\mathcal{D}(i)$, and it can be described as the image in $\mathcal{D}(i)$ of the fibers of the contraction $\mathcal{D}(D) \to \mathcal{D}(D \setminus \{i\})$; abusing notation, we will denote their numerical classes by Γ_i as mentioned before.

3.2 $\mathbb{C}^*$-actions on rational homogeneous varieties

3.2.1 $\mathbb{C}^*$-actions vs. co-characters

Up to composition with a character, an action of $\mathbb{C}^*$ on a rational homogeneous variety X can always be extended to the action of a maximal torus H of an adjoint group G acting transitively on X, so that we may write $X = G/P$ for some P, and assume (after conjugation) that $H \subset B \subset P$. In other words, $\mathbb{C}^*$-actions on G/P are parametrized by group homomorphisms:

$$\sigma : \mathrm{M}(H) \to \mathbb{Z},$$

called co-characters of H. Note that by restricting this map to the root system Φ of G one gets a $\mathbb{Z}$-grading of the Lie algebra $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha = \mathfrak{h} \oplus \underbrace{\bigoplus_{\sigma(\alpha)=0} \mathfrak{g}_\alpha}_{\mathfrak{g}_0} \oplus \bigoplus_{m \in \mathbb{Z}_{\neq 0}} \underbrace{\bigoplus_{\sigma(\alpha)=m} \mathfrak{g}_\alpha}_{\mathfrak{g}_m}$$

The subspace $\mathfrak{g}_0 \subset \mathfrak{g}$ is a Lie subalgebra of $\mathfrak{g}$, that is reductive.

In this paper, we will focus mostly on the case of $\mathbb{C}^*$-actions given by a particular type of co-characters, defined as follows. Given an index $i \in D$ we consider the grading in $\mathfrak{g}$ given by the unique map σ_i sending the positive simple root α_i to 1 and α_j, $j \neq 0$, to 0. In this case, the subalgebra $\mathfrak{g}_0$ is the Lie subalgebra of a (reductive) subgroup G_0 that is a Levi part of a parabolic subgroup $P_i \subset G$, whose Lie algebra is

$$\mathfrak{p}_i = \mathfrak{g}_0 \oplus \bigoplus_{m>0} \mathfrak{g}_m.$$

Note also that the subgroup $\mathbb{C}^* \subset H$ determined by σ_i can be described as the center of the subgroup $G_0 \subset G$.

3.2.2 The transversal group of a $\mathbb{C}^*$-action

Let σ_i be a co-character of H as above, and let $G_0 \subset G$ be the corresponding subgroup determined by the associated $\mathbb{Z}$-grading of $\mathfrak{g}$.

Definition 3.3 With the above notation, the semisimple group

$$G^\perp := [G_0, G_0] \subset G$$

is called the *transversal subgroup* of the co-character σ_i.

Remark 3.4 Note that by construction, for every co-character σ_i as above, $[\mathbb{C}^*, G^\perp] = \{1\}$. In particular, the action of $\mathbb{C}^*$ associated with σ_i on any homogeneous manifold G/P commutes with the action of $G^\perp$. In particular, the action of $G^\perp$ on G/P sends $\mathbb{C}^*$-orbits to $\mathbb{C}^*$-orbits, and leaves the set of $\mathbb{C}^*$-fixed points $(G/P)^{\mathbb{C}^*}$ invariant; since moreover $G^\perp$ is connected, then every component of $\mathbb{C}^*$-fixed points is invariant by $G^\perp$. More precisely:

Proposition 3.5 *Let $X = G/P$ be a rational homogeneous variety, endowed with the $\mathbb{C}^*$-action associated with a co-character σ_i of G as above. Then the fixed point components of X by the action of $\mathbb{C}^*$ are rational homogeneous $G^\perp$-varieties.*

Proof The proof follows the line of argumentation of [35, Theorem 2.6]. We start by considering a very ample G-homogeneous line bundle L on G/P, so that we have an embedding $G/P \subset \mathbb{P}(V)$ where $V = \mathrm{H}^0(X, L)^\vee$, and the action of G on X is the restriction of the action of G on $\mathbb{P}(V)$. In particular we get an action of $\mathbb{C}^* \subset H \subset G$ on V, that provides a $\mathbb{Z}$-grading:

$$V = \bigoplus_{m \geq 0} V_m, \quad V_m := \{v \in V \mid t(v) = t^m v\}.$$

In particular $X^{\mathbb{C}^*} = \bigcup_{m \geq 0} \mathbb{P}(V_m) \cap X$. Note that we may consider V as a $\mathfrak{g}$-module, and then V is a $\mathbb{Z}$-graded module over the $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}$, so that we have

$$\mathfrak{g}_k V_m \subset V_{k+m}, \quad \text{for every } m, k \in \mathbb{Z}. \tag{3}$$

By Remark 3.4, every component of $\mathbb{P}(V_m) \cap X$ is $G^\perp$-invariant, for every m. The proof may then be completed by showing that:

$$T_{X \cap \mathbb{P}(V_m), x} = T_{G^\perp x, x}, \quad \text{for every } x \in \mathbb{P}(V_m) \cap X.$$

Since $G_0 = \mathbb{C}^* G^\perp = G^\perp \mathbb{C}^*$, and x is fixed by $\mathbb{C}^*$, it is enough to show that $T_{X \cap \mathbb{P}(V_m), x}$ is equal to $T_{G_0 x, x}$. The point x is the class modulo homotheties of a vector $v \in V_m$, so we have:

$$T_{G_0 x, x} = \frac{\mathfrak{g}_0 v}{\langle v \rangle}, \quad T_{X \cap \mathbb{P}(V_m), x} = \frac{\mathfrak{g} v \cap V_m}{\langle v \rangle}.$$

We conclude by noting that $\mathfrak{g}_0 v = \mathfrak{g} v \cap V_m$ by Eq. (3). $\qquad\qquad\square$

3.3 The action of a maximal torus on a complete flag manifold

An important ingredient that we will use later on is the action of the Cartan subgroup $H \subset B \subset G$ on the complete flag G/B, whose properties are well known (cf. [8, Section 3.4]), since they are essentially related to the Bruhat decomposition of G. Then via the natural G-equivariant projections $G/B \to G/P$, one may study the action of H (and of any subtorus in it) on any G/P.

3.3.1 Fixed points of the Cartan action on G/B

The following statement describes the fixed points of the H-action on G/B, and its infinites-imal behavior around those points; we include a proof for the reader's convenience.

Proposition 3.6 *Let G be a semisimple algebraic group. The set of fixed points of the action of a maximal torus $H \subset B \subset G$ on the complete flag G/B is:*

$$(G/B)^H = \{wB|\ w \in W\}.$$

For every $w \in W$, the set of weights of the H-action on the tangent space $T_{G/B,wB}$ is $\{w(\alpha)|\ -\alpha \in \Phi^+\}$.

Proof Note first that given an element $w = nH \in W$, the class nB does not depend on the choice of n, so it makes sense to denote nB by wB, as we have done in the statement.

A point $gB \in G/B$ is H-fixed if and only if $g^{-1}Hg \subset B$. Since any two maximal tori in a connected solvable group are conjugated (cf. [15, 19.3]), there exists $b \in B$ such that $g^{-1}Hg = bHb^{-1}$, and we conclude that $gb \in N_G(H)$. By setting $w = gbH \in W$, we may finally write $gB = gbB = wB$.

For the second part it is enough to note that $T_{G/B,wB}$ is ($\mathbb{C}^*$-equivariantly) isomorphic to $\mathfrak{g}/\mathrm{Ad}_w(\mathfrak{b})$, which decomposes in $\mathbb{C}^*$-eigenspaces as:

$$\mathfrak{g}/\mathrm{Ad}_w(\mathfrak{b}) = \sum_{-\alpha \in \Phi^+} \mathfrak{g}_{w(\alpha)}.$$

$\square$

3.3.2 Weights of the Cartan action

Now we will compute the weights of the H-action on G/B on its set of fixed points, with respect to a given line bundle $L \in \mathrm{Pic}(G/B)$, which we identify with a character $\lambda \in \mathrm{M}(H')$ of the maximal torus of the universal covering G' of G, see Proposition 3.2. While the existence of a linearization of the action of H on L is guaranteed in a more general setting by [18, Proposition 2.4], the special behavior of line bundles on rational homogeneous varieties (see Sect. 3.1) allows us to describe easily the associated weights.

In fact, given a line bundle L on G/B associated with a weight $\lambda \in \mathrm{M}(H')$, its description as a homogeneous bundle allows us to define a linearization of the action of H' on L for any character $m \in \mathrm{M}(H')$, as follows:

$$h'[(g', v)] = [(h'g', h'^m v)], \ \text{for any } h' \in H', [(g', v)] \in L.$$

The important point to note here is that for the particular choice $m = \lambda$ the action of H' on L descends to an action of H. In fact, taking into account that the kernel of the map $H' \to H$ is the center of G', we may write, for every $z \in \ker(H' \to H)$:

$$z[(g', v)] = [(zg', z^\lambda v)] = [(g'z, z^\lambda v)] = [(g', v)],$$

for every $[(g', v)] \in L$. We may now compute the weights of the H-action with respect to L.

Proposition 3.7 *Let $L \in \mathrm{Pic}(G/B)$ be a line bundle associated with a weight λ. There exists a linearization of the H-action on G/B whose weights are:*

$$\mu_L^H(wB) = \lambda - w(\lambda) \in \mathrm{M}(H), \ \text{for any } w \in W.$$

Proof For every $h \in H$, $w \in W$, $v \in \mathbb{C}$, we have

$$
\begin{aligned}
h[(w, v)] = [(hw, h^\lambda v)] &= [(ww^{-1}hw, h^\lambda v)] \\
&= [(w, (w^{-1}hw)^{-\lambda} h^\lambda v)] = [(w, h^{-w(\lambda)+\lambda} v)].
\end{aligned}
$$

$\square$

If $\mathbb{C}^* \subset H$ is a subtorus associated with a co-character $\sigma_i : \mathrm{M}(H) \to \mathbb{Z}$, then one knows (cf. [29, Lemma 2.1 (ii)]) that every $\mathbb{C}^*$-fixed component contains at least an H-fixed point. In particular, the following Corollary describes the weights of the $\mathbb{C}^*$-action with respect to any line bundle L.

Corollary 3.8 *Let $L \in \mathrm{Pic}(G/B)$ be a line bundle associated with a weight λ, and $\mathbb{C}^* \subset H$ be a subtorus associated with a co-character $\sigma_i : \mathrm{M}(H) \to \mathbb{Z}$. There exists a linearization of the action whose weights on H-fixed points are:*

$$
\mu_L(wB) = \sigma_i(\lambda - w(\lambda)) \in \mathrm{M}(H), \text{ for any } w \in W.
$$

3.3.3 Fixed point components of $\mathbb{C}^*$-actions on rational homogeneous varieties

We already know that the fixed point components of the $\mathbb{C}^*$-action determined by a co-character σ_i on a variety G/P are $G^\perp$-homogeneous (see Proposition 3.5). We will now describe more precisely these components. We start with the case of the complete flag manifold G/B.

Proposition 3.9 *Let G be a semisimple algebraic group, B a Borel subgroup and $X = G/B$ the corresponding complete flag manifold. Consider the $\mathbb{C}^*$-action on X induced by a co-character of the form σ_i as above. Then the irreducible fixed point components of the action are flag manifolds with respect to the semisimple group $G^\perp$ transversal to the $\mathbb{C}^*$-action.*

Proof By [29, Lemma 2.1(ii)], every irreducible component of $X^{\mathbb{C}^*}$ contains a point of X^H, hence, by Proposition 3.5, every irreducible fixed point component of X by $\mathbb{C}^*$ is of the form $G^\perp wB$ for some $w \in W$. Since the isotropy subgroup of wB by the action of G is $\mathrm{conj}_w(B)$, it follows that

$$
G^\perp wB = \frac{G^\perp}{G^\perp \cap \mathrm{conj}_w(B)}.
$$

Since $G^\perp \cap \mathrm{conj}_w(B)$ is a Borel subgroup of $G^\perp$, the statement follows. $\square$

As a direct consequence (by projecting the $\mathbb{C}^*$-fixed point components in G/B via the natural projection to G/P), we obtain a description of the fixed point components of the $\mathbb{C}^*$-actions introduced above on rational homogeneous spaces of the form G/P.

Corollary 3.10 *Let G be a semisimple algebraic group, B a Borel subgroup, $P \supset B$ a parabolic subgroup. Let $X = G/P$ be the corresponding rational homogeneous space, and consider the $\mathbb{C}^*$-action on X induced by a co-character of the form σ_i as above. Then for every $w \in W$ the irreducible fixed point component passing through the point wP is the rational homogeneous variety:*

$$
\frac{G^\perp}{G^\perp \cap \mathrm{conj}_w(P)}.
$$

Table 1 List of short gradings on simple Lie algebras

$\mathfrak{g}$	A_n	B_n	C_n	D_n	E_6	E_7
σ_i	$\sigma_i \ (1 \leq i \leq n)$	σ_1	σ_n	$\sigma_1, \sigma_{n-1}, \sigma_n$	σ_1, σ_6	σ_7
$\mathfrak{g}^\perp$	$A_{i-1} \times A_{n-i}$	B_{n-1}	A_{n-1}	$D_{n-1}, A_{n-1}, A_{n-1}$	D_5, D_5	E_6

Remark 3.11 It is worth observing that the extremal values of μ_L are achieved at the points eB and w_0B, where $w_0 \in W$ denotes the longest element of the Weyl group of G. In particular, the sink and the source of the $\mathbb{C}^*$-action induced by σ_i are the $G^\perp$-orbits of eB, w_0B, respectively. More generally, if we consider any non-trivial $\mathbb{C}^*$-action induced by a co-character $\sigma : \mathrm{M}(H) \to \mathbb{Z}$, satisfying $\sigma(\alpha_j) \geq 0$ for every $\alpha_j \in \Delta$, the description of the weights of the H-action on the tangent spaces of G/B at eB and w_0B given in Proposition 3.6 tells us that the weights with respect to $\mathbb{C}^*$ at $T_{G/B,eB}$, $T_{G/B,w_0B}$ are all non-positive and non-negative, respectively. This implies that eB, w_0B belong to the sink and the source of this action, respectively.

3.4 Equalized actions with isolated extremal fixed points on rational homogeneous spaces

Later on, we will study the birational maps associated with $\mathbb{C}^*$-actions on rational homogeneous varieties, and we will be interested in the case in which the domain of the birational map is the projective space. For that purpose, we will concentrate on the $\mathbb{C}^*$-actions on rational homogeneous varieties that are equalized and have isolated sink and source. As we will see, these actions are completely classified (see Corollary 3.18). Notice that, using different techniques mainly based on VMRT theory, these varieties have been recently characterized among irreducible Hermitian spaces of "tube type"; we refer to [20] for details.

3.4.1 Equalized actions on rational homogeneous spaces

Via the correspondence of $\mathbb{C}^*$-actions on rational homogeneous varieties with co-characters and $\mathbb{Z}$-gradings (Sect. 3.2), it is known that equalized $\mathbb{C}^*$-actions correspond to *short gradings* (see [10]), that is, to those gradings for which $\mathfrak{g}_m = 0$ if and only if $m \neq 0, \pm 1$. Throughout this section, we will always assume that $\mathfrak{g}$ is simple; in this case the possible short gradings of $\mathfrak{g}$ are known (cf. [35, p. 42]): up to conjugation, they correspond to some co-characters of the form σ_i. The complete list of these co-characters can be read from the Table 1.

Remark 3.12 (Products) In the case in which $\mathfrak{g}$ is semisimple but not simple, a short grading on $\mathfrak{g} = \bigoplus_k \mathfrak{g}^k$ is given by the choice of a short grading on each of its simple direct summands $\mathfrak{g}^k$. If these short gradings are given by co-characters σ_{i_k} in the list of Table 1, then the short grading on $\mathfrak{g} = \bigoplus_k \mathfrak{g}^k$ is given by the co-character denoted by $\sum_k \sigma_{i_k}$. Translated into the language of $\mathbb{C}^*$-actions, an equalized $\mathbb{C}^*$-action on a rational homogeneous variety of the form $\prod_{k=1}^r G^k/P^k$, with $G^1, \ldots, G^r$ simple, is given by the product of an equalized $\mathbb{C}^*$-action on every G^k/P^k.

3.4.2 Actions of $\mathbb{C}^*$ with isolated sink or source

The following statement tells us when a $\mathbb{C}^*$-action associated with a co-character σ_i on a rational homogeneous G-variety has isolated sink.

Proposition 3.13 *The $\mathbb{C}^*$-action associated with the choice of a co-character σ_i on a rational homogeneous G-variety $\mathcal{D}(I)$ has isolated sink if and only if $I = \{i\}$.*

Proof Write $\mathcal{D}(I)$ as G/P for some $P \supset B$. By Remark 3.11, the sink of the action will be the fixed point component passing through eP. Then, by Corollary 3.10, the action has isolated sink if and only if $G^{\perp} \subset P$, and this is possible if and only if $I = \{i\}$. □

Remark 3.14 (**Reversion**) A similar argument allows us to state that our $\mathbb{C}^*$-action has isolated source if and only if $G^{\perp} \subset \text{conj}_{w_0} P$. Now we use the description of w_0 (see Remark 3.1). If the action of w_0 on $M(H)$ equals $-\text{id}$, then $\text{conj}_{w_0} P$ is nothing but the opposite parabolic subgroup of P, which contains $G^{\perp}$ if and only if P does. Therefore, in the cases:

$$B_n, \quad C_n, \quad D_n \ (n \text{ even}), \quad E_7$$

the action of $\mathbb{C}^*$ associated with σ_i on $\mathcal{D}(I)$ has isolated sink and source, if and only if $I = \{i\}$.

In the remaining cases, we use Remark 3.1 to identify the parabolic subgroup $\text{conj}_{w_0}(P)$, and to check when it does contain $G^{\perp}$ obtaining:

- for every i, $A_n(I)$ has isolated sink if and only if $I = \{i\}$ and isolated source if and only if $I = \{n+1-i\}$; in particular, if n is odd and $i = (n+1)/2$, $A_n(i)$ has isolated sink and source;
- for $i = 1$, $D_n(I)$ has isolated sink and source if and only if $I = \{1\}$; for $i = n-1, n$, the action on $D_n(I)$ has isolated sink if and only if $I = \{i\}$, and isolated source if and only if $I = \{n-1, n\} \setminus \{i\}$;
- for $i = 1, 6$, the action on $E_6(I)$ has isolated sink if and only if $I = \{i\}$, and isolated source if and only if $I = \{1, 6\} \setminus \{i\}$.

Remark 3.15 (**Fixed points on products**) Note that, in the setup of Proposition 3.13, even if $\mathfrak{g}$ not simple, a rational homogeneous variety of the form $\mathcal{D}(i)$ is always a quotient of a semisimple algebraic group with simple Lie algebra; more concretely, if we denote by $\mathcal{D}' \subset \mathcal{D}$ the connected component of $\mathcal{D}$ containing the node i, we have $\mathcal{D}(i) = \mathcal{D}'(i)$. Then, if $\mathfrak{g}$ is a direct sum $\sum_{k=1}^{r} \mathfrak{g}^k$ of simple Lie algebras, and we choose a node i_k on the Dynkin diagram $\mathcal{D}_k$ of $\mathfrak{g}^k$, for every k, the $\mathbb{C}^*$-action on the variety $\mathcal{D}(i_1, \ldots, i_r) = \mathcal{D}_1(i_1) \times \cdots \times \mathcal{D}_r(i_r)$ is the product of the actions of $\mathbb{C}^*$ on the $\mathcal{D}_r(i_r)$'s given by the co-characters σ_{i_k}. Then the fixed point components of this action will be the products of the fixed point components of the $\mathbb{C}^*$-actions on each factor and, in particular, the sink of the action will be isolated. For instance, the action induced by the character $\sigma_1 + \sigma_1$ on $A_n(1) \times A_m(1)$ has four fixed components, isomorphic to a point (the sink), $A_{n-1}(1)$, $A_{m-1}(1)$, and $A_{n-1}(1) \times A_{m-1}(1)$.

Proposition 3.13 provides a list of rational homogeneous varieties of Picard number one admitting equalized $\mathbb{C}^*$-actions with isolated sink. Following Remarks 3.12, 3.15, actions of this kind on rational homogeneous varieties of arbitrary Picard number are obtained by considering products of these.

In order to study the remaining fixed point components of each of these actions, one may proceed as follows. First of all one considers the natural embedding of $\mathcal{D}(i)$ on the irreducible representation V_{λ_i} of $\mathfrak{g}$ associated with the fundamental weight $\lambda_i \in M(H)$. Then one considers the short grading of $\mathfrak{g}$ determined by the root α_i, and splits V_{λ_i} as a direct sum of irreducible $\mathfrak{g}_0$-modules; next one studies the intersection of $\mathcal{D}(i)$ with the corresponding projectivizations. The outcome of the process is the following:

G. Occhetta et al.

Table 2 Equalized $\mathbb{C}^*$-actions with isolated extremal fixed points on rational homogeneous spaces of Picard number one

$\mathfrak{g}$	$\mathcal{D}(i)$	Weights j	Fixed point comp. of weight j
A_n	$A_n(i)$	$0, \ldots, \min\{i, n-i+1\}$	$A_{i-1}(i-j) \times A_{n-i}(j)$
	$A_n(n+1-i)$	$\max\{0, n-2i+1\}, \ldots \ldots, n-i+1$	$A_{n-i}(j) \times A_{i-1}(n+1-i-j)$
C_n	$C_n(n)$	$0, \ldots, n$	$A_{n-1}(j)$
B_n	$B_n(1)$	$0, 1, 2$	$B_{n-1}(0), B_{n-1}(1), B_{n-1}(0)$
D_n	$D_n(1)$	$0, 1, 2$	$D_{n-1}(0), D_{n-1}(1), D_{n-1}(0)$
D_n	$D_n(n)$	$0, \ldots, \lfloor n/2 \rfloor$	$A_{n-1}(n-2j)$
	$D_n(n-1)$	$0, \ldots, \lfloor n/2 \rfloor$	$A_{n-1}(n-2(\lfloor n/2 \rfloor - j))$
E_6	$E_6(1)$	$0, 1, 2$	$D_5(6), D_5(5), D_5(1)$
	$E_6(6)$	$0, 1, 2$	$D_5(1), D_5(5), D_5(6)$
E_7	$E_7(7)$	$0, 1, 2, 3$	$E_6(0), E_6(1), E_6(6), E_6(7)$

Table 3 Equalized $\mathbb{C}^*$-actions with isolated sink and source

Variety	$A_{2n-1}(n)$	$C_n(n)$	$B_n(1)$	$D_n(1)$	$D_n(n)$	$E_7(7)$
restrictions	$n \geq 1$	$n \geq 2$	$n \geq 2$	$n \geq 4$	$n \geq 6$, n even	
dimension	n^2	$\binom{n+1}{2}$	$2n-1$	$2n-2$	$\binom{n}{2}$	27
δ	n	n	2	2	$n/2$	3

Proposition 3.16 *Let G be a semisimple group with a simple Lie algebra. The complete list of rational homogeneous G-varieties admitting equalized $\mathbb{C}^*$-actions with an extremal isolated fixed point, and of their irreducible fixed point components, is given in Table* 2.

Remark 3.17 Table 2 must be read with the following conventions: first of all, for any k, we set $\mathcal{D}(k+1) = \mathcal{D}(0)$ to be a point for any diagram with k nodes $\mathcal{D}$. Second, the weights of the fixed point components are taken with respect to a linearization of the ample generator of the Picard group of the corresponding variety with weight 0 at the sink of the action. As a consequence we can see that the criticality and the bandwidth of the action coincide with the maximal weight δ of the action.

3.4.3 $\mathbb{C}^*$-actions with isolated sink and source

Finally, as a consequence of Proposition 3.16, we get a description of all the actions with isolated extremal fixed points, which will have Cremona transformations as their induced birational maps.

Corollary 3.18 *The complete list of rational homogeneous spaces of Picard number one admitting equalized $\mathbb{C}^*$-actions with isolated sink and source is given in Table* 3.

Remark 3.19 In the list above only four cases are geometric realizations of bispecial Cremona transformations (see [26, Definition 4.1]); they are exactly the varieties classified in

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[27, Theorem 8.10], which correspond to the whole list of quadro-quadric special Cremona transformations of Ein and Shepherd-Barron, associated with the four Severi varieties:

$$A_5(3), \quad C_3(3), \quad D_6(6), \quad E_7(7).$$

Note also that the list of quadro-quadric Cremona transformations with smooth fundamental locus (classified in [31, Proposition 5.6]) is completed by adding the birational map induced by the equalized $\mathbb{C}^*$-action on $\mathbb{P}^1 \times Q^n$, where Q^n denotes the n-dimensional smooth quadric. From the description of the fixed point components of the action, it follows that the fundamental locus of the associated birational map in this case is the union of a quadric Q' of dimension $n - 2$ and a point outside of the linear span of Q'.

4 Equivariant Cremona transformations

This section is devoted to the Cremona transformations induced by the equalized $\mathbb{C}^*$-actions with isolated sink and source on rational homogeneous varieties (Corollary 3.18). An important property satisfied by these transformations is that they commute with the action of a semisimple group: the transversal group $G^\perp$. As we will see, under this representation-theoretical point of view one may interpret these Cremona transformations as maps of inversion.

A first important observation is the following:

Corollary 4.1 *Let $X = G/P$ be a rational homogeneous variety equipped with the $\mathbb{C}^*$-action associated to a co-character $\sigma_i : \mathrm{M}(H) \to \mathbb{Z}$, and let $G^\perp$ be the corresponding transversal group. Denote by $Y_\pm$ the sink and source of the $\mathbb{C}^*$-action, and by $\psi : \mathbb{P}(\mathcal{N}_{Y_-|X}) \dashrightarrow \mathbb{P}(\mathcal{N}_{Y_+|X})$ the induced birational transformation. Then the action of $G^\perp$ on X induces actions on $\mathbb{P}(\mathcal{N}_{Y_\pm|X})$ so that ψ is $G^\perp$-equivariant.*

Proof As we noted in Remark 3.4 the action of $G^\perp$ on X leaves invariant $Y_\pm$, hence it extends to an action on $\mathbb{P}(\mathcal{N}_{Y_\pm|X})$. Moreover, since $\mathbb{P}(\mathcal{N}_{Y_\pm|X})$ can be thought of as the geometric quotient parametrizing 1-dimensional $\mathbb{C}^*$-orbits converging to $Y_\pm$, and $G^\perp$ sends closures of $\mathbb{C}^*$-orbits to closures of $\mathbb{C}^*$-orbits (Remark 3.4), then ψ is $G^\perp$-equivariant. □

Applying 2.1.13 to each of the actions described in Table 2—in which the source is a point—we get a birational transformation ψ onto the projectivization of the tangent space of X at the point Y_+. When the sink Y_- is positive dimensional, $\mathbb{P}(\mathcal{N}_{Y_-|X})$ is the projectivization of a homogeneous bundle over Y_-. In particular, with the exception $X = \mathbb{P}^n$ and the sink of the action is a hyperplane, the variety $\mathbb{P}(\mathcal{N}_{Y_-|X})$ has Picard number bigger than one. On the other hand, in the cases of Table 3, we obtain Cremona transformations.

Definition 4.2 Let G be a semisimple group, and consider two projective representations $\mathbb{P}(V_1)$, $\mathbb{P}(V_2)$ of G of the same dimension. A Cremona transformation $\psi : \mathbb{P}(V_1) \to \mathbb{P}(V_2)$ is said to be *G-equivariant* if it commutes with the action of G.

The most obvious examples of equivariant Cremona transformations are those listed in Table 3, that we will now describe geometrically. At this point, Problem 1 that we have already stated in the Introduction naturally arises.

4.1 Example: quadrics

The least interesting of the birational transformations induced by the actions of Table 3 is the case of the non-singular quadrics Q of the form $B_n(1)$ or $D_n(1)$; following [28, Section 5.2], since the criticality of the action is two, and the extremal fixed points are isolated, ψ is forced to be an isomorphism.

This can be also seen as follows: the $\mathbb{C}^*$-action on Q has two extremal fixed points P_-, P_+, which are not connected by a line in Q. Every tangent direction $v_- \in \mathbb{P}(T_{Q,P_-})$ determines a plane π_{v_-} containing P_-, P_+ and the direction v_-. This plane meets Q along a conic C passing through P_-, P_+; this conic is reduced, not necessarily irreducible, and it is non-singular at P_-, P_+. Moreover, since the action is equalized, the plane π_{v_-} is $\mathbb{C}^*$-invariant, hence C is $\mathbb{C}^*$-invariant and we may conclude that $\psi(v_-) = [T_{C,P_+}] \in \mathbb{P}(T_{Q,P_+})$. In particular ψ is well-defined for every $v_- \in \mathbb{P}(T_{Q,P_-})$, so it is regular. The same clearly holds for its inverse.

4.2 Example: balanced Grassmannians

Let us consider two complex vector spaces V_-, V_+ of dimension $n \geq 1$, and the action of $\mathbb{C}^*$ on $V_- \oplus V_+$ that leaves V_-, V_+ invariant, whose weights on these spaces are 0 and 1, respectively. Then we consider the induced action on the Grassmannian $A_{2n-1}(n)$ of n-dimensional linear subspaces of $V_- \oplus V_+$. Its extremal fixed points correspond to the subspaces V_- and V_+. By 2.1.13, the action of $\mathbb{C}^*$ induces a Cremona transformation among the projectivizations of the tangent spaces of $A_{2n-1}(n)$ at the extremal fixed point components:

$$\psi : \mathbb{P}(V_-^\vee \otimes V_+) \dashrightarrow \mathbb{P}(V_+^\vee \otimes V_-).$$

In order to describe this map, we note that a general element of $A_{2n-1}(n)$ is determined by a skew-symmetric tensor:

$$s := (v_1^- + v_1^+) \wedge \cdots \wedge (v_n^- + v_n^+),$$

where $v_1^-, \ldots, v_n^- \in V_-$ and $v_1^+, \ldots, v_n^+ \in V_+$ are linearly independent vectors. Since the orbit of $[s]$ in $A_{2n-1}(n)$ is the set of points of the form:

$$t[s] = \left[(v_1^- \wedge \cdots \wedge v_n^-) + t \sum_{i=1}^n (v_1^- \wedge \cdots \wedge v_i^+ \wedge \cdots \wedge v_n^-) + O(t^2) \right],$$

then the limiting point of the orbit of $[s]$ is the blowup of $A_{2n-1}(n)$ along the source and, as t approaches to 0, the sink is given by:

$$\lim_{t \to 0} t[s] = \left[\sum_{i=1}^n v_1^- \wedge \cdots \wedge v_i^+ \wedge \cdots \wedge v_n^- \right],$$

and analogously:

$$\lim_{t \to \infty} t[s] = \left[\sum_{i=1}^n v_1^+ \wedge \cdots \wedge v_i^- \wedge \cdots \wedge v_n^+ \right].$$

These points can be identified, respectively, with the class of the linear map from V_- to V_+ sending every v_i^- to v_i^+, and its inverse. In other words, as noted by Thaddeus in [37, Section 4], the map ψ is the projectivization of the inversion map.

Remark 4.3 Note that, in the particular case $n = 3$, the exceptional locus of ψ is a Segre variety $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$, which is the Severi variety associated to the division algebra $\mathbb{C}$. The map ψ is one of the special Cremona transformations characterized in [9].

4.3 Example: isotropic Grassmannians

We refer to [21] for more details on this example and its applications. The equalized $\mathbb{C}^*$-action on $C_n(n)$ described in Table 3 can be seen as a restriction of the one we have just studied in the previous Example. In order to see this we start from a direct sum $V_- \oplus V_+$ as above, and consider an isomorphism $\phi : V_- \to V_+^\vee$, which defines a non-degenerate skew-symmetric 2-form ω on $V_- \oplus V_+$:

$$\omega(u^- + u^+, v^- + v^+) := \phi(u^-)(v^+) - \phi^T(u^+)(v^-).$$

The $\mathbb{C}^*$-action on $V_- \oplus V_+$ of the previous example, which acts trivially on V_- and homothetically on V_+, preserves orthogonality with respect to ω, hence leaves invariant the subset $C_n(n) \subset A_{2n-1}(n)$ of ω-isotropic subspaces. Since $[V_-], [V_+] \in C_n(n)$, they are the extremal isolated fixed points of the action.

Note that the tangent space of $C_n(n)$ at $[V_\pm]$ is isomorphic to $S^2 V_\pm$, and that its embedding into $T_{A_{2n-1}(n),[V_\pm]} \simeq V_\pm^\vee \otimes V_\mp$ is induced naturally by the isomorphism ϕ. Then the birational map

$$\psi : \mathbb{P}(S^2 V_-) \dashrightarrow \mathbb{P}(S^2 V_+) \simeq \mathbb{P}(S^2 V_-^\vee)$$

is just the (projectivization of the) inversion map, restricted to symmetric tensors.

Remark 4.4 In the particular case $n = 3$, we obtain another special Cremona transformation: the one whose exceptional locus is the Veronese surface in $\mathbb{P}^5$, which is the Severi variety associated to the division algebra $\mathbb{R}$.

4.4 Example: spinor varieties

The case of $D_n(n)$ (n even) is similar to the one of $C_n(n)$ described above, in the sense that it can also be obtained as a restriction of the case of $A_{2n-1}(n)$ by considering a non-degenerate symmetric 2-form on $V_- \oplus V_+$:

$$q(u^- + u^+, v^- + v^+) = \phi(u^-)(v^+) + \phi^T(u^+)(v^-),$$

induced by a given isomorphism $\phi : V_- \to V_+^\vee$. This defines a quadric $Q \subset \mathbb{P}(V_- \oplus V_+)$ containing $\mathbb{P}(V_-), \mathbb{P}(V_+)$, which are invariant by the action of $\mathbb{C}^*$, and so we get a $\mathbb{C}^*$-action on Q with extremal fixed point components $\mathbb{P}(V_-), \mathbb{P}(V_+)$. As in the case of $C_n(n)$, this induces a $\mathbb{C}^*$-action on the set of linear subspaces contained in Q of maximal dimension, which consists of two (isomorphic) irreducible components, $D_n(n-1), D_n(n)$. Since the dimension of the intersection of two maximal linear subspaces in Q is congruent to $(n-1)$ modulo 2 if and only if they belong to the same irreducible family ([14, Theorem 21.14]), it follows that $\mathbb{P}(V_-), \mathbb{P}(V_+)$ belong to the same family if and only if n is even. Note that in Table 3 we disregard the case $n = 4$ since in this case $D_4(4) \simeq D_4(1)$, and so the action is described in 4.1.

The interpretation of the birational map ψ in this case is analogous to the case of the isotropic Grassmannian. This time we have isomorphisms:

$$T_{\mathrm{D}_n(n),[V_\pm]} \simeq \bigwedge^2 V_\pm,$$

and we may think of

$$\psi : \mathbb{P}\left(\bigwedge^2 V_-\right) \dashrightarrow \mathbb{P}\left(\bigwedge^2 V_+\right) \simeq \mathbb{P}\left(\bigwedge^2 V_-^\vee\right)$$

as the projectivization of the inversion of skew-symmetric tensors.

Remark 4.5 As in the previous examples, in one particular case ($n = 6$) we obtain a special Cremona transformation; in this case ψ is the quadro-quadric transformation whose exceptional locus is the Grassmannian $\mathrm{A}_5(2)$, which is the Severi variety associated to the division algebra $\mathbb{H}$ of quaternions.

4.5 Example: the E_7 case

The last special Cremona transformation, associated to the algebra $\mathbb{O}$ of octonions, makes its appearance in the case of the 27-dimensional variety $\mathrm{E}_7(7)$, which is the unique rational homogeneous variety of exceptional type admitting an equalized $\mathbb{C}^*$-action with isolated extremal fixed points. First of all, a straightforward computation shows that the bandwidth of the action with respect to the ample generator of the Picard group is three. The inner fixed point components of the action are isomorphic to the variety of (E_7-isotropic) lines passing by the sink and the source, which is isomorphic to the Cartan variety $\mathrm{E}_6(1)$, which is the Severi variety associated to the algebra $\mathbb{O}$ of octonions. Then $\mathrm{E}_6(1)$ is the exceptional locus of the Cremona transformation $\psi : \mathbb{P}^{26} \dashrightarrow \mathbb{P}^{26}$. Since moreover one can show that ψ is quadro-quadric (see [27, Section 8]), we conclude that this is precisely the special Cremona transformation associated to that Severi variety.

5 Geometric realizations and rational homogeneous bundles

By restricting $\mathbb{C}^*$-actions on rational homogeneous varieties to invariant subvarieties we may construct more examples of $\mathbb{C}^*$-actions, that will be geometric realizations of other birational transformations. We will focus on birational transformations between the projective space and other homogeneous varieties, so we will restrict the $\mathbb{C}^*$-actions to invariant subsets on which one of the two extremal fixed point components is a point. We will illustrate this procedure here by considering restrictions of $\mathbb{C}^*$-actions to some invariant rational homogeneous bundles over $\mathbb{P}^1$, showing in particular that the homogeneous cases of the list of special transformations of type (2, 1), classified by Fu and Hwang (cf. [12]), can be obtained in this way. Let us start by describing the varieties and $\mathbb{C}^*$-actions that we will use.

Setup 5.1 We start with the adjoint group G of a simple Lie algebra $\mathfrak{g}$ with Dynkin diagram $\mathcal{D}$, a Cartan and a Borel subgroup $H \subset B \subset G$, and a co-character $\sigma_i : \mathrm{M}(H) \to \mathbb{Z}$, $i \in D$, inducing a $\mathbb{C}^*$-action on every rational homogeneous variety of the form $\mathcal{D}(I)$; as we have noted in Proposition 3.13, this action has isolated sink on the variety $\mathcal{D}(i)$. We will denote by P_i the parabolic subgroup containing B such that $\mathcal{D}(i) = G/P_i$. The transversal

Fig. 2 The variety X defined in Setup 5.1 and its fixed point components
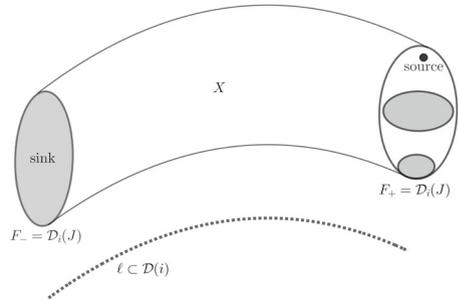
subgroup of the action (which is the semisimple part of P_i) will be denoted by $G^\perp \subset G$; its Dynkin diagram $\mathcal{D}_i$ is obtained deleting the node i on $\mathcal{D}$. As usual, we denote by L_i the ample generator of $\mathrm{Pic}(\mathcal{D}(i))$; the weights of the action with respect to L_i at the sink and the source will be denoted by 0, δ, respectively. We will consider a $\mathbb{C}^*$-invariant irreducible non fixed curve $\ell \subset \mathcal{D}(i)$, and a nonempty subset $J \subset D \setminus \{i\}$. The action of $\mathbb{C}^*$ on $\mathcal{D}(J \cup \{i\})$ restricts to the fiber product

$$X = \mathcal{D}(J \cup \{i\}) \times_{\mathcal{D}(i)} \ell \subset \mathcal{D}(J \cup \{i\}),$$

so that the natural smooth $\mathcal{D}_i(J)$-fibration:

$$\pi : X \longrightarrow \ell$$

is $\mathbb{C}^*$-equivariant (Fig. 2). Since the action on ℓ has two fixed points, denoted by x_-, x_+ (respectively the source and the sink of the action on ℓ), we have two invariant subsets $F_\pm := \pi^{-1}(x_\pm)$, which are isomorphic to $\mathcal{D}_i(J)$. We will assume that $\mathcal{D}$, i, J and ℓ are chosen so that action of $\mathbb{C}^*$ is:

- faithful on ℓ,
- trivial on F_-,
- equalized on F_+, with isolated source.

Proposition 5.2 *In the situation of Setup 5.1, the induced action of $\mathbb{C}^*$ on X is equalized, and it induces a birational map:*

$$\psi : F_- \dashrightarrow \mathbb{P}^{\dim(F_-)}.$$

Proof We have to show that the $\mathbb{C}^*$-action on every 1-dimensional orbit is faithful. Since the action on F_- is trivial, a 1-dimensional orbit $\mathbb{C}^* x$ is either contained in the fiber F_+—where the action is equalized by hypothesis—, or it maps via π onto $\ell \setminus \{x, x_+\}$—and so we conclude by noting that π is $\mathbb{C}^*$-equivariant and the action on ℓ is assumend to be faithful.

We finally note that the sink of the $\mathbb{C}^*$-action on X is the divisor $F_- \subset X$, and that the source coincides with the source of its restriction to F_+, which is an isolated point, say x'_+. Then the birational map induced by the action goes from F_- to $\mathbb{P}(T_{X,x'_+}) \simeq \mathbb{P}^{\dim(F_-)}$. □

In the next subsection, we will see that the (homogeneous) special birational transformations of type $(2, 1)$ can be obtained from some $\mathbb{C}^*$-actions satisfying the above assumptions. Nevertheless, the construction is more general and can be used to construct other birational transformations, such as the one given by in the following example.

Example 5.3 Let us consider the case in which $\mathcal{D}(i)$ is the adjoint variety $E_6(2)$ of the Lie algebra E_6; similar examples can be obtained by starting with other simple Lie algebras.

The action of $\mathbb{C}^*$ on $E_6(2)$ determined by the co-character σ_2 has five fixed point components, which, ordered by their weight with respect to the ample generator of $\mathrm{Pic}(E_6(2))$, are the A_5-homogeneous varieties:

$$Y_- \simeq \{y_-\}, \quad Y_1 \simeq A_5(3), \quad Y_2 \simeq A_5(1,5), \quad Y_3 \simeq A_5(3), \quad Y_+ \simeq \{y_+\}.$$

Note that this action is not equalized.

Let ℓ be a $\mathbb{C}^*$-invariant line passing through $x_- := y_-$ and a point $x_+ \in Y_1$; note that, by the AM vs FM formula (2.1.10), the action on ℓ is faithful. Furthermore, we will consider $J = \{4\}$, so that $E_6(2,4) \to E_6(4)$ is the universal family of lines in $E_6(2)$. Then the variety

$$X = E_6(2,4) \times_{E_6(2)} \ell \subset E_6(2,4)$$

(which is the family of lines in $E_6(2)$ passing by points of ℓ), together with the natural projection $\pi : X \to \ell$, is an $A_5(3)$-bundle over ℓ. The fiber $F_- = \pi^{-1}(x_-)$ parametrizes the lines in $E_6(2)$ passing through x_-, hence it is $\mathbb{C}^*$-fixed. On the other hand, the fiber $F_+ = \pi^{-1}(x_+)$, which parametrizes the lines passing by x_+ is $\mathbb{C}^*$-invariant and that it contains four fixed point components. In fact the $\mathbb{C}^*$-action on $F_+ \simeq A_5(3)$ is the one described in Sect. 4.2 (case $\dim(V_\pm) = 3$), which is equalized and has isolated sink and source (we denote them by $Y'_\pm = \{x'_\pm\}$). It has two inner fixed point components, isomorphic to $\mathbb{P}^2 \times \mathbb{P}^2$. In particular its associated birational transformation is the quadro–quadric Cremona transformation:

$$\psi' : \mathbb{P}(V_-^\vee \otimes V_+) \dashrightarrow \mathbb{P}(V_+^\vee \otimes V_-).$$

Identifying $V_-^\vee \otimes V_+$ with the vector space of complex matrices 3×3, the map ψ' can be written as:

$$\psi'([(x_{ij})]) = [(m^{ij})]$$

where the m^{ij}'s are the quadratic equations given by the 2×2 minors of the matrix (x_{ij}). The indeterminacy loci of ψ' and its inverse are isomorphic to the set of classes of matrices of rank one, which are isomorphic to $\mathbb{P}^2 \times \mathbb{P}^2$, and to the inner fixed point components of the action on F_+. Let us also denote by d the determinant of (x_{ij}), which is a cubic homogeneous polynomial.

We may now describe the birational map induced by the $\mathbb{C}^*$-action on X:

$$\psi : F_- \to \mathbb{P}(T_{X,x'_+}) \simeq \mathbb{P}^9.$$

We consider the Plücker embedding of $F_- \simeq A_5(3)$ into $\mathbb{P}^{19}$ and we take a set of homogeneous coordinates in $\mathbb{P}^9$ of the form:

$$[x_{11} : x_{12} : \cdots : x_{33} : y]$$

where $y = 0$ is the equation of $\mathbb{P}(T_{F_-,x'_+})$ into $\mathbb{P}(T_{X,x'_+})$, and the restrictions of the x_{ij}'s to $\mathbb{P}(T_{F_-,x'_+})$ correspond to the standard homogeneous coordinates in $\mathbb{P}(M_{3\times 3}(\mathbb{C}))$. Then ψ^{-1} is given by:

$$\psi^{-1}([x_{11} : x_{12} : \cdots : x_{33} : y]) = [y^2 x_{11} : \cdots : y^2 x_{33} : y m^{11} : \cdots : y m^{33} : d] \in \mathbb{P}^{19},$$

and $\psi : F_- \subset \mathbb{P}^{19} \to \mathbb{P}^9$ is given by the linear projection of F_- from the projective tangent space of F_- at the point $[\ell]$ corresponding to the line ℓ passing by x_-. In other words, the map ψ^{-1} is a birational transformation of type $(3, 1)$.

We note also that this transformation is not special, but it can be factored as

$$\mathcal{G}X(0,1) \overset{\text{Atiyah flip}}{- - - - - - - - - ->} \mathcal{G}X(1,2)$$
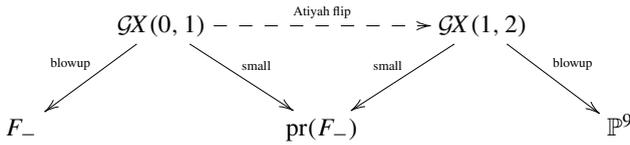
The varieties $\mathcal{G}X(0,1)$, $\mathcal{G}X(1,2)$ are geometric quotients of open sets of X (see [28, Remark 2.14]), and $\mathrm{pr}(F_-)$ is a semi-geometric quotient, isomorphic to the projection of $F_- \subset \mathbb{P}^{19}$ from the inner point $[\ell]$ into $\mathbb{P}^{18}$. The induced birational map $\mathbb{P}^9 \dashrightarrow \mathrm{pr}(F_-) \subset \mathbb{P}^{18}$ is given by:

$$[x_{11} : x_{12} : \cdots : x_{33} : y] \mapsto [yx_{11} : \cdots : yx_{33} : m^{11} : \cdots : m^{33}].$$

This is a birational transformation of type $(2,1)$ that can be resolved with a blowup, but its image is not smooth.

5.1 Special birational transformations of type $(2, 1)$

We will show now how the homogeneous cases in the list of special birational transformations of type $(2,1)$ appear as particular examples of the transformations induced by the $\mathbb{C}^*$-actions described in Setup 5.1. We will work in the following setting:

Setup 5.4 Let G be the adjoint group of a simple Lie algebra $\mathfrak{g}$ with Dynkin diagram $\mathcal{D}$; consider a Cartan subgroup $H \subset G$, and a co-character $\sigma_i : \mathrm{M}(H) \to \mathbb{Z}$, $i \in D$, inducing a $\mathbb{C}^*$-action with isolated sink $\{eP_i\}$ on $\mathcal{D}(i) = G/P_i$. We will assume that the co-character $\sigma_i : \mathrm{M}(H) \to \mathbb{Z}$ induces a short grading of $\mathfrak{g}$ (see the complete list of these characters in Table 1). We will consider a line ℓ in $\mathcal{D}(i)$ passing through the sink of the action, $x_- := eP_i \in c\mathcal{D}(i) = G/P_i$, and the fixed point $x_+ := r_i P_i \in \mathcal{D}(i)$, which belongs to a fixed point component of L_i-weight equal to 1 (as usual, L_i denotes the ample generator of $\mathrm{Pic}(\mathcal{D}(i))$).

Note that the first hypotheses allows us to claim that the action induced by σ_i is equalized on every rational homogeneous space of type $\mathcal{D}$. In particular it is equalized on $\mathcal{D}(i)$, $\mathcal{D}(J \cup \{i\})$ (for any choice of indices J) and, since $\ell \subset \mathcal{D}(i)$ is invariant by the action, also on a subvariety of $\mathcal{D}(J \cup \{i\})$ of the form

$$X := \mathcal{D}(J \cup \{i\}) \times_{\mathcal{D}(i)} \ell,$$

On the other hand, assuming that ℓ passes by the sink eP_i allows us to assert that the fiber F_- is a fixed point component of the action on X. In order to obtain a $\mathbb{C}^*$-action as in Setup 5.1 we need to choose the set J so that the action on the fiber $F_+ \subset X$ over x_+ has isolated source. The following statement guarantees that such a choice is always possible.

Lemma 5.5 *In the situation of Setup 5.4, the induced action of $\mathbb{C}^*$ on F_+ is given by a co-character σ_+ inducing a short grading of the Lie algebra $\mathfrak{g}^\perp$. In particular, we may choose $J \subset D \setminus \{i\}$ so that the induced $\mathbb{C}^*$-action on F^+ has isolated source. With the standard numbering of the nodes in (each of the connected components of) the Dynkin diagram $\mathcal{D}_i$ of $G^\perp$, the values of $\sigma_+ \in \mathrm{M}(H^\perp)^\vee$ and J are described in Table 4.*

Table 4 Induced $\mathbb{C}^*$-action of the fiber F_+

$\mathcal{D}$	$\mathcal{D}(i)$	$\mathcal{D}_i$	σ_+	J
A_n	$A_n(i)$	$A_{i-1} \sqcup A_{n-i}$	$\sigma_{i-1} + \sigma_1$	$\{1, n\}$
C_n	$C_n(n)$	A_{n-1}	σ_{n-1}	$\{1\}$
B_n	$B_n(1)$	B_{n-1}	σ_1	$\{2\}$
D_n	$D_n(1)$	D_{n-1}	σ_1	$\{2\}$
D_n	$D_n(n)$	A_{n-1}	σ_{n-2}	$\{2\}$
E_6	$E_6(1)$	D_5	σ_5	$\{2\}$
E_7	$E_7(7)$	E_6	σ_6	$\{1\}$

Fig. 3 A rational homogeneous variety with isolated sink, and the induced action on X

Proof The fiber F_+ over $x_+ = r_i P_i$ is a quotient of the subgroup $\mathrm{conj}_{r_i}(G^\perp) \subset G$ by the Borel subgroup $\mathrm{conj}_{r_i}(B^\perp)$. In order to study the action of $\mathbb{C}^*$ on F_+ we consider the Cartan subgroup $H^\perp := G^\perp \cap H$, whose character lattice is $\mathrm{M}(H^\perp) = \ker(\sigma_i) \subset \mathrm{M}(H)$. Then the induced action of $\mathbb{C}^*$ on F_+ is given by the co-character σ_+ obtained via the composition:

$$\mathrm{M}(\mathrm{conj}_{r_i}(H^\perp)) = r_i(\mathrm{M}(H^\perp)) \hookrightarrow \mathrm{M}(H) \xrightarrow{\sigma_i} \mathbb{Z}.$$

The image into $\mathrm{M}(H)$ of the base of positive simple roots associated with the choice of the Borel subgroup $\mathrm{conj}_{r_i}(B^\perp) \subset \mathrm{conj}_{r_i}(G^\perp)$ (which is $\mathrm{conj}_{r_i}(\Delta \setminus \{\alpha_i\})$) consists of all the roots of the form $r_i(\alpha_j)$, $j \neq i$. In all the cases described in Table 1 we have that

$$r_i(\alpha_j) = \begin{cases} \alpha_j + \alpha_i & \text{if } j \in \mathrm{ngb}(i), \\ \alpha_j & \text{otherwise,} \end{cases}$$

for $j \neq i$, so that the co-character σ_+ sends every positive simple root α_j to 1 if j neighbors i, and to zero otherwise. Finally a case by case argument provides the list of co-characters in Table 4. The value of the set J follows then by Proposition 3.16. □

With the choice of J as above, the situation of Setup 5.4 has been represented in Fig. 3. By applying now Proposition 5.2, we immediately get:

Corollary 5.6 *In the setting of Setup 5.4, let $J \subset D \backslash \{i\}$ be as in Table 4. Then the $\mathbb{C}^*$-action on X defines a birational transformation:*

$$\mathcal{D}_i(J) \overset{\psi}{\dashrightarrow} \mathbb{P}^{\dim(\mathcal{D}_i(J))}.$$

Let us describe some properties of the $\mathbb{C}^*$-action on X, and of its associated birational transformation. We start by listing in Table 5 the fixed point components of the action, that

Table 5 Fixed point components of the $\mathbb{C}^*$-action on X

$\mathcal{D}$	$\mathcal{D}(i)$	$Y_0 = \mathcal{D}_i(J)$	Y_1	Y_2	Y_3
A_n	$A_n(i)$	$A_{i-1}(1)$	$A_{i-2}(1)$	$A_{i-2}(1)$	$\star$
		$\times$	$\times$	$\sqcup$	
		$A_{n-i}(n-i)$	$A_{n-i-1}(n-i-1)$	$A_{n-i-1}(n-i-1)$	
C_n	$C_n(n)$	$A_{n-1}(1)$	$A_{n-2}(1)$	$\emptyset$	$\star$
B_n	$B_n(1)$	$B_{n-1}(1)$	$\star$	$B_{n-2}(1)$	$\star$
D_n	$D_n(1)$	$D_{n-1}(1)$	$\star$	$D_{n-2}(1)$	$\star$
D_n	$D_n(n)$	$A_{n-1}(2)$	$A_{n-3}(2)$	$A_1(1) \times A_{n-3}(1)$	$\star$
E_6	$E_6(1)$	$D_5(4)$	$A_4(1)$	$A_4(3)$	$\star$
E_7	$E_7(7)$	$E_6(1)$	$D_5(1)$	$D_5(5)$	$\star$

can be obtained out of Lemma 5.5 and Table 2. In the table we use the symbol $\star$ to denote the set consisting of an isolated closed point. We will consider the weights of the action with respect to a linearization on the very ample line bundle

$$L := L_i + \sum_{j \in J} L_j, \qquad (4)$$

and denote by Y_i the union of the fixed components of weight i. As usual, by choosing conveniently the linearization of the action on L, we may assume that $Y_0 = F_-$, and then a case by case inspection shows then that the source of the action, which is a point, is attained at weight 3.

One may easily compute the ranks of the positive and negative parts of the normal bundles of the inner fixed point components of the action, which allow to compute the exceptional locus of ψ and its inverse. As usual, let us denote by $X^\flat \to X$ the blowup of X along its source Y_3, with exceptional divisor $\mathbb{P}^{\dim(\mathcal{D}_i(J))}$; note that the sink $Y_0 = F_-$ of X is already a divisor.

Lemma 5.7 *With the above notation, $v^+(Y_1) = 1$, and $v^-(Y_2) = 1$. In particular, $\mathrm{Exc}(\psi) = \overline{X^+(Y_1)} \cap Y_0 \simeq Y_1$ and $\mathrm{Exc}(\psi^{-1}) = \overline{X^{\flat-}(Y_2)} \cap \mathbb{P}^{\dim(\mathcal{D}_i(J))} \simeq Y_2$.*

Proof First of all, we note that $\overline{X^-(Y_1)} = F_+$ is a divisor in X, hence, by Formula (1), $v^+(Y_1) = 1$. On the other hand, the fact that the source of X is a point easily implies that $v^-(Y_2) = 1$ (in the cases in which $Y_2 \neq \emptyset$). In fact, if this were not the case, we would have a positive dimensional family of curves joining a given point of Y_2 and the point Y_3. This contradicts the fact that, by 2.1.10, these curves are lines in the embedding given by the very ample line bundle L. By Lemma 2.2, this implies that

$$\mathrm{Exc}(\psi) \subseteq \overline{X^{\flat+}(Y_1)} \cap Y_0 \simeq Y_1, \quad \mathrm{Exc}(\psi^{-1}) \subseteq \overline{X^{\flat-}(Y_2)} \cap \mathbb{P}^{\dim(\mathcal{D}_i(J))} \simeq Y_2.$$

Note that, since $Y_- \subset X$ is a divisor, $X^{\flat+}(Y_1) \simeq X^+(Y_1)$,

Note also that in the case in which $\mathcal{D} = C_n$ this already says that ψ^{-1} is a morphism. On the other hand the variety F_+ is a projective space together with the equalized action having as fixed components a hyperplane Y_1 and a point Y_3; in particular we see that also $v^-(Y_1) = 1$ and so also ψ is an isomorphism.

Table 6 (Homogeneous) Special birational transformations of type (2, 1)

$\mathcal{D}_i(J) \overset{\psi}{\dashrightarrow} \mathbb{P}^{\dim(\mathcal{D}_i(J))}$		$\mathrm{Exc}(\psi)$	$\mathrm{Exc}(\psi^{-1})$
$\mathcal{D}_i(J)$	$\dim(\mathcal{D}_i(J))$		
$A_{i-1}(1)$	$n-1$	$A_{i-2}(1)$	$A_{i-2}(1)$
$\times$		$\times$	$\sqcup$
$A_{n-i}(n-i)$		$A_{n-i-1}(n-i-1)$	$A_{n-i-1}(n-i-1)$
$B_{n-1}(1)$	$2n-3$	$\star$	$B_{n-2}(1)$
$D_{n-1}(1)$	$2n-4$	$\star$	$D_{n-2}(1)$
$A_{n-1}(2)$	$2(n-2)$	$A_{n-3}(2)$	$A_1(1) \times A_{n-3}(1)$
$D_5(4)$	10	$A_4(1)$	$A_4(3)$
$E_6(1)$	16	$D_5(1)$	$D_5(5)$

In the rest of the cases ψ^{-1} contracts the divisor $\overline{X^{\flat-}(Y_1)} \cap \mathbb{P}^{\dim(\mathcal{D}_i(J))}$ to $\overline{X^-(Y_1)} \cap F_- \subset F_-$, so that this set, which is isomorphic to Y_1, is necessarily contained in $\mathrm{Exc}(\psi)$. A similar arguments determines $\mathrm{Exc}(\psi^{-1})$. □

Remark 5.8 As we have just seen, in the case of $C_n(n)$, ψ^{-1} and ψ are isomorphisms; in the rest of the cases we obtain examples of non-regular birational transformations. The cases in which $\mathrm{Pic}(Y_0) = 1$ are precisely the homogeneous examples in the list of special birational transformations of type (2, 1) classified by Fu and Hwang, [12]. In a nutshell, the maps ψ are birational linear projections of the minimal embeddings of the rational homogeneous spaces $\mathcal{D}_i(J)$ listed above. The exceptional loci of ψ and its inverse can be computed by using Lemma 5.7 in each case (see Table 6).

5.2 Linear sections

There are still two examples in the list of [12] that do not appear in Table 6. We will show now how to obtain some geometric realizations of them. These realizations will be $\mathbb{C}^*$-invariant subvarieties of the varieties $X = \mathcal{D}(J \cup \{i\}) \times_{\mathcal{D}(i)} \ell$ described above.

We start by considering the projection $\pi : X \to \ell$, and noting that this is a $\mathcal{D}_i(J)$-bundle, which, by Grothendieck's theorem, is determined by a cocycle $\theta \in \mathrm{H}^1(\ell, G^\perp)$, which is completely determined by a co-character $\sigma' : \mathrm{M}(H^\perp) \to \mathbb{Z}$. One may then easily check (see [23, Example 2.3]) that this cocycle is equal to the co-character σ_+ defining the restriction of the $\mathbb{C}^*$-action on X to the fiber F_+ (see Lemma 5.5). The cocycle θ defines then a principal $\mathbb{C}^*$-bundle $\mathcal{E} \to \ell$, so that X may be written as the variety $\mathcal{E} \times^{\mathbb{C}^*} \mathcal{D}_i(J)$, which is defined as the quotient of $\mathcal{E} \times^{\mathbb{C}^*} \mathcal{D}_i(J)$ by the equivalence relation:

$$(e, x) \sim (et, tx), \text{ for every } t \in \mathbb{C}^*.$$

At this point, for any $B \subset \mathcal{D}_i(J)$ smooth $\mathbb{C}^*$-invariant subvariety, we may define a subvariety

$$\overline{B} := \mathcal{E} \times^{\mathbb{C}^*} Y \subset \mathcal{E} \times^{\mathbb{C}^*} \mathcal{D}_i(J) = X,$$

which is a B-bundle over $\mathbb{P}^1$, and inherits the action of $\mathbb{C}^*$. The fixed point locus of the $\mathbb{C}^*$-action on $\overline{B}$ will be $X^{\mathbb{C}^*} \cap \overline{B}$. In particular, if $B \subset \mathcal{D}_i(J)$ contains the source of the

Table 7 The normal bundle $\mathcal{N} := \mathcal{N}_{Y_1|\mathcal{D}_i(J)}$

$\mathcal{D}(i)$	$A_n(i)$	$C_n(n)$	$D_n(n)$	$E_6(1)$	$E_7(7)$
$\mathcal{D}_i(J)$	$A_{i-1}(1) \times A_{n-i}(n-i)$	$A_{n-1}(1)$	$A_{n-1}(2)$	$D_5(4)$	$E_6(1)$
Y_1	$A_{i-2}(1) \times A_{n-i-1}(n-i-1)$	$A_{n-2}(1)$	$A_{n-3}(2)$	$A_4(1)$	$D_5(1)$
$\mathcal{N}$	$\mathcal{O}(1,0) \oplus \mathcal{O}(0,1)$	$\mathcal{O}(1)$	$\mathcal{Q}^{\oplus 2}$	$\bigwedge^2 T(-2)$	$\mathcal{S}$

$\mathbb{C}^*$-action defined by σ^+, $\overline{B}$ will then be a geometric realization of a birational map

$$B \dashrightarrow \mathbb{P}^{\dim(B)}.$$

Furthermore, in order to avoid this map to be an isomorphism, we will require B to meet both $Y_1, Y_2 \subset X^{\mathbb{C}^*}$.

The two non-homogeneous examples in the list of [12, Theorem 1.1] can be achieved by considering complete intersections defined by general elements in the linear system $H^0(\mathcal{D}_i(J), L)_1$. By definition they contain Y_1 and Y_3, and are smooth at its intersection with Y_2 by Bertini's theorem. Furthermore, such a section is smooth at Y_3, since $\mathbb{P}(T^\vee_{\mathcal{D}_i(J), Y_3})$ can be naturally identified with $\langle Y_2 \rangle$, which $\langle B \rangle$ meets in dimension $\dim(B) - 1$, by definition.

Consequentely, since B supports a $\mathbb{C}^*$-action, in order to guarantee its smoothness it is enough to prove that it is smooth at Y_1. For this purpose we use [3, Proposition 1.7.5] together with Proposition 2.5 and Remark 2.6, to claim that a general B of this kind is smooth if $\dim(Y_1) < \dim(\mathcal{D}_i(J))/2$, or $\dim(Y_1) = \dim(\mathcal{D}_i(J))/2$ and the top Chern class of $\mathcal{N}_{Y_1|\mathcal{D}_i(J)} \otimes L$ is zero. The following table contains the description of the normal bundles of Y_1 in $\mathcal{D}_i(J)$ in each case:

This shows that this construction works, and provides examples different from the ones we have already described in the following two cases, which are the ones included in [12, Theorem 1.1]:

(i) $\mathcal{D}_i(J) = A_4(2)$, and $B \subset \mathcal{D}_i(J)$ is a general linear complete intersection of codimension at most two.

(ii) $\mathcal{D}_i(J) = D_5(4)$, and $B \subset \mathcal{D}_i(J)$ is a general linear complete intersection of codimension at most three.

Remark 5.9 The existence of a general linear complete intersection of codimension at most three of (ii) is guaranteed by the existence of an exact sequence of vector bundles in $\mathbb{P}^4$:

$$0 \to \mathcal{O}_{\mathbb{P}^4} \longrightarrow \bigwedge^2 \Omega_{\mathbb{P}^4}(3) \longrightarrow \mathcal{T} \to 0$$

where $\mathcal{T}$ denotes the so-called Tango bundle in $\mathbb{P}^4$ (cf. [17]).

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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